Fluid flow between active elastic plates

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Abstract

We propose a model of a channel flow between actively moving elastic plates as a framework for blood flow in active human arteries. The main difference from extant models is that our model is autonomous. It is a nonlinear partial differential equation governing the deformation of the plates involving 6-th order spatial derivative. The PDE has a similar structure to the equation we proposed earlier to simulate another active system—spinning combustion front.

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1 Introduction

The overwhelming majority of mathematical models of arterial blood flow treat arteries as passive material (see, e.g., [1, 2]). A popular approximation of the flow–artery interaction is a proportional increase of the artery's cross-section to an increase of the flow pressure [3]:

$$p - p_0 \sim \sqrt{A} - \sqrt{A_0}\,,$$

where p_0 and A_0 are the reference pressure and cross-sectional area respectively. The works focused on pulse modelling [4] interpret pulses as a passive response to time-periodic boundary condition imposed at the artery's inlet.

However, the arteries have muscles which actively push the blood. In some models [5] the arteries do actively exert pressure. However, such models are non-autonomous, that is the active component of the pressure, despite in place, is introduced by an explicit function of time and coordinate. Consequently, the pulses take place because this function already has the form of pulses. Effectively it represents some "outside controller" dictating the system's behaviour.

In this note we propose an approach which leads to an autonomous model of the flow between hypothetically active elastic walls. At this stage we do not claim direct connection with biology, but aim to demonstrate the possibility of constructing such a model.

2 Pulses as auto-waves

The two crucial factors governing the blood flow are pressure gradient and viscous resistance from the walls. Because of the viscosity our model must be dissipative; consequently, the pulses must be auto-waves, that is self-sustained dissipative structures. In distinction from conservative waves such as those on water surface, the auto-waves propagate with unique speed and amplitude.

Previously we formulated an auto-wave model for spinning combustion fronts [6]:

$$\frac{\partial F}{\partial t} = \frac{\partial^6 F}{\partial x^6} - \frac{\partial}{\partial x}\left[\left(\frac{\partial F}{\partial x}\right)^3\right] + \left(\frac{\partial F}{\partial x}\right)^4 . \tag{1}$$

$F(x,t)$ stands for the position of the front subject to periodic boundary conditions. The front is thought to be a surface (line in 1D case) separating hot burned products from cold fresh composition. It is important to note that the combustion system is active and dissipative: its activity is due to heat

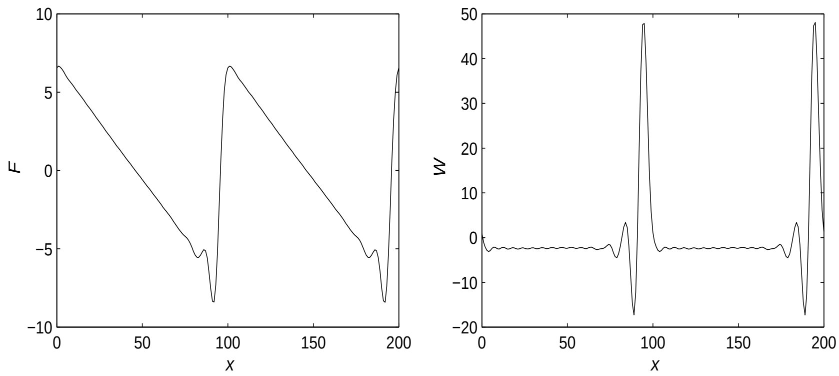

Figure 1: A train of kink-shaped (F) and pulse-shaped (W) auto-waves. The F-wave moves upwards and to the left; the W-wave moves horizontally to the left.

generation in a chemical reaction, and dissipation is due to heat conductivity. These features place the combustion fronts and the arterial blood flow into the same category of active dissipative systems.

Upon differentiating on x, equation (1) can be rewritten in terms of the derivative $\partial F/\partial x \equiv W$:

$$\frac{\partial W}{\partial t} = \frac{\partial^6 W}{\partial x^6} - \frac{\partial^2}{\partial x^2}\left(W^3\right) + \frac{\partial}{\partial x}\left(W^4\right) . \tag{2}$$

A spatially uniform solution F =const of (1) and corresponding uniform solution $W \equiv 0$ of (2) are stable under small perturbations as the linearised equation, $\partial F/\partial t = \partial^6 F/\partial x^6$, is purely dissipative.

For brevity, from now on we will use primes/Roman numerals to denote derivatives on x.

Explaining the mechanism of (1), denote typical amplitude of the perturbation of the uniform solution by $\Delta F > 0$ and typical spatial scale of the perturbation by $\Delta x > 0$. If the perturbation is sufficiently large, it grows because of the pumping effect of the (3-rd order) nonlinear source, $-\left(F'^{\,3}\right)' = \left(-3F'^{\,2}\right) F''$. Effectively, this is an anti-diffusion term with the negative multiplier in front of F''. Evaluate the terms by the order of magnitude in absolute value. We have

$$\left(F'^{\,3}\right)' \sim (\Delta F)^3/(\Delta x)^4 \,.$$

As the perturbation grows, the higher-order nonlinearity comes into play:

$$(F')^4 \sim (\Delta F)^4/(\Delta x)^4 \,.$$

It acts so that certain sections of the F profile become steeper (the W profile locally surges in amplitude) making Δx small. On those sections the dissipation prevails because it is of higher order in Δx:

$$F^{VI} \sim \Delta F/(\Delta x)^6 \,.$$

As a result, the steep sections are smoothed out and the perturbation, instead of turning into a singularity, becomes a smooth self-sustained dissipation structure. It has the form of a train of pulses shown in Fig. 1. Each individual pulse is essentially a stable auto-soliton with the amplitude and speed controlled by the dynamic equation, not initial condition.

We refer to an analogy with damping in a viscous flow between elastic walls. It is known that dissipation for such a flow is represented by 6-th order spatial derivative [7, 8]. This brings an idea to extend the model [7] by source terms representing active motion of the elastic walls in order to make the model structurally similar to (2).

3 Lubrication model for the flow

Consider the flow between elastic walls, assuming symmetry with respect to the middle plane, $z = 0$; hence it will suffice to analyse only half of the flow, $0 < z < H(t)$.

Adopting the lubrication theory [7] we equate the pressure gradient to viscous friction:

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{\eta}\frac{\partial p}{\partial x}\,, \tag{3}$$

where x and z are the coordinates along and across the flow respectively, $v(x, z, t)$ is the flow velocity in the x direction, $p(x, t)$ is the pressure, and η is the viscosity.

The pressure is assumed z-independent, so that integrating (3) on z gives

$$v = \frac{1}{2\eta}\frac{\partial p}{\partial x}\left(z^2 - H^2\right) + v(x, H, t)\,. \tag{4}$$

The mass flux is

$$Q = \int_0^H v\,dz = -\frac{H^3}{3\eta}\frac{\partial p}{\partial x} + v(x, H, t)\,H\,. \tag{5}$$

Define the transversal displacement of the wall, $w(x, t)$, from the neutral position, H_0, by

$$H = H_0 + w\,. \tag{6}$$

Then the continuity equation can be written in the form

$$\frac{\partial w}{\partial t} + \frac{\partial Q}{\partial x} = 0\,. \tag{7}$$

Substituting (5) into (7) gives

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{H^3}{3\eta}\frac{\partial p}{\partial x} - v(x,H,t)\,H\right]. \tag{8}$$

4 Passive elastic walls

Equation (8) links the displacement of the flow boundary, coinciding with the wall, to the flow pressure. The elasticity theory [9, 10] provides the reverse link from the pressure to the displacement:

$$p = D\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right), \tag{9}$$

where

$$N = \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]. \tag{10}$$

In (9) and (10) $u(x,t)$ is the wall's displacement along the flow, D is the flexural rigidity of the wall, E is Young's modulus, h is the thickness of the wall, ν is Poisson's ratio, and N is the force caused by the displacements.

Substituting (10) and (9) into (8), and using the no-slip boundary condition,

$$v(x,H,t) = \frac{\partial u}{\partial t}\,,$$

we obtain

$$\frac{\partial w}{\partial t} = \frac{D}{3\eta}\left(H^3 w^V\right)' - \frac{Eh}{6\eta(1-\nu^2)}\left[H^3\left(w'^{\,3}\right)''\right]' - \frac{Eh}{3\eta(1-\nu^2)}\left[H^3(u'w')''\right]' - \left(\frac{\partial u}{\partial t}H\right)'. \tag{11}$$

The shear stress in the fluid is $T = \eta\partial v/\partial z$, therefore on the boundary, $z = H$, using (4),

$$T = p'\,H\,. \tag{12}$$

This shear stress must be equal to the shear stress produced by the wall,

$$T = N'\,. \tag{13}$$

Equating (12) and (13) with the use of (10), we have

$$\frac{E}{1-\nu^2}\left[u'' + \frac{1}{2}\left(w'^2\right)'\right] = p'\,H\,. \tag{14}$$

The three equations (11), (14) and (9) form the closed system with respect to the three unknown functions $w(x,t)$, $u(x,t)$ and $p(x,t)$. Observe that the pressure can be trivially eliminated.

Now solve the system for small-amplitude perturbations. They satisfy the linearised equations

$$\begin{aligned} &\frac{E}{1-\nu^2}u'' = DH_0 w^V\,, \\ &\frac{\partial w}{\partial t} = \frac{H_0^3}{3\eta}Dw^{VI} - \frac{\partial u'}{\partial t}H_0\,. \end{aligned} \tag{15}$$

Looking for solutions

$$w = A(t)\sin(kx)\,, \quad u = B(t)\cos(kx)\,,$$

we obtain

$$\begin{aligned} &\frac{E}{1-\nu^2}B = -DH_0k^3A\,, \\ &\frac{dA}{dt} = -\frac{H_0^3}{3\eta}Dk^6A + H_0k\frac{dB}{dt}\,. \end{aligned}$$

It is easy to deduce that the amplitudes decay:

$$A(t) \sim B(t) \sim \exp\left[-\frac{H_0^3DEk^6}{3\eta E + 3\eta DH_0^2(1-\nu^2)k^4}\,t\right]\,.$$

5 Active elastic walls

We intend to construct a model assuming that the walls actively move. Suppose that, when deflecting from the neutral position, they exert extra pressure relative to (9):

$$p = Dw^{IV} - (Nw')' + p_1\,, \tag{16}$$

where p_1 depends on w. We postulate that p_1 is proportional to the 4-th power of the vertical displacement (this will lead to an analogy with the term W^3 in (2)):

$$p_1 = -\alpha w^4\,, \quad \alpha > 0\,. \tag{17}$$

Further, we suppose that the walls actively move along the flow, thereby producing extra shear stress relative to (14). We postulate that the wall's motion along the flow, represented by the displacement u and velocity $\partial u/\partial t$, is coupled with w. Specifically, the H-weighted velocity along the flow, $H\partial u/\partial t$, combined with the other u-containing term in (11), is proportional to the 5-th power of w (this will lead to an analogy with the term W^4 in (2)):

$$-\frac{Eh}{3\eta(1-\nu^2)}H^3(u'w')'' - \frac{\partial u}{\partial t}H = \beta w^5\,, \quad \beta > 0\,. \tag{18}$$

From physical viewpoint this relation should reflect an extra (active) shear stress, which we denote T_1. It must be such that the condition of continuity of stress on the boundary is satisfied:

$$T_1 + \frac{E}{1-\nu^2}\left[u'' + \frac{1}{2}\left(w'^{\,2}\right)'\right] = p'\,H\,, \tag{19}$$

where the pressure p is represented by (16).

While we cannot justify specific powers used in (17) and (18), these laws express a biologically motivated property of larger active response from the walls to larger deformations w.

Under the assumptions (16), (17) and (18), the equation (11) governing the dynamics of the vertical displacement becomes u-independent:

$$\frac{\partial w}{\partial t} = \frac{D}{3\eta}\left(H^3 w^V\right)' - \frac{Eh}{6\eta(1-\nu^2)}\left[H^3\left(w'^{\,3}\right)''\right]'$$
$$-\frac{1}{3\eta}\alpha\left[H^3\left(w^4\right)'\right]' + \beta\left(w^5\right)'\,. \tag{20}$$

We arrived at the closed system. The procedure of finding solution is as follows. The function $w(x,t)$ is obtained from (20) under, say, periodic boundary conditions. Then $u(x,t)$ can be found from (18) and the pressure from (16), (17). The extra shear stress $T_1(x,t)$ and total stress are obtained from (19).

Let us analyse the mechanism of (20) in the same way as we did for the combustion equations. Transform (20) into the equation with respect to f defined by $w = f'$:

$$\frac{\partial f}{\partial t} = \frac{D}{3\eta}H^3 f^{VI} - \frac{Eh}{6\eta(1-\nu^2)}H^3\left[(f'')^3\right]'' - \frac{1}{3\eta}\alpha H^3\left[(f')^4\right]' + \beta(f')^5\,. \tag{21}$$

Note that the first two terms in the right-hand side come with the classical theory and, therefore, are dissipative.

Comparing with (1), mark the higher order of nonlinearity of the source (4-th instead of 3-rd); this is necessary to overpower the 3-rd order classical term in (21). Accordingly, the last term in (21) has even higher, 5-th, order of nonlinearity, required to counterbalance the source as the perturbation grows.

Evaluate the terms by the order of magnitude in absolute value. The source,

$$\frac{1}{3\eta}\alpha H^3 \left[(f')^4\right]' \sim (f')^3 f'' \sim (\Delta f)^4/(\Delta x)^5 \,.$$

As the amplitude of the perturbation increases, the 5-th order nonlinearity comes into play:

$$\beta(f')^5 \sim (\Delta f)^5/(\Delta x)^5 \,.$$

It steepens the profile of f at certain sections where the dissipation eventually prevails due to higher order in Δx:

$$\frac{D}{3\eta} H^3 f^{VI} \sim \Delta f/(\Delta x)^6 \,.$$

This mechanism is similar to that of (1), so pulse waves should exist in the hydro-elastic model.

6 Discussion

Our model meets the two major requirements: autonomity and capacity to produce pulses as auto-waves. The pulses are a corollary of the assumption that the walls actively push the fluid against viscous forces.

A living organism is a self-organised system. The property of our model to exhibit, in contrast to all other biofluid models, a self-organised behaviour is new in pulse modelling. It remains to be seen whether or not this approach can lead to practical modelling. However, we predict, based on the classical mechanical laws employed, that, if the walls are coupled with the flow exactly as prescribed by (16) and (18), then the pulses would occur. Theoretically a veryfying experiment can be set up, although it may not be technically simple.

An important issue is self-consistency of the model. It should guarantee that overall displacement of the arteries over some period of time, τ, is zero. This condition can be written as

$$\int_\tau \frac{\partial u}{\partial t}\, dt = 0 \,, \quad \int_\tau \frac{\partial w}{\partial t}\, dt = 0 \,. \tag{22}$$

In the proposed model the zero value of the *transversal* displacement, w, is guaranteed by periodicity of w. Indeed, the wave solution depends on $x/c + t$, where c is the wave speed. Integrating equation (20) on t over the wave period, or many periods, is equivalent to integrating on x/c. This gives zero because all the terms in the equation's right-hand side are full derivatives and the expressions under them are periodic.

Periodicity of the displacement *along the flow*, u, is more difficult to achieve. In view of (18), the first condition (22) becomes

$$\int_\tau \left[\frac{Eh}{3\eta(1-\nu^2)} H^2 (u'w')'' + \frac{\beta}{H} w^5 \right] dt = 0\,. \tag{23}$$

At this stage of the modelling we use the fact that the value of parameter β is at our disposal; it can be tuned so that (23) is satisfied.

Note that, unlike the combustion model (2), the hydro-elastic model (20) generates only pulses travelling to the left. This is caused by the asymmetry: the source term acts as a source only on sections with positive slope, $f' > 0$, because on those sections, $-(f')^3 f''$ is effectively an inti-diffusion with negative coefficient, $-(f')^3 < 0$, in front of f'', whereas on negative slopes, $f' < 0$, it acts as diffusion. Yet, we are satisfied with this property as it implies that our quasi-artery "knows" the direction to "heart", which would be to the right in Fig. 1, and therefore generates pulses travelling in the opposite direction, to the left.

Our next remark goes to the expressions adopted for the extra pressure (17) and extra shear stress (18)–(19). They are empirical and contain the unknown coefficients α and β. The fact that we have only two of them is rather positive because for complex systems such as biological, the more details one takes into account the more coefficients need to be involved. Always, values of such coefficients are difficult to determine.

7 Conclusion

We presented an autonomous model, without directly linking to biology at this stage, of a hypothetical flow between active walls. Lubrication theory is used for the flow, and the walls are supposed to actively exert pressure and shear stress. The analogy with the combustion front equations indicates that the model should have auto-wave solutions in the form of pulses.

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