# Estimation in multiple regression model with elliptically contoured errors under MLINEX loss

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### Abstract

This paper considers estimation of the regression vector of the multiple regression model with elliptically symmetric contoured errors. The generalized least square (GLS), restricted GLS and preliminary test (PT) estimators for regression parameter vector are obtained. The performances of the estimators are studied under multiparameter linear exponential loss function (MLINEX), and the dominance order of the estimators are given.

*Key words and phrases:* Elliptically contoured distribution, GLS estimator, Preliminary test estimator, MLINEX loss function, and Moment generating function.

## 1    Introduction

Multiple regression model is arguably the most widely used statistical tool applied in almost every discipline of modern time. The estimation of parameters of the multiple regression model is a common interest to many users. Often the properties of the

estimators are of prime concern. Selection of any specific statistical property of any estimator often depends on the objective of the study. The choice of any particular estimator may very well be determined by the aim of the end users. It is well known that the ordinary least squares estimators are best linear unbiased. However, if the objective of any study is to minimize some specific risk function then other types of estimators perform better than the ordinary least squares estimator. In recent years there has been growing interest to estimate parameters under different loss functions (cf. Saleh 2006). This paper proposes estimators under the multiparameter linex (MLINEX) loss function. It provides the dominance order of the generalized least squares (GLS), restricted GLS (RGLS) and preliminary test (PT) estimators based on the MLINEX loss.

Consider the multiple regression model

$$y \;=\; X\beta + e, \tag{1.1}$$

where $y$ is an $(n \times 1)$ vector of observations, $X$ is an $(n \times p)$ matrix of full rank $p$, $\beta = (\beta_1, \cdots, \beta_p)'$ is the vector of $p$ parameters and $e = (e_1, \cdots, e_n)'$ is the $n \times 1$ error vector distributed as elliptically contoured distribution (ECD).

The ECD of the error vector $e$ can be denoted by $e \sim E_n(0, \sigma^2 V, g)$, and its density function is written as

$$f(e) \;=\; K_n |\sigma^2 V|^{-1/2} g\left(\frac{e'V^{-1}e}{2\sigma^2}\right), \tag{1.2}$$

where $K_n$ is the normalizing constant, $V$ is known positive definite (p.d.) scale matrix, $\sigma^2 > 0$ is unknown and $g$ is an unknown nonnegative real valued function. Also the characteristic function of $e$ is as follows

$$\Phi_e(t) \;=\; \Psi\left(\frac{t'Vt}{2\sigma^2}\right), \tag{1.3}$$

for $\Psi : [0, \infty) \to \Re$; where $g$ and $\Psi$ determine each other for any specified member of the family of distributions. See Fang et al. (1990) and Gupta and Varga (1993) for more details. Some of the well known members of the multivariate spherically/elliptically contoured family of distributions are the multivariate normal, Kotz Type, Pearson Type VII, Multivariate t, Multivariate Cauchy, Pearson Type II, Logistic, Multivariate Bassel, Scale mixture and Stable laws.

Assume that in addition to the sample information $y$ in the model (1.1), that information also exists in the form of $q$ independent linear hypotheses about the

unknown vector parameter $\beta$ where $q \leq p$. These general restriction can be shown as $H\beta = h$, where $H$ is a $q \times p$ known hypothesis design matrix of rank $q$ and $h$ is a $q \times 1$ vector of prespecified hypothetical values.

In this paper, the generalized least squares (GLS), restricted generalized least squares (RGLS) under the constraint $H\beta = h$ and preliminary test (PT) estimators are obtained for the regression vector parameter $\beta$ when the p.d. scale matrix $V$ is known and $\sigma^2$ is unknown in the model (1.2). Also, under the multivariate linear exponential (MLINEX) loss function, dominance orders of the three generalized estimators have been given. Unlike the quadratic error loss function, the linex loss function assigns unequal weights to the underestimation and overestimation by introducing a shape parameter. For small values of the shape parameter the linex loss function is approximately symmetric and not much different from the quadratic loss function. The linex loss function is more general than the quadratic error loss function as the latter is a special case of the former.

## 2    Estimation of Regression Vector

Given classical conditions, it is well known that for known p.d. scale matrix $V$, the GLS estimator of $\beta$ is

$$\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}y. \tag{2.1}$$

Also the least squares estimator of $\sigma^2$ is

$$\hat{\sigma}^2 = \frac{(y - X\hat{\beta})'V^{-1}(y - X\hat{\beta})}{n}. \tag{2.2}$$

It follows that

$$S^2 = \frac{n\hat{\sigma}^2}{n - p} \tag{2.3}$$

is an unbiased estimator of $\sigma_e^2 = -2\sigma^2\Psi'(0)$.

Obtaining GLS estimator of $\beta$ under the constraint $H_0 : H\beta = h$, using method of Lagrangian multipliers the RGLS estimator of $\beta$, subject to the linear restrictions $H_0 : H\beta = h$ as $\widetilde{\beta}$, is

$$\widetilde{\beta} = \hat{\beta} - (X'V^{-1}X)^{-1}H'[H(X'V^{-1}X)^{-1}H']^{-1}(H\hat{\beta} - h). \tag{2.4}$$

See Ravishanker and Dey (2002) for detailed related discussions in this approach. Consequently

$$S_R^2 = \frac{(y - \widetilde{\beta})'V^{-1}(y - \widetilde{\beta})}{n - p + q} \tag{2.5}$$

is an unbiased estimator for $\sigma_e^2$ under $H\beta = h$.

Set $G_1 = (X'V^{-1}X)^{-1}$ and $G_2 = (HG_1H')^{-1}$, then $E(\widetilde{\beta}) = \beta - G_1H'G_2(H\beta - h)$; which is equal to $\beta$ under $H\beta = h$.

In order to define the preliminary test estimator of $\beta$, first we need obtain the test statistic for testing $H_0 : H\beta = h$. So, we represent the following Theorem from Chu (1973) which is used in obtaining the test statistic.

**Theorem 1**. If $z$ is an $n$-dimensional elliptically contoured random vector with mean equal to $\mu$, scale matrix $\sigma^2 V$ and density function $h(z)$, then, under some regularity conditions, there exists a scalar function $w(t)$ defined on $(0, \infty)$ such that

$$h(z) = \int_0^\infty w(t)\phi_{z|t}(\mu, t^{-1}\sigma^2 V)dt, \tag{2.6}$$

where $\phi_{z|t}(\mu, t^{-1}\sigma^2 V)$ denotes the density function of $N_n(\mu, t^{-1}\sigma^2 V)$, and

$$w(t) = (2\pi)^{n/2}|\sigma^2 V|^{1/2}t^{-n/2}L^{-1}(f(s)), \tag{2.7}$$

in which $L^{-1}(f(s))$ denotes the inverse Laplace transform of $f(s)$ with $f(s) = h(z)$ when $s = \frac{z'V^{-1}z}{2\sigma^2}$. For details on the properties of Laplace transform and its inverse see Gradshteyn and Ryzhik (1980).

On integrating $h(z)$ over $\Re^n$, $w(t)$ integrates to 1. Thus for nonnegative function $w(t)$, it is a density. Some explicit representations of $h(.)$ and $w(t)$ for $s = x'V^{-1}x/2$ are given in the Table below from Cheong (1999).

| Distribution | $h(s)$ | $w(t)$ |
|---|---|---|
| Multivariate Normal | $\frac{|V|^{-1/2}e^{-s}}{(2\pi)^{n/2}}$ | $\delta(t)$ |
| Multivariate Pearson Type $VII$ | $\frac{\Gamma(m)|V|^{-1/2}}{(q\pi)^{n/2}\Gamma(m-n/2)}$ $\times(1 + 2s/q)^{-m}$ | $\frac{t^{m-n/2-1}e^{-qt/2}}{(q/2)^{n/2-m}\Gamma(m-n/2)}$ |
| Multivariate Student-t with $\nu$ d.f. | $\frac{\nu^{\nu/2}\Gamma((\nu+n)/2)|V|^{-1/2}}{\pi^{n/2}\Gamma(\nu/2)}$ $\times(\nu + 2s)^{-(\nu+n)/2}$ | $\frac{\nu(\nu t/2)^{\nu/2-1}e^{-\nu t/2}}{2\Gamma(\nu/2)}$ |
| Generalized Slash | $\frac{\nu s^{-n/2-\nu}|V|^{-1/2}}{(2\pi)^{n/2}}$ $\times[\Gamma(n/2 + \nu) - \Gamma(n/2 + \nu, s)]$ | $\nu t^{\nu-1}$ |

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where $\delta(.)$ is the unit impulse function or the Dirac delta function with the following property

$$\int_0^\infty \delta(t)dt \quad = \quad 1. \tag{2.8}$$

See Spiegel (1989) for more details.

The following Theorem gives the test statistic to test $H\beta = h$ and the sampling distribution of the statistic.

**Theorem 2**. Let $\wp$ is a set in the space of $(\beta, V)$, $V > 0$, such that if $(\beta, V) \in \wp$ then $(\beta, cV) \in \wp$ for all $c > 0$. Assume in the model (1.1), $e \sim E_n(0, \sigma^2 V, g)$. Further suppose $g$ is such that $g(y'y)$ is a density in $\Re^n$ and $t^{n/2}g(t)$ has one finite positive maximum $t_g$. Then the test statistic for testing the hypothesis $H_0 : H\beta = h$ is

$$\zeta \quad = \quad \frac{(H\hat{\beta} - h)'G_2(H\hat{\beta} - h)}{qS^2}, \tag{2.9}$$

which has the following density function:

$$
\begin{aligned}
f(\zeta) \quad = \quad & \sum_{r=0}^\infty (\frac{q}{n-p})^{q/2+r} \frac{2^{n-1}\pi^{n/2}|\sigma^2 V|^{1/2}\Gamma(n/2+r-1)\Gamma(\frac{q+n-p}{2}+r)}{\Delta^{n/2-1}\Gamma(r+1)\Gamma(q/2+r)\Gamma(\frac{n-p}{2})} \\
& \times \frac{\zeta^{q/2+r-1}L^{-1}\left(h(s)\right)}{(1+\frac{q}{n-p}\zeta)^{\frac{q+n-p}{2}+r}},
\end{aligned}
\tag{2.10}
$$

where $h(s) = f(e)$ for $s = \frac{e'V^{-1}e}{2\sigma^2}$, and $f(e)$ is given by (1.2).

**Proof**. Let $\omega = \{\beta : \beta \in E_p, \ H\beta = h, \ \sigma^2 > 0\}$ and $\Omega = \{\beta : \beta \in E_p, \ \sigma^2 > 0\}$. Then using Corollary 1 from Anderson et al. (1986) the likelihood ratio test (LRT) criterion for testing $H_0 : H\beta = h$ is

$$
\begin{aligned}
\zeta \quad &= \quad \frac{|\hat{\sigma}^2 V|^{1/2}}{|\widetilde{\sigma}^2 V|^{1/2}} \\
&= \quad \frac{(H\hat{\beta} - h)'G_2(H\hat{\beta} - h)}{qS^2}.
\end{aligned}
\tag{2.11}
$$

In order to obtain the distribution of $\zeta$, first assume in the linear model (1.1), $e \sim N_n(0, \sigma^2 t^{-1}V)$, then direct computations lead to the test statistic $\zeta$ follows the non-central $F$ distribution with $q$ and $(n-p)$ degrees of freedom and non-centrality parameter $t\Delta = \theta/\sigma^2$ where $\theta = (H\beta-h)'G_2(H\beta-h)$, denoted by $F_{q,n-p,t\Delta}(\cdot)$. Then applying Theorem 1 for the case in which $e \sim E_n(0, \sigma^2 V, g)$ in the model (1.1), using

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weight function $w(t)$ given by (2.7), and Theorem 1.27 from Rudin (1987), we can obtain the density function of $\zeta$ as follows

$$
\begin{aligned}
f_{q,n-p}(\zeta; \Delta, \sigma^2 V) &= \int_0^\infty w(t) F_{q,n-p,t\Delta}(\zeta)\, dt \\
&= \int_0^\infty (2\pi)^{n/2} |\sigma^2 V|^{1/2} t^{-n/2} L^{-1}(h(s)) \\
&\quad \times \sum_{r=0}^\infty \frac{[q/(n-p)]^{q/2+r} \Gamma(\frac{q+n-p}{2}+r)(\frac{t\Delta}{2})^r e^{-t\Delta/2} \zeta^{q/2+r-1}}{\Gamma(r+1)\Gamma(q/2+r)\Gamma(\frac{n-p}{2})(1+\frac{q}{n-p}\zeta)^{\frac{q+n-p}{2}+r}}\, dt \\
&= \sum_{r=0}^\infty \left(\frac{q}{n-p}\right)^{q/2+r} \frac{2^{n-1}\pi^{n/2}|\sigma^2 V|^{1/2}\Gamma(n/2+r-1)\Gamma(\frac{q+n-p}{2}+r)}{\Delta^{n/2-1}\Gamma(r+1)\Gamma(q/2+r)\Gamma(\frac{n-p}{2})} \\
&\quad \times \frac{\zeta^{q/2+r-1} L^{-1}(h(s))}{(1+\frac{q}{n-p}\zeta)^{\frac{q+n-p}{2}+r}}.
\end{aligned} \tag{2.12}
$$

Now following Bancroft (1944), we define the preliminary test estimator (PTE) of $\beta$ as a convex combination of $\hat{\beta}$ and $\widetilde{\beta}$ by

$$
\hat{\beta}^{PT} = \widetilde{\beta} + [1 - I(\zeta \le F_\alpha)](\hat{\beta} - \widetilde{\beta}), \tag{2.13}
$$

where $I(A)$ is the indicator of the set $A$ and $F_\alpha$ is the upper $100\alpha$ percentile of $F_{q,n-p,0}$.

The PTE has the disadvantage that it depends on $\alpha$ $(0 < \alpha < 1)$, the level of significance and also it yields the extreme results, namely $\hat{\beta}$ or $\widetilde{\beta}$ depending on the outcome of the test.

# 3 Bias and Risk Analysis

In such conditions where negative bias and positive bias of the same magnitude have different importance, symmetric loss functions are improper. Also in practical situations, overestimating and underestimating of the same magnitude often have different economic and physical implications and the appropriate loss function is asymmetric. In these conditions for an asymmetric loss function, consider the following multiparameter linear exponential (MLINEX) loss function with the scale parameter $b$ and the $p$-vector shape parameter $a$

$$
L(\beta^*, \beta) = b\{e^{a'(\beta^* - \beta)} - a'(\beta^* - \beta) - 1\}, \tag{3.1}
$$

where $\beta^*$ is an estimator of vector parameter $\beta$.

For more details on the properties of the loss function under study, see Zellner (1986).

Several authors have considered asymmetric loss functions in their studies; for more discussion in this area with different approaches see Ferguson (1967), Varian (1975), Parsian (1990) and Parsian and Kirmani (2002).

In this section, we evaluate the bias and risk function of the three different underlying estimators using the risk function associated with (3.1).

Let $\mu = G_1 H' G_2(h - H\beta)$ then it follows the bias of LS and RLS estimators can be given respectively by

$$
\begin{aligned}
b_1 &= E[\hat{\beta} - \beta] = 0, \\
b_2 &= E[\widetilde{\beta} - \beta] = \mu.
\end{aligned}
\tag{3.2}
$$

By (2.2) and (2.13)

$$
\begin{aligned}
b_3 &= E[\hat{\beta}^{PT} - \beta] \\
&= E[\hat{\beta} - \beta] - E[I(\zeta \leq F_\alpha)(\hat{\beta} - \widetilde{\beta})] \\
&= -G_1 H' G_2^{1/2} E[I(\zeta \leq F_\alpha) G_2^{1/2}(H\hat{\beta} - h)].
\end{aligned}
\tag{3.3}
$$

Let $u = G_2^{1/2}(H\hat{\beta} - h)$, then under normal theory $u/t^{-1/2}\sigma \sim N_q(G_2^{1/2}(H\beta - h)/t^{-1/2}\sigma, I_q)$, $u'u \sim t^{-1}\sigma^2\chi_q^2(t\Delta)$ and using Theorem 4.1 of Judge and Bock (1978) we have

$$
E\left[I\left(\frac{u'u}{qS/(n-p)} \leq F_\alpha\right) G_2^{1/2}(H\hat{\beta} - h)\right] = G_2^{1/2}(H\beta - h)E[I(F_{q+2,n-p,t\Delta} \leq F_\alpha)]
\tag{3.4}
$$

Now applying Theorem 1 to (3.4) and using Theorem 1.27 from Rudin (1987), we get

$$
\begin{aligned}
E[I(\zeta \leq F_\alpha) G_2^{1/2}(H\hat{\beta} - h)] &= G_2^{1/2}(H\hat{\beta} - h)\int_0^\infty w(t) P(F_{q+2,n-p,t\Delta} \leq F_\alpha)\, dt \\
&= G_2^{1/2}(H\hat{\beta} - h)\aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha),
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
\aleph_{q+2,n-p}(\Delta, \sigma^2 V, x) &= \sum_{r=0}^{\infty} \frac{2^{n-1}\pi^{n/2}\Gamma(n/2 + r - 1)|\sigma^2 V|^{1/2}L^{-1}(f(\zeta))}{\Gamma(r+1)\Delta^{n/2-1}} \\
&\quad \times P\left(F_{q+2r,n-p} \leq \frac{qx}{q+2r}\right).
\end{aligned}
\tag{3.6}
$$

Then using (3.3) and (3.5) we obtain

$$
b_3 = -\mu\aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha).
\tag{3.7}
$$

Now we calculate risks of estimators $\hat{\beta}$, $\widetilde{\beta}$ and $\hat{\beta}^{PT}$. Using (2.1) the risk of the GLS estimator under the MLINEX loss is

$$
\begin{aligned}
R_1(\hat{\beta}; \beta) &= E[b\{e^{a'(\hat{\beta}-\beta)} - a'(\hat{\beta}-\beta) - 1\}] \\
&= bE[e^{a'(\hat{\beta}-\beta)} - 1].
\end{aligned}
\tag{3.8}
$$

But applying Theorem 1 to $y = (\hat{\beta}-\beta)$, we get

$$
\begin{aligned}
E(e^{a'y}) &= \int_{-\infty}^{+\infty} e^{a'y} \int_0^{\infty} w(t)\phi_y(0, t^{-1}\sigma^2 G_1) \, dt \, dy \\
&= \int_0^{\infty} w(t) M_x(a) \, dt,
\end{aligned}
\tag{3.9}
$$

where

$$
w(t) = (2\pi)^{p/2} |\sigma^2 G_1|^{1/2} t^{-p/2} L^{-1}(f(s)),
\tag{3.10}
$$

with $f(s) = g(y)$ when $s = \frac{y'G_1^{-1}y}{2\sigma^2}$ and $M_X(\cdot)$ is the moment generating function of $X$ in which $X \sim N_p(0, t^{-1}\sigma^2 G_1)$.

Therefore using (3.8) and (3.9) we get

$$
R_1(\hat{\beta}; \beta) = b\left[\int_0^{\infty} w(t)\exp\left(\frac{\sigma^2 a'G_1 a}{2t}\right) dt - 1\right].
\tag{3.11}
$$

Similarly the risk of the RGLS estimator under the MLINEX loss is

$$
\begin{aligned}
R_2(\widetilde{\beta}; \beta) &= E[b\{e^{a'(\widetilde{\beta}-\beta)} - a'(\widetilde{\beta}-\beta) - 1\}] \\
&= b\left[\int_0^{\infty} w(t)\exp\left(a'\mu + \frac{\sigma^2 a'G_3 a}{2t}\right) dt - a'\mu - 1\right] \\
&= b\left[e^{a'\mu}\int_0^{\infty} w(t)\exp\left(\frac{\sigma^2 a'G_3 a}{2t}\right) dt - a'\mu - 1\right] \\
&= b\left[e^{a'\mu}\int_0^{\infty} w(t)\exp\left\{\frac{\sigma^2 a'G_1 a}{2t} - \frac{\sigma^2 tr[WG_1 H'G_2 HG_1]}{2t}\right\} \right. \\
&\quad \left. \times dt - a'\mu - 1\right],
\end{aligned}
\tag{3.12}
$$

where $W = aa'$ and $tr(\cdot)$ stands for trace operator.

Now consider that $R = G_1^{1/2} H'G_2 HG_1^{1/2}$ is a symmetric idempotent matrix of rank $q \leq p$. Thus, there exists an orthogonal matrix $Q$ $(Q'Q = I_p)$ (Schott, 2005) such that

$$
QRQ' = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix},
\tag{3.13}
$$

$$
QG_1^{1/2}WG_1^{1/2}Q' = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A.
\tag{3.14}
$$

The matrices $A_{11}$ and $A_{22}$ are of order $q$ and $p - q$ respectively.
Now, we can write

$$
\begin{aligned}
tr\{W[G_1H'G_2HG_1]\} &= tr\{QG_1^{1/2}WG_1^{1/2}Q'QRQ'\} \\
&= tr\left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
&= tr(A_{11}).
\end{aligned} \tag{3.15}
$$

Therefore by (3.12) and (3.15), we get

$$
R(\widetilde{\beta}; \beta) = b\left[ e^{a'\mu} \int_0^\infty w(t) \exp\left\{ \frac{\sigma^2 a'G_1 a}{2t} - \frac{\sigma^2 tr(A_{11})}{2t} \right\} dt - a'\mu - 1 \right]. \tag{3.16}
$$

From the definition in (3.1), the risk function of the PTE under MLINEX loss becomes

$$
\begin{aligned}
R_3(\beta^{PT}; \beta) &= E[b\{e^{a'(\beta^{PT} - \beta)} - a'(\beta^{PT} - \beta) - 1\}] \\
&= bE\{\exp(a'[\hat\beta - I(\zeta \le F_\alpha)(\hat\beta - \widetilde\beta) - \beta]) \\
&\quad - a'[\hat\beta - I(\zeta \le F_\alpha)(\hat\beta - \widetilde\beta) - \beta] - 1\} \\
&= bE\{\exp(a'[\hat\beta - \beta] - a'G_1H'G_2^{1/2}[I(\zeta \le F_\alpha)G_2^{1/2}(H\hat\beta - h)]) \\
&\quad - a'[\hat\beta - \beta] + a'G_1H'G_2^{1/2}[I(\zeta \le F_\alpha)G_2^{1/2}(H\hat\beta - h)] - 1\}. \\
&= bE\{\exp(a'[\hat\beta - \beta] - a'G_1H'G_2^{1/2}[I(\zeta \le F_\alpha)G_2^{1/2}(H\hat\beta - h)])\} \\
&\quad - ba'\mu\aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha) - b, \tag{3.17}
\end{aligned}
$$

where $\aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha)$ is given by (3.6). Note that the expression $E\{\exp(a'[\hat\beta - \beta] - a'G_1H'G_2^{1/2}[I(\zeta \le \chi_\alpha)G_2^{1/2}(H\hat\beta - h)])\}$ in the risk function of $\hat\beta^{PT}$ can be solved via numerical computations.

# 4    Comparison of Risks

In this section, analyzing the risks of the estimators under study, the dominance order of $\hat\beta$ and $\widetilde\beta$, under special conditions is proposed. Also making condition on the shape parameter of the MLINEX loss function, it is shown that under some regular conditions $\widetilde\beta$ performs better than $\hat\beta$.

Now, note that for any $t \in (0, \infty)$, $0 < \exp[-\frac{t^{-1}\sigma^2 tr(A_{11})}{2}] < 1$; then direct computations lead to

$$
b[-a'\mu - 1] < R(\widetilde\beta, \beta) < b[e^{a'\mu}(R(\hat\beta, \beta)/b + 1) - a'\mu - 1]. \tag{4.1}
$$

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Therefore, under $H_0$ because $\mu = 0$, $\widetilde{\beta}$ performs better, having smaller risk, than $\hat{\beta}$ ($\widetilde{\beta} \succ \hat{\beta}$). Also for such vector shape parameter $a$ for which $a'\mu = 0$, $\widetilde{\beta} \succ \hat{\beta}$.

Define random variable

$$w = QG_1^{-1/2}\hat{\beta} - QG_1^{-1/2}H'G_2 h, \tag{4.2}$$

it can be considered

$$\eta = E(w) = QG_1^{-1/2}\beta - QG_1^{1/2}H'G_2 h. \tag{4.3}$$

Partitioning the vectors $w = (w_1', w_2')'$ and $\eta = (\eta_1', \eta_2')'$ where $w_1$ and $w_2$ are independent sub-vector of order $q$ and $p - q$ respectively, we obtain

$$\hat{\beta} - \beta = G_1^{1/2}Q'(w - \eta). \tag{4.4}$$

Now, for $c = R(\hat{\beta}, \beta)/b + 1$ define

$$l(a'\mu) = ce^{a'\mu} - a'\mu - 1. \tag{4.5}$$

Clearly $l(a'\mu)$ attains its minimum at $a'\mu = -lnc$. In other words, we have

$$\begin{aligned}
a'\mu &= \pm(a'\mu\mu'a)^{1/2} \\
&= \pm tr^{1/2}(\mu\mu'W) \\
&= \pm(\mu'W\mu)^{1/2} \\
&= \pm[(H\beta - h)'G_2 HG_1 WG_1 H'G_2(H\beta - h)]^{1/2} \\
&= \pm(\eta_1' A_{11}\eta_1)^{1/2}.
\end{aligned} \tag{4.6}$$

By Theorem A.2.4 from Anderson (2003), and using (4.3) for $\theta = \eta_1'\eta_1$ we have

$$\theta ch_1(A_{11}) \leq \eta_1' A_{11}\eta_1 \leq \theta ch_q(A_{11}),$$

or

$$[\theta ch_1(A_{11})]^{1/2} \leq a'\mu \leq [\theta ch_q(A_{11})]^{1/2}, \tag{4.7}$$

or

$$-[\theta ch_q(A_{11})]^{1/2} \leq a'\mu \leq -[\theta ch_1(A_{11})]^{1/2}, \tag{4.8}$$

where $ch_1(A_{11})$ and $ch_q(A_{11})$ are the minimum and maximum eigenvalues of $A_{11}$ respectively.

Using (4.1) and (4.8) we get the lower bound for the risk of $\widetilde{\beta}$ as

$$R_2(\widetilde{\beta}, \beta) > b[(\theta ch_1(A_{11}))^{1/2} - 1]. \tag{4.9}$$

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Also, for a special value of the shape parameter in MLINEX loss function, say $W = G_1^{-1} = X'V^{-1}X$, we get $R_2(\widetilde{\beta}, \beta) > b[\theta^{1/2} - 1]$.

Using (4.7) and the minimum of the equation (4.5), we get $l(-lnc) \leq l([\theta ch_q(A_{11})]^{1/2})$. Therefore solving the following inequality numerically, we can find the upper bound for $R_1(\hat{\beta}, \beta)$.

$$\ln c - c e^{\sqrt{\theta ch_q(A_{11})}} + \sqrt{\theta ch_q(A_{11})} + 1 \leq 0. \tag{4.10}$$

**Theorem 3**. Under the conditions in which $0 \leq a'\mu \leq a'G_4(\hat{\beta} - \beta)$, for $G_4 = G_1 H'G_2 H$, $\hat{\beta}^{PT} \succeq \hat{\beta}$.

Proof: Using (3.11), (3.17) and Cauchy-Schwartz inequality (Chung, 2001), we get

$$
\begin{aligned}
d_1 &= R_3(\hat{\beta}^{PT}; \beta) - R_1(\hat{\beta}; \beta) \\
&= bE\{\exp(a'[\hat{\beta} - \beta] - a'G_1 H'G_2^{1/2}[I(\zeta \leq F_\alpha)G_2^{1/2}(H\hat{\beta} - h)])\} \\
&\quad - ba'\mu \aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha) - b\int_0^\infty w(t) \exp\left(\frac{\sigma^2 a'G_1 a}{2t}\right) dt \\
&\leq bEe^{a'(\hat{\beta}-\beta)} E\exp\{-a'G_1 H'G_2^{1/2}[I(\zeta \leq F_\alpha)G_2^{1/2}(H\hat{\beta} - h)]\} \\
&\quad - ba'\mu \aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha) - b\int_0^\infty w(t) \exp\left(\frac{\sigma^2 a'G_1 a}{2t}\right) dt \\
&= b[E\exp\{-a'G_1 H'G_2^{1/2}[I(\zeta \leq F_\alpha)G_2^{1/2}(H\hat{\beta} - h)]\} - 1] \\
&\quad \times \int_0^\infty w(t) \exp\left(\frac{\sigma^2 a'G_1 a}{2t}\right) dt - ba'\mu \aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha) \\
&= b[E\exp\{-I(\zeta \leq F_\alpha)a'G_4(\hat{\beta} - \beta) + I(\zeta \leq F_\alpha)a'\mu\}] - 1] \\
&\quad \times \int_0^\infty w(t) \exp\left(\frac{\sigma^2 a'G_1 a}{2t}\right) dt - ba'\mu \aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha). \tag{4.11}
\end{aligned}
$$

If $0 \leq a'\mu \leq a'G_4(\hat{\beta} - \beta)$, then $I(\zeta \leq F_\alpha)[a'\mu - a'G_4(\hat{\beta} - \beta)] \leq 0$; which leads $d_1 \leq 0$.

**Theorem 4**. Under one of the following conditions and $tr(WG_1) \leq tr(A_{11})$, $\widetilde{\beta} \succeq \hat{\beta}^{PT}$.

$$(i) \quad : \quad 1 \leq \aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha) \qquad \text{and} \quad a'\mu \geq 0, \tag{4.12}$$

$$(ii) \quad : \quad 0 \leq \aleph_{q+2,n-p}(\Delta, \sigma^2 V, F_\alpha) \leq 1 \quad \text{and} \quad a'\mu \leq 0. \tag{4.13}$$

Proof: Using Theorem 1, (3.16) and (3.17) and Jensen's inequality (Chung,

2001), we have

$$
\begin{aligned}
d_2 &= R(\hat{\beta}^{PT};\beta) - R(\widetilde{\beta};\beta)\\
&= bE\{\exp(a'[\hat{\beta}-\beta] - a'G_1H'G_2^{1/2}[I(\zeta \le F_\alpha)G_2^{1/2}(H\hat{\beta}-h)])\}\\
&\quad -ba'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha) - be^{a'\mu}\int_0^\infty w(t)\exp\left(\frac{\sigma^2 a'G_1 a}{2t} - \frac{\sigma^2 tr(A_{11})}{2t}\right)dt + ba'\mu\\
&\ge b\exp\{E[a'[\hat{\beta}-\beta] - a'G_1H'G_2^{1/2}[I(\zeta \le F_\alpha)G_2^{1/2}(H\hat{\beta}-h)]]\}\\
&\quad -be^{a'\mu}\int_0^\infty w(t)\exp\left(\frac{\sigma^2 a'G_1 a}{2t} - \frac{\sigma^2 tr(A_{11})}{2t}\right)dt\\
&\quad -ba'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha) + ba'\mu\\
&\ge b\{\exp[a'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha)] - a'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha)\}\\
&\quad +ba'\mu - be^{a'\mu}\int_0^\infty w(t)\exp\left(\frac{\sigma^2 a'G_1 a}{2t} - \frac{\sigma^2 tr(A_{11})}{2t}\right)dt. \quad\quad (4.14)
\end{aligned}
$$

Note that for any $t \in (0,\infty)$ and $a'G_1 a \le tr(A_{11})$, $0 < \exp[-\frac{\sigma^2(a'G_1 a - tr(A_{11}))}{2t}] < 1$, therefore by (4.14), we obtain

$$
\begin{aligned}
d_2 &\ge b\Bigg[\{\exp[a'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha)] - a'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha)\}\\
&\quad\quad +a'\mu - e^{a'\mu}\int_0^\infty w(t)dt\Bigg]\\
&= b\left\{e^{a'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha)} - e^{a'\mu} + a'\mu - a'\mu\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha)\right\}.
\end{aligned}
$$

Now consider $f(x) = e^x - x$ is an increasing function for all positive values of $x$. Therefore under the conditions in which $\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha) \ge 1$ and $a'\mu \ge 0$, or $0 \le \aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha) \le 1$ and $a'\mu \le 0$, we have $d_2 \ge 0$.

**Remarks:**

1. If $W = X'V^{-1}X$, then by (3.15), $tr(WG_1) = tr(A_{11}) = q$. Thus the conditions of Theorem 4, reduce to (4.12) or (4.13) for $\widetilde{\beta}$ dominating $\hat{\beta}^{PT}$. Further, for the multivariate normal distribution, because of (2.8),

$$
\begin{aligned}
\aleph_{q+2,n-p}(\Delta,\sigma^2V,F_\alpha) &= \int_0^\infty \delta(t)P(F_{q+2,n-p,t\Delta} \le F_\alpha)\,dt\\
&\le \int_0^\infty \delta(t)\,dt = 1,
\end{aligned}
$$

the conditions of Theorem 4 reduce to that $a'\mu \le 0$.

2. Under the null hypothesis $H_0 : H\beta = h$, using Theorems 3 and 4, the dominance order of $\hat{\beta}$, $\widetilde{\beta}$ and $\hat{\beta}^{PT}$ is as follows.

$$
\widetilde{\beta} \succeq \hat{\beta}^{PT} \succeq \hat{\beta}.
$$

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