Using the history of mathematics for mentoring gifted students: Notes for teachers

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The paper presents a theoretical framework, methodology and practical implications for the work with gifted students using history of mathematics. A teaching-learning model, where history of mathematics is integrated in problem-solving activities, is described. Didactical material based on the concepts of triangle geometry is given in the scope of this model. A beautiful and intriguing piece of geometry – the Lemoine point is the focus of consideration. Its properties are investigated through appropriately designed activities for students. Different examples show the importance of history of mathematics for the development of students' mathematical thinking.

Introduction

Have your students ever asked you about the geometry of the triangle means and what Archimedes could have discovered using dynamic geometry software? History of mathematics connects the content of mathematics with its past and can be a source for further reflections on the ways of teaching and learning mathematics from primary to tertiary levels. Nowadays there is a consensus that the history of mathematics is no longer a remote or tangential issue to the mathematics curriculum and deserves a significant role. Hayes (1991) stated:

> I believe that it is a grave mistake and error of strategy to attempt to teach mathematics without reference to its cultural, social, philosophical and historical background.

Undoubtedly, most teachers are aware of the importance of integrating the history of mathematics in the teaching process and many of them successfully use at least some elements in their work. But, how can this best be done in different classroom situations, with different teachers' expectations and students' beliefs? Undoubtedly, the question needs further specification. The main issue we would like to focus on is integration of history of mathematics, in particular with respect to triangle geometry, in classroom inquiry activities of gifted in mathematics students. This paper is written for teachers and has two main aims:

- to provide teachers with practical advice on how challenging material can be organised in a constructivist framework using history of mathematics;
- to attract teachers' attention to the importance of linkage between mathematical and pedagogical content and demonstrate how difficult or challenging mathematical problems can be addressed through their historical background using generalisation, visualisation and systematisation.

We will describe a fragment of a teaching-learning model, where the Lemoine point takes a central place. The Lemoine point is not the end in itself but a means to show the huge potential in teaching challenging material. It can be presented by the teacher, and seen by students, as a chain of logical reasoning based on the geometry curriculum. We hope teachers will find this paper and the didactical material helpful, either as a direct guide in mentoring gifted students, or as an example to help produce their own materials bringing together history and mathematics.

Theoretical framework

Gray et al. (1999) pointed out that "didactical reversal - constructing a mental object from 'known' properties, instead of constructing properties from 'known' objects causes new kinds of cognitive difficulty" (p.117). We used the idea of "didactical reversal" with respect to historical context of mathematical content. We called it *didactical chronology of a concept*, i.e. we proposed students to build up a successive chain of their argumentation, which would lead them to the discovery of a certain property. In other words, we modelled the same timeline situation as it happened hundreds years ago and modified it slightly so that the final result, i.e. mathematical property, could be discovered. We considered a concept in the context of its historical-mathematical sense (Yevdokimov, 2006a):

- When was a concept posed in a certain problem for the first time?
- Who was the author, and did that author prove/solve the problem on his/her own?
- What other famous mathematicians were interested in the same problem and why?
- How long was that problem known as an unsolved one?
- And the most important issue – how it could be solved.

We called this stage *initial problem situation*. When the initial problem had been solved the next stage began – to find out more what was going on around the context of the problem:

- Which mathematical objects could be taken into further consideration?
- How could those mathematical objects be related to each other?
- Which properties could be suggested to or made known to for students? Or more specifically: Did a certain property follow from the initial problem? How could it be proved?
- Could students introduce some auxiliary elements in order to make the initial problem applicable to a range of situations? (Stoyanova, 2000);

We called this stage *advanced search over situation*. Following Brown and Walter (1990) we proposed a "situation" to mean a localised area of inquiry, within a given historical context, with features that can be taken as given or challenged and modified. We would like to note that there were different directions of students' inquiry work in this stage. To assist the teacher control a "situation" we used Mercer's idea (1995) of "the sensitive, supportive intervention of a teacher in the progress of a learner, who is actively involved in some specific task, but who is not quite able to manage the task alone". We followed Edwards' idea of *conceptual territory before proof* (1997) in conducting the process of students' inquiry activities through historical context in a classroom and took into account that students' exploration and teacher's explanation constituted the main elements that preceded students' discoveries in the scope of inquiring activities.

As advocated by Kronfellner (1996) we distinguished similar patterns in the historical development of the concepts, which could be used to identify the direction of students' inquiry activities in a classroom:

- Implicit use before explicit definition;
- Changeability of mathematical concepts;
- Understandable modification of concepts.

We chose triangle geometry to demonstrate our approach to mentoring gifted students for the following reasons:

- There are more than one hundred distinguished objects (such as points, lines, circles and conics) of a triangle, many of them related to each other (Davis, 1991). Some elements were studied in antiquity.
- Histories of some identifiable objects were recorded and written over a hundred years ago (Mackay, 1892);
- Using technology gives opportunity to model mathematical experimentation through visualisation.

Methodology

All students' activities were carried out as a part of the six months enrichment program for Year 11 students. Fifteen (15) students were selected by their school mathematics teachers to be involved in the program. All students were identified, by their teachers as gifted or, at least, as very good mathematics students. No additional confirmation of that status of each student was required. The teaching experiment was designed for teachers to provide their feedback on the proposed program and encourage them to use the similar ideas in their further work. The methodology consisted of interactions between researcher and teachers, researcher and students. These interactions included interviews with teachers and teaching episodes with students. This teaching experiment was conducted in four parts: observation part, interview part, teaching part, and analysis part. Below we provide the brief details for each part of the experiment.

Observation part

For six months 7 high school mathematics teachers observed 12 teaching episodes carried out with the group of 15 selected students. There were at least 2 students, who were recommended by the same teacher. Teachers were asked to make their notes on 4 different criteria of the program: complexity for teachers, complexity for students, development of students' conceptual constructions, and importance of the historical context. The scale from 1 (strongly disagree) to 10 (fully agree) marks for each index was used. The lowest and the highest marks were removed in each assessment criterion, average values were calculated in the end. Also, we used this part to establish a collaborative relationship with the teachers and the whole class.

Interview part

During the last 2 months of the program each of the seven teachers was individually questioned in 30 minutes interviews. The goal of the interviews was to find out more about teachers' expectations with respect to students' conceptual constructions and problem solving strategies, when history of mathematics was integrated into the teaching process and students were met with challenging problems. In particular, we asked teachers how they could identify that a problem was likely to be hard for students and how historical context could be used in the most effective way in problem solving. Finally, we took short interviews on students' beliefs in the problem solving process, and attitudes to different forms of problems.

Teaching part

There was 90 minutes teaching episode per fortnight. During 8 episodes each of the 15 students worked individually; in the other four episodes, students worked in small groups (3-4 people in each group). All students were given similar problems, but the questions varied in difficulty, depending on their ways of thinking. We paid much attention to researcher-student interaction to keep it in the scope of the theoretical framework. The most important feature of each teaching episode was that instructions were modified continually, according to the students' performance and the ways of thinking. The need for, and nature of, the modifications were decided on the basis of their answers and explanations.

Analysis part

All interviews and teaching episodes were analysed with respect to teachers' comments and their marks for each criterion. Average values of each index were compared with overall marks of teachers' assessment. Feedback from each teacher, for the further possible inclusion of historical background in their problem solving activities with students, was received. We used teachers' protocol sheets of interviews and their comments on the teaching episodes, teachers' notes concerning students' inquiry work through historical context in a classroom and audio-files of the episodes.

Each student, or small group of students, received the first card having the following content:

- initial problem;
- figure;
- description of historical-mathematical neighbourhood for the initial problem;
- solution of the initial problem (optionally);
- questions-hints for students to encourage their work.

Work on the first cards was the first stage of each episode. Students were given 30 minutes for these activities. The first card was the same for all students. We used 12 first cards for 12 teaching episodes. The second stage of each episode was divided in two parts. Students worked for 30 minutes, making different suggestions and conjectures. For the last 30 minutes students received the second card:

- another problem;
- figure;
- description of historical-mathematical neighbourhood for the given problem;
- questions-hints for students to encourage their work.

The main research focus was to find out how students viewed the links between the two "card problems", both in terms of perceived problem properties, and linking relationships between the problems themselves, one to the other.

Also, any help, hints or ideas could be provided by the teacher verbally either in explicit or implicit form. The teacher's role was to conduct students' inquiry work on both card problems and make the transition between them in the most understandable way for every student.

Didactical implications

In this part of the paper we describe four characteristic examples, which were identified by the teachers as the most important. Also, we give short descriptions of two cards of the teaching episode, where the Lemoine point was investigated.

The following two key ideas are the fundamental ones for understanding the nature of students' inquiry activities in a classroom.

Diversity in forms

A mathematical statement (conjecture, question) can be presented in different forms.

Example A: Three forms of the same statement

1. Find all nontrivial integer solutions for the Diophantine equation $x^2+y^2=z^2$.
2. Find all points, which have rational coordinates and lie on a unit circle $x^2+y^2=1$.
3. Find all right-angled triangles, for which lengths of hypotenuses and both sides are integers (Pythagoras triplets).

It is important to note that the first form relates to analytical interpretation of the statement, while the second and third show visual interpretation of the same statement. Teachers should take into account the impact of these interpretations on students' abilities to distinguish different forms of a statement. Moreover, this example shows how different areas of mathematics are intertwined: the first form is taken from the theory of numbers, the second relates to analytical theory of the second order curves and the third form presents particular cases of the famous Pythagoras theorem in geometry.

Our previous research (Yevdokimov, 2003; 2006b) showed that gifted students, in most cases, were successful in making transition from one form of a statement to another in the problem solving process. Nevertheless, one of the teacher's main tasks remained to develop students' abilities to identify and analyse different forms correctly.

The second key idea (property) is more complex: and incorporates the first property:

Diversity in properties

A concept of a mathematical object (point, line, function, etc) can be developed to get different properties, related or otherwise.

The teacher's role was to develop students' mathematical thinking and their inquiring abilities. Students were taught to find some properties, and their possible relations to each other, through the appropriate model of teaching environment. Additionally, we considered this idea through its historical context. See Example B.

Example B: Short history of the Lemoine point

In 1809 the French mathematician L'Huillier tried to find a point inside a triangle such that the sum of the squares of distances from this point to the sides of the triangle is the least. In 1820s Gauss was interested in the properties of this point in relation to his method of the least squares. Later German mathematician Groebe (1847) and French mathematician Aussart (1848) discovered new properties of the same point.

In 1852 Belgian mathematician Catalan was investigating the properties of a point, which was the centre of gravity of the triangle *MNL*. He proved that the distances from this point to the sides of the triangle *ABC* are proportional to the lengths of these sides (Figure 1).

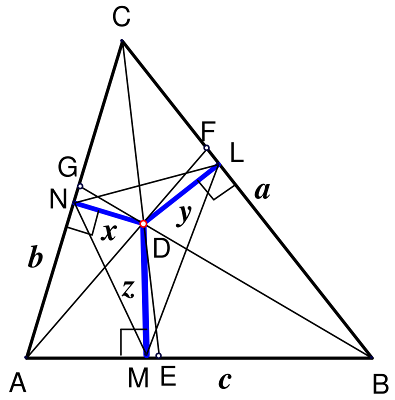

Figure 1.

In 1860 the German mathematician Schlomilch proved that if lines are drawn through the mid-points of the triangle's sides and the mid-points of the corresponding heights, these three lines would intersect in the same point.

Finally in 1873 the French mathematician Lemoine discovered that L'Huillier, Groebe, Aussart, Catalan, Schlomilch and other mathematicians were investigating the same point of a triangle! Since that time it was known as the Lemoine point (Cajori, 1907).

This example shows there is a variety of didactical situations in which a certain point inside a triangle can be investigated.

Inquiry activities, in a constructivist framework, are one of the most effective forms in working with gifted students. Many beautiful results in geometry can be discovered using different kinds of generalisation, both visual and analytical. Understanding how different famous historical theorems and facts can be linked to each other through the different kinds of generalisations is important, and can be developed in a classroom. See example C.

Example C: Visual and analytic generalisations for bisectors' property

We would like to demonstrate how the same geometrical object can be a source for visual and analytic extension and generalisations. Consider the angle bisector *CD* (Figure 2a). If "angle bisector" is changed to "angle trisector" and appropriate intersection points joined, we have as result an equilateral triangle (*MKL*) known as Morley's triangle (Gambier, 1954; Figure 2b).

On the other hand, we can make an analytic generalisation with respect to the point *D* of bisector *CD*. We can use the well-known bisector's *property*: "A bisector of a triangle divides the opposite side onto the parts, which are proportional to the corresponding adjoining sides of this triangle".

For generalising we consider the case, when the parts of each side of a triangle are proportional to the squares of the corresponding adjoining sides of this triangle. The result is the Lemoine point of a triangle, which is a point of intersection of symmedians[1]. Construction is shown in Figure 2c.

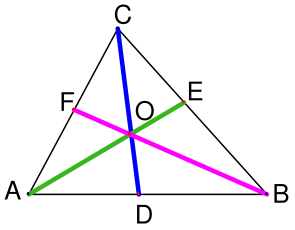

Figure 2a.

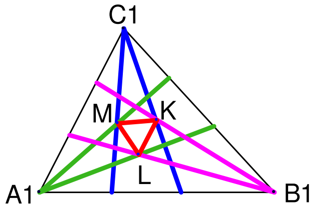

Figure 2b.

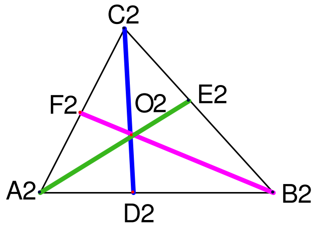

Figure 2c.

AE,BF,CD – triangle's bisectors   Product of visual generalisation   Product of analytic generalisation

Also, we would like to show how different levels of visual generalisations can be used by teachers in their guidance of students' inquiry activities. See example D.

Example D: Levels of visual generalisations

There is a triangle *ABC* inscribed in a circle. Points A_1, B_1, C_1 are feet of the corresponding perpendiculars PA_1, PB_1, PC_1 to the sides of the triangle (Figure 3).

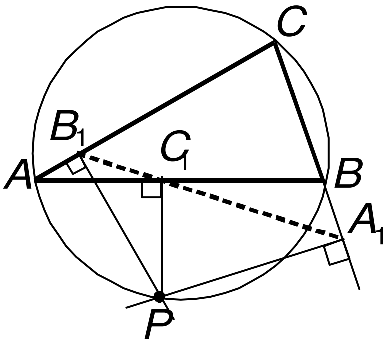

Figure 3.

There are two levels of generalisation here. The first one can be easily presented visually. Any point of the circle has the same *property*: "Feet of the corresponding perpendiculars are in the same line". This line is called the Simson line.

Thus, the first generalisation relates to the circle only. The second and higher level of generalisation is hidden in the figure. However, it is very important to note that
A generalisation might be possible for the points outside a circle. This is the other level of visual generalisation.

A teaching-learning model: Triangle famous points – a case of the Lemoine point

We give an example of two cards of the teaching episode in which a teaching-learning model for the Lemoine point was constructed and used in classroom activities.

[1]A symmedian of a triangle is called a segment inside a triangle, which is symmetric to the median with respect to the corresponding bisector and going through the same vertex of a triangle.

The first card

Initial problem (L'Huillier): Find a point inside a triangle such that the sum of the squares of distances from this point to the sides of the triangle is the least.

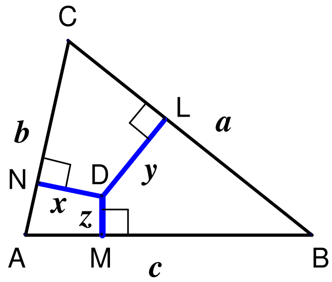

Figure 4.

Very brief description of historical-mathematical neighbourhood was given in Example B.

Questions-hints for students:

- What is the aim of the problem?
- The sum of the squares of distances from this point to the sides of the triangle should be the least. We should find the point. How can we find it?
- What should we begin with?
- "To find a point" means to specify its location with respect to at least two sides of a triangle;
- "To find a point" means to analyse an expression $x^2 + y^2 + z^2$ for a minimal value.

The second card

Problem (Lemoine): A segment BK is called a symmedian, if BK is symmetrical to the median BM with respect to the bisector BL and going through the same vertex B. Show that all symmedians intersect in the same point.

A very brief description of historical-mathematical context was given in Example B.

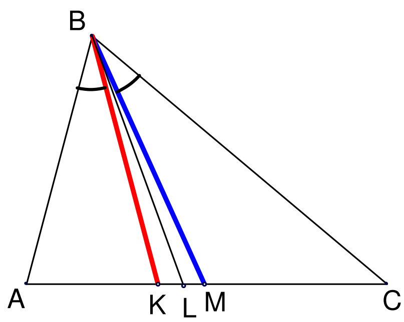

Figure 5.

Questions-hints for students:

- How does the area of the triangle depend on the expressions $x^2+y^2+z^2$ and $a^2+b^2+c^2$ from the first card, if x, y, z are distances from the Lemoine point to the sides of the triangle and a, b, c are lengths of the sides of the triangle?

- Draw the lines connecting corresponding vertices of a triangle with the Lemoine point. How do these lines divide the sides of a triangle?

Concluding remarks

Summing up the results of teachers' response on the enrichment activities we can confirm that almost all teachers approved the program and expressed interest in developing classroom inquiry work within an historical context. Teachers supported the idea that an historical perspective can help us in the work with gifted students. It was found that mentoring gifted students looks very promising and easy to follow, if we use a teaching-learning environment, which provides a logical structure and historical background for inquiry activities. More activity development and research is needed, and more is planned.

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