Optimal selection of topologies for the minimum-weight design of continuum structures with stress constraints

Q Q Liang[1]*, **Y M Xie**[1] and **G P Steven**[2]

[1]School of the Built Environment, Victoria University of Technology, Melbourne, Victoria, Australia

[2]Department of Aeronautical Engineering, The University of Sydney, New South Wales, Australia

Abstract: This paper presents a method for assisting the optimal selection of topologies for the minimum-weight design of continuum structures subject to stress constraints by using the performance index (PI). A performance index is developed for evaluating the efficiency of structural topologies based on the scaling design approach. This performance index is incorporated in the evolutionary structural optimization (ESO) method to monitor the optimization process when gradually removing inefficient material from the structure. The optimal topology can be identified from the performance index history. Various structures with stress and height constraints are investigated by using this performance index, which is also employed to compare the efficiency of structural topologies generated by different optimization methods. It is shown that the proposed performance index is capable of measuring the efficiency of structural topologies obtained by various structural optimization methods and is a valuable design tool for engineers in selecting optimal topologies in structural design.

Keywords: performance index, topology, stress constraint, evolutionary structural optimization, finite element analysis

NOTATION

E	Young's modulus (GPa)
h	height (mm)
$\mathbf{K}^s$	stiffness matrix of a scaled structure (N/mm)
L	length (mm)
$\boldsymbol{P}$	force vector (N)
PI	performance index
t	thickness of elements (mm)
t_e	thickness of the eth element
$\boldsymbol{u}$	actual displacement vector (mm)
$\boldsymbol{u}^s$	scaled displacement vector (mm)
V_i	volume of the current design at the ith iteration (mm³)
V_0	volume of the initial design domain (mm³)
w_e	weight of the eth element (kg)
W	total weight of a structure (kg)
W_i	weight of the current design at the ith iteration (kg)
W_i^s	scaled weight of the current design at the ith iteration (kg)
W_0	weight of the initial design domain (kg)
W_0^s	scaled weight of the initial design domain (kg)
x_e	design variable
x_e^s	scaled design variable
$\boldsymbol{\sigma}$	stress vector of elements (MPa)
$\boldsymbol{\sigma}^s$	scaled stress vector of elements (MPa)
σ^*	prescribed stress limit (MPa)
$\sigma_{i,\max}^{\mathrm{VM}}$	maximum von Mises stress of elements in the current design (MPa)
$\sigma_{\max}^{\mathrm{VM}}$	maximum von Mises stress of elements in the structure (MPa)
$\sigma_{0,\max}^{\mathrm{VM}}$	maximum von Mises stress of elements in the initial design domain (MPa)
φ	scaling factor

1 INTRODUCTION

The topology optimization of structures has increasingly gained popularity in recent years as it is realized that the optimization of topologies can significantly improve the efficiency of the design. Consequently, the reduction in

the weight of structures with optimal topologies is generally more significant than that obtained by sizing optimization [1]. There have been a large number of structural topologies in the literature, produced by various structural optimization methods such as presented by Bendsøe and Mota Soares [2] and by Steven *et al.* [3]. Although optimization methods can be examined by comparing the results with classical solutions such as those presented by Michell [4] and Hemp [5], the final topologies obtained may vary with the criteria and methods used. Since there are no simple rules for measuring the efficiency of structural topologies, engineers face difficulty in selecting these topologies in the design of engineering structures.

Performance indices have been attempted by several researchers for assisting the selection of materials and geometry for structures in design. Boiten [6] has used a performance index to optimize the energy storage device. Ashby [7] has derived a set of performance indices for the selection of materials and cross-section shapes for structural elements. Weaver and Ashby [8] have also employed such performance indices for assisting the selection of material and section shapes. The method outlined by Ashby has recently been used by Burgess [9, 10] to derive performance indices known as form factors for optimizing the structural layouts of trusses and beams with strength and stiffness constraints respectively. Burgess has also used these performance indices to compare the efficiency of structural topologies obtained by different optimization methods. However, it is difficult to extend this approach to the optimization of continuum structures because the objective function can no longer be expressed by the separable functional, geometrical and material parameter functions, as would be the case for single-element and truss structures.

An attempt to derive a performance index (PI) for measuring the efficiency of structural topologies for continuum structures, which are discretized into finite elements, has also been undertaken by Querin [11]. However, the performance index formula given by Querin does not consider any type of constraint and its meaningful application may be very limited. Xie and Steven [12] have measured the efficiency of structural topologies by comparing the volume of the new design with that of the optimized uniform design, which is established by uniformly reducing the thickness of the initial design domain until the prescribed displacement reaches the limit.

This paper presents a methodology for developing a performance index to assist the selection of optimal topologies for the minimum-weight design of continuum structures with stress constraints. The derivation of a performance index is firstly formulated on the basis of the scaling design concept. The evolutionary structural optimization (ESO) method based on the stress ratio criterion is then outlined. The performance index developed is used in the ESO procedure to identify the optimal topologies of various continuum structures with stress and height constraints and to compare the efficiency of structural topologies obtained by different methods. Finally, the factors affecting the performance indices and the optimal topologies of continuum structures are discussed.

2 DERIVATION OF PERFORMANCE INDEX

2.1 Scaling of the design

Scaling of the design has been used in some optimization algorithms after each iteration to obtain the best feasible constrained design [13]. The advantages of scaling the design are that it can trace the history of the reduction in the weight of the structure after each iteration and pick the most active constraints. This method can be applied to structural optimization when the stiffness matrix of the structure is a linear function of the design variable. By scaling the design, the scaled design variable is expressed by

$$x_e^s = \varphi x_e \tag{1}$$

in which x_e^s is the scaled value of the design variable such as the element thickness for the eth element, φ is a scaling factor that is the same for all elements and x_e is the actual design variable of the eth element. The force–displacement relationship in the finite element method can be written as

$$\frac{1}{\varphi} \mathbf{K}^s \boldsymbol{u} = \boldsymbol{P} \tag{2}$$

where $\mathbf{K}^s$ is the stiffness matrix of the scaled structure and is calculated by using the scaled design variable x_e^s. The equilibrium equation for the scaled design can be expressed in terms of the scaled design variable by

$$\mathbf{K}^s \boldsymbol{u}^s = \boldsymbol{P} \tag{3}$$

From equations (2) and (3), the following is obtained:

$$\boldsymbol{u}^s = \frac{1}{\varphi} \boldsymbol{u} \tag{4}$$

From the expressions of the strain–displacement and stress–strain relations in terms of the scaled design variable, the scaled stress vector can be derived as

$$\boldsymbol{\sigma}^s = \frac{1}{\varphi} \boldsymbol{\sigma} \tag{5}$$

in which $\boldsymbol{\sigma}$ is the stress vector of elements. Therefore, in order to satisfy the stress constraint in a structure, the actual design needs to be scaled by

$$\varphi = \frac{\sigma_{max}^{VM}}{\sigma^*} \tag{6}$$

where σ_{max}^{VM} is the maximum von Mises stress of an element in the structure and σ^* is the prescribed stress

limit. By changing the value of the scaling factor φ in the structural optimization process, the best feasible design can be obtained and this is the optimal topology for the structure considered.

2.2 Performance index

The topology optimization of a structure is the selection of the geometry that minimizes the weight of the structure while satisfying the requirements of constraints imposed on the structure. The nature of the material layouts in an optimized structure depends on the type of constraint. A different type of constraint leads to different optimal topology. In this paper the maximum stress constraint in the structure is considered as the most active constraint so that the requirement on the strength limit state needs to be satisfied.

The optimization of continuum structures subject to stress constraint can be expressed as

$$minimize \quad W = \sum_{e=1}^{N} w_e(t_e) \tag{7a}$$

$$subject\ to \quad \sigma_{max}^{VM} \leqslant \sigma^* \tag{7b}$$

where w_e is the weight of the eth element, which varies with the element thickness. For the linear elastic plane stress problems, the stiffness matrix is a linear function of the design variable such as the thickness of elements. By scaling the design with respect to the stress constraint, the scaled weight of the initial design domain can be represented by

$$W_0^s = \left(\frac{\sigma_{0,max}^{VM}}{\sigma^*} \right) W_0 \tag{8}$$

in which W_0 is the actual weight of the initial design domain and $\sigma_{0,max}^{VM}$ is the maximum von Mises stress of an element in the initial design domain under applied loads. In an iterative optimization process, the scaled weight of the current design at the ith iteration is given by scaling the design as

$$W_i^s = \left(\frac{\sigma_{i,max}^{VM}}{\sigma^*} \right) W_i \tag{9}$$

where W_i is the actual weight of the current design at the ith iteration and $\sigma_{i,max}^{VM}$ is the maximum von Mises stress of an element in the current design at the ith iteration.

The performance index at the ith iteration is proposed as

$$PI = \frac{W_0^s}{W_i^s} = \frac{(\sigma_{0,max}^{VM}/\sigma^*)W_0}{(\sigma_{i,max}^{VM}/\sigma^*)W_i} = \frac{\sigma_{0,max}^{VM} W_0}{\sigma_{i,max}^{VM} W_i} \tag{10}$$

If the material density is uniformly distributed within the structure, the performance index can be written in terms of the volume of the structure as

$$PI = \frac{\sigma_{0,max}^{VM} V_0}{\sigma_{i,max}^{VM} V_i} \tag{11}$$

where V_0 is the volume of the initial design domain and V_i is the volume of the current design at the ith iteration.

It can be seen from equation (11) that the performance index is a dimensionless number that measures the efficiency of structural topologies. The performance index reflects the changes in the volume and the maximum stress levels in the structure in an optimization process. For the initial design, the performance index is equal to unity. The efficiency of a structural topology is gained by removing lowly stressed materials from the structure. Since the performance index is inversely proportional to the volume of the current design, minimizing the weight of a structure with stress constraint can be achieved by maximizing the performance index in the optimization process. In addition, it indicates that the optimal topology for the minimum-weight design of a continuum structure with a given support and loading condition is the same for any value of prescribed stress limits. The optimal topology that corresponds to the maximum value of the performance index can be identified from the performance index history. The higher the value of the performance indices, the better is the topology of the structure.

The performance index proposed here is not specific to the optimization methods used, so it can be incorporated in any structural optimization method such as the homogenization [14] and ESO [12] algorithms to trace the performance history and to predict the optimal topology. Moreover, the performance index can also be used to compare the efficiency of structural topologies obtained by different optimization methods.

3 EVOLUTIONARY OPTIMIZATION

The evolutionary structural optimization (ESO) method proposed by Xie and Steven [12, 15, 16] is based on the concept that, by gradually removing inefficient material from a structure, the topology of the remaining design evolves towards an optimum. In this approach, the design domain, which is large enough to cover the final design, is divided into a fine mesh of finite elements. The structure is firstly analysed by undertaking a finite element analysis. For problems with stress constraints, the maximum von Mises stress is used as the element removal criterion, which is expressed by

$$\frac{\sigma_e^{VM}}{\sigma_{i,max}^{VM}} < RR_k \tag{12}$$

where σ_e^{VM} is the von Mises stress of the eth element, and RR_k is the rejection ratio at the kth steady state. All elements that satisfy equation (12) are removed from the structure. The cycle of the finite element analysis and

the element removal is repeated by using the same RR_k until no more elements can be removed from the structure at the current state. At this stage, an evolution rate ER is added to RR_k, and the rejection ratio becomes

$$RR_{k+1} = RR_k + ER \qquad (13)$$

As the element removal and the finite element analysis process is continued, the structure gradually evolves towards a more uniformly stressed design.

The above traditional ESO procedure can generate more efficient structural topologies. However, it is not possible to decide which topology is the optimum for the minimum-weight design of a continuum structure because there is no objective function and stress constraints involved in the optimization process. This problem can be overcome by using the performance index proposed in this paper. By simply recording the maximum von Mises stress of elements and the volume of the current structure at each iteration, the performance index can be calculated using equation (11) for each iteration. Consequently, the optimal topology can be determined from the performance index history.

It has been found that the magnitude of stress constraints might have significant effects on the weight of a final design but not on the optimal topology. Therefore, the structural optimization process can be divided into two steps. The first step is to obtain the optimal topology of the continuum structure using the PI formula and any structural optimization method such as ESO regardless of the value of the stress limit. The second step is to size the obtained optimal topology in order to satisfy the stress constraints. Only the first step is considered in this paper.

4 EXAMPLES

4.1 Deep cantilever beam

The design domain for a deep cantilever beam with maximum stress constraint whose value is not specified here is shown in Fig. 1. The design domain is discretized into a 32×72 mesh using four-node plane stress elements. The support of the cantilever beam is fixed. A point load of 200 N is applied to the centre of the free end. The Young's modulus $E = 200$ GPa, the Poisson's ratio $\nu = 0.3$ and the thickness of elements $t = 1$ mm. Plane stress conditions are assumed. An initial rejection ratio RR_0 of 1 per cent and an evolution rate ER of 1 per cent are used in the optimization process.

The performance index history for the deep cantilever beam is presented in Fig. 2. It can be seen that at the initial iteration the performance index is equal to unity because no elements have been removed at this stage. By gradually removing lowly stressed elements from the structure, the performance index gradually increases. At the final stage, the maximum performance index is 10.86,
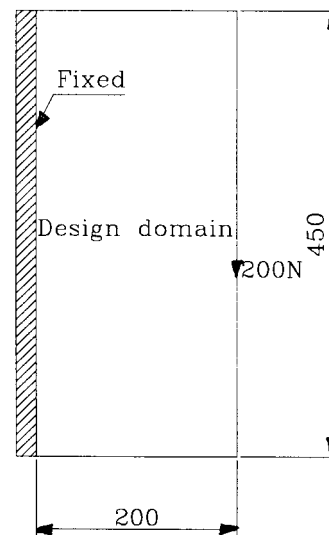

Fig. 1   Design domain for the cantilever beam
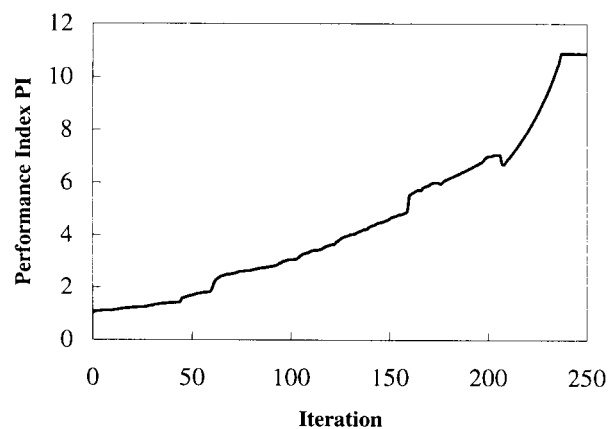

Fig. 2   Performance index history of the cantilever beam

which indicates that the weight of the initial design domain is 10.86 times that of the optimal design obtained if the maximum von Mises stress of elements in the structure reaches the prescribed stress limit. The topologies at iterations 50 and 150 and the optimal topology obtained are shown in Fig. 3. The optimal topology of the deep cantilever beam evolves towards a two-bar truss-like structure which has been proved to be an optimum design under the condition described in this example. Table 1 presents a comparison of material volumes required for the initial design and the three topologies shown in Fig. 3 for various stress limits. It can be seen from the table that the volumes of the optimal topology are always less than those of the other three topologies for each stress limit. This illustrates that the topology shown in Fig. 3c is the best topology, irrespective of the prescribed stress limits.

4.2 Michell-type structures with height constraints

The design domain for the simply supported Michell-type structures with various height constraints is shown
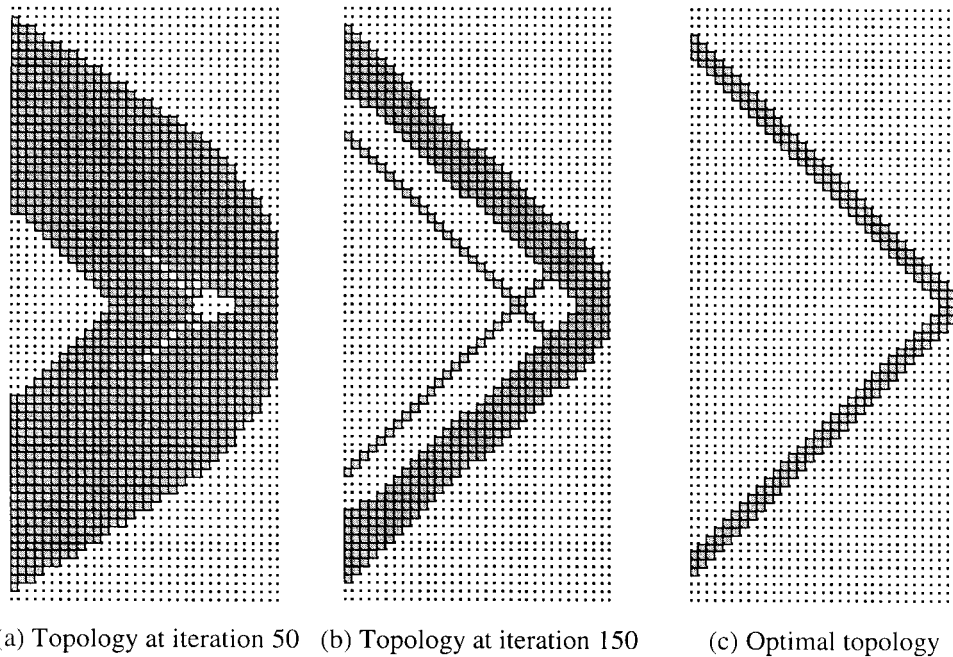
(a) Topology at iteration 50   (b) Topology at iteration 150   (c) Optimal topology

Fig. 3 Evolutionary history of topology optimization for the cantilever beam

Table 1 Material volumes required at different iterations for various stress limits

σ^* (MPa)	V_0^s (mm³)	V_{50}^s (mm³)	V_{150}^s (mm³)	$V_{optimal}^s$ (mm³)	$\mathrm{PI}_{max} = V_0^s/V_{optimal}^s$
100	18 855	11 117	4086	1735	10.86
150	12 570	7412	2724	1157	10.86
250	7542	4447	1634	694	10.86

in Fig. 4. In case (a), the design domain with $h/L = 1/2$ is divided into a 100×50 mesh using four-node plane stress elements and $RR_0 = 1$ per cent and $ER = 0.5$ per cent are used in the optimization process. In case (b), the design domain with $h/L = 1/4$ is divided into a 100×25 mesh. In case (c), the design domain with $h/L = 1/8$ is divided into a 100×13 mesh. In case (d), a 100×9 mesh is used for the structure with $h/L = 1/12$. The material properties for all cases are $E = 200$ GPa, $v = 0.3$ and $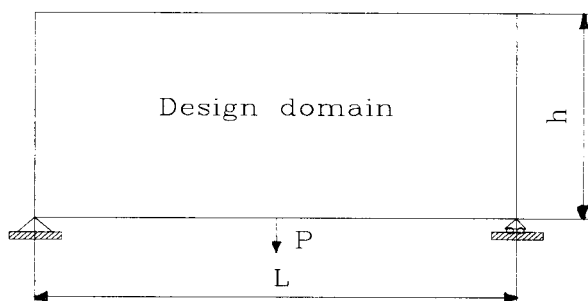t = 2$ mm. A point load $P = 400$ N is applied to the struc-ture. For cases (b) to (d), $RR_0 = 1$ per cent and $ER = 1$ per cent are used. Plane stress conditions are assumed for all cases.

Figure 5 shows the performance index history for case (a). It can be seen that the performance index is increased when inefficient material is removed from the
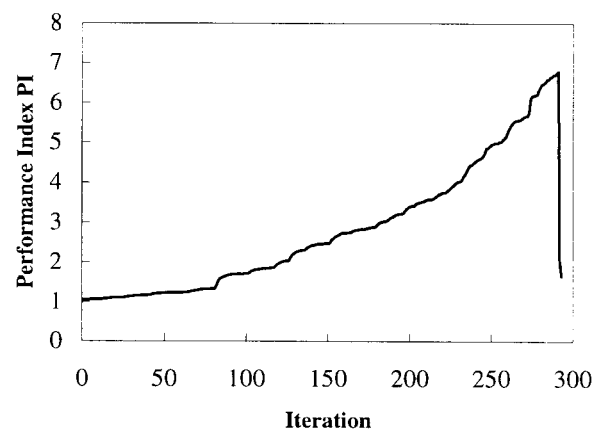

Fig. 4 Design domain for the Michell structures

Fig. 5 Performance index history of case (a)

structure. However, further element removal from the optimal design eventually leads to collapse of the structure. The effects of height constraints on the performance index of the Michell-type structures are illustrated in Fig. 6. It can be seen that the performance index increases with increase in height when compared with the initial design domains. The maximum performance indices for cases (a) to (d) are 6.8, 4.97, 1.89 and 1.44 respectively. The optimal topologies obtained for each case are shown in Fig. 7. The optimal topologies shown

in Figs 7a and b exhibit truss-like structures that can be designed as trusses. When $h \ll L$, the optimal topology as shown in Fig. 7d evolves to a continuum structure.

4.3 Efficiency of structural topology

The performance index developed here can be used to compare the efficiency of structural topologies produced by different optimization methods. A transverse beam of homogeneous material with fixed supports shown in Fig. 8 is optimized by using the ESO procedure. The design domain is discretized into a 90×30 mesh using four-node plane stress elements. A concentrated load of 400 N is applied to the centre of the bottom. The material properties are $E = 200$ GPa, $\nu = 0.3$ and $t = 2$ mm. In this plane stress problem, $RR_0 = 1$ per cent and $ER = 1$ per cent are used.

The evolutionary history of the performance index for this beam is presented in Fig. 9. The maximum performance index is 14.32. The topologies obtained at different iterations are shown in Fig. 10, where the optimal topology is uniformly stressed. Figure 10d shows the final design proposal presented by Mattheck [17] using the soft kill option (SKO) approach. This proposal is regenerated here using the same mesh as used in the ESO. A linear static finite element analysis is carried out to analyse the design proposal. The performance index of the proposal calculated using equation (11) is 1.92, which is
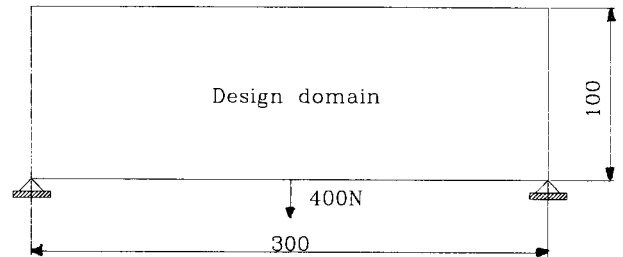

Fig. 6   Effects of height constraints on the performance index

(a) Optimal topology ($h/L = 1/2$)
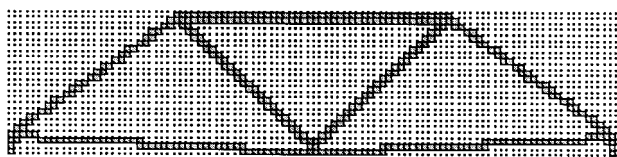
(b) Optimal topology ($h/L = 1/4$)

(c) Optimal topology ($h/L = 1/8$)

(d) Optimal topology ($h/L = 1/12$)

Fig. 7   Effects of height constraints on the optimal topologies

Fig. 8   Design domain for the beam with fixed supports
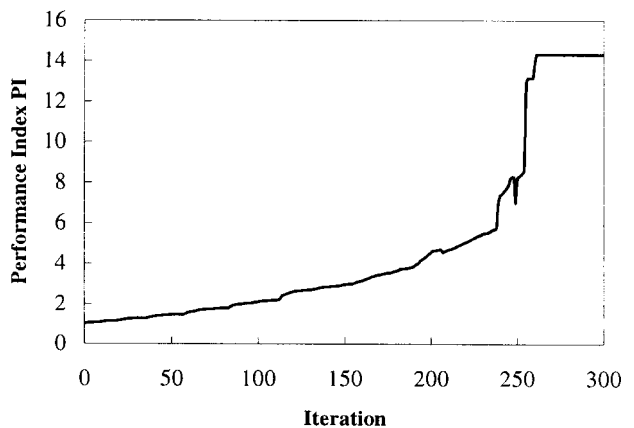
Fig. 9   Performance index history of the beam with fixed supports

(a) Topology at iteration 50

(b) Topology at iteration 95

(c) Optimal topology
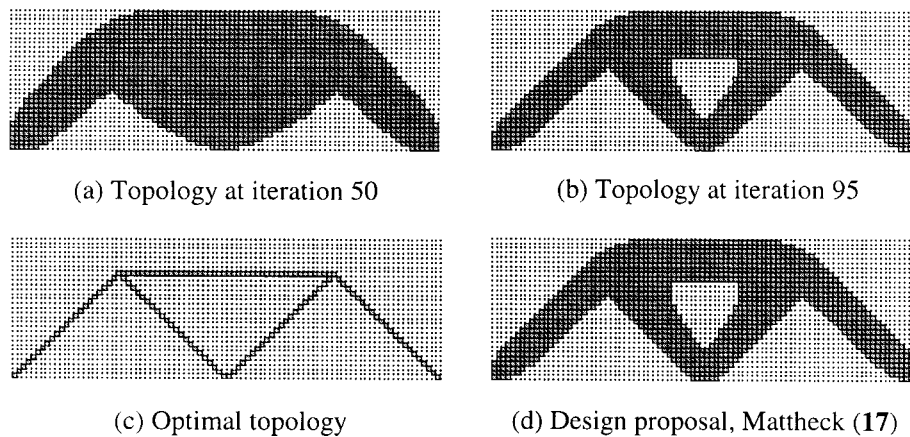
(d) Design proposal, Mattheck (**17**)

Fig. 10 Evolutionary topology optimization for the beam with fixed supports

much less than that obtained by the ESO method. In fact, it can be seen that this final proposal given by Mattheck is very similar to the topology shown in Fig. 10b, which is far from the optimum for the light-weight design of the beam with stress constraint. Therefore, it can be concluded that the proposed performance index is a useful tool for measuring the efficiency of structural topologies and identifying the most efficient topology.

5 DISCUSSION

The examples presented have shown that the proposed performance index can predict the evolutionary history of the structural efficiency in an optimization process and the optimal topology. The performance indices and the optimal topologies of continuum structures with stress constraints are affected by the uniformity of stress within the structure, the height constraints and the position of the loads. These factors need to be considered in the selection of optimal topologies in structural design.

The stress distribution within the initial design domain of a continuum structure is hardly uniform owing to the stress concentration in the region of loading and supports. The performance index formulated by considering the maximum stress constraint can indicate the process of maximizing the uniformity of stresses in the optimal topology. For example, the performance index of the cantilever beam at the optimum is constant at the later stage of iterations as shown in Fig. 2. This means that element stresses in the optimum are approximately uniform. In contrast, for the Michell-type structures with simple supports, the performance index drops sharply after reaching the maximum value, as illustrated in Fig. 5. In this case, there are still lowly stressed elements in the design, but the range of stress levels has been significantly narrowed.

The optimal structural topologies are dependent on

the height constraints imposed on the initial design domains. The efficiency of the final design of a structure is improved when increasing the height of the initial design domains as shown in Fig. 6. For the simply supported Michell structures, the optimal height is close to the value $h/L = 1/2$. When $h \ll L$, the performance index of the optimized design will be close to unity, which is the performance index of the initial design domain.

The magnitude of the load may have a significant impact on the weight of a final design but not on the optimal topology. However, the loading position obviously determines the optimal material layouts of continuum structures. It can be seen from equation (11) that the load applied to the structure is eliminated from the performance index which only depends on the topology and not on the scale of the loads. The proposed performance index can also be used to optimize structures under multiple load cases.

6 CONCLUSIONS

This paper has presented a performance index for assisting the selection of best topologies for the least-weight design of continuum structures with stress constraints. A performance index has been developed using the scaling design approach. This is valid for systems where the stiffness matrix of a structure is a linear function of the design variable. The proposed performance index has been used successfully in the evolutionary structural optimization (ESO) procedure to monitor the optimization process and to identify the optimal topology of various structures with stress and height constraints.

The performance index can indicate the uniformity of stress within the optimal topology of continuum structures. Increasing the height of the initial design domain usually improves the efficiency of the final optimal design. The proposed performance index can be incorporated in any structural optimization method as an indicator of material efficiency and used to compare the

efficiency of structural topologies obtained by different structural optimization methods. Furthermore, the performance index can also be applied to continuum structures under multiple load cases.

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