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POSITIVE SOLUTIONS OF A BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. We consider boundary-value problems studied in a recent paper. We show that some existing theory developed by Webb and Infante applies to this problem and we use the known theory to show how to find improved estimates on parameters μ^*, λ^* so that some nonlinear differential equations, with nonlocal boundary conditions of integral type, have two positive solutions for all λ with $\mu^* < \lambda < \lambda^*$.

1. Introduction

In a recent paper [3], Chasreechai and Tariboon gave some existence theorems for positive solutions of problems of the following type.

$$u''(t) + \lambda g(t) f(u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

with nonlocal boundary conditions (BCs) of the form

$$u(0) = \beta_1 \int_0^\eta u(s)\, ds, \quad u(1) = \beta_2 \int_0^\eta u(s)\, ds. \tag{1.2}$$

Here $\eta \in (0,1)$ is given and β_1, β_2 are to satisfy some inequalities. The authors of [3] actually studied the problem on an interval $[0,T]$ but, since, without loss of generality, most previous theory has considered the interval $[0,1]$, we have changed notation to the usual one.

The authors of [3] were unaware that this equation, and others, with more general BCs had already been studied in [16, 19, 17, 13, 2]. In fact, a general theory which applies to many problems was given in [16]. There the authors studied in detail (1.1) with the nonlocal BCs involving linear functionals $\beta_i[u]$, that is, Riemann-Stieltjes integrals

$$u(0) = \beta_1[u] := \int_0^1 u(s)\, dB_1(s), \quad u(1) = \beta_2[u] := \int_0^1 u(s)\, dB_2(s), \tag{1.3}$$

where B_j can be functions of bounded variation, equivalently dB_1, dB_2 are *signed* measures. Clearly the BCs in (1.2) are special cases of the BCs in (1.3). The paper [17] extended this to allow equations of higher order with BCs of a similar type. Using the theory of [16] reduces the problem of showing there are positive solutions to that of calculating explicitly the constants that occur in that theory.

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Problems with BCs of Riemann-Stieltjes type have been studied by Karakostas and Tsamatos, [7, 8], and by Yang [19, 20], and with a different method, mixed monotone operator theory, by Kong [9], all with (positive) measures. The special case of multipoint BCs has been very extensively studied, for example [4, 5, 11]; for a survey of much work up to 2007 see [12]. The case of BCs of integral type is also well studied, for example, [1, 2].

The idea of having Riemann-Stieltjes type BCs with *sign changing* measures is due to Webb-Infante [15, 16]. A typical examples is when

$$\beta[u] = \sum_{i=1}^{m} \beta_i u(\xi_i) + \int_0^1 b(s)u(s)\,ds,$$

with $\xi_i \in (0,1)$, under some suitable positivity assumptions, but, in the general case, some coefficients β_i can be negative and b can take some negative values.

Let $f^0 := \lim_{u\to 0} f(u)/u$, $f^\infty := \lim_{u\to\infty} f(u)/u$, (with ∞ an allowed 'value').

The results of [3] are of the following type

(1) For every $\lambda > 0$ the boundary value problem (BVP) (1.1), (1.2) has at least one positive solution if

$$\text{either } f^0 = 0 \text{ and } f^\infty = \infty, \quad \text{(superlinear case)},$$
$$\text{or } f^0 = \infty \text{ and } f^\infty = 0, \quad \text{(sublinear case)}.$$

(2) If $f^0 = \infty$ and $f^\infty = \infty$ and there is ρ_1 such that $\lambda f(u) \le \rho_1/\Lambda_1$ for all $u \in [0,\rho_1]$, (Λ_1 is a constant determined by the problem), then the boundary value problem (BVP) (1.1), (1.2) has at least two positive solutions.

(3) If $f^0 = 0$ and $f^\infty = 0$ and there is ρ_2 such that $\lambda f(u) \ge \rho_1/\Lambda_2$ for all $u \in [c\rho_2, \rho_2]$, (c, Λ_2 are constants determined by the problem), then the boundary value problem (BVP) (1.1), (1.2) has at least two positive solutions.

Results of the same type had been given in [16, 17]. In [3] Krasnosel'skiĭ's theorem (see, for example, [10]) is used, whereas [16, 17] uses fixed point index theory, in particular, some results of [18].

For a given nonlinear term f the conditions in (2) are of the form:

> There exists λ^* such that at least two positive solutions exist for every λ with $\lambda \le \lambda^*$.

The conditions in (3) are of the form:

> There exists λ^{**} such that at least two positive solutions exist for every λ with $\lambda \ge \lambda^{**}$.

It is natural to ask whether λ^*, λ^{**} can be closely estimated.

We intend to give a detailed account of the BVP (1.1), (1.2) and we shall show how the results of [16], and some nonexistence results from [14], can be used to determine some explicit *upper and lower* bounds for these quantities. Indeed we show for (2) that there is an explicit μ^* such that there is no positive solution for $\lambda > \mu^*$ and determine an explicit λ^*. Similarly we find explicit upper and lower bounds in case (3).

The BCs (1.2) are rather special, the second BC is a constant multiple of the first, accordingly the calculations using the methods of [16, 17] simplify dramatically, in fact we give a shortcut that applies in this case. We obtain some new estimates of constants related to the constant c in (3) above.

We illustrate our results by revisiting the examples in [3] and show that our estimates on λ in these examples give upper and lower bounds that are quite close together. Our estimates substantially improve those of [3]. We also use a result of [14], which applies in this case, and gives a slightly stronger conclusion on one example.

2. Summary of some known theory

We work throughout in the Banach space $C[0,1]$ of continuous real-valued functions defined on $[0,1]$ endowed with the norm $\|u\| := \max_{t\in[0,1]} |u(t)|$.

We recall the theory developed in [16] as it applies to the following nonlocal BVP,

$$\begin{gathered} u''(t) + g(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \beta_1[u], \quad u(1) = \beta_2[u], \end{gathered} \tag{2.1}$$

where $\beta_j[u]$ are linear functionals on $C[0,1]$, hence given by

$$\beta_1[u] = \int_0^1 u(t)\,dB_1(t), \quad \beta_2[u] = \int_0^1 u(t)\,dB_2(t), \tag{2.2}$$

where B_j are functions of bounded variation. We suppose for simplicity, that f is continuous, g satisfies an integrability condition $g\Phi \in L^1$, so g can have pointwise singularities, where Φ is given below; for more general conditions see [16, 17]. Here we will only consider the case when β_j are positive functionals.

The first step is to find the Green's function G for the problem. A general method for doing this is given in [16, 17] and is recalled below.

Positive solutions of the BVP are then equivalent to positive fixed points of the integral operator

$$Su(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds. \tag{2.3}$$

To find positive solutions we seek fixed points of the operator S in some sub-cone K of the cone

$$P := \{u \in C[0,1] : u(t) \geq 0, \text{ for } t \in [0,1]\}.$$

The theory in [16] requires the following condition.

(C) There exist a subinterval $[a,b] \subseteq [0,1]$, a measurable function Φ, and a constant $c(a,b) > 0$ such that

$$\begin{gathered} G(t,s) \leq \Phi(s) \text{ for } t \in [0,1] \text{ and } s \in [0,1] \\ G(t,s) \geq c(a,b)\Phi(s) \text{ for } t \in [a,b] \text{ and } s \in [0,1]. \end{gathered}$$

This is often proved by showing the following condition for some $c \in P \setminus \{0\}$,

$$c(t)\Phi(s) \leq G(t,s) \leq \Phi(s), \quad \text{for } 0 \leq t,\ s \leq 1. \tag{2.4}$$

If (2.4) is valid we can take any $[a,b] \subset [0,1]$ for which $c(a,b) := \min_{t\in[a,b]} c(t) > 0$. When (C) holds, S maps P into the sub-cone K where

$$K := \{u \in P : u(t) \geq c(a,b)\|u\| \text{ for } t \in [a,b]\}. \tag{2.5}$$

We also use some comparisons with the associated linear operator

$$Lu(t) := \int_0^1 G(t,s)g(s)u(s)\,ds. \tag{2.6}$$

Under the above conditions $L(P) \subset K$. It is known that, under the above conditions, S, L are completely continuous and that the spectral radius of L, denoted $r(L)$, is positive. By the Krein-Rutman theorem, $r(L)$ is an eigenvalue of L with an eigenfunction in P, hence also in K; $r(L)$ is usually called the principal eigenvalue of L. We let $\mu(L) := 1/r(L)$ be the principal characteristic value of L.

The existence result in [16] uses some constants defined as follows.

$$\begin{gathered} m := \Big(\sup_{t\in[0,1]} \int_0^1 G(t,s)g(s)\,ds \Big)^{-1}, \\ M = M(a,b) := \Big(\inf_{t\in[a,b]} \int_a^b G(t,s)g(s)\,ds \Big)^{-1}. \end{gathered} \tag{2.7}$$

We use the following notations.

$$\begin{gathered} f^0 = \limsup_{u\to 0+} f(u)/u, \quad f_0 = \liminf_{u\to 0+} f(u)/u; \\ f^\infty = \limsup_{u\to \infty} f(u)/u, \quad f_\infty = \liminf_{u\to \infty} f(u)/u. \end{gathered}$$

The existence theorem for one positive solution reads as follows.

Theorem 2.1 ([16])**.** *Suppose* (C) *holds for every $[a,b] \subset (0,1)$. Then equation* (2.3) *has a positive solution $u \in K$ if one of the following conditions holds.*

- (S_1) $0 \le f^0 < \mu(L)$ *and* $\mu(L) < f_\infty \le \infty$.
- (S_2) $\mu(L) < f_0 \le \infty$ *and* $0 \le f^\infty < \mu(L)$.

When we want a result for the problem

$$\begin{gathered} u''(t) + \lambda g(t) f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \beta_1[u], \quad u(1) = \beta_2[u], \end{gathered}$$

we replace f by λf in the above. If we want a result that holds for all $\lambda > 0$ then we must impose the sublinear or superlinear conditions on f. The result for at least two positive solutions is the following.

Theorem 2.2 ([16])**.** *Suppose* (C) *holds for some $[a,b] \subset [0,1]$. Then equation* (2.3) *has at least two positive solutions in K if one of the following holds.*

- (D_1) $0 \le f^0 < \mu(L)$, *there is $\rho > 0$ such that $f(u) > M\rho$ for all $u \in [\rho, \rho/c(a,b)]$, and* $0 \le f^\infty < \mu(L)$.
- (D_2) $\mu(L) < f_0 \le \infty$, *for some $\rho > 0$, $f(u) < m\rho$ for all $u \in [0,\rho]$, and* $\mu(L) < f_\infty \le \infty$.

Remark 2.3. Since ρ is at our disposal "$f(u) > M$ for all $u \in [\rho, \rho/c(a,b)]$" is equivalent to "$f(u) > c(a,b)M$ for all $u \in [c(a,b)\rho_1, \rho_1]$". This condition is less restrictive when $c(a,b)$ is as large as possible, the length of the interval on which the condition holds is less, and when $M(a,b)$ is as small as possible, the height to be exceeded by $f(u)$ is less. However, a minimal M could correspond to a very small c so the actual choice of $[a,b]$ depends on properties of the given nonlinearity. We have found that aiming to make $M(a,b)$ as small as possible is often a good choice.

There is another recent result that complements (D_1) and applies when we have the following condition to replace (C).

- (C_0) There exist $\Phi_1(s)$ and a constant $c_0 > 0$ such that $c_0\Phi_1(s) \le G(t,s) \le \Phi_1(s)$, for all $t, s \in [0,1]$.

Theorem 2.4 ([14]). *Suppose* (C_0) *holds. Then equation* (2.3) *has at least two positive solutions in* K *if*

- (D_1') $0 \le f^0 < \mu(L)$, *there exists* $\rho > 0$ *such that* $f(u) > \mu(L)u$ *for* $\rho \le u \le \rho/c_0$, *and* $0 \le f^\infty < \mu(L)$.

The reason why this complements (D_1) is the following. It is known that $m \le \mu(L) \le M(a,b)$ for every $[a,b] \subset [0,1]$ ([18]), and, if $c(a,b)$ and c_0 have been found as large as possible then $c_0 \le c(a,b)$. At the point ρ the condition $f(\rho) > \mu(L)\rho$ is weaker than $f(u) > M(a,b)\rho$, but at the point $u = \rho/c(a,b)$, the condition $f(u) > \mu(L)u$ can be more restrictive than $f(u) > M(a,b)\rho$, depending on the values of $M(a,b)$ and $c(a,b)$. It is easy to give examples where either one of the conditions is applicable but the other is not.

There are also non-existence results see [14, 17], which show that the hypotheses in the Theorem 2.1 are sharp.

Theorem 2.5. *The operator* S *defined in* (2.3) *has no nonzero fixed points in* P *if either*

$$f(t,u) < \mu(L)u \quad \text{for all } u > 0,$$

or

$$f(t,u) > \mu(L)u \quad \text{for all } u > 0.$$

3. The Green's function and related properties

We now consider the BVP

$$u''(t) + \lambda g(t) f(u(t)) = 0, \quad t \in (0,1), \tag{3.1}$$

with the nonlocal BCs

$$u(0) = \beta[u] := \int_0^1 u(s)\, dB(s), \quad u(1) = k\beta[u] = k\int_0^1 u(s)\, dB(s), \quad (k > 0), \tag{3.2}$$

where dB is a Stieltjes measure, that is, B in non-decreasing. A typical example is when

$$\beta[u] = \sum_{i=1}^m \beta_i u(\xi_i) + \int_0^1 b(s)u(s)\, ds.$$

with $\xi_i \in (0,1)$, under positivity conditions on β_i and b.

This is a more general version of the problem studied in [3]. We will determine the Green's function and show how the conditions (C) and (C_0) can be established. These results are new.

The Green's function can be found from the explicit formulae in [16], or, for those who prefer to work with matrices, from [17]. However, there is a shortcut, which uses the *methods* that were used to obtain these formulae, which we now give.

Let $\gamma_1(t) := 1 - t, \gamma_2(t) := t$. The Green's function for the BVP

$$u''(t) + \lambda g(t) f(u(t)) = 0,\ t \in (0,1), \quad u(0) = 0, u(1) = 0, \tag{3.3}$$

is well-known and can be written

$$G_0(t,s) = \begin{cases} \gamma_1(t)\gamma_2(s), & s \le t, \\ \gamma_1(s)\gamma_1(t), & s > t. \end{cases} \tag{3.4}$$

Hence it follows readily that

$$c_0(t)\Phi_0(s) \le G_0(t,s) \le \Phi_0(s), \quad \text{for all } s,t \in [0,1], \tag{3.5}$$

where $\Phi(s) = s(1-s)$, $c_0(t) = \min\{t, 1-t\}$. We write

$$S_0u(t) = \int_0^1 G_0(t,s)g(s)f(u(s))\,ds.$$

It is now easy to see that solutions of the BVP (3.3) are fixed points of the nonlinear operator

$$Tu(t) := \gamma_1(t)\beta[u] + \gamma_2(t)k\beta[u] + S_0u(t) = \gamma(t)\beta[u] + S_0u(t),$$

where $\gamma(t) = 1+(k-1)t$. Hence, if $\beta[\gamma] \neq 1$, by applying the functional β and then replacing $\beta[u]$ (for details see any of [15, 16, 17] in increasing degrees of generality), fixed points of T are fixed points of S where

$$Su(t) = \frac{\gamma(t)}{1-\beta[\gamma]}\int_0^1 \mathcal{G}(s)g(s)f(u(s))\,ds + \int_0^1 G_0(t,s)g(s)f(u(s))\,ds,$$

where

$$\mathcal{G}(s) := \int_0^1 G_0(t,s)\,dB(t). \tag{3.6}$$

Thus, the Green's function for the BVP (3.3) is given by

$$G(t,s) = \frac{\gamma(t)}{1-\beta[\gamma]}\,\mathcal{G}(s) + G_0(t,s). \tag{3.7}$$

We will suppose that $0 \le \beta[\gamma] < 1$ in order that S maps P into P. When sign changing measures are used it is also required that $\mathcal{G}(s) \ge 0$; for (positive) measures that we are now considering this holds automatically. The expression (3.7) can be checked by a longer calculation using the formulae in [16] or [17]. The authors of [3] found their expression for the Green's function in a different form by a direct, longer, calculation. The form we find has several advantages: it is easily determined; it applies to many BCs at once; the properties required follow easily from those of G_0, which are simpler to discuss, for example positivity of G is now obvious.

Let $\gamma_m := \min_{t\in[0,1]} \gamma(t) = \min\{1,k\}$. Clearly we have

$$G(t,s) \le \frac{\|\gamma\|}{1-\beta[\gamma]}\,\mathcal{G}(s) + \Phi_0(s), \tag{3.8}$$

$$G(t,s) \ge \frac{\gamma_m}{1-\beta[\gamma]}\,\mathcal{G}(s) + c_0(t)\Phi_0(s). \tag{3.9}$$

Thus if we let $\Phi(s) := \frac{\|\gamma\|}{1-\beta[\gamma]}\,\mathcal{G}(s) + \Phi_0(s)$ we have

$$G(t,s) \ge \min\Big\{\frac{m_\gamma}{\|\gamma\|}, c_0(t)\Big\}\Phi(s).$$

Thus condition (C) holds with $c(t) = \min\big\{\frac{m_\gamma}{\|\gamma\|}, c_0(t)\big\}$ which in this case is

$$c(t) = \min\Big\{\frac{\min\{1,k\}}{\max\{1,k\}}, t, 1-t\Big\}. \tag{3.10}$$

We will also show that (C_0) holds for the problem (3.1), (3.2). We note that, since $c_0(t)\Phi_0(s) \le G_0(t,s)$, we have

$$\mathcal{G}(s) = \int_0^1 G_0(t,s)\,dB(t) \ge \Phi_0(s)\int_0^1 c_0(t)\,dB(t) = \beta[c_0]\Phi_0(s). \tag{3.11}$$

Therefore, from (3.8), we obtain

$$G(t,s) \le \frac{\|\gamma\|}{1-\beta[\gamma]}\,\mathcal{G}(s) + \frac{1}{\beta[c_0]}\mathcal{G}(s).$$

Hence, taking $\Phi_1(s) := \dfrac{\|\gamma\|}{1-\beta[\gamma]}\,\mathcal{G}(s) + \dfrac{1}{\beta[c_0]}\,\mathcal{G}(s)$, we have

$$G(t,s) \ge \frac{\gamma_m}{1-\beta[\gamma]}\,\mathcal{G}(s) \ge \frac{\gamma_m}{\|\gamma\| + \frac{1-\beta[\gamma]}{\beta[c_0]}}\,\Phi_1(s).$$

Thus, (C_0) holds with

$$c_0 = \frac{\gamma_m}{\|\gamma\| + \frac{1-\beta[\gamma]}{\beta[c_0]}}. \tag{3.12}$$

4. Examples

The authors of [3] studied the problem with BCs

$$u(0) = \beta_1 \int_0^\eta u(s)\,ds, \quad u(1) = \beta_2 \int_0^\eta u(s)\,ds. \tag{4.1}$$

The given conditions on the coefficients β_1, β_2 are

$$0 < \beta_2\eta^2 < 2, \quad \text{and} \quad 0 < \beta_1 < \frac{2-\beta_2\eta^2}{\eta(1-\eta/2)},$$

which is equivalent to the condition $0 < \beta[\gamma] = \beta_2\eta^2/2 + \beta_1(\eta - \eta^2/2) < 1$, in our notation. By concavity arguments, they essentially proved ([3, Lemma 2.4]) a result equivalent to G satisfying condition (C_0) with a constant

$$\widetilde{c}_0 = \min\Big\{\frac{\beta_2\eta(1-\eta)}{2-\beta_1\eta-\beta_2\eta^2}, \frac{\beta_2\eta^2}{2-\beta_1\eta}, \frac{\beta_1\eta(1-\eta)}{2-\beta_1\eta}, \frac{\beta_1\eta^2}{2-\beta_1\eta}\Big\}. \tag{4.2}$$

We shall see that, in the first example we give, this is smaller than the constant found in (3.12), hence would always give a worse result than we could give.

Example 4.1. We first consider Example 4.3 from [3]. We have given several numbers rounded to five decimal places but all numbers in this example can be given with greater accuracy.

Let $f(u) := u^2e^{-u}$. The problem is

$$\begin{gathered} u''(t) + \lambda f(u(t)), \quad 0<t<1, \\ u(0) = (8/5)\int_0^{1/4} u(s)\,ds, \quad u(1) = 16\int_0^{1/4} u(s)\,ds. \end{gathered} \tag{4.3}$$

Hence $k = 10$ in this example. In [3] the problem is posed on $[0, 4/5]$, we have made the simple change of variable to pose the problem on $[0, 1]$.

In [3] it is shown that $\widetilde{c}_0 = 1/16$ and that there are at least two positive solutions for $\lambda \ge 16\exp(32)/25 \approx 5.0536 \times 10^{13}$ (constant modified by us to fit the interval $[0, 1]$). From (3.12) we have $c_0 = 1/13$ in this case, an improvement on $1/16$.

By calculation we have

$$\mathcal{G}(s) = \int_0^1 G_0(t,s)dB(t) = (8/5)\int_0^{1/4} G_0(t,s)\,dt = \frac{1}{20}\begin{cases} 7s - 16s^2, & \text{if } s \le 1/4, \\ 1-s, & \text{if } s > 1/4. \end{cases}$$

We now determine the constants required in Theorem 2.2. We choose $[a, b]$ so as to minimize $M(a, b)$. We note that $c(t) = \min\{m_\gamma/\|\gamma\|, c_0(t)\} = \min\{1/10, t, 1-t\}$

and we can have $c(t) = 1/10$ for all intervals $[a, b] \subset [1/10, 9/10]$. We used Maple to calculate $M(a, b)$ on such intervals and calculated $M(0.273, 0.9) \approx 2.70608$ which is close to minimal for such intervals.

We remark that $f(u)$ is increasing for $0 \le u \le 2$ and is decreasing for $u \ge 2$; also $f(u)/u$ is increasing for $0 < u \le 1$ and decreasing for $u \ge 1$ and $f^0 = f^\infty = 0$. To apply Theorem 2.2 we want to find ρ such that $\lambda f(u) > M\rho$ for $\rho \le u \le \rho/c(a, b) = 10\rho$.

If $10\rho \le 2$ then by the above remarks we must choose ρ so that $\lambda f(\rho) > M\rho$, that is, $\lambda > M \exp(\rho)/\rho$ and the least λ is obtained by choosing $\rho = 1/5$. Thus, for $\rho = 1/5$ we find $\lambda f(u) > \rho M$ for $\rho \le u \le \rho/c(a, b) = 10\rho$ if $\lambda > 5M \exp(1/5) \approx 16.52606$.

If $\rho > 2$ we must have $\lambda f(10\rho) > M\rho$, that is, $\lambda 100\rho^2 \exp(-10\rho) > M\rho$, or $\lambda > \dfrac{M}{10}\dfrac{\exp(10\rho)}{10\rho}$ and the least possibility is $\lambda > \dfrac{M}{10}\dfrac{\exp(20)}{20} \approx 6.56448 \times 10^6$, thus this is not a good choice, though it gives a better constant than 5.0536×10^{13}.

If $\rho < 2 < 10\rho$ then we choose ρ so that $\exp(\rho)/\rho = \exp(10\rho)/(100\rho)$ and find $\rho \approx 0.51169$ and then it suffices to have $\lambda > 8.82185$.

We now give a lower estimate using Theorem 2.5. For this we need the principal characteristic value $\mu(L)$. By considering eigenfunctions of the form $\sin(\omega t + \theta)$ and using the boundary conditions, we find, using Maple to solve some equations involving trigonometric functions, that $\mu(L) \approx 1.36469$. If $\lambda f(u) < \mu(L)u$ for all $u \ge 0$ there is no positive solution, thus there is no positive solution if $\lambda < \exp(1)\mu(L) \approx 3.70963$.

We now see how Theorem 2.4 with condition $(D2')$ applies in this example. We now want to find r such that $\lambda f(u) \ge \mu(L)u$ for all $r \le u \le r/c_0 = 13r$. Since $f(u)/u$ is increasing for $0 < u \le 1$ and decreasing for $u \ge 1$ we choose r so that $f(r)/r = f(13r)/13r$. We calculate $r \approx 0.21375$ and hence we find there are at least two positive solutions if $\lambda > \mu(L)r/f(r) \approx 7.90619$. Thus, in this case, $(D2')$ gives a better result than $(D2)$.

In summary, using the theory of Webb-Infante [16] we have shown that

> for $\lambda > \lambda^{**} \approx 8.82185$ the problem has at least two positive solutions, and for $\lambda < \mu^{**} = \exp(1)\mu(L) \approx 3.70963$ there are no positive solutions.

Using the newer result from [14] we can improve this to get $\lambda^{**} \approx 7.90619$. These estimates of λ^{**} are substantial improvements of that from [3] written above.

Example 4.2. We now consider Example 4.2 from [3], with a changed notation to make the interval $[0, 1]$. Let $f(u) := u^{1/2} + u^2$, let $g(t) = (1 - t)^{1/2}$. The problem is

$$\begin{gathered} u''(t) + \lambda g(t) f(u(t)), \quad 0 < t < 1 \\ u(0) = \frac{2}{5}\int_0^{1/4} u(s)\, ds, \quad u(1) = 8\int_0^{1/4} u(s)\, ds. \end{gathered} \tag{4.4}$$

Hence $k = 20$ in this example. It was shown in [3] that there are at least two positive solutions for $\lambda \le 1/16 = 0.0625$ (constant modified by us to fit the interval $[0, 1]$).

We want to calculate m and $\mu(L)$ for existence and nonexistence results. Firstly $1/m = \max_{t\in[0,1]} \int_0^1 G(t, s)g(s)\, ds$. By a Maple calculation we find $m \approx 6.46779$. Because of the term $\sqrt{1-t}$ we had to use a numerical program, written in C for me by my colleague Prof. K. A. Lindsay, to calculate $\mu(L)$. It gives $\mu(L) \approx 8.375$; we can only give 3 decimal place accuracy here because the numerical program used is written to run on a desktop pc.

We note that f is increasing and that $f(u)/u$ is decreasing for $u \leq p := 2^{-2/3} \approx 0.62996$ and increasing for $u \geq p$. We want ρ such that $\lambda f(\rho) < m\rho$ for $0 \leq u \leq \rho$. Hence we choose $\rho = p$ and then the problem has at least two positive solutions for all $\lambda < \lambda^* = mp/f(p) \approx 3.4223$.

There are no positive solutions if $\lambda f(u) \geq \mu(L)u$ for all $u > 0$, thus there are no positive solutions if $\lambda > \mu^* = \mu(L)p/f(p) \approx 4.431$.

Again the constant λ^* here is a substantial improvement on the result of [3].

Our discussion shows that it is important in examples to choose ρ to fit the behaviour of f.

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