Computer algebra models the inertial dynamics of a thin film flow of power law fluids and other non-Newtonian fluids

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Abstract

Consider the evolution of a thin layer of non-Newtonian fluid. I model the case of a nonlinear viscosity that depends only upon the shear-rate; power law fluids are an important example, but the analysis is for general nonlinear dependence upon the shear-rate. The modelling allows for large changes in film thickness provided the changes occur over a large enough lateral length scale. The modelling is based on two macroscopic modes by fudging the spectrum: here fiddle the surface boundary condition for tangential stress so that, as well as a mode representing conservation of fluid, the lateral shear flow $u \propto y$ is a neutral critical mode. Thus the resultant model describes the dynamics of gravity currents of non-Newtonian fluids when their flow is not very slow. For an introduction I first report on an analogous case of nonlinear diffusive dissipation.

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1 Nonlinear dissipation in a toy problem

Before analysing the flow of a thin fluid with nonlinear rheology, let us play with a toy problem that has the complexity of a nonlinear stress-shear relation but little else. Consider the dynamics of

$$\frac{\partial u}{\partial t} = \frac{\partial \tau}{\partial y} \quad \text{where the `stress'} \quad \tau = \tau(u_y) \tag{1}$$

depends nonlinearly upon the 'shear' $u_y = \frac{\partial u}{\partial y}$; for example, the dependence may be a power law $\tau = (u_y)^s$ for some power s. This PDE is equivalently

$$\frac{\partial u}{\partial t} = \tau'(u_y)\frac{\partial^2 u}{\partial y^2}\,.$$

I aim to solve this nonlinear dissipation equation (1) with boundary conditions

$$u = 0 \text{ on } y = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0 \text{ on } y = 1\,, \tag{2}$$

analogous to 'no slip on the bed' ($y = 0$) and 'no shear on the surface' ($y = 1$), respectively.

Place the system in a form to apply centre manifold theory by modifying the 'no shear' boundary condition to

$$(1 + E\gamma)\frac{\partial u}{\partial y} = (1 - \gamma)u \text{ on } y = 1\,, \tag{3}$$

where E is an Euler parameter that we are free to vary to improve convergence: when $\gamma = 1$ this boundary condition reduces to the original (2) and corresponds to the dynamics of interest; but when $\gamma = 0$ the boundary conditions and PDE have the neutral mode $u = \mathrm{E}y$ of any constant 'shear' E. The marvellous aspect of the boundary condition (3) is that when $\gamma = 0$ the 'shear' solutions $u = \mathrm{E}y$ forms a space of *exact equilibria of the nonlinear problem*. We construct a centre manifold model of the dynamics about the space of equilibria that exists when $\gamma = 0$. Consequently, the centre manifold model is *global* in the order parameter E.

1.1 Toy evolution

A result of the centre manifold modelling is that the evolution of the mean 'shear' E is

$$\frac{\partial \mathrm{E}}{\partial t} \;=\; -(\tfrac{5}{2}\gamma - \tfrac{5}{168}\gamma^3)\nu\mathrm{E} + (\tfrac{5}{6}\gamma^2 - \tfrac{5}{72}\gamma^3)\nu'\mathrm{E}^2$$

$$+\gamma^3\left(\frac{5\nu'^2}{28\nu}-\frac{365\nu''}{1008}\right)E^3+\mathcal{O}\big(\gamma^4\big)\,, \tag{4}$$

where ν and its derivatives are purely dependent upon E. The internal field

$$\begin{aligned} u &= Ey+\gamma E\tfrac{5}{12}y(1-y^2)+\gamma^2E\tfrac{5}{288}y(1-y^2)(7-3y^2) \\ &\quad -\gamma^2\nu'/\nu E^2\tfrac{5}{32}y(1-y^2)^2+\mathcal{O}\big(\gamma^3\big)\,. \end{aligned} \tag{5}$$

As yet, I do not know any significant interpretation of this model.

1.2 Power law dissipation is simpler

Now suppose the stress-shear law is the nondimensional power law $\tau = (u_y)^s$: $s=1$ is Newtonian; $s<1$ is shear thinning; $s>1$ is shear thickening. Restrict the results to this power law with the observation that then $\nu(E) = sE^{s-1}$ and $\nu^{(n)}=(s-n)\nu^{(n-1)}/E$.

With the Euler parameter $E=-1/6$, truncated to errors $\mathcal{O}\big(\gamma^5\big)$, when evaluated at the physical $\gamma=1$, the model gives the field u shown in Figure 1. Observe four aspects: the field u is strictly proportional to the order parameter of mean shear E; the field u closely satisfies the requisite but nontrivial 'no shear' boundary condition $\frac{\partial u}{\partial y}=0$ on $y=1$; the exact curve for $s=1$ is indistinguishable from our approximation; and that shear thinning case has a more plug-like solution, whereas shear thickening case has more uniform shear. All these aspects are very satisfactory.

The corresponding evolution is

$$\begin{aligned} \frac{\partial E}{\partial t} &= E^s\Big[-\gamma\tfrac{5}{2}s+\gamma^2\tfrac{5}{6}s(s-1)-\gamma^3\tfrac{5}{1001}s(27s^2-133s+90) \\ &\qquad +\gamma^4\tfrac{5}{9072}(55s^3-414s^2+935s-570)\Big]+\mathcal{O}\big(\gamma^5\big)\,. \end{aligned} \tag{6}$$

Figure 2 plots the right-hand side coefficient as a function of power s for various truncations of this asymptotic expansion. See that for powers s away from 1 we should seek more terms or a better Euler parameter.

1.3 Recast in terms of the mean field

Rewrite the model in terms of the mean $\bar u = \int_0^1 u\,dy$. Approximately, $E\approx 2\bar u$. But a difficulty is that in the model we need to write the 'viscosity' ν

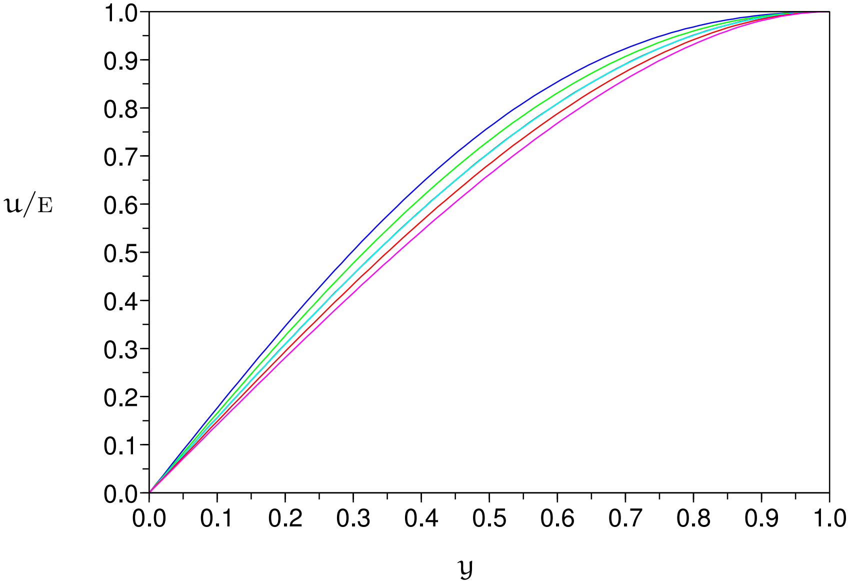

Figure 1: the field u for power laws $s = 0.5 : 0.25 : 1.5$ (from left to right). Also plotted, in black but almost undiscernable under the match with the cyan of the $s = 1$ curve, is $\sin(y\pi/2)$ which is the exact curve for $s = 1$.

and its derivatives as a function of $2\bar{u}$ rather than E; thus denote the viscosity at $2\bar{u}$ by $\bar{\nu}$, whereas we continue to denote the viscosity at E by ν.

I first find that the mean field

$$\begin{aligned}\bar{u} &= \left(\tfrac{1}{2} + \tfrac{5}{48}\gamma + \tfrac{5}{192}\gamma^2 + \tfrac{115}{21504}\gamma^3\right)\mathrm{E} - \left(\tfrac{5}{192}\gamma^2 + \tfrac{155}{9216}\gamma^3\right)\mathrm{E}^2\frac{\nu'}{\nu} \\ &\quad + \gamma^3\left(\frac{5\nu'^2}{8064\nu^2} + \frac{115\nu''}{32256\nu}\right)\mathrm{E}^3 + \mathcal{O}\left(\gamma^4\right). \end{aligned} \tag{7}$$

Revert this series to give the mean shear

$$\begin{aligned}\mathrm{E} &= \left(2 - \tfrac{5}{12}\gamma - \tfrac{5}{288}\gamma^2 + \tfrac{95}{24192}\gamma^3\right)\bar{u} + \left(\tfrac{5}{24}\gamma^2 + \tfrac{5}{1152}\gamma^3\right)\bar{u}^2\frac{\bar{\nu}'}{\bar{\nu}} \\ &\quad + \gamma^3\left(\frac{155\bar{\nu}'^2}{2016\bar{\nu}^2} + \frac{145\bar{\nu}''}{1008\bar{\nu}}\right)\bar{u}^3 + \mathcal{O}\left(\gamma^4\right). \end{aligned} \tag{8}$$

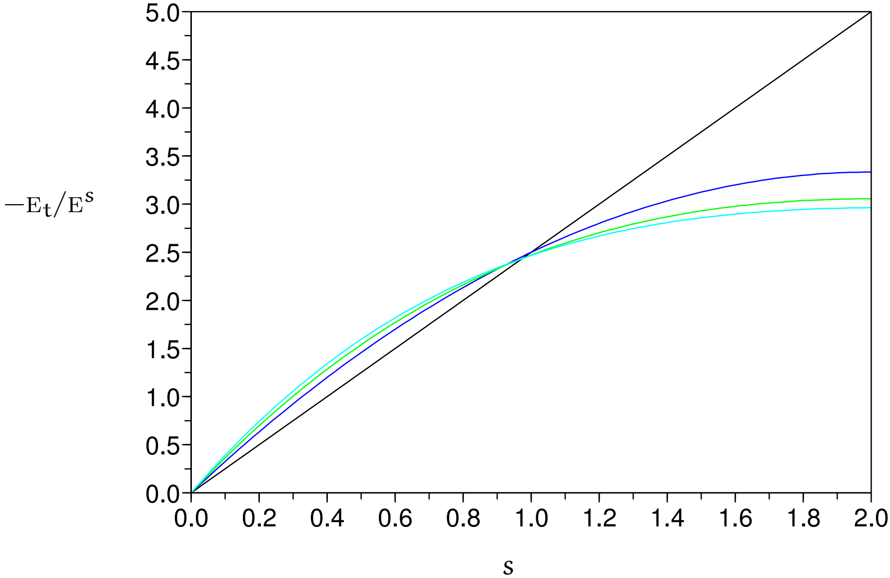

Figure 2: the coefficient of decay for power law dynamics for different truncations of the asymptotic expansions from $\mathcal{O}(\gamma^2)$ errors to $\mathcal{O}(\gamma^5)$ errors: see that although the curves are converging, we could seek quicker convergence or compute to higher order for powers s significantly different from 1.

Finally, this reversion implies we model the corresponding evolution of the mean field $\bar u$ by

$$\begin{aligned}\frac{\partial \bar u}{\partial t} &= -(\tfrac{5}{2}\gamma - \tfrac{5}{168}\gamma^3)\bar v \bar u + (\tfrac{5}{6}\gamma^2 - \tfrac{35}{192}\gamma^3)\bar v' \bar u^2 \\ &\quad - \gamma^3\left(\frac{55\bar v'^2}{168\bar v} - \frac{7415\bar v''}{4032}\right)\bar u^3 + \mathcal{O}(\gamma^4)\,, \end{aligned} \tag{9}$$

Power law dynamics: when $\tau = (u_y)^s$ the model (9) reduces to

$$\begin{aligned}\frac{\partial \bar u}{\partial t} &= (2\bar u)^s\Big[-\tfrac{5}{4}\gamma s - \tfrac{65}{96}\gamma^2 s(1-s) - \tfrac{5}{32256}\gamma^3 s(2840 - 4683s + 1747s^2) \\ &\quad - \tfrac{5}{2322432}\gamma^4 s(149460 - 307655s + 199662s^2 - 42235s^3)\Big]\end{aligned}$$

$$+\mathcal{O}(\gamma^5)\,. \tag{10}$$

1.4 Discussion

This toy problem serves to show how we may overcome the key difficulty in modelling thin films of fluids with nonlinear rheology: namely, adapt the 'surface' boundary condition so that a state of constant 'strain-rate' is a neutral model of the dynamics. Then the algebraic construction of a centre manifold model proceeds routinely. This toy problem also suggests choosing an Euler parameter $E = -1/6$ usefully enhances convergence in the artificial parameter γ. However, more extensive investigation may produce further improvements.

2 Iterate towards a toy model

Consider the dynamics of the toy model (1) with modified boundary condition (3). A construction of the centre manifold model is implemented by the following iteration: the iteration repeatedly updates approximations to the field u and the evolution thereon, $\mathrm{E}_t = g(\mathrm{E})$, until the residuals of the governing equations are zero to some specified order of accuracy.

▷▷ powtoy ◁◁

```
% see power.pdf for documentation
on div; off allfac; on revpri;
◁◁ toy preamble ▷▷
◁◁ toy initialise with linear ▷▷
◁◁ toy truncate the asymptotic expansion ▷▷
it:=1$
repeat begin
◁◁ toy update field from pde and bc ▷▷
showtime;
end until resu=0 and resfs=0 or (it:=it+1)>9;
◁◁ toy power law ▷▷
◁◁ toy recast ▷▷
end;
```

2.1 Toy preamble

Improve printing by factoring with respect to these variables.

▷▷ toy preamble ◁◁

```
factor gam,ee;
```

Denote the mean shear $E(t)$ by ee and its evolution $E_t = g$.

▷▷ toy preamble ◁◁+

```
operator ee;
depend ee,t;
let df(ee,t)=>g;
```

Also define the local diffusion coefficient ν as the derivative of the stress-shear relation: $\nu(u_y) = \tau'(u_y)$. We need various derivatives of ν so denote them by nu(n) when evaluated at the mean shear E. Note: by the chain rule $\frac{\partial}{\partial t}\nu^{(n)}(E) = \nu^{(n+1)}(E)\frac{\partial E}{\partial t}$.

▷▷ toy preamble ◁◁+

```
operator nu;
depend nu,ee;
let df(nu(~n),t)=>nu(n+1)*g;
```

2.2 Toy initialise with linear

Start the iteration from the linear solution that the field $u = Ey$. The evolution of the 'order parameter' is also initially zero: $E_t = g = 0$.

▷▷ toy initialise with linear ◁◁

```
u:=ee*y;
g:=0;
```

2.3 Toy truncate the asymptotic expansion

Just truncate in powers of the fudge factor γ. The parameter `nmax` controls how many orders are computed in the Taylor series expansion of the stress about the mean shear.

```
▷▷ toy truncate the asymptotic expansion ◁◁
let gam^5=>0;
nmax:=deg((1+gam)^9,gam);
```

2.4 Toy update field u from PDE and BC

Compute the residuals of the PDE and the boundary conditions.

```
▷▷ toy update field from pde and bc ◁◁
taud:=nu(0)+(for n:=1:nmax sum
    nu(n)*(df(u,y)-ee)^n/factorial(n));
write resu:=df(u,t)-taud*df(u,y,y);
write resfs:=-sub(y=1,(1+euler*gam)*df(u,y)-(1-gam)*u);
write resbed:=sub(y=0,u);
```

Set the Euler parameter $E = -1/6$ so the γ^2 correction to evolution is zero for the trivial case of constant diffusion coefficent ν. The γ^3 correction is then near smallest too. Later work with power law dynamics indicates that $E = -(2s-1)/(2s+4)$ does the same—since this is not particularly sensitive, I stick with $-1/6$ for now.

```
▷▷ toy preamble ◁◁+
euler:=-1/6;
% let nu(~p)=>0 when p>0; % for constant nu
```

Use these residuals to update the field u and the evolution of the mean shear E.

```
▷▷ toy update field from pde and bc ◁◁+
g:=g+(gd:=3*(-mean(resu*y,y)+nu(0)*resfs));
```

```
u:=u+usolv(resu+y*gd,y)/nu(0);
```

The linear operator `mean` quickly computes the average of some field over the domain.

⊳⊳ toy preamble ⊲⊲+

```
operator mean; linear mean;
let {mean(y^~n,y) => 1/(n+1)
    ,mean(y,y) => 1/2
    ,mean(1,y) => 1 };
```

The linear operator `usolv` solves $\partial_y^2 u' = \text{RHS}$ such that $u' = 0$ on the bed $y = 0$ and on $y = 1$ (this last condition ensures that $\mathcal{E}$ is the average 'shear' across the layer).

⊳⊳ toy preamble ⊲⊲+

```
operator usolv; linear usolv;
let {usolv(y^~n,y) => (y^(n+2)-y)/(n+2)/(n+1)
    ,usolv(y,y) => (y^3-y)/6
    ,usolv(1,y) => (y^2-y)/2 };
```

2.5 Toy power law

Now suppose the stress-shear law is the nondimensional power law $\tau = (u_y)^s$, then $\nu(\mathcal{E}) = s\mathcal{E}^{s-1}$ and $\nu^{(n)} = (s-n)\nu^{(n-1)}/\mathcal{E}$.

⊳⊳ toy power law ⊲⊲

```
pow:={ nu(0)=>s*ee^(s-1)
     , nu(~n)=>(s-n)*nu(n-1)/ee when n>0 };
gp:=(g where pow)/ee^s;
up:=(u where pow)/ee;
```

2.6 Toy recast in terms of mean field

Use uu to denote the mean $\bar{u} = \int_0^1 u\,dy$. Denote the new variable viscosity $\bar{\nu}$, and correspondingly denoting the pth derivative of ν at $2\bar{u}$ by nuu(p).

Write a little iteration to find the reversion of the series and the corresponding evolution for the mean. But the expressions have assorted powers of ν in the denominator, so multiply before transforming and divide aftwerwards.

▷▷ toy recast ◁◁

```
depend uu,t;
operator nuu;
xform:={ ee=>eu , nu(~p)=>nuu(p)+(for n:=1:nmax sum
        nuu(p+n)*(eu-2*uu)^n/factorial(n)) }$
um:=int(u,y,0,1);
eu:=2*uu;
gu:=0;
it:=0$
repeat begin
    resuu:=(um-uu)*nu(0)^(nmax-1);
    resuu:=(resuu where xform)/nuu(0)^(nmax-1);
    eu:=eu-2*resuu;
    resgu:=(gu-df(um,t))*nu(0)^(nmax-1);
    resgu:=(resgu where xform)/nuu(0)^(nmax-1);
    gu:=gu-resgu;
    showtime;
end until resgu=0 and resuu=0 or (it:=it+1)>9;
```

Hmmm, recasting in terms of the mean is not such a "little task" after all. I made several mistakes before getting the above to work.

Derive the form for a power law.

▷▷ toy recast ◁◁+

```
pow:={ nuu(0)=>s*(2*uu)^(s-1)
     , nuu(~n)=>(s-n)*nuu(n-1)/(2*uu) when n>0 };
```

```
gup:=(gu where pow)/(2*uu)^s;
```

3 Iterate to a thin fluid film model

Consider the two-dimensional flow of a thin layer of fluid on a flat substrate. Let coordinate x measure distance along the substrate and coordinate y the distance above the substrate. Let the incompressible fluid have thickness $\eta(x,t)$, constant density ρ, and a nonlinear constitutive relation. The fluid flows with some varying velocity field $\mathbf{q} = (u, v)$ and pressure field p.

For example, Wilchinsky & Feltham (2004) recently used a power-law rheology to model ice flow. Another application is to gravity currents of suspensions with medium to high volume fractions as these are non-Newtonian (Stickel & Powell 2005). Whereas almost all previous analysis on non-Newtonian thin fluid film flow uses a power law dependence, some industrial plastics have a complicated non-monotonic dependence (Bird & Wiest 1995) that cannot be represented by a simple power law.

Denote free surface thickness $\eta(x,t)$ by `h`, modified mean shear $\mathcal{E}(x,t)$ by `ee`,[1] and their evolution $\eta_t = $ `gh` and $\mathcal{E}_t = $ `ge`. The Weber number is `we`, the Reynolds number `re`, and the coefficients of lateral and normal gravitational forcing are `grx` and `gry`. Construct an asymptotic solution of the Navier–Stokes equations in terms of η and $\mathcal{E}$ to some order of nonlinearity in $\mathcal{E}$ and some order of lateral derivatives ∂_x.

Decide upon how the asymptotic expansions of the solution are to be truncated. Then iteratively update the velocity and pressure fields to solve the Navier–Stokes equations and boundary conditions. The iteration continues until the governing equations are satisfied, their residuals are zero, to the order of truncation.

```
                  ▷▷ power ◁◁
% see power.pdf for documentation
on div; off allfac; on revpri;
◁◁ preamble ▷▷
```

[1] Define $\mathcal{E}(x,t)$ to be the strain-rate of a pure shearing flow at the mean shear rate.

```
⊲⊲ initialise with linear ⊳⊳
⊲⊲ truncate the asymptotic expansion ⊳⊳
it:=1$
repeat begin ok:=1;
⊲⊲ solve continuity ⊳⊳
⊲⊲ nonlinear stress-strain relationships ⊳⊳
⊲⊲ solve vertical momentum and normal stress ⊳⊳
⊲⊲ solve horizontal momentum and FS stress ⊳⊳
⊲⊲ update thickness evolution ⊳⊳
showtime;
end until ok or (it:=it+1)>19;
⊲⊲ postprocess ⊳⊳
end;
```

3.1 Preamble

Improve printing by factoring with respect to these variables. It is a matter of taste and may be different depending upon what one wishes to investigate in the algebraic expressions.

```
⊳⊳ preamble ⊲⊲
factor we,re,uu,ee,h;
```

3.1.1 Define order parameters

Use the operator `h(m)` to denote m spatial derivatives of the fluid thickness, $\partial_x^m\eta$, and similarly `ee(m)` denotes m spatial derivatives of the mean shear, $\partial_x^m E$. Also define readable abbreviations for η and its first spatial derivative. Note: use `d` to count the number of lateral x derivatives so we can easily truncate the symptotic expansion.

```
⊳⊳ preamble ⊲⊲+
operator h; operator ee;
```

```
eta:=h(0); etax:=h(1)*d$
```

These operators must depend upon time and lateral space. Then spatial derivatives transform as, for example, ∂_x`h(m)` = `h(m+1)` . Also a time derivative transforms into m spatial derivatives of the corresponding evolution: for example, ∂_t`h(m)` = ∂_x^m`gh` .

⊳⊳ preamble ⊲⊲+

```
depend h,xx,tt;
depend ee,xx,tt;
let { df(h(~m),xx) => h(m+1)
    , df(h(~m),xx,2) => h(m+2)
    , df(h(~m),tt) => df(gh,xx,m)
    , df(ee(~m),xx) => ee(m+1)
    , df(ee(~m),xx,2) => ee(m+2)
    , df(ee(~m),tt) => df(ge,xx,m) };
```

3.1.2 Stretch the coordinates with the free surface

Use stretched coordinates `yy`, `xx` and `tt` to denote $Y = y/\eta(x,t)$, $X = x$ and $T = t$. Note: the free surface is then simply $Y = 1$.

⊳⊳ preamble ⊲⊲+

```
depend xx,x,y,t;
depend yy,x,y,t;
depend tt,x,y,t;
```

Then space-time derivatives transform according to

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} - Y\frac{\eta x}{\eta}\frac{\partial}{\partial Y}\,, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T} - Y\frac{gh}{\eta}\frac{\partial}{\partial Y}\,, \quad \frac{\partial}{\partial y} = \frac{1}{\eta}\frac{\partial}{\partial Y}\,.$$

Note: we neatly insert an automatic count of lateral x derivatives here, with `d`, in between ∂_x and ∂_X.[2]

[2] You might prefer to consider this counting variable equivalent to the multiple scales

▷▷ preamble ◁◁+

```
let { df(~a,x) => df(a,xx)*d-yy*etax/eta*df(a,yy)
    , df(~a,t) => df(a,tt)-yy*gh/eta*df(a,yy)
    , df(~a,y) => df(a,yy)/eta
    , df(~a,x,2) => df(df(a,x),x) };
```

3.2 Initialise with linear

Start the iteration from the linear solution that the lateral velocity $u = \sqrt{2}y\mathcal{E}(x,t) = \sqrt{2}Y\eta\mathcal{E}$ and all other fields are zero, $v = p = 0$. The evolution of the 'order parameters' is also zero: $\mathcal{E}_t = ge = 0$ and $\eta_t = gh = 0$.

Have tried a modification to derive the model directly in terms of the mean velocity field. However, it seems that there are cross dependencies in trying to solve some of the equations, so to keep simple, we will stay with the original case of using the mean shear as the order parameter. In the postprocessing I will try again to recast the model in terms of the mean velocity.

▷▷ initialise with linear ◁◁

```
let r2^2=>2; % r2=sqrt2
u:=r2*ee(0)*eta*yy;
v:=p:=gh:=ge:=0;
```

3.3 Truncate the asymptotic expansion

There are lots of ways to truncate the asymptotic model. The small parameters available are:

---

stretching that $X = dx$ for some small parameter d, and consequently you would view X as the large lateral space scale. However, although equivalent for some purposes, resist the temptation. The multiple scales methodology enforces a highly constraining view that all small parameters must scale with one ordering parameter. The centre manifold approach empowers you to flexibly express models in terms of completely independent small parameters.

- `d` counting the number of x derivatives of the slowly varying lateral spatial structure in any term;
- the homotopy parameter `gam` varying between $\gamma = 0$ for the artificial base problem and $\gamma = 1$ for the physical fluid equations; and
- `grx` and `gry` being the lateral and normal components of gravity.

Note: the Reynolds number is not small; in principle, Re is a parameter of any finite size. Neither is the velocity of the flow small; consider the parameter E finite so that there is no issue as for small E with the mean viscosity $\nu(\text{E})$ being asymptotically zero (shear thickening) or infinity (shear thinning).

Make lateral gravity fairly small by scaling with the magnitude of ∂_x:

```
▷▷ truncate the asymptotic expansion ◁◁
d:=eps^3;
gry:=gy;
grx:=d*gx;
gamma:=eps^2*gam;
factor eps;
```

Truncate to relatively low order in spatial derivatives and boundary condition fudge: errors $\mathcal{O}(\epsilon^6)$ are interesting relatively low order; errors $\mathcal{O}(\epsilon^3)$ are boring low order.

```
▷▷ truncate the asymptotic expansion ◁◁+
let { eps^6=>0 };
nmax:=3;
```

For errors $\mathcal{O}(d^2,\gamma^3)$ or $\mathcal{O}(d^3,\gamma^2)$, need `nmax` ≥ 3.

3.4 Update v with continuity and no flow through bed

The nondimensional PDEs for the incompressible fluid flow include the continuity equation

$$\nabla \cdot \mathbf{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{11}$$

to be solved with no-slip on the bed,

$$\mathbf{q} = 0 \quad \text{on} \quad y = 0\,. \tag{12}$$

Compute the residual of the continuity equation, then update the vertical velocity v by integrating from the bed.

▷▷ solve continuity ◁◁

```
resc:=df(u,x)+df(v,y);
ok:=if ok and(resc=0) then 1 else 0;
v:=v-eta*int(resc,yy,0,yy);
```

3.5 Nonlinear stress-shear relationship

Now the strain-rate tensor (Gratton et al. 1999, Stickel & Powell 2005)

$$\dot\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Then the stress tensor for the fluid is $\sigma_{ij} = -p\delta_{ij} + 2\rho\nu\dot\varepsilon_{ij}$: when the kinematic viscosity ν is constant this models a Newtonian fluid; but when the kinematic viscosity varies with strain-rate then we model shear tickening or shear thinning non-Newtonian fluids.

▷▷ nonlinear stress-strain relationships ◁◁

```
exx:=df(u,x);
eyy:=df(v,y);
exy:=(df(u,y)+df(v,x))/2;
```

Perhaps the simplest class of non-Newtonian fluids have a viscosity which depends upon the magnitude of the strain-rate tensor. According to Betelu & Fontelos (2004) the viscosity depends upon the magnitude $|\dot\varepsilon|$, the second invariant of the strain-rate tensor, where

$$|\dot\varepsilon|^2 = \sum_{i,j} \dot\varepsilon_{ij}^2\,. \tag{13}$$

For example, Bird et al. (1977) (as reported by Betelu & Fontelos 2004) observed that a solution of 0.5% Hydroxyethylcellulose is shear thinning: at 20 °C the solution has viscosity $\mu = m|\dot\varepsilon|^{s-1}$ for exponent $s = 1/1.96$ and coefficient $m = 0.84\,\mathrm{N\,s}^{1/\lambda}/\mathrm{m}^2$. Now in a thin film flow the dominant term in $\dot\varepsilon$ is the shear $u_y/\sqrt{2}$; we seek the asymptotic approximation of the $\sqrt{}$ as part of the iteration.

▷▷ nonlinear stress-strain relationships ◁◁+

```
rese:=exx^2+2*exy^2+eyy^2-ros^2;
ok:=if ok and(rese=0) then 1 else 0;
ros:=ros+rese/(2*ee(0));
```

Also initially approximate the magnitude $\dot\varepsilon$ of the strain-rate tensor.

▷▷ initialise with linear ◁◁+

```
ros:=ee(0);
```

Approximate the viscosity at any point in the fluid as a Taylor series in the difference of the local strain-rate magnitude $\dot\varepsilon$ from its approximate mean E.

▷▷ nonlinear stress-strain relationships ◁◁+

```
vis:=nu(0)+(for n:=1:nmax sum
    nu(n)*(ros-ee(0))^n/factorial(n));
```

Then the deviatoric stress tensor is $2\nu(\dot\varepsilon)\dot\varepsilon_{ij}$.

▷▷ nonlinear stress-strain relationships ◁◁+

```
txx:=2*vis*exx;
tyy:=2*vis*eyy;
txy:=2*vis*exy;
```

In the above we need various derivatives of the kinematic viscosity ν so denote them by `nu(n)` when evaluated at the mean shear E. Note: by the chain rule $\frac{\partial}{\partial t}\nu^{(n)}(\mathrm{E}) = \nu^{(n+1)}(\mathrm{E})\frac{\partial \mathrm{E}}{\partial t}$.

▷▷ preamble ◁◁+

```
operator nu;
depend nu,ee;
let { df(nu(~n),tt)=>nu(n+1)*df(ee(0),tt)
    , df(nu(~n),xx)=>nu(n+1)*df(ee(0),xx) };
```

3.6 Update p from vertical momentum and surface normal stress

The momentum equation is

$$\mathrm{Re}\left(\frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}\cdot\boldsymbol{\nabla}\mathbf{q}\right)=-\boldsymbol{\nabla}p+\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\mathrm{Gr}\,\hat{\mathbf{g}}\,, \tag{14}$$

where Re is an appropriate Reynolds number, $\boldsymbol{\tau}$ is the nondimensional deviatoric stress tensor, and Gr is a nondimensional gravity number measuring the strength of gravity in the direction of the unit vector $\hat{\mathbf{g}}$; when the substrate slopes $\hat{\mathbf{g}}$ is not normal to the substrate.

The vertical momentum equation is solved with boundary condition that the normal stress to the free surface comes from constant environmental pressure and surface tension, that is,

$$-p+\frac{1}{1+\eta_x^2}\left(\tau_{yy}-2\eta_x\tau_{xy}+\eta_x^2\tau_{xx}\right)=\frac{\mathrm{We}\,\eta_{xx}}{(1+\eta_x^2)^{3/2}}\quad\text{on}\quad y=\eta\,, \tag{15}$$

where We is a nondimensional Weber number characterising the importance of surface tension.

Compute the residuals of the vertical momentum equation and the normal pressure on the free surface (multiplied by $1+\eta_x^2$).

▷▷ solve vertical momentum and normal stress ◁◁

```
resv:=re*( df(v,t)+u*df(v,x)+v*df(v,y) )
    +df(p,y) +gry -df(txy,x)-df(tyy,y);
restn:= sub(yy=1,-p*(1+etax^2) +tyy
    -2*etax*txy +etax^2*txx -we*curv );
```

```
ok:=if ok and(resv=0)and(restn=0) then 1 else 0;
```

Note: `curv` is the asymptotic expansion of the free surface curvature multiplied by $1+\eta_x^2$ (we should really find this expansion as an integrated part of the iteration).

```
▷▷ preamble ◁◁+
curv:=h(2)*d^2*(1-etax^2/2+3*etax^4/8-5*etax^6/16)$
```

Then update the pressure field p by integrating down from the free surface $Y = 1$; we use the linear operator `psolv` to solve $\partial_Y p' = -\text{RHS}$ such that $p' = 0$ at $Y = 1$.[3]

```
▷▷ solve vertical momentum and normal stress ◁◁+
p:=p+eta*psolv(resv,yy)+restn;
```

```
▷▷ preamble ◁◁+
operator psolv; linear psolv;
let {psolv(yy^~n,yy) => (1-yy^(n+1))/(n+1)
    ,psolv(yy,yy) => (1-yy^2)/2
    ,psolv(1,yy) => (1-yy) };
```

3.7 Update u from horizontal momentum and surface tangential stress

There must be no tangential stress at the free surface,

$$(1-\eta_x^2)\tau_{xy} + \eta_x(\tau_{yy} - \tau_{xx}) = 0 \quad \text{on} \quad y = \eta\,. \tag{16}$$

This boundary condition of zero tangential stress implicitly and in its effect translates to one of zero shear at the surface; this implicit translation is

[3] Generally it is quicker to use such operators than to use the native integration as we did for the continuity equation.

not appropriate for material with a finite yield stress. We assume the fluid yields for arbitrarily small stress.

To apply centre manifold theory, change this boundary condition (16) on the tangential stress to have an artificial forcing proportional to the local velocity:

$$(1-\tfrac{1}{6}\gamma)\left[(1-\eta_x^2)\tau_{xy}+\eta_x(\tau_{yy}-\tau_{xx})\right]=(1-\gamma)\frac{\nu(\text{E})}{\eta}u \quad\text{on}\quad y=\eta\,. \tag{17}$$

When we evaluate at $\gamma=1$ this artificial right-hand side becomes zero so the boundary condition (17) reduces to the physical boundary condition (16). However, when both the parameter $\gamma=0$ and lateral derivatives negligible, $\partial_x=0$, a neutral mode of the dynamics is the lateral shear $u=\sqrt{2}\text{E}y$.

The Euler parameter of the toy problem suggests introducing the factor $(1-\frac{1}{6}\gamma)$ into the left-hand side of the tangential stress boundary condition (17) in order to improve convergence in the parameter γ when evaluated at the physically relevant $\gamma=1$.

Compute the residuals of the lateral momentum equation, an artificial tangential stress on the free surface, and the boundary condition of no-slip on the bed. See that when $\gamma=0$ the free surface condition is $\partial_y u=u/\eta$, leading to our neutral mode $u=\text{E}(x,t)y/\eta$; whereas when $\gamma=0$ the free surface condition reduces to zero tangential stress.

```
    ▷▷ solve horizontal momentum and FS stress ◁◁
resu:=re*( df(u,t)+u*df(u,x)+v*df(u,y) )
    +df(p,x) -grx -df(txx,x)-df(txy,y);
restt:=-sub(yy=1,
    (1-gamma/6)*((1-etax^2)*txy+etax*(tyy-txx))
    -(1-gamma)*nu(0)*u/eta);
ok:=if ok and(resu=0)and(restt=0) then 1 else 0;
```

Use these residuals to update the lateral velocity field u and the evolution of the mean shear E. First update the evolution with some magic recipe depending upon the boundary residuals and the mean of the residual of the u equation; note the division by `re` is because we express the PDE as $\text{Re}\,\partial_t u$. Second update the lateral velocity field using operator `usolv`.

▷▷ solve horizontal momentum and FS stress ◁◁+
```
ge:=ge+(ged:=3*(-mean(resu*yy,yy)/eta+(r2/2)*restt/eta^2))/re;
u:=u+usolv(resu+r2*eta*yy*ged,yy)*eta^2*rnu;
```

The variable rnu $:= (\nu + E\nu')^{-1}$. Thus $\frac{\partial R_\nu}{\partial E} = -R_\nu^2(2\nu' + E\nu'')$ and $\nu R_\nu = 1 - E\nu' R_\nu$.

▷▷ preamble ◁◁+
```
depend rnu,xx,tt;
let {df(rnu,xx)=>-rnu^2*(2*nu(1)+ee(0)*nu(2))*df(ee(0),xx)
    ,df(rnu,tt)=>-rnu^2*(2*nu(1)+ee(0)*nu(2))*df(ee(0),tt)
    ,nu(0)*rnu=>1-ee(0)*nu(1)*rnu };
```

The linear operator mean quickly computes the average of some field over the fluid thickness.

▷▷ preamble ◁◁+
```
operator mean; linear mean;
let { mean(yy^~n,yy) => 1/(n+1)
    , mean(yy,yy) => 1/2
    , mean(1,yy) => 1 };
```

The linear operator usolv solves $\partial_Y^2 u' = \text{RHS}$ such that $u' = 0$ on the bed $y = 0$ and on $y = \eta$ (this last condition ensures that E is indeed the cross-film average shear).

▷▷ preamble ◁◁+
```
operator usolv; linear usolv;
let { usolv(yy^~n,yy) => (yy^(n+2)-yy)/(n+2)/(n+1)
    , usolv(yy,yy) => (yy^3-yy)/6
    , usolv(1,yy) => (yy^2-yy)/2 };
```

3.8 Update the free surface evolution

The kinematic condition at the free surface,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v \quad \text{on} \quad y = \eta\,, \tag{18}$$

gives the evolution of h.

▷▷ update thickness evolution ◁◁

```
gh:=sub(yy=1,v-u*etax);
```

3.9 Postprocessing

I may use these transformations to check on the dimensionality of various expressions.

▷▷ postprocess ◁◁

```
dims:={ h(~m)=>nh*ll/ll^m
    , ee(~m)=>nne*1/tt/ll^m
    , rnu=>nrn/(ll^2/tt)
    , rnuu=>nrn/(ll^2/tt)
    , uu(~m)=>mu*ll/tt/ll^m
    , gx=>ngx*ll/tt^2
    , nu(~m)=>nnu*ll^2/tt*tt^m
    , nuu(~m)=>nnu*ll^2/tt*tt^m }$
```

3.9.1 Recast model in terms of mean velocity

Rewrite the model in terms of the mean $\bar u = \frac{1}{\eta}\int_0^\eta u\,dy$, denoted by uu. The model evolution is then $\frac{\partial\eta}{\partial t} = \text{ghu}$ and $\frac{\partial\bar u}{\partial t} = \text{gu}$.

Approximately, $\textsc{e} \approx \sqrt{2}\bar u/\eta$. But a difficulty is that we should write the 'viscosity' ν and its derivatives as a function of $\sqrt{2}\bar u/\eta$ rather than $\textsc{e}$; hence define a new variable $\bar\nu$, and correspondingly denoting the pth derivative of ν at $\sqrt{2}\bar u/\eta$ by nuu(p).

▷▷ postprocess ◁◁+

```
operator uu;
depend uu,xx,tt;
shear0:=r2*uu(0)/eta;
operator nuu;
depend nuu,uu;
let { df(nuu(~n),xx)=>nuu(n+1)*df(shear0,xx)
    , df(nuu(~n),tt)=>nuu(n+1)*df(shear0,tt)
    , df(uu(~m),xx) => uu(m+1)
    , df(uu(~m),tt) => df(gu,xx,m) };
```

Also define rnuu to be $R_{\bar\nu} = (\bar\nu + \sqrt{2}\bar\nu'\bar u/\eta)^{-1}$. Thus $\frac{\partial R_{\bar\nu}}{\partial a} = -R_{\bar\nu}{}^2(2\bar\nu' + \sqrt{2}\bar\nu''\bar u/\eta)\frac{\partial}{\partial a}(\sqrt{2}\bar u/\eta)$ and $\bar\nu R_{\bar\nu} = 1 - \sqrt{2}\bar\nu'\bar u/\eta\, R_{\bar\nu}$.

▷▷ postprocess ◁◁+

```
depend rnuu,xx,tt;
let { df(rnuu,xx)=>-rnuu^2*(2*nuu(1)+shear0*nuu(2))*df(shear0,x
    , df(rnuu,tt)=>-rnuu^2*(2*nuu(1)+shear0*nuu(2))*df(shear0,t
    , df(rnuu,uu(0))=>-rnuu^2*(2*nuu(1)+shear0*nuu(2))*df(shea
    , nuu(0)*rnuu=>1-shear0*nuu(1)*rnuu };
```

Write a little iteration to find the reversion of the series and the corresponding evolution for the mean. I do not have to do any funny multiplication and dividing by ν before transforming and aftwerwards, because the variable R_ν encodes the divisions; instead I have to Taylor expand R_ν just as I do for the derivatives of the viscosity ν.[4]

▷▷ postprocess ◁◁+

```
xform:={ ee(~p)=>df(eu,xx,p)
    , rnu=>rnuu+(for n:=1:nmax sum
       df(rnuu,uu(0),n)*(eta/2*(eu-shear0))^n/factorial(n) )
    , nu(~p)=>nuu(p)+(for n:=1:nmax sum
       nuu(p+n)*(eu-shear0)^n/factorial(n) ) }$
```

[4] It seems the time derivatives need to be expanded *before* the xform is done.

```
um:=mean(u,yy)$
eu:=shear0;
gu:=0;
it:=1$
repeat begin
    resuu:=(um-uu(0) where xform);
    eu:=eu-r2*resuu/eta;
    write resuu:=mylength(resuu);
    resgu:=gu-df(um,t);
    resgu:=(resgu where xform);
    gu:=gu-resgu;
    write resgu:=mylength(resgu);
    showtime;
end until {resgu,resuu}={0,0} or (it:=it+1)>9;
ghu:=(gh where xform)$
lengthghu:=mylength(ghu);
```

3.9.2 Power law model is simpler

Now suppose the stress-shear law involves a nondimensional power law for the kinematic viscosity, $\nu = c_s \mathrm{E}^{s-1}$: $s = 1$ is Newtonian; $s < 1$ is shear thinning; $s > 1$ is shear thickening. Such a power law is sometimes called Ostwald's or Norton's constitutive relation (Gratton et al. 1999). Restrict the results to this power law with the observation that then $R_\nu = 1/(c_s s \mathrm{E}^{s-1})$ and $\nu^{(n)} = (s-n)\nu^{(n-1)}/\mathrm{E}$.

```
                    ▷▷ postprocess ◁◁+
pow:={ nu(0)=>cs*ee(0)^(s-1)
     , rnu=>1/(cs*s*ee(0)^(s-1))
     , nu(~n)=>(s-n)*nu(n-1)/ee(0) when n>0
     , nuu(0)=>cs*shear0^(s-1)
     , rnuu=>1/(cs*s*shear0^(s-1))
     , nuu(~n)=>(s-n)*nuu(n-1)/shear0 when n>0 };
% on rounded;
% print_precision 5;
```

```
% gam:=1;
gep:=(re*ge where pow);
ghp:=(gh where pow);
up:=(u where pow)$
gup:=(re*gu where pow);
ghup:=(ghu where pow);
```

3.10 Trace the execution

We like to see how the iteration is proceeding. Thus for each equation, write out the number of terms in its residual in each iteration.

▷▷ preamble ◁◁+

```
procedure mylength(res);
%     res;
    if res=0 then 0 else length(res);
```

▷▷ solve continuity ◁◁+

```
write resc:=mylength(resc);
```

▷▷ nonlinear stress-strain relationships ◁◁+

```
write rese:=mylength(rese);
```

▷▷ solve vertical momentum and normal stress ◁◁+

```
write resv:=mylength(resv);
write restn:=mylength(restn);
```

▷▷ solve horizontal momentum and FS stress ◁◁+

```
write resu:=mylength(resu);
write restt:=mylength(restt);
```

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