Percolation Analysis of the Two-Dimensional Widom-Rowlinson Lattice Model

Sebastian Maurice Carstens

Dissertation an der Fakultät für
Mathematik, Informatik und Statistik
der Ludwig-Maximilians-Universität München
vorgelegt am 16.02.2012

Percolation Analysis of the Two-Dimensional Widom-Rowlinson Lattice Model

Sebastian Maurice Carstens

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig–Maximilians–Universität
München

vorgelegt von
Sebastian Maurice Carstens
aus Marburg

München, den 16.02.2012

Erstgutachter:	Prof. Dr. Hans-Otto Georgii
Zweitgutachter:	Prof. Dr. Franz Merkl
externer Gutachter:	Prof. Dr. Olle Häggström

Tag der Disputation: 01.06.2012

For fun ...

Zusammenfassung

Wir untersuchen das zweidimensionale Widom-Rowlinson-Gittermodell. Dieses diskrete Spin-Modell beschreibt eine Oberfläche, welche mit einem Gasgemisch im Verhältnis eins zu eins besprüht wird. Hierbei soll die Mischung aus zwei sich auf kurzer Distanz stark abstoßenden Gasen bestehen. Die verwendete Gasmenge wird mit einem positiven Parameter beschrieben, den wir Aktivität nennen.

Für unser Hauptergebnis hinterlegen wir den Stern-Graphen $(\mathbb{Z}^2, \boxtimes)$. Wir können zeigen, dass höchstens zwei ergodische Widom-Rowlinson Maße existieren, falls die Aktivität den Wert 2 übersteigt. Diese Aussage lässt sich in zwei Schritten beweisen:

Der erste Schritt verwendet recht allgemeine Argumente. Wir entwickeln eine hinreichende Bedingung für die Existenz von höchstens zwei ergodischen Widom-Rowlinson-Maßen. Die Bedingung besagt, dass mit nicht beliebig kleiner Wahrscheinlichkeit ein $1*$Lasso – ein zum Rand $1*$verbundener $1*$Kreis – existiert. Unser Ansatz basiert auf der sogenannten „infinite cluster method": Wir verhindern die (Ko-)Existenz von gewissen Arten unendlicher Cluster. Hierfür verschärfen wir zuerst die bisherigen Resultate in diese Richtung für allgemeine zweidimensionale abhängige Perkolation.

Im zweiten Schritt zeigen wir, dass die im ersten Schritt hergeleitete hinreichende Bedingung für Aktivitäten größer 2 erfüllt ist. Dazu müssen wir die Wahrscheinlichkeiten von Konfigurationen, die $1*$Lassos aufweisen, mit denen, die 0Lassos aufweisen, vergleichen. Dies erreichen wir durch die Konstruktion einer injektiven Abbildung von dem Raum der Konfigurationen mit einem 0Lasso in den komplementären Raum. Bildlich gesprochen soll die Injektion gewisse Teile eines 0Kreises mit 1Spins füllen und dadurch ein $1*$Lasso bilden.

Abstract

We consider the two-dimensional Widom-Rowlinson lattice model. This discrete spin model describes a surface on which a one to one mixture of two gases is sprayed. These gases shall be strongly repelling on short distances. We indicate the amount of gas by a positive parameter, the so called activity.

The main result of this thesis states that given an activity larger than 2, there are at most two ergodic Widom-Rowlinson measures if the underlying graph is the star lattice $(\mathbb{Z}^2, \boxtimes)$. This falls naturally into two parts:

The first part is quite general and establishes a new sufficient condition for the existence of at most two ergodic Widom-Rowlinson measures. This condition demands the existence of $1*$lassos, i.e., $1*$circuits $1*$connected to the boundary, with probability bounded away from zero. Our approach is based upon the infinite cluster method. More precisely, we prevent the (co)existence of infinite clusters of certain types. To this end, we first have to improve the existing results in this direction, which will be done in a general setting for two-dimensional dependent percolation.

The second part is devoted to verify the sufficient condition of the first part for activities larger than 2. To this end, we have to compare the probabilities of configurations exhibiting $1*$lassos to the ones exhibiting 0lassos. This will be done by constructing an injection that fills certain parts of 0circuits with 1spins and, hereby, forms a $1*$lasso.

Danksagung

Zuerst und vor allem möchte ich mich gerne bei meinem Doktorvater, Herrn Prof. Dr. H.-O. Georgii, bedanken. Seine nahezu perfekt ausgeklügelten Vorlesungen und Bücher fokussierten mein generelles Interesse an Stochastik auf die "random geometry". Zu meinem Glück ist er nicht nur ein vorbildlicher akademischer Lehrer, sondern auch ein weltweit anerkannter Forscher. Das Aufschreiben meiner Gedanken wäre ohne seine wohl überdachten Kritiken, Hilfestellungen bei den Formulierungen und Auffinden mancher Fehler in dieser Form nicht möglich gewesen. Insbesondere möchte ich mich herzlich für das schnelle, gründliche und mehrfache Lesen meiner Dissertation bedanken.

Ohne die Unterstützung von meiner Partnerin Sascha Ines Ulmer wäre nicht nur mein Leben farblos, sondern diese Arbeit auch niemals entstanden. Ich weiß wirklich nicht wie ich mich für ihre Existenz (und Eindeutigkeit) bedanken soll, deswegen werde ich es erst gar nicht versuchen, das erscheint mir das Einfachste zu sein.

Zwar wäre ohne meine Kollegen und Freunde an der Ludwig-Maximilians-Universität München das Arbeiten möglich (vielleicht sogar effizienter) gewesen, ABER es hätte wesentlich weniger Spaß gemacht! Besonders gerne erinnere ich mich an den regelmäßigen Doktoranden-Kaffeeklatsch.

Nicht zuletzt möchte ich mich herzlich bei meiner restlichen Familie für die (nicht nur moralische) Unterstützung bedanken.

Contents

List of Figures

Chapter 1

Introduction

Some of the most interesting natural macroscopic phenomena can be explained by their microscopic dynamics, like ferromagnetism or demixing of gases. Unfortunately, the microscopic structure usually consists of many different aspects and, altogether, is quite complex. Therefore, a natural question arises: which of these microscopic interactions is sufficient or even responsible for the macroscopic phenomenon? The answer is quite difficult to derive from physical experiments, since it is not always possible to exclude all but one microscopic interaction. Fortunately, concentrating on a single aspect is not a problem in mathematics.

For this task Lanford together with Ruelle and independently of them Dobrushin introduced the elegant concept of Gibbs measures in the late sixties, see [Do] and [LR]. From a probabilistic point of view Gibbs measures are "simply" probability measures on a state space of infinitely many particles with some a priori determined conditional probabilities, which implement the microscopic structure. In this setup macroscopic phenomena are tail events, i.e., events that do not depend on the state of finitely many particles. It is the case that Gibbs measures are not necessarily uniquely specified by their microscopic structure. The existence of multiple Gibbs measures – the so called phase transition – corresponds to the existence of several possible distinct macroscopic states. Which macroscopic state really occurs could (for example) depend on the past and not on the microscopic structure. For a thorough introduction in the well-established theory of Gibbs measures see [Geo].

As the title of this thesis alludes to, we approach this question from the perspective of dependent percolation in two dimensions. More precisely, we will consider interacting systems in which each node of the square lattice $\mathbb{Z}^2$ is equipped with a random "spin" taking value either 0 or 1. Two lattice nodes are called adjacent if their Euclidean distance is 1, and $*$adjacent if their distance is 1 or $\sqrt{2}$. The lattice then splits into maximal connected or $*$connected subsets, called clusters resp. $*$clusters, on which the nodes take the same spin. In this way we obtain clusters of 0spins, called 0clusters, and $*$clusters of 1spins, called 1$*$clusters. The analysis of these clusters is in the focus of percolation theory, where some of the most beautiful proofs can be found. The charm of this mathematical area is in its elegant proofs – mostly based upon simple geometric ideas – and in its elementary – easily explained – problems, which a priori seem to be nearly impossible to solve and, afterwards, appear to be so obvious.

In some lattice models of statistical physics the occurrence of interesting macroscopic phenomena can be investigated by percolation methods, since the existence of an infinite cluster equipped with the same spin value is a macroscopic phenomenon itself. This results in a physically rewarding and mathematically beautiful area of research.

Ising Model

A well-known example is the (two-dimensional) Ising model introduced by Wilhelm Lenz [Le], which describes the phenomenon of ferromagnetism. It assumes that the atomic structure of e.g. iron equals a graph. Furthermore, the so called "spins of electrons" of each pair of atoms can either differ or be in agreement. Therefore, each node has spin value $+$ or $-$ and two nodes have the same "spin of electrons" if their spin values coincide. On the one hand, adjacent atoms have the tendency to align their spin values. On the other hand, an increasing temperature and, therefore, an increased movement implies the opposite effect. The Ising model combines both contrary forces to one parameter that describes the level of interaction between adjacent nodes. The parameter is called coupling constant and is reciprocally proportional to the temperature, i.e., a smaller coupling constant means less alignment and more chaos in form of higher entropy.

The Ising measures are modeled as Gibbs measures: Given a finite observation window $\Delta \subset \mathbb{Z}^2$, a fixed outside configuration $\pi \in \{-1,+1\}^{\mathbb{Z}^2}$, and a coupling constant $J > 0$, the probability of a configuration $\sigma \in \{-1,+1\}^{\mathbb{Z}^2}$ is

$$\nu^{\pi}_{\Delta,J}(\sigma) := \frac{1}{Z^{\pi}_{\Delta,J}} \mathbb{1}_{\{\sigma=\pi \text{ off } \Delta\}} \prod_{\substack{x,y\in\mathbb{Z}^2 \\ x \text{ adjacent to } y \\ x \text{ or } y\in\Delta}} \exp[J(-\mathbb{1}_{\{\sigma(x)\neq\sigma(y)\}})]\,,$$

where $\mathbb{1}_{\{\sigma=\pi \text{ off } \Delta\}}$ means that the configuration σ coincides with π in Δ^c and $Z^{\pi}_{\Delta,J}$ is the normalising constant. This formalism describes our above microscopic structure, since each pair of adjacent nodes with different spin values is penalised and, therefore, aligned ones are immediately rewarded. We call a probability measure ν on $\{-1,+1\}^{\mathbb{Z}^2}$ an Ising measure with coupling constant J if it satisfies the DLR equality for $\nu^{\cdot}_{\Delta,J}(.)$, i.e., for all finite $\Delta \subset \mathbb{Z}^2$

$$\nu(.|\mathcal{F}_{\Delta^c})(\omega) = \nu^{\omega}_{\Delta,J}(.)$$

for ν-almost all $\omega \in \Omega$. The abbreviation DLR honors Dobrushin, Lanford, and Ruelle for their fundamental papers [Do] and [LR].

Because of the physical background we can expect the existence of a critical coupling constant J_c below which all +clusters and −clusters are finite. Above this critical coupling constant we expect either a single infinite +cluster or a single infinite −cluster, i.e., two different macrostates. The first of these macrostates can be pictured as an infinite +ocean with finite −∗islands; correspondingly, the other macrostate can be thought of as an infinite −ocean with finite +∗islands. From a more probabilistic point of view, for $J < J_c$ we expect one unique Ising measure exhibiting the above described typical configuration; in the case $J > J_c$ the occurrence of multiple Ising measures can be anticipated, i.e., phase transition. More precisely, the set of Ising measures should be a closed interval, where each of the two extremal points typically exhibits one of the above described macrostates. Even though Ernst Ising assumed otherwise, see [Isi], our expectations are met if the underlying graph is for example the two-dimensional lattice $(\mathbb{Z}^2, \square)$, where $\square$ denotes the horizontal and vertical edges with length one.

Let us recall the historical milestones towards a proof that this is indeed the case. First, Peierls showed in [Pe] that phase transition occurs in the Ising model. Second, the fact that every translation invariant Ising measure is a convex combination of only two extremal Ising measures was first derived for large J by Gallavotti and Miracle-Sole in [GM]; later on, this result was completed for $J > J_c$ by Messager and Miracle-Sole in [MM]. Third, a remarkable approach to extend the result of Messager and Miracle-Sole to all Ising measures was made by Russo in [Ru]. Unfortunately, he did not quite achieve his goal, but, nonetheless, introduced very useful methods. Fourth and last, based upon the seminal work of Russo [Ru], independently of each other Aizenman [Aiz] and Higuchi [Hig79] obtained the existence of at most two extremal Ising measures.

In the year 2000 a simplified approach to the result of Russo, Aizenman, and Higuchi was published by Georgii and Higuchi [GH]. In particular, they developed a new geometrical approach – “the butterfly method“ – for the result of Messager and Miracle-Sole.

Widom-Rowlinson Model

Another well-known example for this beautiful area of research is the Widom-Rowlinson lattice model, which is a discrete version of the continuous Widom-Rowlinson model introduced by Widom and Rowlinson in [WR]. It was first analysed by Lebowitz and Gallavotti in [LG]. Based upon Peierls' method, they showed that phase transition occurs. This model explains the phenomenon of demixing of two strongly repelling gases. Let us describe the situation for two dimensions more precisely. Consider an equal (1:1) mixture of two gases that are strongly repelling on a short distance and spray it on a surface. The Widom-Rowlinson lattice model assumes that the surface equates a graph and that at most one gas particle can be attached to each node. Furthermore, the strong repulsion of the two gases is implemented by suppressing that adjacent nodes have different types of particles. This causes a tendency towards a loosely packed configuration of particles. We call a configuration feasible if it satisfies this condition. The amount of gas sprayed on the surface, which obviously also influences the number of vacant nodes, will be modeled by an activity parameter. This situation is somewhat similar to the one described by the Ising model. We strengthen this analogy by saying a node is equipped with a +spin respectively −spin respectively 0spin if a particle of one type of gas is attached to it, respectively a particle of the other type respectively no particle at all.

Once again, the Widom-Rowlinson measures are modeled as Gibbs measures: For a finite observation window $\Delta \subset \mathbb{Z}^2$, a fixed outside configuration $\pi \in \{-1,0,+1\}^{\mathbb{Z}^2}$, and an activity $\lambda > 0$ the probability of a configuration $\sigma \in \{-1,0,+1\}^{\mathbb{Z}^2}$ is

$$\mu^{\pi}_{\Delta,\lambda}(\sigma) = \frac{1}{Z^{\pi}_{\Delta,\lambda}} \mathbb{1}_{\{\pi=\sigma \text{ off } \Delta\}} \mathbb{1}_F(\sigma) \prod_{x\in\Delta} \lambda^{|\sigma(x)|} \,,$$

where F stands for all **f**easible configurations in $\{-1,0,+1\}^{\mathbb{Z}^2}$ and $Z^{\pi}_{\Delta,\lambda}$ is the normalising constant. This formalism was designed to coincide with the microscopic structure described above. A probability measure μ on $\{-1,0,+1\}^{\mathbb{Z}^2}$ is called a Widom-Rowlinson measure with activity λ if it satisfies the DLR equality regarding the microscopic structure $\mu^{\cdot}_{\Delta,\lambda}(.)$, i.e., for all finite $\Delta \subset \mathbb{Z}^2$

$$\mu(.|\mathcal{F}_{\Delta^c})(\omega) = \mu^{\omega}_{\Delta,\lambda}(.)$$

for μ-almost all $\omega \in \{-1,0,+1\}^{\mathbb{Z}^2}$.

Due to the physical background we would anticipate the existence of a critical activity. More precisely, we would expect that below this critical activity only one infinite Widom-Rowlinson measure exists. Its typical configuration should be one single infinite 0∗ocean with finite +islands and finite −islands. Above the critical activity we would expect two macrostates that exhibit either an infinite +ocean with finite ∗islands or an infinite −ocean with finite ∗islands. Consequently, the set of Widom-Rowlinson measures should have the same topological structure as the set of Ising measures. Interestingly, the existence of a unique critical activity depends on the underlying graph, see [BHW] and [Hä02]. Nonetheless, there is a widespread belief in the above described pattern for the square lattice.

Comparison of Both Models

Let us compare these two somewhat similar and also fundamental different models. Evident differences are that the Ising model exhibits two spin values and no forbidden configurations, whereas the Widom-Rowlinson lattice model has three spin values and forbidden configurations. Also the microscopic interaction of the Widom-Rowlinson measure is significantly more complex: On the one hand, the microscopic interaction of the Ising model only takes place on edges. More precisely, the knowledge which edges connect two nodes with different spin values, together with the spin value of a single node, uniquely determines the whole configuration. On the other hand, the microscopic interaction of the Widom-Rowlinson lattice model takes place on both edges and nodes. More precisely, the knowledge which edges connect two nodes with different spin values, together with the spin value of a single node, does not uniquely determine the whole configuration. Instead, only the nodes with 0spins are known. For the whole configuration we also need to know the spin value of each cluster not equipped with 0spins.

These differences, especially the additional spin value, weaken the methods developed for the Ising model. Consequently, we have considerably less insight in the set of Widom-Rowlinson measures. Nonetheless, some methods can be carried over. For example, Lebowitz and Gallavotti used Peierls' method of [Pe] to show the occurrence of phase transition in the Widom-Rowlinson lattice model in [LG].

The Last Attempt by Higuchi et alii

The last[1] attempt to show that there exist at most two ergodic Widom-Rowlinson measures was undertaken by Higuchi and his PhD-student Takei in [HT] in 2004. Primarily, it was based upon the butterfly method, which was developed by Georgii and Higuchi to simplify the proof of the corresponding statement in the Ising model. Unfortunately, the butterfly method does not provide the existence of at most two ergodic Widom-Rowlinson measures. But it verifies the non-coexistence of an infinite +cluster, an infinite 0cluster, and an infinite −cluster. Fortunately, this is sufficient to compare ergodic Widom-Rowlinson measures with Bernoulli percolation. This results in the existence of at most two ergodic Widom-Rowlinson measures for activities larger than $8p_c/(1-p_c) \approx 12$ if the underlying graph is $(\mathbb{Z}^2, \square)$, where p_c denotes the critical activity for Bernoulli percolation on $(\mathbb{Z}^2, \square)$. Higuchi and Takei's work of 2004 [HT] proceeds with the findings of Higuchi from 1983, see [Hig83], that phase transition occurs if the activity exceeds $8p_c/(1-p_c) \approx 12$ and the underlying graph is $(\mathbb{Z}^2, \square)$. The integer $8 = 2^3$ is a consequence of the fact that at most 4 disjoint +clusters could be combined by adding one +spin.

[1] At least to the best knowledge of the author.

Moreover, in this framework Higuchi also showed the absence of phase transition for activities smaller than $p_c/(1-p_c) \approx 3/2$.

These results could be carried over to the graph $(\mathbb{Z}^2, \boxtimes)$, where $\boxtimes$ denotes the set of horizontal, vertical, or diagonal edges with length 1 or $\sqrt{2}$. This would derive the existence of exactly two different ergodic Widom-Rowlinson measures for activities larger than $8(1-p_c)/p_c \approx 5,5$ as well as the existence of one unique Widom-Rowlinson measure for activities smaller than $(1-p_c)/p_c \approx 0,7$.

Main Result

This thesis shows that there exist at most two ergodic Widom-Rowlinson measures if the activity is at least 2 and the underlying graph is $(\mathbb{Z}^2, \boxtimes)$. More formally, let $\mathrm{WR}^*_{\mathrm{ER}}(\lambda)$ denote the set of ergodic Widom-Rowlinson measures with activity λ and μ^{+*}_λ respectively μ^{-*}_λ the measures with activity λ and $+$ respectively $-$ boundary condition, if the underlying graph is $(\mathbb{Z}^2, \boxtimes)$. The main result of this thesis is the following.

Theorem 1.1 *Let* $\lambda \geq 2$. *Then* $\mathrm{WR}^*_{\mathrm{ER}}(\lambda) = \{\mu^{+*}_\lambda, \mu^{-*}_\lambda\}$.

Notice that this statement does not imply phase transition.

But how to deduce this theorem? Our aim for the next paragraphs is to establish some intuition. Since the reader is probably more familiar with the Ising model, we first argue how one can prove the result of Messager and Miracle-Sole based upon (more or less) the core ideas developed in this thesis for the Widom-Rowlinson model. We do this on an intuitive level, which easily could be made rigorous. Therefore, any reader not familiar with the Ising model is advised to skip the next two paragraphs.

Intuition

Let us begin by applying the Burton-Keane uniqueness theorem to derive the uniqueness of the infinite $+$cluster, infinite $+*$cluster, infinite $-$cluster as well as the infinite $-*$cluster for ergodic Ising measures. It is sufficient to show that an ergodic Ising measure ν that differs from ν^+_λ exhibits an infinite $-$cluster on the upper half plane $\{(x,y) : x \geq 0\}$ ν-almost surely, since, by symmetry, additionally assuming $\nu \neq \nu^-_\lambda$ implies the coexistence of an infinite $-$cluster and an infinite $+$cluster on the upper half plane. This, together with the ergodic theorem, contradicts the uniqueness of the infinite clusters (see [GH, Proof of Cor. 3.2]) and, therefore, proves the existence of at most two ergodic Ising measures.

For contradiction let ν be an ergodic Ising measure that differs from ν^+_λ and that assigns positive ν-probability to the absence of infinite $-$clusters on the upper half plane. Since the absence of an infinite $-$cluster on the upper half plane is invariant under any translation almost surely, see [GH, Shift Lemma 3.4], the ν-almost sure absence of an infinite $-$cluster on the upper half plane follows. Nonetheless, $\nu \neq \nu_\lambda^+$ implies the ν-almost sure existence of an infinite $-$cluster, see [GH, Proof of Lemma 2.1]. Due to extremal decomposition we can exchange the property "ergodic" with the property "extremal", since all considered events are tail events ν-almost surely, like uniqueness and existence of infinite clusters or absence of an infinite cluster in half planes. Our next step towards a contradiction is the application of the following new statement, see [Cars, Theorem 1].

> There exists no probability measure on $\{0,1\}^{\mathbb{Z}^2}$ that satisfies the following conditions: a) positive association, which roughly says that spins of the same type are attracted to each other; b) bounded energy, which is a bit stricter than finite energy; c) a single infinite 0cluster exists almost surely; d) at most one infinite $1*$cluster exists almost surely; e) the probability that a node is contained in the infinite $1*$cluster is bounded from below by a strictly positive constant not depending on the node.

Fortunately, all Ising measures have bounded energy and all extremal Ising measures are positively associated. Consequently, it is sufficient to show condition e) for ν to derive a contradiction. To this end, let $\nu_{\Delta,\lambda}^{\pm}$ denote the finite Ising-measure on the finite reflection invariant observation window Δ with activity λ and $+$spins on the upper and $-$spins on the lower half plane as boundary condition. Note that given any node $(x,0)$ in Δ, by symmetry, the existence of a $+*$circuit around $(x,0)$ $+*$connected to the boundary of Δ – a so called $+*$lasso around $(x,0)$ – has $\nu_{\Delta,\lambda}^{\pm}$-probability at least $1/2$. The ν-almost sure absence of infinite $-$clusters on the upper half plane implies that any box $[-n,n]^2$ is surrounded by a $*$circuit equipped with $+$spins on the upper half plane. This, together with the above observation with respect to $\nu_{\Delta,\lambda}^{\pm}$ and the strong Markov property, shows that an infinite $+*$cluster exists ν-almost surely and that with ν-probability at least $\nu_\lambda^+(\vec{0} \stackrel{+}{\longleftrightarrow} \infty)/2 > 0$, any node of the x-axis is contained in the infinite $+*$cluster. Since there exists no infinite $-$cluster in any translate of the upper half plane ν-almost surely, see [GH, Shift Lemma 3.4], this lower bound holds for all nodes and not only for the ones on the x-axis. Consequently, all conditions of the above theorem are satisfied, which leads to a contradiction and, therefore, to the result of Messager and Miracle-Sole.

Let us take a brief pause to reflect on the core idea of this approach for the Ising model.

> The most crucial (well-known) observation based upon flip-reflection symmetry was that with $\nu_{\Delta,\lambda}^{\pm}$-probability at least $1/2$, a $+*$lasso exists in Δ.

In order to apply this, we assumed the absence of an infinite $-$cluster in the upper half plane. The new result for non-coexistence is also an essential building block, which follows naturally if we come this far and want to achieve the result of Messager and Miracle-Sole. But how to translate this approach into the Widom-Rowlinson model? We try to convey some intuition for the answer in the next paragraphs.

First of all, the Burton-Keane uniqueness theorem – once again – guarantees the uniqueness of all kinds of infinite clusters, see [HT, Prop. 3.5.]. Consequently, we can distinguish eight different scenarios depending on which type of infinite cluster exists. A first fruitless attempt would be to copy the ansatz of the crucial observation above. Hereby, the problem is that because of the 0spins the flip-reflection symmetry lost its edge, since it only compares $+*$lassos to $-*$lassos. We, however, would need to compare $*$lassos equipped with $+$ or $-$spins to 0lassos. But how can we alter the core idea? Let us begin by recalling that the butterfly method precludes the coexistence of an infinite $-*$cluster, an infinite 0cluster, and an infinite $+*$cluster. Furthermore, it is sufficient for our aim to preclude the existence of infinite 0clusters, see [HT, Prop. 3.2.]. Hence, we only have to further exclude three of the eight scenarios, namely

i) the sole existence of a single infinite 0cluster,

ii) the sole coexistence of an infinite 0cluster and an infinite $+*$cluster, and

iii) the sole coexistence of an infinite $-*$cluster and an infinite 0cluster.

By symmetry, eliminating the second scenario also prevents the third scenario. So, how can one preclude the first and the second scenario? Note that in both scenarios each finite subset is encircled by a circuit equipped with 0spins and $+$spins almost surely. This observation, together with the stochastic domination and the strong Markov property, leads us to consider Widom-Rowlinson measures on finite subgraphs with 0spins as boundary conditions denoted by $\mu^{0*}_{\Delta,\lambda}$. Nonetheless, the core idea of the Ising model should stay the same, only interpreted to the new setting, i.e., the $\mu^{0*}_{\Delta,\lambda}$-probability of the existence of a $+*$lasso should be bounded away from zero. But how can we prove this? Let us answer this question in the next paragraph and first check if it enables us to achieve our aim. Fortunately, these $+*$lassos either strangle infinite 0clusters (if the corresponding $+*$circuits are large enough) or imply the existence of an infinite $+*$cluster (if the corresponding $+*$circuits stay small). Consequently, they prevent scenario i), i.e., the existence of a sole infinite 0cluster, and, therefore, we only have to consider the second scenario, i.e., coexistence of a single infinite 0cluster and a single infinite $+*$cluster. Once again, the $+*$lassos help us. They show that condition e) of the above theorem for non-coexistence is satisfied and, therefore, prevent the coexistence. Hence, this approach leads to the proof of Theorem 1.1.

But why are the $\mu^{0*}_{\Delta,\lambda}$-probabilities of the existence of a $+*$lasso bounded away from zero? This is indeed a bit tricky. We have to consider the well-known colorblind version of the finite Widom-Rowlinson measures – called the site-random cluster model, see [GHM, Sec. 6.7] – and prove the corresponding statement with respect to $1*$lassos in this model. The advantages of this measure are that no configurations are forbidden and that it can easily be retransformed into the corresponding Widom-Rowlinson measure. But how to control the probability of the existence of a $1*$lasso in a finite observation window Δ with respect to this measure? First, note that a configuration in $\{0,1\}^{\Delta}$ exhibits either a $1*$lasso or a 0lasso. Second, we will construct an injective map from the set of configurations exhibiting a 0lasso to its complement, i.e., the set of configurations exhibiting a $1*$lasso. This construction is indeed quite complex and, therefore, we describe it in more detail in the next paragraph. For now we are content with the idea that the map fills certain parts of certain 0circuits with 1spins, which results in a configuration with a $1*$lasso. Hereby, the number of $1*$clusters joined together is (more or less) smaller than the number of added 1spins. On the one hand, each finite $1*$cluster in the site-random cluster model originally could have been a $-*$cluster or a $+*$cluster in the underlying finite Widom-Rowlinson model. On the other hand, each added 1spin at least doubles the probability for activities larger than 2. Consequently, because of the map's injectivity, the probability of the existence of a $1*$lasso is larger than the one for a 0lasso.

Let us describe the construction of the injective map from the set of configurations exhibiting a 0lasso to its complement more precisely: First, the map should only add 1spins and never delete them, which makes it easier for us to compare the probabilities of the argument and the mapped configuration. This is the case because, hereby, the decrease of $1*$clusters can be compensated by the increase of 1spins. The injectivity is important because it enables us to compare the probability of the whole set of 0lassos to the probability of the set of $1*$lassos. Unfortunately, the injectivity is also the tricky part. This is the case because the other conditions would be satisfied, for example, by simply equipping the maximal 0circuit with 1spins, which is obviously not injective. A first fruitless approach towards the construction of such an injective map would only fill the parts of the maximal 0circuit that are essential to obtain a new 1circuit and, therefore, a $1*$lasso. Unfortunately, this map is not injective either. The main reason for this is that we consider 1paths in both the exterior and the interior of the maximal 0circuit. Consequently, our next approach would be to fill the parts of the maximal 0circuit such that we obtain a new 1circuit in the union of the interior of the maximal 0circuit and the maximal 0circuit itself. Fortunately, if we sufficiently trim its domain the map is injective and all of its outputs exhibit a $1*$lasso. So, we just have to find a second map on the remaining domain that complements the first one. Recall that for the first map we only considered 1paths in the interior of the maximal 0circuit. Intuition suggests to complement this map by a second map considering only 1paths in the exterior of a certain 0circuit. As we will see later on, this approach indeed works. Unfortunately, this injection only achieves our aim, i.e., the probability of the set of 1$*$lassos is bounded away from zero, for activities larger than $2^{4/3}$. Fortunately, we can present a workaround by comparing the probabilities of two (instead of only one) configurations exhibiting 0lassos and the corresponding configurations with 1$*$lassos for activities larger than 2.

Brief Overview

The remainder of this thesis is organised as follows. In Chapter 2 we introduce basic definitions and notations needed throughout the thesis. Chapter 3 is dedicated to show the non-coexistence of different infinite clusters in the general setting of dependent percolation theory. For our new result regarding non-coexistence in Subsection 3.1.1 the underlying probability measure does not have to be invariant under translation, rotation, or reflection. The general setting tempts us to play a little with infinite clusters, which leads to some other related results[2] presented in the rest of Chapter 3. The main part of Chapter 3 is published in [Cars]. Chapter 4 first introduces the Widom-Rowlinson model. Then a sufficient condition for the absence of phase transition in the two-dimensional Widom-Rowlinson model is derived. Last, we establish the sufficient condition mentioned above for the existence of at most two ergodic Widom-Rowlinson measures. Chapter 5 constructs the non-trivial injective map and, afterwards, establishes a connection to the Widom-Rowlinson model. This already verifies the sufficient condition of Chapter 4 for activities larger than $2^{4/3} \approx 5/2$. Last, we show how to alter the injective map to extend this result to activities larger than 2 and, therefore, verify Theorem 1.1.

All proofs presented in this thesis are based on simple geometric ideas, even though some proofs can get a bit technical.

Further Thoughts

Obviously, there are some important questions that cannot be answered by the author. Nonetheless, in this section the author tries to share his intuition for some issues.

Is it possible to weaken the condition $\lambda \geq 2$? This is a tough question and if the author knew how to achieve this he would have done it. However, this condition is "only" essential for Chapter 5, which compares the probability that a 1$*$lasso occurs with the probability that a 0lasso occurs. So, establishing this comparison

[2] These results are not really necessary for Theorem 1.1

for lower activities would extend the main result of this thesis to these activities. But, from the limited intuition of the author, this seems to be impossible, at least based on the method of Chapter 5.

Could the new method be used for other underlying graphs? Well, this depends. Chapter 3 and Chapter 4 could be gerneralised to other graphs, like the standard square or the triangular lattice. But Chapter 5 crucially depends on the fact that the cardinality of the set of 1∗lassos (more generally, 1lassos with respect to the underlying graph) is larger then the cardinality of the set of 0lassos (more generally, 0lassos with respect to the matching pair of the underlying graph), otherwise an injection is impossible. Consequently, this chapter cannot be carried over to the standard square lattice, where there are less configurations exhibiting 1lassos than configurations exhibiting 0∗lassos. On the bright side, we could use this method for the triangular lattice. The author expects that the main result could be derived for activities larger than 4, since flipping a single 0node of a 0circuit can join three disjoint 1clusters, of which two are inside the 0circuit. On the other hand, this result can also be derived by a simple standard comparison to Bernoulli percolation. But some further new thoughts could decrease this boundary, just like we will decrease the boundary from $2^{4/3}$ to 2 for the star lattice.

What's New?

Since most of this thesis is original research, it is easier to point out which results were already well-known or at least common knowledge:

- The uniqueness of infinite clusters, i.e., the Burton Keane uniqueness theorem;
- Basic facts of the Widom-Rowlinson model, i.e., more or less the whole Sections 4.1 to 4.3;
- The butterfly method, i.e., Subsection 4.5.4;
- Splitting the set of finite configurations regarding lassos, i.e., Lemma 4.28.

In general, good indicators for a well-known statement are both the omission of a proof and the explicit mentioning that it is well-known.

In this context the author would like to state that he and, therefore, this thesis was influenced by many different mathematicians and their works, in fact, too many to list here. Nonetheless, the author would like to explicitly mention Hans-Otto Georgii, Kai Cieliebak, and Thomas Richthammer.

Chapter 2

Preliminaries

First of all, recall the usual order of operations ”BIDMAS“, which stands for brackets, indices, division, multiplication, addition, subtraction. To avoid several brackets we add the rule that intersections apply before unions, i.e.,

$$A \cap B \cup C = (A \cap B) \cup C \qquad A \cup B \cap C = A \cup (B \cap C)\,.$$

In this chapter we establish the fundamental notations of graphs needed throughout the thesis.

Let us begin by recalling that a graph $G = (N, E)$ consists of a set of nodes N and a set of edges E, each connecting two nodes. Furthermore, a set of nodes S is called a cluster regarding G if it is a maximal connected component of this graph, i.e., given any node of S as a starting point, each node of S and only nodes of S can be reached by walking over edges from node to node.

Recall that our main result refers to the (realisation of the) graph $(\mathbb{Z}^2, \boxtimes)$, where

$$\boxtimes := \{\{x, y\} \subset \mathbb{Z}^2 : |x - y| \in \{1, \sqrt{2}\}\}$$

denotes the set of horizontal, vertical, and diagonal edges with length 1 or $\sqrt{2}$. Consequently, we only[1] consider this graph $(\mathbb{Z}^2, \boxtimes)$ and its matching pair $(\mathbb{Z}^2, \square)$, where the set of horizontal and vertical edges with length one is denoted by

$$\square := \{\{x, y\} \subset \mathbb{Z}^2 : |x - y| = 1\}\,.$$

The main reason for this is that we want to stay as elementary as possible, even though some generalisations could be made in Chapter 3 and 4.

These two graphs have a special relation to each other comparable to dual graphs in edge percolation; they are matching pairs. For definitions and a rigorous

[1]The exception that “proves” the rule can be found in Chapter 3 Section 3.2

introduction, we refer the interested reader to [K]. A consequence of this relation is that a cluster with respect to $(\mathbb{Z}^2, \square)$ cannot cross a cluster with respect to $(\mathbb{Z}^2, \boxtimes)$, which is essential if an infinite cluster of one type shall preclude an infinite cluster of the other type. More precisely, if we split $\mathbb{Z}^2$ into two sets V and W, then either there exists an infinite cluster in V with respect to $(\mathbb{Z}^2, \square)$, or every finite subset of $\mathbb{Z}^2$ is encircled by a cluster in W with respect to $(\mathbb{Z}^2, \boxtimes)$.

This thesis deals with interacting systems, in which each node of the square lattice $\mathbb{Z}^2$ is equipped with a random "spin". In particular, we analyse whether infinite clusters equipped with the same spin value exist if the underlying graph is $(\mathbb{Z}^2, \boxtimes)$ or $(\mathbb{Z}^2, \square)$. For convenience, let us introduce a simple notation: we add a star $*$ to any graph theoretical object to indicate that the underlying graph is $(\mathbb{Z}^2, \boxtimes)$; otherwise – if the object refers to $(\mathbb{Z}^2, \square)$ – we refrain from using any index.

The most fundamental term regarding graphs is when two nodes are connected by an edge.

Definition 2.1 (adjacent, $*$adjacent) *A node $x \in \mathbb{Z}^2$ is called adjacent to a set $B \subset \mathbb{Z}^2$ if $x \in \mathbb{Z}^2 \setminus B$ and there exists a node $y \in B$ with $|x - y| = 1$. Likewise, $x \in \mathbb{Z}^2$ is called $*$adjacent if $x \in \mathbb{Z}^2 \setminus B$ and the Euclidean distance to some $y \in B$ is 1 or $\sqrt{2}$.*

In particular, a node is not adjacent or $*$adjacent to itself. We define the boundary and $*$boundary of a subset B as

$$\partial B := \{x \in \mathbb{Z}^2 : x \text{ is adjacent to } B\}$$

and

$$\partial^* B := \{x \in \mathbb{Z}^2 : x \text{ is } * \text{adjacent to } B\} .$$

The following definition of a path includes the self-avoiding property, i.e., a node does not appear twice.

Definition 2.2 (path, $*$path) *We call a finite sequence of nodes $(x_1, \ldots, x_n)$, $n \geq 0$, a path if it is self-avoiding, i.e.,*

$$x_i = x_j \Rightarrow i = j ,$$

and if every pair of successive nodes is connected by an edge, i.e., for all $1 \leq i, j \leq n$

$$|i - j| = 1 \Rightarrow x_i \text{ is adjacent to } x_j .$$

Likewise, a $*$path is defined on $(\mathbb{Z}^2, \boxtimes)$. More precisely, exchanging adjacent with $*$adjacent in the definition of a path leads to $*$paths.

The node x_1 (resp. x_n) is called the starting (resp. ending) node.

Note that by this definition, paths are always finite, which will be extended in the following.

Definition 2.3 ((two-sided) infinite path, (two-sided) infinite $*$path) *A sequence of nodes $(x_i)_{i\geq 1}$ is an infinite path if for all $n \geq 1$, $(x_1, \dots, x_n)$ is a path. A sequence of nodes, $(x_i)_{i\in\mathbb{Z}}$, is called a two-sided infinite path if the sequences $(x_i)_{i\geq 1}$ and $(x_i)_{i<1}$ are two disjoint infinite paths, whose starting nodes are adjacent to each other.*

An infinite $*$path and a two-sided infinite $*$path is defined accordingly.

We say a path hits $\Delta \subset \mathbb{Z}^2$ if one of its nodes belongs to Δ and a path touches Δ if it hits the boundary $\partial\Delta$.

Next, we define a special kind of path that encircles a finite subset of $\mathbb{Z}^2$.

Definition 2.4 (circuit, $*$circuit) *A path $(x_1, \dots, x_n)$ is called a circuit if the starting node x_1 is adjacent to or coincides with the ending node x_n.*

Likewise, a $*$path is called a $*$circuit if its starting node is $*$adjacent to or coincides with its ending node.

The interior of a circuit C, denoted by $\mathrm{int}C$, is the set of nodes in $\mathbb{Z}^2 \setminus C$ that is $*$enclosed by C, i.e., a node is contained in $\mathrm{int}C$ if all infinite $*$paths starting in this node hit C eventually. The exterior of a circuit C, $\mathrm{ext}C$, is defined as $\mathbb{Z}^2 \setminus (C \cup \mathrm{int}C)$. For the sake of completeness, we explicitly define the interior of a $*$circuit D, also denoted by $\mathrm{int}D$, as the set of nodes in $\mathbb{Z}^2 \setminus D$ that is enclosed by D and the exterior of D, $\mathrm{ext}D$, as $\mathbb{Z}^2 \setminus (D \cup \mathrm{int}D)$.

Whenever a set $\Delta \subset \mathbb{Z}^2$ is contained in the union of a circuit (resp. $*$circuit) C and its interior $\mathrm{int}C$ we say C is a circuit (resp. $*$circuit) around Δ. We add the term "strictly" to indicate that Δ lies in the interior of C. Most of the times, circuits will be around the origin $\vec{0}$. Therefore, if we omit the phrase "around x" we usually mean "around the origin". Later on, we will compare $(*)$circuits with respect to their interior, i.e., we say C is larger than D if C is a $(*)$circuit around D.

By misuse of notation, a path or a circuit is often interpreted as a set.

As mentioned earlier, we will consider interacting systems in which each node is equipped with a random "spin" and analyse the occurrence of certain maximal connected or $*$connected subsets on which the nodes take the same spin. To this end, we connect the purely graph theoretical objects to our spaces of configurations, namely $\{0,1\}^{\mathbb{Z}^2}$ and $\{-1,0,1\}^{\mathbb{Z}^2}$.

Definition 2.5 (0path, 1path) *Let $\sigma \in \{0,1\}^{\mathbb{Z}^2}$. We call a path P a 0path$[\sigma]$ if the configuration σ equips its nodes with 0spins, i.e., $P \subset \sigma^{-1}(0)$. Likewise, we say a path P is a 1path$[\sigma]$ if all of its nodes take spin value one.*

Definition 2.6 ($-$path, 0path, $+$path) *Let $\pi \in \{-1,0,1\}^{\mathbb{Z}^2}$. We call a path P a $-$path$[\pi]$ (resp. 0path$[\pi]$ resp. $+$path$[\pi]$) if P is contained in $\pi^{-1}(-1)$ (resp. $\pi^{-1}(0)$ resp. $\pi^{-1}(1)$).*

Usually, we omit the underlying configuration if it is evident within the context.

We extend these definitions in the obvious way to $0*$paths, $1*$paths, $-*$paths, $+*$paths, $0*$circuits, and so on.

Let $A, B, C \subset \mathbb{Z}^2$. We say A is 0connected to B in C and write $A \overset{0}{\longleftrightarrow} B$ in C for the existence of a 0path that belongs to C, starts in A, and ends in B. Analog occurrences will be denoted by $A \overset{1*}{\longleftrightarrow} B$ in C and called A is $1*$connected to B in C, and so on. For $C = \mathbb{Z}^2$ the phrase "in $\mathbb{Z}^2$" is usually omitted. We exchange B with ∞ to express that a corresponding infinite path, which is contained in C, exists and starts in A.

Definition 2.7 (0cluster) *Let $\sigma \in \{0,1\}^{\mathbb{Z}^2}$. A 0cluster$[\sigma]$ is a subset $S \subset \sigma^{-1}(0)$ such that*

a) all nodes of S are 0connected in S, i.e., for all nodes $x, y \in S$

$$x \overset{0}{\longleftrightarrow} y \text{ in } S\,;$$

b) no node of the complement S^c is 0connected to S, i.e., one cannot find a node $z \in S^c$ so that

$$z \overset{0}{\longleftrightarrow} S\,,$$

or equivalently all nodes of the boundary ∂S take spin value 1.

In other words S is a maximal connected component of $\sigma^{-1}(0)$.

For $\pi \in \{-1,0,1\}^{\mathbb{Z}^2}$ we define a 0cluster$[\pi]$ accordingly.

Once again, the configuration is usually omitted and we analogously define $0*$cluster, 1cluster, $1*$cluster, 0circuit, $1*$circuit, $-$cluster, $+$cluster, and so on.

Let Δ be a finite subset of $\mathbb{Z}^2$. If a $1*$circuit around the origin in Δ relative to a configuration σ exists we denote the largest of these by $C_\Delta^{\max 1*}(\sigma)$; otherwise $C_\Delta^{\max 1*}(\sigma)$ is the empty set. Note that $C_\Delta^{\max 1*}(\sigma)$ is indeed well-defined. Analogously, we denote the minimal $1*$circuit by $C_\Delta^{\min 1*}(\sigma)$. Once again, we extend these notations to other spin values and to circuits, e.g., the maximal 0circuit around the origin in Δ for a configuration σ is denoted by $C_\Delta^{\max 0}(\sigma)$.

Last, let us write $\Delta \Subset \Gamma$ to indicate that Δ is a finite subset of Γ.

Chapter 3

Planar Dependent Node Percolation

In this chapter we analyse the non-(co)existence of certain infinite clusters in two-dimensional dependent node percolation. While the first section focuses on the sufficiency of certain conditions for the non-coexistence, the second section provides an example showing that certain conditions are not sufficient for the occurrence of an infinite 1cluster in the triangular lattice.

All probability measures of this chapter will be defined on the same measurable space $(\Omega, \mathcal{F})$, where the sample space Ω is the set of configurations $\{0,1\}^{\mathbb{Z}^2}$ and $\mathcal{F}$ is the σ-algebra generated by the projections $(p_x)_{x\in\mathbb{Z}^2}$.

3.1 Non-Coexistence of Infinite Clusters

This section deals with the question "Under which conditions (on the underlying probability measure) does an infinite 0cluster preclude the occurrence of an infinite 1$*$cluster?". For this task three approaches are outlined in the following.

First, we show that there exists no probability measure on $(\Omega, \mathcal{F})$ with the following four properties. Spins of the same type are in some sense attracted to each other, which later on will be formalised as positive association. A single infinite 0cluster exists almost surely, at most one infinite 1$*$cluster exists almost surely, and certain probabilities are bounded away from zero. The latter condition contains a slightly stricter version of the finite energy, called the bounded energy. Further, the bounded energy enables us to refrain from assuming invariance with respect to translation, reflection or rotation.

Second, we show how to derive the non-coexistence of a sole infinite 1$*$cluster and a sole infinite 0cluster if we assume the finite energy condition, positive association, and a kind of invariance under translation. In contrast to the well-known argument of Zhang, see [GHM, Proof of Theorem 5.18], these assumptions suit the Burton-Keane uniqueness theorem better. Instead of additionally requiring invariance under rotation or reflection, we only need to further assume positive association.

As positive association is often difficult to verify or does not hold at all, the third part analyses the structure of an infinite 0cluster or an infinite 1∗cluster under a quite weak condition on the underlying probability measure. More precisely, under this condition, the existence of infinitely many disjoint infinite 1∗paths follows from the occurrence of an infinite 1∗cluster. The same holds with respect to 0paths and 0clusters. This result could be useful as a first step towards a proof by contradiction of an analogue to Corollary 3.15 that softens or alters the requirement of positive association. As Häggström and Mester showed in [HM], in general dispensing with positive association is not possible. A more detailed discussion of this can be found at the beginning of Section 3.1.3, see page 28.

3.1.1 Dispensing With Invariance Under Translation, Reflection, and Rotation

As described above, we want to dispense with the assumptions of invariance under translation, reflection, and rotation. Nonetheless, spins of the same type have to be attracted to each other, which is formalised in the following definition.

Definition 3.1 (increasing event, positively associated) *An event A is called increasing if $\xi \in A$ and $\eta \geq \xi$ (pointwise) implies $\eta \in A$.*

We say a probability measure μ on $\{0,1\}^{\mathbb{Z}^2}$ is positively associated, if

$$\mu(A \cap B) \geq \mu(A)\mu(B)$$

for all increasing events A and B.

Furthermore, we need to control the probabilities of certain local configurations regardless of their exact position, which leads to the following definition.

Definition 3.2 (bounded energy) *We say a probability measure μ on $\{0,1\}^{\mathbb{Z}^2}$ satisfies the bounded energy condition if for all $n \in \mathbb{N}$, there exists a strictly positive constant c_n such that*

$$\mu(\eta \text{ on } \Delta | \xi \text{ off } \Delta) > c_n$$

for all $\Delta \subset \mathbb{Z}^2$ with $|\Delta| = n$, all $\eta \in \{0,1\}^{\Delta}$, and for μ-almost all $\xi \in \{0,1\}^{\Delta^c}$.

In fact, it is sufficient to verify this condition for $n = 1$, because the general case then follows by induction with $c_n = c_1^n$.

Note that the bounded energy condition is quite weak. For example, it is satisfied by Gibbs measures relative to any shift-invariant and absolutely summable potential; cf. [Geo]. Nonetheless, it is stricter than the finite energy condition that allows c_n to be zero, rigorously defined on page 27.

Now, we are ready to state our first result.

Theorem 3.3 *There does not exist any probability measure μ on $\{0,1\}^{\mathbb{Z}^2}$ satisfying all of the following conditions:*

i) μ is positively associated;

ii) μ satisfies the bounded energy condition;

iii) there exists a single infinite 0cluster μ-almost surely;

iv) there exists at most one infinite $1*$cluster μ-almost surely;

v) there exists a constant $c > 0$ such that $\mu(x \overset{1*}{\longleftrightarrow} \infty) \geq c$ for all $x \in \mathbb{Z}^2$.

Note that conditions iv) and v) imply that with μ-probability at least c, as defined in v), a sole infinite $1*$cluster exists. The occurrence of finite clusters of both types is not precluded by any condition of Theorem 3.3. Moreover, the conditions are modelled on the ones of Sheffield's theorem [Sheff, Theorem 9.3.1], which states that an infinite 1cluster and an infinite 0cluster cannot coexist if the underlying measure satisfies a kind of translation invariance (amongst others).

Proving Theorem 3.3

In the remainder of this subsection, we present a proof by contradiction of this theorem. To this end, let μ be a probability measure on $\{0,1\}^{\mathbb{Z}^2}$ satisfying all assumptions i)-v) of Theorem 3.3.

We derive the contradiction in the following way: Let $\Delta \Subset \mathbb{Z}^2$ be an arbitrary (but fixed) set containing the origin. We show that with probability at least $\epsilon > 0$, the infinite $1*$cluster contains a $1*$circuit around Δ. Moreover, ϵ does not depend on the choice of Δ. So, if $\Delta \Subset \mathbb{Z}^2$ is large enough such that $\mu(\Delta \overset{0}{\longleftrightarrow} \infty) \geq 1-\epsilon/2$, then the impossible event "there exists a $1*$circuit around Δ as well as an infinite 0path starts in Δ" has probability at least $\epsilon/2$, which is a contradiction. Thus, an infinite $1*$cluster prohibits the existence of an infinite 0cluster.

But how do we deduce the existence of ϵ? Our strategy consists of the following three steps: First, if $x, y \in \mathbb{Z}^2$ are sufficiently far away from Δ the event that there exists a $1*$path from x to y in Δ^c occurs with probability at least $c^2/2$, where c is as defined in property v). Second, a $1*$path from x to y in Δ^c could be either clockwise or counterclockwise coiled around the origin and the existence of both types implies the existence of a $1*$circuit around Δ. Third, there exist $x, y \in \Delta^c$ such that with probability at least $c^2/4$, a clockwise $1*$path from x to y in Δ^c exists and with probability at least $c^2/4$ a counterclockwise $1*$path from x to y in Δ^c exists. This, together with the positive association and step two, implies that with probability at least $c^4/2^4 = \epsilon$, a $1*$circuit around Δ exists.

For the first step, we introduce a special $*$circuit, which consists of a 0path and a $1*$path that are connected to form a $*$circuit.

Definition 3.4 (mixed $^{1*}_{0}$circuit) *Let $n, m \geq 0$ and $(x_1, \ldots, x_n)$ be a $1*$path and $(y_1, \ldots, y_m)$ be a 0path with $x_n \overset{*}{\sim} y_1$ and $x_1 \overset{*}{\sim} y_m$. We call the composition $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ a mixed $^{1*}_{0}$circuit.*

Note that a $1*$circuit or a 0circuit is also a mixed $^{1*}_{0}$circuit.

The purpose of this definition is the following: Let $\Delta \subset \Gamma \Subset \mathbb{Z}^2$ and $x, y \in \Gamma^c$. The existence of both a mixed $^{1*}_{0}$circuit in Γ around Δ and a $1*$path from x to y implies that one can also find a $1*$path from x to y not hitting Δ. Therefore, such a circuit "shields" Δ from "outside" $*$paths.

The next definition simplifies the proof of the following lemma and, therefore, will be stated right here, even though it is not really required till the next subsection.

Definition 3.5 (infinite boundary) *Let us consider the event that a sole infinite 0cluster and a sole infinite $1*$cluster coexist. Then fill the finite holes of the infinite $1*$cluster, i.e., flip the spin of all 0clusters $*$encircled by the infinite $1*$cluster. Let each node of this filled infinite $1*$cluster be the centre of a square with side length $3/2$. Given all this, the infinite boundary, which is illustrated in Figure 3.1, is defined as the topological boundary of the union of these squares.*

Note that the infinite boundary is always well-defined, since all nodes $*$adjacent to the infinite $1*$cluster are contained in the infinite 0cluster. Furthermore, by definition, it indicates which side contains the infinite 0cluster. We usually interpret the infinite boundary as a curve.

Lemma 3.6 (Shield lemma) *For all $\Delta \Subset \mathbb{Z}^2$, μ-almost surely there exists a mixed $^{1*}_{0}$circuit around Δ.*

Proof: It is sufficient to take $\Delta = \{-d, \ldots, d\}^2$. We distinguish three cases.

First, we assume that all $1*$clusters meeting $\partial^*\Delta$ are finite. Then there exists a 0circuit around Δ, which, in particular, is a mixed $^{1*}_{0}$circuit in Δ^c.

The second case "only finite 0clusters meet $\partial^*\Delta$" is solved analogously.

Now, we turn our attention to the remaining case that the infinite 0cluster and the infinite $1*$cluster (exist and) meet $\partial^*\Delta$. Thus, the infinite boundary (exists and) splits $\mathbb{Z}^2$ into two sets S_0 and S_{1*} such that the one side S_0 consists of the infinite 0cluster plus all its finite $*$holes, i.e., $*$clusters encircled by the infinite 0cluster, and the other side S_{1*} consists of the infinite $1*$cluster plus all its finite holes, i.e., clusters encircled by the infinite $1*$cluster.

Because of the case assumption the infinite boundary hits $\partial^*\Delta$. Let $x, x' \in \partial^*\Delta \cap S_0$ and $y, y' \in \partial^*\Delta \cap S_{1*}$ be the nodes such that the infinite boundary first enters $\partial^*\Delta$ between x and y and last exits $\partial^*\Delta$ between x' and y'. In particular, x is adjacent to y, x' is adjacent to y', the nodes x, x' belong to the infinite 0cluster,

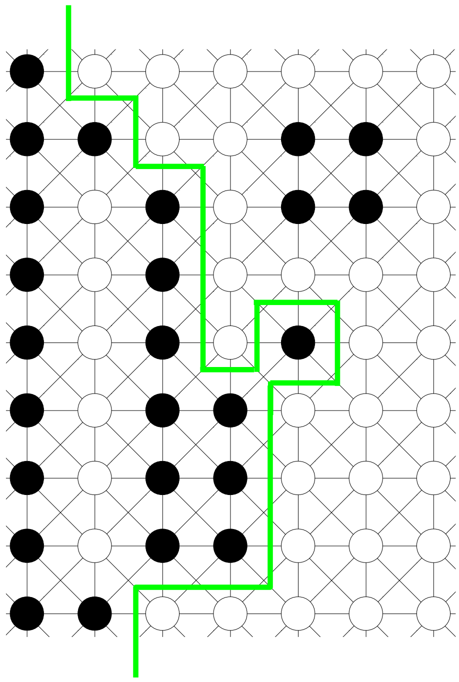

Figure 3.1: Black (resp. white) balls represent the nodes equipped with spins of value one (resp. zero). The horizontal, vertical, and diagonal lines from ball to ball represent the $*$edges. The infinite boundary is illustrated by the green curve.

and y, y' belong to the infinite $1*$cluster. Since all $1*$clusters in S_0 are finite and encircled by the infinite 0cluster, which contains x and x', one can find a 0path from x to x' in $S_0 \cap \Delta^c$. Likewise, there exists a $1*$path from y to y' in $S_{1*} \cap \Delta^c$. The 0path and the $1*$path are the two ingredients of a mixed $^{1*}_{0}$circuit around Δ. Therefore, we have shown the existence in the third case.

The lemma follows from the fact that almost surely one of these three cases occurs. $\square$

Notice that only conditions iii) and iv) were used in this proof. The next lemma, which completes our first step towards proving Theorem 3.3, relies on properties i) and v) in combination with the shield lemma.

Lemma 3.7 *For all $\Delta \Subset \mathbb{Z}^2$, there exists a set $\Gamma \Subset \mathbb{Z}^2$ such that for all $x, y \in \Gamma^c$, the event "x and y are $1*$connected in Δ^c" occurs with probability at least $c^2/2$.*

Proof: Fix an arbitrary $\Delta \Subset \mathbb{Z}^2$. Due to the shield lemma, we can choose $\Gamma \Subset \mathbb{Z}^2$ such that with probability at least $1 - c^2/2$, a mixed $^{1*}_{0}$circuit around Δ in Γ exists. Let $x, y \in \Gamma^c$. The uniqueness of the infinite $1*$cluster yields the existence of a

1$*$path from x to y as soon as x and y belong to this infinite 1$*$cluster. Properties i) and v) imply that the latter event has probability at least c^2. Moreover, by the choice of Γ, we can conclude that with probability at least $c^2/2$, there exists in addition a mixed ${}^{1*}_{0}$circuit around Δ in Γ. Under these conditions, a 1$*$path from x to y in Δ^c can be found. $\square$

In our next step, the $*$paths from x to y off $\Delta \Subset \mathbb{Z}^2$ are distinguished into two classes according to whether they run clockwise or counterclockwise around the origin. If $*$paths of both types exist, one can also find a $*$circuit around Δ. To this end, we introduce the winding number around the origin, which for convenience will only be defined for polygons, i.e., for piecewise linear continuous curves in $\mathbb{R}^2$.

Definition 3.8 (winding number) *Let $n \geq 0$ be a natural number and let $P : [0,1] \to \mathbb{R}^2 \setminus [-n,n]^2$ be a polygon. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and rewrite $P(t)$ in polar form $P(t) = r(t)e^{i\theta(t)}$, where $\theta(.)$ is a continuous function. The winding number*

$$\mathfrak{W}(P) := \frac{\theta(1) - \theta(0)}{2\pi}$$

describes the fractional turns of the polygon around the origin and, therefore, around the box $[-n,n]^2$.

We refer to [Bear] for an alternative definition and elementary properties.

Now, we are ready to define the two classes.

Definition 3.9 (clockwise and counterclockwise polygons) *Let x and y be two nodes and let $P : [0,1] \to \mathbb{R}^2 \setminus [-n,n]^2$ be a polygon from $P(0) = x$ to $P(1) = y$. When $\mathfrak{W}(P)$ is negative P is called a clockwise polygon in $([-n,n]^2)^c$. When $\mathfrak{W}(P)$ is positive P is called a counterclockwise polygon in $([-n,n]^2)^c$.*

The next lemma is a special case of the "Topological Lemma" in [GKR]. It says that a $*$circuit exists if one can find a clockwise $*$path as well as a counterclockwise $*$path. Therefore, it concludes our second step. Obviously, $*$paths can be thought of as polygons.

Lemma 3.10 *Let $\Delta := \{-n, \ldots, n\}^2$ and $x, y \in \Delta^c$. We assume that there exist a clockwise $*$path P from x to y in Δ^c and a counterclockwise $*$path Q from x to y in Δ^c. Then a $*$circuit around Δ in $P \cup Q$ exists.*

Proof: We consider the closed polygon $C(t) := P(2t)\mathbb{1}_{t<1/2} + Q(2-2t)\mathbb{1}_{t\geq 1/2}$. Standard properties of the winding number yield

$$\mathfrak{W}(C) = \mathfrak{W}(P) - \mathfrak{W}(Q),$$

which is negative, because the first summand is negative and the second one is positive. So the origin belongs to a bounded component of $\mathbb{R}^2 \setminus C$. Consequently, there exists a $*$circuit around Δ that walks along a section of P in the direction of y and then a section of Q backwards. $\square$

The aim of our third step is to verify the existence of two nodes x, y such that the probabilities of the events "there exists a clockwise 1$*$path from x to y around Δ", in short $x \overset{1*}{\curvearrowright} y$ around Δ, and "there exists a counterclockwise 1$*$path from x to y around Δ", in short $x \overset{1*}{\circlearrowleft} y$ around Δ, are bounded from below by a strictly positive constant, which does not depend on Δ.

The phrase "x is on the left side of Δ" means that one can find $d \in \mathbb{N}$ such that $x \in \{(i,j) : i \leq -d\}$ and $\Delta \subset [-d,d]^2$ hold. Accordingly, "a node is on the right side of Δ" is used.

First we pursue the following idea: A 1$*$path, that starts on the left side and ends on the right side of the origin, becomes a clockwise polygon when it is sufficiently shifted upwards.

The existential quantifier of the next lemma could be replaced with a universal quantifier, but stating the weaker version simplifies the modification for the theorem in the next subsection.

Lemma 3.11 *For all $\Gamma \Subset \mathbb{Z}^2$, there exist a node x on the left side and a node y on the right side of Γ such that*

$$\exists h > 0 : \mu(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Gamma) \geq c^2/4 \tag{3.1}$$

$$\exists h < 0 : \mu(x_h \overset{1*}{\circlearrowleft} y_h \text{ around } \Gamma) \geq c^2/4\,, \tag{3.2}$$

where $x_h := x + (0,h)$ and $y_h := y + (0,h)$.

Proof: Since the proofs of (3.1) and (3.2) are obviously similar, we only verify (3.1). The idea is more or less the same as in Lemma 3.7.

Fix an arbitrary $\Gamma \Subset \mathbb{Z}^2$ and choose $m \in \mathbb{N}$ such that $\Gamma \subset [-m,m]^2$. Let $x := (-m-1, 0)$ and $y := (m+1, 0)$, which, therefore, are on the left respectively on the right side of Γ, and assume for contradiction that

$$\forall h > 0 : \quad \mu(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Gamma) < c^2/4\,. \tag{3.3}$$

Let $P(h)$ be the shortest path from x_h to y_h, i.e.,

$$P(h) := ((-m-1,h), (-m,h), \ldots, (m,h), (m+1,h))\ .$$

The bounded energy condition ensures the existence of a constant $\delta > 0$ depending on m such that with probability at least δ, for all $h > 0$, all spins of $P(h)$ take the value 1. In particular, for all $h > m$

$$\mu\left(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Gamma\right) \geq \delta\,. \tag{3.4}$$

Let $\delta' := \delta c^2/4$ and let $\Lambda \Subset \mathbb{Z}^2$ be such that Λ contains $\{-m,\dots,m\}^2$ and

$$\mu\left(\Lambda \overset{0}{\longleftrightarrow} \infty\right) > 1 - \delta'/2\,. \tag{3.5}$$

Due to Lemma 3.7, there exists a square $\{-l,\dots,l\}^2$ including Λ such that for all $h > l$

$$\mu\left(x_h \overset{1*}{\longleftrightarrow} y_h \text{ around } \Lambda\right) \geq c^2/2\,.$$

This, together with

$$\{x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Lambda\} \cup \{x_h \overset{1*}{\circlearrowright} y_h \text{ around } \Lambda\} = \{x_h \overset{1*}{\longleftrightarrow} y_h \text{ around } \Lambda\}\,,$$

implies that for all $h > l$

$$\max\left\{\mu\left(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Lambda\right), \mu\left(x_h \overset{1*}{\circlearrowright} y_h \text{ around } \Lambda\right)\right\} \geq c^2/4\,. \tag{3.6}$$

Additionally, considering (3.3) and

$$\forall h > l:\ \{x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Lambda\} \subset \{x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Gamma\}$$

yields that for all $h > l$,

$$\mu\left(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Lambda)\right) < c^2/4\,.$$

Hence, (3.6) implies that for all $h > l$

$$\mu(x_h \overset{1*}{\circlearrowright} y_h \text{ around } \Lambda) \geq c^2/4\,,$$

which, together with (3.4) and the positive association, yields

$$\mu\left(x_{l+1} \overset{1*}{\circlearrowright} y_{l+1} \text{ around } \Lambda, x_{l+1} \overset{1*}{\curvearrowright} y_{l+1} \text{ around } \Lambda\right) \geq \delta c^2/4 = \delta'\,.$$

Given this event, Lemma 3.10 ensures the existence of a $1*$circuit around Λ, a contradiction to (3.5). $\square$

Notice that the proof of this lemma relies on all five conditions of μ, but, fortunately, the bounded energy condition is used only once to verify the existence of a constant $\delta > 0$ such that (3.4) holds. Keeping this in mind will help us by proving the result of the next subsection, where μ' does not satisfy the bounded energy condition. Before we turn towards this, we obtain Theorem 3.3 by applying Lemmas 3.7, 3.10, and 3.11.

Proof of Theorem 3.3: Let $\Delta \Subset \mathbb{Z}^2$ be large enough so that

$$\mu(\Delta \overset{0}{\longleftrightarrow} \infty) \geq 1 - c^4/2^5 . \tag{3.7}$$

Lemma 3.7 allows us to choose a square $\{-m, \dots, m\}^2 =: \Gamma$ with $\Delta \subset \Gamma$ such that with probability at least $c^2/2$, for any two distinct points $x, y \in \Gamma^c$, x and y are $1*$connected in Δ^c. This, together with $\{x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Delta\} \cup \{x_h \overset{1*}{\circlearrowleft} y_h \text{ around } \Delta\} = \{x_h \overset{1*}{\longleftrightarrow} y_h \text{ in } \Delta^c\}$, implies that

$$\max \left\{ \mu(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Delta), \mu(x_h \overset{1*}{\circlearrowleft} y_h \text{ around } \Delta) \right\} \geq c^2/4 \tag{3.8}$$

for all $h \in \mathbb{Z}$. Applying Lemma 3.11 gives the existence of nodes x on the left side and y on the right side of Γ such that

$$\exists h > 0 : \mu(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Delta) \geq c^2/4 \tag{3.9}$$

$$\exists h < 0 : \mu(x_h \overset{1*}{\circlearrowleft} y_h \text{ around } \Delta) \geq c^2/4 . \tag{3.10}$$

The inequalities (3.8), (3.9), and (3.10) yield that there exists a $k \in \mathbb{Z}$ such that

$$\mu\left(x_{k+1} \overset{1*}{\curvearrowright} y_{k+1} \text{ around } \Delta\right), \mu\left(x_k \overset{1*}{\circlearrowleft} y_k \text{ around } \Delta\right) \geq c^2/4 . \tag{3.11}$$

Moreover, since $\{x_{k+1} \overset{1*}{\curvearrowright} y_{k+1} \text{ around } \Delta\}$ and $\{x_k \overset{1*}{\circlearrowleft} y_k \text{ around } \Delta\}$ are increasing events, we can conclude that

$$\mu\left(x_{k+1} \overset{1*}{\curvearrowright} y_{k+1} \text{ around } \Delta, x_k \overset{1*}{\circlearrowleft} y_k \text{ around } \Delta\right) \geq c^4/16 .$$

Thus, because of Lemma 3.10 a $1*$circuit around Δ occurs with probability at least $c^4/16$, a contradiction to (3.7). So, the measure μ cannot exist. □

3.1.2 Exploiting translation invariance

This subsection gives an alternative to Zhang's argument that dispenses with the assumption of invariance under reflection and rotation.

Both ways to verify non-coexistence – Zhang's argument and our alternative – are based upon the uniqueness of an infinite 0cluster and an infinite 1∗cluster, which can be derived by applying the Burton-Keane uniqueness theorem stated later on in this subsection. Our approach to the non-coexistence is (more or less) a version of Theorem 3.3 similar to Sheffield's theorem [Sheff, Theorem 9.3.1]. In order to minimise the assumptions, in this version the conditions ii) and v) of Theorem 3.3 are replaced by a kind of translation invariance.

The following theorem requires the infinite boundary as defined on page 20.

Theorem 3.12 *There does not exist any probability measure μ' on $\{0,1\}^{\mathbb{Z}^2}$ which possesses all of the following properties:*

i') μ' is positively associated;

ii') there exists a single infinite 0cluster μ'-almost surely;

iii') there exists at most one infinite $1*$cluster μ'-almost surely;

iv') the occurrence of an infinite $1*$cluster has positive probability;

v') the distribution of the infinite boundary – conditioned on its existence – is translation-invariant.

This theorem is modelled on Sheffield's theorem [Sheff, Theorem 9.3.1], which proves the non-coexistence of an infinite 0cluster and an infinite 1cluster. Probably Sheffield's proof would be strong enough to verify Theorem 3.12; but since it is a bit involved, we prefer to alter the proof of Theorem 3.3, which can be done with only one small modification.

Proof of Theorem 3.12: The strategy is to show that the conditions i) and iii)-v) of Theorem 3.3 are satisfied and that a sufficiently close analogon to equation (3.4), which is the only point where the bounded energy condition enters the proof, can be verified.

The conditions i),iii) and iv) of Theorem 3.3 are equal to the first three conditions of Theorem 3.12.

Condition v) is a consequence of conditions ii'), iii'), iv') and v'): Since the set of edges is countably infinite and the infinite boundary exists with positive probability, there exists an edge that intersects the infinite boundary with positive probability η. Let a and b be the nodes connected by this edge and assume without loss of generality that with probability $\eta/2$, the infinite 1∗cluster contains a and the infinite 0cluster contains b. Because the infinite boundary is translation-invariant, shifting does not change the probability and, consequently, for all $z \in \mathbb{Z}^2$

$$\mu'(z \overset{1*}{\longleftrightarrow} \infty) \geq \eta/2 > 0\,.$$

Next, we verify a sufficiently close analogue to equation (3.4) with the notation of the proof of Lemma 3.11:

Denote by ζ the probability that an infinite 1∗cluster exists, i.e.,

$$\zeta := \mu'(\mathbb{Z}^2 \overset{1*}{\longleftrightarrow} \infty) > 0\,.$$

Let $\Xi \Subset \mathbb{Z}^2$ be large enough so that with probability at least $3\zeta/4$, the infinite boundary exists and hits Ξ. We recall that Δ was defined in the proof of Lemma 3.11 as an arbitrary (but fixed) finite set of $\mathbb{Z}^2$. Take two translates Ξ' and Ξ'' of Ξ such that every node of Ξ' is on the left side of Δ and every node of Ξ'' is on the right side of Δ.

By subadditivity of μ', the infinite boundary hits both sets Ξ' and Ξ'' with probability at least $\zeta/2$. Moreover, one can find two pairs x, x' and y, y' of adjacent sites in Ξ' resp. Ξ'' such that the event

$$\left\{x \overset{1*}{\longleftrightarrow} \infty, x' \overset{0}{\longleftrightarrow} \infty, y \overset{1*}{\longleftrightarrow} \infty, y' \overset{0}{\longleftrightarrow} \infty\right\}$$

occurs with positive probability ϵ, say.

Take a square $[-i,i]^2$ with $\Xi' \cup \Xi'' \subset [-i,i]^2$ such that with probability at least $\delta := \epsilon/2$, the part of the boundary that starts between x and x' and ends between y and y' exists and is contained in $[-i,i]^2$. Since the distribution of the infinite boundary is translation-invariant, for all $h \in \mathbb{Z}$, the event that the part of the infinite boundary starting between x_h and x'_h and ending between y_h and y'_h exists and is contained in $[-i,i] \times [-i+h, i+h]$ occurs with probability at least δ, where x_h is defined by $x + (0,h)$. Moreover, given this event, one can in fact find a 1∗path from x_h to y_h in $[-i,i] \times [-i+h, i+h]$. This, obviously, implies that for all $h > 2\max\{i, m\}$

$$\mu\left(x_h \overset{1*}{\curvearrowright} y_h \text{ around } \Delta\right) \geq \delta\,,$$

which is sufficiently close to (3.4). □

An important building block for the main result of this subsection, namely the next corollary, is the Burton-Keane uniqueness theorem. One of its assumptions is the finite energy condition, which was discovered by Newman and Schulman in [NewS], is rigorously defined below, and roughly says that every local configuration is compatible with anything that happens elsewhere.

Definition 3.13 (finite energy) *A probability measure μ on $\{0,1\}^{\mathbb{Z}^2}$ satisfies the finite energy condition if for every finite set $\Delta \subset \mathbb{Z}^2$,*

$$\mu(\eta \text{ on } \Delta | \xi \text{ off } \Delta) > 0$$

for all $\eta \in \{0,1\}^\Delta$ and μ-a.e. $\xi \in \{0,1\}^{\Delta^c}$.

Now we are ready to state the Burton-Keane uniqueness theorem.

Theorem 3.14 (Burton-Keane uniqueness theorem) *Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. If $\mathbb{P}$ is invariant under translations and has finite energy, then there exists at most one infinite 0cluster, infinite 1cluster, infinite $0*$cluster, and infinite $1*$cluster.*

For the elegant proof we refer the interested reader to the original paper [BK], which is a must read for anyone interested in random geometry.

The Burton-Keane uniqueness theorem, together with Theorem 3.12, implies our second result, namely the next corollary, which corresponds to the theorem of Gandolfi, Keane and Russo. Instead of any kind of invariance under reflections or rotations, it takes advantage of the finite energy condition.

Corollary 3.15 *Let ρ be an ergodic and positively associated probability measure on $\{0,1\}^{\mathbb{Z}^2}$ that satisfies the finite energy condition. Then, the coexistence of an infinite $1*$cluster and an infinite 0cluster has ρ-probability zero.*

Proof: Because of the Burton-Keane uniqueness theorem at most one infinite 1∗cluster as well as at most one infinite 0cluster exist. We assume for contradiction that both of them coexist with strictly positive probability. The ergodicity yields that this event occurs with ρ-probability one. So, all conditions of Theorem 3.12 are satisfied and the contradiction is shown. $\square$

Moreover, the proofs and definitions can be adapted to generalise this result to a wide range of underlying graphs.

Remark 3.16 *Let $(G, G^*) = ((N, E), (N, E^*))$ be a matching pair of amenable and transitive graphs, in the sense of [K]. Let ρ be an ergodic and positively associated probability measure on $\{0,1\}^N$ that satisfies the finite energy condition. Then, the coexistence of an infinite $1*$cluster with respect to G^* and an infinite 0cluster with respect to G has ρ-probability zero.*

3.1.3 A Single Infinite $1*$Cluster Has Unbounded Width

This section shows that the bounded energy condition and the occurrence of an infinite 1∗cluster is sufficient for the existence of infinitely many disjoint infinite 1∗paths.

The author hopes that this could perhaps lead to a similar statement as Corollary 3.15 with a weakened version of positive association. To this end, one would "only" have to show that the number of disjoint 1∗paths, starting in the subset $\Delta \Subset \mathbb{Z}^2$, is proportional to the cardinality of Δ, in order to reproduce the argument of the Burton-Keane uniqueness theorem. Given the limited imagination of the author, this seems to be impossible. But we will discuss another – more promising – ansatz after the following statement of the subsection's main result.

Theorem 3.17 *Let ν be a probability measure on $\{0,1\}^{\mathbb{Z}^2}$ satisfying the bounded energy condition. Then, ν-almost surely on the event that an infinite $1*$cluster exists, one can find infinitely many disjoint infinite $1*$paths. Analogously, the existence of an infinite 0cluster guarantees the occurrence of infinitely many disjoint infinite 0paths.*

Before proving this theorem, let us first make two remarks.

First, if we assume – in addition to the bounded energy – the coexistence and uniqueness of the infinite 0cluster and the infinite 1∗cluster, then Theorem 3.17 even yields the existence of infinitely many two-sided infinite 1∗paths, see page 15 for the definition. These two-sided infinite 1∗paths exhibit a natural order. The first one P_1 is contained in the boundary of the infinite 0cluster. The second of these two-sided infinite 1∗paths is contained in the boundary of the union of P_1, the infinite 0cluster, and all finite 0clusters adjacent to P_1. Since there exist infinitely many 1∗paths, this procedure can be continued indefinitely. So, the infinite 1∗cluster looks like wall bars. An analogous statement holds for the infinite 0cluster and the lattice splits into one 1∗wall bar and one 0wall bar.

Second, Theorem 3.17 could also be useful as a first step towards a proof by contradiction of an analogue of Corollary 3.15 that weakens or alters the condition of positive association. If the infinite boundary is not too rugged and both the bounded energy condition and ergodicity hold, then it seems to be counterintuitive that unique infinite clusters of both types coexist. For, on the one hand, the infinite 0cluster is not allowed to intersect the intermediate space between the first and the nth two-sided infinite 1∗path as above, which has infinite "length", "width" at least n, and is not too rugged. On the other hand, ergodicity suggests that the infinite 0cluster should be evenly spread over $\mathbb{Z}^2$ and, therefore, fray out the infinite boundary.

This intuition can be made rigorous under the – absurd – further assumption of negative association, which means that any two increasing events are negatively correlated. Namely, subdivide the lattice into squares of the same size such that these squares can be interpreted as nodes of a new lattice. Call two squares adjacent if their distance is one. Furthermore, call a square occupied if it is met by the infinite 0cluster; otherwise it is called vacant. Given the coexistence, the size of the squares can be chosen so large that by exploiting the negative association, a standard path counting argument shows the finiteness of all vacant square-clusters. Let N be a number that exceeds the diameter of the squares. A two-sided infinite square-path is formed by the squares that are hit by the $N+1$th two-sided infinite

1∗path. By choice of N, these squares are contained in the random set of nodes between the first and the $2N+1$th two-sided infinite 1∗path. Therefore, all of them are necessarily vacant, which is impossible because all clusters of vacant squares are finite.

Proving Theorem 3.17

In the rest of this subsection we prove Theorem 3.17. To this end, from now on let ν be a probability measure on $\{0,1\}^{\mathbb{Z}^2}$ satisfying the bounded energy condition.

Our aim is to show that given the existence of an infinite 1∗cluster, one can find infinitely many infinite 1∗paths. To this end, we first have to check the measurability of the latter event, where the corresponding σ-algebra is generated by the cylinder sets.

Lemma 3.18 *The number A of infinite 1∗paths is tail measurable.*

Proof: The statement is a direct consequence of the identity

$$\{A \geq n\} = \bigcup_{l\in\mathbb{N}} \bigcap_{k\geq l} \bigcup_{m\geq k} \bigcap_{i\geq m} \{A_{k,i} \geq n\},$$

which holds for all $n \in \mathbb{N}$. Here, $A_{k,i}$ is the maximal number of disjoint 1∗paths in $\{-i,\dots,i\}^2 \setminus \{-k,\dots,k\}^2$ from $\partial^*\{-k,\dots,k\}^2$ to $\partial^*\{-i+1,\dots,i-1\}^2$. □

Next, we show that configurations with a given number of disjoint infinite 1∗paths exhibit a necklet with this number of 1pearls around any finite set, as is defined now.

Definition 3.19 (necklet with N 1pearls around Γ) *Let $N \in \mathbb{N}$, $\sigma \in \{0,1\}^{\mathbb{Z}^2}$, and $\Gamma \Subset \mathbb{Z}^2$. We call C a necklet with N 1pearls around Γ with respect to σ if C is a circuit around Γ with $|C \cap \sigma^{-1}(1)| = N$.*

The proof of the following existence statement is more or less a direct consequence of the well-known max-flow min-cut theorem of Ford and Fulkerson; cf. [FF]. Since this is the only point where the max-flow min-cut theorem (and its notation) is needed, we use the original notation of [FF] without defining it.

To avoid misunderstandings let us recall that an infinite ∗path is defined as an infinite sequence whereas a two-sided infinite ∗path requires a two-way infinite sequence.

Lemma 3.20 (Bottleneck lemma) *Let $\sigma \in \{0,1\}^{\mathbb{Z}^2}$ be a configuration that possesses exactly N disjoint infinite 1∗paths. Then, for all $\Gamma \Subset \mathbb{Z}^2$, there exists a necklet with N 1pearls around Γ.*

Proof: Fix an arbitrary $\Gamma \Subset \mathbb{Z}^2$ and a configuration σ such that one can find exactly N disjoint infinite 1$*$paths with respect to σ. Let the set S of sources be the $*$boundary of a square $\{-s, \ldots, s\}^2$ large enough so that it contains Γ and N disjoint infinite 1$*$paths starting in this square. Furthermore, the set T of sinks is defined as the $*$boundary of a square $\{-t, \ldots, t\}^2$ large enough so that $S \subset \text{int}T$ and there exist N disjoint 1$*$paths from S to T. The set of intermediate nodes R is $\text{int}T \setminus (S \cup \text{int}S)$. An undirected arc $\{x, y\}$ connects x and y if and only if these two nodes belong to $R \cup S \cup T$ and are $*$adjacent. We define the capacity function $c(.,.)$ of an arc $\{x, y\}$ as

$$c(x,y) = \begin{cases} 1 & \text{if } x, y \subset \sigma^{-1}(1), \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, since there are N disjoint 1$*$paths from S to T, the maximal flow value of this network is N. Applying the max-flow min-cut theorem, see [FF, page 11 plus section 7 and 10], shows the existence of a cut C separating S from T, whose cut capacity is N.

Let B be the set of nodes that are connected to S by a $*$path not intersecting an arc of the cut C. The union of B and all its finite holes is a simple $*$connected set that contains S. Therefore, erasing all loops of this union's $*$boundary leads to a uniquely determined circuit around S, denoted by B^o. By definition of B, the set B^i of all nodes in the interior of B^o $*$adjacent to B^o form a circuit around $\text{int}S$. Furthermore, a node of B^i and a node of B^o $*$adjacent to each other are also connected by an arc of the cut C.

Now we are ready to construct the necklet: First, take the set D of nodes of $B^i \cap \sigma^{-1}(0)$ and combine it with the set E of nodes in $B^i \cap \sigma^{-1}(1)$ $*$adjacent to $B^o \cap \sigma^{-1}(1)$. Since N disjoint 1$*$paths connect S to T and the capacity of C is N, the set E consists of exactly N nodes. Nonetheless, there may be more than N nodes equipped with 1spins in B^i, which is equivalent to the case that $D \cup E$ is not a circuit. Fortunately, we can circumvent these nodes using 0paths in B^o. This is the case because by definition of E, the set F of nodes in B^o $*$adjacent to $B^i \setminus (D \cup E)$ is contained in $\sigma^{-1}(0)$. A moment's thought reveals that $D \cup E \cup F$ is a necklet with N 1pearls around Γ with respect to σ. $\square$

Now, let us gain some insight into the structure of infinite 1$*$clusters under fairly general conditions on the measure.

Proof of Theorem 3.17: Since $(\{0,1\}, \mathcal{P}(\{0,1\}))$ is a perfect space, Theorem 3.3 of [Sok] implies that ν is a Gibbs measure for a suitable specification $(\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^2}$. Since ν satisfies the bounded energy condition, there exist constants $c_n > 0$ such that $\gamma_\Lambda(\eta|\xi) \geq c_n$ for ν-almost all configurations $\xi \in \{0,1\}^{\mathbb{Z}^2}$, whenever $|\Lambda| = n$

and η is a local configuration on Λ. Applying the extremal decomposition [Geo, Theorem (7.26)] yields that the bounded energy condition holds for $\mathbb{P}_\nu$-almost all extremal Gibbs measure specified by $(\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^2}$, where $\mathbb{P}_\nu$ is the unique weight on the set of extremal Gibbs measure with barycentre ν. So, we may assume without loss of generality that ν is trivial on the tail σ-field.

We further assume without loss of generality $\nu(\mathbb{Z}^2 \xleftrightarrow{1*} \infty) > 0$. The triviality of ν on the tail σ-field then implies $\nu(\mathbb{Z}^2 \xleftrightarrow{1*} \infty) = 1$. Consequently, we just have to verify that infinitely many disjoint infinite 1$*$paths ν-almost surely exist. The proof of the other statement is similar.

By assumption, the number A of infinite 1$*$paths is at least one ν-almost surely. We will show that $\nu(A = \infty) = 1$ or, equivalently, that $\nu(A = N) = 0$ for all $N \geq 1$.

Suppose the contrary. Tail triviality, together with Lemma 3.18, implies the existence of some $N \geq 1$ with $\nu(A = N) = 1$. Because ν satisfies the bounded energy condition we can choose an $\epsilon > 0$ such that

$$\nu\left(\eta \text{ on } S \middle| \xi \text{ off } S\right) \geq \epsilon \tag{3.12}$$

for all $S \subset \mathbb{Z}^2$ with $|S| \leq 5N$, $\eta \in \{0\}^S$ and for ν-almost all $\xi \in \{0,1\}^{S^c}$. Let $\Gamma \Subset \mathbb{Z}^2$ be large enough so that

$$\nu(\Gamma \xleftrightarrow{1*} \infty) > 1 - \epsilon/4\,. \tag{3.13}$$

The bottleneck lemma ensures the ν-almost-sure existence of a necklet with N 1pearls around $\Gamma \cup \partial^*\Gamma$. Let $\Delta \Subset \mathbb{Z}^2$ be large enough so that with probability at least $1 - \epsilon/2$, there exists a 0necklet with N 1pearls around $\Gamma \cup \partial^*\Gamma$ in Δ.

Denote by C the maximal 0necklet with N 1pearls around $\Gamma \cup \partial^*\Gamma$ in Δ; if it does not exist C is $\emptyset$. Hence, int$C$ is a well-defined random set, which is determined from outside. Let S be the set of nodes in int$C$ $*$adjacent to a 1pearl of C, under the condition $C \neq \emptyset$. Otherwise S is $\emptyset$. Once again S is a well-defined random set, which is determined from outside of int$C$ and $|S| \leq 5N$ always holds. If $C \neq \emptyset$ and all spins of S take the value zero, a 0circuit around Γ exists. Hence, the inequality (3.12) yields that the existence of a 0circuit around Γ in Δ has probability at least $(1-\epsilon/2)\epsilon$, a contradiction to (3.13). Consequently, $\nu(A \in \mathbb{N}) = 0$. $\square$

3.2 Non-Existence of Infinite Clusters

First of all, let us introduce a new definition exclusively for this section. Given a distance $R \in \mathbb{R}_+$, we say a probability measure $\mathbb{P}$ on $\{0,1\}^{\mathbb{Z}^2}$ is R-independent if the spin values of two regions at least R apart are $\mathbb{P}$-independent of each other.

This section suggests that the conditions R-independence, positive association, ergodicity, and large density $\mu(p_{\vec{0}} = 1)$ are not sufficient for the occurrence of an infinite $1*$cluster. More precisely, we will construct a probability measure μ on $\{0,1\}^{\mathbb{Z}^2}$, whose density $\mu(p_{\vec{0}} = 1)$ is arbitrary close to one, that also satisfies all other conditions, and which exhibits no infinite cluster μ-almost surely. On a first glance, this seems to be contrary to the findings of Liggett et alii in [LSS]. But in fact it only stresses one important aspect in [LSS], namely that the distance R for the R-independence was fixed. In our example, however, R depends on the density $\mu(p_{\vec{0}} = 1)$.

The amenability of the underlying graph plays a crucial role for our counterexample. By dropping this condition Häggström was able to show that a sufficiently large density ensures the existence of an infinite 1cluster, see [Hä96]. More precisely, he considered regular trees and automorphism invariant probability measures.

This measure is based upon the Bernoulli Percolation with density $1/2$ on the triangular lattice graph $(\mathbb{Z}^2, \boxslash)$, where $\boxslash$ is the set of horizontal and vertical edges with length one and the diagonal edges with length $\sqrt{2}$ looking like a slash, i.e.,

$$\boxslash := \left\{\{x,y\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |x-y| = 1 \text{ or } x + (1,1) = y\right\}.$$

Note that the graph $(\mathbb{Z}^2, \boxslash)$ is its own matching pair. To indicate the underlying graph we add $\boxslash$ in front of the corresponding graph theoretical objects, e.g., $\boxslash$adjacent, $\boxslash$circuit or $1\boxslash$circuit. The existence of an infinite cluster on $(\mathbb{Z}^2, \boxslash)$ is denoted by $\{\mathbb{Z}^2 \overset{1\boxslash}{\longleftrightarrow} \infty\}$.

Now we are ready to construct the above described probability measure by filling certain $1\boxslash$circuits with 1spins.

Theorem 3.21 *For all $\epsilon \in]0,1[$, there exists an R_ϵ-independent, positively associated, and ergodic probability measure μ_ϵ on $\{0,1\}^{\mathbb{Z}^2}$ so that with μ_ϵ-probability at least $1-\epsilon$, each node is equipped with a 1spin, i.e.,*

$$\mu_\epsilon(\{\sigma \in \{0,1\}^{\mathbb{Z}^2} : \sigma(\vec{0}) = 1\}) \geq 1 - \epsilon\,,$$

and, nonetheless, an infinite $1\boxslash$cluster does not occur μ_ϵ-almost surely, i.e.,

$$\mu_\epsilon(\mathbb{Z}^2 \overset{1\boxslash}{\longleftrightarrow} \infty) = 0\,.$$

Proof: Our proof starts with some well-known facts of the Bernoulli measure with density p on $\{0,1\}^{\mathbb{Z}^2}$, denoted by ϕ_p. By definition, ϕ_p is translation invariant and, moreover, by Kolmogorov's 0-1-law, the tail events have probability one or zero. It is the case that for each translation invariant event A, there exists a tail event B with $\phi_p(A \setminus B \cup B \setminus A) = 0$, which can be proved by an approximation of A by local events. This fact, together with the tail triviality, verifies that translation invariant events have ϕ_p-probability one or zero and, therefore, the ergodicity of ϕ_p follows. It is also well-known that ϕ_p is also positively associated, which can be proved similar to the positive association of extremal Widom-Rowlinson measures in Chapter 4.

Due to Remark 3.16, $\phi_{1/2}$ exhibits no infinite 0⧄cluster or infinite 1⧄cluster, in short

$$\phi_{1/2}\left(\{\mathbb{Z}^2 \xleftrightarrow{1\boxslash} \infty\} \cup \{\mathbb{Z}^2 \xleftrightarrow{0\boxslash} \infty\}\right) = 0\,.$$

Consequently, for any $\delta \in]0,1[$ there exists a square $\Delta \Subset \mathbb{Z}^2$ with $\vec{0} \in \Delta$ so that with $\phi_{1/2}$-probability at least $1-\delta$, a 1⧄circuit around the origin in Δ exists.

Next, we construct the measure μ_ϵ based upon $\phi_{1/2}$. To this end, fix a square $\Delta \Subset \mathbb{Z}^2$ with $\vec{0} \in \Delta$ so large that with $\phi_{1/2}$-probability at least $1-\epsilon$, there exists a 1⧄circuit around the origin in Δ. We denote the translation by tr_x and call m the map from the space of configurations $\{0,1\}^{\mathbb{Z}^2}$ to $\{0,1\}^{\mathbb{Z}^2}$ that equips a node x with spin value 1 if and only if there exists a 1⧄circuit around x in $\mathrm{tr}_x(\Delta)$, i.e., x has spin value 1 or there exists a 1⧄circuit strictly around x in $\mathrm{tr}_x(\Delta)$. This leads to the probability measure

$$\mu_\epsilon := \phi_{1/2} \circ m^{-1}$$

with $\mu_\epsilon(\{\sigma \in \{0,1\}^{\mathbb{Z}^2} : \sigma(x) = 1\}) \geq 1-\epsilon$.

Let us check that μ_ϵ satisfies the remaining required conditions. Because of the inheritance of ergodicity the measure μ_ϵ is ergodic. The positive association of μ_ϵ follows from the monotonicity of the map m, i.e., the event $m^{-1}(A)$ is increasing if A is increasing. Furthermore, μ_ϵ is R_ϵ-independent with $R_\epsilon = 2\max_{y,z\in\Delta}|y-z|+2$. This is the case because the underlying measure is a Bernoulli measure and the mapped configuration on a set Γ is independent of the underlying configuration in $\{x \in \mathbb{Z}^2 : \mathrm{dist}(x,\Gamma) > \mathrm{diam}(\Delta)\}$, i.e., for $\omega \in \{0,1\}^{\mathbb{Z}^2}$

$$m(\omega)|_\Gamma = m(\sigma)|_\Gamma$$

for all $\sigma \in \{0,1\}^{\mathbb{Z}^2}$ with $\omega = \sigma$ on $\{x \in \mathbb{Z}^2 : \mathrm{dist}(x,\Gamma) \leq \mathrm{diam}(\Delta)\}$.

Last, we have to deduce the μ_ϵ-almost sure absence of an infinite 1⧄cluster. To this end, we will show that each 0⧄circuit with respect to $\sigma \in \{0,1\}^{\mathbb{Z}^2}$ around a finite subset Λ with $\Delta \subset \Lambda$ is also a 0⧄circuit with respect to $m(\sigma)$ around Λ. For this it is sufficient to note that each 0⧄circuit[σ] C around Δ is also a 0⧄circuit[$m(\sigma)$]. This is the case because each node x of C is 0⧄connected[σ]

to the boundary of $\mathrm{tr}_x(\Delta^c)$ by C, which is equivalent to the non-existence of a $1\boxslash$circuit around x in $\mathrm{tr}_x(\Delta)$ and, therefore, x has also 0spin$[m(\sigma)]$. Furthermore, recall that $\phi_{1/2}$-almost surely no infinite $1\boxslash$cluster occurs. Consequently, each $\Gamma \Subset \mathbb{Z}^2$ is encircled by a $0\boxslash$circuit around Γ $\phi_{1/2}$-almost surely, which, together with the above observation, leads to the following fact: For each $\Gamma \Subset \mathbb{Z}^2$, there exists a $0\boxslash$circuit around Γ μ_ϵ-almost surely. This concludes the proof. $\square$

We give an interesting alternative to the last paragraph of the previous proof, which could be called the Chain Mail approach. Instead of showing that the absence of an infinite $1\boxslash$cluster is preserved by the map m, we can verify that the occurrence of an infinite $1\boxslash$cluster is preserved by the map m. More precisely, given the occurrence of an infinite $1\boxslash$cluster, there exists an infinite $1\boxslash$path P and, therefore, each node x of this infinite $\boxslash$path P is encircled by a $1\boxslash$circuit in $\mathrm{tr}_x(\Delta)$ on $m^{-1}(\{\mathbb{Z}^2 \xleftrightarrow{1\boxslash} \infty\})$. By construction, the largest of these $1\boxslash$circuits intersect or are $\boxslash$adjacent to each other and, therefore, form an infinite $1\boxslash$connected set in $\bigcup_{y\in P} tr_y(\Delta)$.

Chapter 4

Two-Dimensional Widom-Rowlinson Lattice Model

For a better understanding of the following brief outline of this chapter, recall that the Widom-Rowlinson measures will be defined as Gibbs-measures, i.e., as probability measures on the whole configuration space equipped with a certain local structure.

The first section defines the local structure by probability kernels, called the finite Widom-Rowlinson measures. As the name already alludes, these kernels are certain probability measures on the configuration space of a finite observation window under arbitrary allowed boundary conditions. Subsequent to this we give reason but no rigorous proof for some well-known properties, like positive association.

The second section is dedicated to define the (infinite) Widom-Rowlinson measures and state some of their well-known properties. More precisely, probability measures on the whole (infinite) configuration space will be called (infinite) Widom-Rowlinson measures if they exhibit the above local structure, i.e., the probability measure conditioned on the outside of any local observation window has to coincide with a finite Widom-Rowlinson measure. Following the definition, we state (but do not prove) some well-known properties of the Widom-Rowlinson measures, which fall naturally into three parts. First the local structure immediately implies some well-known properties. Then further properties follow from the Gibbs theory. Last we link the occurrence of different Widom-Rowlinson measures to percolation, which is also well-known.

After establishing the Widom-Rowlinson model, we turn to define a related well-known finite model in the third section, called the site-random-cluster model. It can be thought of as a color blind version of the finite Widom-Rowlinson measure, where color blind means that it cannot detect the difference between -1 and 1. Another point of view is to interpret it as a perturbation in favour of many clusters of the finite Bernoulli node percolation.

The fourth section is dedicated to a new (but not surprising) sufficient condition for the uniqueness of the Widom-Rowlinson measures.

Last we show a new sufficient condition for the existence of at most two ergodic measures μ_λ^+ and μ_λ^-, which will be done in five subsections.

Nearly all statements of Section 4.1 to 4.3 except Lemma 4.19 and Corollary 4.20 could be universalised to higher dimensions of the underlying mosaic.

Although the results of this chapter do not really depend on the underlying two-dimensional graph, the results of the next chapter do; they were developed for the graphs $(\mathbb{Z}^2, \square)$ and $(\mathbb{Z}^2, \boxtimes)$, especially the latter one. Furthermore, we have to combine the results of Chapter 4 and 5 to reduce the activity, above which the ergodic Widom-Rowlinson measures are known. Consequently, there is no need to state the results of this chapter with respect to a more general matching pair. Especially since they easily can be carried over to other adequate matching pairs.

4.1 The Finite Widom-Rowlinson Model

Since one of our main concerns is to stay as elementary as possible, we only introduce the Widom-Rowlinson model for the graph $(\mathbb{Z}^2, \boxtimes)$ and, thus, will be able to use the notations of Chapter 2 and 3. We choose the graph $(\mathbb{Z}^2, \boxtimes)$ instead of $(\mathbb{Z}^2, \square)$, because our main result Theorem 1.1 refers to this graph.

Let us begin with the measurable space $(\Omega, \mathcal{F})$ on which both the finite and infinite Widom-Rowlinson measures will be defined. The sample space Ω is the set of configurations $\{-1, 0, 1\}^{\mathbb{Z}^2}$, where, by misuse of notation, we often refer to the -1spins (resp. 1spins) as $-$spins (resp. $+$spins). The σ-algebra $\mathcal{F}$ is generated by the projections $(p_x)_{x \in \mathbb{Z}}$. But sometimes coarser σ-algebras $\mathcal{F}_\Lambda$ that are generated by $(p_x)_{x \in \Lambda}$, $\Lambda \subset \mathbb{Z}^2$, we be needed and called local if the observation window is finite, i.e., $\Lambda \Subset \mathbb{Z}^2$. Extending this terminology, an event is called local if it is measurable with respect to a local σ-algebra.

Now let us turn to the first task of this section: the definition of the finite Widom-Rowlinson measures on $(\mathbb{Z}^2, \boxtimes)$. A configuration $\pi \in \{-1, 0, 1\}^{\mathbb{Z}^2}$ is said to be feasible if all $*$adjacent nodes $x, y \in \mathbb{Z}^2$ take spin values of the same algebraic sign, i.e., they satisfy $\pi(x)\pi(y) \neq -1$. The set of feasible configurations is denoted by F.

Definition 4.1 (finite Widom-Rowlinson measure) *Let $\Lambda \Subset \mathbb{Z}^2$, $\lambda > 0$ and $\omega \in F$. Then the probability kernel from $(\Omega, \mathcal{F}_{\Lambda^c})$ to $(\Omega, \mathcal{F})$*

$$\mu_{\Lambda,\lambda}^{\omega*}(\sigma) = \mathbb{1}_{\{\omega=\sigma \text{ off } \Lambda\}} \frac{\mathbb{1}_F(\sigma)}{Z_{\Lambda,\lambda}^{\omega*}} \prod_{x \in \Lambda} \lambda^{|\sigma(x)|}$$

is called the finite Widom-Rowlinson measure for the observation window Λ*, activity* λ *and boundary condition* ω*, where* $\sigma \in \Omega$ *is the locally modified configuration,* $\mathbb{1}_{\{\omega=\sigma \text{ off } \Lambda\}}$ *is short for* $\prod_{x\in\Lambda^c} \mathbb{1}_{\{\sigma(x)\}}(\omega(x))$ *and*

$$Z^{\omega *}_{\Lambda,\lambda} = \sum_{\substack{\sigma\in F: \\ \omega=\sigma \text{ off } \Lambda}} \prod_{x\in\Lambda} \lambda^{|\sigma(x)|}$$

is the normalising constant, often called the partition function.

Obviously, the three pure boundary conditions $-(.) = -1$, $0(.) = 0$ and $+(.) = +1$ are going to play an important role and, therefore, we denote the corresponding finite Widom-Rowlinson measures by $\mu^{+*}_{\Lambda,\lambda}$, $\mu^{0*}_{\Lambda,\lambda}$ and $\mu^{-*}_{\Lambda,\lambda}$ and call them the finite Widom-Rowlinson measure with +boundary condition, with 0boundary condition and with $-$boundary condition.

Let us adjust two definitions of the first chapter to the new setting.

Definition 4.2 (increasing events) *An event* $E \in \mathcal{F}$ *is called increasing, if* $\sigma \in E$ *and* $\pi \geq \sigma$ *(pointwise) implies* $\pi \in E$*.*

Note that for all nodes $x \in \mathbb{Z}^2$ and $i \in \{-1, 0, 1\}$ the event $\{\omega \in \Omega : \omega(x) \geq i\}$ is local and increasing and, therefore, the σ-algebra $\mathcal{F}$ is also generated by the set of local increasing events, which is closed under intersections.

Based upon the previous definition, we can redefine the "positive association" property.

Definition 4.3 (positively associated) *Let* μ *be a probability measure on* $(\Omega, \mathcal{F})$ *and* $\Delta \subset \mathbb{Z}^2$*. We say* μ *is positively associated if for all increasing sets* $A, B \in \mathcal{F}$

$$\mu(A \cap B) \geq \mu(A)\mu(B)\,.$$

The next definition is crucial for comparing probability measures, which is more or less the content of this chapter.

Definition 4.4 (stochastically dominated) *Let* μ *and* ν *be probability measures on* $(\Omega, \mathcal{F})$*. We say* μ *stochastically dominates* ν *if for all increasing sets* $A \in \mathcal{F}$

$$\nu(A) \leq \mu(A)\,.$$

Note that two probability measures coincide if and only if they stochastically dominate each other.

A closely related idea is the coupling of two measures.

Definition 4.5 (coupling) *Let X and Y be two Ω-valued random variables, μ and μ' their distributions and $\mathbb{P}$ a probability measure on $\Omega \times \Omega$. We say $\mathbb{P}$ is a coupling of X and Y or of μ and μ' if*

$$\begin{aligned}\mathbb{P}(\{(\omega, \omega') : \omega \in .\}) &= \mu(.)\\ \mathbb{P}(\{(\omega, \omega') : \omega' \in .\}) &= \mu'(.)\,,\end{aligned}$$

i.e., $\mathbb{P}$ has marginals μ and μ'.

The stochastic domination $\mu \leq \mu'$ is in fact equivalent to the existence of a coupling $\mathbb{P}$ of μ and μ', whose first coordinate is $\mathbb{P}$-almost surely smaller than its second. This characterisation is called Strassen's Theorem and the fact that the existence of such a coupling can be derived from $\mu \leq \mu'$ is too involved to sketch it here in full generality, see [Li, page 72] for a proof.

Now we are ready to list some well-known properties of the finite Widom-Rowlinson measures.

Lemma 4.6 *The finite Widom-Rowlinson measures satisfy the following conditions:*

i) $\mu^{\omega *}_{\Lambda,\lambda}$ *is Markovian, i.e., for $\Delta \subset \Lambda \Subset \mathbb{Z}^2$ the $\mu^{\omega *}_{\Lambda,\lambda}$-probability of the configuration in Δ only depends on the $*$boundary of Δ, which means for any $B \in \mathcal{F}_\Delta$*

$$\begin{aligned}\mu^{\omega *}_{\Lambda,\lambda}(B|\mathcal{F}_{\Delta^c})(\xi) &= \mu^{\omega *}_{\Lambda,\lambda}(B|\mathcal{F}_{\partial^* \Delta})(\xi)\\ &= \mu^{\xi *}_{\Delta,\lambda}(B)\end{aligned}$$

for $\mu^{\omega *}_{\Lambda,\lambda}$-almost all ξ.

ii) there exists a coupling $\mathbb{P}_\Lambda$ of X and Y with distributions $\mu^{\omega *}_{\Lambda,\lambda}$ and $\mu^{\omega' *}_{\Lambda,\lambda}$ with $\mathbb{P}(\{(\sigma, \sigma') : \sigma \leq \sigma'\}) = 1$, in short $\mathbb{P}_\Lambda(X \leq Y) = 1$, if $\omega \leq \omega'$ (pointwise);

iii) $\mu^{\omega *}_{\Lambda,\lambda} \leq \mu^{\omega' *}_{\Lambda,\lambda}$ *for $\omega \leq \omega'$;*

iv) $\mu^{\omega *}_{\Lambda,\lambda}$ *is positively associated;*

v) $\mu^{+*}_{\Lambda,\lambda} \leq \mu^{+*}_{\Lambda',\lambda}$ *for $\Lambda' \subset \Lambda \Subset \mathbb{Z}^2$;*

vi) $\mu^{-*}_{\Lambda,\lambda} \leq \mu^{-*}_{\Lambda',\lambda}$ *for $\Lambda \subset \Lambda' \Subset \mathbb{Z}^2$;*

vii) $\mu^{+*}_{\Lambda,\lambda} \circ f = \mu^{-*}_{\Lambda,\lambda}$ *for any $\Lambda \Subset \mathbb{Z}^2$,*

where $f : \{-1, 0, 1\}^{\mathbb{Z}^2} \to \{-1, 0, 1\}^{\mathbb{Z}^2}; \omega \mapsto -\omega$ denotes the spin-flip.

As the lemma is well-known, we only present the core ideas of the proof and refer the interested reader to [GHM, Proof of Theorem 4.8 and Theorem 4.11].

Idea of the proof: By definition, the first property is obvious.

The second property is a consequence of Holley's inequality in [GHM], i.e., it follows from the ergodic Markov theorem, the observation that for $a \in \{-1, 0, 1\}$

$$\mu^{\xi *}_{\Lambda,\lambda}\left(p_x \geq a\right) \tag{4.1}$$

is increasing in $\xi \in F$ and the following construction of two Markov chains $(X_n)_{n\in\mathbb{N}}$ with values in $\{\sigma \in \Omega : \sigma = \omega \text{ off } \Lambda\}$ and $(Y_n)_{n\in\mathbb{N}}$ with values in $\{\sigma \in \Omega : \sigma = \omega' \text{ off } \Lambda\}$. Pick X_0 according to $\mu^{\omega *}_{\Lambda,\lambda}$ and Y_0 according to $\mu^{\omega' *}_{\Lambda,\lambda}$. In each time-step the values of both X_n and Y_n at some random node x is altered according to the conditioned probability $\mu^{X_n *}_{x,\lambda}$ respectively $\mu^{Y_n *}_{x,\lambda}$, where the same dice is rolled for both modifications. Obviously, the Markov chain $(Y_n)_{n\in\mathbb{N}}$ eventually hits the maximal feasible configuration ξ, which is larger than the maximal configuration ν with $\mu^{\omega' *}_{\Lambda,\lambda}(\nu) > 0$. From that moment on, the construction, together with the monotonicity of (4.1), guarantees $Y_n \geq X_n$. Since the limiting distributions of X_n (resp. Y_n) stays $\mu^{\omega *}_{\Lambda,\lambda}$ (resp. $\mu^{\omega' *}_{\Lambda,\lambda}$) as n tends to infinity, we have constructed a coupling $\mathbb{P}_\Lambda$ to $\mu^{\omega *}_{\Lambda,\lambda}$ and $\mu^{\omega' *}_{\Lambda,\lambda}$ respectively to $\lim_n X_n$ and $\lim_n Y_n$ that satisfies $\mathbb{P}_\Lambda(\lim_n X_n \leq \lim_n Y_n) = 1$.

The third property is a direct consequence of the second one.

The fourth property follows from the idea used for the second property with one little modification, namely instead of comparing $\mu^{\omega *}_{\Lambda,\lambda}$ to $\mu^{\omega' *}_{\Lambda,\lambda}$ we compare $\mu^{\omega *}_{\Lambda,\lambda}$ to $g\mu^{\omega *}_{\Lambda,\lambda}$ for any increasing positive local function g such that $g\mu^{\omega *}_{\Lambda,\lambda}$ is a probability measure. Note that for $\mu^{\omega *}_{\Lambda,\lambda}\left(p_. = \xi(.) \text{ off } x\right) > 0$

$$\frac{\mu^{\omega *}_{\Lambda,\lambda}\left(p_x \geq a | p_. = \xi(.) \text{ off } x\right)}{\mu^{\omega *}_{\Lambda,\lambda}\left(p_x < a | p_. = \xi(.) \text{ off } x\right)} \leq \frac{g\mu^{\omega *}_{\Lambda,\lambda}\left(p_x \geq a | p_. = \xi(.) \text{ off } x\right)}{g\mu^{\omega *}_{\Lambda,\lambda}\left(p_x < a | p_. = \xi(.) \text{ off } x\right)}$$

holds and implies

$$\mu^{\omega *}_{\Lambda,\lambda}\left(p_x \geq a | p_. = \xi(.) \text{ off } x\right) \leq g\mu^{\omega *}_{\Lambda,\lambda}\left(p_x \geq a | p_. = \xi(.) \text{ off } x\right) .$$

Analogously to the previous idea we obtain $\mu^{\omega *}_{\Lambda,\lambda} \leq g\mu^{\omega *}_{\Lambda,\lambda}$ for any such g, which, in particular, is already sufficient if we consider $g = \frac{\mathbb{1}_B(.)+1}{\mu^{\omega *}_{\Lambda,\lambda}(B)+1}$ for any local and increasing B.

The next two properties are direct consequences of the previous two and the last one follows immediately from the definition. ☺

Calling $\mu^{\omega *}_{\Lambda,\lambda}$ Markovian refers to the possibility of interpreting $\mu^{\omega *}_{\Lambda,\lambda}(.|\mathcal{F}_{\Delta^c})$ as a Markov chain indexed by $\Delta \Subset \mathbb{Z}^2$, where $\Delta^c \setminus \partial^*\Delta$ is the past, $\partial^*\Delta$ the presence, and Δ the future.

4.2 The Infinite Widom-Rowlinson Model

In this section we first define the infinite Widom-Rowlinson measures and then state some basic well-known properties.

4.2.1 Definition and Direct Consequences

Recall that an infinite Widom-Rowlinson measure should exhibit the accurate local structure, which justifies the following definition.

Definition 4.7 (Widom-Rowlinson measure) *Let $\lambda > 0$. We call a probability measure μ on $(\Omega, \mathcal{F})$ an infinite Widom-Rowlinson measure with activity λ if it satisfies the DLR equality regarding the finite Widom-Rowlinson measures, i.e., for all $\Lambda \Subset \mathbb{Z}^2$*

$$\mu(.|\mathcal{F}_{\Lambda^c})(\omega) = \mu^{\omega *}_{\Lambda,\lambda}(.)$$

for μ-almost all $\omega \in \Omega$.

The set of infinite Widom-Rowlinson measures with activity λ on $(\mathbb{Z}^2, \boxtimes)$ is denoted by $\mathrm{WR}^*(\lambda)$. Recall that we omit the star if the underlying graph is $(\mathbb{Z}^2, \square)$. In particular, $\mathrm{WR}(\lambda)$ denotes the set of Widom-Rowlinson measures with activity λ on $(\mathbb{Z}^2, \square)$.

The abbreviation DLR honors Dobrushin, Lanford and Ruelle for their fundamental papers [Do] and [LR]. This approach was developed in a more general framework and leads to the theory of Gibbs measures. For a thorough introduction see [Geo].

The existence of an infinite Widom-Rowlinson measure is in fact well-known, but since it can be done on an elementary level establishing some insight, we will discuss it in the next paragraph.

Due to property v) of Lemma 4.6, for any local increasing event B, the probability $\mu^{+*}_{\Lambda_n,\lambda}(B)$ converges to the same limit for all increasing sequences Λ_n of finite subsets with $\bigcup_{n\in\mathbb{N}} \Lambda_n = \mathbb{Z}^2$, which we refer to by writing $\lim_{\Lambda \nearrow \mathbb{Z}^2} \mu^{+*}_{\Lambda,\lambda}(B)$. Since $\mathcal{F}$ is generated by local increasing events and Ω is compact, there exists a unique probability measure on $(\Omega, \mathcal{F})$, denoted by $\mu^{+*}_{\lambda}(.)$, that satisfies

$$\mu^{+*}_{\lambda}(B) = \lim_{\Lambda \nearrow \mathbb{Z}^2} \mu^{+*}_{\Lambda,\lambda}(B)$$

for all local events B. The independence from the explicit sequence Λ_n implies the automorphism invariance of $\mu^{+*}_{\lambda}(.)$. Moreover, by property i) of Lemma 4.6, for all local events $B \in \mathcal{F}_\Lambda$ and $C \in \mathcal{F}_\Gamma \subset \mathcal{F}_{\Lambda^c}$

$$\begin{aligned}\mu^{+*}_{\lambda}(\mathbb{1}_C(.)\mu^{.*}_{\Lambda,\lambda}(B)) &= \lim_{\Delta \nearrow \mathbb{Z}^2} \mu^{+*}_{\Delta,\lambda}(\mathbb{1}_C(.)\mu^{.*}_{\Lambda,\lambda}(B)) \\ &= \lim_{\Delta \nearrow \mathbb{Z}^2} \mu^{+*}_{\Delta,\lambda}(B \cap C) = \mu^{+*}_{\lambda}(B \cap C)\end{aligned}$$

holds, which implies that $\mu_\lambda^{+*}(.)$ is in fact an infinite Widom-Rowlinson measure. Consequently, the set of infinite Widom-Rowlinson measures with activity λ on $(\mathbb{Z}^2, \boxtimes)$ is non-empty.

The local structure entails that some properties of the finite Widom-Rowlinson measures can be carried over to the infinite Widom-Rowlinson measures. For example the spin-flip relation between between $\mu_{\Lambda,\lambda}^{+*}$ and $\mu_{\Lambda,\lambda}^{-*}$, see the seventh property of Lemma 4.6, can be extended to the corresponding infinite Widom-Rowlinson measures

$$\mu_\lambda^{+*} \circ f = \mu_\lambda^{-*}$$

and also leads to

$$\mu_\lambda^{-*}(B) := \lim_{\Lambda \nearrow \mathbb{Z}^2} \mu_{\Lambda,\lambda}^{-*}(B)$$

for all local events B. Furthermore, the third property of Lemma 4.6, together with the DLR equation, ensures the next well-known lemma.

Lemma 4.8 (Sandwich property) *Each infinite Widom-Rowlinson measure $\mu \in \mathrm{WR}^*(\lambda)$ is stochastically dominated by μ_λ^{+*} and stochastically dominates μ_λ^{-*}, i.e.,*

$$\mu_\lambda^{-*} \leq \mu \leq \mu_\lambda^{+*}.$$

This leads to the following well-known characterisation of the existence of different infinite Widom-Rowlinson measures, which we called phase transition.

Lemma 4.9 *The following four statements are equivalent:*

i) $|\mathrm{WR}^*(\lambda)| > 1$

ii) $\mu_\lambda^{+*} \neq \mu_\lambda^{-*}$

iii) $\exists x \in \mathbb{Z}^2 : \ \mu_\lambda^{+*}(p_x = +) > \mu_\lambda^{-*}(p_x = +)$

iv) $\exists x \in \mathbb{Z}^2 : \ \int \mu_\lambda^{+*}(d\omega)\omega(x) > 0$

Additionally, the third condition is equivalent to

iii') $\exists x \in \mathbb{Z}^2 : \ \mu_\lambda^{+*}(p_x = -) < \mu_\lambda^{-*}(p_x = -)$

and the fourth item is equivalent to

iv') $\exists x \in \mathbb{Z}^2 : \ \int \mu_\lambda^{-*}(d\omega)\omega(x) < 0.$

Once again, we refer to [GHM, Proof of Theorem 4.15] and only describe the core ideas.

Idea of the proof: The equivalence of the first two conditions follows from the Sandwich property in combination with the fact that $\mathcal{F}$ is also generated by local increasing events.

Symmetry implies that for all nodes $x \in \mathbb{Z}^2$

$$\mu_\lambda^{+*}(p_x = i) = \mu_\lambda^{-*}(p_x = j)$$

holds if $i = -j$. Hence, the equivalence of the third and fourth condition follows.

The implication of the third to the second condition is obvious. The reverse implication could be shown by a coupling argument, but we prefer a more elementary argument, used by Lebowitz and Martin-Löf in [LeMa-L, Proof of Lemma 2]. To this end, let us assume condition ii), $\mu_\lambda^{+*} \neq \mu_\lambda^{-*}$, and, therefore, the existence of an event $B \in \mathcal{F}$ with $\mu_\lambda^{+*}(B) \neq \mu_\lambda^{-*}(B)$. Because $\mathcal{F}$ is generated by local increasing events, we can even pick a local increasing event $A \in \mathcal{F}_\Delta$ with $\Delta \Subset \mathbb{Z}^2$ and $\mu_\lambda^{+*}(A) \neq \mu_\lambda^{-*}(A)$. Note that $f_A(\sigma) := \sum_{x\in\Delta} \sigma(x) - \mathbb{1}_A(\sigma)$ is also increasing. The Sandwich property leads to both

$$\mu_\lambda^{+*}(A) > \mu_\lambda^{-*}(A)$$

and

$$\mathbb{E}_{\mu_\lambda^{+*}}(f_A) \geq \mathbb{E}_{\mu_\lambda^{-*}}(f_A)\,.$$

A short calculation shows that the latter observation is equivalent to

$$\frac{\mu_\lambda^{+*}(A) - \mu_\lambda^{-*}(A)}{|\Delta|} \leq \mathbb{E}_{\mu_\lambda^{+*}}(p_{\vec{0}}) - \mathbb{E}_{\mu_\lambda^{-*}}(p_{\vec{0}})\,.$$

By symmetry, the right side equals

$$2\mu_\lambda^{+*}(p_{\vec{0}} = +) - 2\mu_\lambda^{+*}(p_{\vec{0}} = -)$$

and the third condition follows. ☺

The first statement of Lemma 4.6 is similar to the DLR equation, but there also exists a somewhat stronger version, which is based upon the following definition.

Definition 4.10 (determined from outside) *A finite random subset S of $\mathbb{Z}^2$ is said to be determined from outside if $\{S = \Lambda\} \in \mathcal{F}_{\Lambda^c}$ for any $\Lambda \Subset \mathbb{Z}^2$. The corresponding outside σ-algebra is*

$$\mathcal{F}_{S^c} := \{A \in \mathcal{F} : A \cap \{S = \Lambda\} \in \mathcal{F}_{\Lambda^c} \text{ for all } \Lambda \Subset \mathbb{Z}^2\}$$

Due to the countability of the finite subsets of $\mathbb{Z}^2$, the – above announced – next well-known property follows from the local structure.

Lemma 4.11 (Strong Markov property) *Let μ be an infinite Widom-Rowlinson measure and S a μ-almost surely finite random set determined from outside. Then μ satisfies the strong Markov property, i.e., for μ-almost all $\omega \in \Omega$*

$$\mu(.|\mathcal{F}_{S^c})(\omega) = \mu^{\omega*}_{S(\omega),\lambda}(.)\,.$$

4.2.2 Consequences of the Gibbs-Theory

Since we have defined the infinite Widom-Rowlinson measures as a special Gibbs measures, we can benefit from the well-established Gibbs theory. In particular, it is well-known that $\mathrm{WR}^*(\lambda)$ is a compact and convex set, if we consider the topology of local convergence, see [Geo, page 59]. Hence, analysing this set can be done by considering its boundary.

Definition 4.12 (extremal) *We say an infinite Widom-Rowlinson measure with activity λ is extremal if it is not the non-trivial convex combination of two different infinite Widom-Rowlinson measures with activity λ.*

We denote the set of extremal infinite Widom-Rowlinson measures with activity λ by

$$\mathrm{WR}^*_{\mathrm{EX}}(\lambda)\,.$$

Furthermore, if for all local events A the limit of $\mu^{\omega*}_{\Lambda,\lambda}(A)$ is unique, then $\lim_{\Lambda=\mathbb{Z}^2}\mu^{\omega*}_{\Lambda,\lambda}$ denotes the unique infinite Widom-Rowlinson measure that coincides with these limits. In other words, if we can write $\lim_{\Lambda\nearrow\mathbb{Z}^2}\mu^{\omega*}_{\Lambda,\lambda}(A)$ for all local events A, then $\lim_{\Lambda=\mathbb{Z}^2}\mu^{\omega*}_{\Lambda,\lambda}$ denotes the unique infinite Widom-Rowlinson measure with

$$\lim_{\Lambda=\mathbb{Z}^2}\mu^{\omega*}_{\Lambda,\lambda}(A) = \lim_{\Lambda\nearrow\mathbb{Z}^2}\mu^{\omega*}_{\Lambda,\lambda}(A)$$

for all local events A, e.g.,

$$\lim_{\Lambda=\mathbb{Z}^2}\mu^{+*}_{\Lambda,\lambda} = \mu^{+*}_{\lambda}\,.$$

The next characterisation follows from general Gibbs theory, see [Geo, Theorem (7.7) and (7.12)].

Lemma 4.13 *The conditions i)-iii) are equivalent:*

i) $\mu \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$

ii) $\mu \in \mathrm{WR}^*(\lambda)$ *is tail-trivial, i.e., all tail events have μ-probability one or zero.*

iii) $\lim_{\Lambda=\mathbb{Z}^2} \mu^{\omega *}_{\Lambda,\lambda} = \mu$ *for* μ*-almost all* ω

Idea of the proof: If A is a non-trivial tail event, then the identity

$$\mu(.) = \mu(A)\mu(.|A) + \mu(A^c)\mu(.|A^c)$$

implies that μ is not extremal because $\mu(.|A)$ and $\mu(.|A^c)$ are two different Widom-Rowlinson measures.

The third condition follows from the second one on account of the following two observations. By the definition of Widom-Rowlinson measures and the reverse martingal theorem, for any local event B

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \mu^{\omega *}_{\Lambda,\lambda}(B) = \mu(B| \bigcap_{\Delta\Subset\mathbb{Z}^2} \mathcal{F}_{\Delta^c})(\omega)$$

for μ-almost all ω and by the tail triviality, $\mu(.|\bigcap_{\Delta\Subset\mathbb{Z}^2} \mathcal{F}_{\Delta^c}) = \mu(.)$.

Summing up, we know that

$$\lim_{\Lambda=\mathbb{Z}^2} \mu^{\omega *}_{\Lambda,\lambda}(.) = \mu(.)$$

for μ-almost all ω.

The reverse implication iii)⇒ii) is also a consequence of

$$\mu(.| \bigcap_{\Delta\Subset\mathbb{Z}^2} \mathcal{F}_{\Delta^c}) = \lim_{\Lambda=\mathbb{Z}^2} \mu^{\omega *}_{\Lambda,\lambda}(.) = \mu(.)\,, \tag{4.2}$$

which still holds. More precisely, $\mu(.|\bigcap_{\Delta\Subset\mathbb{Z}^2} \mathcal{F}_{\Delta^c}) = \mu(.)$ implies that any event A is μ-stochastically independent of the tail-σ-algebra, which implies that any tail event B satisfies $\mu(B) = \mu(B)\mu(B)$ and, therefore, μ is evidently tail trivial.

The implication of the second to the first condition follows from the fact that an extremal Widom-Rowlinson measure is determined by the probability of its tail events. More precisely, if we assume that a measure $\mu \in \mathrm{WR}^*(\lambda)$ is tail trivial and $\mu = s\mu' + (1-s)\mu''$, then the Radon-Nikodym theorem guarantees the existence of a density f with $\mu' = f\mu$, which in fact is tail measurable. For the exact calculations see [Geo, Proof 2) of Prop. (7.3)]. Hence, μ-almost surely $f = 1$ and, consequently, $\mu = \mu' = \mu''$. ☺

The equality of Lemma (4.13) iii) especially holds for increasing events and, therefore, the next remark follows from the positive association of finite Widom-Rowlinson measures.

Remark 4.14 *All extremal Widom-Rowlinson measures are positively associated.*

Recall that we are interested in the subset of all translation-invariant infinite Widom-Rowlinson measures and, in particular, in the extremal points of this convex subset.

Definition 4.15 (ergodic) *We say a translation-invariant Widom-Rowlinson measure with activity λ is ergodic if it is not the non-trivial convex combination of two different translation-invariant Widom-Rowlinson measures with activity λ. Let*

$$\mathrm{WR}^*_{\mathrm{ER}}(\lambda)$$

denote the set of ergodic infinite Widom-Rowlinson measures with activity λ.

In Chapter 3 a probability measure was called ergodic if it is translation-invariant and trivial on the translation-invariant σ-algebra, i.e., all translation-invariant events have probability one or zero. Fortunately, because of the following remark we do not have to break with that habit.

Corollary 4.16 *An infinite Widom-Rowlinson measure μ is ergodic if and only if it is translation-invariant as well as trivial on the translation-invariant σ-algebra.*

For the proof we refer the interested reader to [Geo, Theorem (14.15)] and note its similarity to the equivalence ii) $\Longleftrightarrow$ i) of Lemma 4.13.

For example, the Widom-Rowlinson measures μ_λ^{+*} and μ_λ^{-*} are extremal as well as ergodic, since the Sandwich property leads to

$$\mu_\lambda^{+*}, \mu_\lambda^{-*} \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda)\,,$$

which, together with the translation-invariance of these measures, implies

$$\mu_\lambda^{+*}, \mu_\lambda^{-*} \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda) \cap \mathrm{WR}^*_{\mathrm{ER}}(\lambda)\,.$$

It is the case that similarly to Lemma 4.13 iii), there also exists a limit theorem for ergodic Widom-Rowlinson measures, see [Geo, Theorem (14.20) (b)]. It states that if μ is an ergodic Widom-Rowlinson measure, then for any local event A,

$$\lim_{\Lambda \nearrow \mathbb{Z}^2} 1/|\Lambda| \sum_{x\in\Lambda} \mu^{\mathrm{tr}_x(\omega)*}_{\mathrm{tr}_x(\Lambda),\lambda}(A) = \mu(A)$$

holds for μ-almost all ω, where tr_x denotes the translation by x. However, this does not imply positive association, as it is the case with extremal Widom-Rowlinson measures, since this property can be lost by the arithmetic mean.

Another similarity of extremal and ergodic Gibbs-measures is that each Widom-Rowlinson measure (resp. translation-invariant Widom-Rowlinson measure) is a weighted average of extremal (resp. ergodic) Widom-Rowlinson measures, see [Geo, Theorem (7.26) resp. Theorem (14.17)]. More precisely, any Widom-Rowlinson measure (resp. translation-invariant Widom-Rowlinson measure) with activity λ is represented as the barycentre of a mass distribution on $\mathrm{WR}^*_{\mathrm{EX}}(\lambda)$ (resp. $\mathrm{WR}^*_{\mathrm{ER}}(\lambda)$), which is called the extremal (resp. ergodic) decomposition.

For the moment this similarity is the last statement that uses the general theory of Gibbs measures and we return to more elementary arguments.

4.2.3 Connecting Phase Transition to Percolation

Before we begin with the characterisation of phase transition in terms of percolation, let us first verify the uniqueness of pure infinite clusters regarding ergodic Widom-Rowlinson measures, i.e., for any ergodic Widom-Rowlinson measure μ, there exists at most one infinite $-*$cluster (resp. infinite 0cluster resp. infinite $+*$cluster) μ-almost surely. To this end, we introduce the following notation.

Definition 4.17 *For $\sqcup \in \{-*, 0, +*, -0, 0+\}$ the event that a sole infinite $\sqcup$cluster occurs is denoted by $E^{\sqcup}$, e.g., E^{-*}. Accordingly, for $\sqcup \in \{-*, 0, +*, -0, 0+\}$ the event of finiteness of all $\sqcup$clusters is denoted by $F^{\sqcup}$.*

The letter E alludes to the **e**xistence of one corresponding infinite cluster and the letter F alludes to the **f**initeness of all corresponding clusters. Furthermore, all events of this definition are translation-invariant.

For the next statement recall that intersections apply before unions.

Lemma 4.18 *For each type of spin, there exists at most one pure infinite cluster $\mathrm{WR}^*_{\mathrm{ER}}(\lambda)$-almost surely.*

Proof: Fix a measure $\mu \in \mathrm{WR}^*_{\mathrm{ER}}(\lambda)$ and recall the Burton-Keane uniqueness theorem, which assumes ergodicity and the finite energy condition. Unfortunately, because a $+$spin and a $-$spin may not be $*$adjacent, μ does not satisfy the latter assumption.

For a workaround of this problem, consider the following maps from $\{-1,0,1\}^{\mathbb{Z}^2}$ to $\{0,1\}^{\mathbb{Z}^2}$. Denote by $m_{-\mapsto 0}$ the change of all $-$spins to 0spins, i.e.,

$$m_{-\mapsto 0} : \{-1,0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}; \sigma \mapsto \mathbb{1}_{\sigma^{-1}(1)}$$

and denote by $m_{\substack{0\mapsto 1\\ -\mapsto 0}}$ the change of all 0spins to 1spins and, afterwards, of all $-$spins to 0spins, i.e.,

$$m_{\substack{0\mapsto 1\\ -\mapsto 0}} : \{-1,0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}; \sigma \mapsto \mathbb{1}_{\sigma^{-1}(1)\cup\sigma^{-1}(0)}\,.$$

The inheritance of ergodicity guarantees that the two resulting probability distributions $\mu \circ m_{-\mapsto 0}^{-1}$ and $\mu \circ m_{\substack{0\mapsto 1\\ -\mapsto 0}}^{-1}$ are also ergodic. Moreover, both probability distributions satisfy the finite energy condition, which will be proved rigorously in the next paragraph. The intuitive reason for this is that the μ-probability of inserting a 0spin at a given node x conditioned on the outside is bounded away from zero, i.e., one can find an $\epsilon > 0$ so that

$$\mu(p_x = 0 | \mathcal{F}_{\partial^* x}) \geq \epsilon$$

μ-almost surely.

By symmetry, it is sufficient to show that $\mu \circ m^{-1}_{-\mapsto 0}$ satisfies the finite energy condition, i.e., for all $x \in \mathbb{Z}^2$ and $j = 0, 1$

$$\mu \circ m^{-1}_{-\mapsto 0}\big(\{\sigma \in \{0,1\}^{\mathbb{Z}^2} : \sigma(x) = j\} \big| \mathcal{F}'_{x^c}\big) > 0\,, \tag{4.3}$$

where $\mathcal{F}'_{x^c}$ stands for the σ-algebra on $\{0,1\}^{\mathbb{Z}^2}$ generated by projections on x^c. It suffices to verify that for an arbitrary node $x \in \mathbb{Z}^2$, for $j = 0, 1$, and for all $A \in \mathcal{F}'_{x^c}$ with $\mu \circ m^{-1}_{-\mapsto 0}(A) > 0$

$$\int_A \mathbb{1}_{\{\sigma(x)=j\}}\; d\mu \circ m^{-1}_{-\mapsto 0} > 0 \tag{4.4}$$

holds. To this end, fix an arbitrary node $x \in \mathbb{Z}^2$ and an arbitrary set $A \in \mathcal{F}'_{x^c}$ with $\mu \circ m^{-1}_{-\mapsto 0}(A) > 0$. The case $j = 0$ follows from:

$$\begin{aligned}
\int_A \mathbb{1}_{\{\sigma(x)=0\}}\; d\mu \circ m^{-1}_{-\mapsto 0} &= \int_{m^{-1}_{-\mapsto 0}(A)} \mathbb{1}_{\{\sigma(x)\in\{-1,0\}\}}\; d\mu \\
&= \int_{m^{-1}_{-\mapsto 0}(A)} \mu\Big(\sigma(x) \in \{-1,0\} \Big| m^{-1}_{-\mapsto 0}(\mathcal{F}'_{x^c})\Big) d\mu \\
&\geq \int_{m^{-1}_{-\mapsto 0}(A)} \underbrace{\mu^{+*}_{x^c,\lambda}\big(\sigma(x) \in \{-1,0\}\big)}_{>0}\, d\mu > 0\,.
\end{aligned}$$

The more complicated case $j = 1$ remains to be shown, i.e.,

$$\int_A \mathbb{1}_{\{\sigma(x)=1\}}\; d\mu \circ m^{-1}_{-\mapsto 0} \overset{!}{>} 0\,. \tag{4.5}$$

Let us first assume

$$\mu\left(m^{-1}_{-\mapsto 0}(A) \cap \big\{\sigma \in \{-1,0,1\}^{\mathbb{Z}^2} : \sigma(y) \geq 0 \text{ for all } y \in \partial^* x\big\}\right) > 0 \tag{4.6}$$

and prove it rigorously later on. The intuitive reason for (4.6) is that the set $m^{-1}_{-\mapsto 0}(A)$ cannot distinguish between $-$spins and 0spins and, moreover, inserting

0spins is always possible. Considering $\mu\left(\sigma(x)=1 \mid \mathcal{F}_{x^c}\right)>0$ on $\{\sigma \in\{-1,0,1\}^{\mathbb{Z}^2}:$ $\sigma(y) \geq 0$ for all $y \in \partial^* x\}$ leads to our aim (4.5):

$$
\begin{aligned}
0 & <\int_{m_{-\mapsto 0}^{-1}(A) \cap\{\sigma(y) \geq 0 \text { for all } y \in \partial^* x\}} \underbrace{\mathbb{1}_{\{\sigma(x)=1\}}}_{>0} \, d\mu \\
& \leq \int_{m_{-\mapsto 0}^{-1}(A)} \mathbb{1}_{\underbrace{\{\sigma \in\{-1,0,1\}^{\mathbb{Z}^2}: \sigma(x)=1\}}_{=m_{-\mapsto 0}^{-1}(\sigma \in\{0,1\}^{\mathbb{Z}^2}: \sigma(x)=1)}} d\mu \\
& =\int_A \mathbb{1}_{\{\sigma(x)=1\}} \, d\mu \circ m_{-\mapsto 0}^{-1} .
\end{aligned}
$$

It remains to show (4.6). To this end, recall that the set $m_{-\mapsto 0}^{-1}(A)$ has positive μ-probability. Hence, we can fix an $\omega \in\{-1,0,1\}^{\partial^* x}$ with

$$
\mu\left(m_{-\mapsto 0}^{-1}(A) \cap \overline{\omega}\right)>0,
$$

where $\overline{\omega}$ denotes $\{\sigma \in\{-1,0,1\}^{\mathbb{Z}^2}: \sigma(y)=\omega(y)$ for all $y \in \partial^* x\}$. After fixing the configuration on $\partial^* x$, we forget all restrictions on

$$
\Delta:=\omega^{-1}(\{-1,0\}),
$$

i.e., we consider the set

$$
B:=m_{-\mapsto 0}^{-1}(A) \cap \overline{\mathbb{1}_{(\partial^* x) \backslash \Delta}} \in \mathcal{F}_{\Delta^c} .
$$

By definition, $m_{-\mapsto 0}^{-1}(A) \cap \overline{\omega} \subset B$ and, therefore, $\mu(B)>0$. Moreover, it is the case that

$$
\begin{aligned}
B \cap\{\sigma \in & \{-1,0,1\}^{\mathbb{Z}^2}: \sigma(y)=0 \text { for all } y \in \Delta\} \\
& \subset m_{-\mapsto 0}^{-1}(A) \cap\{\sigma \in\{-1,0,1\}^{\mathbb{Z}^2}: \sigma(y) \geq 0 \text { for all } y \in \partial^* x\} .
\end{aligned}
$$

Now we are ready to verify (4.6):

$$
\begin{aligned}
\mu & \left(m_{-\mapsto 0}^{-1}(A) \cap\{\sigma \in\{-1,0,1\}^{\mathbb{Z}^2}: \sigma(y) \geq 0 \text { for all } y \in \partial^* x\}\right) \\
& \geq \mu\left(B \cap\{\sigma \in\{-1,0,1\}^{\mathbb{Z}^2}: \sigma(y)=0 \text { for all } y \in \Delta\}\right) \\
& =\int_B \underbrace{\mu(\{\sigma \in\{-1,0,1\}^{\mathbb{Z}^2}: \sigma(y)=0 \text { for all } y \in \Delta\} \mid \mathcal{F}_{\Delta^c})}_{>0} \, d\mu>0 .
\end{aligned}
$$

Consequently, we can apply the Burton-Keane uniqueness theorem to both probability distributions $\mu \circ m_{-\mapsto 0}^{-1}$ and $\mu \circ m_{\substack{0 \mapsto 1 \\ -\mapsto 0}}^{-1}$.

Note that the $+*$clusters (resp. -0clusters) for μ coincide with the $1*$clusters (resp. 0clusters) for $\mu \circ m_{-\mapsto 0}^{-1}$ and the $-*$clusters (resp. $0+$clusters) regarding μ coincide with the $0*$clusters (resp. 1clusters) regarding $\mu \circ m_{\substack{0\mapsto 1\\ -\mapsto 0}}^{-1}$. Therefore, the Burton-Keane uniqueness theorem ensures the μ-almost sure uniqueness of the infinite $+*$cluster, the infinite -0cluster, the infinite $-*$cluster and the infinite $0+$cluster.

The μ-almost sure uniqueness of the infinite 0cluster follows from the uniqueness of the infinite -0cluster and the infinite $0+$cluster. More precisely, assume for contradiction that with positive μ-probability, the uniqueness of the infinite 0cluster fails. Then, by ergodicity of μ, at least two infinite 0clusters exist μ-almost surely. Two non-exclusive scenarios can occur; these two infinite 0clusters are separated by an infinite $-*$cluster or by an infinite $+*$cluster. If the first scenario occurs, then the infinite $-*$cluster has at least two infinite holes, i.e., at least two infinite $0+$clusters exist; otherwise (the second scenario) the infinite $+*$cluster has at least two infinite holes, i.e., at least two infinite -0clusters exist. Both scenarios are contrary to the uniqueness of the infinite -0cluster and the infinite $0+$cluster. □

The $*$connectedness of the boundary of a simply connected set is essential for our argument that the infinite 0cluster is unique. An alternative proof that modifies the original proof of the Burton-Keane uniqueness theorem and does not depend on the $*$connectedness of the boundary can be found in [HT, Prop. 3.5].

We can strengthen this uniqueness result for two-dimensional extremal and ergodic Widom-Rowlinson measures, which was partially done in [HT, Prop. 3.6]. The planarity of the underlying mosaic is essential for the application of Corollary 3.15.

Lemma 4.19 *There exists at most one pure infinite cluster* $\mathrm{WR}^*_{\mathrm{ER}}(\lambda)\cap\mathrm{WR}^*_{\mathrm{EX}}(\lambda)$-*almost surely, i.e., for any ergodic as well as extremal Widom-Rowlinson measure* $\mu \in \mathrm{WR}^*_{\mathrm{ER}}(\lambda) \cap \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$, *we know that*

a) $\mu(E^{+*} \cap E^{-0}) = 0$;

b) $\mu(E^{0+} \cap E^{-*}) = 0$.

Proof: Fix an arbitrary $\mu \in \mathrm{WR}^*_{\mathrm{ER}}(\lambda) \cap \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$ and recall Corollary 3.15 of Chapter 3.

Note that the map $m_{-\mapsto 0}$ of the proof of Lemma 4.18 is increasing, so that $\{m_{-\mapsto 0} \in A\}$ is an increasing event if A was an increasing event. Thus, the measure $\mu \circ m_{-\mapsto 0}^{-1}$ is – in addition to the finite energy condition and the ergodicity – positively associated. Applying Corollary 3.15 in the same way as the Burton-Keane uniqueness theorem in the proof of Lemma 4.18, we can conclude that the coexistence of an infinite $+*$cluster and an infinite -0cluster has μ-probability zero, i.e.,

$$\mu(E^{+*} \cap E^{-0}) = 0\,.$$

Similar arguments based upon the monotone map $m^{-1}_{\substack{0\mapsto 1\\ -\mapsto 0}}$ yield

$$\mu(E^{0+} \cap E^{-*}) = 0\,.$$

This concludes the proof of Lemma 4.19. □

Taking the statements of Lemma 4.19 and Lemma 4.18 together, we know that μ-almost surely for any ergodic as well as extremal Widom-Rowlinson measure $\mu \in \mathrm{WR}^*_{\mathrm{ER}}(\lambda) \cap \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$, there exists either a sole pure infinite cluster or no pure infinite cluster at all.

Based upon symmetry, this statement can be strengthend regarding the measure μ^{-*}_λ, μ^{+*}_λ.

Corollary 4.20 *Let $\lambda > 0$. Then μ^{+*}_λ-almost surely there exists either a single infinite $+*$cluster or a single infinite 0cluster or no pure infinite cluster at all, i.e.,*

$$\mu^{+*}_\lambda(E^0 \cup E^{+*} \cup F^{-*} \cap F^0 \cap F^{+*}) = 1$$
$$\mu^{+*}_\lambda(E^{-0} \cap E^{+*}) = 0\,.$$

The analogous statements regarding μ^{-*}_λ also hold, i.e.,

$$\mu^{-*}_\lambda(E^{-*} \cup E^0 \cup F^{-*} \cap F^0 \cap F^{+*}) = 1$$
$$\mu^{-*}_\lambda(E^{-*} \cap E^{0+}) = 0\,.$$

Proof: It is sufficient to prove the first part of the corollary regarding μ^{+*}_λ. The second part then follows by symmetry.

Because μ^{+*}_λ is both ergodic and extremal, Lemma 4.19 and Lemma 4.18 guarantee that μ^{+*}_λ-almost surely there exists either a sole infinite pure cluster or no pure infinite cluster at all. Consequently, it is sufficient to show that an infinite $-*$cluster occurs with μ^{+*}_λ-probability zero.

Assume the contrary, i.e., with positive μ^{+*}_λ-probability, there exists a sole infinite $-*$cluster. By ergodicity, this cluster occurs μ^{+*}_λ-almost surely. This, together with symmetry and stochastical domination, gives

$$1 = \mu^{+*}_\lambda(E^{-*}) = \mu^{-*}_\lambda(E^{+*}) \le \mu^{+*}_\lambda(E^{+*})$$

and verifies the μ^{+*}_λ-almost sure coexistence of an infinite $-*$cluster and an infinite $+*$cluster, which have to be separated by an infinite 0cluster. This is a contradiction to Lemma 4.19. □

The following well-known lemma states that the existence of a certain cluster is in fact equivalent to phase transition. For convenience, we include the items of Lemma 4.9 and refer to [HT, Prop. 5.2.].

Lemma 4.21 *Let $\lambda > 0$. The following statements are equivalent:*

i) $|\mathrm{WR}^*(\lambda)| > 1$

ii) $\mu_\lambda^{+*} \neq \mu_\lambda^{-*}$

iii) $\exists x \in \mathbb{Z}^2 : \ \mu_\lambda^{+*}(p_x = +) > \mu_\lambda^{-*}(p_x = +)$

iii') $\exists x \in \mathbb{Z}^2 : \ \mu_\lambda^{+*}(p_x = -) < \mu_\lambda^{-*}(p_x = -)$

iv) $\exists x \in \mathbb{Z}^2 : \ \int \mu_\lambda^{+*}(d\omega)\omega(x) > 0$

iv') $\exists x \in \mathbb{Z}^2 : \ \int \mu_\lambda^{-*}(d\omega)\omega(x) < 0$

v) $\mu_\lambda^{+*}(\mathbb{Z}^2 \overset{+*}{\longleftrightarrow} \infty) = 1$

v') $\mu_\lambda^{-*}(\mathbb{Z}^2 \overset{-*}{\longleftrightarrow} \infty) = 1$

Idea of the proof: By symmetry, the equivalence of v) and v') is obvious.

Corollary 4.20 states that there μ_λ^{+*}-almost surely exists either

a) a sole infinite $+*$cluster or

b) a sole infinite 0cluster or

c) no pure infinite clusters at all.

Consequently, for the equivalence i)-iv) $\Longleftrightarrow$ v), it is sufficient to show that only case a) implies phase transition.

Let us begin with i)$\Leftarrow$v), i.e., case a) implies phase transition: By symmetry, the μ_λ^{+*}-almost sure existence of an infinite $+*$cluster is equivalent to the μ_λ^{-*}-almost sure existence of an infinite $-*$cluster and, therefore, by Lemma 4.19, $\mu_\lambda^{+*} \neq \mu_\lambda^{-*}$.

The implication $\neg$iii)$\Leftarrow$ $\neg$v), i.e., case b) and c) precludes phase transition, can be verified as follows: The μ_λ^{+*}-almost sure existence of an infinite 0cluster (case b)) or the non-existence of a pure infinite cluster (case c)) guarantees that any finite subset of $\mathbb{Z}^2$ is encircled by a 0circuit μ_λ^{+*}-almost surely. Hence, for all $x \in \mathbb{Z}^2$ and $\epsilon > 0$ there exists a $\Gamma_{x,\epsilon}$ with $x \in \Gamma_{x,\epsilon}$ so that with μ_λ^{+*}-probability at least $1-\epsilon$, there exists a 0circuit in $\Gamma_{x,\epsilon}$ strictly around x. This, together with the strong Markov property and symmetry, verifies the following identities:

$$\begin{aligned}
\mu_\lambda^{+*}(p_x=-) &\geq \mu_\lambda^{+*}(p_x=-, C_{\Gamma_{x,\epsilon}}^{\max 0} \neq \emptyset) \\
&= \int \mu_\lambda^{+*}(d\omega) \mathbb{1}_{\{C_{\Gamma_{x,\epsilon}}^{\max 0} \neq \emptyset\}}(\omega) \mu_\lambda^{+*}(p_x = - | \mathcal{F}_{\left(\mathrm{int} C_{\Gamma_{x,\epsilon}}^{\max 0}\right)^c})(\omega) \\
&= \int \mu_\lambda^{+*}(d\omega) \mathbb{1}_{\{C_{\Gamma_{x,\epsilon}}^{\max 0} \neq \emptyset\}}(\omega) \mu^{0*}_{\mathrm{int} C_{\Gamma_{x,\epsilon}}^{\max 0}(\omega),\lambda}(p_x=-) \\
&= \int \mu_\lambda^{+*}(d\omega) \mathbb{1}_{\{C_{\Gamma_{x,\epsilon}}^{\max 0} \neq \emptyset\}}(\omega) \mu^{0*}_{\mathrm{int} C_{\Gamma_{x,\epsilon}}^{\max 0}(\omega),\lambda}(p_x=+) \\
&= \mu_\lambda^{+*}(p_x=+, C_{\Gamma_{x,\epsilon}}^{\max 0} \neq \emptyset) \\
&\geq \mu_\lambda^{+*}(p_x=+) - \epsilon\,.
\end{aligned}$$

By letting ϵ tend to zero, we have

$$\mu_\lambda^{+*}(p_x=-) \geq \mu_\lambda^{+*}(p_x=+)\,.$$

This, together with symmetry, implies that for any node $x \in \mathbb{Z}^2$,

$$\mu_\lambda^{+*}(p_x=+) \leq \mu_\lambda^{+*}(p_x=-) = \mu_\lambda^{-*}(p_x=+)$$

holds ($\neg$iii)) and, therefore, the absence of phase transition $|\mathrm{WR}^*(\lambda)| = 1$. ☺

Note that already the existence of a Widom-Rowlinson measure μ with

$$\mu(\mathbb{Z}^2 \overset{+*}{\longleftrightarrow} \infty) > 0$$

implies $\mu_\lambda^{+*}(\mathbb{Z}^2 \overset{+*}{\longleftrightarrow} \infty) = 1$ and, therefore, phase transition.

4.3 Site-Random-Cluster Measure

In this section we define and briefly analyse the (finite) site-random-cluster measure with activity λ and free (resp. wired) boundary condition on $(\mathbb{Z}^2, \boxtimes)$, denoted by $\phi^{f*}_{\Lambda,\lambda}$ (resp. $\phi^{w*}_{\Lambda,\lambda}$).

Definition 4.22 (free site-random-cluster measure) *Let $\lambda > 0$ and $\Lambda \Subset \mathbb{Z}^2$. Then*

$$\phi^{f*}_{\Lambda,\lambda} : \{0,1\}^{\mathbb{Z}^2} \to [0,1]; \sigma \mapsto \mathbb{1}_{\{\sigma=0 \text{ off } \Lambda\}} \frac{2^{\kappa^{f*}(\sigma)}}{Z^{f*}_{\Lambda,\lambda}} \prod_{x\in\Lambda} \lambda^{\sigma(x)}$$

is called the free site-random-cluster measure with activity λ *on* $\{0,1\}^\Lambda$, *where* $\mathbb{1}_{\{\sigma=0 \text{ off } \Lambda\}}$ *is short for* $\prod_{x\in\Lambda^c}\mathbb{1}_{\{0\}}(\sigma(x))$, $\kappa^{f*}(\sigma)$ *is the number of* $*$*clusters in* $\sigma^{-1}(1)$ *and* $Z^{f*}_{\Lambda,\lambda}$ *is the normalising constant, i.e.,*

$$Z^{f*}_{\Lambda,\lambda} = \sum_{\substack{\sigma\in\{0,1\}^{\mathbb{Z}^2}\\ \sigma=0 \text{ off } \Lambda}} 2^{\kappa^{f*}(\sigma)} \prod_{x\in\Lambda} \lambda^{\sigma(x)} .$$

We can link this distribution to the finite Widom-Rowlinson measure with zero boundary condition, see [HT, Lemma 5.1].

Remark 4.23 *Let* $\lambda > 0$ *and* $\Lambda \Subset \mathbb{Z}^2$.

a) It is the case that

$$\mu^{0*}_{\Lambda,\lambda}(\{\omega \in \{-1,0,1\}^{\mathbb{Z}^2} : \mathbb{1}_{\omega^{-1}(1)\cup\omega^{-1}(-1)} \in .\}) = \phi^{f*}_{\Lambda,\lambda}(.) .$$

In other words, select $X \in \{-1,0,1\}^{\mathbb{Z}^2}$ *according to* $\mu^{0*}_{\Lambda,\lambda}$. *The distribution of* X^2 *then coincides with the free site-random-cluster measure with activity* λ *on* $\{0,1\}^\Lambda$, $\phi^{f*}_{\Lambda,\lambda}$.

b) Let $\beta^{f*}_\Lambda(.|.)$ *denote a stochastic kernel that conditioned on* $\sigma \in \{0,1\}^{\mathbb{Z}^2}$ *with* $\sigma = 0$ *off* Λ, $\beta^{f*}_\Lambda(.|\sigma)$, *is uniformly distributed on* $\{\omega \in \{-1,0,1\}^{\mathbb{Z}^2} : \omega \text{ is feasible}, \mathbb{1}_{\omega^{-1}(1)\cup\omega^{-1}(-1)} = \sigma\}$. *Then*

$$\int \phi^{f*}_{\Lambda,\lambda}(d\omega)\beta^{f*}_\Lambda(.|\omega) = \phi^{f*}_{\Lambda,\lambda}\beta^{f*}_\Lambda(.) = \mu^{0*}_{\Lambda,\lambda}(.)$$

holds.

In other words, choose $Y \in \{0,1\}^\Lambda$ *according to* $\phi^{f*}_{\Lambda,\lambda}$. *If we flip the spin values of all* $*$*clusters of* $Y(.)^{-1}(1)$ *independently of each other with probability* $\frac{1}{2}$, *then the resulting distribution is the finite Widom-Rowlinson measure with activity* λ *and free boundary condition on* $\{0,1\}^\Lambda$, $\mu^{0*}_{\Lambda,\lambda}$.

After establishing this connection, it is natural to ask if we can define a site-random-cluster measure with a link to the distribution $\mu^{+*}_{\Lambda,\lambda}$ and, therefore, to the occurrence of phase transition.

Definition 4.24 (wired site-random-cluster measure) *Let* $\lambda > 0$ *and* $\Lambda \Subset \mathbb{Z}^2$. *Then*

$$\phi^{w*}_{\Lambda,\lambda} : \{0,1\}^{\mathbb{Z}^2} \to [0,1]; \sigma \mapsto \mathbb{1}_{\{\sigma=1 \text{ off } \Lambda\}} \frac{2^{\kappa^{w*}(\sigma)}}{Z^{w*}_{\Lambda,\lambda}} \prod_{x\in\Lambda} \lambda^{\sigma(x)}$$

is called the wired site-random-cluster measure with activity λ *on* $\{0,1\}^\Lambda$*, where* $\mathbb{1}_{\{\sigma=1 \text{ off } \Lambda\}}$ *is short for* $\prod_{x\in\Lambda^c}\mathbb{1}_{\{1\}}(\sigma(x))$*,* $\kappa^{w*}(\sigma)$ *is the number of* $*$*clusters in* $\sigma^{-1}(1)$ *not* $*$*adjacent to* Λ^c *and* $Z^{w*}_{\Lambda,\lambda}$ *the normalising constant, i.e.,*

$$Z^{w*}_{\Lambda,\lambda} = \sum_{\substack{\sigma\in\{0,1\}^\Lambda \\ \sigma=1 \text{ off } \Lambda}} 2^{\kappa^{w*}(\sigma)} \prod_{x\in\Lambda} \lambda^{\sigma(x)} .$$

In analogy to Remark 4.23 this measure corresponds to the finite Widom-Rowlinson measure with plus boundary condition, see [HT, Lemma 5.1].

Remark 4.25 *Let* $\lambda > 0$ *and* $\Lambda \Subset \mathbb{Z}^2$*.*

a) It is the case that

$$\mu^{+*}_{\Lambda,\lambda}(\{\omega \in \{-1,0,1\}^{\mathbb{Z}^2} : \mathbb{1}_{\omega^{-1}(1)\cup\omega^{-1}(-1)} \in .\}) = \phi^{w*}_{\Lambda,\lambda}(.) .$$

In other words, select $X \in \{-1,0,1\}^{\mathbb{Z}^2}$ *according to* $\mu^{+*}_{\Lambda,\lambda}$*. If we flip all spin values of* $X(.)^{-1}(-1)$*, then the distribution of the resulting configuration equals the wired site-random-cluster measure with activity* λ *on* $\{0,1\}^\Lambda$*,* $\phi^{w*}_{\Lambda,\lambda}$*.*

b) We know that

$$\phi^{w*}_{\Lambda,\lambda}\beta^{f*}_\Lambda(.) = \mu^{+*}_{\Lambda,\lambda}(.)$$

holds, where $\beta^{w*}_\Lambda(.|.)$ *is the stochastic kernel that conditioned on a configuration* $\sigma \in \{0,1\}^{\mathbb{Z}^2}$ *with* $\sigma = 1$ *off* Λ^c*, i.e.,* $\beta^{w*}_\Lambda(.|\sigma)$*, is uniformly distributed on* $\{\omega \in \{-1,0,1\}^{\mathbb{Z}^2} : \omega \text{ is feasible}, \omega = 1 \text{ off } \Lambda, \mathbb{1}_{\omega^{-1}(1)\cup\omega^{-1}(-1)} = \sigma\}$*.*

In other words, choose Y from $\{0,1\}^{\mathbb{Z}^2}$ *according to* $\phi^{w*}_{\Lambda,\lambda}$*. If we flip the spin values of all* $*$*clusters of* $Y(.)^{-1}(1)$ *that are not* $*$*adjacent to* Λ^c *independently of each other with probability* $\frac{1}{2}$*, then the distribution of the resulting configuration equals the finite Widom-Rowlinson measure with activity* λ *and plus boundary condition on* $\{0,1\}^\Lambda$*,* $\mu^{+*}_{\Lambda,\lambda}$*.*

Note that this remark would justify defining an infinite wired site-random-cluster measure on $\{0,1\}^{\mathbb{Z}^2}$ by μ^{+*}_λ with flipped -1spins. This seems to be interesting on its own right, but we prefer to directly analyse the infinite Widom-Rowlinson measure.

As intended, we can add another equivalent condition to Lemma 4.21.

Lemma 4.26 *The following statements are equivalent:*

i) $|\mathrm{WR}^*(\lambda)| > 1$

ii) $\exists\, x \in \mathbb{Z}^2 : \limsup_{\Lambda\nearrow\mathbb{Z}^2} \phi^{w*}_{\Lambda,\lambda}(x \overset{1*}{\longleftrightarrow} \infty) > 0.$

The proof is a direct consequence of the identity

$$\begin{aligned}
\mu_\lambda^{+*}(x \xleftrightarrow{+*} \infty) &= \inf_{\Gamma\nearrow\mathbb{Z}^2} \mu_\lambda^{+*}(x \xleftrightarrow{+*} \Gamma^c) \\
&= \inf_{\Gamma\nearrow\mathbb{Z}^2} \inf_{\Lambda\nearrow\mathbb{Z}^2} \mu_{\Lambda,\lambda}^{+*}(x \xleftrightarrow{+*} \Gamma^c) \\
&= \inf_{\Lambda\nearrow\mathbb{Z}^2} \inf_{\Gamma\nearrow\mathbb{Z}^2} \mu_{\Lambda,\lambda}^{+*}(x \xleftrightarrow{+*} \Gamma^c) \\
&= \inf_{\Lambda\nearrow\mathbb{Z}^2} \mu_{\Lambda,\lambda}^{+*}(x \xleftrightarrow{+*} \infty) \\
&= \limsup_{\Lambda\nearrow\mathbb{Z}^2} \mu_{\Lambda,\lambda}^{+*}(x \xleftrightarrow{+*} \infty) \\
&= \limsup_{\Lambda\nearrow\mathbb{Z}^2} \phi_{\Lambda,\lambda}^{w*}(x \xleftrightarrow{1*} \infty)
\end{aligned}$$

and Lemma 4.21.

A site-random-cluster measure can be thought of as a color-blind finite Widom-Rowlinson measure and, therefore, some information is lost by considering a site-random-cluster measure in comparison to the finite Widom-Rowlinson measure. The advantage of this model is that all configurations are possible and, moreover, it even has bounded energy, which enables us to change spin values without loosing our grip on the probability, see chapter 5.

4.4 A Condition for the Absence of Phase Transition

This section presents a new sufficient condition for the absence of phase transition, closely related to $\mu_\lambda^{+*}(x \xleftrightarrow{+*} \infty)$. At the end the analogous result regarding the absence of phase transition on the graph $(\mathbb{Z}^2, \square)$ will be stated.

First, let us recall the following notation. In order to specify the underlying configuration of a mathematical object we add the configuration in brackets, e.g., we write "the 0circuit[σ]" instead of "the 0circuit w.r.t. σ".

To prepare the proof of the main result of this section, we first split the configuration space $\{0,1\}^\Lambda$ with $\Lambda \Subset \mathbb{Z}^2$ into two disjoint sets depending on which circuit around the origin is larger, the 0circuit or the 1$*$circuit. To this end, we introduce the following definition.

Definition 4.27 (lasso, $*$lasso) *Let Δ be a simply $*$connected finite subset of $\mathbb{Z}^2$ and fix a node $x \in \Delta$. Further, let C be a circuit around x contained in Δ and let P be a path starting in C, ending in $\partial(\Delta^c)$ and contained in Δ. We call the union $C \cup P$ a lasso around x in Δ.*

Analogously, a $*$lasso around x in Δ consists of a $*$path attached to a $*$circuit.

We extend this definition to 0lassos[σ], 1lassos[σ], 0$*$lassos[σ], and so on, where σ stands for the underlying configuration. Most of the times, lassos will be around the origin $\vec{0}$. Therefore, if we omit the phrase "around x" we usually mean "around the origin". Furthermore, we omit "in Δ" if it clear from the context.

Now, we verify a slightly more general relation as the one announced above, namely that a configuration $\sigma \in \{0,1\}^\Lambda$ on a local observation window $\Lambda \Subset \mathbb{Z}^2$ exhibits either a 1$*$lasso as well as a 0$*$lasso or a 0lasso or a 1lasso.

Lemma 4.28 *For all simply $*$connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the set of configurations $\{0,1\}^\Lambda$ is a disjoint union of the following three sets*

$$\begin{aligned}
&\{\sigma \in \{0,1\}^\Lambda : \exists \text{ 1lasso}[\sigma]\} \\
&\{\sigma \in \{0,1\}^\Lambda : \exists \text{ 0lasso}[\sigma]\} \\
&\{\sigma \in \{0,1\}^\Lambda : \exists \text{ 1}{*}\text{lasso}[\sigma] \wedge \exists \text{ 0}{*}\text{lasso}[\sigma]\}\,.
\end{aligned}$$

Proof: First, we argue why it is sufficient to show

$$\{0,1\}^\Lambda = \{\sigma \in \{0,1\}^\Lambda : \exists \text{ 0lasso}[\sigma]\} \uplus \{\sigma \in \{0,1\}^\Lambda : \exists \text{ 1}{*}\text{lasso}[\sigma]\}\,. \tag{4.7}$$

Since flipping all spins is bijective, equality (4.7) is equivalent to

$$\{0,1\}^\Lambda = \{\sigma \in \{0,1\}^\Lambda : \exists \text{ 1lasso}[\sigma]\} \uplus \{\sigma \in \{0,1\}^\Lambda : \exists \text{ 0}{*}\text{lasso}[\sigma]\}\,. \tag{4.8}$$

Now Lemma 4.28 follows from the intersection of (4.7) and (4.8), because a lasso is also a $*$lasso and it is impossible for a 1lasso and a 0lasso to coexist. Therefore, it is sufficient to verify (4.7), which will be done in the sequel.

Note that the origin can be interpreted as a 1$*$circuit if it takes value 1; otherwise it can be interpreted as a 0circuit. Hence, we can always find a 1$*$circuit or a 0circuit and compare the size of the maximal 1$*$circuit to the size of the maximal 0circuit. This is the case because the non-existence of a 1$*$circuit, i.e., $C^{\max 1*} = \emptyset$, implies that even the smallest 0circuit – the origin – is larger than every 1$*$circuit. Further, the absence of a 0circuit implies that the origin has spin value 1 and – interpreted as a $*$circuit – is larger than every 0circuit.

There are two types of configurations depending on whether the maximal 1$*$circuit is larger than the maximal 0circuit or the other way around. In the first case, by case assumption, the maximal 1$*$circuit is 1$*$connected to $\partial^*(\Lambda^c)$ and therefore, is part of a 1$*$lasso. In the second case, by case assumption, the maximal 0circuit is 0connected to $\partial(\Lambda^c)$ and therefore, is part of a 0lasso. Consequently,

$$\{0,1\}^\Lambda = \{\sigma \in \{0,1\}^\Lambda : \exists \text{ 0lasso}[\sigma]\} \cup \{\sigma \in \{0,1\}^\Lambda : \exists \text{ 1}{*}\text{lasso}[\sigma]\} \tag{4.9}$$

holds. Since $(\mathbb{Z}^2, \square)$ and $(\mathbb{Z}^2, \boxtimes)$ are matching pairs, the existence of a $1{*}$lasso prevents the existence of a 0lasso, i.e.,

$$\{\sigma \in \{0,1\}^\Lambda : \exists\ 1{*}\text{lasso}[\sigma]\} \subset \{\sigma \in \{0,1\}^\Lambda : \nexists\ 0\text{lasso}[\sigma]\}\,. \tag{4.10}$$

Combining the equations (4.9) and (4.10) yields (4.7). $\square$

Note that as an immediate consequence a configuration $\sigma \in \{-1,0,1\}^\Lambda$ on a local observation window $\Lambda \Subset \mathbb{Z}^2$ exhibits either a 0lasso or a $-+*$lasso, i.e., a $*$lasso in $\sigma^{-1}(\{-1,1\})$. This implies that a configuration $\mathrm{WR}^*(\lambda)$-almost surely exhibits either a 0lasso or a $+*$lasso or a $-*$lasso in $\Lambda \Subset \mathbb{Z}^2$, i.e., for any $\mu \in \mathrm{WR}^*(\lambda)$, the previous property holds μ-almost surely.

Up to now, all statements of this chapter were more or less well-known. This does not apply to the next theorem.

Theorem 4.29 *Let* $\lambda > 0$. *If*

$$\limsup_{\Lambda \nearrow \mathbb{Z}^2} \mu^{+*}_{\Lambda,\lambda}(\exists\ 0\text{lasso in } \Lambda) > 0\,,$$

then phase transition does not occur, i.e.,

$$|WR^*(\lambda)| = 1\,.$$

Proof: Assume that the condition holds and for contradiction that phase transition occurs.

We note three direct consequences. First, we can fix a sequence of finite subsets Λ_n of $\mathbb{Z}^2$ containing the origin with $\Lambda_n \nearrow \mathbb{Z}^2$ and an $\epsilon > 0$ such that

$$\limsup_{n\to\infty} \mu^{+*}_{\Lambda_n,\lambda}(\exists\ 0\text{lasso in } \Lambda_n) \geq \epsilon\,.$$

Second, due to Lemma 4.21 there exists an infinite $+*$cluster μ^{+*}_λ-almost surely. Third, the μ^{+*}_λ-almost sure finiteness of all -0clusters follows from Corollary 4.20.

Because of the last two statements we can find two integers k, m with $k \leq m$ so that with μ^{+*}_λ-probability at least $1-\epsilon/2$, the infinite $+*$cluster hits Λ_k and all -0clusters intersecting Λ_k are contained in Λ_m. Note that this event is increasing and implies that for all $i \geq m$ with μ^{+*}_λ-probability at least $1-\epsilon/2$, a $+*$lasso in Λ_i exists, which is a local increasing event. Thus, for all $i \geq m$ the $\mu^{+*}_{\Lambda_i,\lambda}$-probability of the existence of a $+*$lasso is at least $1-\epsilon/2$, which leads to

$$\limsup_{n\to\mathbb{Z}^2} \mu^{+*}_{\Lambda_n,\lambda}(\exists\ 0\text{lasso in } \Lambda_n) \leq \epsilon/2\,,$$

a contradiction. $\square$

Next, we translate this statement to the underlying graph $(\mathbb{Z}^2, \square)$.

Theorem 4.30 *Let $\lambda > 0$. If*

$$\limsup_{\Lambda \nearrow \mathbb{Z}^2} \mu^{+}_{\Lambda,\lambda}(\exists\ 0{*}\text{lasso in } \Lambda) > 0\,,$$

then phase transition does not occur, i.e.,

$$|WR(\lambda)| = 1\,.$$

The proof of this theorem is almost a copy of the previous one.

4.5 A Condition for the Existence of at Most Two Ergodic Measures

In this section the following new sufficient condition for the existence of at most two ergodic Widom-Rowlinson measures, namely μ_λ^{-*} and μ_λ^{+*}, will be shown. We call a $*$lasso a $\pm*$lasso if all of its nodes have $-$spin or $+$spin.

Theorem 4.31 *Let $\lambda > 0$. If*

$$\liminf_{\substack{\Lambda \nearrow \mathbb{Z}^2 \\ \partial^*\Lambda \text{ is a circuit}}} \mu^{0*}_{\Lambda,\lambda}(\exists\ \pm{*}\text{lasso in } \Lambda) > 0\,,$$

then

$$\mathrm{WR}^*_{\mathrm{ER}}(\lambda) = \{\mu_\lambda^{-*}, \mu_\lambda^{+*}\}\,.$$

Note that this theorem does not imply phase transition.

The proof falls naturally into four cases according to how many pure infinite clusters can (co)exist, where by saying pure infinite cluster we mean the infinite $-*$cluster, the infinite 0cluster and the infinite $+*$cluster.

First of all, let us establish a sound basis for the following distinction of cases.

Lemma 4.32 *Let $\Delta \Subset \mathbb{Z}^2$ be the interior of a circuit and $\lambda > 0$. Then for all events $A \in \mathcal{F}_\Delta$*

$$\mu^{+*}_{\Delta,\lambda}(A) = \mu^{\omega*}_{\Delta,\lambda}(A)$$

holds for all $\omega \in F$ with $+$spins on $\partial\Delta$. In particular, nothing more than feasibility is required of the spins$[\omega]$ in $\partial^*\Delta \setminus \partial\Delta$.

Proof: The key observation is that each node in Δ $*$adjacent to $\partial^*\Delta$ is also $*$adjacent to $\partial\Delta$, i.e.,

$$\{x \in \Delta : x \text{ *adjacent to } \partial^*\Delta\} = \{x \in \Delta : x \text{ *adjacent to } \partial\Delta\}\,. \tag{4.11}$$

The inclusion " $\supset$ " is obvious and " $\subset$ " is a consequence of the following. If there would exist a node x in Δ $*$adjacent to $\partial^*\Delta$ and not to $\partial\Delta$ then we could connect x to Δ^c without hitting $\partial\Delta$, a contradiction to the definition of $\partial\Delta$.

The same reasoning implies that all nodes of $(\partial^*\Delta)\setminus(\partial\Delta)$ are adjacent to $\partial\Delta$. Consequently, for an $\omega \in F$ with +spins on $\partial\Delta$ we know that

$$\omega(x) = \begin{cases} + & \text{, if } x \in \partial\Delta \\ 0,+ & \text{, if } x \in (\partial^*\Delta)\setminus(\partial\Delta)\,. \end{cases} \tag{4.12}$$

The lemma follows from the identity (4.11) , the observation (4.12) , and the definition of the finite Widom-Rowlinson measures. $\square$

A direct consequence is a useful sufficient condition for identifying a Widom-Rowlinson measure as μ_λ^{+*} respectively μ_λ^{-*}.

Corollary 4.33 *Let $\mu \in \mathrm{WR}^*(\lambda)$. If each finite subset $\Delta \Subset \mathbb{Z}^2$ is μ-almost surely surrounded by a $+*$circuit (resp. a $-*$circuit), then $\mu = \mu_\lambda^{+*}$ (resp. $\mu = \mu_\lambda^{-*}$).*

Proof: Since, by symmetry, the statements are equivalent, we only show the first one. To this end, assume that each finite subset $\Delta \Subset \mathbb{Z}^2$ is encircled by a $+*$circuit μ-almost surely. Further, let A be an arbitrary increasing local event and $\Delta \Subset \mathbb{Z}^2$ so that $\vec{0} \in \Delta$ and $A \in \mathcal{F}_\Delta$.

By our assumptions, for all $\epsilon > 0$ there exists a finite subset $\Gamma \Subset \mathbb{Z}^2$ that contains a $+*$circuit in Γ around Δ with μ-probability at least $1-\epsilon$. Therefore, the strong Markov property leads to

$$\begin{aligned} \mu(A) &\geq \mu(A, C_{\Gamma\setminus\Delta}^{\max +*} \neq \emptyset) \\ &\geq \int \mu(d\omega) 1_{\{C_{\Gamma\setminus\Delta}^{\max +*}\neq\emptyset\}}(\omega)\mu^{\omega *}_{\mathrm{int}C_{\Gamma\setminus\Delta}^{\max +*}}(A) \\ &= \int \mu(d\omega) 1_{\{C_{\Gamma\setminus\Delta}^{\max +*}\neq\emptyset\}}(\omega)\mu^{+*}_{\mathrm{int}C_{\Gamma\setminus\Delta}^{\max +*}}(A) \\ &\geq \int \mu(d\omega) 1_{\{C_{\Gamma\setminus\Delta}^{\max +*}\neq\emptyset\}}(\omega)\mu^{+*}_{\lambda}(A) \\ &\geq (1-\epsilon)\mu_\lambda^{+*}(A)\,, \end{aligned}$$

where the identity follows from Lemma 4.32.

By letting ϵ tend to zero, we obtain that for all increasing local events A,

$$\mu(A) \geq \mu_\lambda^{+*}(A)$$

holds and the lemma follows from the sandwich property. $\square$

4.5.1 No Pure Infinite Cluster

This subsection analyses Widom-Rowlinson measures without pure infinite cluster.

Recall that we denote the absence of an infinite $+*$cluster (resp. $-*$cluster resp. 0cluster) by F^{+*} (resp. F^{-*} resp. F^0), where the letter F alludes to the **f**initeness of all corresponding clusters.

Because of Corollary 4.33 it is not surprising that absence of pure infinite clusters implies absence of phase transition.

Proposition 4.34 *Let $\lambda > 0$. If there exists a $\mu \in \mathrm{WR}^*(\lambda)$ with $\mu(F^{-*} \cap F^0 \cap F^{+*}) > 0$, then $|\mathrm{WR}^*(\lambda)| = 1$.*

Proof: The basic idea is to show that each finite subset is μ-almost surely surrounded by both a $-*$circuit and a $+*$circuit and, therefore, the conclusion $\mu_\lambda^+ = \mu = \mu_\lambda^-$ follows from the previous lemma.

Note that the event $F^{-*} \cap F^0 \cap F^{+*}$ is tail measurable. By extremal decomposition, without loss of generality we can assume $\mu \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$ with $\mu(F^{-*} \cap F^0 \cap F^{+*}) = 1$.

We first prove that each finite subset is μ-almost surely surrounded by a 0circuit. To this end, fix an arbitrary finite subset $\Delta \Subset \mathbb{Z}^2$. Because of the μ-almost sure finiteness of all $-*$clusters and $+*$clusters, the union of Δ, all $-*$clusters meeting Δ, and all $+*$clusters meeting Δ is μ-almost surely finite. Note that the union's outer $*$boundary, defined as the set of nodes of the $*$boundary that are $*$adjacent to infinite $*$paths never hitting the $*$boundary, is a 0circuit surrounding Δ.

Next, we show that each finite subset of $\mathbb{Z}^2$ is μ-almost surely surrounded by a $+*$circuit or a $-*$circuit. Therefore, fix an arbitrary finite subset $\Delta \Subset \mathbb{Z}^2$. Because of $\mu(F^0) = 1$, the union of Δ and all 0clusters meeting Δ is μ-almost surely finite and its boundary is either a $+*$circuit or a $-*$circuit surrounding Δ. Since the events "each finite subset is surrounded by a $-*$circuit" and "each finite subset is surrounded by a $+*$circuit" are tail events, at least one of them occurs μ-almost surely.

It remains to verify that both tail events occur μ-almost surely. To this end, fix $\Delta \Subset \mathbb{Z}^2$ with $\vec{0} \in \Delta$ and without loss of generality assume that each finite subset is μ-almost surely surrounded by a $+*$circuit. Consequently, for all $\epsilon > 0$ there exist finite subsets $\Gamma, \Lambda \Subset \mathbb{Z}^2$ so that with μ-probability at least $1 - \epsilon$, we can find a $+*$circuit in Γ surrounding Δ as well as a 0circuit in Λ surrounding Γ. The strong Markov property, together with the symmetry of $\mu^{0*}_{\mathrm{int}C^{\max 0}_{\Lambda \setminus \Gamma},\lambda}$, ensures that with μ-probability at least $1-\epsilon$, a $-*$circuit in Γ encircles Δ. More precisely, the strong Markov property guarantees

$$
\begin{aligned}
1-\epsilon &\le \mu(C^{\max +*}_{\Gamma\setminus\Delta} \neq \emptyset, C^{\max 0}_{\Lambda\setminus\Gamma} \neq \emptyset) \\
&= \int \mu(d\omega)\mu(C^{\max +*}_{\Gamma\setminus\Delta} \neq \emptyset, C^{\max 0}_{\Lambda\setminus\Gamma} \neq \emptyset | \mathcal{F}_{(\mathrm{int}C^{\max 0}_{\Lambda\setminus\Gamma})^c})(\omega) \\
&= \int \mu(d\omega)\mathbb{1}_{\{C^{\max 0}_{\Lambda\setminus\Gamma}\neq\emptyset\}}(\omega)\mu^{0*}_{\mathrm{int}C^{\max 0}_{\Lambda\setminus\Gamma}(\omega),\lambda}(C^{\max +*}_{\Gamma\setminus\Delta} \neq \emptyset) \\
&= \int \mu(d\omega)\mathbb{1}_{\{C^{\max 0}_{\Lambda\setminus\Gamma}\neq\emptyset\}}(\omega)\mu^{0*}_{\mathrm{int}C^{\max 0}_{\Lambda\setminus\Gamma}(\omega),\lambda}(C^{\max -*}_{\Gamma\setminus\Delta} \neq \emptyset) \\
&= \int \mu(d\omega)\mu(C^{\max -*}_{\Gamma\setminus\Delta} \neq \emptyset, C^{\max 0}_{\Lambda\setminus\Gamma} \neq \emptyset | \mathcal{F}_{(\mathrm{int}C^{\max 0}_{\Lambda})^c})(\omega) \\
&= \mu(C^{\max -*}_{\Gamma\setminus\Delta} \neq \emptyset, C^{\max 0}_{\Lambda\setminus\Gamma} \neq \emptyset)\,,
\end{aligned}
$$

where the third equality follows from symmetry. Since Δ was arbitrary, the tail-triviality of μ concludes the proof. □

4.5.2 One Pure Infinite Cluster

We devote this subsection to analyse a Widom-Rowlinson measure with a sole pure infinite cluster.

Recall that we denote the occurrence of one infinite $+*$cluster (resp. $-*$cluster resp. 0cluster) by E^{+*} (resp. E^{-*} resp. E^0), where the letter E alludes to the **e**xistence of one corresponding infinite cluster.

Recall Corollary 4.33 on page 61 and let us begin with two immediate consequences.

Proposition 4.35 *Let $\mu \in \mathrm{WR}^*(\lambda)$. If there exists a sole infinite $-*$cluster as well as all 0clusters and $+*$clusters are finite μ-almost surely, then the underlying probability measure μ coincides with the Widom-Rowlinson measure with $-$boundary condition, in short*

$$\mu(E^{-*}\cap F^0 \cap F^{+*}) = 1 \quad \Rightarrow \quad \mu = \mu^{-*}_\lambda\,.$$

The analogous result $\mu = \mu^{+*}_\lambda$ holds for the μ-almost sure occurrence of a sole infinite $+*$cluster as well as the μ-almost sure finiteness of all 0clusters and $-*$clusters, in short

$$\mu(F^{-*}\cap F^0 \cap E^{+*}) = 1 \quad \Rightarrow \quad \mu = \mu^{+*}_\lambda\,.$$

Proof: Our proof of the first implication starts with the observation that the μ-almost sure finiteness of all 0+clusters follows from $\mu(E^{-*}\cap F^0 \cap F^{+*}) = 1$. This is the case because the $*$boundary of the infinite $-*$cluster is only equipped with 0spins. Consequently, the existence of an infinite 0+cluster would imply an infinite hole in the infinite $-*$cluster and, therefore, an infinite 0path (contained in both the infinite hole and the $*$boundary of the infinite $-*$cluster) would occur, a contradiction to $\mu(F^0) = 1$.

Hence, for any $\Delta \Subset \mathbb{Z}^2$, the union of Δ and all 0+clusters meeting Δ is μ-almost surely finite and, therefore, any finite subset of $\mathbb{Z}^2$ is surrounded by a $-*$circuit μ-almost surely. By Corollary 4.33, $\mu = \mu_\lambda^{-*}$ follows.

The second implication is a consequence of the first one and symmetry. $\square$

Unfortunately, we are not able to show that μ is a convex combination of μ_λ^{-*} and μ_λ^{+*} if only a sole infinite 0cluster exists. This leads us to the search of a condition that precludes this event.

Proposition 4.36 *Let $\lambda > 0$. If*

$$\liminf_{\substack{\Lambda \nearrow \mathbb{Z}^2 \\ \partial^*\Lambda \text{ is a circuit}}} \mu_{\Delta,\lambda}^{0*}(\exists\ \pm*\text{lasso in } \Delta) > 0\,,$$

then $F^{-*} \cap E^0 \cap F^{+*}$ is impossible $\mathrm{WR}^*(\lambda)$*-almost-surely, i.e., for any measure $\mu \in \mathrm{WR}^*(\lambda)$, this set $F^{-*} \cap E^0 \cap F^{+*}$ has μ-probability zero.*

Proof: We fix $\epsilon \in]0,1[$ so that

$$\liminf_{\substack{\Lambda \nearrow \mathbb{Z}^2 \\ \partial^*\Lambda \text{ is a circuit}}} \mu_{\Delta,\lambda}^{0*}(\exists\ \pm*\text{lasso in } \Delta) \geq \epsilon$$

holds. Let $\Gamma \Subset \mathbb{Z}^2$ with $\vec{0} \in \Gamma$ be so large that for any Δ with $\partial^*\Delta$ is a circuit encircling Γ

$$\mu_{\Delta,\lambda}^{0*}(\exists\ \pm*\text{lasso in } \Delta) \geq \epsilon/2$$

holds, which due to Lemma 4.28 is equivalent to

$$\mu_{\Delta,\lambda}^{0*}(\exists\ 0\text{lasso in } \Delta) \leq 1 - \epsilon/2\,. \tag{4.13}$$

We proceed by showing that $F^{-*} \cap E^0 \cap F^{+*}$ is a tail event $\mathrm{WR}^*(\lambda)$-almost-surely. To this end, note that the existence of an infinite 0cluster and the event "any finite subset of $\mathbb{Z}^2$ is encircled by a 0circuit" are tail-events and, moreover, the intersection of the previous two events equals $F^{-*} \cap E^0 \cap F^{+*}$ $\mathrm{WR}^*(\lambda)$-almost-surely.

Consequently, it is sufficient to show that for any extremal Widom-Rowlinson measure, the event $F^{-*} \cap E^0 \cap F^{+*}$ has probability zero. To this end, let us suppose the contrary, i.e., assume the existence of an extremal Widom-Rowlinson measure

$\mu \in WR^*_{\text{EX}}(\lambda)$ with $\mu(F^{-*} \cap E^0 \cap F^{+*}) = 1$ and fix it. By case assumption, we also know $\mu(\mathbb{Z}^2 \overset{-+*}{\longleftrightarrow} \infty) = 0$. Thus, we can find $\Gamma_1, \Gamma_2, \Gamma_3 \Subset \mathbb{Z}^2$ with $\Gamma \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma_3$ so that with μ-probability $1 - \epsilon/4$, the infinite 0cluster meets Γ_1, which is encircled by a 0circuit contained in Γ_2, and a 0circuit contained in Γ_3 encircles Γ_2, in short

$$\mu(\Gamma_1 \overset{0}{\longleftrightarrow} \infty, C^{\max 0}_{\Gamma_2 \setminus \Gamma_1} \neq \emptyset, C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset) \geq 1 - \epsilon/4\,. \tag{4.14}$$

Given the slight generalisation $\{C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset, \Gamma_1 \overset{0}{\longleftrightarrow} C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}, C^{\max 0}_{\Gamma_2 \setminus \Gamma_1} \neq \emptyset\}$ of the event in (4.14) one can always find a 0lasso in $\text{int}C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}$, i.e.,

$$\begin{aligned}\{C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset, \Gamma_1 \overset{0}{\longleftrightarrow} C^{\max 0}_{\Gamma_3 \setminus \Gamma_2},\, C^{\max 0}_{\Gamma_2 \setminus \Gamma_1} \neq \emptyset\}& \\ \subset \{C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset, \exists 0\text{lasso in int}C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}\}&\,. \end{aligned}\tag{4.15}$$

The strong Markov property ensures the identity in the following chain of inequalities

$$\begin{aligned}1 - \epsilon/4 &\overset{(4.14)}{\leq} \mu(\Gamma_1 \overset{0}{\longleftrightarrow} \infty, C^{\max 0}_{\Gamma_2 \setminus \Gamma_1} \neq \emptyset, C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset) \\ &\leq \int \mu(d\omega) \mathbb{1}_{C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset}(\omega) \mu(\Gamma_1 \overset{0}{\longleftrightarrow} C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}, C^{\max 0}_{\Gamma_2 \setminus \Gamma_1} \neq \emptyset | \mathcal{F}_{(\text{int}C^{\max 0}_{\Gamma_3 \setminus \Gamma_2})^c})(\omega) \\ &= \int \mu(d\omega) \mathbb{1}_{C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset}(\omega) \mu^{0*}_{\text{int}C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}}(\Gamma_1 \overset{0}{\longleftrightarrow} C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}, C^{\max 0}_{\Gamma_2 \setminus \Gamma_1} \neq \emptyset) \\ &\overset{(4.15)}{\leq} \int \mu(d\omega) \mathbb{1}_{C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset}(\omega) \mu^{0}_{\text{int}C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}}(\exists 0\text{lasso in int}C^{\max 0}_{\Gamma_3 \setminus \Gamma_2}) \\ &\overset{(4.13)}{\leq} \int \mu(d\omega) \mathbb{1}_{C^{\max 0}_{\Gamma_3 \setminus \Gamma_2} \neq \emptyset}(\omega)(1 - \epsilon/2) \\ &\leq 1 - \epsilon/2\,,\end{aligned}$$

which is a contradiction. □

4.5.3 Two Pure Infinite Clusters

This subsection analyses a Widom-Rowlinson measure with two pure infinite clusters.

Let us begin with the absurd case.

Lemma 4.37 *Let $\lambda > 0$. The event $E^{-*} \cap F^0 \cap E^{+*}$ is impossible* $\text{WR}^*(\lambda)$*-almost-surely, i.e., for any Widom Rowlinson measures μ, the event $E^{-*} \cap F^0 \cap E^{+*}$ has μ-probability zero.*

Proof: Assume for contradiction that there exists an extremal Widom-Rowlinson measure $\mu \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$ with $\mu(E^{-*} \cap F^0 \cap E^{+*}) > 0$ and fix it. By tail triviality of μ, the tail-event F^0 occurs μ-almost surely, which implies that any finite subset $\Delta \Subset \mathbb{Z}^2$ is encircled by a $-+*$circuit, i.e., a $*$circuit equipped with $-$spins or $+$spins. Since two $*$adjacent nodes never have strict opposite spin values μ-almost surely, we can even state that any finite subset $\Delta \Subset \mathbb{Z}^2$ is encircled by a $-*$circuit or a $+*$circuit μ-almost surely. But this is a contradiction either to the existence of an infinite $-*$cluster or to the existence of an infinite $+*$cluster depending on which $*$circuit occurs infinitely often. □

Since the remaining two cases are similar, we deal with them in one proposition.

Proposition 4.38 *Let $\lambda > 0$. If*

$$\liminf_{\substack{\Lambda \nearrow \mathbb{Z}^2 \\ \partial^*\Lambda \text{ is a circuit}}} \mu^{0*}_{\Delta,\lambda}(\exists \ \pm *\text{lasso in } \Delta) > 0\,,$$

then the set $E^{-*} \cap E^0 \cap F^{+*} \cup F^{-*} \cap E^0 \cap E^{+*}$ is impossible $\mathrm{WR}^*(\lambda)$-almost surely.

Proof: We restrict ourselves to show that the set $F^{-*} \cap E^0 \cap E^{+*}$ is impossible $\mathrm{WR}^*(\lambda)$-almost surely, since the rest of the statement follows by symmetry. By extremal decomposition, it is sufficient to show that for any extremal Widom-Rowlinson measure $\mu \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$, the event $F^{-*} \cap E^0 \cap E^{+*}$ has μ-probability zero.

Let us begin by verifying that $F^{-*} \cap E^0 \cap E^{+*}$ is a tail-event $\mathrm{WR}^*(\lambda)$-almost surely, which will be done by applying the Shield Lemma of Chapter 3, see page 20. To this end, we map the configurations of $\{-1,0,1\}^{\mathbb{Z}^2}$ to $\{0,1\}^{\mathbb{Z}^2}$ by flipping all $-$spins and denote this map by $m_{-\mapsto+}$. Since all finite $-*$clusters are encircled by 0circuits $\mathrm{WR}^*(\lambda)$-almost surely, the event $m_{-\mapsto+}(F^{-*} \cap E^0 \cap E^{+*})$ exhibits a sole infinite 0cluster and a sole infinite 1$*$cluster. Therefore, all conditions – the uniqueness of both infinite clusters – of the Shield Lemma are met and, given the "flipped" event $m_{-\mapsto+}(F^{-*} \cap E^0 \cap E^{+*})$, any finite subset $\Delta \Subset \mathbb{Z}^2$ is encircled by a mixed $^{1*}_{0}$circuit. Consequently, given the event $F^{-*} \cap E^0 \cap E^{+*}$, any finite subset $\Delta \Subset \mathbb{Z}^2$ is encircled by a mixed $^{+*}_{0}$circuit $\mathrm{WR}^*(\lambda)$-almost surely, since any sequence of subsets tending to $\mathbb{Z}^2$ is met by the infinite $+*$cluster eventually. Summing up, we can characterise the event $F^{-*} \cap E^0 \cap E^{+*}$ by the intersection of the tail-events "an infinite 0cluster occurs", "an infinite $+*$cluster occurs" and "any finite subset $\Delta \Subset \mathbb{Z}^2$ is encircled by a mixed $^{+*}_{0}$circuit", which implies that $F^{-*} \cap E^0 \cap E^{+*}$ itself is a tail-event $\mathrm{WR}^*(\lambda)$-almost surely.

Hence, fix a measure $\mu \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$ and let us assume for contradiction $\mu(F^{-*} \cap E^0 \cap E^{+*}) = 1$. Recall that there exists an $\epsilon > 0$ so that

$$\liminf_{\substack{\Lambda \nearrow \mathbb{Z}^2 \\ \partial^*\Lambda \text{ is a circuit}}} \mu^{0*}_{\Delta,\lambda}(\exists \ \pm *\text{lasso in } \Delta) \geq \epsilon\,. \tag{4.16}$$

Fix an arbitrary node $x \in \mathbb{Z}^2$. Inequality (4.16), together with symmetry, enables us to fix a $\Gamma \Subset \mathbb{Z}^2$ with $\vec{0}, x \in \Gamma$ so that for any Δ with $\partial^*\Delta$ is a circuit encircling Γ

$$\mu^{0*}_{\Delta,\lambda}(\exists\ +*\text{lasso around } x \text{ in } \Delta) \geq \epsilon/3\,. \tag{4.17}$$

Since all $-*$clusters are finite μ-almost surely, for any $\Delta \Subset \mathbb{Z}^2$ with $\partial^*\Delta$ is a circuit encircling Γ, there exists a $\Delta' \Subset \mathbb{Z}^2$ larger than Δ so that with probability at least $1/2$, one can find a $0+$circuit in Δ' around Δ. Considering the strong Markov property entails the following identities

$$\begin{aligned}
0 < c :&= \mu^{+*}_\lambda(0 \xleftrightarrow{+*} \infty)\epsilon/6 \\
&\leq \mu^{+*}_\lambda(x \xleftrightarrow{+*} \infty)(\epsilon/3)\mu(C^{\max 0+}_{\Delta'\setminus\Delta} \neq \emptyset) \\
&= \mu^{+*}_\lambda(x \xleftrightarrow{+*} \infty) \int \mu(d\omega)\mathbb{1}_{\{C^{\max 0+}_{\Delta'\setminus\Delta} \neq \emptyset\}}(\omega)(\epsilon/3) \\
&\overset{(4.17)}{\leq} \mu^{+*}_\lambda(x \xleftrightarrow{+*} \infty) \int \mu(d\omega)\mathbb{1}_{\{C^{\max 0+}_{\Delta'\setminus\Delta} \neq \emptyset\}}(\omega) \\
&\qquad\qquad \mu^{\omega*}_{\mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta},\lambda}(\exists\, +*\text{lasso around } x) \\
&= \int \mu(d\omega)\mathbb{1}_{\{C^{\max 0+}_{\Delta'\setminus\Delta} \neq \emptyset, \exists +*\text{lasso around } x \text{ in } \mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}\}}(\omega)\mu^{+*}_\lambda(x \xleftrightarrow{+*} \infty) \\
&\leq \int \mu(d\omega)\mathbb{1}_{\{C^{\max 0+}_{\Delta'\setminus\Delta} \neq \emptyset, \exists +*\text{lasso around } x \text{ in } \mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}\}}(\omega) \\
&\qquad\qquad \mu^{+*}_\lambda(x \xleftrightarrow{+*} C^{\max +*}_{\mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}(\omega)}(\omega)) \\
&\leq \int \mu(d\omega)\mathbb{1}_{\{C^{\max 0+}_{\Delta'\setminus\Delta} \neq \emptyset, \exists +*\text{lasso around } x \text{ in } \mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}\}}(\omega) \\
&\qquad\qquad \mu^{\omega*}_{\mathrm{int}C^{\max +*}_{\mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}}}(x \xleftrightarrow{+*} C^{\max +*}_{\mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}}) \\
&= \mu(C^{\max 0+}_{\Delta'\setminus\Delta} \neq \emptyset, \exists\, +*\text{lasso around } x \text{ in } \mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}, x \xleftrightarrow{+*} C^{\max +*}_{\mathrm{int}C^{\max 0+}_{\Delta'\setminus\Delta}}) \\
&\leq \mu(x \xleftrightarrow{+*} \Delta^c)\,,
\end{aligned}$$

where the last but one inequality follows from Lemma (4.32) and positive association of μ^{+*}_λ. Summing up, if Δ tends to $\mathbb{Z}^2$, then with probability at least $c > 0$, any x is contained in the infinite $+*$cluster, i.e., for any $x \in \mathbb{Z}^2$

$$\mu(x \xleftrightarrow{+*} \infty) \geq c\,.$$

To apply Theorem 3.3 on page 19 we have to map $\{-1, 0, 1\}^{\mathbb{Z}^2}$ to $\{0, 1\}^{\mathbb{Z}^2}$ monotonously, which will be done by exchanging $-$spins for 0spins, denoted by

$m_{-\mapsto 0}$. The advantage of this kind of mapping is that the event $\{m_{-\mapsto 0} \in A\}$ is increasing if A is increasing and, therefore, the measure $\mu' := \mu \circ m_{-\mapsto 0}^{-1}$ has positive associations. Furthermore, there still exists one infinite 1∗cluster as well as one infinite 0cluster μ'-almost surely, since every finite $-*$cluster is encircled by a 0circuit μ-almost surely. By construction of μ', the bounded energy property is also satisfied. This is the case because of the following two facts:

a) every 0spin$[m_{-\mapsto 0}(\xi)]$ of a mapped configuration $m_{-\mapsto 0}(\xi)$ could be a 0spin$[\xi]$ in the underlying configuration ξ;

b) for any node y, the μ-probability that y takes spin value 0 or + is strictly positive if we condition on the event that $\partial^* y$ is equipped with 0 or +spins.

Thus, we can apply Theorem 3.3 and obtain a contradiction to our assumption $\mu(F^{-*} \cap E^0 \cap E^{+*}) = 1$. □

4.5.4 Three Pure Infinite Clusters

In this section we prove that the coexistence of all three pure infinite clusters has probability zero $\mathrm{WR}^*_{\mathrm{ER}}(\lambda)$-almost surely. This can be achieved by the butterfly-method, which first appeared in [GHM] and later on in [GH], was developed by Georgii and Higuchi for an analogous statement in the two-dimensional Ising-model, and is originally based upon Zhang's argument. It is straight forward to apply this method to the Widom-Rowlinson model, as done in [HT, Lemma 6.5, Prop. 6.6]. For self containment, we will modify this approach in such a way that it is based upon our Theorem 3.12.

Our general aim is to show that given the event $E^{-*} \cap E^0 \cap E^{+*}$, Corollary 3.15 ensures the existence of a special kind of infinite -0cluster, which is contrary to the uniqueness of the pure infinite clusters and, therefore, to ergodicity.

Let us begin by stating a sufficient condition for flip-reflection-invariance, which obviously is stronger than translation-invariance. To this end, we need the following definitions.

Definition 4.39 (halfplane, boundary line, +∗arc, −∗arc) *For* $n \in (\mathbb{N} \cup \mathbb{N} + 1/2) = \{r \in \mathbb{R} : 2r \in \mathbb{N}\}$, *we call*

$$H^{\mathrm{up}}(n) := \{(x, y) \in \mathbb{Z}^2 : y \geq n\}$$

the upper halfplane of level n *and*

$$l_{\mathrm{hor}}(n) := \{(x, y) \in \mathbb{R}^2 : y = n\}$$

the horizontal boundary line of level n. *We define the lower, left, right halfplane of level* n, *and the vertical boundary line of level* n *accordingly.*

For any $\Delta \Subset \mathbb{Z}^2$, a $+*$path P either in $H^{\mathrm{up}}(n) \cap \Delta^c$ or in $H^{\mathrm{down}}(n) \cap \Delta^c$ that satisfies the following conditions is called a horizontal $+*$arc of level n around Δ. The starting node of P lies on the left side[1] *of Δ and has Euclidean distance at most $1/2$ from $l_{\mathrm{hor}}(n)$. The ending node of P lies on the right side of Δ and has Euclidean distance at most $1/2$ from $l_{\mathrm{hor}}(n)$.*

A vertical $+*$arc of level n around Δ is defined accordingly.

The above mentioned sufficient condition falls naturally in two parts: the flip-reflection-stochastic-domination and the flip-reflection-stochastic-subordination.

Lemma 4.40 *Let $\lambda > 0$, $n \in \mathbb{N} \cup (\mathbb{N} + 1/2)$ and $\mu \in \mathrm{WR}^*(\lambda)$. If for any arbitrary $\Delta \Subset \mathbb{Z}^2$ there exists μ-almost surely an*

$$\begin{cases} \text{horizontal} & +*\text{arc of level } n \text{ around } \Delta \\ \text{horizontal} & -*\text{arc of level } n \text{ around } \Delta \\ \text{vertical} & +*\text{arc of level } n \text{ around } \Delta \\ \text{vertical} & -*\text{arc of level } n \text{ around } \Delta, \end{cases}$$

then

$$\mu(.) \begin{cases} \geq \mu \circ f \circ R_{\mathrm{hor}}(n)(.) \\ \leq \mu \circ f \circ R_{\mathrm{hor}}(n)(.) \\ \geq \mu \circ f \circ R_{\mathrm{vert}}(n)(.) \\ \leq \mu \circ f \circ R_{\mathrm{vert}}(n)(.) \end{cases},$$

where f flips the spin values, $R_{\mathrm{hor}}(n)$ is the reflection in the horizontal boundary line of level n, $l_{\mathrm{hor}}(n)$, and so on.

Proof: We restrict ourselves to the first case for $n = 0$, since the proofs of the other cases are similar.

First of all, fix an arbitrary finite subset $\Delta \Subset \mathbb{Z}^2$ with $\vec{0} \in \Delta$ and an arbitrary increasing event $A \in \mathcal{F}_\Delta$.

Next, we introduce two nomenclatures. We say a circuit C is reflection-invariant if C equals its reflection $R_{\mathrm{hor}}(0)(C)$. Furthermore, we say a random circuit C, whose realisations $C(\omega)$ are reflection-invariant circuits, stochastically dominates its flipped reflection if $\omega|_{C(\omega)} \geq f \circ R_{\mathrm{hor}}(0)(\omega)|_{C(f \circ R_{\mathrm{hor}}(0)(\omega))}$ and, therefore,

$$\mu^{\omega *}_{\mathrm{int} C(\omega)}(.) \geq \mu^{f \circ R_{\mathrm{hor}}(0)(\omega)*}_{\mathrm{int} C(f \circ R_{\mathrm{hor}}(0)(\omega))}(.)\,.$$

Note that for two circuits C' and C'' that stochastically dominate their flipped reflection, the maximal circuit in $C' \cup C''$ is also such a circuit. This is the

[1]The phrase "on the left side" was introduced on page 23.

case because a node of $C' \cup C''$ belongs to C' if and only if the reflected node $R_{\text{hor}}(0)(x)$ belongs to C', which follows from the fact that both circuits C' and C'' are reflection-invariant. Consequently, given a finite subset $\Gamma \Subset \mathbb{Z}^2$ and the existence of such a circuit in Γ, there exists a maximal circuit in Γ stochastically dominating its flipped reflection. By case assumption, for every $\epsilon > 0$ there exists a $\Gamma \Subset \mathbb{Z}^2$ so that with μ-probability at least $1-\epsilon$, one can find a horizontal $+*$arc of level 0 around Δ in Γ, denoted by $P = P(\epsilon, \Delta, \Gamma)$; otherwise let P be the empty set. Without loss of generality assume that P is contained in $H^{\text{up}}(0)$. The union of P and its reflection $R_{\text{hor}}(0)(P)$, denoted by C, is a $*$circuit around Δ in Γ, if $P \neq \emptyset$. In case $P \neq \emptyset$ the expansion $C' := \partial^*(\text{int}C)$ is a random circuit around Δ in Γ that stochastically dominates its flipped reflection, which follows from Lemma 4.6 by an analogous argument as in Lemma 4.32. More precisely (briefly recalling the reasoning of Lemma 4.32), all nodes in $\text{int}C$ $*$adjacent to C' are also adjacent to C and all nodes in $C' \setminus C$ are adjacent to C. Therefore, for any $\omega \in \{C \neq \emptyset\}$ and

$$\omega'(x) = \begin{cases} +1 & \text{for } x \in C^c \cap C' \cap H^{\text{up}}(0) \\ \omega(x) & \text{else} \end{cases},$$

it is the case that

$$\mu^{\omega *}_{\text{int}C(\omega)}(.) = \mu^{\omega' *}_{\text{int}C(\omega)}(.)\,.$$

Consequently, the construction of C and Lemma 4.6 ensures that $C' := \partial^*(\text{int}C)$ is a random circuit around Δ in Γ that stochastically dominates its flipped reflection.

Let $C^{\max}$ denote the maximal circuit around Δ in Γ that is reflection-invariant and stochastically dominates its flipped reflection. Note that $\text{int}C^{\max}$ is determined from outside and, by choice of Γ, exists with probability at least $1-\epsilon$.

The strong Markov property guarantees the following first and last identity

$$\begin{aligned}
\mu(A) &\geq \mu(A, C^{\max} \neq \emptyset) \\
&= \int \mu(d\omega) \mathbb{1}_{\{C^{\max} \neq \emptyset\}}(\omega) \mu^{\omega *}_{\text{int}C^{\max}(\omega)}(A) \\
&\geq \int \mu(d\omega) \mathbb{1}_{\{C^{\max} \neq \emptyset\}}(\omega) \mu^{f \circ R_{\text{hor}}(0)(\omega) *}_{\text{int}C^{\max}(f \circ R_{\text{hor}}(0)(\omega))}(A) \\
&= \int \left(\mu \circ f \circ R_{\text{hor}}(0)\right)(d\omega) \mathbb{1}_{\{C^{\max} \neq \emptyset\}}(f \circ R_{\text{hor}}(0)(\omega)) \mu^{\omega *}_{\text{int}C^{\max}(\omega)}(A) \\
&= \left(\mu \circ f \circ R_{\text{hor}}(0)\right)\left(A, (f \circ R_{\text{hor}}(0)(C^{\max})) \neq \emptyset\right) \\
&\geq \mu \circ f \circ R_{\text{hor}}(0)(A) - \epsilon\,,
\end{aligned}$$

where the second inequality follows, since $C^{\max}$ stochastically dominates its flipped reflection. The lemma thus follows by letting ϵ tend to zero and Δ to $\mathbb{Z}^2$. $\square$

Note that the sufficient conditions of Lemma 4.40 are tail-events and, therefore, the opposite events – the occurrence of butterflies – are also tail-events.

Definition 4.41 (butterfly) *We say a butterfly exists if one can find two half-planes H and H' with $H \uplus H' = \mathbb{Z}^2$ so that there exists either an infinite -0cluster in both H and in H', or an infinite $0+$cluster in both H and in H'.*

The next step is to show the existence of at least one butterfly under certain conditions.

Lemma 4.42 (Butterfly Lemma) *Let $\lambda > 0$. Given the event $E^{-*} \cap E^0 \cap E^{+*}$, a butterfly exists* $\mathrm{WR}^*(\lambda)$*-almost surely.*

Proof: Pick a measure $\mu \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda)$ with $\mu(E^{-*} \cap E^0 \cap E^{+*}) = 1$. Moreover, assume for contradiction that the absence of a butterfly has positive μ-probability, which – in fact – by tail triviality, has μ-probability one.

Since all conditions of Lemma 4.40 are satisfied, μ is flip-reflection invariant at each level and, therefore, invariant under translations. This leads to $\mu \in \mathrm{WR}^*_{\mathrm{EX}}(\lambda) \cap \mathrm{WR}^*_{\mathrm{ER}}(\lambda)$, which, together with Lemma 4.19 on page 51, is contrary to $\mu(E^{-*} \cap E^0 \cap E^{+*}) = 1$. $\square$

Corollary 4.43 *Let $\lambda > 0$. The event $E^{-*} \cap E^0 \cap E^{+*}$ is impossible* $\mathrm{WR}^*_{\mathrm{ER}}(\lambda)$*-almost surely.*

Proof: Suppose the contrary, i.e., let $\mu \in \mathrm{WR}^*_{\mathrm{ER}}(\lambda)$ be so that the translation-invariant event $E^{-*} \cap E^0 \cap E^{+*}$ has positive μ-probability, which – in fact – by ergodicity, has μ-probability one. By extremal decomposition and Lemma 4.42, there exists a butterfly μ-almost surely. Without loss of generality assume that one can find a -0cluster in $H^{\mathrm{up}}(0)$ as well as in $H^{\mathrm{down}}(-1)$.

We denote by $E^{-0}_{\text{up, down}}(z, 0)$ the event that the node $(z, 0)$ is contained in an infinite -0cluster in $H^{\mathrm{up}}(0)$ as well as in an infinite -0cluster in $H^{\mathrm{down}}(0)$ and say that given this event, $(z, 0)$ pins the "up,down -0butterfly".

Next, we show that $E^{-0}_{\text{up, down}}(0, 0)$ has positive μ-probability. To this end, fix $\Delta \Subset \mathbb{Z}^2$ with $\vec{0} \in \Delta$ so large that with μ-probability at least $1/2$, the infinite -0cluster in $H^{\mathrm{up}}(0)$ as well as the infinite -0cluster in $H^{\mathrm{down}}(-1)$ meet $\partial\Delta$ and denote this event by A. Note that A is measurable with respect to $\mathcal{F}_{\Delta^c}$ and the $\mu^{\omega *}_{\Delta}$-probability of the event B that all nodes of Δ are equipped with 0spins is uniformly (in $\omega \in F$) bounded away from zero by a positive constant δ. Thus, the Markov property ensures the following identity

$$0 < c := \delta/2 \leq \int (d\omega) \mathbb{1}_A(\omega) \mu^{\omega *}_{\Delta}(B) = \mu(A, B) \leq \mu(E^{-0}_{\text{up, down}}(z, 0)) \,.$$

The ergodicity of μ, together with the Ergodic Theorem, verifies that infinitely many nodes of the positive x-axis pin the "up,down -0butterfly" μ-almost surely. But since the Ergodic Theorem also implies that infinitely many nodes of the positive x-axis are contained in the infinite $+*$cluster, this is contrary to the uniqueness of the infinite $+*$cluster. $\square$

4.5.5 Proof of Theorem 4.31

For the sake of completeness, we give the formal proof of Theorem 4.31. To this end, recall that Lemma 4.18 states that for each spin, there exists $\mathrm{WR}^*_{\mathrm{ER}}(\lambda)$-almost surely at most one infinite pure cluster.

Proof: Fix $\mu \in \mathrm{WR}^*_{\mathrm{ER}}(\lambda)$. Since each pure infinite cluster is unique, see Lemma 4.18, we can distinguish cases according to the number of pure infinite clusters. These cases are obviously translation-invariant and by ergodicity, occur with μ-probability one or zero.

First, let us consider the case that no pure infinite cluster $F^{-*} \cap F^0 \cap F^{+*}$ exists at all. To this end, recall Proposition 4.34 and note that $\mu = \mu_\lambda^{+*} = \mu_\lambda^{-*}$ follows from $\mu(F^{-*} \cap F^0 \cap F^{+*}) = 1$.

Next, we consider the events with at least one pure infinite cluster and discover that all of them except $E^{-*} \cap F^0 \cap F^{+*}$ and $F^{-*} \cap F^0 \cap E^{+*}$ are μ-null sets:

- Corollary 4.43 yields that the existence of three pure infinite clusters $E^{-*} \cap E^0 \cap E^{+*}$ has μ-probability zero;
- Lemma 4.37 and Proposition 4.38 imply that the existence of two pure infinite clusters $F^{-*} \cap E^0 \cap E^{+*} \cup E^{-*} \cap F^0 \cap E^{+*} \cup E^{-*} \cap E^0 \cap F^{+*}$ has μ-probability zero;
- Proposition 4.36 ensures that the occurrence of one sole infinite cluster, namely the infinite 0cluster, has μ-probability zero.

Consequently, we have either $\mu(F^{-*} \cap E^0 \cap F^{+*}) = 1$, which by Proposition 4.35 implies $\mu = \mu_\lambda^{+*}$, or $\mu(E^{-*} \cap F^0 \cap F^{+*}) = 1$, which gives $\mu = \mu_\lambda^{-*}$. $\square$

Chapter 5

A Combinatorial Approach to the Sufficient Condition

This chapter is dedicated to analyse the structure of the space of configurations that are equipped with 0spins outside of a set $\Lambda \Subset \mathbb{Z}^2$, in short $\{\sigma \in \{0,1\}^{\mathbb{Z}^2} : \sigma = 0 \text{ off } \Lambda\}$, when Λ is simply $*$connected. Recall that a configuration exhibits either a 1$*$lasso or a 0lasso in Λ, see Lemma 4.28 on page 58, and, therefore, this configuration space can be split into $\{\exists \text{ 0lasso in } \Lambda\}$ and $\{\exists \text{ 1}{*}\text{lasso in } \Lambda\}$. We connect these two subsets with a non-trivial injection that (more or less) adds more 1spins than it reduces the number of 1$*$clusters. This property enables us to compare the $\phi^{f*}_{\Lambda,\lambda}$-probability of the domain with the $\phi^{f*}_{\lambda,\Lambda}$-probability of the image if the activity is at least two.

First let us recall and extend a useful notation. In order to specify the underlying configuration of an object we add the configuration in brackets, e.g., we write "the maximal 0circuit[σ] in Λ" instead of "the maximal 0circuit in Λ w.r.t. σ", which is denoted by $C^{\max 0}_{\Lambda}(\sigma)$. Multiple configurations in a bracket express that a certain property holds for these configuration, e.g., C is a 0circuit[σ, π].

For convenience, we only consider configurations in $\{0,1\}^{\Lambda}$, which could easily be extended to $\{0,1\}^{\mathbb{Z}^2}$ equipped with 0spins off Λ. For example by writing $\{\exists \text{ 0lasso}\} = \{\sigma \in \{0,1\}^{\Lambda} : \exists \text{ 0lasso}[\sigma]\}$ we actually mean $\{\sigma \in \{0,1\}^{\mathbb{Z}^2} : \sigma = 0 \text{ off } \Lambda, \exists \text{ 0lasso}[\sigma] \text{ in } \Lambda\}$.

5.1 A Non-Trivial Injection

This section's goal is to construct an injection from

$$\{\exists \text{ 0lasso in } \Lambda\} = \{\sigma \in \{0,1\}^{\mathbb{Z}^2} : \sigma = 0 \text{ off } \Lambda, \exists \text{ 0lasso}[\sigma] \text{ in } \Lambda\}$$

into

$$\{\exists \text{ 1}{*}\text{lasso in } \Lambda\} = \{\sigma \in \{0,1\}^{\mathbb{Z}^2} : \sigma = 0 \text{ off } \Lambda, \exists \text{ 1}{*}\text{lasso}[\sigma] \text{ in } \Lambda\}$$

in such a manner that we only add 1spins and are able to control the decrease in the number of 1∗clusters. Note that we cannot control the $\phi^{f*}_{\Lambda,\lambda}$-probability by simply flipping all spin values in Λ.

First of all, let us define a special path P that is a unique connection from the starting node x to the ending node y in P.

Definition 5.1 (induced path) *We call a path $(x_1,\dots,x_n)$ an induced path if there exists only one path from x_1 to x_n in the set $\{x_1,\dots,x_n\}$, i.e., for all $1 \leq i,j \leq n$*

$$|i-j| = 1 \iff x_i \text{ is adjacent to } x_j\,.$$

Accordingly, we define an induced circuit, which is uniquely defined by its interior, in the following way.

Definition 5.2 (induced circuit) *A circuit $(c_1,\dots,c_n)$ is called an induced circuit if $(c_1,\dots,c_n)$ is a path and*

$$1 \leq i,j \leq n \text{ with } i-j = 1 \bmod n \iff x_i \text{ is adjacent to } x_j\,.$$

Furthermore, we denote by ${}^{\mathrm{i}}\mathfrak{C}\,(\Lambda)$ the set of induced circuits around $\vec{0}$ in Λ.

We extend these definitions in the obvious way to induced 0paths[σ], induced 1paths[σ], induced 0circuits[σ], induced 1circuits[σ] for a configuration σ. An induced circuit ${}^{\mathrm{i}}C$ is also a circuit and, therefore, the interior of ${}^{\mathrm{i}}C$, denoted by $\mathrm{int}\,{}^{\mathrm{i}}C$, is the set of nodes encircled by ${}^{\mathrm{i}}C$.

Recall that $C^{\max 0}_{\Lambda}(\sigma)$ denotes the maximal 0circuit[σ] in Λ. From now on, we add the index i to the left upper side of such an object to indicate that we mean the maximal induced 0circuit[σ] in Λ, e.g.,

$${}^{\mathrm{i}}C^{\max 0}_{\Lambda}(\sigma) := \max {}^{\mathrm{i}}\mathfrak{C}\left(\sigma^{-1}(0) \cap \Lambda\right)\,.$$

Let us state some obvious notes.

Remark 5.3 *As opposed to a circuit, an induced circuit is uniquely defined by its interior, which is – in contrast to the interior of a circuit – always simply* $*$*connected.*

Furthermore, for any configuration $\sigma \in \{0,1\}^\Lambda$*, the maximal induced* 0*circuit*[σ] ${}^{\mathrm{i}}C_\Lambda^{\max 0}(\sigma)$ *is always smaller than and contained in the maximal* 0*circuit*[σ] $C_\Lambda^{\max 0}(\sigma)$*.*

Last, the interior of the maximal induced 0*circuit*[σ] *around* $\vec{0}$ *in* Λ*,* $\mathrm{int}{}^{\mathrm{i}}C_\Lambda^{\max 0}(\sigma)$*, is a* $*$*cluster that contains the origin or is* $*$*adjacent to it.*

Next, let us consider the domain of our future injection: For an arbitrary simply $*$connected set $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, let

$$\begin{aligned}
A_\Lambda^0 &= A^0 := \{\sigma \in \{0,1\}^\Lambda : \exists \text{ 0lasso}[\sigma] \text{ in } \Lambda,\ \vec{0} \in C_\Lambda^{\max 0}(\sigma)\} \\
A_\Lambda^1 &= A^1 := \{\exists \text{ 0lasso in } \Lambda, \vec{0} \notin C_\Lambda^{\max 0}, \partial^* C_\Lambda^{\max 0} \overset{1*}{\not\longleftrightarrow} \partial^* C_\Lambda^{\max 1}\} \\
A_\Lambda^2 &= A^2 := \{\exists \text{ 0lasso in } \Lambda, \vec{0} \notin C_\Lambda^{\max 0}, \partial^* C_\Lambda^{\max 0} \overset{1*}{\longleftrightarrow} \partial^* C_\Lambda^{\max 1}\}\,,
\end{aligned}$$

and note that these sets decompose the set $\{\exists \text{ 0lasso}\}$, i.e.,

$$\{\sigma \in \{0,1\}^\Lambda : \exists \text{ 0lasso}[\sigma]\} = A_\Lambda^0 \uplus A_\Lambda^1 \uplus A_\Lambda^2\,.$$

Now we define "the injection" and comment it afterwards:

Definition 5.4 (m_Λ**,** ${}^{\mathrm{i}}C^{\mathbf{fill}}(\sigma)$**)** *For an arbitrary simply* $*$*connected set* $\Lambda \Subset \mathbb{Z}^2$ *with* $\vec{0} \in \Lambda$*, let*

$$\begin{aligned}
m_\Lambda : \{\sigma \in \{0,1\}^\Lambda : \sigma = 0 \text{ off } \Lambda, \exists \text{ 0lasso in } \Lambda\} &\to \\
&\to \{\sigma \in \{0,1\}^\Lambda : \sigma = 0 \text{ off } \Lambda, \exists \text{ 1}*\text{lasso in} \Lambda\}; \\
\sigma \mapsto \mathbb{1}_{\sigma^{-1}(1) \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)}
\end{aligned}$$

be "the injection", where

$${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) = \begin{cases} \vec{0} & \text{for } \sigma \in A_\Lambda^0 \\ \min {}^{\mathrm{i}}\mathfrak{C}\left(C_\Lambda^{\max 0}(\sigma) \cup \sigma^{-1}(1) \cap \mathrm{ext} C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma)\right) & \text{for } \sigma \in A_\Lambda^1 \\ \max {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext} C_\Lambda^{\max 1}(\sigma)} \cup \sigma^{-1}(1)\right) & \text{for } \sigma \in A_\Lambda^2\,. \end{cases}$$

The above presentation of the injection might suggest that a whole 0circuit[σ] is filled with 1spins, but a closer look reveals that ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is often equipped with both 0spins[σ] and 1spins[σ]. This definition turns out to be convenient, since the induced circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is uniquely defined by its interior.

The first part of the following lemma roughly says that for $i = 1, 2$ and $\sigma \in A_\Lambda^i$, we could define ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ with respect to the **induced** circuit ${}^{\mathrm{i}}C_\Lambda^{\max 0}(\sigma)$ respectively ${}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext} C_\Lambda^{\max 1}(\sigma)}$. The second part states some obvious consequences of the definition.

Remark 5.5 *For an arbitrary simply ∗connected set $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the map m_Λ is well defined as well as*

$$
\begin{aligned}
{}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) &= \min {}^{\mathrm{i}}\mathfrak{C}\left({}^{\mathrm{i}}C^{\max 0}_{\Lambda}(\sigma) \cup \sigma^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}_{\Lambda}(\sigma)}(\sigma)\right) && \text{for } \sigma \in A^1_\Lambda \quad (5.1)\\
{}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) &= \max {}^{\mathrm{i}}\mathfrak{C}\left({}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)} \cup \sigma^{-1}(1)\right) && \text{for } \sigma \in A^2_\Lambda \quad (5.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma|_{({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma))^c} &= m(\sigma)|_{({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma))^c} && (5.3)\\
\sigma^{-1}(1) \cup {}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma) &= m(\sigma)^{-1}(1) && (5.4)
\end{aligned}
$$

hold.

Proof: Identity (5.1) follows from ${}^{\mathrm{i}}C^{\max 0}_{\Lambda}(\sigma) \leq C^{\max 0}_{\Lambda}(\sigma)$, Equation (5.2) from ${}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)} \geq C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}$. The last two equalities are evident. □

It remains to show that the map m_Λ is injective. For this, the difference between σ and $m_\Lambda(\sigma)$ plays an important role, which is the content of the next definition.

Definition 5.6 (special paths) *By definition, the set $\{x \in \Lambda : \sigma(x) \neq m_\Lambda(\sigma)(x)\}$ is part of an induced circuit in $\sigma^{-1}(0)$. We interpret $\{x \in \Lambda : \sigma(x) \neq m_\Lambda(\sigma)(x)\}$ as the union of* 0*paths*[σ]*, which we denote by $P_1(\sigma), \ldots, P_{N(\sigma)}(\sigma)$, and call special paths.*

Note that

$$m(\sigma) = \mathbb{1}_{\sigma^{-1}(1) \uplus \biguplus_{1 \leq i \leq N(\sigma)} P_i(\sigma)} \,.$$

The proof of the injectivity of m_Λ falls naturally into three steps by considering the maps $m_\Lambda|_{A^0_\Lambda}$, $m_\Lambda|_{A^1_\Lambda}$ and $m_\Lambda|_{A^2_\Lambda}$ separately. Each step consists of two parts. First we show some properties of the image of the map in question, which will later on imply that the three images are disjoint. Then we prove that the respective map is injective.

Let us begin with the first map $m_\Lambda|_{A^0_\Lambda}$.

Proposition 5.7 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the map $m_\Lambda|_{A^0_\Lambda}$ is injective and the image of $m_\Lambda|_{A^0_\Lambda}$ is a subset of*

$$\{\exists\ 1{*}\mathrm{lasso}, \vec{0} \in C^{\max 1*}_{\Lambda}\} \,.$$

This proposition simply relies on flipping the origin's spin value and will be proved in the Subsection 5.1.1.

Next, we consider $m_\Lambda|_{A^1_\Lambda}$.

Proposition 5.8 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the map $m_\Lambda|_{A^1_\Lambda}$ is injective and the image of $m_\Lambda|_{A^1_\Lambda}$ is a subset of*

$$\{\exists\ 1{*}\text{lasso}, C^{\max 1}_\Lambda \neq \emptyset, \vec{0} \notin C^{\max 1}_\Lambda \stackrel{1*}{\not\longleftrightarrow} C^{\max 1}_{\text{int}C^{\max 1}_\Lambda}\}\,.$$

Idea of the proof: Fix a configuration $\sigma \in A^1_\Lambda$.

The main idea is that we fill parts of the maximal induced 0circuit[σ] to turn σ into $m(\sigma)$ and we empty parts of a "maximal" induced 1circuit[$m(\sigma)$] to change $m(\sigma)$ to σ. These parts are the intersections of both mentioned circuits.

More precisely, consider the maximal 0circuit[σ], add all 1clusters[σ] adjacent to it and take the smallest induced circuit of this set, which equals ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$. By the choice of σ, there exists a node contained in both the maximal 0circuit[σ] and ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$. Recall that $m(\sigma)$ is the configuration that matches σ with the one exception that all spin values in ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ are one. Since ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma) \cap C^{\max 0}(\sigma) \neq \emptyset$, the configuration $m(\sigma)$ exhibits a 1∗lasso. We will show that ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ is the minimal induced 1circuit[$m(\sigma)$] in the exterior of the maximal 0circuit[$m(\sigma)$], which enables us to locate ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ by considering only the configuration $m(\sigma)$. The maximal circuit in the union of ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ and all 0clusters[$m(\sigma)$] adjacent to ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ equals the maximal 0circuit[σ]. Therefore, the map $m_\Lambda|_{A^1_\Lambda}$ is invertible. ☺

The complete proof will be given in Subsection 5.1.2.

Only the analysis of $m_\Lambda|_{A^2_\Lambda}$ is left.

Proposition 5.9 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the map $m_\Lambda|_{A^2_\Lambda}$ is injective and the image of $m_\Lambda|_{A^2_\Lambda}$ is a subset of*

$$\{\exists\ 1{*}\text{lasso}, \vec{0} \notin C^{\max 1}_\Lambda \stackrel{1*}{\longleftrightarrow} C^{\max 1}_{\text{int}C^{\max 1}_\Lambda}\}\,.$$

Idea of the proof: Fix a configuration $\sigma \in A^2_\Lambda$.

The main idea is similar to the one for Proposition 5.8. The only difference is that we fill parts of a "minimal" induced 0circuit[σ] to turn σ into $m(\sigma)$ and we empty parts of the maximal induced 1circuit[$m(\sigma)$] to turn $m(\sigma)$ into σ.

More precisely, consider the minimal 0circuit[σ] in the exterior of the maximal 1circuit[σ], add all 1clusters[σ] adjacent to it and take the largest induced circuit of this set. The resulting induced circuit equals ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$. By the choice of σ, there exists a node contained in both the maximal 0circuit[σ] and ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$. Let $m(\sigma)$ be the configuration matching σ except that all spin values in ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ are one. Since ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma) \cap C^{\max 0}(\sigma) \neq \emptyset$, the configuration $m(\sigma)$ exhibits a 1∗lasso. Obviously, ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ is the maximal induced 1circuit[$m(\sigma)$], which enables us to locate ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ by looking at the configuration $m(\sigma)$. Consider the minimal induced circuit in the union of ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$ and all 0clusters[$m[\sigma)$] adjacent to ${}^{\text{i}}C^{\text{fill}}_\Lambda(\sigma)$. We will show that this circuit equals the minimal induced 0circuit[σ] in the exterior of the maximal 1circuit[σ]. Therefore, the map $m_\Lambda|_{A^2_\Lambda}$ is invertible. ☺

For the complete proof we refer to Subsection 5.1.3.

Given these propositions, the following theorem and, therefore, the injectivity of m_Λ is quite evident.

Theorem 5.10 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the map m_Λ is injective and $\{\sigma \neq m_\Lambda(\sigma)\} \subset \sigma^{-1}(0)$. In particular, we can interpret m_Λ as a filling of some parts of a 0circuit.*

Proof: Theorem 5.10 is a direct consequence of Proposition 5.7, 5.8, and 5.9. In particular, $\{\sigma \neq m_\Lambda(\sigma)\} \subset \sigma^{-1}(0)$ directly follows from the definition of the map m_Λ. □

Since flipping all spins is bijective, we can state a very close version of Theorem 5.10.

Theorem 5.11 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the map*

$$f \circ m_\Lambda \circ f : \{\exists \text{ 1lasso}\} \to \{\exists \text{ 0}{*}\text{lasso}\}$$

is injective and $\{\sigma \neq f \circ m_\Lambda \circ f(\sigma)\} \subset \sigma^{-1}(1)$, where f flips the spin values. In particular, we can interpret $f \circ m_\Lambda \circ f$ as emptying some parts of a 1circuit.

A direct consequence of this Theorem is the following.

Corollary 5.12 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$ and for all $\lambda \leq 1$*

$$\mu^+_{\Lambda,\lambda}(\exists \text{ 1lasso in } \Lambda) \leq 2\mu^+_{\Lambda,\lambda}(\exists \text{ 0}{*}\text{lasso in } \Lambda)$$

holds.

Proof: Choose an arbitrary configuration $\sigma \in \{\exists \text{1lasso}\}$. Let $\kappa^w(\sigma)$ be the number of 1clusters[σ] not adjacent to Λ^c. Note that

$$\kappa^w(\sigma) \leq 1 + \kappa^w(f \circ m_\Lambda \circ f(\sigma))$$

holds, since $f \circ m_\Lambda \circ f$ empties some parts of one 1circuit[σ]. Consequently, we know that for any $\lambda \leq 1$,

$$\begin{aligned} Z^w_{\Lambda,\lambda}\phi^w_{\Lambda,\lambda}(\sigma) =& \lambda^{\sum_{x\in\Lambda}\sigma(x)}2^{\kappa^w(\sigma)} \\ \leq& \lambda^{\sum_{x\in\Lambda}\sigma(x)}2^{1+\kappa^w(f\circ m_\Lambda\circ f(\sigma))} \\ \leq& \lambda^{\sum_{x\in\Lambda}f\circ m_\Lambda\circ f(\sigma)(x)}2^{1+\kappa^w(f\circ m_\Lambda\circ f(\sigma))} \\ =& Z^w_{\Lambda,\lambda}2\phi^w_{\Lambda,\lambda}(f\circ m_\Lambda\circ f(\sigma)) \end{aligned}$$

holds, where the latter inequality is a consequence of both $\sigma^{-1}(0) \subset (f \circ m_\Lambda \circ f(\sigma))^{-1}(0)$ and $\lambda \leq 1$ This concludes the proof. $\square$

This, together with Theorem 4.30, implies the absence of phase transition of the Widom-Rowlinson model on $(\mathbb{Z}^2, \square)$, i.e., $|\mathrm{WR}(\lambda)| = 1$ for $\lambda \leq 1$. But the absence of phase transition is already known for activities smaller than $p_c/(1-p_c) \approx 3/2$, see [Hig83], where p_c denotes the critical probability of Bernoulli node-percolation on $(\mathbb{Z}^2, \square)$.

5.1.1 Proof of Proposition 5.7

We prove Propositions 5.7, 5.8 and 5.9 using the same approach: First some properties of the map's image are shown. Then we verify that the considered map is injective.

Lemma 5.13 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the image of $m_\Lambda|_{A^0_\Lambda}$ is equal to*

$$\{\exists\ 1{*}\text{lasso}, \vec{0} \in C^{\max 1*}_\Lambda\}\,.$$

Proof: Let $\Lambda \Subset \mathbb{Z}^2$ be given and recall that the domain A^0_Λ of the map $m_\Lambda|_{A^0_\Lambda}$ equals

$$\begin{aligned}\{\exists\ 0\text{lasso}, \vec{0} \in & C^{\max 0}_\Lambda\} = \\ &=\{\exists\ 0\text{lasso}, \vec{0} \in C^{\max 0}_\Lambda, \partial^*\vec{0} \xleftrightarrow{1*} \partial^*(\Lambda^c)\} \\ &=\{\exists\ 0\text{lasso}, \vec{0} \in C^{\max 0}_\Lambda, \partial^*\vec{0} \xleftrightarrow{1*} \partial^*(\Lambda^c), \partial\vec{0} \xleftrightarrow{0} \partial(\Lambda^c)\} \\ &=\{\vec{0} \in C^{\max 0}_\Lambda, \partial^*\vec{0} \xleftrightarrow{1*} \partial^*(\Lambda^c), \partial\vec{0} \xleftrightarrow{0} \partial(\Lambda^c)\}\,,\end{aligned}$$

where the first identity holds because the origin is contained in the maximal 0circuit, whose nodes are all ∗weakly 1∗connected to Λ^c. By saying a node x is ∗weakly 1∗connected to Λ^c we mean that $\partial^* x$ is 1∗connected to $\partial^*(\Lambda^c)$. The second equality follows from the existence of a 0lasso and the fact that the maximal 0circuit contains the origin.

Since the map $m_\Lambda|_{A^0_\Lambda}$ simply flips the spin of the origin, the image of $m_\Lambda|_{A^0_\Lambda}$ is

$$\begin{aligned}\{\vec{0} \in C^{\max 1*}_\Lambda, & \partial^*\vec{0} \xleftrightarrow{1*} \partial^*(\Lambda^c), \partial\vec{0} \xleftrightarrow{0} \partial(\Lambda^c)\} = \\ &= \{\exists\ 1{*}\text{lasso}, \vec{0} \in C^{\max 1*}_\Lambda, \partial^*\vec{0} \xleftrightarrow{1*} \partial^*(\Lambda^c), \partial\vec{0} \xleftrightarrow{0} \partial(\Lambda^c)\}\,.\end{aligned}$$

The same arguments as above – used in reverse – verify

$$m_\Lambda|_{A^0_\Lambda}(A^0_\Lambda) = \{\exists\ 1{*}\text{lasso}, \vec{0} \in C^{\max 1*}_\Lambda\}$$

and therefore the lemma. □

It remains to observe that m_Λ is injective, which will be done in the next lemma. Here this is quite obvious, whereas in the next subsections the injectivity requires more involved arguments, partly based upon observations made in the first lemmas describing the image of m_Λ.

Lemma 5.14 *For all simply ∗connected sets $\Lambda \Subset \mathbb{Z}^2$ with $\vec{0} \in \Lambda$, the map $m_\Lambda|_{A^0_\Lambda}$ is injective.*

Proof: Let $\Lambda \Subset \mathbb{Z}^2$ be given and recall that m_Λ simply flips the spin of the origin. This is obviously injective. □

5.1.2 Proof of Proposition 5.8

It may help the reader to refresh the core idea of this subsection, see page 77.

Let $\Lambda \Subset \mathbb{Z}^2$ be a simply $*$connected set with $\vec{0} \in \Lambda$. We only prove the statement of Proposition 5.8 for this fixed set Λ. Sometimes, we even omit the index Λ, e.g., m denotes the map m_Λ in the sequel.

Recall that the domain A^1 of the map m is defined as

$$\{\exists \text{ 0lasso}, \vec{0} \notin C_\Lambda^{\max 0}, \partial^* C_\Lambda^{\max 0} \overset{1*}{\not\longleftrightarrow} \partial^* C_\Lambda^{\max 1}\}.$$

The existence of a 0lasso implies that the maximal 0circuit, which is weakly 0connected to Λ^c, is larger than every 1circuit (if a 1circuit exists at all), where by saying is weakly 0connected to Λ^c we mean that its boundary is 0connected to $\partial(\Lambda^c)$. Moreover, one can find a second 0circuit in $\text{int}C_\Lambda^{\max 0}$ again larger than any 1circuit, since the origin is not contained in the maximal 0circuit and the maximal 0circuit is not $*$weakly 1$*$connected to the maximal 1circuit. This implies (in fact it is equivalent) that the maximal 0circuit in $\text{int}C_\Lambda^{\max 0}$ exists and is 0$*$connected to $C_\Lambda^{\max 0}$. Summing up, we can state

$$\begin{aligned} A^1 = \{&\exists \text{0lasso}, \vec{0} \notin C_\Lambda^{\max 0}, \partial^* C_\Lambda^{\max 0} \overset{1*}{\not\longleftrightarrow} \partial^* C_\Lambda^{\max 1}, \\ & C_\Lambda^{\max 0} > C_\Lambda^{\max 1}, \emptyset \neq C^{\max 0}_{\text{int}C_\Lambda^{\max 0}} \overset{0*}{\longleftrightarrow} C_\Lambda^{\max 0}, C^{\max 0}_{\text{int}C_\Lambda^{\max 0}} > C_\Lambda^{\max 1}\} \end{aligned} \tag{5.5}$$

Figure 5.1 illustrates the properties of a configuration of A^1.

After analysing the domain, we state some useful fundamental relations between $\sigma \in A^1$ and $m(\sigma)$.

For this task recall the definition of the circuit

$${}^{\text{i}}C^{\text{fill}}(\sigma) = \min {}^{\text{i}}\mathfrak{C}\left(C_\Lambda^{\max 0}(\sigma) \cup \sigma^{-1}(1) \cap \text{ext}C^{\max 0}_{\text{int}C_\Lambda^{\max 0}(\sigma)}(\sigma)\right) \tag{5.6}$$

for $\sigma \in A^1$, see page 75. In other words, this circuit, ${}^{\text{i}}C^{\text{fill}}(\sigma)$, is the minimal induced circuit that satisfies the following two conditions:

- It is contained in the union of the maximal 0circuit[σ], $C_\Lambda^{\max 0}(\sigma)$, and the set of nodes equipped with 1spins[σ], $\sigma^{-1}(1)$;

- It is strictly larger than the "second largest" 0circuit[σ], $C^{\max 0}_{\text{int}C_\Lambda^{\max 0}(\sigma)}(\sigma)$.

Note that this description equates our definition.

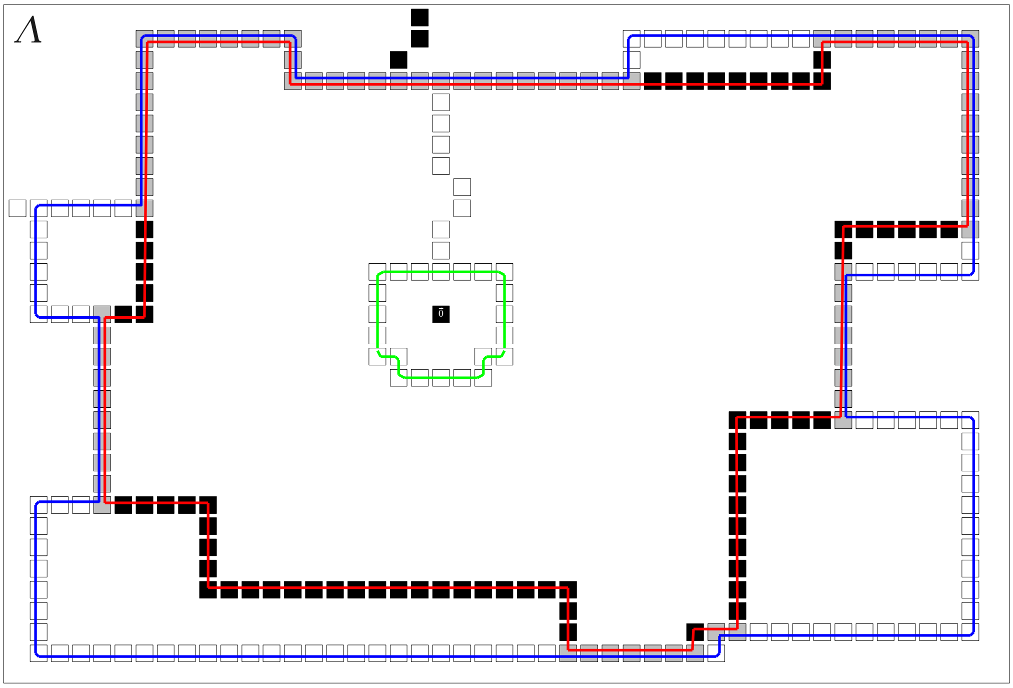

Figure 5.1: In this figure the white squares represent nodes with 0spins$[\sigma, m(\sigma)]$ and the black squares are nodes with 1spins$[\sigma, m(\sigma)]$. The gray squares are nodes equipped with 0spins$[\sigma]$ and 1spins$[m(\sigma)]$. The maximal 0circuit$[\sigma]$ is indicated by a blue curve and consists of white and gray squares. The circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is indicated by a red curve and consists of black and gray squares. The "second largest" 0circuit$[\sigma]$, $C^{\max 0}_{\mathrm{int}C^{\max 0}_{\Lambda}(\sigma)}(\sigma)$, is indicated by a green curve.

Remark 5.15 *Let $\sigma \in A^1$. Then the following properties hold*

$$\sigma^{-1}(1) \cup C^{\max 0}_{\Lambda}(\sigma) \supset m(\sigma)^{-1}(1) \tag{5.7}$$
$$\sigma^{-1}(0) = m(\sigma)^{-1}(0) \cup C^{\max 0}_{\Lambda}(\sigma) \tag{5.8}$$
$${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap C^{\max 0}_{\Lambda}(\sigma) \neq \emptyset \tag{5.9}$$
$$\partial^{*\mathrm{i}}C^{\mathrm{fill}}(\sigma) \overset{\sigma^{-1}(1)*}{\longleftrightarrow} \partial^*(\Lambda^c) \tag{5.10}$$
$$C^{\max 0}_{\mathrm{int}C^{\max 0}_{\Lambda}(\sigma)}(\sigma) \subset \sigma^{-1}(0) \cap m(\sigma)^{-1}(0) \tag{5.11}$$
$$C^{\max 0}_{\Lambda}(\sigma) \geq C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma))\,. \tag{5.12}$$

Proof: Fix a configuration $\sigma \in A^1$. By the definitions of both the configuration

$$m(\sigma) = \mathbb{1}_{\sigma^{-1}(1) \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)} \tag{5.13}$$

and the circuit

$${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) = \min {}^{\mathrm{i}}\mathfrak{C}\left(C_\Lambda^{\max 0}(\sigma) \cup \sigma^{-1}(1) \cap \mathrm{ext} C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma)\right),$$

the statements (5.7) and (5.8) are evident.

For the next properties let us first define the "half-open" respectively "open" annulus specified by the maximal 0circuit[σ] and the "second largest" 0circuit[σ] by

$$\left]C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma), C_\Lambda^{\max 0}(\sigma)\right] := C_\Lambda^{\max 0}(\sigma) \cup \mathrm{int} C_\Lambda^{\max 0}(\sigma) \cap \mathrm{ext} C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma)$$

respectively

$$\left]C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma), C_\Lambda^{\max 0}(\sigma)\right[:= \mathrm{int} C_\Lambda^{\max 0}(\sigma) \cap \mathrm{ext} C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma).$$

Next, recall the description of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ after (5.6). In particular, the second condition of this description said that the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is strictly larger than the "second largest" 0circuit[σ], $C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma)$. Moreover, the first condition, together with minimality of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$, immediately implies that ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is smaller than the maximal 0circuit[σ], $C_\Lambda^{\max 0}(\sigma)$. Summing up, we know that ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ lies in the "half-open" annulus, i.e.,

$${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \subset \left]C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma), C_\Lambda^{\max 0}(\sigma)\right]. \tag{5.14}$$

Further, by definition, we know that all nodes of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ not contained in the maximal 0circuit[σ] are equipped with 1spins[σ], i.e.,

$${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap (C_\Lambda^{\max 0}(\sigma))^c \subset \sigma^{-1}(1). \tag{5.15}$$

In other words, a node of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ has 1spin[σ] if and only if it belongs to the "open" annulus, in short for all $x \in {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$

$$\sigma(x) = 1 \iff x \in \left]C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\sigma)}(\sigma), C_\Lambda^{\max 0}(\sigma)\right[.$$

Property (5.9) is a consequence of these two Observations (5.14) and (5.15), together with the choice of

$$\sigma \in A^1 \overset{(5.5)}{\subset} \{\pi \in \{0,1\}^\Lambda : C^{\max 0}_{\mathrm{int} C_\Lambda^{\max 0}(\pi)}(\pi) \overset{0*}{\longleftrightarrow} C_\Lambda^{\max 0}(\pi), \},$$

i.e., the maximal 0circuit[σ] and the "second largest" 0circuit[σ] are 0$*$connected. Since every node of $C_\Lambda^{\max 0}(\sigma)$ is $*$weakly 1$*$connected[σ] to Λ^c, Property (5.10) follows from Property (5.9).

As before ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is strictly larger than the "second largest" 0circuit[σ], see (5.14). Hence, since we only change spin values in ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$, see (5.3), inclusion (5.11) follows.

The last Statement (5.12) is a direct consequence of the facts that ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is smaller than the maximal 0circuit[σ], see (5.14), and that ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is a 1circuit[$m(\sigma)$] $*$weakly 1$*$connected[$m(\sigma)$] to Λ^c, see (5.13) and (5.10). $\square$

These remarks enable us to describe the image of $m|_{A^1}$.

Lemma 5.16 *Let $\sigma \in A^1$. Then $m_\Lambda(\sigma)$ is an element of*

$$\{\exists\ 1{*}\text{lasso}, \vec{0} \notin C_\Lambda^{\max 1} \neq \emptyset, C_\Lambda^{\max 1} \overset{1*}{\not\longleftrightarrow} C^{\max 1}_{\mathrm{int}C_\Lambda^{\max 1}}\}\,.$$

Proof: Let $\sigma \in A^1$ and recall that

$${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) = \min {}^{\mathrm{i}}\mathfrak{C}\left(C_\Lambda^{\max 0}(\sigma) \cup \sigma^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C_\Lambda^{\max 0}(\sigma)}(\sigma)\right)\,.$$

A direct consequence of the definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is that the $*$boundary of every node of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is 0$*$connected[$\sigma, m(\sigma)$] in $({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma))^c$ to $C^{\max 0}_{\mathrm{int}C_\Lambda^{\max 0}(\sigma)}(\sigma)$, i.e., for all nodes $z \in {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$

$$\partial^* z \overset{0*}{\longleftrightarrow} C^{\max 0}_{\mathrm{int}C_\Lambda^{\max 0}(\sigma)}(\sigma) \text{ in } ({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma))^c \tag{5.16}$$

holds.

Since the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is a 1circuit[$m(\sigma)$], see (5.4), which is $*$weakly 1$*$connected[$m(\sigma)$] to Λ^c, see (5.9) and (5.4), the existence of a 1$*$lasso[$m(\sigma)$] follows, i.e.,

$$m(\sigma) \in \{\exists\ 1{*}\text{lasso}\}\,.$$

But we have already verified the existence of a 0circuit[$m(\sigma)$], e.g. $C^{\max 0}_{\mathrm{int}C_\Lambda^{\max 0}(\sigma)}(\sigma)$, see (5.10), which, therefore, has to be smaller than the 1circuit[$m(\sigma)$] ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ that is $*$weakly 1$*$connected[$m(\sigma)$] to Λ^c. Consequently, the origin cannot be contained in the maximal 1circuit[$m(\sigma)$], i.e.,

$$m(\sigma) \in \{\vec{0} \notin C_\Lambda^{\max 1} \neq \emptyset\}\,.$$

Recall that the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ intersects the maximal 0circuit[σ], see (5.9), which is 0connected[σ] to the boundary of Λ^c, see (5.5), and that the configuration σ equals $m(\sigma)$ outside of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$. Hence, there exists a node $y \in {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ with

$$\partial y \overset{0}{\longleftrightarrow} \partial(\Lambda^c) \text{ in } ({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma))^c\,,$$

i.e., we can find a node $y \in {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ that is weakly 0connected$[\sigma, m(\sigma)]$ to Λ^c. This, together with (5.16), implies that $y \in {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ satisfies

$$\partial(\Lambda^c) \overset{0}{\longleftrightarrow} \partial^* y \overset{0*}{\longleftrightarrow} C^{\max 0}_{\mathrm{int}C^{\max 0}_{\Lambda}(\sigma)}(\sigma) \text{ in } ({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma))^c \,. \tag{5.17}$$

Thus, one cannot find two disjoint 1circuits$[m(\sigma)]$ in $\mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}_{\Lambda}(\sigma)}(\sigma)$, since both would have to intersect the node y. This, together with the fact that ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is a 1circuit$[m(\sigma)]$, see (5.4), strictly larger than the 0circuit$[\sigma, m(\sigma)]$ $C^{\max 0}_{\mathrm{int}C^{\max 0}_{\Lambda}(\sigma)}(\sigma)$, see (5.11), implies

$$m(\sigma) \in \{C^{\max 1}_{\Lambda} \overset{1*}{\not\longleftrightarrow} C^{\max 1}_{\mathrm{int}C^{\max 1}_{\Lambda}}\}\,,$$

which concludes the proof. □

It may help the reader to – once again – refresh the core idea of this subsection described on page 77. Our next step is to "connect" a configuration $m(\sigma)$ to the original configuration $\sigma \in A^1$. More precisely, we want to determine both ${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$ from $m(\sigma) \in m(A^1)$ and

$$\begin{aligned} {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) &:= \max {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) \cup m(\sigma)^{-1}(0)\right) \\ &= \max {}^{\mathrm{i}}\mathfrak{C}\left({}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) \cup m(\sigma)^{-1}(0)\right) \end{aligned} \tag{5.18}$$

from $\sigma \in A^1$, where the identity follows from

$${}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) \geq C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma))$$

and

$${}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) \subset C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma))\,.$$

Little is known, but ${}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))$ is the circuit that changes $m(\sigma)$ into σ if it is emptied.

Lemma 5.17 *Let $\sigma \in A^1$. Then the minimal induced 1circuit$[m(\sigma)]$ in the exterior of the maximal 0circuit$[m(\sigma)]$, ${}^{i}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma))$, equals ${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$ and the maximal induced 0circuit$[\sigma]$ ${}^{i}C^{\max 0}_{\Lambda}(\sigma)$ equals ${}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))$, i.e.,*

$${}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) = {}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma) \tag{5.19}$$

$${}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) = {}^{\mathrm{i}}C^{\max 0}_{\Lambda}(\sigma)\,. \tag{5.20}$$

Proof: Since ${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$ is a 1circuit$[m(\sigma)]$, see (5.4), that is $*$weakly 1$*$connected$[\sigma, m(\sigma)]$ to Λ^c, see additionally (5.10), we know that $C^{\max 0}(m(\sigma)) \subset \mathrm{int}{}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$, which, together with (5.4), leads to

$${}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) \leq {}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)\,.$$

On the other hand

$$\begin{aligned}
{}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) &= \min{}^{\mathrm{i}}\mathfrak{C}\left(C_\Lambda^{\max 0}(\sigma) \cup \sigma^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C_\Lambda^{\max 0}(\sigma)}(\sigma)\right) \\
&= \min{}^{\mathrm{i}}\mathfrak{C}\left((C_\Lambda^{\max 0}(\sigma) \cup \sigma^{-1}(1)) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C_\Lambda^{\max 0}(\sigma)}(\sigma)\right) \\
&\overset{(5.11)}{\leq} \min{}^{\mathrm{i}}\mathfrak{C}\left((C_\Lambda^{\max 0}(\sigma) \cup \sigma^{-1}(1)) \cap \mathrm{ext}C_\Lambda^{\max 0}(m(\sigma))\right) \\
&\overset{(5.7)}{\leq} \min{}^{\mathrm{i}}\mathfrak{C}\left(m(\sigma)^{-1}(1) \cap \mathrm{ext}C_\Lambda^{\max 0}(m(\sigma))\right) \\
&= {}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C_\Lambda^{\max 0}(m(\sigma))}(m(\sigma))\,.
\end{aligned}$$

Both inequalities together give (5.19). This in turn implies the following:

$$\begin{aligned}
m(\sigma)^{-1}(1) &\overset{(5.4)}{=} \sigma^{-1}(1) \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(m(\sigma)) \\
&\overset{(5.19)}{\iff} m(\sigma)^{-1}(0) = \sigma^{-1}(0) \setminus {}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C_\Lambda^{\max 0}(m(\sigma))}(m(\sigma))\,,
\end{aligned}$$

which verifies

$$\sigma^{-1}(0) \subset m(\sigma)^{-1}(0) \cup C^{\min 1}_{\mathrm{ext}C_\Lambda^{\max 0}(m(\sigma))}(m(\sigma))\,. \tag{5.21}$$

On the one hand

$$\begin{aligned}
{}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) &= \max{}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 1}_{\mathrm{ext}C_\Lambda^{\max 0}(m(\sigma))}(m(\sigma)) \cup m(\sigma)^{-1}(0)\right) \\
&\overset{(5.21)}{\geq} \max{}^{\mathrm{i}}\mathfrak{C}\left(\sigma^{-1}(0)\right) \\
&= {}^{\mathrm{i}}C_\Lambda^{\max 0}(\sigma)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
{}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) &= \max{}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 1}_{\mathrm{ext}C_\Lambda^{\max 0}(m(\sigma))}(m(\sigma)) \cup m(\sigma)^{-1}(0)\right) \\
&\overset{(5.12)}{\leq} \max{}^{\mathrm{i}}\mathfrak{C}\left(C_\Lambda^{\max 0}(\sigma) \cup m(\sigma)^{-1}(0))\right) \\
&\overset{(5.8)}{=} {}^{\mathrm{i}}C_\Lambda^{\max 0}(\sigma)\,.
\end{aligned}$$

Considering both inequalities yields (5.20). □

Having established the above "connection" between σ and $m(\sigma)$, we are ready to see that

$$m^{-1} : m(A^1) \overset{!}{\to} A^1; \quad m(\sigma) \mapsto 1 - \mathbb{1}_{m(\sigma)^{-1}(0) \cup {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))} \overset{!}{=} \sigma$$

is a well-defined promising candidate for the inverse map, which is illustrated in Figure 5.1.

Lemma 5.18 *Let* $\sigma \in A^1$. *Then* $1 - \mathbb{1}_{m(\sigma)^{-1}(0) \cup {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))} = \sigma$, *i.e., the map* $m_\Lambda|_{A^1}$ *is injective.*

Proof: It is sufficient to show

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap \sigma^{-1}(0) \stackrel{!}{=} {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) \cap m(\sigma)^{-1}(1) $$

if we want to prove $1 - \mathbb{1}_{m(\sigma)^{-1}(0) \cup {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))} = \sigma$.

"$\subset$" This inclusion is a consequence of

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \stackrel{(5.19)}{=} {}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) \subset m(\sigma)^{-1}(1) $$

and

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap \sigma^{-1}(0) \stackrel{(5.1)}{\subset} {}^{\mathrm{i}}C^{\max 0}_{\Lambda}(\sigma) \stackrel{(5.20)}{=} {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) \, . $$

"$\supset$" This inclusion is a consequence of

$$ {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) \stackrel{(5.20)}{=} {}^{\mathrm{i}}C^{\max 0}_{\Lambda}(\sigma) \subset \sigma^{-1}(0) $$

and

$$ {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) \cap m(\sigma)^{-1}(1) \stackrel{(5.18)}{\subset} {}^{\mathrm{i}}C^{\min 1}_{\mathrm{ext}C^{\max 0}_{\Lambda}(m(\sigma))}(m(\sigma)) \stackrel{(5.19)}{=} {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \, , $$

which concludes the proof. $\square$

5.1.3 Proof of Proposition 5.9

The structure of this subsection is similar to the previous two subsections. It may help the reader to refresh the core ideas of this subsection, described on page 77.

Now, let us consider our domain

$$\begin{aligned} A^2 &= \{\exists\ 0\text{lasso}, \partial^* C_\Lambda^{\max 0} \xleftrightarrow{1*} \partial^* C_\Lambda^{\max 1}\} \\ &= \{\exists\ 0\text{lasso}, \partial^* C_\Lambda^{\max 0} \xleftrightarrow{1*} \partial^* C_\Lambda^{\max 1}, C_\Lambda^{\max 0} > C_\Lambda^{\max 1}\}\,, \end{aligned} \tag{5.22}$$

where the second identity follows from the fact that the maximal 0circuit is weakly 0connected to Λ^c and, therefore, is larger than every 1circuit.

As before, we state some fundamental relations between σ and $m(\sigma)$.

Remark 5.19 *Let $\sigma \in A^2$. Then the following properties hold:*

$$\sigma^{-1}(1) \cup C^{\min 0}_{\text{ext}C_\Lambda^{\max 1}(\sigma)}(\sigma) \supset m(\sigma)^{-1}(1) \tag{5.23}$$

$$\sigma^{-1}(0) = m(\sigma)^{-1}(0) \cup C^{\min 0}_{\text{ext}C_\Lambda^{\max 1}(\sigma)}(\sigma) \tag{5.24}$$

$${}^{\mathrm{i}}C^{\text{fill}}(\sigma) \cap C^{\min 0}_{\text{ext}C_\Lambda^{\max 1}(\sigma)}(\sigma) \cap C^{\max 0}(\sigma) \neq \emptyset \tag{5.25}$$

Proof: First of all, recall that

$$m(\sigma) = \mathbb{1}_{\sigma^{-1}(1)\cup {}^{\mathrm{i}}C^{\text{fill}}(\sigma)}$$

and that for our case

$${}^{\mathrm{i}}C^{\text{fill}}\,(\sigma) = \max {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\text{ext}C_\Lambda^{\max 1}(\sigma)}(\sigma) \cup \sigma^{-1}(1)\right)\,. \tag{5.26}$$

These definitions immediately imply the first two Properties (5.23) and (5.24).

Recall that the maximal 0circuit[σ] is larger than the maximal 1circuit[σ], since it is weakly 0connected[σ] to Λ^c. Because of this, the minimal 0circuit[σ] in the exterior of the maximal 1circuit[σ], $C^{\min 0}_{\text{ext}C_\Lambda^{\max 1}(\sigma)}(\sigma)$, exists and is contained in the annulus

$$\left]C_\Lambda^{\max 1}(\sigma), C_\Lambda^{\max 0}(\sigma)\right] := \text{ext}C_\Lambda^{\max 1}(\sigma) \cap \text{int}C_\Lambda^{\max 0}(\sigma) \cup C_\Lambda^{\max 0}(\sigma)\,.$$

Consequently, each node of the maximal 0circuit[σ] that is $*$weakly 1$*$connected[σ] to the maximal 1circuit[σ] also belongs to the minimal 0circuit[σ] in the exterior of the maximal 1circuit[σ]. Because of

$$\sigma \overset{(5.22)}{\in} \{\partial^* C_\Lambda^{\max 0}(\sigma) \xleftrightarrow{1*} \partial^* C_\Lambda^{\max 1}(\sigma)\}$$

such a node exists and, therefore,

$$C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap C^{\max 0}(\sigma) \neq \emptyset$$

follows. Moreover, at least one of these nodes in

$$C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap C^{\max 0}(\sigma)$$

has to be weakly 0connected[σ] to Λ^c in

$$\Big]C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma), \partial^* \Lambda\Big[:= \mathrm{ext}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap \Lambda \,.$$

This is the case because a 0lasso[σ] exists and, therefore, we could follow a 0path from $\partial(\Lambda^c)$ to the maximal 0circuit[σ] and then through the maximal 0circuit[σ] to $C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma)$; the first node x in $C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma)$ reached this way satisfies the required feature.

If a node of $C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma)$ is $*$weakly 0$*$connected to Λ^c in the annulus

$$\Big]C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma), \partial^* \Lambda\Big[\,,$$

then it also belongs to ${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$, since it cannot be "circumvented" by a 1path of

$${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma) \overset{\mathrm{Def.}}{=} \max {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cup \sigma^{-1}(1)\right) \,.$$

Consequently, the node x also belongs to the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$.

Summing up, the node x belongs to

$${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma) \cap C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap C^{\max 0}(\sigma) \,,$$

which implies the last Property (5.25). $\square$

Once again, we need further (more involved) relations between a configuration $\sigma \in A^2$ and the corresponding configuration $m(\sigma)$. More precisely, we have to find a circuit ${}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))$ that changes $m(\sigma)$ to σ if it is emptied. Our candidate is

$$\begin{aligned} {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) &:= \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\max 1}_{\Lambda}(m(\sigma)) \cup m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\right) \\ &= \min {}^{\mathrm{i}}\mathfrak{C}\left({}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma)) \cup m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\right) \,, \end{aligned} \tag{5.27}$$

where the identity follows from ${}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma)) \leq C^{\max 1}_{\Lambda}(m(\sigma))$ and ${}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma)) \subset C^{\max 1}_{\Lambda}(m(\sigma))$.

Lemma 5.20 *Let $\sigma \in A^2$. Then the maximal induced 1circuit$[m(\sigma)]$ equals the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$ and the minimal induced 0circuit$[\sigma]$ in the exterior of the maximal 1circuit$[\sigma]$ equals ${}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))$, in short*

$$ {}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma)) = {}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma) \tag{5.28} $$

$$ {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) = {}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma)\,. \tag{5.29} $$

Proof: On the one hand, the inequality ${}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma)) \geq {}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$ follows from

$$ \sigma^{-1}(1) \cup {}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma) = m(\sigma)^{-1}(1)\,, $$

see (5.4). On the other hand, we also know that

$$ \begin{aligned} {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) &\overset{\text{Def.}}{=} \max {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cup \sigma^{-1}(1)\right) \\ &\overset{(5.23)}{\geq} {}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma)) \end{aligned} $$

holds. Both inequalities together imply (5.28), which immediately leads to

$$ C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \subset {}^{\mathrm{i}}C^{\max 1}(m(\sigma)) \subset C^{\max 1}(m(\sigma))\,. \tag{5.30} $$

We further know that

$$ C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \leq {}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \leq {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)\,. \tag{5.31} $$

holds. Indeed, the first inequality is obvious and the second one follows from the definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ as $\max {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cup \sigma^{-1}(1)\right)$. A further fact is

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \overset{(5.25)}{\neq} \emptyset\,. \tag{5.32} $$

These two Relations (5.31) and (5.32) prove that all $1*$circuits in $\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ have to be contained in $\mathrm{int}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma)$, which gives us the following inclusion

$$ \begin{aligned} C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \subset \mathrm{ext}C^{\max 1*}_{\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)}(\sigma) &\overset{(5.3)}{=} \mathrm{ext}C^{\max 1*}_{\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)}(m(\sigma)) \\ &\overset{(5.28)}{=} \mathrm{ext}C^{\max 1*}_{\mathrm{int}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma)) = \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\,. \end{aligned} $$

Therefore,

$$ C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap ({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma))^{c} \subset m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\,, \tag{5.33} $$

since the inclusion

$$C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap \left({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)\right)^c \subset m(\sigma)^{-1}(0)$$

follows from $C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \subset \sigma^{-1}(0)$ and (5.3).

Next, by definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ and $m(\sigma)$, we know that ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \subset m(\sigma)^{-1}(1) \cap \mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)$, which immediately implies

$$C^{\max 1}_{\Lambda}(\sigma) \leq C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma)) \,. \tag{5.34}$$

So, on the one side,

$$\begin{aligned}
{}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) &= \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\max 1}_{\Lambda}(m(\sigma)) \cup m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\right) \\
&\overset{(5.30),(5.33)}{\leq} \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cup C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cap \left({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)\right)^c\right) \\
&= {}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma)
\end{aligned}$$

holds. On the other side, it is the case that

$$\begin{aligned}
{}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) &\overset{(5.27)}{=} \min {}^{\mathrm{i}}\mathfrak{C}\left({}^{\mathrm{i}}C^{\max 1}_{\Lambda}(m(\sigma)) \cup m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\right) \\
&\overset{(5.28)}{=} \min {}^{\mathrm{i}}\mathfrak{C}\left({}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cup m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\right) \\
&\overset{(5.31)}{\geq} \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cup m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1*}_{\mathrm{int}C^{\max 1}_{\Lambda}(m(\sigma))}(m(\sigma))\right) \\
&\overset{(5.34)}{\geq} \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cup m(\sigma)^{-1}(0) \cap \mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)\right) \\
&\overset{m(\sigma)^{-1}(0)\subset\sigma^{-1}(0)}{\geq} \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \cup \sigma^{-1}(0) \cap \mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)\right) \\
&= {}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C^{\max 1}_{\Lambda}(\sigma)}(\sigma) \,.
\end{aligned}$$

Taking both inequalities together yields (5.29). □

Now, we have gathered enough to finally analyse the image of $m|_{A^2}$.

Lemma 5.21 *Let $\sigma \in A^2$. Then*

$$m(\sigma) \in \{\exists\ 1{*}\mathrm{lasso}, C^{\max 1}_{\Lambda} \overset{1*}{\longleftrightarrow} C^{\max 1}_{\mathrm{int}C^{\max 1}_{\Lambda}}\} \,.$$

Proof: By (5.4), the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}\,(\sigma)$ is a 1circuit$[m(\sigma)]$, which, by (5.25), intersects the maximal 0circuit$[\sigma]$. The intersection is ∗weakly 1∗connected$[\sigma]$ to Λ^c. Since also $\sigma^{-1}(1) \subset m(\sigma)^{-1}(1)$, we can conclude that

$$m(\sigma) \in \{\exists\ 1{*}\mathrm{lasso}\} \,.$$

So, it only remains to show that

$$m(A^2) \overset{!}{\in} \{C_\Lambda^{\max 1} \overset{1*}{\longleftrightarrow} C^{\max 1}_{\mathrm{int} C_\Lambda^{\max 1}}\}\,. \tag{5.35}$$

Since $C^{\min 0}_{\mathrm{ext} C_\Lambda^{\max 1}(\sigma)}(\sigma)$ is the minimal 0circuit[σ] outside of the maximal 1circuit[σ], every node of $C^{\min 0}_{\mathrm{ext} C_\Lambda^{\max 1}(\sigma)}(\sigma)$ is $*$weakly 1$*$connected[$\sigma, m(\sigma)$] to the maximal 1circuit[σ], i.e., for all $x \in C^{\min 0}_{\mathrm{ext} C_\Lambda^{\max 1}(\sigma)}(\sigma)$

$$\partial^* x \overset{1*}{\longleftrightarrow} C_\Lambda^{\max 1}(\sigma) \tag{5.36}$$

with respect to σ and, therefore, also with respect to $m(\sigma) = \mathbb{1}_{\sigma^{-1}(1) \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)}$. A further fact is

$$^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap C^{\min 0}_{\mathrm{ext} C_\Lambda^{\max 1}(\sigma)}(\sigma) \overset{(5.25)}{\neq} \emptyset\,. \tag{5.37}$$

A consequence of (5.36) and (5.37) is

$$\partial^{*\mathrm{i}} C^{\mathrm{fill}}(\sigma) \overset{1*}{\longleftrightarrow} C_\Lambda^{\max 1}(\sigma) \tag{5.38}$$

with respect to σ and $m(\sigma)$. Therefore, by (5.28), the statement (5.35) will be proved once we have shown that the right side of (5.38) equals $C^{\max 1}_{\mathrm{int} C_\Lambda^{\max 1}(m(\sigma))}(m(\sigma))$. Summing up, we know everything but the last identity in the following:

$$\begin{aligned} C_\Lambda^{\max 1}(m(\sigma)) \overset{1*}{\longleftrightarrow} {}^{\mathrm{i}}C_\Lambda^{\max 1}(m(\sigma)) \overset{(5.28)}{=} \partial^{*\mathrm{i}} C^{\mathrm{fill}}(\sigma) \overset{\substack{(5.38)\\ 1*}}{\longleftrightarrow} \\ \overset{1*}{\longleftrightarrow} C_\Lambda^{\max 1}(\sigma) \overset{!}{=} C^{\max 1}_{\mathrm{int} C_\Lambda^{\max 1}(m(\sigma))}(m(\sigma))\,, \end{aligned}$$

where the first 1$*$connection is obvious. The latter identity will be proven in the remainder.

"$\leq$": By the definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$, we know that $C_\Lambda^{\max 1}(\sigma) \subset \mathrm{int}^{\mathrm{i}} C^{\mathrm{fill}}(\sigma) \overset{(5.28)}{=} \mathrm{int}^{\mathrm{i}} C_\Lambda^{\max 1}(m(\sigma))$. A consequence of this, together with $m(\sigma) = \mathbb{1}_{\sigma^{-1}(1) \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)}$, is $C_\Lambda^{\max 1}(\sigma) \leq C^{\max 1}_{\mathrm{int} C_\Lambda^{\max 1}(m(\sigma))}(m(\sigma))$.

"$\geq$": It is the case that

$$\begin{aligned} C^{\max 1}_{\mathrm{int} C_\Lambda^{\max 1}(m(\sigma))}(m(\sigma)) &= C^{\max 1}_{\mathrm{int}^{\mathrm{i}} C_\Lambda^{\max 1}(m(\sigma))}(m(\sigma)) \\ &\overset{(5.28)}{=} C^{\max 1}_{\mathrm{int}^{\mathrm{i}} C^{\mathrm{fill}}(\sigma)}(m(\sigma)) \\ &\overset{(5.3)}{=} C^{\max 1}_{\mathrm{int}^{\mathrm{i}} C^{\mathrm{fill}}(\sigma)}(\sigma) \\ &\leq C_\Lambda^{\max 1}(\sigma)\,, \end{aligned}$$

which conclude the proof. $\square$

Last, we have to show the invertibility of $m|_{A^2}$. Our candidate for the inverse map is

$$m^{-1} : m(A^2) \overset{!}{\to} A^2; \quad m(\sigma) \mapsto 1 - \mathbb{1}_{m(\sigma)^{-1}(0) \cup {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))} \overset{!}{=} \sigma\,,$$

which is illustrated in Figure 5.2.

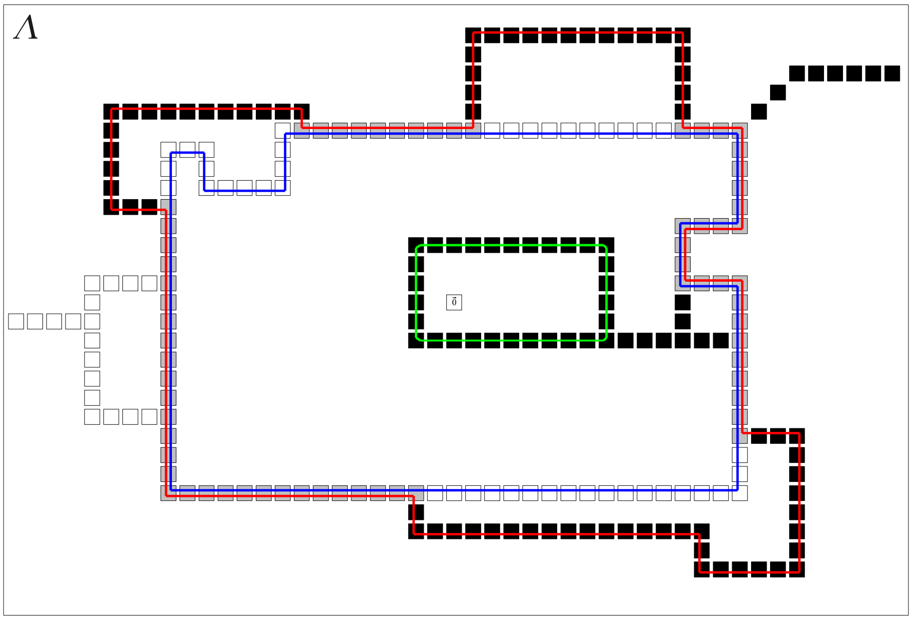

Figure 5.2: In this figure the white squares represent nodes with 0spin[$\sigma, m(\sigma)$]. The gray squares stand for nodes with 0spin[σ] and 1spin[$m(\sigma)$]. The black squares are nodes with 1spin[$\sigma, m(\sigma)$]. The circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$ is indicated by a red curve. The circuit ${}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))$ is indicated by a blue curve. The maximal 1circuit[σ] is indicated by a green curve.

Lemma 5.22 *Let* $\sigma \in A^2$. *Then* $1 - \mathbb{1}_{m(\sigma)^{-1}(0) \cup {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma))} = \sigma$, *i.e., the map* $m_\Lambda|_{A^2}$ *is injective.*

Proof: It is sufficient to show that

$${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap \sigma^{-1}(0) \overset{!}{=} {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) \cap m(\sigma)^{-1}(1)$$

for each $\sigma \in A^2$.

"$\subset$" This direction is a consequence of

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \overset{(5.28)}{=} {}^{\mathrm{i}}C_{\Lambda}^{\max 1}(m(\sigma)) \subset m(\sigma)^{-1}(1) $$

and

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap \sigma^{-1}(0) \overset{(5.2)}{\subset} {}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C_{\Lambda}^{\max 1}(\sigma)}(\sigma) \overset{(5.29)}{=} {}^{\mathrm{i}}C^{\mathrm{empty}}\,(m(\sigma))\,. $$

"$\supset$" This implication follows from

$$ {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) \overset{(5.29)}{=} {}^{\mathrm{i}}C^{\min 0}_{\mathrm{ext}C_{\Lambda}^{\max 1}(\sigma)}(\sigma) \subset \sigma^{-1}(0) $$

and

$$ {}^{\mathrm{i}}C^{\mathrm{empty}}(m(\sigma)) \cap m(\sigma)^{-1}(1) \overset{(5.27)}{\subset} {}^{\mathrm{i}}C_{\Lambda}^{\max 1}(m(\sigma)) \overset{(5.28)}{=} {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)\,, $$

which concludes the proof. $\square$

Consequently, we have proved the third and, therefore, all propositions.

5.2 The Connection to the Widom-Rowlinson Model

The section is dedicated to establish a connection between the injection m and the Widom-Rowlinson model. More precisely, we look at the number of 1$*$clusters that are joined by filling the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$.

In the sequel we often consider nodes of the $*$boundary of a node x. For this we introduce the following notation that intuitively describes the relative location of these nodes to x.

Definition 5.23 *Let $x \in \mathbb{Z}^2$. The nodes $*$adjacent to the node x are denoted by*

$$ {}^{\bullet}x, \quad \overset{\bullet}{x}, \quad x^{\bullet}, \quad \bullet x, \quad x\bullet, \quad {}_{\bullet}x, \quad \underset{\bullet}{x}, \quad x_{\bullet}\,, $$

where the bullet shall indicate the position of the node (in question) relative to x.

Note that we refrain from using the standard orientation, i.e., the node $\bullet x$ does not have to be $(x_1 - 1, x_2)$. Instead we specify the orientation by explicitly determining one of these nodes, e.g., set $\bullet x := (x_1 + 1, x_2)$.

Up to now we interpreted the injective map m of the last section as a filling of the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$, which consists of 0paths[σ] and 1paths[σ]. These 0paths[σ] are within a 0circuit[σ], $C^{empty}(m(\sigma))$. From another point of view this map m fills 0paths[σ] that are contained in the 0circuit[σ] $C^{empty}(m(\sigma))$ and whose beginning and ending nodes are adjacent to 1paths[σ]. These 1paths[σ] combined with the above 0paths[σ] form the circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$. The difference of this to the former approach is that we start with the 0circuit[σ] $C^{empty}(m(\sigma))$ instead of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$.

The setting of the next lemma describes a 0path[σ] of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma) \cap C^{empty}(m(\sigma))$ and its surroundings. The statement, however, investigates how many 1$*$clusters are combined by filling such a 0path[σ]. Before approaching this question rigorously, let us establish some intuition: Obviously, at most every second node of the $*$boundary of the to-be-filled path can belong to a different 1$*$cluster. Moreover, if we interpret the $*$boundary as a path, then it seems plausible that this path makes turns as little as possible, since otherwise two non-consecutive nodes are $*$adjacent. In the next lemma we pursue this intuition. To this end, we need paths and circuits that may intersect themselves.

Definition 5.24 (non-self-avoiding ($*$)path, non-self-avoiding ($*$)circuit) *We call a finite sequence of nodes $(x_1, \ldots, x_n)$, $n \geq 1$, a non-self-avoiding path if every pair of successive nodes is connected by an edge, i.e., for all $1 \leq i, j \leq n$*

$$ |i - j| = 1 \Rightarrow x_i \text{ is adjacent to } x_j\,. $$

Once again, the node x_1 (resp. x_n) is called the starting (resp. ending) node.

A non-self-avoiding path $(x_1, \ldots, x_n)$ *is called a non-self-avoiding circuit if the starting node* x_1 *is adjacent to the ending node* x_n.

Accordingly, we define a non-self-avoiding ∗path and a non-self-avoiding ∗circuit.

Let us prepare the first part of the following lemma by interpreting its setting and, hereby, introducing some notation: Let $k \geq 3$, $(p_1, \ldots, p_k)$ an induced path, $P := (p_2, \ldots, p_{k-1})$ and $\Delta := \partial^* P \cup P$. Set $p_2\bullet = p_3$ and $p_{k-2} = \bullet p_{k-1}$. We further assume that $p_1 \in \{\bullet p_2, \overset{\bullet}{p}_2\}$ and $p_k \in \{\overset{\bullet}{p}_{k-1}, p_{k-1}\bullet\}$. We interpret $\partial^* P$ as a non-self-avoiding ∗circuit and $\partial^* P \setminus \{p_1, p_k\}$ as two non-self-avoiding ∗paths. The non-self-avoiding ∗path containing $\underset{\bullet}{p}_2$ will be referred to as the "lower" part and denoted by Δ_{down}; the other non-self-avoiding ∗path as the "upper" part, Δ_{up}. Consequently, we can say the induced path $(p_1, \ldots, p_k)$ splits Δ into three disjoint parts, an "upper" part Δ_{up}, a "lower" part Δ_{down}, and $(p_1, \ldots, p_k)$.

Lemma 5.25 a) *As above, let* $k \geq 3$ *and* $(p_1, \ldots, p_k)$ *an induced path with* $p_1 \in \{\bullet p_2, \overset{\bullet}{p}_2\}$ *and* $p_k \in \{\overset{\bullet}{p}_{k-1}, p_{k-1}\bullet\}$ *if we set* $p_2\bullet = p_3$ *and* $\bullet p_{k-2} = p_{k-1}$. *Further, let* $P := (p_2, \ldots, p_{k-1})$, $\Delta := \partial^* P \cup P$, Δ_{up} *the "upper" part of* Δ *and* Δ_{down} *the "lower" part of* Δ. *Further, let* $\sigma \in \{0,1\}^{\Delta}$ *be such that*

i) $\{p_1, \ldots, p_k\} \subset \sigma^{-1}(0)$;

ii) $\exists\, x, y \in \Delta_{\text{down}}$ with $x \neq y$, $x \sim p_2$, $y \sim p_{k-1}$, $\{x, y\} \subset \sigma^{-1}(1)$, and $x \overset{1}{\not\leftrightarrow} y$.

If there are at least $|P| + 2$ *disjoint* 1∗*clusters* ∗*adjacent to* P, *then*

1) P *forms a straight line, i.e., for all* j *with* $2 \leq j \leq k-1$

$$p_j = p_2 + (j-2)(p_3 - p_2)\,; \tag{5.39}$$

2) the length of P *is odd and at least 3;*

3) if we set $\bullet p_i := p_{i-1}$ *for all* i *with* $3 \leq i \leq k-1$, *then it is the case that*

$$\begin{array}{l} {}^{\bullet}p_2, \underset{\bullet}{p}_2, \overset{\bullet}{p}_3, \underset{\bullet}{p}_4, \overset{\bullet}{p}_5, \ldots, \underset{\bullet}{p}_{k-3}, \overset{\bullet}{p}_{k-2}, \underset{\bullet}{p}_{k-1}, p_{k-1}{}^{\bullet} \subset \sigma^{-1}(1) \\ \bullet p_2, \overset{\bullet}{p}_2, \underset{\bullet}{p}_3, \overset{\bullet}{p}_4, \ldots, \underset{\bullet}{p}_{k-4}, \overset{\bullet}{p}_{k-3}, \underset{\bullet}{p}_{k-2}, \overset{\bullet}{p}_{k-1}, p_{k-1}\bullet \subset \sigma^{-1}(0) \end{array}\,, \tag{5.40}$$

which is illustrated in Figure 5.3.

b) *Let* C *an induced circuit strictly around the origin. Furthermore, let* $\pi \in \{0,1\}^{C \cup \partial^* C}$ *be a configuration such that* C *is a* 0*circuit*$[\pi]$. *Then the number of* 1∗*clusters*$[\pi]$ ∗*adjacent to* C *is at most* $|C| + 2$.

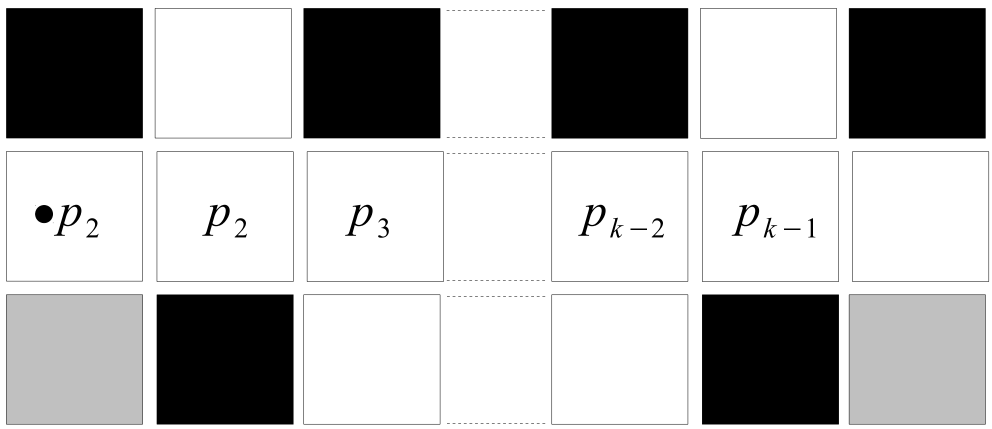

Figure 5.3: This graphic illustrates $\Delta \setminus ({}_\bullet p_2 \cup p_{k-1\bullet})$. The black squares are nodes that have taken spin value 1. The white squares are nodes with spin value 0. The gray squares are nodes, which spin values cannot be specified in generality.

Proof: Our strategy consists of three steps, where the second step decomposes into four cases. First, given an arbitrary induced path Q, we define a non-self-avoiding circuit $R(Q)$ such that its cardinality is $2|Q|+6$ and it contains the $*$boundary of Q. Second, we prove the first part of the lemma. More precisely, we (more or less) take away both nodes x and y from $R(P)$, together with all nodes $*$adjacent to them. Note that the rest of $R(P)$ decomposes into at most two non-self-avoiding paths, called P_1 and P_2. The lemma (more or less) follows from the fact that at most every second node of P_1 or P_2 can be hit by a "new" 1$*$cluster. Third, with the aid of the non-self-avoiding circuit $R(Q)$ defined in the first step, we prove the second part of the lemma.

Let us note, right here, that instead of defining $R(Q)$ we could use $\partial^* Q$ to derive this lemma. This would be easier at the beginning, namely step one, but later on we would have to argue more carefully. This is the case because $\partial^* Q$ depends more on the whole path Q than $R(Q)$.

First Step: First we recursively define $R(Q)$ and simultaneously prove

$$|R(Q)| = 2|Q| + 6$$

for all induced paths Q.

If the path Q consists of one node (base case) we define $R(Q)$ by $\partial^* Q$. A moment's thought reveals $|\partial^* Q| = 2|Q| + 6 = 8$. Furthermore, $R(Q) = (r_1, \ldots, r_8)$ satisfies the following additional (technical) property, called the "unique index": For all nodes z such that (Q, z) is an induced path, there exists exactly one index i such that $r_i = z$ and $\{r_{i-1}, r_{i+1}\} = \{\overset{\bullet}{z}, \underset{\bullet}{z}\}$, where $\bullet z$ shall be the ending node of Q and the indices $i-1$ and $i+1$ are to be understood modulo 8.

Figure 5.4 may help the reader to understand the inductive step $n-1 \to n$: Let $Q' := (q_1, \ldots, q_{n-1})$ be an arbitrary induced path and let $R(Q') := (r_1, \ldots, r_m)$

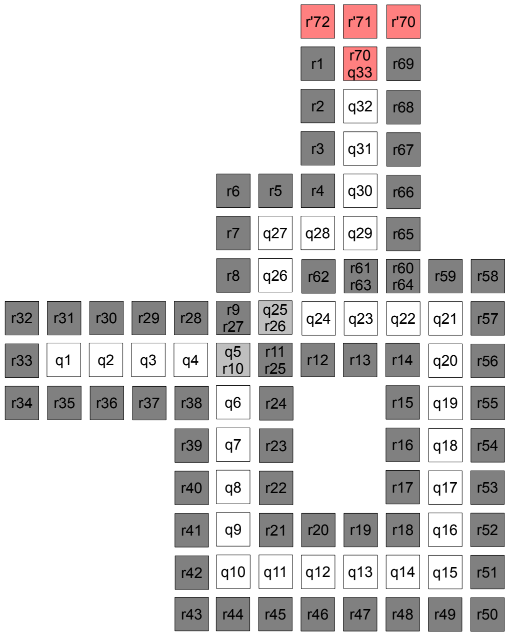

Figure 5.4: This graphic illustrates the non-self-avoiding circuits $R(Q') = (r1, \ldots, r70)$ and $R(Q) = (r1, \ldots, r69, r'70, r'71, r'72)$. The red node $q33 = r70 = r_i$ is the node $q_n = Q \setminus Q'$, which is subtracted from $R(Q')$, and the red nodes $r'70$, $r'71$ and $r'72$ are added to the result to get $R(Q)$.

be the corresponding non-self-avoiding circuit, which contains $\partial^* Q'$, satisfies

$$|R(Q')| = 2|Q'| + 6\,,$$

and exhibits the "unique index" property. Further, let q_n be a node so that adding the node q_n to $Q' = (q_1, \ldots, q_{n-1})$ results in an induced path $Q = (Q', q_n) =$

$(q_1, \ldots, q_n)$. Set $\bullet q_n = q_{n-1}$ and let i be the unique index such that $r_i = q_n$ and $\{r_{i-1}, r_{i+1}\} = \{\overset{\bullet}{q}_n, \underset{\bullet}{q_n}\}$. The cardinality of the path $(r_{i+1}, \ldots, r_m, r_1, \ldots, r_{i-1})$ is $2|Q'| + 5 = 2|Q| + 3$. We define $R(Q)$ by the non-self-avoiding circuit

$$C := (q_n{}^{\bullet}, q_n\bullet, q_{n\bullet}, r_{i+1}, \ldots, r_m, r_1, \ldots, r_{i-1})\,,$$

whose cardinality is $2|Q| + 6$. To check that C exhibits the unique index property take any z adjacent to q_n and set $\bullet z = q_n$, then, by definition of $R(Q)$, the existence of two different indices i, j with $\{r_{i-1}, r_{i+1}\} = \{r_{j-1}, r_{j+1}\} = \{\overset{\bullet}{z}, \underset{\bullet}{z}\}$ implies that the node $z\bullet$ also belongs to Q. Therefore, (Q, z) cannot be an induced path, since the supposedly ending node z would have two adjacent nodes of Q, namely $\bullet z$ and $z\bullet$. Consequently, the "unique index" property is satisfied. Moreover, $R(Q)$ contains both $\partial^* Q' \setminus q_n$ and $\partial^* Q \setminus \partial^* Q'$, which, taken together, equals $\partial^* Q$. Thus, $R(Q)$ is well-defined for all induced paths Q.

Interlude: Before turning towards the second step, we exclude $|P| = 1, 2$ by testing all possibilities, which is done in Figure 5.5.

Second Step: Recall the setting of the first part of the lemma. From now on assume $|P| \geq 3$ and $R(P) = (r_1, \ldots, r_m)$ and set $p_2\bullet = p_3$ and $\bullet p_{k-1} = p_{k-2}$. Recall that the node p_2 is adjacent to $x \in \Delta_{down}$, p_{k-1} is adjacent to $y \in \Delta_{down}$, and x, y have spin value 1. Further, there are only two possible locations for x, namely $\underset{\bullet}{p_2}$ or $\bullet p_2$, and two possible locations for y, namely $\underset{\bullet}{p_{k-1}}$ or $p_{k-1}\bullet$.

Before we subtract certain nodes of $R(P)$ and, afterwards, split the remaining set into two non self-avoiding paths, we try to establish some intuition by describing this in an easier setting. To this end, assume that $\partial^* P$ is a circuit $C = (c_1, \ldots, c_m)$. A consequence of this assumption is $R(P) = C$. Pick the indices i and j so that $c_i = x$ and $c_j = y$, which are uniquely determined in this setting. Recall that P has at least three nodes and that we set $p_2\bullet = p_3$ and $\bullet p_{k-1} = p_{k-2}$. Without loss of generality let C be enumerated so that $3 \leq i < j \leq m - 2$. Consequently, the intersection

$$\{c_{i-2}, c_{i-1}, c_i = x, c_{i+1}, c_{i+2}\} \cap \{c_{j-2}, c_{j-1}, c_j = y, c_{j+1}, c_{j+2}\}$$

is contained in

$$\{c_i = x, c_{i+1}, c_{i+2}, c_{j-2}, c_{j-1}, c_j = y\}\,.$$

Now, in a first step, if $x = \bullet p_2$ then subtract the nodes $c_{i-2} = \overset{\bullet}{p_2}$, $c_{i-1} = {}^{\bullet}p_2$, $c_{i+1} = {}_{\bullet}p_2$, and $c_{i+2} = \underset{\bullet}{p_2}$ from $C \setminus c_i$, which are all $*$adjacent to $c_i = x$, and denote the resulting set by C', i.e.,

$$C' = C'(x = \bullet p_2) := C \setminus \{c_{i-2}, c_{i-1}, c_i, c_{i+1}, c_{i+2}\} = C \setminus \{\overset{\bullet}{p_2}, {}^{\bullet}p_2, \bullet p_2, {}_{\bullet}p_2, \underset{\bullet}{p_2}\}\,;$$

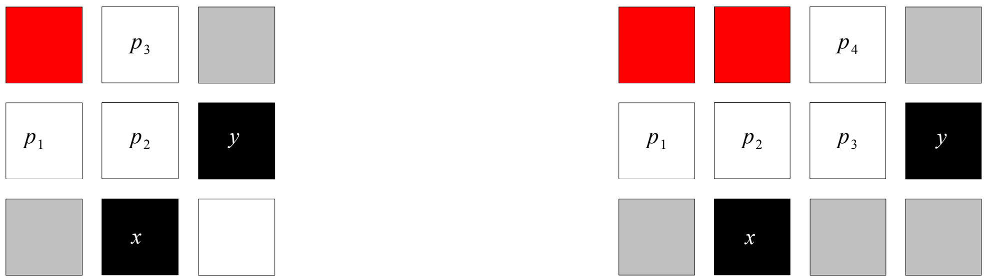

Figure 5.5: The figure illustrates all possibilities – up to rotation and reflection – for $|P| = 1, 2$. The black (resp. white) squares represent nodes that take spin value 1 (resp. 0). The spin values of the gray and red squares cannot be specified in generality. The path (p_1, p_2, p_3) resp. (p_1, p_2, p_3, p_4) splits the left resp. the right illustration in two parts, namely the upper half consisting of the red nodes and the lower half consisting of the lower and right nodes.
Obviously, in both graphics at most one 1∗cluster can hit the upper part, which is the case if one of the red squares has spin value 1. Moreover, in the left graphic at most one 1∗cluster can hit the lower part, since the nodes x and y are ∗adjacent and all other nodes of the lower part are adjacent to them. In the right graphic at most two 1∗clusters can hit the lower part. This is the case if and only if the node $\underset{\bullet}{p_3}$ has 0spin; otherwise one 1∗cluster hits the lower part, because the nodes x and y are 1∗connected by $\underset{\bullet}{p_3}$. Summing up, on the left side there are at most two 1∗clusters ∗adjacent to $P = (p_2)$ and on the right side there are at most three 1∗clusters ∗adjacent to $P = (p_2, p_3)$.

in the alternative case $x = \underset{\bullet}{p_2}$ subtract the nodes $c_{i-2} = {\bullet}p_2$, $c_{i-1} = {}_{\bullet}p_2$, and c_{i+1} from $C \setminus c_i$, which are all ∗adjacent to $c_i = x$, and denote the resulting set by C', i.e.,

$$C' = C'(x = \underset{\bullet}{p_2}) := C \setminus \{c_{i-2}, c_{i-1}, c_i, c_{i+1}\} = C \setminus \{{\bullet}p_2, {}_{\bullet}p_2, \underset{\bullet}{p_2}, a\}\,.$$

Note that the set C' has lost at least four and at most five nodes in comparison to C depending on the exact position of x relative to p_2 with $p_2{\bullet} = p_3$. In a second step, in the same manner we subtract the nodes $c_{j-2} = \underset{\bullet}{p_{k-1}}$, $c_{j-1} = p_{k-1}{}_{\bullet}$, $c_{j+1} = p_{k-1}{}^{\bullet}$, and $c_{j+2} = \overset{\bullet}{p}_{k-1}$ from $C' \setminus c_j$ if $y = c_j = p_{k-1}{\bullet}$ and denote the resulting set by C''; in the case $y = \underset{\bullet}{p_{k-1}}$ we subtract the nodes c_{j-1}, $c_{j+1} = p_{k-1}{}_{\bullet}$, and $c_{j+2} = p_{k-1}{\bullet}$

from $C' \setminus c_j$ and denote the resulting set by C''; Note that $2 \leq |C' \setminus C''| \leq 5$, since the nodes c_{j+1} and c_{j+2} are subtracted in any case. Moreover, the nodes x and y may not be adjacent and, therefore, c_{i+1} cannot coincide with y. Note that $|C' \setminus C''| = 2$ can only occur if only the nodes c_{j+1} and c_{j+2} are subtracted from C'. This happens if c_{i+2} coincides with y and is already subtracted from C to get C', i.e., $x = \bullet p_2$ and $y = \underset{\bullet}{p_{k-1}}$. In particular, in this case, $c_{i+1} = c_{j-1}$ and in total seven nodes are subtracted from C to get C''. Summing up, the decrease of nodes from C to C'' is at least seven and at most ten, in short and more precisely

$$|C \setminus C''| = |C| - 7 - \mathbb{1}_{c_{i+1} \neq c_{j-1}} - \mathbb{1}_{\bullet p_2 = x, c_{i+1} \neq c_{j-1}, c_{i+2} \neq c_{j-1}} - \mathbb{1}_{p_2 \bullet = y, c_{i+1} \neq c_{j-1}, c_{i+1} \neq c_{j-2}}$$

Now let us return to our more general setting.

Similar as above, we determine the shape of certain parts of $R(P) = (r_1, \ldots, r_m)$ "around" x and y depending on the exact location of x and y:

1. If $x = \bullet p_2$, then there exists an index i such that $r_{i-2} = \overset{\bullet}{p_2}$, $r_{i-1} = {}^{\bullet}p_2$, $r_i = x = \bullet p_2$, $r_{i+1} = {}_{\bullet}p_2$, and $r_{i+2} = \underset{\bullet}{p_2}$, where the indices should be understood modulo $|R(P)| = m$. Consequently,

$$R(P) = (r_1, \ldots, \overset{\bullet}{\mathbf{p_2}}, {}^{\bullet}\mathbf{p_2}, \bullet\mathbf{p_2}, {}_{\bullet}\mathbf{p_2}, \underset{\bullet}{\mathbf{p_2}}, \ldots, r_m)$$

holds;

2. If $x = \underset{\bullet}{p_2}$, then there exists an index i such that $r_{i-2} = \bullet p_2$, $r_{i-1} = {}_{\bullet}p_2$, and $r_i = x = \underset{\bullet}{p_2}$, where the indices should be understood modulo $|R(P)| = m$. Consequently,

$$R(P) = (r_1, \ldots, \bullet\mathbf{p_2}, {}_{\bullet}\mathbf{p_2}, \underset{\bullet}{\mathbf{p_2}}, \mathbf{r_{i+1}}, \ldots, r_m)$$

holds;

3. If $y = p_{k-1}\bullet$, then there exists an index i such that $r_{i-2} = \underset{\bullet}{p_{k-1}}$, $r_{i-1} = p_{k-1\bullet}$, $r_i = x = p_{k-1}\bullet$, $r_{i+1} = p_{k-1}{}^{\bullet}$, and $r_{i+2} = \overset{\bullet}{p_{k-1}}$, where the indices should be understood modulo $|R(P)| = m$. Consequently,

$$R(P) = (r_1, \ldots, \underset{\bullet}{\mathbf{p_{k-1}}}, \mathbf{p_{k-1\bullet}}, \mathbf{p_{k-1}}\bullet, \mathbf{p_{k-1}}{}^{\bullet}, \overset{\bullet}{\mathbf{p_{k-1}}}, \ldots, r_m)$$

holds;

4. If $y = \underset{\bullet}{p_{k-1}}$, then there exists an index j such that $r_j = x = \underset{\bullet}{p_{k-1}}$, $r_{j+1} = p_{k-1\bullet}$, and $r_{i+1} = \underset{\bullet}{p_{k-1}}$, where the indices should be understood modulo $|R(P)| = m$. Consequently,

$$R(P) = (r_1, \dots, \mathbf{r_{j-1}}, \underset{\bullet}{\mathbf{p_{k-1}}}, \mathbf{p_{k-1\bullet}}, \mathbf{p_{k-1}}^{\bullet}, \dots, r_m)$$

holds.

Once again, without loss of generality assume $3 \leq i < j \leq m-2$, which implies that the intersection

$$\{r_{i-2}, r_{i-1}, r_i, r_{i+1}, r_{i+2}\} \cap \{r_{j-2}, r_{j-1}, r_j, r_{j+1}, r_{j+2}\}$$

has to be contained in

$$\{r_i, r_{i+1}, r_{i+2}, r_{j-2}, r_{j-1}, r_j\}.$$

Note that $r_i = x$ and $r_j = y$ cannot coincide or be adjacent to each other. Now subtract these at least seven entries from $R(P)$, specified by the four statements above and printed in bold type. This splits $R(P)$ into at most two non-self-avoiding paths P_1 and P_2 that satisfy the following:

$$\begin{aligned}
|P_1| + |P_2| &= |R(P)| - 7 - \mathbb{1}_{r_{i+1} \neq r_{j-1}} - \mathbb{1}_{x = {}^\bullet p_2, r_{i+1} \neq r_{j-1}, r_{i+2} \neq r_{j-1}} \\
&\qquad\qquad\qquad\qquad - \mathbb{1}_{y = p_{k-1}^\bullet, r_{i+1} \neq r_{j-1}, r_{i+1} \neq r_{j-2}} \\
&= 2|P| - 1 - \mathbb{1}_{r_{i+1} \neq r_{j-1}} - \mathbb{1}_{x = {}^\bullet p_2, r_{i+1} \neq r_{j-1}, r_{i+2} \neq r_{j-1}} \\
&\qquad\qquad\qquad\qquad - \mathbb{1}_{y = p_{k-1}^\bullet, r_{i+1} \neq r_{j-1}, r_{i+1} \neq r_{j-2}},
\end{aligned}$$

where one of these paths is the empty set if $r_{i+1} = r_{j-1}$ or r_{i+1} is adjacent to r_{j-1}.

There are at most $\lceil |P_1|/2 \rceil$ resp. $\lceil |P_2|/2 \rceil$ disjoint $1*$clusters hitting P_1 resp. P_2, since at most every second node can be hit by a "new" $1*$cluster.

We distinguish between four cases according to how many entries we subtract from $R(P)$:

<u>*First Case:*</u> If we subtract 10 entries, then $|P_1| + |P_2| = 2|P| - 4$ follows. Therefore, at most

$$\lceil |P_1|/2 \rceil + \lceil |P_2|/2 \rceil \leq \lceil |P_1|/2 + |P_2|/2 \rceil + 1 = \underbrace{\lceil |P| - 2 \rceil}_{=|P|-2} + 1 = |P| - 1$$

$1*$clusters can hit P_1 or P_2. Recall that the nodes x and y have 1spin and, therefore, by construction, at most two disjoint $1*$clusters can hit the nodes that were subtracted from $R(P)$. Summing up, one can find at most $|P| + 1$ disjoint $1*$clusters hitting $\partial^* P$. This is a contradiction to the assumption of the lemma that there are $|P| + 2$ disjoint $1*$clusters hitting $\partial^* P$, which means that this case is impossible.

<u>*Second Case:*</u> If we subtract 9 entries from $R(P)$, then $|P_1| + |P_2| = 2|P| - 3$. Note that either $|P_1|$ or $|P_2|$ is odd and

$$\lceil |P_1|/2 \rceil + \lceil |P_2|/2 \rceil \leq \lceil |P_1|/2 + |P_2|/2 \rceil + 1 = \underbrace{\lceil |P| - 3/2 \rceil}_{=|P|-1} + 1 = |P|$$

is an upper bound for the number of disjoint 1$*$clusters hitting $P_1 \cup P_2$. This upper bound can only be reached if both $|P_1|$ and $|P_2|$ are odd, which is not the case. Therefore, at most $|P| - 1$ disjoint 1$*$clusters hit $P_1 \cup P_2$. Once again, since at most two disjoint 1$*$clusters can hit the nodes that were subtracted from $R(P)$, one can find at most $|P| + 1$ disjoint 1$*$clusters hitting $\partial^* P$. This is a contradiction to the assumption of the lemma that there are $|P| + 2$ disjoint 1$*$clusters hitting $\partial^* P$. Again, this case is impossible.

<u>*Third Case:*</u> If we subtract 8 entries from $R(P)$, then $|P_1| + |P_2| = 2|P| - 2$. Thus, at most

$$\lceil |P_1|/2 \rceil + \lceil |P_2|/2 \rceil \leq \lceil |P_1|/2 + |P_2|/2 \rceil + 1 = \underbrace{\lceil |P| - 1 \rceil}_{=|P|-1} + 1 = |P|$$

1$*$clusters can hit $P_1 \cup P_2$. Note that the inequality above can only be an equality, if both $|P_1|$ and $|P_2|$ are odd. In particular, in this case neither P_1 nor P_2 are the empty set and, therefore, the nodes r_{i+1} and r_{j-1} do not coincide and are not adjacent to each other. Consequently, we know that $\underset{\bullet}{p_2} = x$ and $\underset{\bullet}{p_{k-1}} = y$ holds. Further, assuming that $|P|$ disjoint 1$*$clusters hit $P_1 \cup P_2$ implies that every second node of both P_1 and P_2 has to be hit by a "new" 1$*$cluster, beginning at the starting nodes and ending at the ending nodes.

Note that at most two disjoint 1$*$clusters can hit the nodes subtracted from $R(P)$. So, if at most $|P| - 1$ disjoint 1$*$clusters hit $P_1 \cup P_2$ then this is a contradiction to our assumption that at least $|P| + 2$ disjoint 1$*$clusters hit P. Thus, we only have to consider the event that $|P|$ disjoint 1$*$clusters hit $P_1 \cup P_2$ in the sequel. Recall all consequences of this, stated in the last paragraph.

Remember that we set $p_2{\bullet} = p_3$ and ${\bullet}p_{k-1} = p_{k-2}$ at the beginning of the second step. Furthermore, we know that $x = \underset{\bullet}{p_2}$, $y = \underset{\bullet}{p_{k-1}}$, $r_{i+1} \neq r_{j-1}$, and the nodes r_{i+1} and r_{j-1} are not adjacent to each other. Therefore, we subtracted the following 8 nodes from $R(P)$ to get $P_1 \cup P_2$:

$${\bullet}p_2, {}_{\bullet}p_2, \underset{\bullet}{p_2}, r_{i+1}, r_{j-1}, \underset{\bullet}{p_{k-1}}, {p_{k-1}}_{\bullet}, \text{ and } p_{k-1}{\bullet}\ .$$

Hence, P_1 or P_2 starts in ${}^{\bullet}p_2$ and ends in ${p_{k-1}}^{\bullet}$; without loss of generality say P_1. Recall that both nodes, namely ${}^{\bullet}p_2$ and ${p_{k-1}}^{\bullet}$, have 1spin, since every second node of P_1 has to be hit by a "new" 1$*$cluster, beginning at the starting node and ending at the ending node. Because of the same reasoning the second node $\overset{\bullet}{p}_2$ and the second to last node $\overset{\bullet}{p}_{k-1}$ of P_1 have 0spin. We can even determine the position of p_4 as $p_3{\bullet} = p_4$ if we set ${\bullet}p_3 = p_2$: Assume for contradiction $p_4 \in \{\overset{\bullet}{p}_3, \underset{\bullet}{p}_3\}$: If $p_4 = \overset{\bullet}{p}_3$ then the starting node of P_1, namely ${}^{\bullet}p_2$, is $*$adjacent to or coincides with the third node of P_1. This is a contradiction to the fact that every second node of P_1 has to be hit by a "new" 1$*$cluster, beginning at the starting node ${}^{\bullet}p_2$. Otherwise, $p_4 = \underset{\bullet}{p}_3$ implies that the third node $\overset{\bullet}{p}_2$ of P_1 is $*$adjacent to both the fourth node $p_2^{\bullet}$ and fifth node $p_2{\bullet}$ of P_1. Once again, we have derived a contradiction to the fact that every second node of P_1 has to be hit by a "new" 1$*$cluster, beginning at the starting node. Since now the position of p_4 is known, we can determine the position of r_{i+1} as $p_{2\bullet} = \underset{\bullet}{p}_3$ and, therefore, $\underset{\bullet}{p}_3 \subset \sigma^{-1}(0)$, as the starting node of P_2 has to be hit by a new 1$*$cluster, which, in particular, must not be 1connected to x by r_{i+1}. Moreover, $\overset{\bullet}{p}_3$, being the third node of P_1, is equipped with a 1spin.

The last paragraph gives the induction base, $i = 3$, of the following proof by induction that P forms a straight line, i.e., (5.39), and that the configuration of $\partial^* P \setminus \{{}_{\bullet}p_2, p_{k-1\bullet}\}$ is as in (5.40).

Our induction hypothesis is that for all $3 \leq j \leq i$ with $i < k - 1$,

$$p_{j+1} = p_2 + (j-1)(p_3 - p_2)$$

and that for ${\bullet}p_j = p_{j-1}$,

i) if j is even then the node $\overset{\bullet}{p}_j$ has 0spin and $\underset{\bullet}{p}_j$ has 1spin;

ii) if j is odd then the node $\overset{\bullet}{p}_j$ has 1spin and $\underset{\bullet}{p}_j$ has 0spin.

We prove our induction step, $i \to i+1$ with $i+1 < k-1$, only for even i, since the case for odd i is obviously similar: Set $p_{i-1} = {\bullet}p_i$. By induction hypothesis, the node $\overset{\bullet}{p}_i$ has 0spin and $\underset{\bullet}{p}_i$ has 1spin. So, the node p_{i+1} being equipped with a 0spin cannot coincide with $\underset{\bullet}{p}_i$, which has 1spin. For contradiction assume $p_{i+1} = \overset{\bullet}{p}_i$. This implies that the (i-3)-th node of P_2, namely $\underset{\bullet}{p}_i$, is $*$adjacent to both the (i-2)-th and the (i-1)-th node of P_2, namely $p_{i\bullet}$ and $p_i{\bullet}$. But this is a contradiction, since the (i-3)-th node of P_2 is equipped with a 1spin and every second node of P_1 has to be hit by a "new" 1$*$cluster, beginning at the starting node.

Summing up, we have proved that P forms a straight line, i.e., (5.39), and the configuration of $\partial^* P \setminus \{{}_{\bullet}p_2, p_{k-1\bullet}\}$ evidently is as in (5.40).

<u>*Fourth Case:*</u> If we subtract 7 entries of $R(P)$, then $r_{i+1} = r_{j-1}$ and one path, say P_2, is the empty set. Once again, one can find at most

$$\lceil |P_1|/2 \rceil = \lceil |P| - 1/2 \rceil = |P|$$

$1{*}$clusters in P_1. Note that this bound can only be reached if every second node of P_1 is hit by a "new" $1{*}$cluster, beginning at the starting node. If we further assume that the nodes x and y are $*$adjacent then at most $|P|+1$ disjoint $1{*}$clusters can hit $\partial^* P$, which is contrary to the assumption that at least $|P|+2$ disjoint $1{*}$clusters hit $\partial^* P$. Hence, the node x is not $*$adjacent to the node y and, therefore, the path $(x, r_{i+1} = r_{j-1}, y)$ forms a straight line. We have four possibilities to check: $r_{i+1} \in \{p_2, p_{2\bullet}, \underset{\bullet}{\underset{\bullet}{p_2}}, {}_\bullet p_2\}$.

i) If $r_{i+1} = p_2$ then, by construction of $R(P)$, a node p_l of P with $l > 3$ coincides with the node ${}^\bullet p_2$. This is a contradiction to our condition that P is an induced path.

ii) If $r_{i+1} = p_{2\bullet}$ then the first part of the lemma follows from an easy sketch, which is left for the reader to draw.

iii) If $r_{i+1} \in \{\underset{\bullet}{\underset{\bullet}{p_2}}, {}_\bullet p_2\}$ then at most $|P|+1$ disjoint $1{*}$clusters are $*$adjacent to P. This is the case because the third node $\dot{p}_3 = p_2^\bullet$ of P_1 is $*$adjacent to the fourth and fifth node of P_1, namely $p_3^\bullet$ and $p_{3\bullet}$. Therefore, not every second node of P_1 can be hit by a "new" $1{*}$cluster, beginning at the starting node. It may help the reader to draw a sketch.

Consequently, the first part of the lemma follows.

<u>*Third Step:*</u> Fix an induced circuit C and a configuration π as required in the second part of the lemma. Our aim is to verify that at most $|C|+2$ disjoint $1{*}$clusters$[\pi]$ can be $*$adjacent to C. For this, it suffices to check that

$$|\partial^* C| \leq 2|C| + 1 \tag{5.41}$$

holds, since at most $1 + \lceil \frac{|\partial^* C|}{2} \rceil$ disjoint $1{*}$clusters$[\pi]$ can be $*$adjacent to C.

Recall that $C =: (c_1, \ldots, c_n)$ is an induced circuit strictly around the origin and, therefore, C consists of at least 8 nodes, in short $n \geq 8$. Hence, the circuit C has to make a turn, say at c_2. More precisely, set $c_1 := c_{2\bullet}$ and without loss of generality assume that $c_3 = \underset{\bullet}{c_2}$.

Figure 5.6 may help the reader in the following: The set $C \setminus c_2$ can be interpreted as an induced path and the non-self-avoiding circuit $R(C \setminus c_2)$ of step one is well-defined. Moreover, it is the case that

$$|R(C \setminus c_2)| = 2|C \setminus c_2| + 6 = 2|C| + 4\,.$$

By construction of $R(C \setminus c_2) =: (r_1, \dots, r_{2|C|+4})$, the node c_2 appears at least twice in $R(C \setminus c_2)$ and the nodes c_1 and c_3 each appear once in $R(C \setminus c_2)$. Hence, it is the case that

$$|R(C \setminus c_2) \setminus C| \leq 2|C| \,.$$

The only node of $\partial^* C$ not contained in $R(C \setminus c_2)$ can be ${}^{\bullet}c_2$, in short

$$\partial^* C \setminus R(C \setminus c_2) \subset \{{}^{\bullet}c_2\} \,.$$

Summing up, we know that

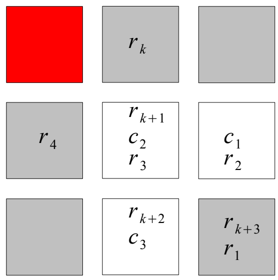

Figure 5.6: The graphic illustrates $\partial^* c_2$. The gray squares are not contained in C. Whereas, the white squares belong to C. The red square can but does not have to belong to $R(C \setminus c_2)$. Without loss of generality we assume that $R(C \setminus c_2) = (r_1, \dots, r_k, \dots r_{2|C|+4})$ starts in $c_{2\bullet}$.

$$|\partial^* C| \leq |R(C \setminus c_2) \setminus C| + 1 \leq 2|C| + 1$$

holds and, therefore, (5.41) follows, which concludes the proof. □

We will need a special observation regarding this lemma:

Corollary 5.26 *In the situation of Lemma 5.25 a) there are at most* $|P| + 1$ *disjoint* $1*$*clusters in* Δ_{down} $*$*adjacent to* P.

Proof: Once again, we exclude $|P| = 1, 2$ by checking all possibilities. Since for $|P| \geq 3$ the node $\overset{\bullet}{p_3}$ lies in Δ_{up}, our statement follows directly from Lemma 5.25. □

Now, we are ready to establish the direct connection to the Site-Random-Cluster model and, therefore, to the Widom-Rowlinson model. Hereby, the activity has to be at least $2^{4/3}$, which will be reduced to 2 later on. Nonetheless, the following proof points out the problem for activities in $[2, 2^{4/3}[$.

Proposition 5.27 *Let $\Lambda \Subset \mathbb{Z}^{2*}$ be a simply connected set with $0 \in \Lambda$. Then the injective map m_Λ of Theorem 5.10 satisfies*

$$\phi^{f*}_{\Lambda,\lambda}(\sigma) \leq \begin{cases} 8\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\sigma)) & \forall \sigma \in A^0_\Lambda, \lambda \geq 2 \\ 4\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\sigma)) & \forall \sigma \in A^1_\Lambda, \lambda \geq 2^{4/3} \\ 8\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\sigma)) & \forall \sigma \in A^2_\Lambda, \lambda \geq 2. \end{cases} \tag{5.42}$$

In particular,

$$\mu^{0*}_{\Lambda,\lambda}(\exists \text{ 0lasso in } \Lambda) \leq 8\mu^{0*}_{\Lambda,\lambda}(\exists \ -{+}{*}\text{lasso in } \Lambda) \qquad \forall \lambda \geq 2^{4/3}\,.$$

Proof: The proof falls naturally into three parts depending on whether σ belongs to A^0_Λ, A^1_Λ, or A^2_Λ.

First Part: Let $\sigma \in A^0_\Lambda$ and recall that $m|_{A^0_\Lambda}$ flips the spin of the origin. Therefore, at most four disjoint 1∗clusters[σ] are combined, which verifies the statement for $m|_{A^0_\Lambda}$.

Second Part: The statement for $m|_{A^1_\Lambda}$ is more involved: Let $\sigma \in A^1_\Lambda$. Recall that the induced 0paths[σ] $P_1(\sigma), \ldots, P_{N(\sigma)}(\sigma)$, $N(\sigma) \geq 1$, are contained in the maximal induced 0circuit[σ] and let them be numbered clockwise. Moreover, $m|_{A^1_\Lambda}$ flips the spin values of these special paths. For $1 \leq j \leq N$, the starting node of P_j is weakly 1connected[σ] to the ending node of P_{j-1} in $\text{int}C^{\max 0}(\sigma)$, where $P_0 = P_N$.

In the following three paragraphs we will distinguish three cases to prove that

$$\kappa(\sigma) - \kappa(m(\sigma)) \leq 2 + \sum_{\substack{1 \leq i \leq N(\sigma) \\ |P_i(\sigma)| \leq 2}} |P_i(\sigma)| + \sum_{\substack{1 \leq i \leq N(\sigma) \\ |P_i(\sigma)| \geq 3}} (|P_i(\sigma)| + 1)\,, \tag{5.43}$$

where $\kappa(\sigma)$ is the number of 1∗clusters[σ].

If $N(\sigma) = 1$ and $P_1(\sigma)$ is the maximal 0circuit[σ], i.e., the map m fills the whole circuit $C^{\max 0}_\Lambda(\sigma) = {}^{\mathrm{i}}C^{\mathrm{fill}}(\sigma)$, then there are at most $|P_1(\sigma)| + 2$ disjoint 1∗clusters ∗adjacent to $P_1(\sigma)$, see Lemma 5.25 b). These 1∗clusters are combined into one 1∗cluster by applying the map m. In this case Inequality (5.43) follows.

If $N(\sigma) = 1$ and $P_1(\sigma)$ is the maximal 0circuit[σ] minus one or two nodes, then at most $|P_1(\sigma)| + 4$ disjoint 1∗clusters are ∗adjacent to $C^{\max 0}_\Lambda(\sigma)$, see Lemma 5.25 b), and, therefore, to $P_1(\sigma)$. Because of $\sigma \in A^1 \subset \{\vec{0} \notin C^{\max 0}_\Lambda\}$ the maximal 0circuit[σ] consists of at least 8 nodes and, therefore, $|P_1(\sigma)| \geq 6$. Consequently, in this case Inequality (5.43) follows.

In the alternative case when $|C^{\max 0}_\Lambda(\sigma) \setminus \bigcup_{1 \leq i \leq N(\sigma)} P_i(\sigma)| > 2$, for each path $P_j(\sigma)$ we are in the setting of Lemma 5.25 a). Thus, for each $1 \leq j \leq N(\sigma)$ there are at most $|P_j| + 2$ disjoint 1∗clusters[σ] ∗adjacent to P_j and this bound can only be reached if $|P_j| \geq 3$. Moreover, one can find at least one 1cluster[σ] adjacent to both P_j and P_{j-1} for each $1 \leq j \leq N(\sigma)$. An evident consequence is that there are at most

$$\sum_{\substack{1\leq i\leq N\\ |P_i|\leq 2}} |P_i| + \sum_{\substack{1\leq i\leq N\\ |P_i|\geq 3}} (|P_i|+1)$$

disjoint 1∗clusters[σ] ∗adjacent to $\bigcup_{1\leq i\leq N} P_i$. Consequently, Inequality (5.43) holds in all three cases.

This inequality, together with $\lambda \geq 2^{4/3}$, guarantees the following inequalities for all $\sigma \in A^1_\Lambda$:

$$\begin{aligned}
Z^{f*}_{\Lambda,\lambda}\phi^{f*}_{\Lambda,\lambda}(\sigma) &= \lambda^{\sum_{x\in\Lambda}\sigma(x)}2^{\kappa(\sigma)}\\
&\overset{(5.43)}{\leq} \lambda^{\sum_{x\in\Lambda}\sigma(x)}2^{\kappa(m_\Lambda(\sigma))+2+\sum_{\substack{1\leq i\leq N\\ |P_i|\leq 2}}|P_i|+\sum_{\substack{1\leq i\leq N\\ |P_i|\geq 3}}(|P_i|+1)}\\
&\overset{\lambda\geq 2}{\leq} 4\lambda^{\sum_{x\in\Lambda}\sigma(x)+\sum_{\substack{1\leq i\leq N\\ |P_i|\leq 2}}|P_i|}2^{\kappa(m_\Lambda(\sigma))+\sum_{\substack{1\leq i\leq N\\ |P_i|\geq 3}}(|P_i|+1)}\\
&\overset{\lambda\geq 2^{4/3}}{\leq} 4\lambda^{\sum_{x\in\Lambda}\sigma(x)+\sum_{\substack{1\leq i\leq N\\ |P_i|\leq 2}}|P_i|+\sum_{\substack{1\leq i\leq N\\ |P_i|\geq 3}}|P_i|}2^{\kappa(m_\Lambda(\sigma))}\\
&= 4\lambda^{\sum_{x\in\Lambda}m_\Lambda(\sigma)(x)}2^{\kappa(m_\Lambda(\sigma))}\\
&= 4Z^{f*}_{\Lambda,\lambda}\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\sigma))
\end{aligned}$$

This proves the corresponding inequality in (5.42).

<u>*Third Part:*</u> Last, we prove Inequality (5.42) for the map $m|_{A^2_\Lambda}$: Let $\sigma \in A^2_\Lambda$. Recall that

$$A^2 \subset \{C^{\max 1}_\Lambda \neq \emptyset\}$$

and $m|_{A^2_\Lambda}$ flips the spins of the induced 0paths[σ] $P_1(\sigma), \ldots, P_{N(\sigma)}(\sigma)$, $N(\sigma) \geq 1$[1], which are contained in $C^{\min 0}_{\mathrm{ext}C^{\max 1}}$.

In the case $N = 1$ and $|C^{\min 0}_{\mathrm{ext}C^{\max 1}} \setminus P_1| \leq 2$ we can verify

$$\kappa(\sigma) - \kappa(m(\sigma)) \leq |P_1| + 3 \tag{5.44}$$

similar to the corresponding statement in the second part of this proof.

For the remaining cases we are in the setting of Lemma 5.25 a) and all nodes of

$$\partial^* C^{\min 0}_{\mathrm{ext}C^{\max 1}} \cap \mathrm{int}C^{\min 0}_{\mathrm{ext}C^{\max 1}} \cap \sigma^{-1}(1)$$

are 1∗connected[σ] to $C^{\max 1}_\Lambda$ and, therefore, to each other. In other words, there exists only one 1∗cluster[σ] in $\mathrm{int}C^{\min 0}_{\mathrm{ext}C^{\max 1}}$ ∗adjacent to $\bigcup_{1\leq i\leq N} P_i$.

[1]From now on we omit mentioning σ if the underlying configuration is uniquely determined by the context.

By definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}$, for each $1 \leq j \leq N$ there exists at least one 1cluster[σ] in $\mathrm{ext}C^{\min 0}_{\mathrm{ext}C^{\max 1}}$ adjacent to both P_j and P_{j+1} with $P_{N+1} := P_1$. This, together with Corollary 5.26, implies that there are at most $\sum_{1\leq i\leq N} |P_i|$ disjoint 1$*$clusters[σ] in

$$\partial^* C^{\min 0}_{\mathrm{ext}C^{\max 1}} \cap \mathrm{ext}C^{\min 0}_{\mathrm{ext}C^{\max 1}}$$

$*$adjacent to $\bigcup_{1\leq i\leq N} P_i$. Summing up, there are at most $1 + \sum_{1\leq i\leq N} |P_i|$ disjoint 1$*$clusters $*$adjacent to $\bigcup_{1\leq i\leq N} P_i$, one inside $C^{\min 0}_{\mathrm{ext}C^{\max 1}}$ and $\sum_{1\leq i\leq N} |P_i|$ outside $C^{\min 0}_{\mathrm{ext}C^{\max 1}}$, in short

$$\kappa(\sigma) - \kappa(m_\Lambda(\sigma)) \leq \sum_{1\leq i\leq N} |P_i| \,, \tag{5.45}$$

where $\kappa(\sigma)$ is the number of 1$*$clusters[σ]. These inequalities (5.44) and (5.45) lead to

$$\begin{aligned} Z_{\Lambda,\lambda}\phi^{f*}_{\Lambda,\lambda}(\sigma) &= \lambda^{\sum_{x\in\Lambda}\sigma(x)} 2^{\kappa(\sigma)} \\ &\overset{(5.44),(5.45)}{\leq} \lambda^{\sum_{x\in\Lambda}\sigma(x)} 2^{\kappa(m_\Lambda(\sigma))+3+\sum_{1\leq i\leq N}|P_i|} \\ &\overset{\lambda\geq 2}{\leq} 8\lambda^{\sum_{x\in\Lambda}\sigma(x)+\sum_{1\leq i\leq N}|P_i|} 2^{\kappa(m_\Lambda(\sigma))} \\ &= 8\lambda^{\sum_{x\in\Lambda} m_\Lambda(\sigma)(x)} 2^{\kappa(m_\Lambda(\sigma))} \\ &= 8 Z_{\Lambda,\lambda}\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\sigma)) \end{aligned}$$

for all $\sigma \in A^2_\Lambda$ and $\lambda \geq 2$, which proves the last part of (5.42). □

After this proof we can identify the exact point that undoes our above approach to

$$\phi^{f*}_{\Lambda,\lambda}(\exists\ 0\mathrm{lasso}) \leq 8\phi^{f*}_{\Lambda,\lambda}(\exists\ 1{*}\mathrm{lasso})$$

for $\lambda \in [2, 2^{4/3}[$, namely the configurations that exhibit special paths $P_i(\sigma)$ with $|P_i(\sigma)| + 2$ disjoint 1$*$clusters[σ] $*$adjacent to them. The next section is dedicated to bypass this problem.

5.3 Compensation of Outliers

Recall that for all simply connected sets $\Lambda \Subset \mathbb{Z}^2$, the map

$$m_\Lambda : \{\omega \in \{0,1\}^{\mathbb{Z}^2} : \omega = 0 \text{ off } \Lambda, \exists \text{ a 0lasso}[\omega] \text{ in } \Lambda\} \to$$
$$\to \{\omega \in \{0,1\}^{\mathbb{Z}^2} : \omega = 0 \text{ off } \Lambda, \exists \text{ a } 1{*}\text{lasso}[\omega] \text{ in } \Lambda\}$$

is an injection, which satisfies

$$\phi^{f*}_{\Lambda,\lambda}(\pi) \leq 8\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\pi))$$

for all $\lambda \geq 2^{4/3}$. If we consider activities $\lambda \in [2, 2^{4/3}[$, then, in general, this inequality is wrong, since the number of 1∗clusters joined by m_Λ could be significantly larger than the number of 1spins added. Fortunately, the subset

$$\{\pi : \phi^{f*}_{\Lambda,\lambda}(\pi) > 8\phi^{f*}_{\Lambda,\lambda}(m(\pi))\}$$

can be determined a bit more precisely. It is contained in A^1_Λ as Proposition 5.27 shows. Moreover, it is very small in comparison to A^1_Λ and, therefore, these outliers will be compensated by other configurations. In particular, for every outlier π there exists a configuration σ, which has a one-to-one correspondence to π, such that

$$\phi^{f*}_{\Lambda,\lambda}(\pi) \leq 8\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\sigma))$$

and

$$\phi^{f*}_{\Lambda,\lambda}(\sigma) \leq 8\phi^{f*}_{\Lambda,\lambda}(m_\Lambda(\pi))$$

for all $\lambda \geq 2$. In this section we will make this rigorous.

5.3.1 Nullification Paths and Their Impact

Let $\Lambda \Subset \mathbb{Z}^2$ be such that $\partial^*\Lambda$ can be interpreted as a circuit. Without loss of generality we assume the existence of a configuration $\pi \in A^1 \subset \{0,1\}^{\mathbb{Z}^2}$ with at least $|P(\pi)| + 5$ disjoint 1∗clusters ∗adjacent to $P(\pi) = P_1(\pi) \cup \ldots \cup P_{N(\pi)}(\pi)$ and fix it. A moment's thought, together with Lemma 5.25, reveals $N(\pi) > 2$, see the second part of the proof of Proposition 5.27. Moreover, it is the case that

$$\phi^{f*}_{\Lambda,2}(\pi) > 8\phi^{f*}_{\Lambda,2}(m(\pi)),$$

where the configuration $m(\pi)$ coincides with π off ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ and equips each node of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ with 1spins. The circuit ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ is the minimal induced circuit in the union of the maximal 0circuit[σ] and all 1clusters[σ] adjacent to the maximal 0circuit[σ] (vide Definition 5.4). Remember that we called $C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi)$ the "second largest"

0circuit[π] and defined it as the maximal 0circuit[π] in int$C_\Lambda^{\max 0}(\pi)$. By definition, ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ is contained in the “half-open“ annulus

$$\Big]C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi), C^{\max 0}_\Lambda(\pi)\Big] := \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi) \cap \mathrm{int}C^{\max 0}_\Lambda(\pi) \cup C^{\max 0}_\Lambda(\pi)$$

and its nodes are equipped with 1spins[π] if and only if they belong to the ”open” annulus

$$\Big]C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi), C^{\max 0}_\Lambda(\pi)\Big[:= \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi) \cap \mathrm{int}C^{\max 0}_\Lambda(\pi)\,.$$

From now on we omit mentioning π if the context uniquely determines the underlying configuration.

Recall that the special paths $P_1, \ldots, P_N$ are the induced 0paths[π] of maximal length in

$$\{x \in \Lambda : \pi(x) \neq m(\pi)(x)\} = \pi^{-1}(0) \cap m(\pi)^{-1}(1) \overset{(5.1)}{\subset} {}^{\mathrm{i}}C^{\max 0}(\pi)\,.$$

Note that the right side of the latter inclusion is contained in $C^{\max 0}(\pi)$.

Definition 5.28 (fixed paths) *The induced* 1*paths*[π] *of maximal length in* ${}^{\mathrm{i}}C^{\mathrm{fill}} \setminus P$ *are called the fixed* 1*paths and denoted by* $Q_1, \ldots, Q_N$. *The induced* 0*paths*[π] *of maximal length in* $C^{\max 0}(\pi) \setminus P$ *are called the fixed* 0*paths and denoted by* $O_1, \ldots, O_N$.

Calling these paths “fixed paths” hints to the fact that $\pi = m(\pi)$ along these paths. The fixed and the special paths are illustrated in Figure 5.7.

In this paragraph we enumerate the above fixed paths and the special paths: First of all, order them clockwise such that

$$(P_1, Q_1, P_2, Q_2, \ldots, P_N, Q_N) = {}^{\mathrm{i}}C^{\mathrm{fill}}$$

and

$$(P_1, O_1, P_2, O_2, \ldots, P_N, O_N) = C^{\max 0}\,,$$

i.e., the starting nodes of Q_i and O_i are adjacent to the ending nodes of P_i. Now we specify which path shall be Q_1 and, therefore, disambiguate the indices. To this end, take the $*$paths that start in $\vec{0}$, end in

$$Q := \bigcup_{1 \le i \le N} Q_i\,,$$

and are contained in int${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \cup Q$. Interpret them as polygons and consider the ones of minimal (euclidean) length. Note that these $*$paths "of minimal length"

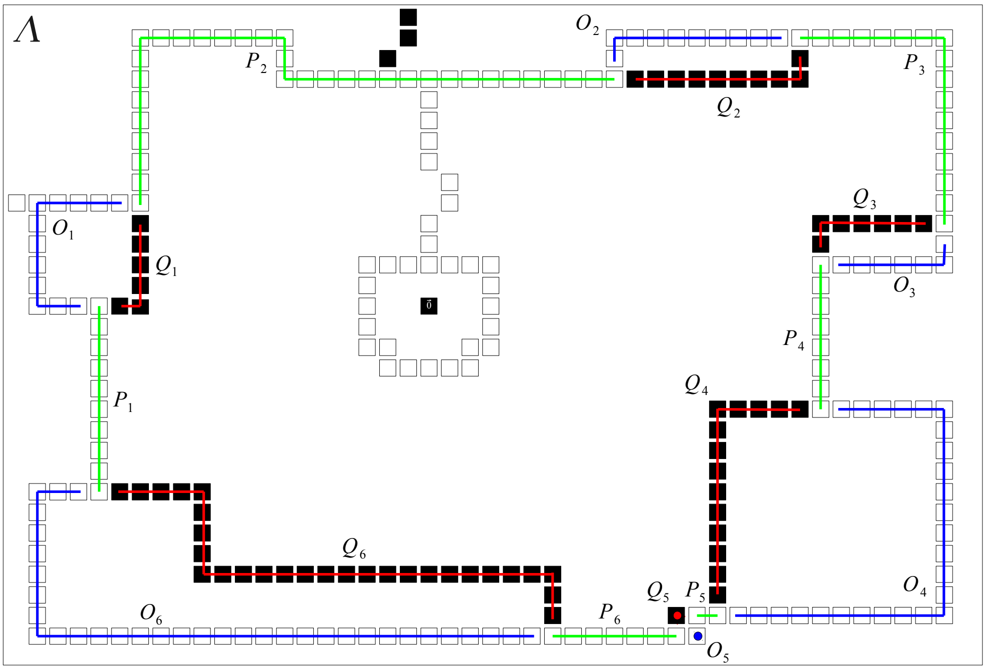

Figure 5.7: In this figure the white squares represent nodes with 0spin[π]. The black squares are nodes with 1spin[π]. The 0paths[π] $P_1, \ldots, P_6$ are indicated by green curves. The 1paths[π] $Q_1, \ldots, Q_6$ are indicated by red curves respectively a red dot. The 0paths[π] $O_1, \ldots, O_6$ are indicated by blue curves respectively a blue dot. The "blue and green" circuit is the maximal induced 0circuit[π].

intersect Q only at their ending nodes. Consider these ending nodes and let Q_1 be the 1path in Q containing the minimal (w.r.t. the lexicographic order) node of the considered ones.

Let $B(\Gamma)$ the number of 1∗clusters ∗adjacent to $\Gamma \subset \Lambda$. The map m_Λ fills $P = \bigcup_{1\leq i\leq N} P_i$ and, hereby, $B(P)$ 1∗clusters[π] ∗adjacent[π] to P are merged into one 1∗cluster[$m_\Lambda(\pi)$]. Since only some special paths cause problems, we want to fill one special path after another instead of filling them all at once, which should give us a better leverage on the probability of a single special path. Thus, we are interested in the number of 1∗clusters that are assigned to a special path P_j and, therefore, merged by filling P_j. A first (fruitless) attempt would be to consider the number of 1∗clusters ∗adjacent to P_j. This would imply that $\sum_{1\leq j\leq N} B(P_j)$ 1∗clusters are joined by filling one special path after another. But since one can find 1∗clusters ∗adjacent to several special paths (e.g. Q_1), the real number of 1∗clusters $B(P)$ merged by filling P is smaller than this sum. Consequently, this approach overestimates the number of merged 1∗clusters. The lesson of this is that we have to assign each 1∗cluster ∗adjacent to P to a special path such that each 1∗cluster is only considered once.

Definition 5.29 (assignment of 1∗clusters) *A 1∗cluster ∗adjacent to the special paths $P_{j_1}, \ldots, P_{j_k}$ is assigned to $P_{\max\{j_1,\ldots,j_k\}}$.*

Lemma 5.25 implies that no more than $|P_i| + 2$ disjoint 1∗clusters can be ∗adjacent to a special path P_i. In other words, there are at most $|P_i| + 2$ disjoint 1∗clusters ∗adjacent to P_i. Moreover, for $j \neq N$, the 1∗cluster containing Q_j is always ∗adjacent to P_{j+1} and, therefore, is never assigned to P_j. Consequently, the maximal possible number of 1∗clusters assigned to the special path P_j is $|P_j|+1$. As for the excluded case $j = N$, the maximal possible number of 1∗clusters assigned to P_N is $|P_N| + 2$.

Definition 5.30 (bad path) *We call a special path P_i with $i \neq N$ a bad path if $|P_i| + 1$ disjoint 1∗clusters are assigned to it.*

Note that $|P_i| + 2$ disjoint 1∗clusters are ∗adjacent to a bad path P_i and, therefore, P_i forms a straight line (vide Lemma 5.25). Furthermore, since filling a bad path joins more assigned 1∗clusters than it adds 1spins and the number of bad paths is not known, up to now we were not able to control the probability for any activity in $[2, 2^{4/3}[$.

Due to the choice of π at least one special path is bad. Without loss of generality we assume $P_i =: (p_1, \ldots, p_n)$ is bad[π], where the nodes of P_i are enumerated clockwise, and set $p_j^{\bullet} = p_{j+1}$ for all j with $1 \leq j \leq n-1$.

The following lemma has two main tasks: First, it shall identify the configuration in $\partial(p_1, p_2, p_3) \cup \partial(\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3)$. Second, it ensures that there is some "distance" between the bad path P_i and $Q \cup (P \setminus P_i)$. Later on, we need this "space" to alter the i-th special path without influencing the other special paths.

Lemma 5.31 *The following statements are true:*

a) The bad[π] induced path $P_i = (p_1, \ldots, p_n)$ consists of at least three nodes, i.e., $n \geq 3$.

b) The ∗boundary of P_i belongs to Λ, in short

$$\partial^* P_i \subset \Lambda\,. \tag{5.46}$$

c) *If the node* $\overset{\substack{\bullet\\\bullet}}{p_2}$ *belongs to* Λ *and is equipped with a* 0*spin, then the nodes* $\bullet\left(\overset{\substack{\bullet\\\bullet}}{p_1}\right), \overset{\substack{\bullet\\\bullet}}{p_1}, \overset{\substack{\bullet\\\bullet\\\bullet}}{p_1}$ *or the nodes* $\overset{\substack{\bullet\\\bullet\\\bullet}}{p_3}, \overset{\substack{\bullet\\\bullet}}{p_3}, \left(\overset{\substack{\bullet\\\bullet}}{p_3}\right)\bullet$ *are not contained in* Λ*, in short*

$$\overset{\substack{\bullet\\\bullet}}{p_2} \in \pi^{-1}(0) \cap \Lambda \Rightarrow \begin{cases} \bullet\left(\overset{\substack{\bullet\\\bullet}}{p_1}\right), \overset{\substack{\bullet\\\bullet}}{p_1}, \overset{\substack{\bullet\\\bullet\\\bullet}}{p_1} \in \Lambda^c \\ \qquad\qquad \text{or} \\ \overset{\substack{\bullet\\\bullet\\\bullet}}{p_3}, \overset{\substack{\bullet\\\bullet}}{p_3}, \left(\overset{\substack{\bullet\\\bullet}}{p_3}\right)\bullet \in \Lambda^c\,. \end{cases} \tag{5.47}$$

Moreover, the spin values of $\partial(p_1, p_2, p_3) \cup \partial(\overset{\bullet}{p_1}, \overset{\bullet}{p_2}, \overset{\bullet}{p_3})$ *are illustrated in Figure 5.8.*

d) *The union* Q *of all fixed* 1*paths is not adjacent to the nodes* $\overset{\bullet}{p_1}$*,* $\overset{\bullet}{p_2}$*, or* $\overset{\bullet}{p_3}$*, in short*

$$Q(\pi) \cap \partial(\overset{\bullet}{p_1}, \overset{\bullet}{p_2}, \overset{\bullet}{p_3}) = \emptyset\,. \tag{5.48}$$

e) *No special path except* P_i *meets* $\partial(p_1, p_2, p_3) \cup \partial(\overset{\bullet}{p_1}, \overset{\bullet}{p_2}, \overset{\bullet}{p_3})$*, in short*

$$(P \setminus P_i) \cap \Big(\partial(p_1, p_2, p_3) \cup \partial(\overset{\bullet}{p_1}, \overset{\bullet}{p_2}, \overset{\bullet}{p_3})\Big) = \emptyset\,. \tag{5.49}$$

Proof: Our strategy consists of three steps: First, we state some direct consequences of Lemma 5.25, which prove Lemma 5.31 a) and b). Next, Implication (5.47) and, therefore, Figure 5.8 will be verified. Last, we turn to the Identities (5.48) and (5.49).

First Step: First of all, the path P_i is bad and, therefore, $|P_i| + 2$ disjoint $1*$clusters are $*$adjacent to P_i. Hence, by definitions of P_i, all assumptions of Lemma 5.25 are satisfied and we can state some immediate consequences:

i) The length of the path P_i is at least three, i.e., $n \geq 3$;

ii) The path P_i forms a straight line, i.e., for all j with $1 \leq j \leq n$

$$p_j = p_1 + (j-1)(p_2 - p_1)\,;$$

iii) The configuration in $(p_1, p_2, p_3) \cup \partial^*(p_1, p_2, p_3) \setminus ({}_{\bullet}p_1 \cup p_{3\bullet})$ is known. More precisely, if we set ${\bullet}p_n = p_{n-1}$ and $p_j{\bullet} = p_{j+1}$ for $1 \leq j < n$ then it is the case that

$$\begin{aligned} &{}^{\bullet}p_1, \underset{\bullet}{p_1}, \overset{\bullet}{p_2}, \underset{\bullet}{p_3}, \overset{\bullet}{p_4}, \dots, \underset{\bullet}{p_{n-3}}, \overset{\bullet}{p_{n-2}}, \underset{\bullet}{p_{n-1}}, p_{n-1}{}^{\bullet} \subset \pi^{-1}(1) \\ &{\bullet}p_1, \overset{\bullet}{p_1}, \underset{\bullet}{p_2}, \overset{\bullet}{p_3}, \dots, \underset{\bullet}{p_{n-4}}, \overset{\bullet}{p_{n-3}}, \underset{\bullet}{p_{n-2}}, \overset{\bullet}{p_{n-1}}, p_{n-1}{\bullet} \subset \pi^{-1}(0) \end{aligned},$$

which is illustrated in Figure 5.8;

iv) Each node of $\pi^{-1}(1) \cap \partial^* P \setminus ({}_{\bullet}p_1 \cup p_{n\bullet})$ is contained in a different 1∗clusters. In other words, the nodes of $\pi^{-1}(1) \cap \partial^* P \setminus ({}_{\bullet}p_1 \cup p_{n\bullet})$ are contained in disjoint 1∗clusters. In particular, the nodes ${}^{\bullet}p_1$, $\underset{\bullet}{p_1}$, $p_3{}^{\bullet}$, and $\underset{\bullet}{p_3}$ are not 1∗connected and the nodes $\overset{\bullet}{\overset{\bullet}{p_1}}$ and $\overset{\bullet}{\overset{\bullet}{p_3}}$ are equipped with 0spins if they belong to Λ at all.

Next we show that a further consequence of Lemma 5.25 is $\partial^* P_i \subset \Lambda$: On the one side, by the clockwise enumeration, the nodes

$${\bullet}p_1, {}_{\bullet}p_1, \underset{\bullet}{p_2}, \dots, \underset{\bullet}{p_n}, p_{n\bullet}, p_n{\bullet}$$

are contained in $C^{\max 0}(\pi) \cup \mathrm{int} C^{\max 0}(\pi)$ and, therefore, in Λ. On the other side, due to item iii) the nodes

$${}^{\bullet}p_1, \overset{\bullet}{p_2}, \overset{\bullet}{p_4}, \dots, \overset{\bullet}{p_{n-3}}, \overset{\bullet}{p_{n-1}}, p_n{}^{\bullet}$$

are contained in $\pi^{-1}(1)$, which is a subset of Λ. We still have to show that the remaining nodes

$$\overset{\bullet}{p_1}, \overset{\bullet}{p_3}, \dots, \overset{\bullet}{p_{n-2}}, \overset{\bullet}{p_n}$$

of $\partial^* P_i$ are also contained in Λ. To this end, note that each of these nodes is adjacent to a node in Λ, namely $p_1, p_3, \dots, p_{n-2}, p_n$. Hence, if $\overset{\bullet}{p_1}, \overset{\bullet}{p_3}, \dots, \overset{\bullet}{p_{n-2}}$, or $\overset{\bullet}{p_n}$ belongs to Λ^c it is contained in the circuit $\partial^*\Lambda$. But this is impossible, because these nodes are dead ends for paths in Λ^c, i.e., $\partial \overset{\bullet}{p_i} \cap \Lambda^c \subset \overset{\bullet}{\overset{\bullet}{p_i}}$ for $i = 1, 3, \dots, n$.

<u>Second Step:</u> The aim of this step is to prove Implication (5.47), which because of the item iii) and iv) of the first step confirms Figure 5.8.

Let us begin by proving that at least one of the two nodes $\overset{\bullet}{\overset{\bullet}{p_1}}$ and $\overset{\bullet}{\overset{\bullet}{p_3}}$ is not contained in Λ if the node $\overset{\bullet}{\overset{\bullet}{p_2}}$ belongs to Λ and is equipped with a 0spin: To this end, recall that by definition, the nodes p_1, p_2, and p_3 are numbered clockwise and contained in the maximal 0circuit. Now, assume for contradiction that the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ takes spin value 0 and that the nodes $\overset{\bullet}{\overset{\bullet}{p}}_1$, $\overset{\bullet}{\overset{\bullet}{p}}_2$, and $\overset{\bullet}{\overset{\bullet}{p}}_3$ belong to Λ. On the one hand, by the latter assumption and the fact $\partial^* P_i \subset \Lambda$ (proved in step one), the clockwise path

$$(p_1, \overset{\bullet}{p}_1, \overset{\bullet}{\overset{\bullet}{p}}_1, \overset{\bullet}{\overset{\bullet}{p}}_2, \overset{\bullet}{\overset{\bullet}{p}}_3, \overset{\bullet}{p}_3, p_3)$$

lies in Λ. It is also a 0path, which follows immediately from the statements iii) and iv) of step one and the assumption that the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ has 0spin. On the other hand, we can interpret $C_\Lambda^{\max 0} \setminus p_2$ as a counterclockwise 0path starting in p_1 and ending in p_3. The construction of both 0paths, together with the opposite algebraic sign of the winding numbers of both 0paths, guarantees the existence of a 0circuit in the union of both 0paths, whose interior contains the node p_2, a contradiction to the fact that p_2 belongs to the maximal 0circuit. Summing up, at least one of the two nodes $\overset{\bullet}{\overset{\bullet}{p}}_1$ and $\overset{\bullet}{\overset{\bullet}{p}}_3$ is not contained in Λ if the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ belongs to Λ and is equipped with a 0spin, see Figure 5.8.

Now, we are ready to conclude this step by proving Implication (5.47). To this end, for the remainder of this paragraph assume that $\overset{\bullet}{\overset{\bullet}{p}}_2$ is contained in Λ and equipped with a 0 spin. By this assumption and step one, the following two statements hold:

$$\{\overset{\bullet}{p}_1, \overset{\bullet}{\overset{\bullet}{p}}_2\} \subset \partial \overset{\bullet}{\overset{\bullet}{p}}_1 \cap \Lambda \tag{5.50}$$

$$\{\overset{\bullet}{p}_3, \overset{\bullet}{\overset{\bullet}{p}}_2\} \subset \partial \overset{\bullet}{\overset{\bullet}{p}}_3 \cap \Lambda \tag{5.51}$$

In particular, both nodes $\overset{\bullet}{\overset{\bullet}{p}}_1$ and $\overset{\bullet}{\overset{\bullet}{p}}_3$ are adjacent to a node in Λ, namely $\overset{\bullet}{\overset{\bullet}{p}}_2$. Hence, the statement of the last paragraph implies that at least one of the two nodes $\overset{\bullet}{\overset{\bullet}{p}}_1$ and $\overset{\bullet}{\overset{\bullet}{p}}_3$ belongs to the circuit $\partial^*\Lambda$. If $\overset{\bullet}{\overset{\bullet}{p}}_1$ belongs to the circuit $\partial^*\Lambda$ then because of (5.50) the circuit $\partial^*\Lambda$ has to hit $\bullet\left(\overset{\bullet}{\overset{\bullet}{p}}_1\right)$ and $\overset{\bullet}{\overset{\bullet}{\overset{\bullet}{p}}}_1$ before and after it hits $\overset{\bullet}{\overset{\bullet}{p}}_1$. More precisely, this is the case because the above observation (5.50) ensures that the nodes $\overset{\bullet}{p}_1$ and $\overset{\bullet}{\overset{\bullet}{p}}_2$ belong to Λ and, therefore, are never contained in the circuit

$\partial^*\Lambda$. Analogously, $\overset{\bullet}{\overset{\bullet}{p}}_3 \in \partial^*\Lambda$ and (5.51) implies $\overset{\bullet}{\overset{\bullet}{\overset{\bullet}{p}}}_3$, $\left(\overset{\bullet}{\overset{\bullet}{p}}_3\right)\bullet \in \partial^*\Lambda$. The Implication (5.47) follows.

<u>*Third Step:*</u> Next, we prove Identity (5.48): Recall that Q is equipped with 1spins and contained in $\mathrm{int}^{\mathrm{i}} C_\Lambda^{\max 0}(\pi)$. Consequently, the nodes in $\partial(\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3)$ that are equipped with 0spins, namely p_1, p_2, p_3, $\overset{\bullet}{\overset{\bullet}{p}}_3$, and $\overset{\bullet}{\overset{\bullet}{p}}_1$, cannot be contained in Q. Moreover, the node $\overset{\bullet}{\overset{\bullet}{p}}_2 \in \partial(\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3)$ adjacent to $\overset{\bullet}{p}_2 \in \mathrm{ext}^{\mathrm{i}} C_\Lambda^{\max 0}(\pi)$ cannot belong to $Q \subset \mathrm{int}^{\mathrm{i}} C_\Lambda^{\max 0}(\pi)$, either. If the node ${}^\bullet p_1$ was in Q it would belong to Q_{i-1}, see Figure 5.7, and, therefore, is weakly 1connected to the node p_1, which is contrary to observation iv) above. Accordingly, $p_3{}^\bullet \in Q$ implies $p_3{}^{\overset{\bullet}{\bullet}} \in Q_i$ and, therefore, is also contrary to the fourth observation above. This concludes the proof of the desired Identity (5.48).

Last, we verify Identity (5.49): Because of $\{p_1, p_2, p_3\} \cup \partial(p_1, p_2, p_3) \subset \partial P_i \cup P_i$ and ${}^\bullet p_1 \cup p_3{}^\bullet \subset \pi^{-1}(1)$

$$(P \setminus P_i) \cap \big(\partial(p_1, p_2, p_3) \cup \{p_1, p_2, p_3\} \cup {}^\bullet p_1 \cup p_3{}^\bullet\big) = \emptyset$$

holds.

The following three paragraphs are dedicated to verify $\overset{\bullet}{\overset{\bullet}{p}}_1 \notin P \setminus P_i$ by assuming the contrary $\overset{\bullet}{\overset{\bullet}{p}}_1 \in P \setminus P_i$. This assumption, together with $\overset{\bullet}{\overset{\bullet}{p}}_1 \xleftrightarrow{0} p_1$ in $\mathrm{ext}^{\mathrm{i}} C^{\mathrm{fill}}(\pi)$ by $\overset{\bullet}{p}_1$, implies that $\overset{\bullet}{\overset{\bullet}{p}}_1$ is the ending node of P_{i-1} or the starting node of P_{i+1}. Recall that we enumerated the nodes and paths clockwise, which, together with the location of p_2 in P_i, yields that $\overset{\bullet}{\overset{\bullet}{p}}_1$ is the ending node of P_{i-1}, whose boundary is 1connected to $\underset{\bullet}{p_1}$ in $\mathrm{int} C^{\max 0}(\pi)$ by Q_{i-1}. We distinguish two cases, namely $\left(\overset{\bullet}{\overset{\bullet}{p}}_1\right)\bullet = \overset{\bullet}{\overset{\bullet}{p}}_2 \in \pi^{-1}(1) \cup \Lambda^c$ and $\left(\overset{\bullet}{\overset{\bullet}{p}}_1\right)\bullet \in \pi^{-1}(0) \cap \Lambda$, and show that a contradiction can be derived from both case assumptions. Figure (5.8) may help the reader in the following:

<u>*First Case:*</u> Assume $\left(\overset{\bullet}{\overset{\bullet}{p}}_1\right)\bullet \in \pi^{-1}(1) \cup \Lambda^c$. The third observation of the first step, together with the fact that P_i is clockwise enumerated, ensures that the node $\overset{\bullet}{p}_2$ is equipped with a 1spin[π], see Figure 5.8, and belongs to $\mathrm{ext} C^{\max 0}(\pi)$. By

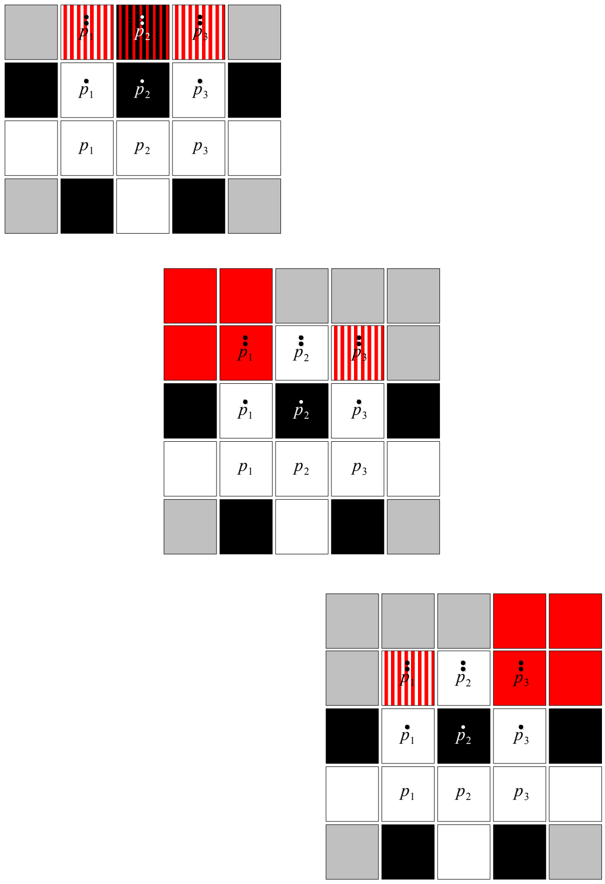

Figure 5.8: The upper figure illustrates the surroundings of the first three nodes of the bad path $P_i(\pi)$ if the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ belongs to $\pi^{-1}(1) \cup \Lambda^c$. The (non-exclusive) lower two figures illustrate the surroundings of the first three nodes of the bad path $P_i(\pi)$ if the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ belongs to $\pi^{-1}(0)$. Black squares are nodes equipped with 1spins. White squares are nodes with spin value 0. The spin value of the grey squares cannot be specified in general. Red squares are nodes that are contained in Λ^c. The striped red and black respectively white squares are either nodes with 1spin respectively 0spin or contained in Λ^c.

case assumption, the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ either has 1spin or does not lie in Λ at all. If it has 1spin then the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ belongs to $\mathrm{ext}C_\Lambda^{\max 0}(\pi)$, since it has 1spin and is adjacent to $\overset{\bullet}{p_2} \in \mathrm{ext}C_\Lambda^{\max 0}(\pi)$. Summing up, the node $\overset{\bullet}{\overset{\bullet}{p_2}}$ is contained in $\mathrm{ext}C_\Lambda^{\max 0}(\pi) \cup \Lambda^c$. Consequently, the nodes $\overset{\bullet}{\overset{\bullet}{\overset{\bullet}{p_1}}}$ and $\overset{\bullet}{p_1}$ have to be the predecessor and successor of $\overset{\bullet}{\overset{\bullet}{p_1}}$ in the maximal 0circuit and the node $\bullet\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)$ has to be the starting node of the 1path Q_{i-1} ending in $\underset{\bullet}{p_1}$. This is a contradiction to remark iv) above, which says "${}^{\bullet}p_1$ and $\underset{\bullet}{p_1}$ are not 1$*$connected".

<u>*Second Case:*</u> Assume $\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)\bullet \in \pi^{-1}(0)\cap\Lambda$. This, together with $\overset{\bullet}{\overset{\bullet}{p_1}} \in P\setminus P_i \subset \Lambda$ and our above Implication (5.47), implies $\overset{\bullet}{\overset{\bullet}{p_3}} \in \Lambda^c$. Consequently, there are two possible locations for the starting node of the 1path Q_{i-1}, namely $\bullet\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)$ and $\overset{\bullet}{\overset{\bullet}{\overset{\bullet}{p_1}}}$:

i) The 1path Q_{i-1} ending in $\underset{\bullet}{p_1}$ starts in $\bullet\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)$: As above, this is a contradiction to remark iv).

ii) The 1path Q_{i-1} ending in $\underset{\bullet}{p_1}$ starts in $\overset{\bullet}{\overset{\bullet}{\overset{\bullet}{p_1}}}$: An immediate consequence of this assumption is that the node $\overset{\bullet}{\overset{\bullet}{\overset{\bullet}{p_1}}}$ – as a part of Q – belongs to the interior of the maximal 0circuit. This, together with the facts that the maximal 0circuit is numbered clockwise and goes through $\overset{\bullet}{p_1}$ from $\overset{\bullet}{\overset{\bullet}{p_1}}$ to p_1, gives that the node $\bullet\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)$ also belongs to the interior of the maximal 0circuit. Summing up, both nodes $\overset{\bullet}{\overset{\bullet}{\overset{\bullet}{p_1}}}$ and $\bullet\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)$ belong to the interior of the maximal 0circuit. Therefore, because of $\overset{\bullet}{\overset{\bullet}{p_1}} \in \mathrm{int}C_\Lambda^{\max 0}$ the node $\overset{\bullet}{\overset{\bullet}{p_2}}$ is belongs to the maximal

0circuit. Moreover, because of $\ddot{p}_3 \in \Lambda^c$ and $\dot{p}_2 \in \pi^{-1}(1)$ the node $\dddot{p}_2$ also has to be contained in the maximal 0circuit. But, by definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$, the node $\ddot{p}_1$ cannot be the ending node of P_{i-1} as assumed, since $\dddot{p}_2$ belongs to the special path P_{i-1} and is also adjacent to Q_{i-1}.

The proof of $\ddot{p}_3 \notin P \setminus P_i$ is analogous.

It remains to prove $\ddot{p}_2 \notin P \setminus P_i$ and, therefore, Equality (5.49), which concludes the lemma. Let us assume the contrary, i.e., $\ddot{p}_2 \in P \setminus P_i$. Recall that we have already shown that $\ddot{p}_2$ is a dead end for special paths, i.e., $\partial(\ddot{p}_2) \cap (P \setminus P_i) \subset \dddot{p}_2$. Hence, it is the ending or starting node of a special path. Moreover, because of Implication (5.47) one of the nodes $\ddot{p}_1$ or $\ddot{p}_3$ lies in Λ^c; without loss of generality say $\ddot{p}_3$. This, together with $\dot{p}_2 \in \mathrm{ext}C^{\max 0}(\pi)$, implies that the remaining two nodes adjacent to $\ddot{p}_2$, namely $\dddot{p}_2$ and $\ddot{p}_1$, are the nodes in the maximal 0circuit[π] before and after $\ddot{p}_2$. Thus, there exists no node adjacent to $\ddot{p}_2$ that could be the starting or ending node of a fixed 1path, a contradiction to the fact that $\ddot{p}_2$ has to be the ending or starting node of a special path. ☐

As mentioned earlier, we want to find a configuration σ such that transforming π into $m(\sigma)$ and σ into $m(\pi)$ adds roughly speaking more 1spins than 1$*$clusters are merged. Our first step towards this is to define a configuration π' satisfying this condition for the i-th special path.

Before rigorously defining π' in the next proposition, we first describe some of its required properties to get a better understanding of its purpose: The configuration π' shall be a local modification of π, more precisely π' will be an element of A^1 and will coincide with π outside of $\{p_1, \dot{p}_1, \dot{p}_2, \dot{p}_3, p_3\}$, i.e.,

$$\pi' \in A^1 \tag{R1}$$

$$\{x \in \Lambda : \pi(x) \neq \pi'(x)\} \subset \{p_1, \dot{p}_1, \dot{p}_2, \dot{p}_3, p_3\} . \tag{R2}$$

Furthermore, ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi')$ should bypass p_2 using $\{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\}$, i.e.,

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi') = \{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \setminus p_2 \,. \tag{R3} $$

Note that because of (5.49) and (5.48) the right side of Equation (R3) is an induced circuit. Let $B_j(\pi)$ the number of 1∗clusters[π] assigned[π] to $P_j(\pi)$ and $B_j(\pi')$ the number of 1∗clusters[π] assigned[π'] to $P_j(\pi')$. By changing π to π' only the bad path $P_i(\pi)$ should be influenced, i.e.,

$$ P(\pi') = \bigcup_{j \neq i} P_j(\pi) \cup P_i(\pi') \tag{R4} $$

$$ B_j(\pi) = B_j(\pi') \qquad \forall j \neq i \,, \tag{R5} $$

where $P(\pi)$ was defined as $\bigcup_{1\leq j \leq n} P_j(\pi)$. For the next property define $m^i_{loc}(\pi)$ as the configuration $\mathbb{1}_{\pi^{-1}(1)\cup P_i(\pi)}$ and $m^i_{loc}(\pi')$ as the configuration $\mathbb{1}_{\pi'^{-1}(1)\cup P_i(\pi')}$. We can interpret $m^i_{loc}(.)$ as a map that fills the special path $P_i(.)$. If we change π to $m^i_{loc}(\pi')$, then we "add" more 1spins than we "join" assigned 1∗clusters, i.e.,

$$ |m^i_{loc}(\pi')^{-1}(1)| - |\pi^{-1}(1)| \geq B_i(\pi) = |P_i(\pi)| + 1 \,. \tag{R6} $$

If we change π' to $m^i_{loc}(\pi)$, then we "add" more 1spins than we "join" assigned 1∗clusters, i.e.,

$$ |m^i_{loc}(\pi)^{-1}(1)| - |\pi'^{-1}(1)| \geq B_i(\pi') \,. \tag{R7} $$

The last required property is that

$$ m^i_{loc}(\pi) \text{ and } m^i_{loc}(\pi') \text{ have the same number of 1}{*}\text{clusters} \,. \tag{R8} $$

The properties (R6), (R7), and (R8) confirm our intention to compare $m^i_{loc}(\pi)$ with π' and $m^i_{loc}(\pi')$ with π, since we "add" more 1spins than we join assigned 1∗clusters (at least) regarding the i-th special path.

Now, let us define π': To this end set $\pi'' := \mathbb{1}_{\pi^{-1}(1)\setminus \overset{\bullet}{p}_2}$.

Proposition 5.32 *Let*

$$ \pi' = \mathbb{1}_{\pi''^{-1}(1)\cup\{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\}\setminus (C^{\max 0}(\pi'')\cup \mathrm{ext}C^{\max 0}(\pi''))} \,. $$

This configuration satisfies

$$ \pi' = \begin{cases} \mathbb{1}_{\pi''^{-1}(1)} & \text{if } p_1, \overset{\bullet}{p}_1, p_3, , \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup p_1} & \text{if } p_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup p_3} & \text{if } p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup \{p_1, p_3\}} & \text{if } p_1, p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup \{p_1, \overset{\bullet}{p}_1\}} & \text{if } p_1, \overset{\bullet}{p}_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup \{\overset{\bullet}{p}_3, p_3\}} & \text{if } p_3, \overset{\bullet}{p}_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1 \notin \mathrm{int}C^{\max 0}(\pi'') \end{cases} \tag{5.52} $$

and the requirements (R1),..., (R8).

We demonstrate the change of the configuration π into π' in Figure 5.9 and Figure 5.10. The accuracy of the illustrations will be proved in Lemma (5.36).

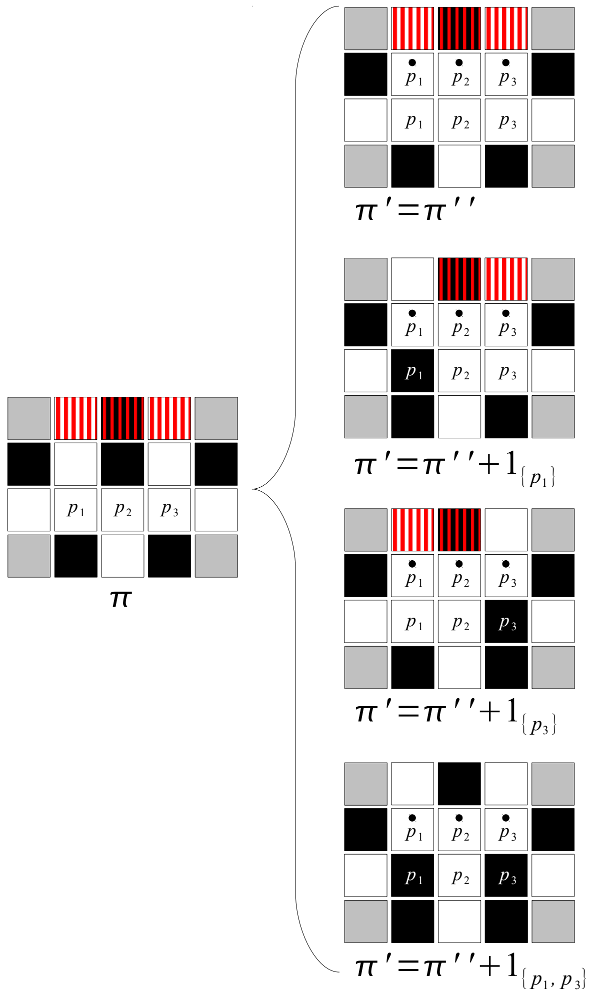

Figure 5.9: This figure illustrates the case $\overset{\bullet}{\overset{\bullet}{p}}_2 \in \pi^{-1}(1) \cup \Lambda^c$: The left side illustrates π. The right side illustrates the four possibilities of π'. Black squares are nodes with spin value 1. White squares are nodes with spin value 0. Grey squares are nodes, which will not concern us. The striped red and black respectively white squares are either nodes with 1spin respectively 0spin or contained in Λ^c

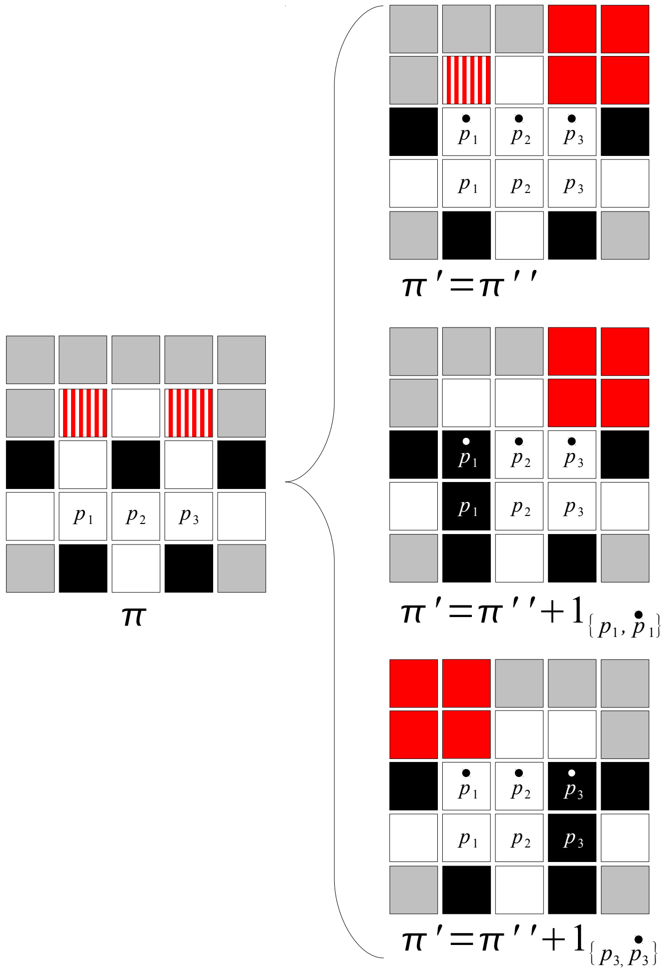

Figure 5.10: This figure illustrates the case $\dot{p}_2 \in \pi^{-1}(0) \cap \Lambda$: The left side illustrates π. The right side illustrates the three possibilities of π'. Black squares are nodes with spin value 1. White squares are nodes with spin value 0. Grey squares are nodes, in which we are not interested. Red squares are nodes in Λ^c. The striped red and black respectively white squares are either nodes with 1spin respectively 0spin or contained in Λ^c

For now we take this proposition for granted and verify it in the next subsection, since the proof is longish and technical. The downside of this is that until then we have to believe the statement after "on the other hand" in the next paragraph and convince ourselves of its correctness in the next subsection.

On the one hand, the configuration π' is uniquely determined by π, since by construction, π' is uniquely determined by π'' and π'' is uniquely determined by π. On the other hand (this is the part we have to believe in until we read the next subsection), π is uniquely determined by π', since the shape of $P_i(\pi') = R$ uniquely defines π (vide Corollary 5.40). So there exists a one-to-one correspondence between $P_i(\pi')$ and $P_i(\pi)$ and the following is well-defined:

Definition 5.33 (nullification path) *A nullification path is a special path like $P_i(\pi')$, which is the result of the transformation of a bad path $P_i(\pi)$, described in Proposition 5.32.*

The Inequality (R6) ensures that a bad path can never be a nullification path and vice versa.

Definition 5.34 (very special paths) *A very special path is a special path that is either a bad path or a nullification path.*

The above Properties (R2), (R3), (R4), and (R5) ensure that a nullification path can always be changed into a bad path (and vice versa) without influencing the other special paths. Hence, the configuration σ in the next paragraph is indeed well-defined.

The configuration π uniquely determines a corresponding configuration σ in the following way: All nullification[π] paths are changed into the corresponding bad[σ] paths and all bad[π] paths are changed into the corresponding nullification[σ] paths.

Note that π and σ could differ in more than one special path and all above Inequalities (R6), (R7), (R8), deal with configurations differing in only one special path. So, if we want to compare these configurations we need to "connect" them by configurations differing only in one special path: Let $1 \le i_1, \ldots, i_L \le N(\pi)$ the indices of the very special paths[π]. With the help of these indices we will inductively define $L+1$ configurations "connecting" π and σ. Let $\pi_0 := \pi$. We define π_j such that the only difference of π_j and π_{j-1} is the i_j-th special path: If $P_{i_j}(\pi_{j-1})$ is a bad[π_{j-1}] path, then $P_{i_j}(\pi_j)$ is a nullification[π_j] path. If $P_{i_j}(\pi_{j-1})$ is a nullification[π_{j-1}] path, then $P_{i_j}(\pi_j)$ is a bad[π_j] path. In particular $\pi_L = \sigma$.

Recall (vide the remark after Definition 5.29 on page 113)

$$B_N(\pi_0) \le |P_N(\pi_0)| + 2\,. \tag{5.53}$$

Next, we can use a telescope-argument:

$$\begin{aligned}
Z^{f*}_{\Lambda,\lambda}(\pi)\phi^{f*}_{\Lambda,\lambda}(\pi) &= \lambda^{|\pi^{-1}(1)|}2^{\kappa(\pi)}\\
&= \lambda^{|\pi_0^{-1}(1)|}2^{\kappa(\pi_0)-\sum_{1\le i\le N(\pi_0)} B_i(\pi_0)} \prod_{1\le i\le N(\pi_0)} 2^{B_i(\pi_0)}\\
&= \lambda^{|\pi_0^{-1}(1)|}2^{\kappa(m(\pi_0))} \prod_{1\le i\le N(\pi_0)} 2^{B_i(\pi_0)}\\
&\overset{(\mathrm{R8}),\lambda\ge 2}{\le} 2^{\kappa(m(\pi_L))}\lambda^{|\pi_0^{-1}(1)|+B_{i_1}(\pi_0)+\dots B_{i_L}(\pi_0)} \prod_{\substack{1\le j\le N(\pi_0):\\ j\ne\{i_1,\dots,i_L\}}} 2^{B_j(\pi_0)}\\
&\overset{(\mathrm{R6}),(\mathrm{R7})}{\le} 2^{\kappa(m(\sigma))}\lambda^{|P_{i_1}(\sigma)|+|\pi_1^{-1}(1)|+B_{i_2}(\pi_0)+\dots B_{i_L}(\pi_0)} \prod_{\substack{1\le j\le N(\pi_0):\\ j\ne\{i_1,\dots,i_L\}}} 2^{B_j(\pi_0)}\\
&\overset{(\mathrm{R6}),(\mathrm{R7})}{\le} \dots \le 2^{\kappa(m(\sigma))}\lambda^{|P_{i_1}(\sigma)|+\dots|P_{i_L}(\sigma)|+|\pi_L^{-1}(1)|} \prod_{\substack{1\le j\le N(\pi_0):\\ j\ne\{i_1,\dots,i_L\}}} 2^{B_j(\pi_0)}\\
&\overset{\lambda\ge 2}{\le} 2^{\kappa(m(\sigma))}\lambda^{|P_1(\sigma)|+\dots+|P_{N-1}(\sigma)|+|\pi_L^{-1}(1)|}2^{B_N(\pi_0)}\\
&\overset{(5.53)}{\le} 2^{\kappa(m(\sigma))}\lambda^{|P_1(\sigma)|+\dots+|P_{N-1}(\sigma)|+|\pi_L^{-1}(1)|}2^{P_N(\pi_0)+2}\\
&\overset{\lambda\ge 2}{\le} 2^{\kappa(m(\sigma))}\lambda^{|m(\sigma)^{-1}(1)|}2^2\\
&= Z^{f*}_{\Lambda,\lambda}(m(\sigma))4\phi^{f*}_{\Lambda,\lambda}(m(\sigma))\,.
\end{aligned}$$

Summing up, for all $\lambda \ge 2$,

$$\phi^{f*}_{\Lambda,\lambda}(\pi) \le 4\phi^{f*}_{\Lambda,\lambda}(m(\sigma)) \tag{5.54}$$

holds. For all $\lambda \ge 2$,

$$\phi^{f*}_{\Lambda,\lambda}(\sigma) \le 4\phi^{f*}_{\Lambda,\lambda}(m(\pi)) \tag{5.55}$$

can be derived in the same way. This, together with (5.54), yields

$$\phi^{f*}_{\Lambda,\lambda}(\pi) + \phi^{f*}_{\Lambda,\lambda}(\sigma) \le 4\phi^{f*}_{\Lambda,\lambda}(m(\sigma)) + 4\phi^{f*}_{\Lambda,\lambda}(m(\pi)) \qquad \forall\lambda \ge 2\,. \tag{5.56}$$

Now we are ready to prove the next theorem.

Theorem 5.35 *Let $\lambda \ge 2$ and $\Lambda \Subset \mathbb{Z}^2$ so that $\partial^*\Lambda$ can be interpreted as a circuit. Then eight times the $\phi^{f*}_{\Lambda,\lambda}$-probability of the set $\{\exists\ 1{*}\mathrm{lasso}\}$ is larger than the $\phi^{f*}_{\Lambda,\lambda}$-probability of the set $\{\exists\ 0\mathrm{lasso}\}$, i.e.,*

$$\phi^{f*}_{\Lambda,\lambda}(\exists\ 0\mathrm{lasso}) \le 8\phi^{f*}_{\Lambda,\lambda}(\exists\ 1{*}\mathrm{lasso})\,.$$

Proof: We split A^1 into three subsets A^1_1, A^1_2, and A^1_3: Let A^1_1 the set of configurations without a very special path. This definition immediately implies that there exists no bad path in A^1_1. Consequently, there are at most $|P|+2$ $1{*}$clusters ${*}$adjacent to P and, therefore, for all $\lambda \geq 2$

$$\phi^{f*}_{\Lambda,\lambda}(A^1_1) \leq 4\phi^{f*}_{\Lambda,\lambda}(m(A^1_1)) \tag{5.57}$$

holds. Let A^1_2 the set of configurations such that the very special path with the lowest index is a bad path. Let A^1_3 the set of configurations such that the very special path with the lowest index is a nullification path.

Let g be the bijective map from A^1_2 to A^1_3 changing all bad paths to nullification paths and all nullification paths to bad paths. So, for all $\lambda \geq 2$

$$\begin{aligned}
\phi^{f*}_{\Lambda,\lambda}(\exists 0\text{lasso}) &= \phi^{f*}_{\Lambda,\lambda}(A_0) + \phi^{f*}_{\Lambda,\lambda}(A_1) + \phi^{f*}_{\Lambda,\lambda}(A_2) \\
&= \phi^{f*}_{\Lambda,\lambda}(A_0 \cup A_2) + \phi^{f*}_{\Lambda,\lambda}(A^1_1) + \phi^{f*}_{\Lambda,\lambda}(A^1_2) + \phi^{f*}_{\Lambda,\lambda}(A^1_3) \\
&\overset{(5.42)}{\leq} 8\phi^{f*}_{\Lambda,\lambda}(m(A_0 \cup A_2)) + \phi^{f*}_{\Lambda,\lambda}(A^1_1) + \sum_{\xi \in A^1_2} \Big(\phi^{f*}_{\Lambda,\lambda}(\xi) + \phi^{f*}_{\Lambda,\lambda}(g(\xi))\Big) \\
&\overset{(5.57)}{\leq} 8\phi^{f*}_{\Lambda,\lambda}(m(A_0 \cup A_2)) + 4\phi^{f*}_{\Lambda,\lambda}(m(A^1_1)) + \sum_{\xi \in A^1_2} \Big(\phi^{f*}_{\Lambda,\lambda}(\xi) + \phi^{f*}_{\Lambda,\lambda}(g(\xi))\Big) \\
&\overset{(5.56)}{\leq} 8\phi^{f*}_{\Lambda,\lambda}(m(A_0 \cup A_2 \cup A^1_1)) + 4\sum_{\xi \in A^1_2} \Big(\phi^{f*}_{\Lambda,\lambda}(m(\xi)) + \phi^{f*}_{\Lambda,\lambda}(m(g(\xi)))\Big) \\
&= 8\phi^{f*}_{\Lambda,\lambda}(m(A_0 \cup A_2 \cup A^1_1)) + 4\phi^{f*}_{\Lambda,\lambda}(m(A^1_2 \cup A^1_3)) \\
&\leq 8\phi^{f*}_{\Lambda,\lambda}(\exists 1{*}\text{lasso})
\end{aligned}$$

holds, where the last inequality is a consequence of $m(\exists\ 0\text{lasso}) \subset \{\exists\ 1{*}\text{lasso}\}$. This concludes the proof of the theorem. □

5.3.2 The Proof of Theorem 1.1

Finally we are ready to prove the main result of this thesis:

Theorem 1.1 *Let* $\lambda \geq 2$. *Then* $\mathrm{WR}^*_{\mathrm{ER}}(\lambda) = \{\mu^{+*}_\lambda, \mu^{-*}_\lambda\}$.

Proof: Theorem 5.35, together with Lemma 4.28, Theorem 4.31, and Remark 4.23, verifies Theorem 1.1. □

Nonetheless, we still have to verify Proposition 5.32.

5.3.3 The Proof of Proposition 5.32

First of all, we recall the statement of the proposition. We set $\pi'' := \mathbb{1}_{\pi^{-1}(1)\setminus \overset{\bullet}{p}_2}$.

Proposition 5.32 *Let*

$$\pi' = \mathbb{1}_{\pi''^{-1}(1)\cup\{p_1,\overset{\bullet}{p}_1,p_3,\overset{\bullet}{p}_3\}\setminus(C^{\max 0}(\pi'')\cup \mathrm{ext}C^{\max 0}(\pi''))}\,.$$

This configuration satisfies

$$\pi' = \begin{cases} \mathbb{1}_{\pi''^{-1}(1)} & \text{if } p_1,\overset{\bullet}{p}_1,p_3,,\overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup p_1} & \text{if } p_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1,p_3,\overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup p_3} & \text{if } p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1,\overset{\bullet}{p}_1,\overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup\{p_1,p_3\}} & \text{if } p_1,p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1,\overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup\{p_1,\overset{\bullet}{p}_1\}} & \text{if } p_1,\overset{\bullet}{p}_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_3,\overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \mathbb{1}_{\pi''^{-1}(1)\cup\{\overset{\bullet}{p}_3,p_3\}} & \text{if } p_3,\overset{\bullet}{p}_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1,\overset{\bullet}{p}_1 \notin \mathrm{int}C^{\max 0}(\pi'') \end{cases} \tag{(5.52)}$$

and the Requirements (R1),..., (R8).

We illustrated the change of the configuration π into the configuration π' in Figure 5.9 and Figure 5.10.

First of all, let us recall some basic facts: We called $C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi)$ the "second largest" 0circuit[π] and defined it as the maximal 0circuit[π] in $\mathrm{int}C^{\max 0}_\Lambda(\pi)$. Because of $\pi \in A^1$ the "second largest" 0circuit[π] is 0∗connected to the maximal 0circuit[π], i.e.,

$$C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi) \overset{0*}{\longleftrightarrow} C^{\max 0}_\Lambda(\pi)\,.$$

By definition, ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ is contained in the "half-open" annulus

$$\Big]C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi), C^{\max 0}_\Lambda(\pi)\Big] := \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi) \cap \mathrm{int}C^{\max 0}_\Lambda(\pi) \cup C^{\max 0}_\Lambda(\pi)$$

and its nodes are equipped with 1spins[π] if and only if they belong to the "open" annulus

$$\Big]C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi), C^{\max 0}_\Lambda(\pi)\Big[:= \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}_\Lambda(\pi)}(\pi) \cap \mathrm{int}C^{\max 0}_\Lambda(\pi)\,.$$

Moreover, by definition each node of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ is ∗weakly 0∗connected to the "second largest" 0cicuit[π] in $\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$. The first three nodes of P_i, namely p_1, p_2, and

p_3, are contained in $C^{\max 0}_{\Lambda}(\pi)$, the node $\overset{\bullet}{p}_2$ belongs to $\text{ext}C^{\max 0}_{\Lambda}(\pi)$, and the path $(\underset{\bullet}{p_1}, \underset{\bullet}{p_2}, \underset{\bullet}{p_3})$ is contained in the "open" annulus

$$\Big] C^{\max 0}_{\text{int}C^{\max 0}_{\Lambda}(\pi)}(\pi), C^{\max 0}_{\Lambda}(\pi) \Big[\,.$$

The latter statement follows from the clockwise enumeration and $\{\underset{\bullet}{p_1}, \underset{\bullet}{p_3}\} \subset \pi^{-1}(1)$ and $\{p_1, p_2, p_3\} \subset C^{\max 0}_{\Lambda}(\pi)$.

The rest of this subsection is dedicated to prove the proposition. For convenience, we divide the proof into a sequence of 8 lemmas.

First of all, we note that π' obviously satisfies

$$\{x \in \Lambda : \pi(x) \neq \pi'(x)\} \subset \{p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3\} \,. \tag{R2}$$

Furthermore,

$$C^{\max 0}(\pi') = C^{\max 0}(\pi'') \tag{5.58}$$

follows from $C^{\max 0}(\pi'') \subset \pi'^{-1}(0) \subset \pi''^{-1}(0)$, where the first inclusion implies "$\geq$" in (5.58) and the second one implies "$\leq$".

Now we tend to verify Identity (5.52) and, hereby, establish some intuition.

Lemma 5.36 *The configuration π' can be described in the following way:*

$$\pi' = \begin{cases} \mathbb{1}_{\pi''^{-1}(1)} & \text{if } p_1, \overset{\bullet}{p}_1, p_3, , \overset{\bullet}{p}_3 \notin \text{int}C^{\max 0}(\pi''); \\ \mathbb{1}_{\pi''^{-1}(1) \cup p_1} & \text{if } p_1 \in \text{int}C^{\max 0}(\pi''), \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3 \notin \text{int}C^{\max 0}(\pi''); \\ \mathbb{1}_{\pi''^{-1}(1) \cup p_3} & \text{if } p_3 \in \text{int}C^{\max 0}(\pi''), p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \text{int}C^{\max 0}(\pi''); \\ \mathbb{1}_{\pi''^{-1}(1) \cup \{p_1, p_3\}} & \text{if } p_1, p_3 \in \text{int}C^{\max 0}(\pi''), \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \text{int}C^{\max 0}(\pi''); \\ \mathbb{1}_{\pi''^{-1}(1) \cup \{p_1, \overset{\bullet}{p}_1\}} & \text{if } p_1, \overset{\bullet}{p}_1 \in \text{int}C^{\max 0}(\pi''), p_3, \overset{\bullet}{p}_3 \notin \text{int}C^{\max 0}(\pi''); \\ \mathbb{1}_{\pi''^{-1}(1) \cup \{\overset{\bullet}{p}_3, p_3\}} & \text{if } p_3, \overset{\bullet}{p}_3 \in \text{int}C^{\max 0}(\pi''), p_1, \overset{\bullet}{p}_1 \notin \text{int}C^{\max 0}(\pi''). \end{cases} \tag{(5.52)}$$

Moreover, we know that for $i = 1, 3$,

$$p_i \in \text{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_i \notin \text{int}C^{\max 0}(\pi'') \Rightarrow \overset{\bullet}{\overset{\bullet}{p}}_2 \in \pi^{-1}(1) \cup \Lambda^c \tag{5.59}$$

$$\overset{\bullet}{p}_i \in \text{int}C^{\max 0}(\pi'') \Rightarrow \overset{\bullet}{\overset{\bullet}{p}}_2 \in \pi^{-1}(0) \cap \Lambda \,. \tag{5.60}$$

In particular, the Illustrations 5.9 and 5.10 are correct.

Proof: First of all, note that there are $2^4 = 16$ cases which nodes of $\{p_1, \overset{\bullet}{p}_1, p_3, , \overset{\bullet}{p}_3\}$ belong to $\mathrm{int}C^{\max 0}(\pi'')$. Our strategy consists of three steps: First, we show that for $i = 1, 3$, $\overset{\bullet}{p}_i \in \mathrm{int}C^{\max 0}(\pi'')$ implies $p_i \in \mathrm{int}C^{\max 0}(\pi'')$ and, therefore, seven cases are excluded. Second, for $\{j, i\} = \{1, 3\}$, we derive $p_i \notin \mathrm{int}C^{\max 0}(\pi'')$ from $\overset{\bullet}{p}_j \in \mathrm{int}C^{\max 0}(\pi'')$, which excludes another three cases. A side product is the proof of Implication (5.60). Note that these two steps already verify Identity (5.52). Our last step shows our remaining Addendum (5.59).

<u>*First Step:*</u> Because of $\pi^{-1}(0) \subset \pi''^{-1}(0)$ the maximal 0circuit$[\pi'']$ is larger than the maximal 0circuit$[\pi]$, i.e.,

$$C^{\max 0}(\pi) \leq C^{\max 0}(\pi'')\,.$$

Note that the nodes $\overset{\bullet}{p}_1$ and $\overset{\bullet}{p}_3$ belong to $C^{\max 0}(\pi) \cup \mathrm{ext}C^{\max 0}(\pi)$. So, if the node $\overset{\bullet}{p}_1$ respectively $\overset{\bullet}{p}_3$ is contained in $\mathrm{int}C^{\max 0}(\pi'')$ then p_1 respectively p_3, being a node of $C^{\max 0}(\pi)$, also belongs to $\mathrm{int}C^{\max 0}(\pi'')$.

<u>*Second Step:*</u> For the second step it is sufficient to show the following two implications:

i) $p_3 \notin \mathrm{int}C^{\max 0}(\pi'')$ follows from $p_1, \overset{\bullet}{p}_1 \in \mathrm{int}C^{\max 0}(\pi'')$;

ii) $p_1 \notin \mathrm{int}C^{\max 0}(\pi'')$ is a consequence of $p_3, \overset{\bullet}{p}_3 \in \mathrm{int}C^{\max 0}(\pi'')$.

Both implications can be shown analogously and, therefore, we only prove the first one in the sequel. Furthermore, as we will see, the proof of i) verifies Implication (5.60) for $i = 1$ and, moreover, the proof of ii) would analogously verify Implication (5.60) for $i = 3$.

If the node $\overset{\bullet}{\overset{\bullet}{p}}_3$ is contained in Λ^c then the nodes $p_3, \overset{\bullet}{p}_3$ are $*$weakly 1$*$connected$[\pi, \pi'']$ to Λ^c by $p_3{}^{\bullet}$, which immediately gives us $p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'')$. So, it is sufficient to show the following:

i') A consequence of $p_1, \overset{\bullet}{p}_1 \in \mathrm{int}C^{\max 0}(\pi'')$ is $\overset{\bullet}{\overset{\bullet}{p}}_3 \in \Lambda^c$.

Because the node p_2 belongs to the maximal 0circuit$[\pi]$, there exists a 1$*$path$[\pi]$ in $\mathrm{ext}C^{\max 0}(\pi)$ starting $*$adjacent to p_2 and ending $*$adjacent to Λ^c. The only possible starting node for this 1$*$path$[\pi]$ is $\overset{\bullet}{p}_2 = \partial^* p_2 \cap \mathrm{ext}C^{\max 0}(\pi) \cap \pi^{-1}(1)$. Further note that $\overset{\bullet}{\overset{\bullet}{p}}_2$ is the only node that could have a 1spin$[\pi]$ and is $*$adjacent to $\overset{\bullet}{p}_2$. These observations, together with $\{x \in \Lambda : \pi(x) \neq \pi''(x)\} = \overset{\bullet}{p}_2$, implies that one of the following three scenarios has to occur:

a) At least one of the nodes $\overset{\bullet}{\overset{\bullet}{p}}_1$ and $\overset{\bullet}{\overset{\bullet}{p}}_2$ is contained in Λ^c;

b) The node $\overset{\bullet}{\overset{\bullet}{p}}_2$ is the starting node of a $1{*}\text{path}[\pi,\pi'']$ ending $*$adjacent to Λ^c;

c) The node $\overset{\bullet}{\overset{\bullet}{p}}_3$ is contained in Λ^c.

From now on assume $p_1, \overset{\bullet}{p}_1 \in \text{int}C^{\max 0}(\pi'')$: The first two scenarios a) and b) are impossible, since in both scenarios the node $\overset{\bullet}{p}_1$, which is contained in $\text{int}C^{\max 0}(\pi'')$, is $*$weakly $1*$connected$[\pi'']$ to Λ^c, a contradiction. Thus, the third scenario has to occur and we have verified i'). Moreover, we have also proved Implication (5.60) for $i = 1$ by precluding scenarios a) and b) under the assumption $p_1, \overset{\bullet}{p}_1 \in \text{int}C^{\max 0}(\pi'')$.

<u>*Third Step:*</u> We now turn to show Implication (5.59) for $i = 1$, since in the other case $i = 3$ the implication can be proved analogously. To this end, for contradiction assume that $p_1 \in \text{int}C^{\max 0}(\pi'')$, $\overset{\bullet}{p}_1 \notin \text{int}C^{\max 0}(\pi'')$, and $\overset{\bullet}{\overset{\bullet}{p}}_2 \in \pi^{-1}(0) \cap \Lambda$. By definition of π' and $p_1 \in C^{\max 0}(\pi) \leq C^{\max 0}(\pi'')$, the first two assumptions are equivalent to $p_1 \in \pi'^{-1}(1)$ and

$$\overset{\bullet}{p}_1 \in C^{\max 0}(\pi'') = C^{\max 0}(\pi') \subset \pi'^{-1}(0)\,. \tag{5.61}$$

Moreover, the nodes $\overset{\bullet}{p}_2$ and $\overset{\bullet}{\overset{\bullet}{p}}_1$ have to be the nodes before and after $\overset{\bullet}{p}_1$ in the maximal 0circuit$[\pi'',\pi']$. This is the case because the other two nodes ${}^{\bullet}p_1$ and p_1 adjacent to $\overset{\bullet}{p}_1$ are equipped with 1spins$[\pi'',\pi']$, i.e., $\partial\overset{\bullet}{p}_1 \cap \pi''^{-1}(0) = \{\overset{\bullet}{\overset{\bullet}{p}}_1, \overset{\bullet}{p}_2\}$. But since the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ is also equipped with a 0spin, belongs to Λ, and is adjacent to the nodes $\overset{\bullet}{p}_2$ and $\overset{\bullet}{\overset{\bullet}{p}}_1$, the maximal 0circuit$[\pi'',\pi']$ would go through $\overset{\bullet}{\overset{\bullet}{p}}_2$ from $\overset{\bullet}{p}_2$ to $\overset{\bullet}{\overset{\bullet}{p}}_1$ instead of going through $\overset{\bullet}{p}_1$ from $\overset{\bullet}{p}_2$ to $\overset{\bullet}{\overset{\bullet}{p}}_1$. This is the case because this way the node $\overset{\bullet}{p}_1$ belongs to the interior of the maximal 0circuit$[\pi'',\pi']$ instead of being part of it. Hence, we know that the node $\overset{\bullet}{p}_1$ does not belong to $C^{\max 0}(\pi'')$, which contradicts (5.61). Implication (5.59) for $i = 1$ follows, which concludes the proof. □

One of our aims is to show

$${}^{\mathrm{i}}C^{\text{fill}}(\pi') = \{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup {}^{\mathrm{i}}C^{\text{fill}}(\pi) \setminus p_2 \tag{R3}$$

and an important building block of the definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}$ is the maximal 0circuit. Consequently, it will be useful that a great part of the maximal 0circuit[π] is contained in the maximal 0circuit[π', π'']. To this end and because we need the following set a lot, we define it as

$$K := \overset{\bullet}{p}_2 \cup C^{\max 0}(\pi) \setminus \partial^* \overset{\bullet}{p}_2 \,.$$

Note that the node $\overset{\bullet}{p}_2$ does not belong to $\partial^* \overset{\bullet}{p}_2$.

Lemma 5.37 *The set K is contained in the maximal* 0*circuit*[π'', π'], *in short*

$$K \subset C^{\max 0}(\pi'') = C^{\max 0}(\pi') \,. \tag{5.62}$$

Proof: Because of (5.58) only

$$K \subset C^{\max 0}(\pi'')$$

remains to be shown for (5.62). This will done by verifying the following two conditions:

i) All nodes of K are $*$weakly 1$*$connected[π''] to Λ^c;

ii) K is contained in a 0circuit[π''].

Note that the second statement does not demand a 0circuit[π'].

Let us begin with i): It may help the reader to look at Figures 5.9 and 5.10 while reading the following paragraph, whose aim is to show that the node $\overset{\bullet}{p}_2$ is $*$weakly 1$*$connected[π''] to Λ^c.

Let us assume without loss of generality that the nodes $\overset{\bullet}{\overset{\bullet}{p}}_1$, $\overset{\bullet}{\overset{\bullet}{p}}_2$, and $\overset{\bullet}{\overset{\bullet}{p}}_3$ belong to Λ; otherwise our aim is evident. Note that p_2, as a node of $C^{\max 0}(\pi)$, is $*$weakly 1$*$connected[π] to Λ^c. Consequently, there exists a 1$*$path[π] in ext$C^{\max 0}(\pi)$ starting $*$adjacent to p_2 and ending $*$adjacent to Λ^c. The only possible starting node for this 1$*$path[π] is $\overset{\bullet}{p}_2 = \partial^* p_2 \cap \mathrm{ext} C^{\max 0}(\pi) \cap \pi^{-1}(1)$. Further, since the only node equipped with a 1spin[π] $*$adjacent to $\overset{\bullet}{p}_2$ is $\overset{\bullet}{\overset{\bullet}{p}}_2$, this node $\overset{\bullet}{\overset{\bullet}{p}}_2$ is also a starting node of a 1$*$path[π] in ext$C^{\max 0}(\pi) \setminus \overset{\bullet}{p}_2$ to $\partial^*(\Lambda^c)$. This, together with the fact that the configurations π and π'' coincide off $\overset{\bullet}{p}_2$, implies that the node $\overset{\bullet}{\overset{\bullet}{p}}_2$ is a starting node of a 1$*$path[π''] to $\partial^*(\Lambda^c)$. Consequently, the node $\overset{\bullet}{p}_2$ is $*$weakly 1$*$connected[π''] to Λ^c.

For i) we still have to show that $K \setminus \overset{\bullet}{p}_2$ is $*$weakly 1$*$connected[π''] to Λ^c. But this is the case because of the following four observations:

a) By definition, $K\setminus\overset{\bullet}{p}_2 = C^{\max 0}(\pi)\setminus\partial^*\overset{\bullet}{p}_2 = C^{\max 0}(\pi)\setminus(\partial^*\overset{\bullet}{p}_2\cup\overset{\bullet}{p}_2)$. Therefore, each node of $K\setminus\overset{\bullet}{p}_2$ is $*$weakly $1*$connected$[\pi]$ to Λ^c by a $1*$path$[\pi]$ not starting in $\overset{\bullet}{p}_2$;

b) By Figures 5.9 and 5.10, the node $\overset{\bullet}{p}_2$ is a dead end for $1*$paths$[\pi]$, i.e., $\partial^*\overset{\bullet}{p}_2\cap\pi^{-1}(1)\subset\overset{\bullet}{\overset{\bullet}{p}}_2$. Thus, if a $1*$path$[\pi]$ of a) meets the node $\overset{\bullet}{p}_2$ then it ends in $\overset{\bullet}{p}_2$. Consequently, subtracting the node $\overset{\bullet}{p}_2$ from such a $1*$path$[\pi]$ always results in a $1*$path$[\pi]$ that ends in $\overset{\bullet}{\overset{\bullet}{p}}_2$;

c) By definition, the configuration π coincides with π'' off $\overset{\bullet}{p}_2$. Hence, a shortened $1*$path$[\pi]$ of b) is also a $1*$path$[\pi'']$;

d) Because of $\partial^*\overset{\bullet}{p}_2\cap\Lambda^c\subset\{\overset{\bullet}{\overset{\bullet}{p}}_1,\overset{\bullet}{\overset{\bullet}{p}}_2,\overset{\bullet}{\overset{\bullet}{p}}_3\}$, which follows from Lemma 5.31 b), these shortened $1*$paths$[\pi]$ ending in $\overset{\bullet}{\overset{\bullet}{p}}_2$ still end $*$adjacent to Λ^c.

The proof of condition ii) is a bit more involved, although it is straightforward to identify the 0circuit$[\pi'']$ containing K:

$$K' := K\cup\{\overset{\bullet}{p}_1,\overset{\bullet}{p}_3\}\cup p_1\mathbb{1}_{\overset{\bullet}{p}_1\notin C^{\max 0}(\pi)}\cup p_3\mathbb{1}_{\overset{\bullet}{p}_3\notin C^{\max 0}(\pi)},$$

where by misuse of notation

$$p_1\mathbb{1}_{\overset{\bullet}{p}_1\notin C^{\max 0}(\pi)} = \begin{cases} p_1 & \overset{\bullet}{p}_1\notin C^{\max 0}(\pi);\\ \emptyset & \text{otherwise.}\end{cases}$$

Obviously, $K'\subset\pi''^{-1}(0)$ holds. It remains to show that K' is a circuit: Because of $\underset{\bullet}{p_1}\subset\pi^{-1}(1)$ the node x in $C^{\max 0}(\pi)\setminus p_2$ successive to p_1 is either ${\bullet}p_1$ or $\overset{\bullet}{p}_1$. Analogously, the node y in $C^{\max 0}(\pi)\setminus p_2$ successive to p_3 is either $p_3{\bullet}$ or $\overset{\bullet}{p}_3$. On the one hand, there exists a (counterclockwise) path in $C^{\max 0}(\pi)\setminus\{p_1,p_2,p_3\}$ starting in x and ending in y, which never hits $\overset{\bullet}{p}_2$ and which could hit, if at all, the nodes $\overset{\bullet}{p}_1$ and $\overset{\bullet}{p}_3$ only in $\{x,y\}$. On the other hand, by case-by-case analysis, the path $\{x,y,\overset{\bullet}{p}_1,\overset{\bullet}{p}_2,\overset{\bullet}{p}_3\}\cup p_1\mathbb{1}_{\overset{\bullet}{p}_1\notin C^{\max 0}(\pi)}\cup p_3\mathbb{1}_{\overset{\bullet}{p}_3\notin C^{\max 0}(\pi)}$ is a (clockwise) path starting in x and ending in y, which hits the other (counterclockwise) path only in its starting and ending node. Hence, the union of both (clockwise and counterclockwise) paths is a circuit, see Lemma 3.10, and, by definition, equals K'. This implies the second condition. $\square$

Now (more precisely, because of $\overset{\bullet}{p}_2 \in C^{\max 0}(\pi''))$ we are ready to prove the first requirement:

Lemma 5.38 *The configuration π' of Proposition 5.32 satisfies*

$$\pi' \in A^1 \overset{\text{Def.}}{=} \{\exists \text{ 0lasso in } \Lambda, \vec{0} \notin C_\Lambda^{\max 0}, \partial^* C_\Lambda^{\max 0} \overset{1*}{\not\leftrightarrow} \partial^* C_\Lambda^{\max 1}\}\,. \tag{R1}$$

Proof: First, let us verify the existence of a 0lasso[π']. Because of $\pi^{-1}(0) \subset \pi''^{-1}(0)$ the maximal 0circuit[π''] is larger than the maximal 0circuit[π], i.e.,

$$C_\Lambda^{\max 0}(\pi'') \geq C_\Lambda^{\max 0}(\pi)\,. \tag{5.63}$$

Since we have chosen π so that there exists a 0lasso[π], the circuit $C_\Lambda^{\max 0}(\pi)$ is weakly 0connected[π] to Λ^c and, therefore, weakly 0connected[π''] to Λ^c. Further, π' coincides with π'' off int$C^{\max 0}(\pi'')$, see (5.52). Thus, the larger maximal 0circuit[π', π''] $C_\Lambda^{\max 0}(\pi'') \overset{(5.58)}{=} C_\Lambda^{\max 0}(\pi')$ is also weakly 0connected[π', π''] to Λ^c and we know that there exists a 0lasso[π'].

Second, the origin does not belong to $C_\Lambda^{\max 0}(\pi')$ because

$$\vec{0} \overset{\pi \in A^1}{\notin} C_\Lambda^{\max 0}(\pi) \overset{(5.63)}{\leq} C_\Lambda^{\max 0}(\pi'') \overset{(5.58)}{=} C_\Lambda^{\max 0}(\pi')\,.$$

At last, it remains to show that the maximal 1circuit[π'] is not $*$weakly 1$*$connected to the maximal 0circuit[π'], which means that a 0circuit is squeezed in between $C_\Lambda^{\max 1}(\pi')$ and $C_\Lambda^{\max 0}(\pi')$. Little is known, but the "second largest" 0circuit[π], $C^{\max 0}_{\text{int}C_\Lambda^{\max 0}(\pi)}(\pi)$, does this job: It is the case that the "second largest" 0circuit[π] is also a 0circuit[π'], since by definition, π coincides with π' in the interior of the maximal 0circuit[π]. Second, by definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$, all nodes of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ are $*$weakly 0$*$connected to the "second largest" 0circuit[π] in int${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$. In particular, the node p_2 has this feature. The only node $*$adjacent to p_2 that both belongs to the interior of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ and is equipped with a 0spin[π] is $\underset{\bullet}{p_2}$, Consequently, a 0$*$path[π, π'] in int${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ starts in $\underset{\bullet}{p_2}$ and ends $*$adjacent to the "second largest" 0circuit[π]. This 0$*$path can be extended by $p_2 \in \pi'^{-1}(0)$ and, therefore, the node $\overset{\bullet}{p}_2 \in K \overset{(5.62)}{\subset} C^{\max 0}(\pi')$ is $*$weakly 0$*$connected[π'] to the "second largest" 0circuit[π], which is also a 0circuit[π']. Property (R1) follows, which concludes this lemma. $\square$

The next lemma may be a bit technical, but for certain cases it provides insight into the shape of $P_i(\pi)$.

Lemma 5.39 *If the node p_3 has 1spin[π'], then $P_i(\pi)$ consists of $\{p_1, p_2, p_3\}$, i.e.,*

$$p_3 \in \pi'^{-1}(1) \Rightarrow |P_i(\pi)| = n = 3\,. \tag{5.64}$$

Proof: For the proof, assume $p_3 \in \pi'^{-1}(1)$ and, therefore, $p_3 \in \mathrm{int}C^{\max 0}(\pi'') = \mathrm{int}C^{\max 0}(\pi')$. Recall that there are $|P_i(\pi)| + 2$ disjoint 1∗clusters[π] ∗adjacent to the bad[π] path $P_i(\pi)$, which, due to Lemma 5.25, determines the spin values[π] in $\partial^* P_i(\pi) \setminus \{{}_\bullet p_1, p_{n\bullet}\}$ and further implies that $|P_i(\pi)|$ is odd, $P_i(\pi)$ is a straight line segment, and $|P_i(\pi)| = n \geq 3$.

We prove $n = 3$ by assuming the contrary $n > 3$ and, therefore, $n \geq 5$: Our clockwise enumeration of $P_i(\pi)$, together with $p_4 = p_3{\bullet} \in C_\Lambda^{\max 0}(\pi)$, implies that the node $\overset{\bullet}{p}_3$ belongs to $\mathrm{ext}C^{\max 0}(\pi)$.

On the one hand, this paragraph shows that the node p_3 is ∗weakly 1∗connected[π'] to Λ^c: To this end, note that the definition of K, together with $p_4 \in C_\Lambda^{\max 0}(\pi)$ and

$$K \overset{(5.62)}{\subset} C^{\max 0}(\pi'')\,,$$

implies $p_4 \in C^{\max 0}(\pi'')$. Consequently, the node p_4 is ∗weakly 1∗connected[π''] to Λ^c by a 1∗path[π'']. In other words, there exists a 1∗path[π''] that is contained in $\mathrm{ext}C_\Lambda^{\max 0}(\pi'')$ and starts ∗adjacent to p_4. The starting node of this ∗path has to be $\overset{\bullet}{p}_4$ because

$$\partial^* p_4 \cap \pi^{-1}(1) \cap \mathrm{ext}C_\Lambda^{\max 0}(\pi'') = \overset{\bullet}{p}_4$$

holds, which follows from these three facts:

a) By $\{p_3, p_4, p_5\} \subset C^{\max 0}(\pi)$, the nodes ${}_\bullet p_4$, $\underset{\bullet}{p_4}$, and $p_{4\bullet}$ are contained in the interior of the maximal 0circuit[π] and, therefore in the interior of the maximal 0circuit[π''] due to $\pi^{-1}(0) \subset \pi''^{-1}(0)$, i.e.,

$$\{{}_\bullet p_4, \underset{\bullet}{p_4}, p_{4\bullet}\} \subset \mathrm{int}C^{\max 0}(\pi) \subset \mathrm{int}C^{\max 0}(\pi'')\,.$$

b) By case assumption, the nodes ${\bullet}p_4 = p_3$ and $p_4{\bullet} = p_5$ are contained in the maximal 0circuit[π] and, therefore, in the union of the maximal 0circuit[π] and its interior, i.e.,

$$\{{\bullet}p_4, p_4{\bullet}\} = \{p_3, p_5\} \subset C^{\max 0}(\pi) \subset C^{\max 0}(\pi'') \cup \mathrm{int}C^{\max 0}(\pi'')\,.$$

c) Since $P_i(\pi)$ is a bad[π] path, the nodes ${}^\bullet p_4$ and $p_4{}^\bullet$ have 0spin[π] and, therefore, have 0spin[π''], in short

$$\{{}^\bullet p_4, p_4{}^\bullet\} \subset \pi^{-1}(0) \subset \pi''^{-1}(0)\,.$$

Because of $\pi''^{-1}(1) \subset \pi'^{-1}(1)$ all 1∗paths[π''] are also 1∗paths[π']. Consequently, the node p_3 is ∗weakly 1∗connected[π'] to Λ^c, since p_3 is ∗adjacent to $\overset{\bullet}{p}_4$.

On the other hand, because of $p_3 \in \pi'^{-1}(1)$ the definition of π' implies that $p_3 \in \mathrm{int}C^{\max 0}(\pi'') = \mathrm{int}C^{\max 0}(\pi')$, which is impossible, since p_3 is ∗weakly 1∗connected[π'] to Λ^c. □

It will turn out that the set

$$R := (p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3, \ldots, p_n) \cap \pi'^{-1}(0)$$

equals $P_i(\pi')$. Right now, we are just able to describe R in an explicit way. To this end, recall that by construction of π', only the nodes p_1, $\overset{\bullet}{p}_1$, p_3, and $\overset{\bullet}{p}_3$ could be contained in $\pi'^{-1}(1)$.

Corollary 5.40 *The exact shape of R is determined by π':*

$$R = \begin{cases} (p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3, \ldots, p_n) & p_1, p_3 \in \pi'^{-1}(0) \\ (\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3, \ldots, p_n) & p_1 \in \pi'^{-1}(1), p_3 \in \pi'^{-1}(0) \\ (p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3) & p_1 \in \pi'^{-1}(0), p_3 \in \pi'^{-1}(1) \\ (\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3) & p_1, p_3 \in \pi'^{-1}(1) \\ (\overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3, \ldots, p_n) & p_1, \overset{\bullet}{p}_1 \in \pi'^{-1}(1) \\ (p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2) & p_3, \overset{\bullet}{p}_3 \in \pi'^{-1}(1) \end{cases}$$

In particular, the set $R =: (r_1, \ldots, r_m)$ can be interpreted as a path and for $i = 1, 3$

$$p_i \in R \iff p_i \in C^{\max 0}(\pi') \tag{5.65}$$

$$\overset{\bullet}{p}_i \in R \iff \overset{\bullet}{p}_i \in C^{\max 0}(\pi')\,. \tag{5.66}$$

Proof: The exact shape of R is a direct consequence of (5.64) and the description of π', see (5.52).

Note that for $x \in \{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\}$, we know that $x \notin R$ is equivalent to $x \in \pi'^{-1}(1)$, which is equivalent to $x \in \mathrm{int}C^{\max 0}(\pi')$, see Description (5.52) of π'. Summing up, it is a fact that

$$x \notin R \iff x \in \mathrm{int}C^{\max 0}(\pi')$$

and, therefore,

$$x \in R \iff x \in C^{\max 0}(\pi') \cup \mathrm{ext}C^{\max 0}(\pi')\,.$$

So, on the one hand,

$$x \in R \Leftarrow x \in C^{\max 0}(\pi')$$

is obvious.

On the other hand, assume for contradiction that

$$x \in R \cap \mathrm{ext}C^{\max 0}(\pi')$$

and, therefore, $x \notin C^{\max 0}(\pi)$ because $C^{\max 0}(\pi) \leq C^{\max 0}(\pi')$. This, together with $p_1, p_3 \in C^{\max 0}(\pi)$, implies $x \in \{\overset{\bullet}{p}_1, \overset{\bullet}{p}_3\}$. We restrict ourselves to the case $x = \overset{\bullet}{p}_1$, since deriving the contradiction for $x = \overset{\bullet}{p}_3$ is similar. Recall that the node $x = \overset{\bullet}{p}_1$ does not belong to the maximal 0circuit$[\pi]$, in short $x = \overset{\bullet}{p}_1 \notin C^{\max 0}(\pi)$. Therefore, because of $p_1 \in \pi^{-1}(1)$, the nodes $\bullet p_1$ and $p_1 \bullet = p_2$ have to be the nodes in $C^{\max 0}(\pi) \setminus \overset{\bullet}{p}_1$ adjacent to p_1. Recall that the node $\underset{\bullet}{p_3}$ is equipped with a 1spin$[\pi]$ and distinguish two cases whether $\overset{\bullet}{p}_3$ is a node in $C^{\max 0}(\pi)$ or not:

i) If $\overset{\bullet}{p}_3 \in C^{\max 0}(\pi)$, then the set $\{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup C^{\max 0}(\pi) \setminus (p_2 \cup p_3)$ can be interpreted as a 0circuit$[\pi'']$.

ii) If $\overset{\bullet}{p}_3 \notin C^{\max 0}(\pi)$, then the set $\{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup C^{\max 0}(\pi) \setminus p_2$ can be interpreted as a 0circuit$[\pi'']$.

This holds (in both cases) because the set $\{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup C^{\max 0}(\pi)$ is contained in $\pi''^{-1}(0)$ and the maximal 0circuit$[\pi]$, $C^{\max 0}(\pi)$, hits the nodes p_1, p_2, and p_3 one after another, but never hits the nodes $\overset{\bullet}{p}_1$ and $\overset{\bullet}{p}_2$. In both cases $x = \overset{\bullet}{p}_1$ belongs to a 0circuit$[\pi'']$ larger than $C^{\max 0}(\pi)$ and, therefore, has to be contained in $C^{\max 0}(\pi'') \cup \mathrm{int}C^{\max 0}(\pi'')$. This is a contradiction to the assumption $x \in \mathrm{ext}C^{\max 0}(\pi') = \mathrm{ext}C^{\max 0}(\pi'')$. Consequently, the implication

$$x \in R \Rightarrow x \in C^{\max 0}(\pi')$$

holds, which concludes the proof. □

Now let us define the right side of (R3) as M, i.e.,

$$M := \{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \setminus p_2 = (P(\pi) \setminus P_i(\pi)) \cup R \cup Q(\pi) \cup \{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\},$$

where the second identity follows from ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) = P(\pi) \cup Q(\pi)$ and the explicit description of R. Our aim is to show condition (R3), i.e., $M = {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi')$. To this end, we need two fundamental relations between the set M and π':

Lemma 5.41 *Take the "half-open" annulus*

$$\Big]C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi), C^{\max 0}(\pi')\Big] := \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi) \cap \mathrm{int}C^{\max 0}(\pi') \cup C^{\max 0}(\pi')$$

and thin it by subtracting all nodes of the open annulus

$$\Big]C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi), C^{\max 0}(\pi')\Big[:= \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi) \cap \mathrm{int}C^{\max 0}(\pi')$$

equipped with 0*spins*$[\pi]$*. Then the resulting set contains* M*, in short*

$$M \subset C^{\max 0}(\pi') \cup \pi'^{-1}(1) \cap \mathrm{int}C^{\max 0}(\pi') \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)\,. \tag{5.67}$$

Furthermore, a node of M *has* 0*spin*$[\pi']$ *value if and only if it belongs to* $P(\pi)\setminus P_i(\pi)$ *or* R*, in short*

$$M \cap \pi'^{-1}(0) = (P(\pi) \setminus P_i(\pi)) \cup R\,. \tag{5.68}$$

Moreover, the paths $P_1, \dots, P_{i-1}$*,* R*,* $P_{i+1}, \dots, P_n$ *are not adjacent to each other. In other words they can be interpreted as clusters in* $(P(\pi) \setminus P_i(\pi)) \cup R$*.*

Proof: The definition of K, together with the explicit description of R and the fact that the set $P(\pi) \setminus P_i(\pi)$ is not $*$adjacent to $\overset{\bullet}{p}_2$, see (5.49), gives

$$(P(\pi) \setminus P_i(\pi)) \cup R \subset K \cup \{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\} \cap R\,.$$

Since the maximal 0circuit$[\pi']$ contains K, see (5.62), and also $\{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\} \cap R$, see (5.65), we even know that the maximal 0circuit$[\pi']$ contains the right side, $K \cup \{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\} \cap R$, of the latter inclusion. Summing up, we know that

$$(P(\pi) \setminus P_i(\pi)) \cup R \subset C^{\max 0}(\pi')\,. \tag{5.69}$$

Recall that the part of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ equipped with 1spins$[\pi]$, denoted by $Q(\pi)$, lies between the maximal 0circuit$[\pi]$ and the "second largest" 0circuit$[\pi]$. Further, the maximal 0circuit$[\pi', \pi'']$ is larger than the maximal 0circuit$[\pi]$. These observations, together with the fact that the nodes p_1, $\overset{\bullet}{p}_1$, p_3, and $\overset{\bullet}{p}_3$ are always in $\mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}$ and either in R or in $\pi'^{-1}(1) \cap \mathrm{int}C^{\max 0}(\pi')$, imply

$$\{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\} \cap R^c \cup Q(\pi) \subset \pi'^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)} \cap \mathrm{int}C^{\max 0}(\pi')\,. \tag{5.70}$$

These two Observations (5.69) and (5.70) prove (5.67) and (5.68).

It remains to show that $P_1, \dots, P_{i-1}$, R, $P_{i+1}, \dots, P_n$ are not adjacent to each other. By definition, we already know that the special$[\pi]$ paths $P_1, \dots, P_{i-1}$, P_i,

$P_{i+1}, \dots, P_n$ are not adjacent to each other. Consequently, it is sufficient to show that R is not adjacent to P_j for all $j \neq i$. But this becomes obvious if we consider the explicit description of R, see Corollary 5.40, and the fact that the nodes $\overset{\bullet}{p}_1$, $\overset{\bullet}{p}_2$, and $\overset{\bullet}{p}_3$ are not adjacent to $P \setminus P_i$, see Identity (5.49) of Lemma (5.31) on page 114. □

Now we are ready to verify the next two requirements:

Lemma 5.42 *The configuration π' of Proposition 5.32 satisfies*

$$ {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi') = \{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \setminus p_2 \tag{R3} $$

and

$$ P(\pi') = \bigcup_{j \neq i} P_j(\pi) \cup P_i(\pi') \,, \tag{R4} $$

where $P(\pi)$ is defined as $\bigcup_{1 \leq j \leq n} P_j(\pi)$.

Proof: We begin by outlining the structure of the proof: First, we state two items connecting π with π' and characterising M. Then, we show that Equality (R3) follows from these statements. Next, we prove the two items of the first step. As a last step, we give the proof that the enumeration of special paths of π and π' starts at the same node, which implies Requirement (R4).

First Step: Let us state the following two observations: First, the "second largest" 0circuit$[\pi]$ is also the "second largest" 0circuit$[\pi']$, in short

$$ C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi')}(\pi') = C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi) \tag{5.71} $$

Second, adding nodes of the exterior of the "second largest" 0circuit$[\pi]$ equipped with 1spins$[\pi]$ to the circuit M will not result in a smaller induced circuit than M itself, in short

$$ M = \min {}^{\mathrm{i}}\mathfrak{C} \left(M \cup \pi^{-1}(1) \cap \mathrm{ext} C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi) \right) \,. \tag{5.72} $$

The next two steps verify that the Observations (5.71) and (5.72), which will be proved afterwards in the fourth step, imply the lemma.

Second Step: Recall that M equals the right side of (R3). Keeping this in mind, the Identity (R3) is a direct consequence of

$$
\begin{aligned}
M &= \{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \setminus p_2 && (5.73)\\
&\overset{(5.67)}{\geq} \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\max 0}(\pi') \cup \pi'^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)\right)\\
&\overset{(5.71)}{=} \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\max 0}(\pi') \cup \pi'^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi')}(\pi')\right)\\
&\overset{\mathrm{Def.}}{=} {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi')\\
&\overset{(5.71)}{=} \min {}^{\mathrm{i}}\mathfrak{C}\left(C^{\max 0}(\pi') \cup \pi'^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)\right)\\
&\geq \min {}^{\mathrm{i}}\mathfrak{C}\left(M \cup \pi^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)\right)\\
&\overset{(5.72)}{=} M\,, && (5.74)
\end{aligned}
$$

where the second inequality is a consequence of $M \leq C^{\max 0}(\pi')$, which follows from (5.67), and $\pi'|_{\mathrm{int}M} = \pi|_{\mathrm{int}M}$, which is a consequence of the definition of M and the fact that π coincides with π' off $\{p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3\}$.

<u>*Third Step:*</u> Now let us turn towards proving our Assumptions (5.71) and (5.72) of the first step.

We begin with (5.71), i.e., our first aim is to show that the "second largest" 0circuit[π] coincides with the "second largest" 0circuit[π']: There exists no circuit in the "open" annulus

$$\big]M, C^{\max 0}(\pi')\big[:= \mathrm{ext}M \cap \mathrm{int}C^{\max 0}(\pi')\,,$$

because $M \overset{(5.67)}{\leq} C^{\max 0}(\pi')$, $\overset{\bullet}{p}_2 \overset{(5.62)}{\in} C^{\max 0}(\pi')$, and $\overset{\bullet}{p}_2 \overset{\mathrm{Def.}}{\in} M$. This observation, together with $M \cap \mathrm{int}C^{\max 0}(\pi') \subset \pi'^{-1}(1)$, see (5.67), implies

$$C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi')}(\pi') = C^{\max 0}_{\mathrm{int}M}(\pi')\,. \tag{5.75}$$

Note that p_2 is a dead end for paths in $\mathrm{int}M$, i.e., $|\partial p_2 \cap M| = 3$. Consequently, one cannot find a circuit that is contained in $\mathrm{int}M$ and hits the node p_2. Additionally considering $\mathrm{int}M \overset{\mathrm{Def.}}{=} \mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \cup p_2$ gives

$$C^{\max 0}_{\mathrm{int}M}(\pi') = C^{\max 0}_{\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)}(\pi')\,. \tag{5.76}$$

Moreover, we can also state

$$C^{\max 0}_{\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)}(\pi') \leq C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi') = C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi) = C^{\max 0}_{\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)}(\pi')\,,$$

where the inequality is a consequence of $\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \subset \mathrm{int}C^{\max 0}(\pi)$ and the first identity follows from $\pi|_{\mathrm{int}C^{\max 0}(\pi)} = \pi'|_{\mathrm{int}C^{\max 0}(\pi)}$, which, together with the definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ for $\pi \in A^1$, also gives the last equality. In particular, it is a fact that

$$C^{\max 0}_{\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)}(\pi') = C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)\,. \tag{5.77}$$

Thus, Identity (5.71) follows from Equalities (5.75), (5.76), and (5.77).

In order to prove (5.72), we first claim that the set M, which was defined as the right side of (R3), i.e.,

$$M = \{\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3\} \cup {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \setminus p_2\,,$$

is an induced circuit. Indeed, this is a consequence from the following three facts:

a) By definition, ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ is an induced circuit;

b) By definition, ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$ contains the nodes p_1, p_2, and p_3;

c) By (5.49) and (5.48), ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \setminus \{p_1, p_2, p_3\}$ does not hit the set $\partial(\overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3)$.

Now we are ready to verify (5.72), i.e., M is the minimal element of

$${}^{\mathrm{i}}\mathfrak{C}\left(M \cup \pi^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)\right)\,.$$

First of all, let us characterise the minimal element, before we show that it coincides with M: The fact $\pi'|_{\mathrm{int}M} = \pi|_{\mathrm{int}M}$, together with the – already verified – Identity (5.71), implies

$$\begin{aligned}&\min{}^{\mathrm{i}}\mathfrak{C}\left(M \cup \pi^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)\right)\\ &\qquad = \min{}^{\mathrm{i}}\mathfrak{C}\left(M \cup \pi'^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi')}(\pi')\right)\,.\end{aligned} \tag{5.78}$$

It remains to prove that this minimal element (5.78) is indeed M: To this end, note that by definition of ${}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$, each of its nodes is $*$weakly $0*$connected$[\pi]$ to the second largest circuit $C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi)$ in $\mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi)$. As $p_2 \in \pi'^{-1}(0) \cap \pi^{-1}(0)$ and $\pi'|_{\mathrm{int}M} = \pi|_{\mathrm{int}M}$, this implies that each node of M is $*$weakly $0*$connected$[\pi']$ to the "second largest" 0circuit$[\pi, \pi']$, $C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi)}(\pi) \overset{(5.71)}{=} C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi')}(\pi')$, in $\mathrm{int}M = \mathrm{int}^{\mathrm{i}}C^{\mathrm{fill}}(\pi) \cup p_2$. Hence, M is the smallest circuit in

$${}^{\mathrm{i}}\mathfrak{C}\left(M \cup \pi'^{-1}(1) \cap \mathrm{ext}C^{\max 0}_{\mathrm{int}C^{\max 0}(\pi')}(\pi')\right)\,,$$

which, together with (5.78), implies (5.72).

<u>*Fourth Step:*</u> The following paragraphs show that the same node x is responsible for the enumeration (vide the paragraph after Definition 5.28 on page 111) of the special paths[π] and the special paths[π'].

To this end, we introduce some useful notation: Let $W(.)$ be the set of $*$paths that start in $\vec{0}$, end in $Q(.)$, and are contained in $\mathrm{int}^{\mathrm{i}} C^{\mathrm{fill}}(.) \cup Q(.)$. Further, let $W^{\min}(.)$ be the set of $*$paths of $W(.)$ with minimal (euclidean) length. Obviously, the $*$paths of $W^{\min}(.)$ hit $Q(.)$ only in their ending nodes. Now we are ready to note that in order to show that the same node x is responsible for the enumeration of the special paths[π] and the special paths[π'], it is sufficient to verify

$$W^{\min}(\pi) = W^{\min}(\pi'),$$

which will be done in the sequel:

This paragraph shows $W^{\min}(\pi) \subset W(\pi')$: First of all, recall

$$Q(\pi') = Q(\pi) \cup \{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\} \cap R^c \tag{5.79}$$

and note that

$$\mathrm{int}^{\mathrm{i}} C^{\mathrm{fill}}(\pi') = \mathrm{int}^{\mathrm{i}} C^{\mathrm{fill}}(\pi) \cup p_2 \tag{5.80}$$

follows from (R3), which was proved in the second step. An immediate consequence of both Identities (5.79) and (5.80) is

$$W(\pi) \subset W(\pi').$$

This, together with $W^{\min}(\pi) \subset W(\pi)$, which follows from the definition, verifies

$$W^{\min}(\pi) \subset W(\pi'). \tag{5.81}$$

This paragraph shows $W^{\min}(\pi') \subset W(\pi)$: By definition, it is the case that

$$W^{\min}(\pi') \subset W(\pi').$$

We take the fact for granted that a $*$path of $W(\pi') \setminus W(\pi)$ is never a $*$path with "minimal length" of $W(\pi')$, in short

$$(W(\pi') \setminus W(\pi)) \cap W^{\min}(\pi') = \emptyset, \tag{5.82}$$

and prove it in the next but one paragraph. The Identity (5.82) and the Observation $W^{\min}(\pi') \subset W(\pi')$ lead to our aim of this paragraph

$$W^{\min}(\pi') \subset W(\pi). \tag{5.83}$$

The two Observations (5.83) and (5.81) imply that the $*$paths of $W^{\min}(\pi)$ have the same length as the $*$paths of $W^{\min}(\pi')$ and, therefore, they verify our aim

$$W^{\min}(\pi) = W^{\min}(\pi')\,.$$

It remains to prove Identity (5.82), i.e., $*$paths of $W(\pi') \setminus W(\pi)$ do not belong to $W^{\min}(\pi')$: Let us assume for contradiction that there exists a $*$path S belonging to $W(\pi') \cap (W(\pi))^c \cap W^{\min}(\pi')$, in short

$$S \in (W(\pi') \setminus W(\pi)) \cap W^{\min}(\pi')\,.$$

So, because of (5.79) and (5.80) the $*$path S has to hit p_2 or has to end in $Q(\pi') \setminus Q(\pi)$, which is a subset of $\{p_1, \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3\}$. In both cases ($S$ hits p_2 and S ends in $Q(\pi') \setminus Q(\pi)$) we are going to derive a contradiction: To this end, we first note that

$$\partial^* p_2 \cap {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi') = \{\underset{\bullet}{p}_1, p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3\} \cup \underset{\bullet}{p}_3 \mathbb{1}_{\underset{\bullet}{p}_3 \in Q(\pi)} \tag{5.84}$$

follows from (R3), which was proved in the second step.

i) Assume that S hits p_2. Due to (5.84) the $*$path S has to go through $\underset{\bullet}{p}_2$ or $p_{2\bullet}$ to hit p_2. But such a $*$path does not belong to $W^{\min}(\pi')$, since we can construct a strictly shorter path of $W(\pi')$: To this end, recall that the $*$paths were interpreted as polygons and we considered the ones with minimal euclidean length, see page 111. Now, shorten S by all nodes after p_2 and p_2 itself. Note that we cut off at least two nodes. Because of (5.84) the shortened $*$path has to end in $\underset{\bullet}{p}_2$ or $p_{2\bullet}$. We consider these two cases in the following and derive a contradiction in each case.

   First, assume that the shortened $*$path ends in $p_{2\bullet}$. Obviously, its length is at least $\sqrt{2}+1$ shorter than the length of S. Hence, if we prolong the shortened $*$path by $\underset{\bullet}{p}_2$ and $\underset{\bullet}{p}_1 \in Q(\pi')$, the resulting $*$path is strictly shorter than S. Moreover, it belongs to $W(\pi')$, a contradiction to $S \in W^{\min}(\pi')$.

   Second, assume that the shortened $*$path ends in $\underset{\bullet}{p}_2$. Recall that this shortened $*$path has at least two nodes less than S. Prolonging this shortened $*$path by one node, namely $\underset{\bullet}{p}_1 \in Q(\pi')$, leads to a $*$path that belongs to $W(\pi')$. Moreover, it is strictly shorter than S, a contradiction to $S \in W^{\min}(\pi')$.

   Summing up, S does not hit p_2.

ii) It remains to show that S does not end in $\overset{\bullet}{p}_1, p_1, \overset{\bullet}{p}_3$, or p_3. Because of (5.84) and the fact that S does not hit p_2 we can exclude the nodes $\overset{\bullet}{p}_1, \overset{\bullet}{p}_3$ as ending nodes of S.

Further, the node p_1 can be eliminated: If S ends in p_1 then, due to (5.84) and $S \cap p_2 = \emptyset$, the node $\underset{\bullet}{p_2}$ has to be the last but one node of S. Thus, shortening S by its ending node cuts away an edge of length $\sqrt{2}$. Prolonging the shortened $*$path by $\underset{\bullet}{p_1} \in Q(\pi')$ adds an edge of length 1. Obviously, this prolonged $*$path is an element of $W(\pi')$ and, moreover, is strictly shorter than S, a contradiction.

Last, p_3 cannot be the ending node of S. This is the case because $p_3 \in Q(\pi')$ and (5.64) imply $\underset{\bullet}{p_3} \in Q(\pi')$, which, together with (5.84), verifies

$$\partial^* p_2 \cap {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi') = \{\underset{\bullet}{p_1}, p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, \overset{\bullet}{p}_3, p_3, \underset{\bullet}{p_3}\}\,.$$

So, if p_3 is the ending node of S then the second but last node of S has to be $\underset{\bullet}{p_2}$, since we already know $S \cap p_2 = \emptyset$. Hence, as before shortening S by its ending node and prolonging the resulting $*$path by $\underset{\bullet}{p_1} \in Q(\pi')$ produces a $*$path of $W(\pi')$ strictly shorter than S, a contradiction.

Thus, the enumeration of the special paths$[\pi]$ and the special paths$[\pi']$ starts in the same node x.

The identity $M \overset{(5.74)}{=} {}^{\mathrm{i}}C^{\mathrm{fill}}(\pi')$, together with Lemma (5.41), implies Requirement (R4). □

Next we consider the other special paths and analyse if our modification of the bad path influences the number of 1$*$clusters assigned to them.

Lemma 5.43 *The configuration π' of Proposition 5.32 satisfies* (R5)*, i.e., the numbers of* 1$*$*clusters assigned to the special paths $P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n$ do not change if π is replaced by π'.*

Proof: The Observation $\{p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_2, p_3, \overset{\bullet}{p}_3\} \subset \{\pi \neq \pi'\}$, see (5.52), and Figure 5.8 guarantee that at most five 1$*$clusters$[\pi]$ are affected. Hereby, we can distinguish three non-exclusive scenarios[2]:

i) In the case $\overset{\bullet}{\overset{\bullet}{p}}_2 \in \Lambda^c \cup \pi^{-1}(0) \cap \Lambda$ the set $\overset{\bullet}{p}_2$ is the whole 1$*$cluster$[\pi]$ containing $\overset{\bullet}{p}_2$, which implies that changing π to π' deletes this 1$*$cluster$[\pi]$ $\overset{\bullet}{p}_2$ completely.

[2]It may help the reader to compare these scenarios with Figures 5.9 and 5.10.

ii) If $p_1 \in \pi'^{-1}(1)$, then changing π to π' joins two 1∗clusters[π] that contain the nodes $\underset{\bullet}{p_1}$ and ${}^{\bullet}p_1$.

iii) If $p_3 \in \pi'^{-1}(1)$, then changing π to π' joins two 1∗clusters[π] that contain the nodes $\underset{\bullet}{p_3}$ and $p_3{}^{\bullet}$.

All other 1∗clusters are not affected. More precisely, their shapes stay the same and they still touch the same special paths after changing π to π'. Consequently, we only have to consider the behaviour of these 1∗clusters[π] ∗adjacent to p_1 or p_3. Recall that these 1∗clusters[π] are disjoint, since $P_i(\pi)$ is bad[π].

If the first scenario occurs, then the 1∗cluster ∗adjacent to the other special paths are not affected, since $\overset{\bullet}{p_2}$ is not ∗adjacent to $\bigcup_{j\neq i} P_j$ (vide (5.49)).

Let us consider the second scenario: We call K_1 resp. K_2 the 1∗cluster[π] containing the node ${}^{\bullet}p_1$ resp. $\underset{\bullet}{p_1}$. Changing π to π' merges K_1 and K_2 into a new 1∗cluster[π']

$$K' := K_1 \cup p_1 \cup K_2 \cup \overset{\bullet}{p_1} \mathbb{1}_{\overset{\bullet}{p_1} \in \pi'^{-1}(1)}\,.$$

Note that $P_i(\pi)$ is the special path[π] with the highest index ∗adjacent to K_1 and K_2 and, therefore, ${}_{\bullet}p_1$ and $\bullet\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)$ are not contained in $\bigcup_{j>i} P_j$. Since also the set $\partial^*\{p_1, \overset{\bullet}{p_1}\} \setminus \left\{{}_{\bullet}p_1, \bullet\left(\overset{\bullet}{\overset{\bullet}{p_1}}\right)\right\}$ is not ∗adjacent to $\bigcup_{j\neq i} P_j$ (vide (5.49)) and K' is adjacent to $\overset{\bullet}{p_2} \in P_i(\pi')$, this implies that $P_i(\pi')$ is still the special path with the highest index ∗adjacent to the new 1∗cluster[π'] K'. Again the 1∗clusters assigned to the other special paths are not affected.

In a final step we consider the third scenario: Let's call K_3 resp. K_4 the 1∗cluster[π] containing $p_3{}^{\bullet}$ resp. $\underset{\bullet}{p_3}$. Since $P_i(\pi)$ is bad, we know that $i \neq N(\pi)$ and that $P_i(\pi)$ is the special path[π] with the highest index ∗adjacent to K_3 and, therefore, the node $\left(\overset{\bullet}{\overset{\bullet}{p_3}}\right)\bullet$ is not contained in $\bigcup_{j>i} P_j$. By definition and (5.64), K_4 contains $Q_i(\pi)$ and, therefore, is adjacent to $P_{i+1}(\pi) = P_{i+1}(\pi')$ and is not assigned[π] to $P_i(\pi)$. Recall that the 1∗cluster[π']

$$K'' := K_3 \cup p_3 \cup K_4 \cup \overset{\bullet}{p_3} \mathbb{1}_{\underset{\bullet}{p_3} \in \pi'^{-1}(1)}$$

is assigned[π'] to the special path with the highest index and, therefore, is assigned[π'] to the same special path K_4 was assigned[π] to, whose index is at least

$i+1$. Once again this is the case because the set $K_3 \cup p_3 \cup \overset{\bullet}{p_3}\mathbb{1}_{\overset{\bullet}{p_3}\in\pi'^{-1}(1)}$ is disjoint and not $*$adjacent to a special path with higher index than the special path K_4 was assigned[π] to. This follows from the following four facts:

a) K_3 was assigned[π] to $P_i(\pi)$;

b) the node $p_{3\bullet}$ is adjacent to or contained in K_4;

c) the node $\left(\overset{\bullet}{\overset{\bullet}{p_3}}\right)\bullet$ is not contained in $\bigcup_{j>i} P_j$, since it is adjacent to or contained in K_3;

d) the set $\partial^*\{p_3, \overset{\bullet}{p_3}\} \setminus \left\{ \left(\overset{\bullet}{\overset{\bullet}{p_3}}\right)\bullet, p_{3\bullet} \right\}$ is disjoint to $\bigcup_{j\neq i} P_j$ (vide (5.49)).

So, the number of 1$*$clusters assigned to any other special path is not affected by changing π to π'. $\square$

After we negated our previous question, whether the number of 1$*$clusters assigned to the other special paths is influenced by our modification, it remains to prove the inequalities regarding the number of 1$*$clusters assigned to the bad path. This will enable us to compare the $\phi^{f*}_{\Lambda,\lambda}$-probabilities.

Lemma 5.44 *The configuration π' of Proposition 5.32 satisfies*

$$|m^i_{loc}(\pi')^{-1}(1)| - |\pi^{-1}(1)| \geq B_i(\pi) = |P_i(\pi)| + 1 \tag{R6}$$

and

$$|m^i_{loc}(\pi)^{-1}(1)| - |\pi'^{-1}(1)| \geq B_i(\pi') \tag{R7}$$

and

$$m^i_{loc}(\pi) \text{ and } m^i_{loc}(\pi') \text{ have the same number of } 1*\text{clusters}\,. \tag{R8}$$

Proof: By construction of π', all 1$*$clusters[π] not $*$adjacent[π] to $P_i(\pi)$ are also 1$*$clusters[π'] not $*$adjacent[π'] to $P_i(\pi')$ (and vice versa), which proves Property (R8).

It is the case that

$$|m^i_{loc}(\pi')^{-1}(1) \setminus \pi^{-1}(1)| = |\{p_1, \overset{\bullet}{p_1}, \overset{\bullet}{p_3}, p_3, \ldots, p_n\}| = n+1 = |P_i(\pi)| + 1$$

follows from Corollary (5.40), together with the fact that R equates $P_i(\pi')$, see (5.68) in combination with (R3) and the definition of M. Therefore, Property (R6) is satisfied.

Next, we verify Property (R7): By construction of π'', the identity

$$m^i_{loc}(\pi)^{-1}(1) \setminus \pi''^{-1}(1) = P_i(\pi) \cup \overset{\bullet}{p}_2$$

holds and, therefore,

$$|m^i_{loc}(\pi)^{-1}(1)| - |\pi''^{-1}(1)| = |P_i(\pi)| + 1\,.$$

This, together with $\pi''^{-1}(1) \subset \pi'^{-1}(1)$ and (Description (5.52) may help the reader to understand the following cases)

$$\pi'^{-1}(1) \setminus \pi''^{-1}(1) = \begin{cases} \emptyset & \text{if } p_1, \overset{\bullet}{p}_1, p_3, , \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ p_1 & \text{if } p_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ p_3 & \text{if } p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \{p_1, p_3\} & \text{if } p_1, p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \{p_1, \overset{\bullet}{p}_1\} & \text{if } p_1, \overset{\bullet}{p}_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ \{\overset{\bullet}{p}_3, p_3\} & \text{if } p_3, \overset{\bullet}{p}_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1 \notin \mathrm{int}C^{\max 0}(\pi'') \end{cases}$$

implies that

$$\begin{aligned} &|m^i_{loc}(\pi)^{-1}(1)| - |\pi'^{-1}(1)| = \\ &= \begin{cases} |P_i(\pi)| + 1 & \text{if } p_1, \overset{\bullet}{p}_1, p_3, , \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| & \text{if } p_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| & \text{if } p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| - 1 & \text{if } p_1, p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| - 1 & \text{if } p_1, \overset{\bullet}{p}_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| - 1 & \text{if } p_3, \overset{\bullet}{p}_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1 \notin \mathrm{int}C^{\max 0}(\pi'') \end{cases} \end{aligned} \tag{5.85}$$

If $p_1 \in \mathrm{int}C^{\max 0}(\pi'')$, then $p_1 \in \pi'^{-1}(1)$ and two disjoint 1∗clusters[π] are merged into one 1∗cluster[π'] (this equates scenario 2 in the proof of the last lemma). If $p_3 \in \mathrm{int}C^{\max 0}(\pi'')$, then $p_3 \in \pi'^{-1}(1)$ and again two disjoint 1∗clusters[π] are merged into one 1∗cluster[π'] (this equates to scenario 3 in the proof of the last lemma). A consequence of such a merging is that the number of 1∗clusters assigned[π'] to $P_i(\pi')$ decreases by one in comparison to $B_i(\pi)$. Moreover, $\overset{\bullet}{p}_1$ or

$\overset{\bullet}{p}_3$ contained in $\pi'^{-1}(1)$ implies that $\overset{\bullet}{p}_2 \in \pi^{-1}(0)$, see (5.60), and, therefore, that a $1{*}$cluster$[\pi]$ vanishes, namely the one only consisting of the node $\overset{\bullet}{p}_2$, see Figures 5.9 and 5.10. Summing up, since $B_i(\pi) = |P_i(\pi)| + 1$, we know that

$$B_i(\pi') \le \begin{cases} |P_i(\pi)| + 1 & \text{if } p_1, \overset{\bullet}{p}_1, p_3, , \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| & \text{if } p_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| & \text{if } p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| - 1 & \text{if } p_1, p_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } \overset{\bullet}{p}_1, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| - 1 & \text{if } p_1, \overset{\bullet}{p}_1 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_3, \overset{\bullet}{p}_3 \notin \mathrm{int}C^{\max 0}(\pi'') \\ |P_i(\pi)| - 1 & \text{if } p_3, \overset{\bullet}{p}_3 \in \mathrm{int}C^{\max 0}(\pi'') \text{ and } p_1, \overset{\bullet}{p}_1 \notin \mathrm{int}C^{\max 0}(\pi'') \end{cases} . \tag{5.86}$$

We write "$\le$", since the $1{*}$cluster only consisting of $\overset{\bullet}{p}_2$ could also completely vanish by changing π to π' if $\overset{\bullet}{p}_2 \in \Lambda^c$ (which could occur in the first four cases). However, even without taking this into consideration, (5.85) and (5.86) imply (R7). $\square$

At last all required properties of π' are shown and, therefore, Proposition 5.32 is verified.

... and frustration!

Bibliography

[Aiz] Aizenman, M. (1980) Translation invariance and instability of phase coexistence in the two-dimensional Ising system, Comm. Math. Phys. **73**, 83-94.

[AKN] Aizenman, M. and Kesten, H. and Newman, C. M. (1987) Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation, Comm. Math. Phys. **111**, 505-531.

[Bear] Beardon, A. F. (1979) Complex Analysis, Chapter 7, Wiley, New York.

[BR] Bollobas, B. and Riordan O. (2008) Percolation on dual lattices with k-fold symmetry, Random Struct. Algorithms **32(4)**, 463-472.

[BHW] Brightwell G., Häggström O., and Winkler P. (1999) Nonmonotonic behavior in hard-core and Widom–Rowlinson models, J. Statist. Phys. **94**, 415-435.

[BK] Burton, R.M. and Keane, M. (1989) Density and uniqueness in percolation, Comm. Math. Phys. **121**, 501-505.

[Cars] Carstens, S.M. (2011) Non-coexistence of Infinite Clusters in Two-Dimensional Dependent Site Percolation, J. Statist. Phys. **144**, 1223-1237.

[Do] Dobrushin, R.L. (1968) Gibbsian random fields for lattice systems with pairwise interactions, Funct. Anal. Appl. **2**, 292-301.

[FF] Ford, Jr. ,L. R. and Fulkerson D. R. (1962) Flows in Networks, Princeton University Press, Princeton, N. J.

[GM] Gallavotti, G. and Miracle-Sole, S. (1972) Equilibrium states of the Ising model in the two-phase region. Phys. Rev. B **5**, 2555-2559.

[GKR] Gandolfi, A. and Keane, M. and Russo, L. (1988) On the Uniqueness of the Infinite Occupied Cluster in Dependent Two- Dimensional Site Percolation, Ann. Probab. **16** , 1147-1157.

[Geo] Georgii, H.-O. (1988) Gibbs Measures and Phase Transitions, de Gruyter, Berlin New York.

[GHM] Georgii, H.-O. and Häggström, O. and Maes, C. (2000) The random geometry of equilibrium phases, In: C. Domb and J.L. Lebowitz (eds.) Phase Transitions and Critical Phenomena, Vol. 18, pp. 1-142, Academic Press.

[GH] Georgii, H.-O. and Higuchi, Y. (2000) Percolation and number of phases in the 2D Ising model, J.Math.Phys. **41**, 1153-1169 (special issue on Probabilistic Methods in Statistical Physics)

[Gr] Grimmett, G.R. (1999) Percolation, 2. edition, Springer, New York.

[Hä96] Häggström, O. (1997) Infinite clusters in dependent automorphism invariant percolation on trees, Ann. Probab. **25**, 1423-1436.

[Hä02] Häggström, O. (2002) A Monotonicity Result for Hard-core and Widom-Rowlinson Models on Certain d-dimensional Lattices, Elec. Commun. Prob. **7**, 67-78

[HM] Häggström, O. and Mester, P. (2009) Some two-dimensional finite energy percolation processes, Elec. Commun. Prob. **14**, 42-54.

[Hig79] Higuchi, Y. (1979) On the absence of non-translation invariant Gibbs states for the two-dimensional Ising model, in : J. Fritz, J.L. Lebowitz and D. Szász (eds.), Random Fields, Esztergom (Hungary) 1979. Amsterdam: North-Holland, Vol. **I**, 517-534.

[Hig83] Higuchi, Y. (1983) Applications of a stochastic inequality to two-dimensional Ising and Widom-Rowlinson models, Probability theory and mathematical statistics, Lecture Notes in Math., **1021**, 230-237.

[HT] Higuchi Y. and Takei, M., (2004) Some results on the phase structure of the two-dimensional Widom-Rowlinson model, Osaka J. Math. **41**, 237-255

[Isi] Ising, E. (1925) Beitrag zur Theorie der Ferromagnetismus, Z. Phys. **31**, 253-258

[K] Kesten, H. (1982) Percolation theory for mathematicians, Boston MA, Birkhauser.

[LR] Lanford, O. and Ruelle, D. (1969) Observables at infinity and states with short range correlations in statistical mechanics, Commun. Math. Phys. **13**, 194-215.

[LG] Lebowitz, J.L. and Gallavotti, G. (1971) Phase transitions in binary lattice gases, J. Math. Phys. **12**, 1129-1133.

[LeMa-L] Lebowitz, J.L. and Martin-Löf, A. (1972) On the uniqueness of the equilibrium state for Ising spin systems, Commun. Math. Phys. **25**, 276-282.

[Le] Lenz, W. (1920) Beitrag zum verständnis der magnetischen Erscheinungen in festen Körpern, Phys. Zeitschr. **21**, 613-615.

[Li] Liggett, T. (1985) Interacting Particle Systems, Springer, New York.

[LSS] Liggett, T., Schonman, R.H., and Stacey, A.M. (1997) Domination by product measures, Ann, Probab. **25**, 71-96.

[MM] Messager, A. and Miracle-Sole, S. (1975) Equilibrium States of the Two-dimensional Ising Model in the Two-Phase Region, Comm. Math. Phys. **40**, 187-196

[NewS] Newman, C.M. and Schulman, L.S. (1981) Infinite clusters in percolation models, J. Stat. Phys. **26**, 613-628.

[Pe] Peierls, R. (1936) On Ising's model of ferromagnetism, Proc. Cambridge Phil. Soc. **32**, 477-481.

[Ru] Russo, L. (1979) The infinite cluster method in the two-dimensional Ising model. Commun. Math. Phys. **67**, 251-266.

[Sheff] Sheffield, S. (2005) Random Surfaces, Asterisque **304**

[Sok] Sokal, A.D. (1981) Existence of compatible families of proper regular conditional probabilities, ZWvG **56**, 537-548.

[WR] Widom, B. and Rowlinson, J.S. (1970) New model for the study of liquid-vapor phase transition, J. Chem. Phys. **52**, 1670-1684.

Index