Resolution of Curvature Singularities in Black Holes and the Early Universe

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Zusammenfassung

Diese Dissertation beschäftigt sich in zwei Ansätzen mit dem Problem von Singularitäten in der Allgemeinen Relativitätstheorie. Im ersten gehen wir von der einsteinschen Theorie aus und stellen eine Vermutung für eine asymptotisch äquivalente aber nichtsinguläre Theorie auf. Im zweiten Ansatz beginnen wir bei der Stringtheorie als fundamentaler Beschreibung der Welt und untersuchen die aus dieser Annahme resultierende effektive Theorie bei niedrigen Energien.

Der erste Ansatz stellt eine Anwendung der Hypothese über die Krümmungsbegrenzung („limiting curvature hypothesis“) auf anisotrope Kosmologien dar. Dies erweitert die Betrachtung isotroper Kosmologien von BRANDENBERGER et al. Diese konstruierten eine Theorie, in der alle homogenen und isotropen Lösungen frei von Singularitäten sind. Auf Grund der Nichtanalytizität der Gleichungen gelang es uns nicht, diesen Beweis im anisotropen Fall zu wiederholen. Dennoch deutet die analytische und numerische Untersuchung auf eine Auflösung der Singularitäten auch in diesem Fall hin. Generisch scheint die Auflösung nicht wie erwartet durch eine DE-SITTER-Phase zu erfolgen. Stattdessen verbindet die Lösung ein kontrahierendes anisotropes Universum mit einem in zeitsymmetrischer Weise expandierenden. Der Übergang erfolgt in einer näherungsweise flachen MINKOWSKI-Phase. Diese Lösung könnte eine Alternative zu den sogenannten Bounce-Lösungen darstellen, wie sie in Pre-Big-Bang-Modellen vorkommen.

Im zweiten Ansatz konstruieren wir ein einfaches Modell in der Typ-IIA-Superstring-Theorie. Mit einer D7- oder D9-Bran, welche die BPS-Bedingung nicht erfüllt, führen wir einen tachyonischen Freiheitsgrad ein. Dessen Potenzial wird durch den kompakten Hintergund, auf den die Bran gewickelt ist, beeinflusst. In gewissem Sinn kann die Masse durch die Größe der kompakten Dimension eingestellt werden. Wir verwenden eine trunkierte Wirkung, welche so konstruiert ist, dass das Verhalten der vollen Stringtheorie bei dynamischer Erzeugnung und Zerfall von Nicht-BPS-Branen möglichst gut reproduziert wird. In niedrigster Ordnung von Metrik und Dilaton sowie der tachyonischen Anregung finden wir Bounce-Lösungen. Diese werden ermöglicht durch die Tatsache, dass das Tachyon in der verwendeten Wirkung stets mit positivem Druck auftritt. Sowohl Krümmung als auch die Zeitableitungen des Dilatons sind während des Bounces klein, so dass die Gravitation vollständig klassisch betrachtet werden kann. Die gefundenen Bounce-Lösungen nähern sich asymptotisch den Pre-Big-Bang- oder Post-Big-Bang-Lösungen an, so dass Singularitäten in Krümmung und Dilaton vor oder nach dem Bounce verbleiben. Diese Singularitäten im String-Bezugssystem können durch ein ad hoc eingeführtes, zusätzliches Potenzial aufgelöst werden. Ein solches könnte durch α'-Korrekturen im Offenen-String-Sektor herrühren, deren exakte Berechnung für belastbare Aussagen erforderlich wäre.

Abstract

This thesis is concerned with two approaches on the singularity problem of the general theory of relativity. The first is of bottom-up nature. We start from EINSTEIN's well established general relativity and make an educated guess for an asymptotically equivalent but non-singular theory. In the second approach we take the top-down perspective starting with the assumption that string theory gives the fundamental description of nature and analyse the resulting low energy effective theory.

Our bottom-up approach is an application of the limiting curvature hypothesis to anisotropic cosmologies. This extends the success for isotropic cosmologies of BRANDENBERGER et al. Applying the LCH, they constructed a theory in which all homogeneous and isotropic solutions are singularity free. Due to the non-analytic nature of the equations we were unable repeat the proof in the anisotropic case, but analytical and numerical analysis produce circumstantial evidence for a resolution of the singularity in this case as well. Generically this resolution seems not to involve a DE SITTER phase as expected. Instead it would interpolate between a contracting anisotropic universe and a universe, that time-symmetrically expands anisotropically. During this transition spacetime evolves through a nearly flat, MINKOWSKI phase. This solution could represent an alternative to the so-called bounce solutions as they appear in pre-big-bang scenarios.

In our top-down approach we construct a simple model in type IIA super string theory. With a non-BPS D7 or D9 brane we introduce a tachyonic degree of freedom. Its potential is influenced by the compact background wrapped by the brane. In a way the mass can be tuned by the size of the compact dimension. We use a truncated action which was constructed in order to approximate the full string theory result for the dynamical creation and decay of non-BPS branes quite accurately. Taking the lowest order effective action for metric, dilaton and an effective action for the open tachyonic mode, we obtaine bounce solutions. The bounce results from the positivity of the pressure of the tachyon field in our Lagrangian. Both curvature and time derivative of the dilaton remain small during our bounce so that the gravitational sector behaves entirely classical. Asymptotically our bounce solutions are similar to pre-big bang and post-big bang solutions respectively. Thus there remain singularities in the curvature and the dilaton before or after the bounce. These asymptotic string frame curvature singularities can be resolved by the ad hoc addition of a potential term, that might result from α' corrections in the open string sector. Exact calculation of the corrections would be necessary in order to give a more precise picture.

1. Introduction

Since almost a century EINSTEIN's general theory of relativity provides a wonderful description of physics on the large scale. Based on this theory of gravitation, an astonishingly simple model of an isotropic and homogeneous universe fits all observational data from solar system dynamics to the microwave background radiation and large scale structure.

But despite the observational confirmation, the gravitational theory exhibits an unwanted feature. As soon as a universe is not absurdly symmetric it must, as PENROSE and HAWKING showed, contain singular points. At a singular point spacetime ends. On the one hand any object or information reaching a future singularity drops, in a manner of speaking, off the universe and is lost forever. On the other hand at a singular point in our past arbitrary initial conditions could be given, that are in no way restricted by what we could ever find out about the rest of the universe. This fundamentally limits our predictive power and results in the fact that a description of the universe based on general relativity can never be complete.

A further shortcoming of the geometric theory of gravitation is that it is a classical theory, but we already know that the microscopic world is governed by the rules of quantum mechanics. A fully unified theory of everything has to be either classical or quantum, since a classical and a quantum theory cannot be combined without exhibiting contradictions. At the latest since the strikingly clear experiments of GREENBERGER, HORNE, SHIMONY, and ZEILINGER [10] we know that a classical description will not work for atomic or particle physics. We are thus convinced that the 'final' theory is of quantum mechanical nature. And are hence in need of a quantum theory of gravity.

Time evolution of a quantum mechanical state is expected to be unitary. But this is obviously incompatible with the aforementioned information loss at spacetime singularities. Obviously the effective low energy description of the theory of everything we know as general relativity is invalid in the vicinity of these singularities. We expect that the fundamental theory deviates from the general relativistic prescription and does not exhibit singular behaviour there. Taking the low energy point of view, which is somehow natural for us, the singularities are thus resolved by the fundamental theory.

To investigate the resolution of singularities further there are two different approaches one can take. The first is a bottom-up approach starting from what we know and asking the question 'What changes could be made to general relativity in order to resolve the singularities?' This might lead us to an improved low energy effective theory where some or even all singular points are resolved, but we cannot expect to gain deeper understanding of the nature and origin of the corrections along this path. The other approach is of top-down nature. If we can somehow guess the fundamental theory, we can derive the low energy effective limit, which then should not exhibit the singular behaviour. The main candidate for this approach is of course string theory.

An approach of the first kind would be for instance the pre-big bang scenario, in which the singular solutions are regularised with the help of an ad hoc potential for the dilaton. Essentially of a bottom-up nature as well is the approach of loop quantum cosmology, which postulates a discretisation of the derivative operator, leading to a modification of the dynamics at small scales and thus introducing bounce solutions resolving the big bang singularity. The most well known approach of the second kind is the KKLT scenario, which is a very elaborate model of branes in string theory that could reproduce the properties of the standard model of elementary particle physics in its low energy regime. But due to its complexity actual calculations are very hard.

This work starts with an overview of the theory of general relativity and string theory in chapter 2. In the following chapters we then develop two different approaches on the singularity problem – one of bottom-up nature and one top-down approach.

In chapter 3 we try to extend the success of the 'limiting curvature hypothesis' that allowed BRANDENBERGER et al [5] to resolve the big-bang singularity of a homogeneous and isotropic cosmology in an elegant way. Our extension of their approach to homogeneous but anisotropic cosmologies covers many generic singularities. For instance the interior part of the SCHWARZSCHILD solution is locally indistinguishable from an anisotropic cosmology with one special direction. In a theory where these cosmologies are non-singular, black holes would become non-singular as well. In the analytic exploration of the modified EINSTEIN field equations it cannot be excluded that there remain non-analytic, singular solutions. Indeed a numerical analysis shows that generic initial conditions develop singular coordinate functions, but a closer look suggests that this is only a coordinate singularity and spacetime is non-singular, even more, locally flat at these points. We are lead to the interpretation that the singularity was replaced by a solution bouncing through flat MINKOWSKI spacetime.

In chapter 4 we explore a path of the top-down approach. Starting from type IIA superstring theory we build a cosmological model that is sufficiently simple to actually calculate the low energy effective action and solve the equations of motion at least numerically. Our model makes use of the tachyonic degree of freedom arising from an unstable D-brane, which does not meet the BPS condition. The dynamics are strongly influenced by the compactification of the background geometry, which we choose to be on a torus. A $\mathbb{Z}_2$ orbifold introduces symmetry conditions on the field. By these the purely tachyonic ground state is projected out. Henceforth only the lowest KALUZA KLEIN mode remains, which leads to a field where the mass is tunable through the size of the extra dimensions from tachyonic ($m^2 = -1$) to arbitrarily heavy. In this setup there exist bouncing solutions in the string frame which resolve the cosmological singularity. But the numerical solutions still contain singularities in the past or the future of the bounce. These singularities can be resolved introducing an ad hoc potential into the action. Such a potential term might result from α' corrections or quantum loop corrections.

2. Review of current theories

2.1. General Relativity

Gravitation is described by EINSTEIN's general theory of relativity. A more detailed description of gravity can be found in [23]. The fundamental concept of general relativity is, that the apparent forces of gravity and inertia are solely an effect of spacetime curvature. Spacetime is described as a four-dimensional LORENTZian (or pseudo-RIEMANNian) manifold. This means that at any point there exist charts that map the manifold into flat MINKOWSKI space with three (or maybe more) spacelike and one timelike direction. Differentiability and other concepts can thus be lifted from the well known flat space to an arbitrary complicated manifold.

While formulating a physical theory in the manifold one only has to make sure that a change of coordinates does not alter the predictions of the theory but only its description in terms of coordinates. For this tensor calculus has to be lifted to the manifold. In order to do so, the essential step is the definition of a vector since higher rank tensors can then be constructed from vectors and their duals. The mathematical definition of a vector is astonishingly close its everyday realisation as the velocity of an object. Vectors are defined as the equivalence classes of directional derivatives along curves through the point at which the tangent vector space shall be defined. Given a parametrised curve within the manifold, which is a differentiable map of the real line into the manifold, the tangent vector to this curve is defined as the operator giving the directional derivative along this curve. This definition is obviously independent of the chart. With this notion of a vector it is easy to introduce dual vectors, which are linear maps from vectors to scalars. Similarly tensors of higher rank are introduced such that a (n, m)-tensor is defined as a linear map of n dual vectors and m vectors to scalars.

The spacetime manifold is characterised by a metric defining lengths of curves in the manifold. It is given as a symmetric $(0, 2)$-tensor $g_{\alpha\beta}$

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta.$$

This replaces the $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, \ldots, 1)$ known from flat MINKOWSKI space.

At any single point of the manifold we thus can make use of a one-to-one correspondence of vectors and more general tensors of arbitrary order within the manifold (at that point) and corresponding objects in a flat spacetime. The laws of physics which are formulated in tensor language of MINKOWSKI space can now be lifted to the spacetime manifold, while the remapping rules for different charts make sure that the lifting procedure is independent of the arbitrarily chosen coordinate chart.

In this way we can lift the laws of physics onto the curved manifold at any point. The chart remapping translates the laws into arbitrary coordinates. Thus we have established EINSTEIN's general principle of relativity, that states: *All systems of reference are equivalent with respect to the formulation of the fundamental laws of physics*.

Up to now we only have dealed with tensors at a single point. In order to compare objects at different points in spacetime we need to specify how to transport a vector from one point to another. It turns out that due to arbitrarily chosen coordinates it doesn't make sense to simply choose the vector with the 'same' coordinates at the target point as we would have done in EUKLIDean space. One therefore defines the notion of parallel transport along a curve pulling the vector from one point to another, while keeping it at any instant parallel to itself. Since vectors are defined as directional derivatives, this then defines a covariant derivative. The change in coordinates is described by the connection $\Gamma^{\delta}_{\alpha\beta}$, which could in principle be chosen arbitrarily. But in a manifold with a metric there is a unique connection, the LEVI CIVITA connection, that is *compatible* with this metric. This means that the metric is constant under covariant derivatives. The connection coefficients are given by

$$\Gamma^{\delta}_{\alpha\beta} = g^{\delta\gamma}\frac{1}{2}\left(g_{\alpha\gamma,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right).$$

Here we used the shorthand notation for the partial derivative

$$g_{\alpha\gamma,\beta} := \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}}.$$

Despite the suggestive index structure the object obtained from a vector or more general a tensor of non-zero rank through the partial derivative is not a tensor itself. We thus introduce the covariant derivative resulting in a well defined tensor

$$\begin{aligned} V^{\alpha}{}_{;\beta} &= V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\delta\beta}V^{\delta}, \\ V_{\alpha;\beta} &= V_{\alpha,\beta} - \Gamma^{\delta}_{\alpha\beta}V_{\delta}. \end{aligned}$$

For higher rank tensor one has to add a correction term as above for each vector and dual vector index of the original tensor.

Effects of curvature are the non-commutativity of covariant derivatives, the path-dependence of parallel transport, and the fact that the three inner angles of a triangle do not sum up to 180° – which are all equivalent. Mathematically these effects are all described by the RIEMANN CHRISTOFFEL or curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$. This $(1,3)$-tensor maps three vectors onto a fourth vector. For a closed path along an infinitesimal parallelogram loop the input vectors are the vector to be parallel transported along the loop and the two vectors describing the parallelogram's edges. The difference between the original vector and its copy after the parallel transport along the loop is the output of the curvature tensor. Expressed in terms of CHRISTOFFEL symbols the curvature tensor is given by

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma}.$$

Note that the metric does not appear explicitely. The notion of parallel transport can thus already be defined on manifolds featuring a connection but no metric.

A natural choice for the action governing the evolution of geometry is the mean curvature of spacetime, given through the integral over the RICCI scalar

$$R = R^{\alpha\beta}{}_{\alpha\beta},$$

the full contraction of the RIEMANN CHRISTOFFEL tensor. In fact this is the unique invariant that can be formed out of metric components and first derivatives of metric components that is of quadratic order in these.

The invariant volume element is given by $\sqrt{-g}\,\mathrm{d}^4x$, where $g = \det\left(g_{\alpha\beta}\right)$ is the metric determinant. In addition to this EINSTEIN HILBERT action we have to add the action describing matter evolution to obtain the total action

$$S = \int R\,\sqrt{-g}\,\mathrm{d}^4x + \int \mathscr{L}_\mathrm{m}\,\sqrt{-g}\,\mathrm{d}^4x.$$

Variation of this action w. r. t. the metric components or equivalently the components of the metric inverse $g^{\alpha\beta}$ gives the EINSTEIN field equations. The variation of the matter action $S_\mathrm{m} = \int \mathscr{L}_\mathrm{m}\,\sqrt{-g}\,\mathrm{d}^4x$ is taken as the definition of the energy momentum tensor

$$T_{\alpha\beta} = -\frac{1}{\sqrt{-g}}\frac{\delta\,S_\mathrm{m}}{\delta g^{\alpha\beta}}.$$

The field equations then read

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = T_{\alpha\beta}.$$

In many cases one is not interested in the microscopical origin of the energy momentum tensor. Especially in cosmology one often models the matter as an ideal fluid, which is solely described by energy density ρ and pressure p. Both are functions of cosmological time t only and their relation is given by the equation of state

$$p(t) = w\rho(t),$$

where w is, for ordinary fluids, a constant. For dust one has $w = 0$ and radiation or more generally highly relativistic matter gives $w = \frac{1}{3}$. Furthermore we will choose comoving coordinates in which the cosmic fluid is at rest at any coordinate point, or in coordinates: the fluid's velocity shall be given as $(u^\mu) = (1, 0, 0, 0)$. The energy momentum tensor of an ideal fluid in comoving coordinates is then given by

$$(T_{\alpha\beta}) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} & \\ 0 & & & \end{pmatrix} = \left(pg_{\alpha\beta} + (p+\rho)u_\mu u_\nu\right).$$

The EINSTEIN HILBERT action can be supplemented by another, even simpler term: a constant Λ. Originally EINSTEIN introduced the cosmological constant in order to make a stationary universe possible. Since gravity is always attractive no universe containing matter could ever be static, except there would be something which exerts negative pressure. The cosmological constant can generate negative pressure. Looking at the field equations one can reinterpret the cosmological constant as a component of the energy momentum tensor, which is proportional to the metric. The cosmological constant is thus an energy density of the vacuum and gives an energy momentum tensor of an ideal fluid with equation of state given by $w = -1$.

It might surprise that quite simple systems exhibit such a strange equation of state $p = -\rho$. The energy momentum tensor of a scalar field ϕ with a potential $V(\phi)$ can be written as one of an ideal fluid characterised by energy density

$$\rho = \frac{1}{2}(\partial\phi)^2 + V(\phi)$$

and pressure

$$p = \frac{1}{2}(\partial\phi)^2 - V(\phi).$$

Henceforth the equations of state parameter $w = \frac{p}{\rho}$, which is now non-constant, is bounded from below by -1 for any non-negative potential $V(\phi)$, which guarantees the validity of the weak energy dominance condition $\rho + p \geq 0$. If the potential has a local minimum at ϕ_0, then $\phi = \phi_0$ is a solution of the equations of motion, that under violation of the strong energy dominance condition $\rho + 3p \geq 0$ exhibits the quation of state

$$p = -\rho = -V(\phi_0)$$

and thus mimics a cosmological constant.

2.1.1. FRIEDMANN ROBERTSON WALKER Cosmologies

Modern cosmology is based on the Copernican Principle stating that earth is not in any special or central position in the universe. Furthermore the Cosmological Principle assumes that the universe is homgeneous, which is more or less the only chance for a single observer to make statements on the universe as a whole. This results in the requirement that a cosmological model is homogeneous and isotropic. The most general ansatz for the spacetime metric obeying these symmetries is the FRIEDMANN ROBERTSON WALKER line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \right),$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$ is the surface element on a two-dimensional sphere. Here k determines, if the spatial slices of constant time are negatively curved ($k = -1$), flat ($k = 0$), or positively curved ($k = +1$).

With this ansatz and the energy momentum tensor of an ideal fluid the field equations reduce to the FRIEDMANN equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{2.1a}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{2.1b}$$

These equations are obviously time reflection invariant. With positive energy density ρ and non-negative pressure $p \geq 0$ it follows immediately from (2.1a) that $\ddot{a} < 0$ hence an expanding universe (as currently observed) should decelerate – as expected from the fact that gravity is always attractive. This means that expansion was faster at earlier times and that the universe can be traced back to a singular point in the finite past with $a = 0$. This singularity is called the Big Bang. Of course since energy density diverges at this point as well, we do not expect the classical theory to hold at this stage. But from a classical point of view this singularity is unavoidable and has to be resolved in a more fundamental theory.

A positive cosmological constant Λ on the other hand is equivalent to energy density with negative pressure and results in a maximally symmetric spacetime. This spacetime is called DE SITTER space and due to its symmetry one can choose different spacelike slicing without spoiling homogeneity. In this way it presents itself to the observer as an open ($k = -1$), flat ($k = 0$), or closed ($k = 1$) universe with a scale factor of

$$a = \sqrt{\tfrac{3}{\Lambda}}\sinh\left(\sqrt{\tfrac{\Lambda}{3}}t\right), \quad a \propto \exp\left(\pm\sqrt{\tfrac{\Lambda}{3}}t\right), \quad a = \sqrt{\tfrac{3}{\Lambda}}\cosh\left(\sqrt{\tfrac{\Lambda}{3}}t\right). \tag{2.2}$$

For large times t – choosing the $+$-branch – these forms are indistinguishable and exponentially expanding. Thus after sufficiently long DE SITTER expansion every universe looks flat – independent of the curvature that was present at the beginning of the expansion. This is how inflation solves the flatness problem of cosmology and explains why our observed universe looks flat, which would require a tremendous amount of fine tuning otherwise.

2.1.2. SCHWARZSCHILD Black Holes

Another exact solution which is very important is the solution outside a spherical matter source. The most general ansatz for a stationary, spherically symmetric spacetime is given by

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r,t)}\mathrm{d}t^2 + \mathrm{e}^{\lambda(r,t)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right).$$

Note that there is no $\mathrm{d}t\mathrm{d}r$-term mixing timelike and spacelike coordinates, since it can always be removed by a coordinate transformation $t \mapsto \tilde{t}(t,r)$. The components of the CHRISTOFFEL symbol

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\mu} \left(g_{\alpha\mu,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} \right)$$

are given below. We use a dot for partial derivatives w. r. t. t and prime for the ones w. r. t. r and omit components that can be obtained from the symmetry property of the CHRISTOFFEL symbol

$$\begin{aligned}
\Gamma^t_{tt} &= \frac{\dot{\nu}}{2}, & \Gamma^r_{tr} &= \frac{\dot{\lambda}}{2}, & \Gamma^\theta_{r\theta} &= \frac{1}{r}, \\
\Gamma^t_{tr} &= \frac{\nu'}{2}, & \Gamma^r_{rr} &= \frac{\lambda'}{2}, & \Gamma^\theta_{\phi\phi} &= -\cos\theta\sin\theta, \\
\Gamma^t_{rr} &= \mathrm{e}^{\lambda-\nu}\frac{\dot{\lambda}}{2}, & \Gamma^\phi_{r\phi} &= \frac{1}{r}, & \Gamma^r_{\theta\theta} &= -\mathrm{e}^{-\lambda} r, \\
\Gamma^r_{tt} &= \mathrm{e}^{\nu-\lambda}\frac{\nu'}{2}, & \Gamma^r_{\phi\phi} &= -\mathrm{e}^{-\lambda} r \sin^2\theta, & \Gamma^\phi_{\theta\phi} &= \cot\theta.
\end{aligned}$$

Since the metric is diagonal, the off-diagonal components of the EINSTEIN tensor are identical to the corresponding components of the RICCI tensor.

$$\begin{aligned}
G_{tr} = R_{tr} &= \Gamma^\lambda_{tr,\lambda} - \Gamma^\lambda_{t\lambda,r} + \Gamma^\lambda_{\mu\lambda}\Gamma^\mu_{tr} - \Gamma^\lambda_{\mu r}\Gamma^\mu_{\lambda t} = \\
&= \Gamma^t_{tr,t} + \underline{\Gamma^r_{tr,r}} - \Gamma^t_{tt,r} - \underline{\Gamma^r_{tr,r}} + \overline{\Gamma^t_{tt}\Gamma^t_{tr}} + \Gamma^t_{rt}\Gamma^r_{tr} + \underbrace{\Gamma^r_{tr}\Gamma^t_{tr}} + \overbrace{\Gamma^r_{rr}\Gamma^r_{tr}} - \overbrace{\Gamma^r_{rr}\Gamma^r_{rt}} \\
&\quad + \Gamma^\theta_{r\theta}\Gamma^r_{tr} + \Gamma^\phi_{r\phi}\Gamma^r_{tr} - \underbrace{\Gamma^r_{tr}\Gamma^t_{rt}} - \Gamma^t_{rr}\Gamma^r_{tt} - \overline{\Gamma^t_{tr}\Gamma^t_{tt}} = \\
&= \frac{\dot{\nu}'}{2} - \frac{\dot{\nu}'}{2} + \frac{\dot{\lambda}}{2}\frac{\nu'}{2} + \frac{1}{r}\frac{\dot{\lambda}}{2} + \frac{1}{r}\frac{\dot{\lambda}}{2} - \frac{\dot{\lambda}}{2}\frac{\nu'}{2} = \\
&= \frac{\dot{\lambda}}{r}.
\end{aligned}$$

We directly see from $G_{tr} = 0$ that $\lambda(r,t) = \lambda(r)$. $G_{tt} = 0$ and $G_{rr} = 0$ are equivalent to

$$\begin{aligned}
r\lambda' &= 1 - \mathrm{e}^\lambda \\
r\nu' &= \mathrm{e}^\lambda - 1.
\end{aligned}$$

Thus $\nu(r,t) = \nu(r) + f(t)$. By rescaling the coordinate t we can set $f(t) = 0$ without perturbing the other metric coefficients. Furthermore

$$\lambda' + \nu' = 0 \quad \Rightarrow \quad \left(\lambda' + \nu'\right)\mathrm{e}^{\lambda+\nu} = 0 \quad \Rightarrow \quad \mathrm{e}^{\lambda+\nu} = \text{const.}$$

again by rescaling the coordinate t we can choose this constant to be 1 and obtain

$$\mathrm{e}^\nu = \mathrm{e}^{-\lambda}\,.$$

We can integrate $G_{tt} = 0$, e. g. by separation of variables to obtain λ

$$\begin{aligned} r\frac{\mathrm{d}\,\lambda}{\mathrm{d}r} &= 1 - \mathrm{e}^{\lambda} \\ \int \frac{1}{r}\,\mathrm{d}r &= \int \frac{1}{1-\mathrm{e}^{\lambda}}\,\mathrm{d}\lambda \\ \ln\frac{r}{r_{\mathrm{S}}} &= \ln\frac{\mathrm{e}^{\lambda}}{1-\mathrm{e}^{\lambda}} \\ \mathrm{e}^{\lambda} &= \frac{1}{1-\frac{r_{\mathrm{S}}}{r}}. \end{aligned}$$

This leaves us with the SCHWARZSCHILD metric

$$\mathrm{d}s^2 = -\left(1-\frac{r_{\mathrm{S}}}{r}\right)\mathrm{d}t^2 + \left(1-\frac{r_{\mathrm{S}}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.$$

Although we derived the line element thinking of the spacetime outside a spherical matter distribution, it contains more than just that. The surface $r = r_{\mathrm{S}}$ is called event horizon. From the perspective of an infinitely far observer a freely falling object will never reach the horizon; it will slow down approaching the horizon and at the same time the signals become more and more redshifted. In this way the object will quite soon disappear from the view of the infinitely far observer. Since neither light nor matter can escape objects described by this metric are called black holes.

At the horizon the SCHWARZSCHILD metric becomes singular, but closer inspection shows that this is a coordinate problem only. Curvature is finite at the horizon and in fact for very massive black holes not even large. One can introduce coordinates which are perfectly regular at the horizon surface, e. g. KRUSKAL SEKERES coordinates. Using these coordinates the spacetime can be extended over the horizon. The resulting spacetime can be described using the above line element with $r < r_{\mathrm{S}}$ but in this regime r is a timelike coordinate and t is spacelike.

The points $r = 0$ form a spacelike singularity in the future of any timelike or lightlike geodesic starting within the event horizon. For any object including light it is therefore impossible to avoid the singularity as soon as it passed the horizon. The singularity is reached in finite time (or at finite geodesic parameter) and the geodesic ends at this point since it cannot be extended beyond. Furthermore curvature blows up at this point, resulting in infinitely large tidal forces on test bodies.

Further solutions describing rotating black holes (KERR solution) or such that carry charges (REISSNER NORDSTRÖM solution) exist. They exhibit a more complex geometry than the very simple SCHWARZSCHILD solution, but share the general feature of singularities.

2.1.3. Standard Cosmology

According to the concordance model of astrophysics the history of the universe is described as follows.

The universe begins in a spacelike singularity. This cosmological singularity is followed by an era of inflation, where spacetime is approximately DE SITTER and the scale factor grows exponentially. While the background inflates forever, locally isolated regions thermalise and form bubbles of FRIEDMANN ROBERTSON WALKER cosmologies. Although these bubbles expand into the surrounding background they will not collide or merge with each other since the inflationary expansion separates them faster than they can expand.

Inflation can explain the observed homogeneity and isotropy of our universe. Initially present inhomogeneities are simply streched out so far that they are behind our HUBBLE horizon. Our observable universe originates from such a small patch of the initial state that there are essentially no inhomogeneities or anisotropies. For this to be the case inflation has to take sufficiently long in order for the scale factor to grow by at least a factor of e^{60}.

Quantum fluctuations in the initial state source density fluctuations in the energy distribution in this patch of the universe. Due to their origin in the freezing out of modes as soon as they cross the HUBBLE horizon, these fluctuations have an almost scale-invariant spectrum.

Gravity is always attractive and thus matter will float to higher density regions increasing their density. In this way the density fluctuations are contrast enhanced by gravity and finally form dust clouds, stars and planets.

Currently we observe that the expansion rate is again increasing. The combined observations [12] of cosmic microwave background anisotropies by WMAP and luminosity distances of type I a supernovae are best explained in the ΛCDM model which is a FRIEDMANN ROBERTSON WALKER cosmology with critical energy density dominated by 72,6% vacuum energy (cosmological constant), 22,8% cold dark matter, and 4,56% baryonic matter. This results in a model where the universe undergoes several phases during the expansion after inflation. At the beginning radiation or relativistic matter dominates. Since radiation density is thinned out more strongly ($\propto \frac{1}{a^4}$) during expansion, at some point non-relativistic matter, with a density that goes $\propto \frac{1}{a^3}$, takes over and dominates the expansion. Finally vacuum energy with constant density will dominate. Actually we seem to live just at this point where vacuum energy begins to take over.

A theory of everything has to address and explain all aspects of the above model. In this work we will focus on the cosmological singularity and how this could be resolved.

2.2. String Theory

Originally intended as a theory describing strong interactions, string theory developed to the most advanced candidate for a theory of quantum gravity – even if it is still far from it. This review is mainly inspired by [26]. A more detailed introduction can be found in text books like [30] or [2].

2.2.1. Closed Strings

In String Theory the fundamental elements are not point particles but one-dimensional objects called strings. For the motion of a point particle the action that is extremalised is given by the invariant length of the worldline swept out during the motion. Since the string itself is already a one-dimensional object, it sweeps out a two-dimensional surface, the worldsheet, moving in spacetime. The natural choice for an action is thus the worldsheet volume (or surface area). This leads to the NAMBU GOTO action

$$S_{\mathrm{NG}}[X] = -\frac{1}{2\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \ \sqrt{-h}, \tag{2.3}$$

where $h = \det h_{ab}$ is the determinant of the induced world sheet metric $h_{ab} = \partial_a X^\alpha \partial_b X^\beta \eta_{\alpha\beta}$. In this way one can equivalently treat a string moving in a d-dimensional spacetime on the one hand or d fields X^α evolving within the two-dimensional world sheet.

In order to avoid the non-linearities arising from the fact that the NAMBU GOTO action involves the dynamical variables in the square root of the metric determinant, one introduces an auxiliary internal metric on the world sheet γ_{ab} and obtains the POLYAKOV action

$$S_{\mathrm{Pol}}[\gamma, X] = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \ \sqrt{-\gamma}\gamma^{ab}\partial_a X^\alpha \partial_b X^\beta \eta_{\alpha\beta}, \tag{2.4}$$

which is equivalent to (2.3) but results in linear equations of motion. This can be seen by calculating the variation with respect to the metric components using the simple relation[1] $\delta\gamma = -\gamma\gamma_{ab}\delta\gamma^{ab}$ to be

$$h_{ab} = \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd}. \tag{2.5}$$

Solving for $\sqrt{-\gamma}\gamma^{ab}$ and inserting into the POLYAKOV action (2.4) the world-sheet metric can be eliminated and we indeed return to the NAMBU-GOTO-action.

The POLYAKOV action (2.4) obeys a number of symmetries. It is invariant under reparametrizations of the world-sheet and WEYL rescalings. These symmetries of the POLYAKOV action give in total three gauge degrees of freedom, which allow to choose all three degrees of freedom in the worldsheet metric arbitrarily. Using

[1] The variation of γ may be computed directly from LAPLACE's formula for the determinant.

these gauge transformations the world-sheet metric can always be brought into the MINKOWSKIan form

$$\gamma_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.6}$$

In this particular gauge, the POLYAKOV action assumes the form of an action for D free bosonic fields on the cylinder,

$$S_{\mathrm{FF}}[X] = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{2.7}$$

But due to our gauge fixing the action principle has to be accompanied by a constraint equation

$$\partial_a X_\mu \partial_b X^\mu = \frac{1}{2} \eta_{ab} \partial_c X_\mu \partial^c X^\mu, \tag{2.8}$$

following from the constraint equation (2.5) for the world sheet metric with the gauge choice $\gamma_{ab} = \eta_{ab}$ inserted.

The equations of motion for the fields X^μ following from (2.7) are

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0. \tag{2.9}$$

These are a two-dimensional wave equations for each field X^μ. Its solution can be decomposed into the center of mass motion and oszillation that are left-moving and oscillations that are right-moving. Expressed in light-cone coordinates the fiels are thus given by

$$X^\mu(\sigma_+, \sigma_-) = x^\mu + \alpha' p^\mu \tau + \mathrm{i} \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(a_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma_-} + \bar{a}_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma_+} \right). \tag{2.10}$$

As usual we introduce canonically conjugated momenta $\Pi^\mu = \dot{X}^\mu$, where the dot denotes the derivative with respect to τ, and observe the POISSON structure and equal times

$$\begin{aligned} \left\{ X^\mu(\tau,\sigma); X^\nu(\tau,\sigma') \right\}_{\mathrm{P.B.}} &= \left\{ \Pi^\mu(\tau,\sigma); \Pi^\nu(\tau,\sigma') \right\}_{\mathrm{P.B.}} = 0 \\ \left\{ X^\mu(\tau,\sigma); \Pi^\nu(\tau,\sigma') \right\}_{\mathrm{P.B.}} &= 2\pi\alpha' \eta^{\mu\nu} \delta(\sigma - \sigma'). \end{aligned} \tag{2.11}$$

Equivalently we can choose the center of mass variables x^μ, p^μ and the oscillators a_n^μ, $\bar{a}_n^\mu$ as our variables and obtain the POISSON structure

$$\begin{aligned} \left\{ a_n^\mu; a_m^\nu \right\}_{\mathrm{P.B.}} &= \left\{ \bar{a}_n^\mu; \bar{a}_m^\nu \right\}_{\mathrm{P.B.}} = \mathrm{i}n\eta^{\mu\nu} \delta_{n+m,0}, \\ \left\{ a_n^\mu; \bar{a}_m^\nu \right\}_{\mathrm{P.B.}} &= 0, \\ \left\{ x^\mu; a_n^\nu \right\}_{\mathrm{P.B.}} &= \left\{ x^\mu; \bar{a}_n^\nu \right\}_{\mathrm{P.B.}} = 0 \qquad \forall n \neq 0, \\ \left\{ p^\mu; x^\nu \right\}_{\mathrm{P.B.}} &= \eta^{\mu\nu}. \end{aligned} \tag{2.12}$$

The energy-momentum tensor of the two-dimensional field theory is given by

$$T_{ab} = \frac{\delta \mathscr{L}}{\delta \partial^a X^\mu} \partial_b X^\mu - \eta_{ab} \mathscr{L} = \frac{1}{2\pi\alpha'} \partial_a X^\mu \partial_b X_\mu - \frac{1}{4\pi\alpha'} \eta_{ab} \partial_c X^\mu \partial^c X_\mu. \tag{2.13}$$

It is symmetric $T_{ab} = T_{ba}$, traceless $\eta^{ab}T_{ab} = 0$, and divergence free $\partial_a T^{ab} = 0$, which reduces the number of independent components to two. Especially in lightcone coordinates there are only the diagonal components which each depend on one coordinate only:

$$
\begin{aligned}
T_{++} &= T_{++}(\sigma_+) = \sum_{n\in\mathbb{Z}} l_n \mathrm{e}^{-\mathrm{i}n\sigma_-} &\quad \text{with} \quad \bar{l}_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} \bar{a}^\mu_m \bar{a}^\nu_{n-m}\eta_{\mu\nu}, \\
T_{--} &= T_{--}(\sigma_-) = \sum_{n\in\mathbb{Z}} l_n \mathrm{e}^{-\mathrm{i}n\sigma_-} &\quad \text{with} \quad l_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} a^\mu_m a^\nu_{n-m}\eta_{\mu\nu}.
\end{aligned}
$$

The constraint (2.8) we need to impose in addition to the action principle is equivalent to $T_{ab} = 0$. Expressed through the modes l_n and $\bar{l}_n$ it takes the simple form

$$l_n = \bar{l}_n = 0 \qquad \forall n \in \mathbb{Z}. \tag{2.14}$$

Using the POISSON structure (2.12) for the modes of X^μ we obtain, that the l_n and $\bar{l}_n$ obey the WITT algebra

$$\{l_n; l_m\}_{\text{P.B.}} = \mathrm{i}(n-m)l_{n+m}, \tag{2.15a}$$

$$\{\bar{l}_n; \bar{l}_m\}_{\text{P.B.}} = \mathrm{i}(n-m)\bar{l}_{n+m}, \tag{2.15b}$$

and POISSON commute with each other $\{l_n; \bar{l}_m\}_{\text{P.B.}} = 0$.

In the quantisation procedure for a classical theory with constraints one can either impose the constraints already on the classical level or after the actual quantisation on the quantum phase space. Both paths eventually lead to the same quantum theory. We will sketch the second approach in the following.

In the canonical quantization procedure the commutation relations of the operators are obtained from the POISSON brackets by replacing them with commutators

$$\{\cdot;\cdot\}_{\text{P.B.}} \mapsto \mathrm{i}\,[\cdot;\cdot]\,. \tag{2.16}$$

Through this replacement we obtain from the POISSON structure (2.12) the following commutation relations of the basic operators $a^\mu_n, \bar{a}^\mu_m, p^\mu$ and x^μ, where we do not introduce new symbols for the operators but continue to use the symbols introduced for the classical coefficients.

$$
\begin{aligned}
&\left[a^\mu_n; a^\nu_m\right] == \left[\bar{a}^\mu_n; \bar{a}^\nu_m\right] = n\eta^{\mu\nu}\delta_{n+m,0}, \\
&\left[a^\mu_n; \bar{a}^\nu_m\right] = \left[x^\mu; a^\nu_n\right] = \left[x^\mu; \bar{a}^\nu_n\right] = 0 \quad \forall n \neq 0, \\
&\left[x^\mu; p^\nu\right] = \mathrm{i}\eta^{\mu\nu}.
\end{aligned} \tag{2.17}
$$

We call operators a^μ_{-n} and $\bar{a}^\mu_{-n}$ with negative index (i. e. $n > 0$) creation operators and the remaining a^μ_n and $\bar{a}^\mu_n$ with positive index annihilation operators. We define ground states $|\,k\rangle$ as the states that are annihilated by all annihilation operators and that are eigenstates of the momentum operators p^μ with eigenvalue k^μ, i. e.

$$
\begin{aligned}
&a^\mu_n\,|\,k\rangle = \bar{a}^\mu_n\,|\,k\rangle = 0 \quad \forall n > 0, \\
&p^\mu\,|\,k\rangle = k^\mu\,|\,k\rangle\,.
\end{aligned} \tag{2.18}
$$

The state space can now be constructed by acting with any number of creation operators on each of the ground states $|\,k\rangle$ and taking the direct sum, which is, since k^μ is continuous, a generalised "'direct integral."'

The state space comes equipped with a natural bilinear form, which is unique as soon as we give the following normalisation and 'reality' condition

$$\langle k\,|\,k'\rangle = \delta^{(D)}(k-k'), \tag{2.19a}$$

$$\left(a_n^\mu\right)^* = a_{-n}^\mu, \qquad \left(\bar{a}_n^\mu\right)^* = \bar{a}_{-n}^\mu. \tag{2.19b}$$

This bilinear form is not positive definite and does thus not promote our state space to a HILBERT space.

While promoting the constraints (2.14) to operators there is an ordering ambiguity in l_0. We choose normal ordering for the operator versions of the constraints

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :\ \hat{a}_m^\mu \hat{a}_{n-m}^\nu\ : \eta_{\mu\nu}. \tag{2.20}$$

This results in the fact that L_0 may be shifted from the "'true"' operator $\hat{l}_0 = L_0 + a$ by some constant number a and the algebra obtains a central extension

$$[L_n; L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{2.21}$$

The constraints are implemented on the state space only in the weak form making sure that all matrix elements $\langle\psi\,|\,L_n\,|\,\phi\rangle$ vanish. We require $L_n\,|\,\psi\rangle = \bar{L}_n\,|\,\psi\rangle = 0$ for $n > 0$ and $L_0\,|\,\psi\rangle = \bar{L}_0\,|\,\psi\rangle = a\,|\,\psi\rangle$. It turns out that this requirement removes all states of negative norm from the state space as long as $a = 1$ and $D = 26$ as proven in a theorem by BROWER, GODDARD, and THORN. The still remaining null states of vanishing norm do not allow a sensible interpretation as wave functions. But it is consistent to simply set them to zero and factor them out to obtain the HILBERT space of closed bosonic string theory.

To obtain the mass spectrum of closed bosonic string theory one observes that

$$L_0 = \frac{\alpha'}{4}p^2 + \sum_1^\infty n\hat{N}_n \tag{2.22}$$

and we thus obtain for the mass

$$M^2 = -p^2 = -\frac{4}{\alpha'} + \frac{4}{\alpha'}\sum_1^\infty nN_n = -\frac{4}{\alpha'} + \frac{4}{\alpha'}\sum_1^\infty n\bar{N}_n. \tag{2.23}$$

The lowest mass state is $M^2 = -\frac{4}{\alpha'}$, which is tachyonic, but all excited states have non-negative mass. A closer look reveals that the tachyon $|\,k\rangle$ is a scalar particle. The first excited level are massless states of the form $|\,\Omega, k\rangle = \Omega_{\mu\nu}(k)a_{-1}^\mu \bar{a}_{-1}^\mu\,|\,k\rangle$. From the constraints L_0, L_1, $\bar{L}_1$ one obtains the conditions $k^\mu\Omega_{\mu\nu} = \Omega_{\nu\mu}k^\mu = 0$, and $k^2 = 0$. The decomposition into irreducible representations of SO(24) we obtain a symmetric traceless part representing a spin 2 particle, the graviton $G_{\alpha\beta}$, an antisymmetric part representing a spin 1 particle, the 2-form or KALB RAMOND field $B_{\alpha\beta}$, and a scalar particle, the dilaton Φ.

2.2.2. Open Strings

The description of open strings is very similar to the one of closed strings. Formally they obey the same action (2.7)

$$S_{\mathrm{FF}}[X] = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\,\mathrm{d}\sigma\,\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

accompanied by the same constraint equation (2.8)

$$\partial_a X_\mu \partial_b X^\mu = \frac{1}{2}\eta_{ab}\partial_c X_\mu \partial^c X^\mu.$$

But since now the world sheet swept out by the string propagating through spacetime has a boundary, namely the world lines of the string's ends, we have to impose boundary conditions. During the variation of (2.7) we now encounter a boundary term vanishing only if we impose either VON NEUMANN or DIRICHLET boundary conditions. While with VON NEUMANN boundary conditions the string's ends move freely and the condition $\partial_\sigma X^\mu|_{\text{string's end}} = 0$ makes sure that no momentum is flowing off the string, DIRICHLET boundary conditions pin the string's end at a fixed position $X^\mu|_{\text{string's end}} = x_0^\mu$. At first glance the VON NEUMANN condition seems to be the only natural choice since DIRICHLET conditions break the LORENTZ symmetry of spacetime and appear to be unphysical. But branes, which are higher dimensional solitons of string theory which extend along a $(p+1)$-dimensional hyperplane, break LORENTZ symmetry in exactly the same way. Thus branes get an alternative interpretation as objects on which open strings can end.

2.2.3. Super String

So far there were only bosonic degrees of freedom in our theory. In order to describe the real world, we need to add fermions. We thus supplement the bosonic fields X^μ already included in our two-dimensional field theory with D two-component spinor fields

$$\psi = \begin{pmatrix}\psi_-^\mu \\ \psi_+^\nu\end{pmatrix}, \qquad\qquad \bar\psi^\mu = (\psi^\mu)^*\,\rho^0,$$

where ρ^a are a set of matrices obeying the DIRAC algebra

$$\{\rho a, \rho b; =\} - 2\eta^{ab}.$$

The action reads

$$S[\psi] = \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\,\bar\psi\rho^a\partial_a\psi^\nu\eta_{\mu\nu}. \tag{2.24}$$

For the open string the coordinates again take values on the cylinder $\sigma \in [0, 2\pi]$ and $\tau \in \mathbb{R}$. The standard representation of the DIRAC matrices is given by

$$\rho^0 = \begin{pmatrix}0 & -\mathrm{i} \\ \mathrm{i} & 0\end{pmatrix}, \qquad\qquad \rho^1 = \begin{pmatrix}0 & \mathrm{i} \\ \mathrm{i} & 0\end{pmatrix}. \tag{2.25}$$

Using light cone cooridnates $\sigma_\pm = \tau \pm \sigma$ we thus obtain

$$S[\psi] = \frac{\mathrm{i}}{2\pi\alpha'} \int \mathrm{d}^2\sigma\, \eta_{\mu\nu} \left(\psi^\mu_- \partial_+ \psi^\nu_- + \psi^\mu_+ \partial_- \psi^\nu_+ \right). \tag{2.26}$$

The bosonic fields had the interpretation of coordinates of the string in spacetime and where thus required to be periodic $X(\sigma) = X(\sigma + 2\pi)$. This is not the case for the fermionic fields. From the variation of action (2.26) one obtains

$$\delta S[\psi] = -\frac{\mathrm{i}}{\pi\alpha'} \int \mathrm{d}^2\sigma\, \eta_{\mu\nu} \left(\partial_+ \psi^\mu_- \delta\psi^\nu_- + \partial_- \psi^\mu_+ \delta\psi^\nu_+ \right) + \frac{\mathrm{i}}{2\pi\alpha'} \int \mathrm{d}t\, \eta_{\mu\nu} \left(\psi^\mu_- \delta\psi^\nu_- + \psi^\mu_+ \delta\psi^\nu_+ \right) \Big|_{\sigma=0}^{\sigma=\pi}. \tag{2.27}$$

We thus conclude that either one of the following two boundary conditions is sufficient

$$\psi_\pm(\sigma + 2\pi) = \begin{cases} \psi_\pm(\sigma) & \text{Ramond sector,} \\ -\psi_\pm(\sigma) & \text{Neveau Schwarz sector.} \end{cases} \tag{2.28}$$

Furthermore left and right moving waves on the string are independent so that we can choose from four sets of boundary conditions, dividing our state space into four sectors: R-R, NS-NS, R-NS, and NS-R.

The equations of motion derived from (2.27) are again quite simple

$$\partial_+ \psi^\mu_- = 0, \qquad \partial_- \psi^\mu_+ = 0. \tag{2.29}$$

The fields thus depend on one light cone coordinate only. We again expand the solution in terms of modes b^μ_r, where we now allow for half integer indizes $r \in \frac{1}{2}\mathbb{Z}$ in order to allow for the two possible boundary conditions:

$$\psi^\mu_- = \begin{cases} \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} b^\mu_n \mathrm{e}^{-\mathrm{i}n(\tau-\sigma)} & \text{R sector,} \\ \sqrt{\alpha'} \sum_{s\in\mathbb{Z}+\frac{1}{2}} b^\mu_s \mathrm{e}^{-\mathrm{i}s(\tau-\sigma)} & \text{NS sector,} \end{cases} \tag{2.30a}$$

$$\psi^\mu_+ = \begin{cases} \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} \bar{b}^\mu_n \mathrm{e}^{-\mathrm{i}n(\tau+\sigma)} & \text{R sector,} \\ \sqrt{\alpha'} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \bar{b}^\mu_s \mathrm{e}^{-\mathrm{i}s(\tau+\sigma)} & \text{NS sector.} \end{cases} \tag{2.30b}$$

The canonically conjugated momenta to the fields $\psi^\mu_\pm$ can be read off (2.26) to be $2\pi\mathrm{i}\alpha'\Pi^\mu_\pm = \psi^\mu_\pm$ and we obtain the equal time Poisson anti-brackets

$$\left\{ \psi^\mu_\mp(\tau, \sigma); \psi^\nu_\mp(\tau, \sigma') \right\}_{\text{P.B.},+} = 2\pi\mathrm{i}\alpha' \eta^{\mu\nu} \delta(\sigma - \sigma'). \tag{2.31}$$

This translates into anti-commutation rules that take the same form in both sectors

$$\left\{ \bar{b}^\mu_r; \bar{b}^\nu_{r'} \right\} = \left\{ b^\mu_r; b^\nu_{r'} \right\} = \eta^{\mu\nu} \delta_{r+r',0}. \tag{2.32}$$

The stress-energy tensor is given by

$$T_{ab} = \frac{\mathrm{i}}{4\pi\alpha'}\eta_{\mu\nu}\left(\bar{\psi}^{\mu}\rho_a\partial_b\psi^{\nu} + \bar{\psi}^{\mu}\rho_b\partial_a\psi^{\nu}\right).$$

As in the bosonic case it is symmetric $T_{ab} = T_{ba}$, traceless $\eta^{ab}T_{ab} = 0$, and divergence-free $\partial_a T^{ab} = 0$. Again only two independent components. Especially in light cone coordinates the only non-vanishing components are he diagonal ones. Furthermore each of these depend on one coordinate only: $T_{\pm\pm} = T_{\pm\pm}(\sigma_\pm)$. We thus can again expand the energy momentum tensor components into FOURIER modes

$$T_{\pm\pm}(\sigma_\pm) = \frac{\mathrm{i}}{4\pi\alpha'}\psi^{\mu}_{\pm}\partial_{\pm}\psi^{\nu}_{\pm}\eta_{\mu\nu} = \sum_{n=-\infty}^{\infty} l_n \mathrm{e}^{-\mathrm{i}n\sigma_\pm}. \tag{2.33}$$

The modes l_n again obey the WITT algebra

$$\{l_n; l_m\}_{\text{P.B.}} = (n-m)l_{n+m}.$$

As in the case of the bosonic string, quantisation is done by promoting the oscillator modes b^{μ}_r to operators and building the state space from the ground states by acting with an arbitrary number of creation operators. While the ground state in the NS sector is unique, the creation operators from the R sector include zero modes b^{μ}_0.

The constraints are again promoted to the VIRASORO generators

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m \left(m + \frac{n}{2}\right) : b_{-m}bn + m : + \frac{D}{16}\delta_{m,0} &\qquad& \text{R sector,} \\ L_n &= \frac{1}{2}\sum_r \left(r + \frac{n}{2}\right) : b_{-r}bn + r : && \text{NS sector,} \end{aligned} \tag{2.34}$$

which obey the VIRASORO algebra with a central charge of $\frac{D}{2}$.

The mass spectrum for the superstring is obtained by implementing the constraints L_n and their superpartners, which are the modes of the superpartner of the energy momentum tensor, the supercurrent

$$\mathcal{G}_a = \frac{1}{4\pi\alpha'}\rho^b\rho_a\psi^{\mu}\partial_b X_{\mu}. \tag{2.35}$$

Each of the components $\mathcal{G}_a$ is a two component spinor but the current conservation $\partial_a\mathcal{G}^a$ and the relation $\rho^a\mathcal{G}_a = 0$ reduce the number of independent degrees of freedom to two, which are functions of one light cone coordinate each: $\mathcal{G}_- = \mathcal{G}_-(\sigma_-)$ and $\mathcal{G}_+ = \mathcal{G}_+(\sigma_+)$. As for the energy momentum tensor, we define modes of the supercurrent by an expansion:

$$\mathcal{G}_- = \begin{cases} \sum_{n\in\mathbb{Z}} G_n \mathrm{e}^{-\mathrm{i}\sigma_-(n+\frac{1}{2})}, \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r \mathrm{e}^{-\mathrm{i}\sigma_-(r+\frac{1}{2})}, \end{cases} \qquad \mathcal{G}_+ = \begin{cases} \sum_{n\in\mathbb{Z}} \bar{G}_n \mathrm{e}^{-\mathrm{i}\sigma_+(n+\frac{1}{2})}, \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{G}_r \mathrm{e}^{-\mathrm{i}\sigma_+(r+\frac{1}{2})}, \end{cases} \tag{2.36}$$

where the upper line refers to the R-sector and the lower line to the NS-sector.

The energy momentum tensor modes L_n and the suppercurrent modes G_n and G_r together obey the super-VIRASORO algebra

$$\begin{aligned}[L_n; G_r] &= \left(\frac{1}{2}n - r\right) G_{m+r}, \\ \{G_r; G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},\end{aligned} \tag{2.37}$$

and analogous relations in the NS sector, where r, s are replaced by integer valued m, l. In both the R and the NS sector applying these constraints on the state space we obtain a HILBERT space without negative norm states if $D = 10$. The lowest mass states in the R sector are two massless spinor representations $(\underline{8})_S$ and $(\underline{8})_C$ of $\mathrm{Spin}(8)$ corresponding to two chiralities. The NS sector ground state is a tachyon and the first excited states are a massless vector representation $(\underline{8})_V$ of $\mathrm{SO}(8)$. Interestingly despite the obvious supersymmetry within the worldsheet the spacetime particle spectrum is not supersymmetric.

In order to get rid of the NS sector tachyon and allow for spacetime supersymmetry, one introduces GSO projection short for GLIOZZI, SCHERK and OLIVE. The possibility results from the fact that the action is invariant under the transformation $b^\mu \mapsto -b^\mu$. This allows for the definition of two parity operators $\Gamma_\pm$ acting on the ground states of the NS- and the R-sector by

$$\Gamma_\pm \,|\, 0\rangle_{\mathrm{NS}} = -\,|\, 0\rangle_{\mathrm{NS}}, \qquad \Gamma_\pm \,|\, 0\rangle_{\mathrm{R}} = \pm \,|\, 0\rangle_{\mathrm{R}} \tag{2.38}$$

and anti-commuting with the oscillators:

$$\Gamma_\pm b^\mu_{n,r} = -b^\mu_{n,r}\Gamma_\pm. \tag{2.39}$$

In the NS-sector $\Gamma_\pm$ simply determine wether the number of fermionic oscillators is even (eigenvalue -1) or odd (eigenvalue $+1$). In the R-sector the operators are given by the chirality operator multiplied with the oscillator number parity operator. Due to the symmetry of the action we can now consistently truncate the spectrum to those states with eigenvalue $+1$ under the action of either Γ_+ or Γ_-. In the NS-sector this truncation is unique and removes the tachyonic ground state so that the new ground state is the massless vector. In the R-sector we have two choices which chirality to keep. This becomes relevant as soon as we combine the left- and right-moving sectors, since we obtain inequivalent theories either choosing the same or different projectors in both sectors:

$$\Gamma_{\mathrm{A}} \coloneqq \Gamma_+\bar{\Gamma}_- \equiv \Gamma_-\bar{\Gamma}_+, \qquad \Gamma_{\mathrm{B}} \coloneqq \Gamma_+\bar{\Gamma}_+ \equiv \Gamma_-\bar{\Gamma}_-. \tag{2.40}$$

Projecting onto the $+1$ eigenstates of either one of these parity operators gives the state space of type II superstring theory. From the NS-NS-sector there remain as massless states the dilaton Φ, the KALB-RAMOND field $B^{\alpha\beta}$ and the graviton $G^{\alpha\beta}$.

For type IIA the massless R-R-sector consists of a 1-form C_α and a 3-form field $C_{\alpha\beta\gamma}$ while the massless states from the R-NS- and the NS-R-sector form dilatino and gravitino, which are right- and left-handed respectively. For type IIB on the other hand the R-R-sector contains a 0-form C, a 2-form $C_{\alpha\beta}$, and a 4-form $C_{\alpha\beta\gamma\delta}$ while R-NS- and NS-R-sector now both contain the same left-handed multiplets forming dilatino and gravitino giving a chiral spectrum.

The existence of massless n-form fields allows, generalising ordinary MAXWELL theory, higher dimensional objects like Dp-branes to carry charges with respect to these fields. This allows the brane to be stabilised by the conserved charge. Since a n-form with even n couples to odd p-dimensional charged objects and vice versa, we obtain that type IIA theory contains even dimensional stabel branes while type IIB theory contains stable branes of odd dimension. Dp-branes with odd dimension in type IIA and even dimension in IIB respectively break supersymmetry. They do not carry conserved charges and are thus unstable. This results in the existence of tachyonic excitations.

2.2.4. Compactification

In contrast to the string theory prediction, every experiment from everyday observations to the highest precision collider measurements suggest that spacetime is $(3+1)$-dimensional. If string theory describes the real world, additional 6 spacelike dimensions must exist and somehow have to be hidden from our view. There are two main mechanims leading to a spacetime appearing as if it had less dimensions.

If all motion of particles that we and our experimental equipment consist of is bound to a $(3+1)$-dimensional manifold (a D3-brane) none of our probes, photons for instance, could move in the extra dimensions and could signal its presence. Only gravity as curvature of spacetime itself could always 'leave' the brane. But measurements concerning gravity are far less accurate than the ones concerning standard model interactions. Therefore it is quite easy to invent curved geometries surrounding the brane that prevent the gravitational detection of the extra dimensions. As was shown in [25], one can have more than four non-compact dimensions but nevertheless have NEWTONian and general relativistic gravity reproduced to sufficient precision.

The other possibility is compactification of some of the spacelike dimensions. If for instance a dimension closes in itself in a toroidal fashion and it's radius shrinks, already from basic quantum mechaincs it is clear that motion in this direction is strongly restricted. Essentially any particle existing in such a spacetime is confined in a quantum well with periodic boundary conditions. Thus there is a discrete number of values the momentum in this direction might take. But for a sufficiently small extra dimension the first excited state in this quantum well is beyond any energy range probed by current accelerators.

A simple example for compactification

To illustrate the concept of compactification and the effects thereof we look at a very simple model. Let there be one extra-dimension y in addition to the $(3+1)$ dimensions x^μ of ordinary spacetime. We will now linearise EINSTEIN's theory of gravity around the flat MINKOWSKI geometry on our 'brane' at $y=0$.

A massless particle in d dimensions is a representation of $SO(d-2)$, which exactly gives the 2 helicity states of the graviton in $d=4$ from SO(2). From the decomposition of SO(5) into irreducible representations of SO(2) we expect for $d=5$ the spin 2 graviton to reappear but in addition there should be a massless spin 1 vector particle and a massless spin 0 scalar particle.

In the procedure of linearisation we consider a one-parameter group of spacetimes given by the metric tensor g^ϵ_{AB} which are all solutions to the EINSTEIN field equations. We choose the parametrisation in such a way that the background-solution is obtained for $\epsilon = 0$; in our case this means $g^0_{AB} = \eta_{AB}$. In the same way all geometric objects A^ϵ are now functions of ϵ. For small ϵ we can TAYLOR expand this function in ϵ. The linearisation δA of the object A^ϵ is the first-order coefficient of this series:

$$\delta A := \left.\frac{\partial}{\partial \epsilon} A^\epsilon \right|_{\epsilon=0} .$$

As the linearisation of the EINSTEIN field equations we obtain a quasi-linear partial differential equation for the metric-perturbations. It's coefficients are given as functions of the background metric coefficients g^0_{AB}.

Gauge-transformations are ϵ-dependant coordinate-transformations. The gauge-field is given as the linearisation of the transformation:

$$\xi(x) = \delta x' = \left.\frac{\partial\, x'_\epsilon(x)}{\partial \epsilon}\right|_{\epsilon=0} .$$

Linearised fields are transformed with the LIE-derivative of the background-field:

$$\delta A'(x') = \delta A(x) - \mathscr{L}_\xi A^\epsilon(x).$$

For the metric this gives $\delta g'_{AB} = \delta g_{AB} - \xi_{(A;B)}$.

The most general perturbation of the MINKOWSKI-background can be written as

$$(\delta g_{AB}) = \begin{pmatrix} -2A & -\boldsymbol{S}^T & -B \\ & (h_{ij}) & \boldsymbol{E} \\ & & 2C \end{pmatrix}$$

The vectors $\boldsymbol{S}$ and $\boldsymbol{E}$ can be decomposed into a divergence and a divergence-free vector:

$$S_i = \nabla_i S + \bar{S}_i$$
$$E_i = \nabla_i E + \bar{E}_i$$

the same is done for the spatial part of the gauge-field:

$$(\xi_A) = (T, \nabla_i L + \bar{L}_i, L_y)$$

the symmetric tensor h can be decomposed:

$$\begin{aligned} h_{ij} &= 2H\delta_{ij} + 2F_{ij} = \\ &= 2H\delta_{ij} + 2\left(\nabla_{(i}F_{j)} + H_{ij}\right) \\ F_i &= \nabla_i F + \bar{F}_i \end{aligned}$$

where $\bar{\boldsymbol{F}}$ is divergence-free and H_{ij} is traceless and divergence-free.

$$\begin{aligned} A &\mapsto A + \partial_t T & \bar{S}_i &\mapsto \bar{S}_i - \partial_t \bar{L}_i \\ H &\mapsto H & \bar{E}_i &\mapsto \bar{E}_i + \partial_y \bar{L}_i \\ S &\mapsto S + T - \partial_t L & \bar{F}_i &\mapsto \bar{F}_i + \bar{L}_i \\ E &\mapsto E + \partial_y L + L_y \\ B &\mapsto B - \partial_t L_y + \partial_y T \\ C &\mapsto C + \partial_y L_y \\ F &\mapsto F + L & H_{ij} &\mapsto H_{ij} \end{aligned}$$

In addition to the invariant tensor H_{ij} the following 4 scalars and 2 vectors do not change under gauge-transformations:

$$\begin{aligned} \Psi &= A - \partial_t \left(S + \partial_t F\right) \\ \Phi &= -H \\ \mathcal{B} &= B - \partial_y \left(S + \partial_t F\right) + \partial_t \left(E - \partial_y F\right) \\ \mathcal{C} &= C - \partial_y \left(E - \partial_y F\right) \\ \Sigma_i &= \bar{S}_i + \partial_t \bar{F}_i \\ \mathcal{E}_i &= \bar{E}_i - \partial_t \bar{F}_i \end{aligned}$$

The condition

$$F = S = E = 0 \qquad \bar{\boldsymbol{F}} = \boldsymbol{0}$$

fixes the gauge completely. The remaining metric-perturbations are equal to the gauge-invariant variables. In this generalised longitudinal gauge the line-element is

$$\begin{aligned} \mathrm{d}s^2 = -(1+2\Psi)\mathrm{d}t^2 - 2\Sigma_i \mathrm{d}t\mathrm{d}x^i - 2\mathcal{B}\mathrm{d}t\mathrm{d}y + 2\mathcal{E}_i \mathrm{d}x^i \mathrm{d}y \\ + \left((1-2\Phi)\delta_{ij} + 2H_{ij}\right)\mathrm{d}x^i \mathrm{d}x^j + (1+2\mathcal{C})\mathrm{d}y^2 \end{aligned} \quad (2.41)$$

The perturbation of the energy-momentum-tensor ($T^0_{AB} = 0$) for a NEWTONian source on the brane is given by

$$(\delta T_{AB}) = \begin{pmatrix} \rho & \frac{1}{2}\boldsymbol{v}^T & 0 \\ \frac{1}{2}\boldsymbol{v} & (T_{ij}) & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \cdot \delta(y), \tag{2.42}$$

where $T_{ij} = P\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\triangle\right)\Pi + \partial_{(i}\Pi_{j)} + \Pi_{ij}$. In the NEWTONian limit we have $\rho = |T_{00}| \gg |T_{0i}| \sim \rho V$ and $|T_{00}| \gg |T_{ij}| \sim \rho V^2$, where $V \ll 1$ is a typical velocity of the system in units of the speed of light. The linearised EINSTEIN field equations are given by

$$\begin{aligned}
G_{tt} &= \triangle(2\Phi - \mathcal{C}) + 3\partial_y^2\Phi \\
G_{ti} &= \partial_i\left(\partial_t\left(2\Phi - \mathcal{C}\right) - \frac{1}{2}\partial_y\mathcal{B}\right) + \frac{1}{2}\partial_y\left(\partial_y\Sigma_i + \partial_t\mathcal{E}_i\right) + \frac{1}{2}\triangle\Sigma_i \\
G_{ij} &= (\delta_{ij}\triangle - \partial_i\partial_j)(\Psi - \Phi + \mathcal{C}) + \delta_{ij}\left(\partial_t^2(2\Phi - \mathcal{C}) - \partial_y^2(2\Phi - \Psi) - \right. \\
&\quad \left. -\partial_t\partial_y\mathcal{B}\right) + \partial_t\partial_{(i}\Sigma_{j)} + \partial_y\partial_{(i}\mathcal{E}_{j)} + \left(\partial_t^2 - \triangle - \partial_y^2\right)H_{ij} \\
G_{ty} &= \frac{1}{2}\triangle\mathcal{B} + 3\partial_t\partial_y\Phi \\
G_{iy} &= \partial_i\left(\partial_y\left(2\Phi - \Psi\right) + \frac{1}{2}\partial_t\mathcal{B}\right) + \frac{1}{2}\left(\partial_t\partial_y\Sigma_i + \partial_t^2\mathcal{E}_i - \triangle\mathcal{E}_i\right) \\
G_{yy} &= -\triangle\left(2\Phi - \Psi\right) + 3\partial_t^2\Phi
\end{aligned}$$

Since the length L of the extra dimension y shall be much smaller than any scale of the system in consideration only zero-modes with respect to y have to be considered. Furthermore since the additional dimension cannot be observed, we integrate it out. This removes the DIRAC-delta from the energy momentum tensor and gives a factor L. The four dimensional NEWTON constant is $G_4 = \frac{\pi}{2L}G_5$. We thus obtain

$$\begin{aligned}
\triangle\left(2\Phi - \mathcal{C}\right) &= 8\pi G_4\rho & \frac{1}{2}\triangle\Sigma_i &= 8\pi G_4\bar{v}_i \\
\partial_t\mathcal{C} - 2\partial_t\Phi &= 8\pi G_4 v & \partial_t\Sigma_i &= 8\pi G_4\Pi_i \\
\partial_t^2\left(2\Phi - \mathcal{C}\right) &= 8\pi G_4\left(P + \frac{2}{3}\triangle\Pi\right) & \triangle\mathcal{E}_i - \partial_t^2\mathcal{E}_i &= 0 \\
\Phi - \Psi - \mathcal{C} &= 8\pi G_4\Pi \\
\triangle\mathcal{B} &= 0 \\
\partial_t\mathcal{B} &= 0 \\
-\triangle\left(2\Phi - \Psi\right) + 3\partial_t^2\Phi &= 0 & -\triangle H_{ij} + \partial_t^2 H_{ij} &= 8\pi G_4\Pi ij
\end{aligned}$$

We observe that $\mathcal{B}$ and $\boldsymbol{\mathcal{E}}$, which parametrise $\delta g_{\mu y}$, are completely decoupled. Linear combination gives three wave equations for Φ, Σ_i, H_{ij}

$$\left(\partial_t^2 - \triangle\right)\Phi = -8\pi G_4\left(\rho - \triangle\Pi\right)$$
$$\left(\partial_t^2 - \triangle\right)\Sigma_i = 8\pi G_4\left(\dot{\Pi} + 2\bar{v}_i\right)$$
$$\left(\partial_t^2 - \triangle\right)H_{ij} = 8\pi G_4 \Pi_{ij}$$

The other variables are determined by constraint equations.

The compactification of a vacuum spacetime on a circle resulted in an effective theory in a spacetime of lower dimension containing a massless vector field and a massless scalar field. The latter is called *modulus* field and its appearance is typical for any compactification scheme. The modulus describes the size of the extra dimension.

From the lower dimensional point of view the motion in the compactified direction is unobservable. The momentum in this direction is quantised, since the motion along the circle is essentially given by a particle in a well with periodic boundary conditions. The quanta are inverse proportional to the size R of the dimension. For the low dimensional observer the momentum in the extra dimension appears as a rest-mass-like contribution to the mass.

While these KALUZA KLEIN modes appear already for point particles, strings exhibit an additional feature, when compactified. Due to the fact that strings are one-dimensional objects, they can wrap the extra dimension. From lower dimensional point of view a string wrapped around a compact dimension has again a rest-mass-like contribution to its mass from the tension of the string winding around the small extra dimension, which is now proportional to R.

For open strings it depends on wether the boundary conditions in the compactified dimension ist of the DIRICHLET or VON NEUMANN type which of the aforementioned effects appears. For closed strings we will encounter both. The effective low-dimensional mass formula for a closed string then reads

$$M^2 = \frac{l^2}{R^2} + \frac{m^2R^2}{\alpha'^2} - \frac{4}{\alpha'} + \frac{4}{\alpha'}\sum nN, \tag{2.43a}$$

and for the open string with DIRICHLET boundary conditions

$$M^2 = \frac{m^2R^2}{\alpha'^2} - \frac{1}{\alpha'} + \frac{1}{\alpha'}\sum nN, \tag{2.43b}$$

and for the open string with VON NEUMANN boundary conditions

$$M^2 = \frac{l^2}{R^2} - \frac{1}{\alpha'} + \frac{1}{\alpha'}\sum nN. \tag{2.43c}$$

Here we immediately see that the spectrum is invariant under a transformation $R \mapsto R' = \frac{\alpha'}{R}$ as long as we exchange DIRICHLET and VON NEUMANN boundary conditions at the same time. This is called T-duality.

2.2.5. Orbifold

Another very simple geometry with interesting features is an *orbifold*, which means that points in spacetime that are mapped by the action of some discrete group onto each other are identified. In a $\mathbb{Z}_2$-orbifold the mapping is given by $y \mapsto -y$ resulting in interesting symmetry conditions on all fields.

2.3. Resolution of singularities

Through the last century EINSTEIN's theory of gravity has proven itself a reliable and accurate description of the universe on large scales. This simple description of the geometry of spacetime up to now passed any precision test it was posed and there is no indication of failure. Nevertheless there is good reason to believe that it is only an effective description of nature.

The singularity theorems of PENROSE [24] and HAWKING [11] prove that – under some reasonable conditions on causality and energy – in any spacetime, which is not extremely symmetric and thus capable of describing our universe, EINSTEIN's field equations predict that incomplete, inextendable time-like or null geodesics exist. The end point of such an inextendable geodesic is a SCHMIDT singularity. If it is time-like and in the future light cone an observer could in principle reach such a point in finite time and would cease to exist. Located in the past light cone at the singularity we would be forced to choose arbitrary initial conditions reducing the predictive power of the theory.

Hence the existence of a complete description for the whole universe based on EINSTEIN's theory of gravity would imply that nature itself breaks down at some points in space and/or time. As physicists we do not believe in this but rather conclude that EINSTEIN's theory of gravity successful as it is has to be incomplete. There has to be a 'better' theory in which singularities in the SCHMIDT sense are avoided at all. There are several ways to approach the search for such an enhanced theory of nature. Guided by higher principles we could try to write down the theory of everything from scratch and then prove that it reduces to EINSTEIN's theory of gravity with a $(3+1)$-dimensional spacetime in a certain limit. In contrast to this top-down approach we could proceed bottom-up by trying to guess the leading corrections added to the known field theory as soon as we depart from this limit.

In general the properties of SCHMIDT singularities in a spacetime are quite difficult to analyse. The aforementioned singularity theorems do not predict the concrete nature of the singularities so that they are hard to pin down. At least it seems to be a quite generic feature of any singularity in spacetime that some curvature invariant diverges. This coincides with our naive intuition of a singularity but singularities in the above sense without singular behaviour of curvature invariants are conceivable. As an example we might take the tip of a cone which is singular in the sense that geodesics stop without the possibility to be extended. But at any point in the vicinity of the tip the surface of the cone is exactly flat and thus all curvature invariants vanish.

For singularities that are reached on time-like curves this intuitive statement can be refined. If there exists an end point of time-like curves along which all the components of the RIEMANN tensor do tend to a limit then CLARKE proves in [6] that the spacetime can be extended beyond this point. Hence at any essential singularity reached on a time-like curve at least one component of the RIEMANN tensor has to diverge. As we pointed out earlier this is clearly the case for the most common singularities of SCHWARZSCHILD spacetime and the FRIEDMANN ROBERTSON WALKER cosmologies.

According to the concordance model of cosmology for our universe there was an inflationary phase in the early history. In the 'new inflation' model the expansion is driven by a self-coupled scalar field, the inflaton. The inflaton potential has a maximum at the origin but is very flat there. The slow rolling away from the origin guarantees inflation to last for a sufficiently long period. But finally the inflaton reaches the potential minimum. During oscillations around the potential minimum the energy is transferred from the inflaton field into ordinary matter fields. Henceforth spacetime turns from approximately DE SITTER at the potential maximum to a FRIEDMANN ROBERTSON WALKER cosmology with ordinary matter.

DE SITTER universe is a perfectly regular and singularity free spacetime. Thus one might hope that this inflationary epoch avoids the initial singularity, the Big Bang. This is not the case. A spacetime that is eternally inflating to the future as in the old inflationary scenario and obeys the weak energy condition as well as some technical assumptions cannot be past null geodesically complete (see [3]). While the past singularities might not form a spacelike surface as the Big Bang, the general problem of the past singularities remains.

2.3.1. Mixmaster cosmology

The Mixmaster cosmology is an example for a more general singularity compared to the highly symmetric examples like the Big Bang in FRW cosmology or the singularity of a SCHWARZSCHILD black hole. It generalises the KASNER metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{p_1}(\mathrm{d}x^1)^2 + t^{p_2}(\mathrm{d}x^2)^2 + t^{p_3}(\mathrm{d}x^3)^2, \tag{2.44}$$

whith $p_1 + p_2 + p_3 = (p_1)^2 + (p_2)^2 + (p_3)^2 = 1$. While approaching the singularity, space collapses along two axes and expands along the third. Introducing space curvature in the Mixmaster model results in the fact that this KASNER collapse cannot continue to the singularity and is diverted into a different KASNER regime exchanging expanding and contracting directions. In this way while approaching the singularity, spacetime oscillates chaotically. For generic initial conditions there are infinitely many epochs of different KASNER regimes in the vicinity of the singularity.

Although the singularity remains at finite proper time in the past of the Mixmaster cosmology, its history is infinite in the sense that the universe had undergone infinitely many previous, discrete events. This resembles the argument that even in the FRW cosmology reasonable clocks would assign an infinite number of ticks to the time period since the Big Bang. For any real clock giving the time since the Big Bang is an operation of mathematical extrapolation, since the clock could not exist during the first ticks it should have counted. It would have to be replaced by a clock that can exist at higher energies, which means that its period is smaller and thus the new clock gives a higher tick number. This infinite series of clocks might thus assign the universe an infinite past in the appropriate sense.

2.3.2. Pre-big bang

There are several ways to address the singularity problem. In string theory corrections to the EINSTEIN HILBERT action arise in many different ways. The pre-big bang scenario (see [8] for a review) is a consequence of a symmetry of the tree-level equations of motion of string theory. The equations for the scale factor $a(t)$ and the dilaton $\phi(t)$ are not only symmetric under time reflection $t \mapsto -t$ as the FRIEDMANN equations but also symmetric under the scale-factor duality transformation

$$a \mapsto \frac{1}{a}, \qquad\qquad \phi \mapsto \phi - 2D \ln a, \tag{2.45}$$

where D is the number of spacetime dimensions. The 'post-big bang' solution of standard cosmology with decelerated expansion defined for positive times is by these dualities connected to an inflationary 'pre-big bang' solution for negative times. In this way the cosmic evolution is extended to times prior to the big bang in a self-dual way but the solution is still singular. One can obtain regular self-dual solutions by imposing a suitable potential for the dilaton. But albeit a potential of this form might be the result of higher-loop quantum corrections, its actual appearance is not proven and its exact form is not derived rigorously.

2.3.3. Ekpyrotic and cyclic universe

In the ekpyrotic scenario [16, 15] the hot big bang is the result of the collision of two branes, which might be the boundaries of an extra dimension. The cyclic universe of [28] is a refined version of this model, where the universe undergoes and endless or at least very high number of ekpyrotic transitions in an oscillatory way. Up to now explicit cyclic models are only given as effective four-dimensional models that can only be motivated by 'inspiration' from heterotic M-theory. Furthermore the bounce in the ecpyrotic scenario occurs at a real curvature singularity, which makes it impossible to follow the evolution of perturbations through the bounce. This seems to be discouraged by explicit calculations in simple models for a bouncing universe as in [1].

Due to the necessary violation of the null energy condition during the bounce, there is the generic danger of introducing matter with negative energy density,

i. e. ghosts. This will in general result in an instability of the vacuum. The new ekpyrotic scenario in which the problems were addressed by the addition of a Ghost condensate, suffers from this problem (see [14]).

2.3.4. KKLT scenario

The scenario of KACHRU, KALLOSH, LINDE, and TRIVEDI (see [13]) is based on brane anti-brane inflation. The inflation in the four dimensional geometry is driven by the interaction of a Dp-$\bar{\text{D}}p$ pair. The interbrane distance modulus plays the role of the inflaton. In order to satisfy the slow-roll conditions for inflation and obtain a sufficiently flat potential, the brane anti-brane pair is placed in a warped throat with approximately ADS geometry. The model requires the stabilisation of all moduli apart from the interbrane distance, which is a highly non-trivial issue since generically stabilisation mechanisms affect the interbrane distance as well an thus spoil inflation.

Furthermore the KKLT scenario utilises a rather complex geometry with a CALABI-YAU manifold with an approximately ADS throat, which is expected to have a smooth tip of finite size, where the anti-brane is placed.

3. Limiting Curvature Through Higher Derivatives

3.1. Motivation

Having observed that the blowing up of some curvature invariant at a singularity seems to be a generic feature we conclude that a promising step on the way towards a singularity free theory seems to effectively limit the curvature that can occur in spacetime. In an analogous way the step from NEWTONian mechanics to EINSTEIN's special theory of relativity can be viewed as simply limiting the speed of propagation to the speed of light c.

Based on these observations we are led to construct approximations to the low energy effective theory of full quantum gravity as minimal deformations of EINSTEIN's general relativity. A generic feature of low energy effective actions as limiting cases of fundamental theories of gravity is the appearance of higher derivative terms as well as non-local terms.

Higher derivative corrections to EINSTEIN's theory in general tend to induce even more singularities than there were present before. Hence we have to choose the higher derivative terms very carefully in order to remove singularities from solutions and on the other hand not to spoil the NEWTONian limit. The *Limiting Curvature Hypothesis* proposed by Markov [22] provides a simple scheme to achieve this.

3.2. Limiting Curvature Hypothesis

Several ground-breaking new developments in theoretical physics are accompanied by some limiting process. In NEWTONian mechanics any two bodies can in principle be accelerated to arbitrary high relative velocities, while in EINSTEIN's special theory of relativity the relative velocity of any two objects is limited by the finite and – independently of the observer – constant propagation speed of light.

Furthermore in quantum mechanics the minimal phase space volume a particle can be localised in is given by PLANCK's constant $\hbar$ whereas its phase space position could in principle have been given arbitrarily precise in classical mechanics.

But there is still one fundamental constant left that has not been used for a limiting procedure in the current standard model of theoretical physics: NEWTON's gravitational constant. In a limiting procedure for spacetime curvature NEWTON's constant plays the analogous role as the speed of light in special relativity and PLANCK's constant in quantum mechanics.

In the step from NEWTONian mechanics of a point particle to the special theory of relativity there is only one scalar function, namely the spatial velocity squared v^2, which has to be bounded to obtain a theory where all velocities are bounded. The general theory of relativity is more difficult in the sense that there are infinitely many curvature invariants that all have to be bounded. From the RIEMANN tensor we can build an arbitrary number of spacetime scalars, e. g.

$$R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu;\lambda}R^{\mu\nu;\lambda}, \ldots$$

which all have to be bounded in a non-singular theory. It does not make sense to introduce an infinite number of LAGRANGE multiplier fields. Thus it is impossible to bound each spacetime scalar explicitly.

The idea of the Limiting Curvature Hypothesis is now to impose an explicit bound on a finite subset of invariants only and make sure, that their limiting value uniquely determines a non-singular spacetime. In this way we ensure that at any would be singular spacetime is driven to a non-singular solution and henceforth all infinitely many invariants are finite.

3.3. Limiting procedure

Let us first review the general procedure allowing us to obtain a new, modified theory, in which certain invariants are bounded, starting from the LAGRANGEian formulation of a well established theory. Let us start with the action of classical NEWTONian mechanics of a point particle

$$S_{\text{old}}[x] = \int \mathrm{d}t \, \frac{m}{2}\dot{x}^2. \tag{3.1}$$

and try to 'derive' the special theory of relativity from the fact that the particle's velocity cannot exceed the speed of light c.

So we want to limit the particle's velocity $\dot{x}$, which is a constant of motion for the free point particle we are looking at. We can achieve this by adding a LAGRANGE multiplier field $\phi(t)$ multiplying the to-be-bounded function $\dot{x}^2$ and an appropriate potential $V(\phi)$ to obtain

$$S[x,\phi] = m \int \mathrm{d}t \, \left(\frac{1}{2}\dot{x}^2 + \phi\dot{x}^2 - V(\phi)\right). \tag{3.2}$$

The field ϕ is an auxiliary field since there are not any derivatives. It can be removed from the theory without changing the dynamical degrees of freedom. This would leave us with an action that is given by NEWTON's and some correction given in terms of some complicated function of $\dot{x}$. But the field equation for ϕ,

$$\dot{x}^2 = V'(\phi), \tag{3.3}$$

allows us to impose constraints on $\dot{x}^2$ by the choice of our potential. If we choose the potentials such that $|V'(\phi)| \leq c^2$, this enforces the limit $\dot{x}^2 \leq c^2$ on the particle's velocity.

But we are not completely free in the choice of the potential. We have to make sure that we do not spoil our well established low-velocity regime of the theory. For particles with velocities small against the velocity of light c NEWTON's theory should be recovered. In order to derive appropriate constraints on the potential, we do a TAYLOR expansion of the potential

$$V(\phi) = V_1\phi^n + \mathcal{O}(\phi^{n+1}) \tag{3.4}$$

for small ϕ. This simplifies the field equation for ϕ significantly and we can easily solve it in leading order

$$\phi = \left(\frac{\dot{x}^2}{nV_1}\right)^{\frac{1}{n+1}} + \ldots.$$

Using this solution, we can eliminate the auxiliary field ϕ from the action $S[x, \phi]$ and obtain

$$S[x] = m \int \mathrm{d}t \, \left(\frac{1}{2}\dot{x}^2 + a\dot{x}^{\frac{2n}{n-1}}\right), \tag{3.5}$$

with some constant a. The correction term to the NEWTONian action is thus of higher order and therefore negligible at low velocities as long as $n > 1$. A potential satisfying this constraint is $V(\phi) = \frac{2\phi^2}{2\phi+1}$. In fact this potential is not chosen arbitrarily but does exactly reproduce the action for a point particle in special relativity

$$S[x] = m \int \mathrm{d}t \, \sqrt{1 - \dot{x}^2}. \tag{3.6}$$

3.4. Cosmological singularity

We now review the application of the Limiting Curvature Hypothesis from [5] to EINSTEIN's theory of gravity for isotropic, homogeneous spacetimes. These strong symmetry assumptions simplify the calculation very much and allow a full analysis of the problem. But they are only applicable to the Big Bang singularity of the FRIEDMANN ROBERTSON WALKER cosmology and not for a generic singularity. We will extend the analysis for anisotropic singularities in section 3.5. For explicit calculations we choose comoving coordinates for the FRW cosmology in the usual way

$$\mathrm{d}s^2 = -\nu(t)^2\mathrm{d}t^2 + a(t)^2\left(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right), \tag{3.7}$$

and define the HUBBLE parameter $H(t) = \frac{\dot{a}(t)}{a(t)}$. The lapse function $\nu(t)$ is introduced for convenience and set to 1 after derivation of the field equations.

First we choose a curvature invariant that vanishes if and only if spacetime is DE SITTER. DE SITTER spacetime is non-singular. In the vicinity of a would-be-singularity the invariant takes its limiting value 0. Hence in this region spacetime is approximately DE SITTER and thus non-singular. An invariant that singles out DE SITTER spacetime from all isotropic, homogeneous spacetimes is given by

$$I_2 = 4R_{\mu\nu}R^{\mu\nu} - R^2 = 12\dot{H}^2, \tag{3.8}$$

where we took the name I_2 from [5]. I_2 clearly vanishes if and only if $H(t) = \text{const.}$ in which case (3.7) reduces to a DE SITTER metric. We furthermore observe that I_2 is non-negative for a wide class of spacetimes, e. g. for all spherically symmetric ones.

Our modified version of the EINSTEIN HILBERT action then reads

$$S[g, \phi] = \int \mathrm{d}^4x \ \sqrt{-\det g}\,(R + \phi I_2 - V(\phi)). \tag{3.9}$$

Instead of the term ϕI_2 we could have chosen $\phi f(I_2)$ with a rather arbitrary function f. But a suitable field redefinition will then recover the above form with a slightly adjusted potential.

The choice of the potential $V(\phi)$ is restricted by the requirement that

$$\lim_{|\phi| \to \infty} V'(\phi) = 0.$$

For large ϕ we assume the general form of the potential to be

$$V(\phi) = H_0^2 - \phi^{-n} + \mathcal{O}(\phi^{-n-\epsilon}) \tag{3.10}$$

Further conditions arise from the requirement that for small values of ϕ the experimentally well tested NEWTONian limit is recovered. For this we have to make sure that we only introduce higher order corrections in the low curvature regime. TAYLOR expanding the potential

$$V(\phi) = V_1\phi^m + \mathcal{O}(\phi^{m+1}) \tag{3.11}$$

for small ϕ and eliminating ϕ from the action $S[x, \phi]$, we obtain—keeping only the lowest order terms—

$$S \approx \begin{cases} \int \left(R + A I_2^{\frac{m}{m-1}}\right) \sqrt{-g}\,\mathrm{d}^4x & \text{for } m > 1 \\ \int (R + B + C I_2) \sqrt{-g}\,\mathrm{d}^4x & \text{for } m = 1 \end{cases} \tag{3.12}$$

with some constants A, B, C. Since I_2 is itself quadratic in metric components, the correction term is of the desired order provided $m > 1$. The case $m = 1$ reduces to EINSTEIN gravity with cosmological constant at low curvature. From the constraint equation (3.24) and the fact that the invariant I_2 is strictly non-negative it follows that V has to be a monotonically increasing function.

For practical purpose we now explicitly choose the potential to be

$$V(\phi) = H_0^2 \frac{\phi^2 \operatorname{sgn}(\phi)}{\left(1 + \sqrt{|\phi|}\right)^4} \tag{3.13}$$

but the qualitative features of the solutions will not depend on this explicit choice.

Since we are only interested in cosmological solutions, the field equations can be derived by explicitly plugging in the metric ansatz (3.7) into the action (3.9) and calculate the EULER LAGRANGE equations w. r. t. $\nu(t)$, $a(t)$ and $\phi(t)$. We obtain

$$-4\ddot{H}\phi H - 4\dot{H}\dot{\phi}H + 2\dot{H}^2\phi - 12\dot{H}H^2\phi + H^2 + \frac{V(\phi)}{6} = 0, \tag{3.14a}$$

$$-4\dddot{H}\phi - 4\ddot{\phi}\dot{H} - 8\ddot{H}\dot{\phi} - 24\ddot{H}H\phi - 18\dot{H}^2\phi - 24\dot{H}\dot{\phi}H \tag{3.14b}$$

$$-36\dot{H}H^2\phi + 2\dot{H} + 3H^2 + \frac{V(\phi)}{2} = 0,$$

$$\dot{H}^2 - \frac{1}{12}V'(\phi) = 0. \tag{3.14c}$$

Where equation (3.14b) resulting from the variation w. r. t. $a(t)$ can be easily shown to be a consequence of the other two.

We are looking at contracting cosmological solutions with $H < 0$ with the big crunch singularity in the future. Big bang singularities with expanding cosmology can be obtained from this by time reversal. The system can be solved for $\dot{H}$ and $\dot{\phi}$ and we obtain

$$\frac{\dot{H}}{\dot{\phi}} = \frac{\mathrm{d}\,H}{\mathrm{d}\phi} = \frac{H\,(2V' + \phi V'')}{6H^2 + V \pm 12\,\sqrt{3}H^2\phi\,\sqrt{V'} + \phi V'}. \tag{3.15}$$

This expressions tends to 0 for large ϕ with any potential of the general form (3.10). Figure 3.1 reveals the general structure of the phase space of solutions. The phase space is separated into two parts. A trajectory starting on the left will reach $\phi \to -\infty$ approaching a horizontal line asymptotically while trajectories starting on the right will go to $\phi \to \infty$.

Thus for each individual trajectory the absolute curvature $|H| <$ const. is bounded but there is no general bound $|H| < H_0$ valid for all of the trajectories. Brandenberger et al show in [5] that this can be overcome by introducing a second auxiliary field ϕ_2 coupled to the invariant $I_1 = R - \sqrt{3}\,\sqrt{4R_{\mu\nu}R^{\mu\nu} - R^2}$. Since for a homogeneous, spatially flat universe $I_1 = 12H^2$ it is clear that a suitable potential with limited derivative will result in a hard limit on the curvature.

3.5. General singularities

More general singularities in EINSTEIN's general theory of relativity involve less symmetry than the homogeneous and isotropic FRIEDMANN ROBERTSON WALKER

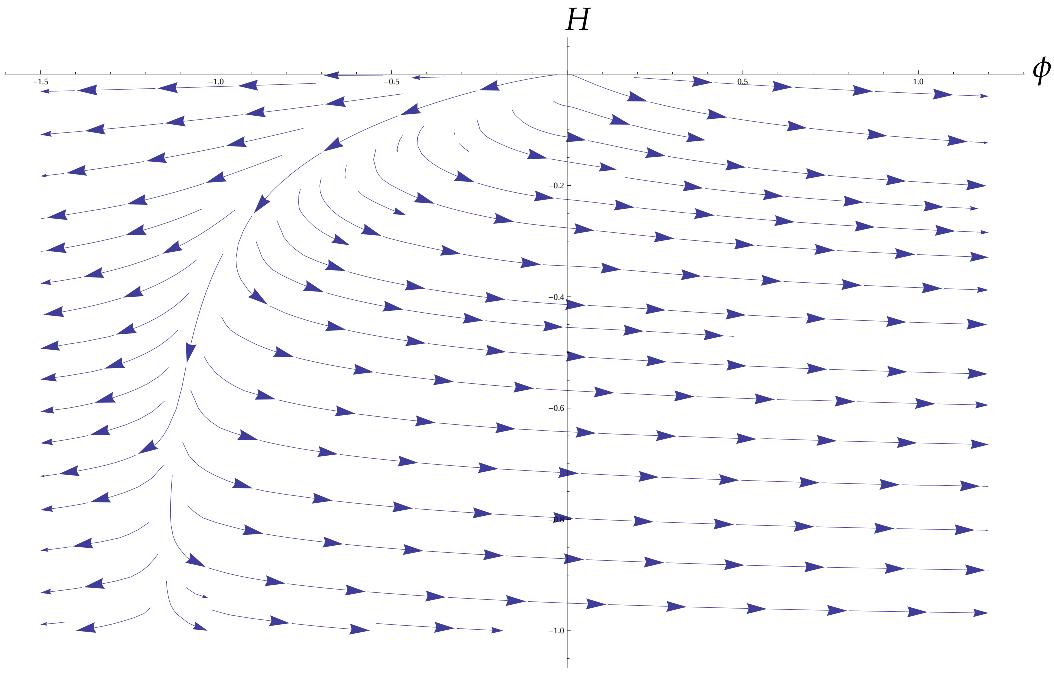


Figure 3.1.: Stream lines of the vector field for $\frac{\dot{H}}{\dot{\phi}}$.

spacetime. We will thus analyse more general spacetimes. Our parametrisation is very similar to the Cartesian coordinates of FRW spacetime. We introduce scale factors $\alpha(t)$, $\beta(t)$, $\gamma(t)$ for each direction in space. Our line element hence reads

$$\mathrm{d}s^2 = -\nu(t)^2\mathrm{d}t^2 + \alpha(t)^2\mathrm{d}x^2 + \beta(t)^2\mathrm{d}y^2 + \gamma(t)^2\mathrm{d}z^2, \tag{3.16}$$

where we introduced the lapse function $\nu(t)$. This is convenient to derive the field equations and will be set to $\nu(t) = 1$ thereafter. Otherwise our ansatz would not give the 00-component of the EINSTEIN field equations.

In the vicinity of a general spacelike singularity spacetime should be well approximated by a metric of this form. The SCHWARZSCHILD spacetime of an eternal black hole has the line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{3.17}$$

This form of the metric is valid in- and outside of the horizon but inside the horizon the naming of the coordinates is misleading, since $r < 2GM$ and thus t is a spacelike coordinate whereas r is timelike. We thus introduce comoving coordinates with proper time $\tau = \int \frac{1}{\sqrt{\frac{2GM}{r} - 1}}\mathrm{d}r$ and $x = t$. The line element now reads

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \left(\frac{2GM}{r}(\tau) - 1\right)\mathrm{d}x^2 + r(\tau)^2\mathrm{d}\Omega^2. \tag{3.18}$$

Locally we can approximately choose cartesian coordinates on the sphere such that $d\Omega^2 = dy^2 + dz^2$ and we obtain the metric of an anisotropically contracting homogeneous universe. The metric is of the form (3.16) with $\beta = \gamma$.

In order to apply the limiting curvature hypothesis in the same way as for the isotropic case, we need to find an invariant which for all spacetimes of the form (3.16) is zero if and only if the spacetime is non-singular. As we already pointed out earlier, the invariant I_2 used in the isotropic case is non-negative in a wide class of spacetimes including (3.16). We thus only have to add another non-negative invariant that is non-zero in spacetimes which are not DE SITTER but have vanishing I_2. An invariant which measures the anisotropy of a spacetime is the squared WEYL tensor $C^2 = C_{\mu\nu\tau\sigma}C^{\mu\nu\tau\sigma}$.

To single out DE SITTER spacetime from the class of homogeneous and anisotropic spacetimes we can thus choose the sum of C^2, who's vanishing guarantees isotropy, and the non-negative invariant I_2 selecting DE SITTER from the isotropic solutions. We thus choose the higher derivative action to be of the form

$$S = -\frac{1}{16\pi G}\int \left(R - I\phi + V(\phi)\right)\sqrt{-g}\,d^4x. \tag{3.19}$$

where the invariant I is given by

$$I = I_2 + 3C^2 = 4R_{\mu\nu}R^{\mu\nu} - R^2 + 3C_{\mu\nu\tau\sigma}C^{\mu\nu\tau\sigma}. \tag{3.20}$$

Here I_2 is the invariant chosen in the isotropic case. The factor 3 in front of the WEYL tensor squared is to simplify numerical factors, it does not alter the equations substantially, if one uses a different (but positive) factor.

For an isotropic Universe the invariant I is, in fact, a perfect square which simplified the equations a lot. For the metric of an anisotropic and homogeneous cosmology with line element (3.16) the invariant I cannot be written as perfect square but as the sum of four squares

$$
\begin{aligned}
I = 4\nu^{-4}\Bigg(& 2\tfrac{\ddot{\alpha}^2}{\alpha^2} + 2\tfrac{\ddot{\beta}^2}{\beta^2} + 2\tfrac{\ddot{\gamma}^2}{\gamma^2} - \tfrac{\ddot{\alpha}\ddot{\beta}}{\alpha\beta} - \tfrac{\ddot{\alpha}\ddot{\gamma}}{\alpha\gamma} - \tfrac{\ddot{\beta}\ddot{\gamma}}{\beta\gamma} + \tfrac{\ddot{\alpha}}{\alpha}\left(-4\tfrac{\dot{\alpha}\dot{\nu}}{\alpha\nu} + \tfrac{\dot{\beta}\dot{\nu}}{\beta\nu} + \tfrac{\dot{\gamma}\dot{\nu}}{\gamma\nu} - \tfrac{\dot{\alpha}\dot{\beta}}{\alpha\beta} - \tfrac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma}\right) \\
& + \tfrac{\ddot{\beta}}{\beta}\left(\tfrac{\dot{\alpha}\dot{\nu}}{\alpha\nu} - 4\tfrac{\dot{\beta}\dot{\nu}}{\beta\nu} + \tfrac{\dot{\gamma}\dot{\nu}}{\gamma\nu} - \tfrac{\dot{\alpha}\dot{\beta}}{\alpha\beta} - \tfrac{\dot{\beta}\dot{\gamma}}{\beta\gamma}\right) + \tfrac{\ddot{\gamma}}{\gamma}\left(\tfrac{\dot{\alpha}\dot{\nu}}{\alpha\nu} + \tfrac{\dot{\beta}\dot{\nu}}{\beta\nu} - 4\tfrac{\dot{\gamma}\dot{\nu}}{\gamma\nu} - \tfrac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma} - \tfrac{\dot{\beta}\dot{\gamma}}{\beta\gamma}\right) \\
& + 2\tfrac{\dot{\alpha}^2\dot{\nu}^2}{\alpha^2\nu^2} + 2\tfrac{\dot{\beta}^2\dot{\nu}^2}{\beta^2\nu^2} + 2\tfrac{\dot{\gamma}^2\dot{\nu}^2}{\gamma^2\nu^2} - \tfrac{\dot{\alpha}\dot{\beta}\dot{\nu}^2}{\alpha\beta\nu^2} - \tfrac{\dot{\alpha}\dot{\gamma}\dot{\nu}^2}{\alpha\gamma\nu^2} - \tfrac{\dot{\beta}\dot{\gamma}\dot{\nu}^2}{\beta\gamma\nu^2} + \tfrac{\dot{\alpha}^2\dot{\beta}\dot{\nu}}{\alpha^2\beta\nu} + \tfrac{\dot{\alpha}^2\dot{\gamma}\dot{\nu}}{\alpha^2\gamma\nu} + \tfrac{\dot{\alpha}\dot{\beta}^2\dot{\nu}}{\alpha\beta^2\nu} \\
& + \tfrac{\dot{\beta}^2\dot{\gamma}\dot{\nu}}{\beta^2\gamma\nu} + \tfrac{\dot{\alpha}\dot{\gamma}^2\dot{\nu}}{\alpha\gamma^2\nu} + \tfrac{\dot{\beta}\dot{\gamma}^2\dot{\nu}}{\beta\gamma^2\nu} + 2\tfrac{\dot{\alpha}^2\dot{\beta}^2}{\alpha^2\beta^2} + 2\tfrac{\dot{\alpha}^2\dot{\gamma}^2}{\alpha^2\gamma^2} + 2\tfrac{\dot{\beta}^2\dot{\gamma}^2}{\beta^2\gamma^2} - \tfrac{\dot{\alpha}^2\dot{\beta}\dot{\gamma}}{\alpha^2\beta\gamma} - \tfrac{\dot{\alpha}\dot{\beta}^2\dot{\gamma}}{\alpha\beta^2\gamma} - \tfrac{\dot{\alpha}\dot{\beta}\dot{\gamma}^2}{\alpha\beta\gamma^2}\Bigg) \\
= 4\Bigg(& \left(\dot{H}_1 + H_1^2 - H_2H_3\right)^2 + \left(\dot{H}_2 + H_2^2 - H_1H_3\right)^2 + \left(\dot{H}_3 + H_3^2 - H_1H_2\right)^2\Bigg) \\
& + 2\Bigg(\left(\dot{H}_1 - \dot{H}_2 + H_1^2 - H_2^2 - (H_1 - H_2)H_3\right)^2 \\
& \quad + \left(\dot{H}_1 - \dot{H}_3 + H_1^2 - H_3^2 - (H_1 - H_3)H_1\right)^2 \\
& \quad + \left(\dot{H}_2 - \dot{H}_3 + H_2^2 - H_3^2 - (H_2 - H_3)H_1\right)^2\Bigg).
\end{aligned}
\tag{3.21}
$$

Here we introduced HUBBLE parameters $H_1(t) = \frac{\dot{\alpha}}{\alpha}$, $H_2(t) = \frac{\dot{\beta}}{\beta}$ and $H_3(t) = \frac{\dot{\gamma}}{\gamma}$ and set $\nu(t) = 1$ in the second line.

The invariant I vanishes if and only if all six squares vanish separately. There are two ways for this to happen. Either

$$H_1(t) = H_2(t) = H_3(t) = \bar{H} = \text{const.}, \tag{3.22a}$$

which corresponds to the usual metric of spatially flat DE SITTER spacetime (or MINKOWSKI spacetime for $\bar{H} = 0$), or

$$H_i(t) = \frac{1}{t} \quad \text{for one } i \in \{1, 2, 3\} \text{ and } H_j(t) = 0 \quad \text{for } j \neq i. \tag{3.22b}$$

Choosing without loss of generality $i = 1$, this last solution results in the metric $\alpha = t$, $\beta = \gamma = 1$ with line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^2\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Using coordinates labeling null geodesics in negative and positive x-direction $x_\pm = x \pm \ln t$ we obtain the line element $-\mathrm{d}t^2 + t^2\mathrm{d}x^2 = \mathrm{e}^{x_+ - x_-}\mathrm{d}x_+\mathrm{d}x_-$. A rescaling $\tilde{x}_\pm = \pm\mathrm{e}^{\pm x_\pm}$ gives $\mathrm{e}^{x_+ - x_-}\mathrm{d}x_+\mathrm{d}x_- = \mathrm{d}\tilde{x}_+\mathrm{d}\tilde{x}_-$, which can be cartesian coordinates $\tilde{t} = \frac{1}{2}(\tilde{x}_- - \tilde{x}_+)$ and $\tilde{x} = \frac{1}{2}(\tilde{x}_- + \tilde{x}_+)$ of flat MINKOWSKI spacetime

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{x}^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Putting everything together the coordinate transformation is given by

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} -t\cos x \\ t\sinh x \end{pmatrix}.$$

We mainly focus on the singularity of the interior SCHWARZSCHILD spacetime. As seen from the line element (3.18) we have thus two interchangeable spacelike directions. In the following we therefore restrict ourselves to metrics (3.16) with $\gamma = \beta$.

3.6. Field equations

Variation of the higher derivative action (3.19) with respect to ϕ gives (after setting $\nu = 1$)

$$\left(\dot{F} + F^2 - H^2\right)^2 + 2\left(\dot{H} + H^2 - HF\right)^2 + \left(\dot{F} - \dot{H} + F^2 - HF\right)^2 = \frac{1}{4}V'(\phi), \tag{3.23a}$$

while the 00-component of the EINSTEIN equation takes the form

$$\begin{aligned} 0 = {} & 8\dot{\phi}\left((H+F)\dot{H} + H\dot{F} + 2H(H-F)^2\right) \\ & + 4\phi\Big(2(H+F)\ddot{H} + 2H\ddot{F} - \dot{H}^2 - 2\dot{H}\dot{F} + 4H(H+2F)\dot{H} \\ & \qquad + 2H(H+2F)\dot{F} - H^4 + 3H^2F^2 - 2HF^3\Big) \\ & - 2H^2 - 4HF - V, \end{aligned} \tag{3.23b}$$

where, again, we set $\nu = 1$ after variation.

The equations arising from variation w.r.t. the spatial components of the metric α and β contain third derivatives of F and H but given (3.23a) and (3.23b) only one of the two spatial equations is independent. Thus we can choose a linear combination such that no third derivatives of F appear (after setting $\nu(t) = 1$):

$$\begin{aligned} 0 = {} & V + 4\dot{H} + 6H^2 + 8\ddot{\phi}\left(-\dot{H} + H(H-F)\right) \\ & + 4\dot{\phi}\left(-4\ddot{H} - 2\left(2H + 3F\right)\dot{H} - 2H\dot{F} + 4H^3 - 6FH^2 + 2F^2H\right) \\ & + 4\phi\Bigg(-2\dddot{H} - 2\left(5H + F\right)\ddot{H} - 7\dot{H}^2 - 2\dot{H}\dot{F} - 2\left(8H^2 + 4HF - F^2\right)\dot{H} \\ & \qquad - 2\left(H^2 - 3HF\right)\dot{F} - 3H^4 - 2H^3F + 5H^2F^2 + 2HF^3\Bigg). \end{aligned} \tag{3.23c}$$

3.7. NEWTONian limit

The fact that laboratory tests perfectly fit EINSTEIN's theory of gravity – or in many cases even NEWTON's – results in a very important constraint on our higher derivative theory. After eliminiation of the unphysical and non-dynamical field ϕ from the action it should remain the EINSTEIN HILBERT term R and some corrections of higher order that are negligible at small curvature.

The auxiliary field is determined by the constraint equation, which is algebraic or – depending on the potential – at least not a differential equation,

$$I = V'(\phi). \tag{3.24}$$

From this and the fact that the invariant I is strictly non-negative we directly see that V has to be a monotonically increasing function.

In order to obtain the limiting curvature effect we again require that V' is bounded and tends to zero for large ϕ.

For small ϕ we assume that the potential can be expanded into a power series $V = V_0\phi^m + V_1\phi^{m+1} + \ldots$, where we allow for non-integer m. Equation (3.24) can be solved by inverting the power series of V'

$$I = \begin{cases} V_0 + 2V_1\phi + \mathcal{O}(\phi^2) & \text{for } m = 1 \\ V_0 m\phi^{m-1} + V_1(m+1)\phi^m + \mathcal{O}(\phi^{m+1}) & \text{for } m > 1, \end{cases} \tag{3.25}$$

resulting in a power series expansion for ϕ

$$\phi = \begin{cases} \frac{1}{2V_1}(I - V_0) - \frac{3V_2}{8V_1^3}(I - V_0)^2 + \mathcal{O}\left((I - V_0)^3\right) & \text{for } m = 1 \\ V_0 m\phi^{m-1} + V_1(m+1)\phi^m + \mathcal{O}(\phi^{m+1}) & \text{for } m > 1. \end{cases} \tag{3.26}$$

Plugging this solution of the constraint equation (3.24) back into the action (3.19) we obtain the higher derivative corrected EINSTEIN HILBERT action. Keeping only the lowest order terms yields

$$S \approx \begin{cases} \int \left(R + AI^{\frac{m}{m-1}}\right) \sqrt{-g}\,\mathrm{d}^4x & \text{for } m > 1 \\ \int (R + B + CI) \sqrt{-g}\,\mathrm{d}^4x & \text{for } m = 1 \end{cases} \tag{3.27}$$

with some constants A, B, C. Since I is itself quadratic in metric components, the correction term is of higher order than and negligible against R from the EINSTEIN HILBERT term if $m > 1$. The case $m = 1$ reduces to EINSTEIN gravity with cosmological constant at low curvature.

Let us first disuss the conditions for approaching a DE SITTER universe. If $V(\phi)$ rises slower than linear in ϕ for $\phi \to \infty$ the equations of motion derived from (3.19) will enforce DE SITTER spacetime as soon as ϕ grows large. This is certainly achieved by a potential which approaches a constant value as ϕ goes to infinity.

To be more explicit we now choose a potential satisfying the conditions given above. This is only done to be able to do numerical calculations. The qualitative behaviour of solutions should not depend on this explicit choice. In the explicit solutions given later we chose

$$V(\phi) = H_0^2 \frac{\phi^2 \operatorname{sgn}(\phi)}{\left(1 + \sqrt{|\phi|}\right)^4}. \tag{3.28}$$

This potential has the asymptotic power series expansion at infinity of

$$V(\phi) = H_0^2 \left(1 - \frac{4}{\phi^{\frac{1}{2}}} + \frac{10}{\phi} - \frac{20}{\phi^{\frac{3}{2}}} + \dots\right). \tag{3.29}$$

The appearance of fractional powers in the asymptotic expansion will be a source for non-analytic behaviour. For a more general discussion we will assume a potential with an expansion of the form

$$V(\phi) = H_0^2 \left(1 - \frac{A}{\phi^n} + \frac{B}{\phi^{n+1}} - \frac{C}{\phi^{n+2}} + \dots\right). \tag{3.30}$$

It will turn out in section 3.8.1 that we cannot choose $n = 1$ but have to choose $0 < n < 1$ in order to have DE SITTER as attractor.

A closer look at the equations (3.23) reveals that the total differential order is 5. It is possible to eliminate the LAGRANGE multiplier field ϕ with the help of (3.23a), which is algebraic in ϕ. The resulting system of two differential equations for $\ddot{F}$ and $\dddot{H}$ (or $\dddot{F}$ and $\ddot{H}$) contains the inverse function of the potential. Since for this general discussion we do not want to specify the potential beyond its asymptotic properties we choose to keep ϕ in the equations.

3.8. Large ϕ limit

The case of large ϕ bears special interest since this limit corresponds to the regime where the invariant has to be small. We will discuss the two configurations (3.22a) and (3.22b) with vanishing invariant I separately.

3.8.1. Approximate DE SITTER solution

Asymptotic analysis

In the first case we will assume that at a given time spacetime is approximately DE SITTER. Assuming the existence of a solution $\phi(t)$ we can replace the independent variable t by ϕ, at least locally. The HUBBLE parameters and their derivatives can now be expanded in powers of $\frac{1}{\phi}$. We have to allow for fractional powers in this generalised expansion. The ϕ-equation (3.23a) restricts the lowest order that can appear in these expansions, since each square ist at most of the order of $\left(\frac{1}{\phi}\right)^{n+1}$

$$H = \bar{H} + \frac{\bar{h}}{\phi^{\frac{n+1}{4}}} + \frac{h}{\phi^{\frac{n+1}{2}}} + \cdots, \tag{3.31}$$

$$F = \bar{H} + \frac{\bar{f}}{\phi^{\frac{n+1}{4}}} + \frac{f}{\phi^{\frac{n+1}{2}}} + \cdots, \tag{3.32}$$

$$\dot{H} = \frac{\tilde{h}}{\phi^{\frac{n+1}{2}}} + \cdots, \tag{3.33}$$

$$\dot{F} = \frac{\tilde{f}}{\phi^{\frac{n+1}{2}}} + \cdots \tag{3.34}$$

We assume a contracting universe with negative HUBBLE parameter $\bar{H} < 0$. The expanding FRIEDMANN ROBERTSON WALKER universe is non-singular in the future and the big bang singularity in the past can be discussed by time reversal.

Obviously these expansions are not independent, since $\dot{H} = \frac{\partial H}{\partial t}$, which will be used later to determine the coefficients $\tilde{h}$ and $\tilde{f}$. Substituting this expansion into (3.23a) gives in leading order $\bar{f} = \bar{h}$.[1] The second derivatives of the HUBBLE parameters can be expressed through derivatives of ϕ:

$$\ddot{H} = -\frac{\tilde{h}\frac{n+1}{2}}{\phi^{\frac{n+1}{2}}}\frac{\dot{\phi}}{\phi} + \cdots \tag{3.35a}$$

$$\ddot{F} = -\frac{\tilde{f}\frac{n+1}{2}}{\phi^{\frac{n+1}{2}}}\frac{\dot{\phi}}{\phi} + \cdots \tag{3.35b}$$

With this expansion the 00-component (3.23b) of the field equations reduces to:

$$\begin{aligned} 0 =& 4\dot{\phi}\left(\bar{H}\frac{n+3}{2}(\tilde{f}+2\tilde{h})\frac{1}{\phi^{\frac{n+1}{2}}} + \mathcal{O}\left(\frac{1}{\phi^{\frac{3(n+1)}{4}}}\right)\right) \\ &+4\phi\left(3\bar{H}^2(\tilde{f}+2\tilde{h})\frac{1}{\phi^{\frac{n+1}{2}}} + \mathcal{O}\left(\frac{1}{\phi^{\frac{3(n+1)}{4}}}\right)\right) \\ &-3\bar{H}^2 - \frac{H_0^2}{2} + \mathcal{O}\left(\frac{1}{\phi^{\frac{n+1}{4}}}\right). \end{aligned} \tag{3.36}$$

Neglecting higher order terms we get

$$\dot{\phi} + \bar{H}\frac{3}{2}(n+3)\phi - \frac{\frac{3}{4}\bar{H} + \frac{3}{8}\frac{H_0^2}{\bar{H}}}{\tilde{f}+2\tilde{h}}\phi^{\frac{n+1}{2}} = 0. \tag{3.37}$$

[1] Actually selfconsistency later requires $\bar{h} = 0$.

For $n < 1$ this has an asymptotically exponential solution

$$\phi(t) \propto \exp\left(-\frac{3(n+3)}{2}\bar{H}t\right), \tag{3.38}$$

so that ϕ increases forever since $\bar{H} < 0$. For $n = 1$ there is a second term of order ϕ in (3.37), which leads to an asymptotic behaviour of the form

$$\phi(t) \approx \exp\left(\left(\frac{3\bar{H}^2 + \frac{1}{2}H_0^2}{8\bar{H}(\tilde{f} + 2\tilde{h})} - \frac{3}{2}\bar{H}\right)t\right). \tag{3.39}$$

This solution is decaying for certain values of $\tilde{f}$ and $\tilde{h}$. This is inconsistent with the assumption $\phi \gg 1$. This means for $n = 1$ DE SITTER spacetime at $\phi = \infty$ is an attractor only if $\tilde{f} + 2\tilde{h} < 0$ or $\tilde{f} + 2\tilde{h} > \frac{1}{4} + \frac{1}{24}\frac{H_0^2}{\bar{H}}$. These conditions can however not be assumed a priori. They can only be verified once the asymptotic solution is known. This is in contrast to the isotropic model [5] where exponential growth of ϕ is generic.

With the help of the asymptotic equation for $\dot{\phi}$

$$\dot{\phi} = -3\bar{H}\phi + \mathcal{O}\left(\phi^{\frac{3}{4}}\right) \tag{3.40}$$

we can check the consistency of our ansatz (3.31) for H and $\dot{H}$ (and analogously for F and $\dot{F}$), since

$$\dot{H} = \frac{\partial H}{\partial \phi}\dot{\phi} = -\frac{\frac{n+1}{4}\bar{h}}{\phi^{\frac{n+1}{4}}}\frac{\dot{\phi}}{\phi} - \frac{\frac{n+1}{2}h}{\phi^{\frac{n+1}{2}}}\frac{\dot{\phi}}{\phi} + \mathcal{O}\left(\frac{1}{\phi^{\frac{3(n+1)}{4}}}\right) \tag{3.41}$$

By comparing coefficients we get

$$\bar{h} = 0 \qquad \tilde{h} = \frac{3}{2}(n+1)\bar{H}h \tag{3.42a}$$

$$\bar{f} = 0 \qquad \tilde{f} = \frac{3}{2}(n+1)\bar{H}f \tag{3.42b}$$

therefore the leading terms in the expansion of $H, F, \dot{H}, \dot{F}$ in (3.31) are all determined in terms of h and f.

Linearisation

To analyse perturbations around an isotropic background we cannot simply linearise the field equations (3.23), since the constraint (3.23a) is of quadratic order and will thus always vanish. Instead we use the ansatz $H_i(t) = H(t) + \epsilon\delta H_i(t)$ with $i \in \{1, 2\}$, where $H(t)$ satisfies the isotropic field equations and we take the auxiliary field as background field, which is calculated from the constraint equation (3.23a)

$$V'(\phi) = 12\dot{H}(t).$$

Solving the constraint equation involves the inverse function to V'. For our choice of potential (3.28) this is equivalent to finding the roots of a polynomial of fifth degree and can hence be done only numerical. From the remaining field equations (3.23b) and (3.23c) we obtain a system of quasilinear equations for the perturbations δH_i. The linearised system of field equations decouples after introduction of the new dependent variables $x(t) = \delta H_2(t) - \delta H_1(t)$ and $y(t) = \delta H_2(t) + 2\delta H_1(t)$. The equation for the linearised anisotropy $x(t)$ is

$$\dddot{x} = -2\left(3H + \frac{\dot{\phi}}{\phi}\right)\ddot{x} - \left(15H^2 + 6\dot{H} + 3H\frac{\dot{\phi}}{\phi} + \frac{\ddot{\phi}}{\phi} - \frac{1}{2\phi}\right)\dot{x}$$
$$+ 3\left(-6H^3 - 7\dot{H}H + \frac{H}{2\phi} - \ddot{H} - H^2\frac{\dot{\phi}}{\phi} + H\frac{\ddot{\phi}}{\phi}\right)x. \tag{3.43a}$$

$$\ddot{y} = -\left(3H - \frac{\dot{H}}{H} + \frac{\dot{\phi}}{\phi}\right) - \left(\frac{\ddot{H}}{H} + 6\dot{H} + \frac{\dot{H}\dot{\phi}}{H\phi} - \frac{1}{2\phi}\right)y. \tag{3.43b}$$

Hence a hint on the linear stability of the isotropic solution can be derived from the eigenvalues of the differential equation (3.43a) for the anisotropical perturbation x. A linear instability is indicated by an eigenvalue with positive real part, corresponding to an exponentially growing mode. With the help of the background field equations, we can express the eigenvalues in terms of H and ϕ. The domain of existence of an eigenvalue with positive real part for the given potential (3.28) is plotted in figure 3.2. In a contracting universe ($H < 0$) there exists an eigenvalue with positive real part only for sufficiently small ϕ. Hence we conclude that the isotropic solution is stable if ϕ is large.

3.8.2. Appoximate MINKOWSKI solution

Let us now turn to the second possibility (3.22b) for vanishing invariant I at large ϕ. Here the HUBBLE parameters approach MINKOWSKI spacetime in singular coordinates (3.22a), i. e.

$$H(t) \to 0, \qquad F(t)^2 + \dot{F}(t) \to 0, \qquad t \to t_0. \tag{3.44}$$

Without restricting the generality we can assume $t_0 = 0$. In order to see if this behaviour can be realised in a global solution we are then looking for a solution of the field equations approaching $H = 0$ and $F = \frac{1}{t}$ for $t \to 0$. In this case the field equations immediately imply $\phi \to \frac{1}{t}$. To continue we observe that the field equations are invariant under the discrete transformation simultaneously sending $t \to -t$, $H(t) \to -H(t)$, $F(t) \to -F(t)$, and $\phi(t) \to \phi(t)$. Approaching $t_0 = 0$ from below with these asymptotics, we thus suggest the behaviour sketched in figure 3.3.

To make this discussion more precise let us rewrite the equations of motion in terms of $G = 1/F$, H, and $\psi = 1/\phi$ which are more suitable for analysing the small

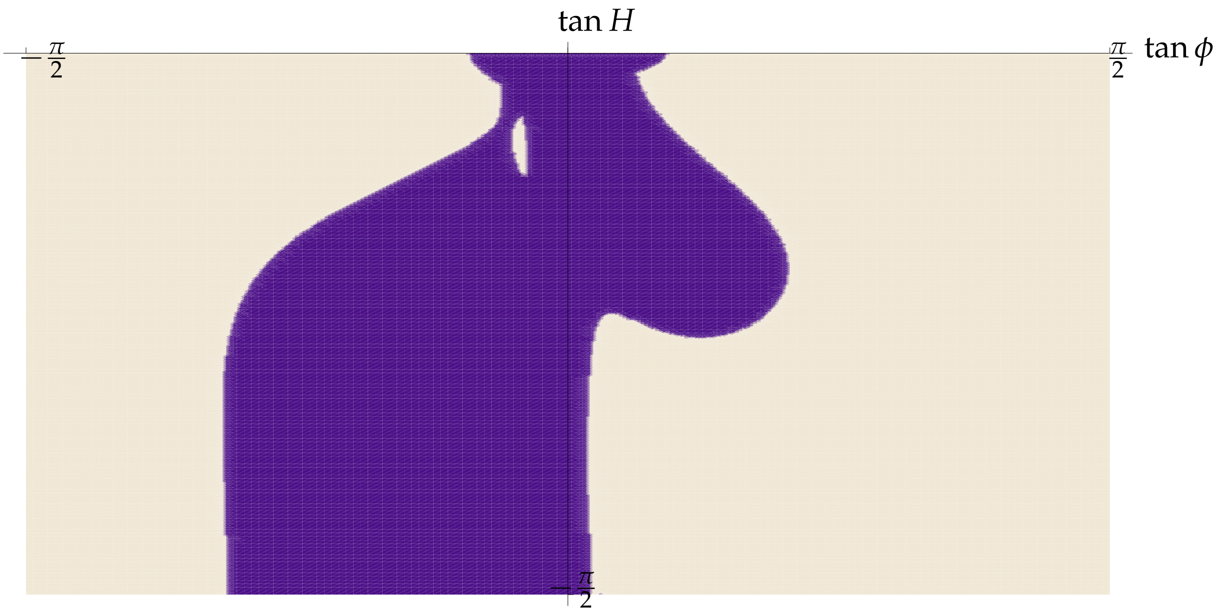


Figure 3.2.: Domain of existence of an eigenvalue of (3.43a) with positive real part in the ϕ-H-plane for contracting universe.

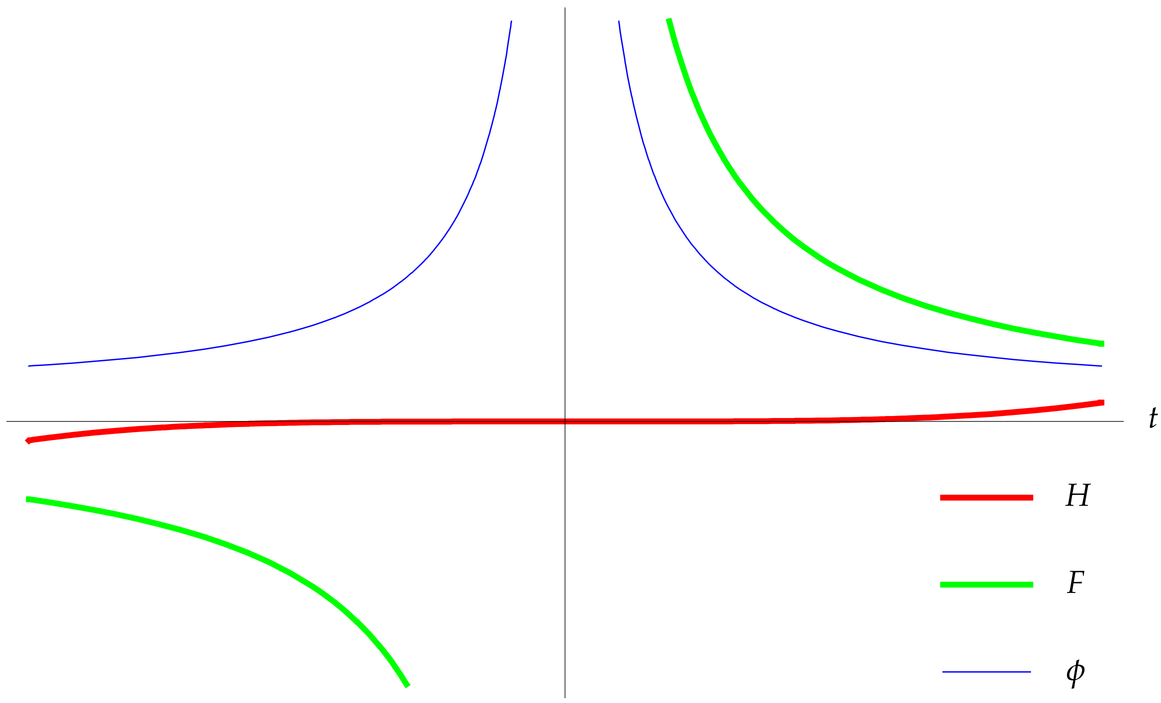


Figure 3.3.: Qualitative behaviour of a possible solution bouncing through MINKOWSKI spacetime. $\phi(t)$ diverges symmetrically around $t = 0$ while $F(t)$ diverges asymmetrically. $H(t)$ is smooth at $t = 0$.

t behaviour. The proposed asymptotic solution is now $G = t$, $\psi = |t|$, and $H = 0$. Therefore we seek approximate solutions of the form

$$G = t + a_1 t^{1+\alpha_1} + a_2 t^{1+\alpha_1+\alpha_2} + a_3 t^{1+\alpha_1+\alpha_2+\alpha_3} + \dots, \tag{3.45}$$

$$H = b_1 t^{\beta_1} + b_2 t^{\beta_1+\beta_2} + b_3 t^{\beta_1+\beta_2+\beta_3} + \dots, \tag{3.46}$$

$$\psi = \pm t \pm c_1 t^{1+\gamma_1} \pm c_2 t^{1+\gamma_1+\gamma_2} \pm c_3 t^{1+\gamma_1+\gamma_2+\gamma_3} + \dots, \tag{3.47}$$

where α_i, β_i, and γ_i are positive constants and the two signs correspond to $t \gtrless 0$. In the following we will only discuss the case $t > 0$. For convenience we choose a specific potential with asymptotic expansion

$$V(\phi) = V_1 - \frac{4V_2}{\phi^{\frac{1}{2}}} + \frac{10V_3}{\phi} - \frac{20V_4}{\phi^{\frac{3}{2}}} + \frac{35V_5}{\phi^2} \mp \dots \tag{3.48}$$

The potential $V(\phi) = H_0^2 \frac{\phi^2}{(1+\sqrt{\phi})^4}$ corresponds to the choice $V_i = H_0^2$. Plugging this ansatz into the equations of motion and neglecting all obviously subleading terms we obtain

$$\begin{aligned} 0 &= -\frac{W}{4} t^{\frac{3}{2}} + b_1^2 (3 - 2\beta_1 + 3\beta_1^2) t^{2(\beta_1 - 1)} && (3.49a) \\ &\quad + 2a_1 b_1 (1+\alpha_1)(1+\beta_1) t^{-3+\alpha_1+\beta_1} + 2a_1^2 (1+\alpha_1)^2 t^{2\alpha_1 - 4}, \\ 0 &= V_1 + 8b_1 (2 + \beta_1 - \beta_1^2) t^{\beta_1 - 4}, && (3.49b) \\ 0 &= V_1 - 8b_1 (6 + \beta_1 - 4\beta_1^2 + \beta_1^3) t^{\beta_1 - 4}. && (3.49c) \end{aligned}$$

In (3.49b) and (3.49c) we can either choose $\beta_1 = 4$ to cancel both terms against each other. Otherwise we have to choose $\beta_1 < 4$ as a zero of the two coefficients. We are left with two admissible solutions obeying $\beta_1 > 0$

$$\alpha_1 = \frac{11}{4}, \qquad \beta_1 = 2, \qquad a_1 = \pm\frac{\sqrt{V_2}}{13}, \qquad \text{or} \tag{3.50a}$$

$$\alpha_1 = \frac{11}{4}, \qquad \beta_1 = 4, \qquad a_1 = \pm\frac{\sqrt{V_2}}{13}, \qquad b_1 = \frac{V_1}{80}. \tag{3.50b}$$

Proceeding in this way we obtain a perturbative solution such that the equations of motion are satisfied up to linear order at $t = 0$ with

$$G = t - \frac{2V_2}{77b_1} t^{\frac{7}{2}}, \tag{3.51a}$$

$$H = b_1 t^2 + (48 b_1 c_1 + V_1) t^4 + \frac{2(1155 b_1 c_2 - 38 V_2)}{4235} t^{\frac{9}{2}}, \tag{3.51b}$$

$$\psi = \pm t \pm c_1 t^3 \pm c_2 t^{\frac{7}{2}} \qquad \text{for } t \gtrless 0. \tag{3.51c}$$

We have not been able to extend the solution as a power series in fractional powers of t beyond this order suggesting that if a solution exists for a finite range of t, non-analytic behaviour of a different kind will be required. Note also that all components of the RIEMANN tensor (but not all its derivatives) are finite at $t = 0$.

To summarise, we found that if ϕ is already large while the two HUBBLE parameters are similar the proposed limiting curvature procedure works perfectly well. In the case where the leading order correction to the potential has an exponent $n < 1$, large ϕ implies that spacetime is nearly DE SITTER and the evolution of ϕ extends to infinite future while growing exponentially and thus forcing spacetime to approach DE SITTER even more. Due to the exponential growth of ϕ it takes an infinite time to actually reach the DE SITTER end stage. Henceforth the addition of the WEYL tensor squared to the invariant in deed results in the anticipated way. The solution is forced to be isotropic and in this way the problem is reduced to the isotropic case.

On the other hand if the difference of the two HUBBLE parameters is of order one, while ϕ is large, the metric may approach MINKOWSKI spacetime in singular coordinates in finite time. Although this solution, if it exists, is non-analytic at $t = 0$ it can be continued symmetrically through the 'MINKOWSKI phase' at $t = 0$.

We thus have shown, that any would-be singularity where ϕ diverges is resolved. Either this point lies infinitely far in the future and spacetime is approximately DE SITTER or the solution bounces through flat MINKOWSKI spacetime at finite time while we observe a coordinate singularity in the chosen set of coordinates. In section 3.10 we will further analyse the existence of a global solution using numerical methods.

3.9. Phase Space Analysis

In order to exclude the possibility of singularities of global solutions of the equations of motion, where ϕ does not diverge, we have to do a phase space analysis. The equations of motion (3.23) form a system of ordinary differential equations with total differential order of 5. To make this obvious we can eliminate derivatives of F from the equations (3.23b) and (3.23c) with the help of (3.23a) and all derivatives of ϕ from equation (3.23c) by (3.23b). In this way we solve for the highest remaining derivatives $\dot{F}$, $\dot{\phi}$, and $\dddot{H}$. Since (3.23a) is a quadratic equation in $\dot{F}$, we obtain two sets of explicit, ordinary differential equations. As usual we rewrite the equations into a system of first order differential equations for the vector $\boldsymbol{u} := \left(\phi, F, H, \dot{H}, \ddot{H}\right)$ and obtain

$$\dot{\boldsymbol{u}} = \boldsymbol{V}_{\pm}(\boldsymbol{u}), \tag{3.52}$$

where $\boldsymbol{V}_{\pm}$ is obtained from the field equations (3.23) as described above. The components $V^1_{\pm}$ and $V^5_{\pm}$ are to complicated to explicitely give here, but the remaining components are

$$\begin{aligned}
V^2_\pm &= -u_2^2 + \frac{1}{2}u_2u_3 + \frac{1}{2}u_3^2 + \frac{1}{2}u_4 \\
&\quad \pm \frac{1}{2}\sqrt{-5u_2^2u_3^2 + 10u_2u_3^3 - 5u_3^4 + 6u_2u_3u_4 - 5u_4^2 + \tfrac{1}{2}V'}, \\
V^3_\pm &= u_4, \\
V^4_\pm &= u_5.
\end{aligned}$$

Since (3.23a) contains $\dot{F}^2$ solving the equations involves a square root of

$$B = V'(\phi) - 10F^2H^2 + 20FH^3 - 10H^4 + 12H(F-H)\dot{H} - 10\dot{H}^2. \tag{3.53}$$

The reality of the solution is thus not guaranteed a priori. However, it turns out that the hypersurface $B = 0$ is left invariant under the flow of the vectorfield $\boldsymbol{V}$, since the directional derivative

$$\left.\frac{\partial B}{\partial u^i}V^i\right|_{B=0} = 0 \tag{3.54}$$

along the vector field flow vanishes. Thus no solution following the vector field can ever cross the boundary $B = 0$ into or from the region in which B is negative. Hence if we restrict the initial conditions $u^i_{(0)}$ to the domain of $B > 0$, i. e. real vector field $\boldsymbol{V}(\boldsymbol{u}_{(0)})$, the solution will remain real for all times.

Let us now discuss the existence of singular points. Critical points where $\boldsymbol{V}(\boldsymbol{u})$ vanishes are problematic if they correspond to fixed points away from the asymptotic isotropic regime. At such a fixed point the solutions following the vector field flow might stop at finite time and thus be singular. We observe that $V^3 = u^4$ and $V^4 = u^5$. Furthermore $V^2 = 0$ can be transformed into a polynomial expression of degree four in u^2, which can be explicitely solved. We thus can reduce the problem of searching for critical points $\boldsymbol{V} = \boldsymbol{0}$ to two dimensions by replacing u^4 and u^5 with 0, and u^2 by one of the four solutions of the equation $V^2 = 0$. It can be shown that the remaining two functions $V^1(u^1, u^2)$ and $V^5(u^1, u^2)$ do not vanish simultaneously. But $V^5(u^1, u^2)$ has a one-dimensional set of fractional zeros and a one-dimensional set of fractional poles. The poles typically correspond to branching points of the solution. If more than one real solution meet at a branching point, then the CAUCHY problem is not well defined. If furthermore the set of poles and zeroes intersect then further complications arise since the details will now depend on how the solution approaches the singularity. While we cannot exclude the existence of such points a numerical determination of loci of zeroes and poles indicates $V^5(u^1, u^2)$ whether the intersection is zero or not depends on the details of the potential $V(\phi)$. Even if generic choices of potential lead to non-empty intersection, this allows for the existence of potentials that do not exhibit such critical points.

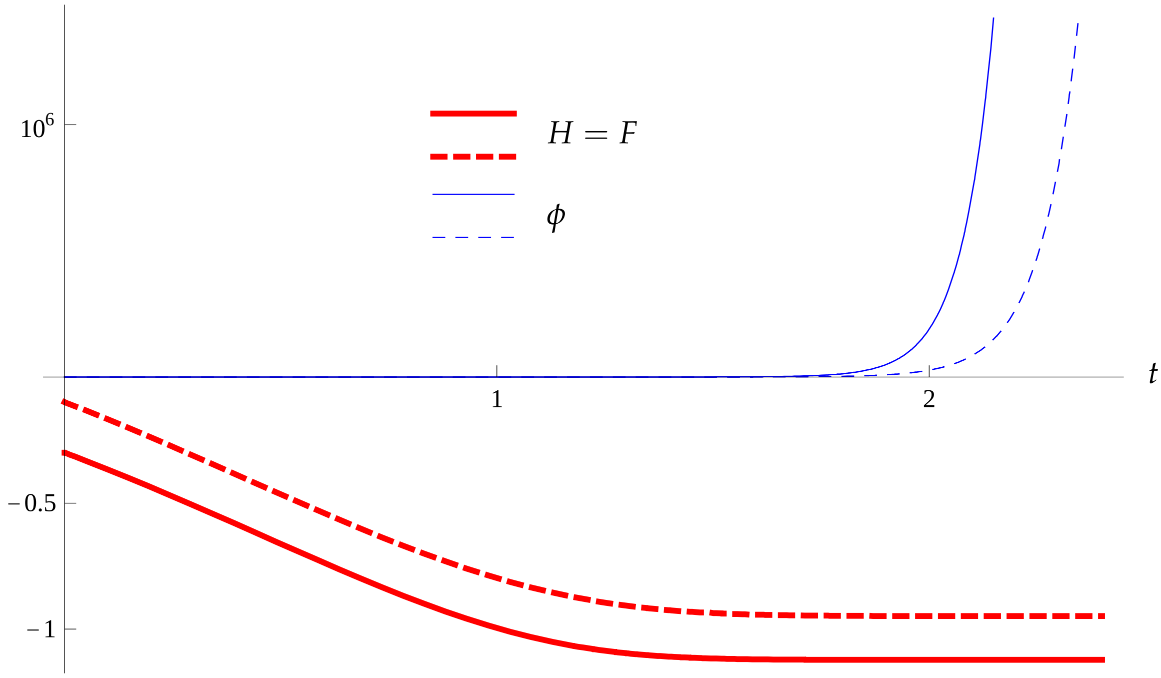


Figure 3.4.: Two numerical solutions (solid and dashed lines; $H = F$ (thick, red), ϕ (blue)) starting with isotropic initial conditions.

3.10. Numerical solutions

In this section we supplement the analytic discussion of the set of solution to the higher derivative action by a numerical investigation. Of particular interest is the question whether the non-analytic local solution passing through the MINKOWSKI stage can be embedded in a global solution and to determine the fate of a generic anisotropic Universe. While we are not able to give a conclusive answer to these questions we find that the numerical analysis supports the idea that a generic anisotropic initial condition will go through a Minkowski phase before returning to the anisotropic Universe in a time symmetric fashion.

The numerical calculations were done using MATHEMATICA. For the numerical solutions we chose the potential to be of the form (3.28) with the constant set to $H_0 = 10$.

Let us however begin with isotropic initial conditions. The behaviour of the solutions of [5] is recovered – the HUBBLE parameter $H(t) = F(t)$ approaches a constant while ϕ grows exponentially. This is not surprising, since for an isotropic ansatz the square of the WEYL tensor vanishes so that $I = I_2$ and the equations of the isotropic case are recovered. Numerical differences to the explicit solutions given in [5] are expected, since we use a slightly different potential. But the qualitative features of the solutions are unchanged. The two numerical solutions plotted in

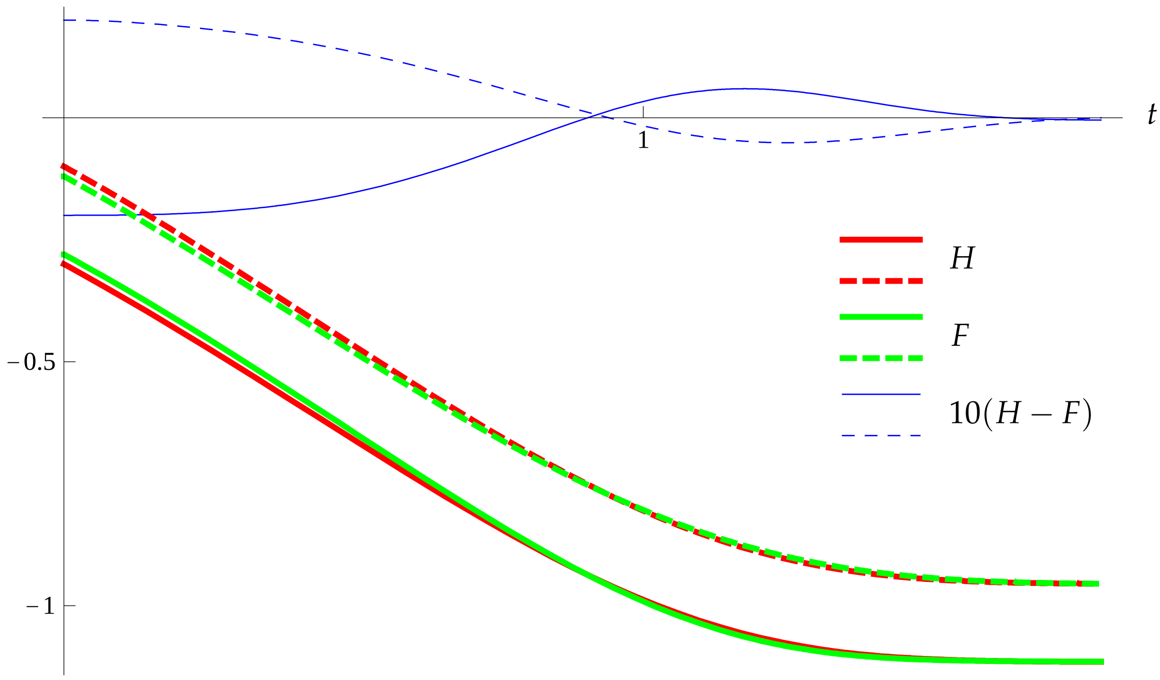


Figure 3.5.: Two numerical solutions (H (thick, red), F (thick, green), and $H - F$ (blue) where the latter is rescaled with a factor of 10) starting with initial conditions slightly perturbed from isotropy.

figure 3.4 were obtained using initial conditions

$$H(0) = F(0) = \begin{cases} -0.1 \\ -0.3 \end{cases}, \qquad \phi(0) = 0.1.$$

Initial conditions prescribing a slightly perturbed isotropic universe evolve to an isotropic solution – the perturbations are damped and the HUBBLE parameters and ϕ show the same asymptotic behaviour as in the isotropic case. Figure 3.5 shows the difference $H - F$ decay while the two HUBBLE parameters approach a constant. The auxiliary field ϕ is not plotted since its behaviour is similar to the isotropic case. The initial conditions used were

$$H(0) = \begin{cases} -0.1 \\ -0.3 \end{cases}, \qquad F(0) = \begin{cases} -0.12 \\ -0.28 \end{cases}, \qquad \phi(0) = 0.1.$$

The numerical evolution of generic initial conditions breaks down at a pole-like singularity in F, while H approaches 0, and ϕ has just passed a minimum and thus seems to be approximately constant. But from (3.23a) we imply that $\phi \propto \frac{1}{t}$ as soon as F approaches the singularity further. This behaviour resembles the aforementioned singular coordinate system of MINKOWSKI spacetime. We therefore suggest that these numerical solutions could be patched to the approximate solution (3.51) describing a kind of bounce through flat MINKOWSKI spacetime.

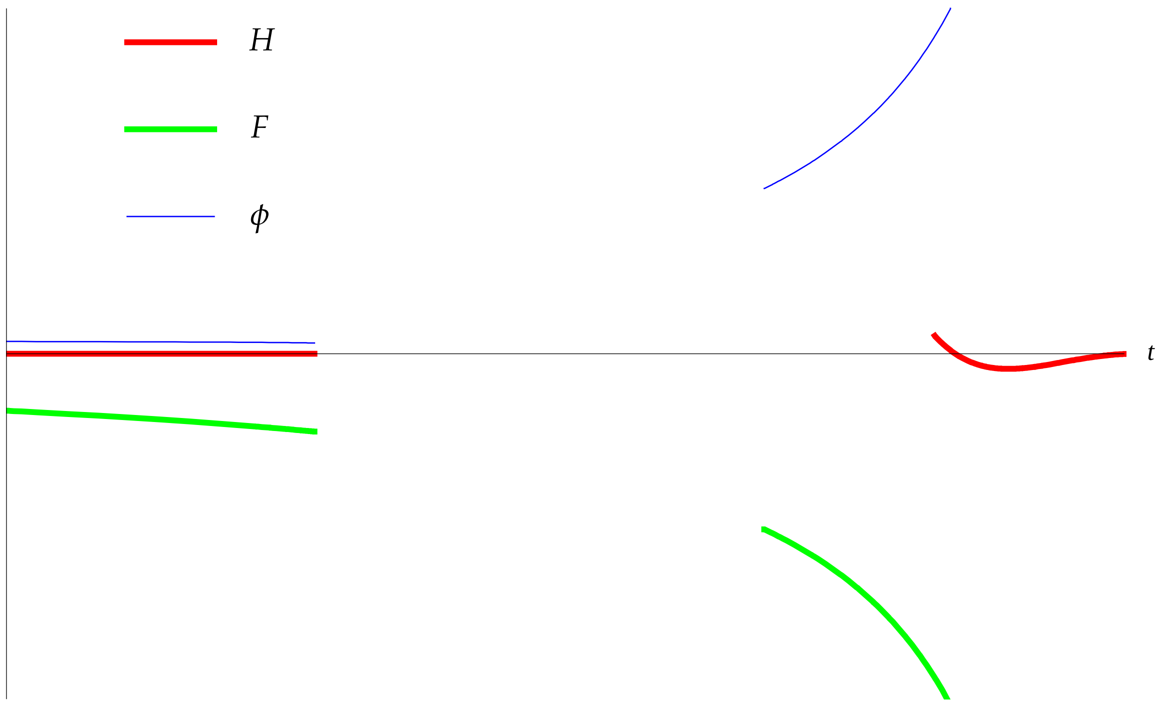


Figure 3.6.: Sketch of the proposed matching of the numerical solution on the left and the possible approximate continuation on the right. There is no overlap between the respective regions of validity of the two approximations.

In figure 3.6 this continuation is sketched. The numerical solution is integrated from the left till it stops. Using the free parameters that are left in (3.51) the asymptotical solution is fitted approximately. We do not expect to be able to patch the two sets of functions together smoothly since the there is no overlap between the range of validity of the numerical solution and the approximate solution (3.51) at $t = t_0$. In particular H obviously does not meet. Nevertheless the picture hints on the existence of a continued solution.

3.11. Discussion

By application of the limiting curvature hypothesis we have made an educated guess for a higher derivative theory of gravity, in which some curvature invariants are bounded. This results in the resolutiuon of a class of curvature singularities. The vanishing of the chosen curvature invariants, the square of the WEYL tensor and the invariant I_2, enforce DE SITTER spacetime. In the construction of the theory the invariants are bounded with the help of a LAGRANGE multiplier field. A generic model of this type has two qualitatively different endpoints of a contracting, anisotropic universe. Following the evolution in the linearised regime small initial anisotropies remain small sufficiently long, allowing the Lagrange multiplier field to grow. We found convincing evidence that for this set of initial configurations the non-linear effects will then suppress the anisotropies leading to an asymptotically contracting DE SITTER spacetime at late times. Since the WEYL tensor appears quadratically in the higher derivative theory the anisotropies are not suppressed in the linear regime. A possible way to improve on this might be the introduction of separate LAGRANGE multiplier fields for the WEYL tensor and I_2 respectively. This would probably help to enlarge the regions of applicability of the linearised theory.

On the other hand if the initial anisotropies are of order 1, then linearisation is not applicable for our higher derivative theory. Analysis of the equations of motion with analytical methods, where possible, and numerical integration produce circumstantial evidence for the existence of a global solution and thus a resolution of the singularity. This solution would interpolate between a contracting anisotropic universe and a universe, that time-symmetrically expands anisotropically. This transition occurs at some finite time and evolves through a nearly flat, MINKOWSKI phase. While we are not able to proof the existence of such a solution globally at present, we think that a better understanding of this solution, if it exists, could have important applications in the resolution of cosmological singularities as an alternative to the so-called bounce solutions as they appear in pre-big-bang scenarios.

Furthermore the same mechanism could be used to resolve the space-like singularity inside a SCHWARZSCHILD black hole, since the black hole interior is locally isomorphic to an anisotropically contracting universe as we discussed here. In our model we find evidence that the surrounding of the singularity is replaced by a region which approaches flat MINKOWSKI spacetime connecting the black hole with a corresponding white hole solution in a causally disconnected spacetime region. In an alternative scenario proposed in [7] the black hole singularity is replaced by a region which is approximately DE SITTER spacetime. While we do not find evidence for this scenario in our model we cannot exclude it at present.

4. String Cosmology

4.1. Motivation

In this part, we want to take a different approach than in chapter 3. In a top-down way we want to construct a simple but explicit model in string theory and derive the effective theory of gravity.

String theory offers a variety of scalar fields that might serve to resolve the singularities of EINSTEIN's theory or play the role of the inflaton field. In the pre-big bang scenario the impact of the dilaton field ϕ alone was analysed. Many models, such as some based on the famous KKLT scenario, study moduli fields. The ekpyrotic and cyclic scenarios as well as other models of brane inflation study inter-brane distances as scalar fields.

Another candidate is the tachyon field that displays the instability of a non-BPS D-brane. While tachyonic inflation originating from non-BPS branes seems to be unable to achieve inflation over long enough time scales resulting in a sufficient number of e-foldings (see [17]) it may nevertheless play a role in resolving cosmological singularities.

In the following sections 4.2 and 4.3 a simple model is constructed. The brane wraps a number of compact dimensions. The size of the internal space then influences the tachyon potential. To give an illustration, we assume that the extra dimension is compactified on a torus, which is Z_2 orbifolded by the identification $y \mapsto -y$. In this case the GSO projection only allows for modes of the tachyon, which are antisymmetric in y. Thus the ground state, which is symmetric in y and has a mass of $m^2 = -1$, is projected out. The lowest order term in the action is then the first KALUZA KLEIN mode with a mass of $m^2 = -1 + \frac{1}{R^2}$, where R is the radius of the extra dimension. But the radius of the extra dimension is given through a dynamical modulus field. If the modulus would be fixed independently of the tachyon dynamics, then one could tune the effective mass of the tachyon field from tachyonic to massless and finally massive. The coupled dynamics of modulus and tachyon could also lead to an interesting interplay. A simple toy model is described in appendix A.

4.2. Non-BPS D9-Brane in isotropic background

In order to analyse the backreaction of the tachyon and modulus field on a curved universe, we obviously need a more elaborate model than the toy model of appendix A. We construct such a model using a space-filling D-brane that shall finally wrap a number of compactified dimensions. The tachyon field itself originates in the fact that we choose an unstable D-brane, which does not fullfill the BPS condition.

Let us ignore for now the need for compactification and analyse a true 10-dimensional model. We place a space-filling D9-brane into our spacetime of type IIA superstring, which introduces a tachyon due to its instability. The action thus reads

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\ \sqrt{-g}\mathrm{e}^{-2\Phi}\left(R+4(\partial\Phi)^2\right) + \int \mathrm{d}^{10}x\ \sqrt{-g}\mathrm{e}^{-\Phi}L(T,(\partial T)^2), \quad (4.1)$$

where for now we only assume that the tachyon action is a function of T and its first derivatives only.

We define the energy-momentum tensor in the usual way as

$$T^\mu_\nu = \frac{2}{\sqrt{-g}}\frac{\delta S_\mathrm{T}(T,\partial_\mu T\partial^\mu T,\Phi)}{\delta g^\nu_\mu}. \quad (4.2)$$

As long as the tachyon is minimally coupled to gravity, the metric appears in the LAGRANGE density through $\partial_\mu T\partial^\mu T$ only, and we can write the energy-momentum tensor as

$$T^\mu{}_\nu = 2\mathrm{e}^{-\Phi}\frac{\partial L(T,(\partial T)^2)}{\partial\left((\partial T)^2\right)}\partial^\mu T\partial_\nu T - g^\mu_\nu\mathrm{e}^{-\Phi}L. \quad (4.3)$$

We are again interested in the cosmological aspects of this model und thus consider homogeneous and isotropic universes. With all fields depending on time only, the energy density ϵ and pressure p are given as the diagonal components of the energy momentum tensor

$$\epsilon = T^0_0, \qquad p = -T^i_i = \mathrm{e}^{-\Phi}L, \quad (4.4)$$

where we do not sum over i here.

In contrast to a BORN-INFELD type action within the restriction to first derivative actions an approximate effective action for the open string tachyon of an unstable brane was derived in [20] and further studied in [21]. It is given in terms of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}t\,\mathrm{e}^{-t^2}$$

by

$$L = -\sqrt{2}\tau_9\mathrm{e}^{-\frac{T^2}{2\alpha'}}\left(\mathrm{e}^{-(\partial T)^2} + \sqrt{\pi(\partial T)^2}\,\mathrm{erf}\left(\sqrt{(\partial T)^2}\right)\right), \quad (4.5)$$

where (as derived in [27]) the tension of a non-BPS 9-brane is given as $\sqrt{2}\tau_9$ with the tension of a BPS 9-brane τ_9.

For constant tachyon, e. g. $(\partial T)^2 \equiv 0$, the potential is given by

$$V(T) = \sqrt{2}\tau_9\mathrm{e}^{-\frac{T^2}{2\alpha'}}, \quad (4.6)$$

which corresponds to the open string tachyon potential found in boundary superstring field theory [19, 18, 29]. $V(T)$ is minimal at infinity and at $T = \infty$ the energy is degenerate with the closed string vacuum which can be interpreted in the way that there is no non-BPS brane present at all.

4.2.1. Isotropic solutions without orbifold

For a first look at the equations we treat all spacelike dimensions equally and assume isotropy. Furthermore we do not apply the orbifolding procedure for now. The equations of motion now read

$$72H^2 = 36H\dot{\Phi} - 4\dot{\Phi}^2 + \frac{1}{\lambda^2}e^{2\Phi}\epsilon, \tag{4.7a}$$

$$2\ddot{\Phi} = 8\dot{H} + 2\dot{\Phi}^2 - 16H\dot{\Phi} + 36H^2 + \frac{1}{2\lambda^2}e^{2\Phi}p, \tag{4.7b}$$

$$2\ddot{\Phi} = 9\dot{H} + 2\dot{\Phi}^2 - 18H\dot{\Phi} + 45H^2 + \frac{1}{4\lambda^2}e^{2\Phi}p, \tag{4.7c}$$

$$\dot{\epsilon} = -9H(\epsilon + p) + \dot{\Phi}p. \tag{4.7d}$$

The energy and pressure are given by

$$\epsilon = e^{-\Phi}e^{-T^2+\dot{T}^2}, \tag{4.8a}$$

$$p = -e^{-\Phi}e^{-T^2}\left(e^{\dot{T}^2} + i\sqrt{\pi\dot{T}^2}\,\mathrm{erf}\,i\sqrt{\dot{T}^2}\right). \tag{4.8b}$$

From (4.7b) and (4.7c) we can eliminate $\ddot{\Phi}$ and obtain

$$\dot{H} + 9H^2 - 2H\dot{\Phi} = \frac{1}{4\lambda^2}e^{2\Phi}p. \tag{4.9}$$

In order to obtain a bounce we need to have a change of sign in the HUBBLE parameter from negative to positive values which requires $\dot{H} > 0$ at the point where $H = 0$. For this to be possible a necessary condition is obviously $p > 0$. In our case with the error function action this is indeed possible but would *not* be possible with a BORN-INFELD type action, where the pressure cannot be positive.

Asymptotic analysis for large times t

For large times t there are three cases where different terms dominate the equations of motion (4.7).

1. $\dot{\Phi}$ and H are of similar order so that we can write approximately $\dot{\Phi} = cH$ with some number c. Eliminating p from (4.7b) and (4.7c) we obtain

$$(c-5)\dot{H} = (27 - 10c + c^2)H^2.$$

Since the right hand side can only vanish for $H^2 = 0$, $c = 5$ implies $\dot{H} \gg H^2$ but in this case the constraint equation (4.7a) requires negative energy density

$$-8H^2 = \frac{1}{\lambda^2} e^{2\Phi} \epsilon.$$

Since this does not correspond to a physical solution, we conclude that $\dot{H} \propto H^2$ and thus

$$H = \frac{h}{t}, \qquad \Phi = f \log(t).$$

With this ansatz we obtain from (4.7b) and (4.7c) that

$$f = 5h - \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 8h^2} \tag{4.10}$$

implying that $|h| \leq \frac{1}{2\sqrt{2}}$. Using (4.7a) positiveness of the energy density requires

$$18h^2 - 9hf + f^2 \geq 0. \tag{4.11}$$

One can check that the above bound on $|h|$ is already more stringent.

From (4.7a–4.7c) we can calculate the equation of state parameter

$$w = \frac{h(9h - 2f - 1)}{f^2 - 9fh + 18h^2}. \tag{4.12}$$

On the other hand we can determine the equation of state from (4.7d) using $\epsilon = e^{-\Phi} V(T^2 - \dot{T}^2)$, which implies

$$\dot{V} + (9H - \dot{\Phi})(1 + w)V = 0.$$

We thus obtain

$$V \propto t^{-\beta},$$

where $\beta = (1 + w)\left(4h + \frac{1}{2}\left(1 \mp \sqrt{1 - 8h^2}\right)\right)$. In the case where the energy density is subdominant in the equations of motion, i. e. the relation (4.11) holds exactly, β has to be larger than 2 in order for ϵ vanishing sufficiently fast. This is the case for $h = \frac{1}{3}$, $f = 1$ and $h = -\frac{1}{3}$, $f = -2$; for the latter we obtain $\beta = -(1 + w) \leq 0$ while the first gives $\beta = 2(1 + w)$ which is acceptable as long as $w > \frac{1}{2}$. But $V(T^2 - \dot{T}^2) \propto t^{-\beta}$ with $\beta \geq 3$ implies $T^2 - \dot{T}^2 \sim \beta \log t$ and thus

$$T \sim e^t + \frac{\beta}{4} \log(t) e^{-t},$$

which has $w \sim 0$ contradicting the assumption. Thus the case of subdominant energy density is fully excluded. Self consistency of our ansatz still requires that $f - \beta = -2$. Assuming $w = 0$, which appears to be generic at late times, we obtain $h = -\frac{2}{9}$ and $f = -2$. This gives $V(T^2 - \dot{T}^2) \sim$ const. and thus either $w = 0$ (with $T \sim \mathrm{e}^t$) or $w = -1$ (with $T =$ const.) both contradicting the value of $w = -\frac{1}{4}$ obtained from (4.12).

Summarising this analysis excludes the case $\dot{\Phi} = cH$.

2. $\dot{\Phi}$ is much smaller than H so that the term with H^2 dominates equation (4.7a), which reads asymptotically

$$72H^2 = \frac{1}{\lambda^2}\mathrm{e}^{2\Phi}\epsilon.$$

On the other hand the dominant part of the equation where we eliminated p from (4.7b) and (4.7c) gives

$$5\dot{H} + 27H^2 = 0,$$

where we used that $\ddot{\Phi}$ cannot grow large while $\dot{\Phi}$ remains small for $t \to \infty$. Thus we obtain $H = \frac{5}{27t}$ which is inconsistent with equation (4.7c).

3. $\dot{\Phi}$ is much larger than H so that the term with $\dot{\Phi}$ dominates equation (4.7a), which reads asymptotically

$$4\dot{\Phi}^2 = \frac{1}{\lambda^2}\mathrm{e}^{2\Phi}\epsilon.$$

Again we obtain a second equation from (4.7b) and (4.7c) giving

$$-\ddot{\Phi} + \dot{\Phi}^2 = 0,$$

which is solved by $\Phi \propto -\log(t - t_0)$. From (4.7b) and (4.7a) we obtain

$$\dot{H} - 2H\dot{\Phi} = \frac{1}{4\lambda^2}\mathrm{e}^{2\Phi}p = \frac{1}{4\lambda^2}\mathrm{e}^{2\Phi}w\epsilon = w\dot{\Phi}^2.$$

Neglecting the subdominant term $2H\dot{\Phi}^2$ this gives the relation

$$\dot{H} = w\dot{\Phi}^2.$$

If w was non-zero, this would result in $H \propto \frac{1}{t-t_0} \propto \dot{\Phi}$, which is in contradiction with our assumption that $\dot{\Phi} \gg H$. Hence $w = 0$ and thus $H \propto \frac{1}{(t-t_0)^2}$ at most.

To summarise the asymptotic analysis, we have found one asymptotic solution described by

$$H \propto \frac{1}{(t - t_0)^2}, \qquad \Phi \propto -\log(t - t_0). \tag{4.13}$$

Numerical analysis

To obtain numerical solutions we used the following scheme: from equations (4.7b–4.7d) we extract two differential equations for the tachyon T and the dilaton Φ that are algebraic in the HUBBLE parameter H. The constraint equation (4.7a) is then used to eliminate H from the differential equations. After the numerical integration H is calculated using (4.7a). Initial data is then given either as the set $\{\Phi_0, \dot{\Phi}_0, T_0, \dot{T}_0\}$ or using $\{H_0, \Phi_0, \dot{\Phi}_0, T_0\}$. In the latter case we calculate the necessary initial value for the derivative of the tachyon field $\dot{T}_0$ using the constraint equation. In this way the constraint is guaranteed to hold. Since the constraint is quadratic in H, for each set of initial data we obtain two numerical solutions.

We easily find bouncing solutions starting with $H_0 = 0$ integrating to positive and negative times. While these solutions provide a non-singular replacement for the Big Bang singularity, they still exhibit singular behaviour either to future (figure 4.1) or to past times (figure 4.2). But these new singularities are of a completely different kind. As we will discuss in section 4.2.2 these could be resolved by higher order terms in the action.

Another possible scheme is to ignore the constraint (4.7a) and solve equations (4.7b–4.7d) as a system of differential equations for H, Φ, and T. Integration in this alternate scheme confirms the results described above. But in general the errors are bigger in this scheme and integration tends to stop earlier due to numerical artifacts.

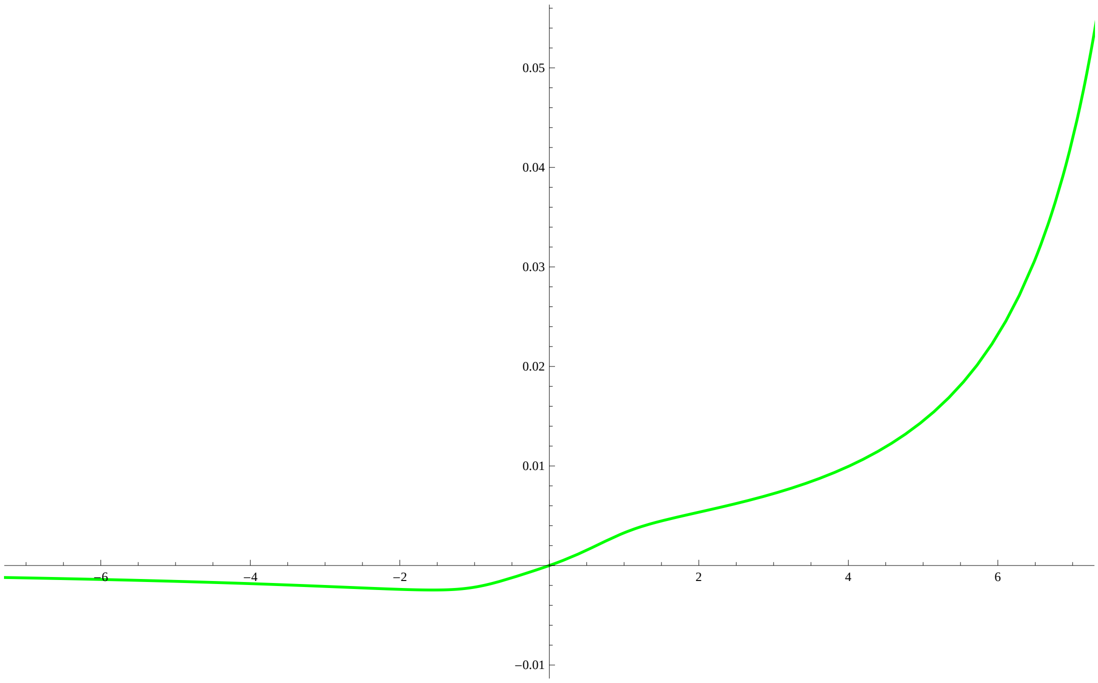
4.2.2. Regularisation

The singular behaviour in our solutions after the bounce might be resolved by α' corrections or quantum loop corrections or alternatively using a dilaton potential (see [9] for some explicit examples). The resolution of this kind of asymptotic singularities appears less difficult than the resolution of the big-bang singularity. In this section we will give an example of resolution of this asymptotic singularity.

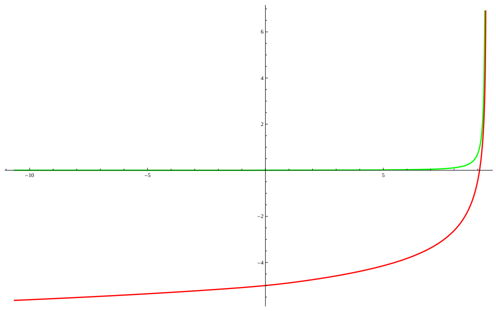
As we pointed out earlier in section 2.1, there exists only one scalar invariant which is quadratic in metric components. We can thus predict that there will be a term proportional to the RICCI curvature scalar R in the tree level α' correction for the open string coupling constant. The prefactor of course cannot be determined by such simple arguments but remains to be calculated from explicit string α' corrections. We take this to justify the addition of a term

$$\frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \ \sqrt{-g}\mathrm{e}^{-2\Phi} W(\Phi) R \tag{4.14}$$

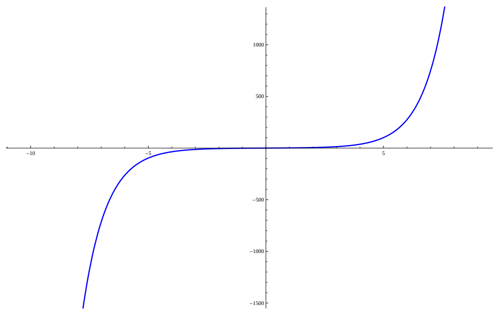
with the potential $W(\Phi) = -\mathrm{e}^{\Phi-\Phi_0}$ to the action (4.1). Since curvature and dilaton blow up in our scenario such a term might be capable of smoothing out the singularity. A similar potential has been motivated in the context of string gas cosmology in [4] as a Casimir-type potential.



(a) HUBBLE parameter $H(t)$

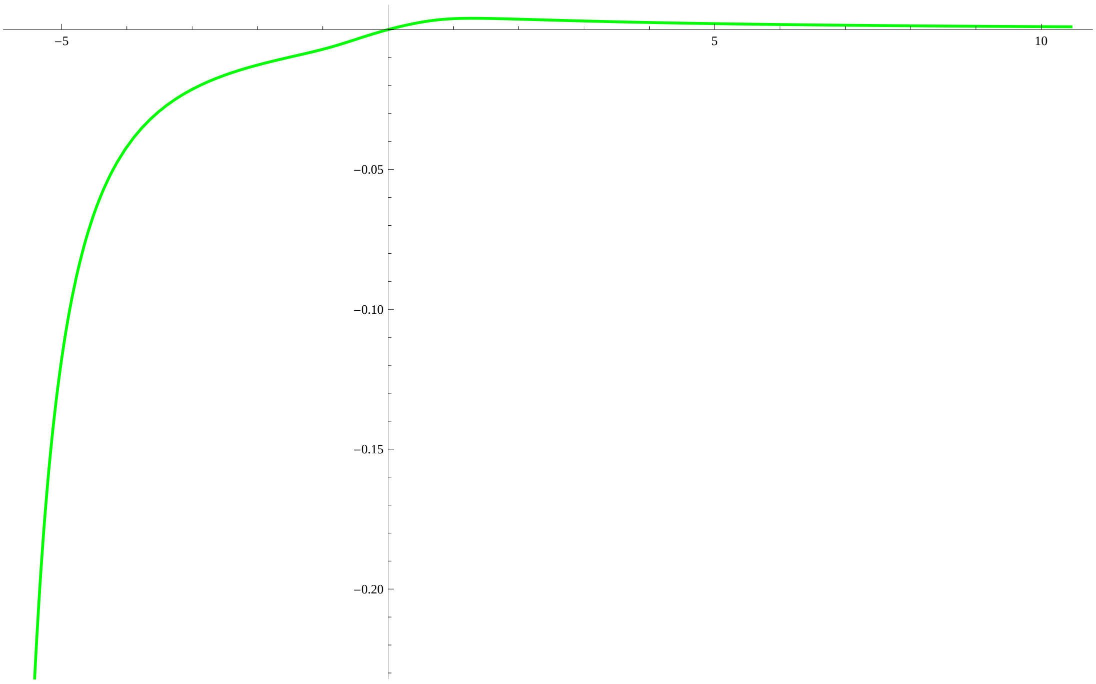


(b) HUBBLE parameter $H(t)$ (green) and Dilaton $\Phi(t)$ (red)

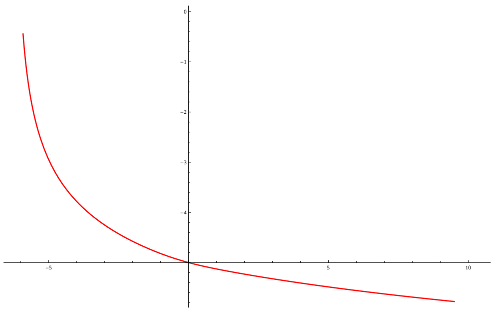


(c) Tachyon $T(t)$

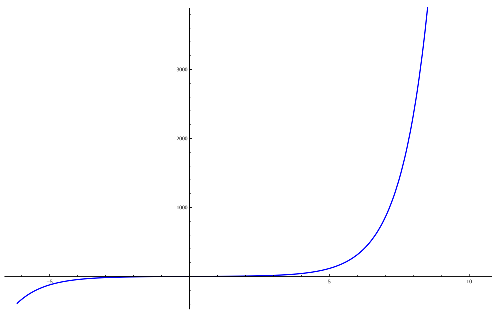
Figure 4.1.: Numerical solution with bounce and singular future, integrated from $H(0) = 0$, $\Phi(0) = -5$, $\dot{\Phi}(0) = 0.1$, and $T(0) = 0$.



(a) HUBBLE parameter $H(t)$

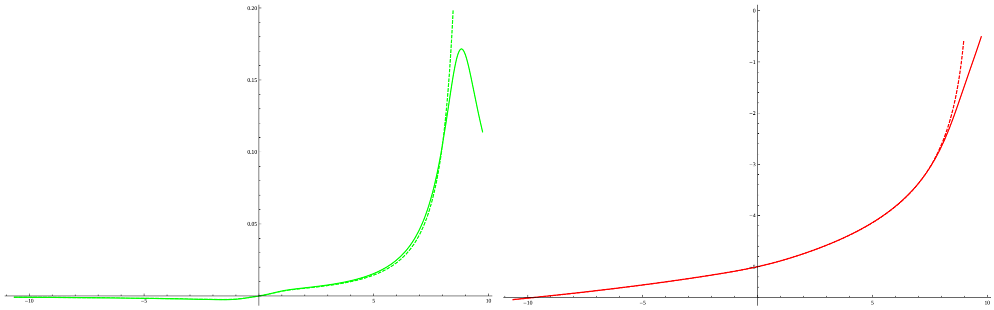


(b) Dilaton $\Phi(t)$



(c) Tachyon $T(t)$

Figure 4.2.: Numerical solution with bounce and singular past, integrated from $H(0) = 0$, $\Phi(0) = -5$, $\dot{\Phi}(0) = -0.15$, and $T(0) = 0$.



(a) HUBBLE parameter $H(t)$ (b) Dilaton $\Phi(t)$

Figure 4.3.: Numerical solution with bounce and regularised future, integrated from $H(0) = 0$, $\Phi(0) = -5$, $\dot{\Phi}(0) = 0.1$, and $T(0) = 0$ with potential parameter $\Phi_0 = 1.5$. The singular solution is plotted for comparison (dashed lines).

With the addition of the potential term the equations of motion now read

$$72H^2\left(1 + W(\Phi)\right) + 4\dot{\Phi}^2 - 36H\dot{\Phi}\left(1 + W(\Phi) - \frac{1}{2}W'(\Phi)\right) - 2\kappa_{10}^2 e^{2\Phi}\epsilon = 0 \tag{4.15a}$$

$$2\ddot{\Phi} - 2\dot{\Phi}^2 + 18H\dot{\Phi} - \left(9\dot{H} + 45H^2\right)\left(1 + W(\Phi) - \frac{1}{2}W'(\Phi)\right) - \kappa_{10}^2\frac{e^{2\Phi}}{2}p = 0 \tag{4.15b}$$

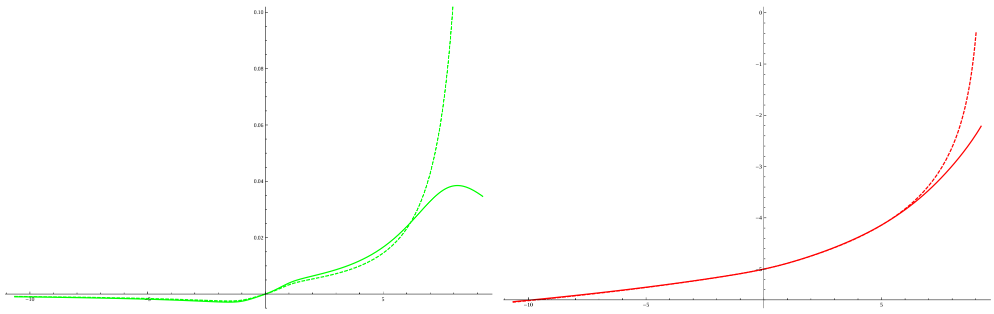
$$\begin{aligned}(\ddot{\Phi} + 8H\dot{\Phi})\left(W'(\Phi) - 2W(\Phi) - 2\right) + \dot{\Phi}^2\left(W''(\Phi) - 4W'(\Phi) + 4W(\Phi) + 2\right)\\ + \left(1 + W(\Phi)\right)\left(36H^2 + 8\dot{H}\right) + \kappa_{10}^2 e^{2\Phi}p = 0\end{aligned} \tag{4.15c}$$

$$\dot{\epsilon} + 9H(\epsilon + p) - \dot{\Phi}p = 0. \tag{4.15d}$$

We need to make sure that the additional term does not spoil the bounce obtained in the previous section. Therefore we require that it is important for large curvature only. Effectively this is a requirement on the numerical prefactor contained in the potential W. For the numerical analysis we choose $\Phi_0 = 1.5 \approx \frac{e^5}{33}$ and $\Phi_0 = 3 \approx \frac{e^5}{7.4}$. Translating this into a shift $\Delta\Phi = 5$ and a prefactor $W_0 = \frac{1}{33}$ and $W_0 = 0.14$ for the potential $W = -W_0 e^{\Phi+\Delta\Phi}$ we see, that there is no heavy fine-tuning at work.

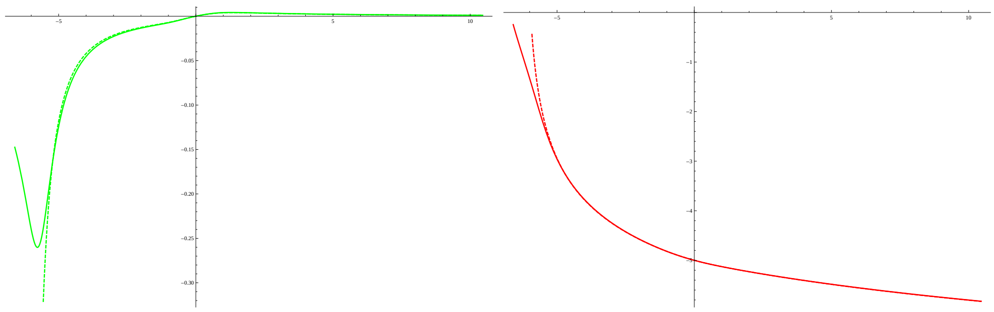
The numerical solutions are obtained in the same integration scheme and with the same initial conditions as in section 4.2.1. Figures 4.3–4.6 indeed show that the singularity in the HUBBLE parameter is resolved. In both cases of the bouncing solutions with singularity in the past or the future the blowing up of the curvature is eliminated due to the additional term (4.14).

In all cases a definite resolution of the singularity takes place only in the HUBBLE parameter. The numerical solutions suggest that the dilaton might remain singular.



(a) HUBBLE parameter $H(t)$ (b) Dilaton $\Phi(t)$

Figure 4.4.: Numerical solution with bounce and regularised future, integrated from $H(0) = 0$, $\Phi(0) = -5$, $\dot{\Phi}(0) = 0.1$, and $T(0) = 0$ with potential parameter $\Phi_0 = 3$. The singular solution is plotted for comparison (dashed lines).



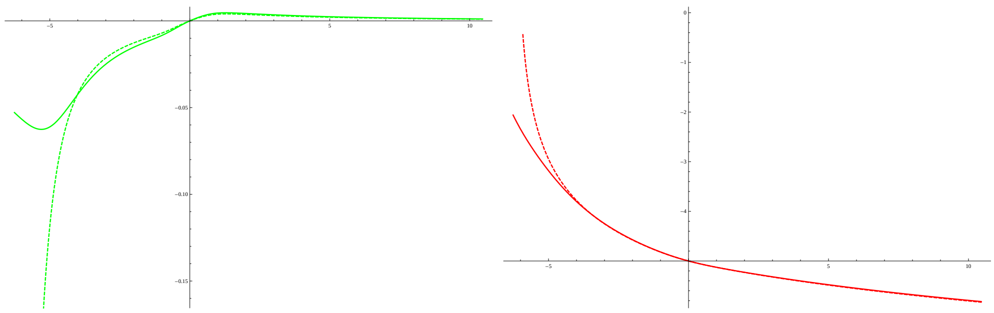
(a) HUBBLE parameter $H(t)$ (b) Dilaton $\Phi(t)$

Figure 4.5.: Numerical solution with bounce and regularised future, integrated from $H(0) = 0$, $\Phi(0) = -5$, $\dot{\Phi}(0) = -0.15$, and $T(0) = 0$ with potential parameter $\Phi_0 = 1.5$. The singular solution is plotted for comparison (dashed lines).

This is in contrast to the case where the tachyon sector is absent (i. e. $\epsilon = 0$ and $p = 0$). A numerical solution for this case is shown in figures 4.7. Here the pre-Big Bang singularity can be resolved in the dilaton Φ as well. Qualitatively our solutions exhibit the same singular behaviour as the standard pre-big bang scenario. We thus expect that the dilaton singularity can be resolved as well, but this may require fine-tuning.

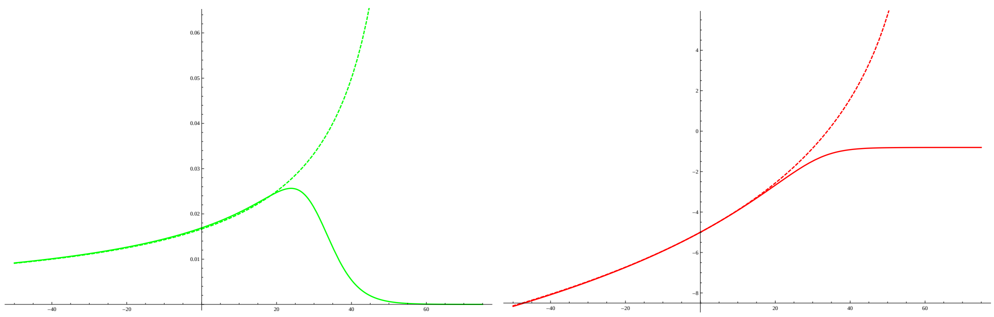
4.2.3. Discussion

In this chapter we obtained bounce scenarios, in which a non-BPS space-filling D9-brane in type IIA superstring theory drives a bounce of the scale factor in the string frame. In EINSTEIN frame these solutions are not bouncing but expanding or



(a) HUBBLE parameter $H(t)$ (b) Dilaton $\Phi(t)$

Figure 4.6.: Numerical solution with bounce and regularised future, integrated from $H(0) = 0$, $\Phi(0) = -5$, $\dot{\Phi}(0) = -0.15$, and $T(0) = 0$ with potential parameter $\Phi_0 = 3$. The singular solution is plotted for comparison (dashed lines).



(a) HUBBLE parameter $H(t)$ (b) Dilaton $\Phi(t)$

Figure 4.7.: Non-singular pre-Big Bang solution, integrated from $\Phi(0) = -5$, and $\dot{\Phi}(0) = 0.1$ with potential parameter $\Phi_0 = 1.5$. The singular solution is plotted for comparison (dashed lines).

contracting for all times.

Taking the lowest order effective action for metric, dilaton and an effective action for the open tachyonic mode as a result of the instability of the non-BPS D-brane, we obtained bounce solutions. The bounce results from the positivity of the pressure of the tachyon field in the error function Lagrangian for the brane mode. In contrast to this, the DBI action for instance can not drive a bounce. Both curvature and time derivative of the dilaton remain small during our bounce so that the gravitational sector behaves entirely classical.

Asymptotically our bounce solutions are similar to pre-big bang and post-big bang solutions respectively. There remain singularities in the curvature and the dilaton before or after the bounce. These asymptotic string frame curvature singularities can be resolved by the ad hoc addition of a potential, proportional to $Re^{-\Phi}$. Such a term might result from α' corrections in the open string sector. Exact calculation of the corrections would be necessary in order to give a more precise picture of the effects resulting from the corrections. With our choice for the sign of the prefactor the gravitational coupling changes sign in the string frame at some time after or before the bounce. After transformation to the EINSTEIN frame, this turns into a bounce without violating the null energy condition.

While our phenomenological potential clearly stabilises the dilaton within the perturbative regime without the tachyon, the numerical analysis hints, that this is no longer the case once the tachyonic sector is included. The obvious question is then, whether a modified potential exists which stabilises the dilaton in our model. Furthermore it would be very interesting to see whether the string theory α' corrections result in such a modified potential.

For the numerical solutions, we have obtained, we assumed that while the bounce takes place, the nine space dimensions are isotropic – apart from the compactification of six dimensions, but the actual details of the compactification did not play a role here. For further analysis it might be worthwhile to consider seperate dynamics for the scale factors of the compact and non-compact dimensions. This might result in phenomenologically interesting solutions, since finally the connection to the currently observed four-dimensional FRIEDMANN ROBERTSON WALKER cosmology has to be made.

4.3. Non-BPS D7-brane in compactified background

Let us now adress the issue of compactification. In our model the 10-dimensional spacetime with metric G_{10} shall now be given as a direct product of ordinary $(3+1)$-dimensional spacetime with a compact space. To keep the model simple and calculable we assume toric geometry in the compact space. We introduce a non-BPS 7-brane that fills spacetime and wraps a torus T^4 with circumference $L_4 \mathrm{e}^{\sigma}$, where we assume isotropic modulus within the torus. The remaining two dimensions shall form a 2-Torus T^2 of fixed circumference L_2. The line element of the metric in the string frame then reads

$$\mathrm{d}s^2 = g_{\mu\nu}(x^\rho)\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\sigma(x^\rho)}\delta_{ij}\,\mathrm{d}x^i \mathrm{d}x^j + \delta_{ab}\,\mathrm{d}x^a \mathrm{d}x^b, \tag{4.16}$$

where greek indices $\mu, \nu \in \{0, \ldots, 3\}$ label spacetime coordinates and latin indices $i, j \in \{4, \ldots, 7\}$ and $a, b \in \{8, 9\}$ label the compact space dimensions. To get rid of the tachyon ground state the torus T^4 shall be Z_2-orbifolded identifying $x^4 \to -x^4$. This orbifold does not change the line element – only the ranges of variables have to be adjusted.

The action in string frame is given by

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int \mathrm{e}^{-2\Phi} \sqrt{-G} \left(R + 4(\partial\Phi)^2 \right) \mathrm{d}^{10}x \\ - T_7 \int \mathrm{e}^{-\Phi} V(\tilde{T}) \sqrt{-\det\left(g_{AB} + \partial_A \tilde{T} \partial_B \tilde{T}\right)}\, \mathrm{d}^8 x, \end{aligned} \tag{4.17}$$

where we have for now used a DIRAC BORN INFELD type of action for the D-brane, with the potential

$$V(\tilde{T}) = \exp\left(-\frac{\tilde{T}^2}{4\alpha'}\right).$$

Brane tension T_7, and 10-dimensional gravitational constant κ are

$$T_7 = \frac{1}{g_s(2\pi)^7(\alpha')^4}, \qquad 2\kappa^2 = (2\pi)^7(\alpha')^4 g_s^2.$$

We will simplify the action further by TAYLOR expansion of the square root from the DBI action and the exponential in the integral for small tachyon and small derivatives of the tachyon.

We assume that the fields (metric of ordinary spacetime $g_{\mu\nu}$, modulus field σ, dilaton Φ) have non-trivial dependence only on the four coordinates x^μ of ordinary spacetime. The compactification on the tori T_4 and T_2 then in the first integral of (4.17) simply results in the replacement

$$\int \mathrm{d}^{10}x\,\sqrt{-G} \to \frac{1}{2}L_4^4 L_2^2 \int \mathrm{d}^4x\,\sqrt{-g}\mathrm{e}^{4\sigma}. \tag{4.18a}$$

When the Z_2-orbifold identification $x^4 \to -x^4$ is introduced, this results in an antisymmetry condition on the tachyon field so that the lowest mode is given as $\tilde{T}(x^\mu, x^4) = \sqrt{2}\sin\left(\frac{2\pi x^4}{L_4}\right)T(x^\mu)$, where T is independent of x^4. We then TAYLOR expand the square root and the exponential in the integral for small T and small $\dot{T}$.

The block diagonal form of the metric (4.16) can be used to simplify the expression for the ten dimensional RICCI scalar and express it in terms of the four dimensional curvature. Obviously the flat T_2 factor does not contribute to the curvature. We calculate the RICCI curvature scalar for the slightly more general case of a $D = d + \tilde{d}$ dimensional manifold with line element

$$\mathrm{d}s^2 = g^d_{\mu\nu}(x^\rho)\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\sigma(x^\rho)}\delta^{\tilde{d}}_{ij}\,\mathrm{d}x^i \mathrm{d}x^j. \tag{4.19}$$

The CHRISTOFFEL symbols are

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= g^{\rho\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) = {}^d\Gamma^\rho_{\mu\nu},\\
\Gamma^i_{\mu\nu} &= -\frac{1}{2}g^{ij}g_{\mu\nu,j} = 0,\\
\Gamma^i_{\mu j} &= \frac{1}{2}g^{ik}g_{kj,\mu} = \delta^i_j\partial_\mu\sigma,\\
\Gamma^i_{jk} &= 0,\\
\Gamma^\rho_{ij} &= -\frac{1}{2}g^{\rho\mu}g_{ij,\mu} = -\delta_{ij}g^{\rho\mu}\mathrm{e}^{2\sigma}\partial_\mu\sigma,\\
\Gamma^\rho_{\mu j} &= 0.
\end{aligned}$$

The RICCI scalar

$$R = R^{\alpha\beta}{}_{\alpha\beta} = \Gamma^{\alpha}_{\gamma\alpha}\Gamma^{\gamma}_{\beta\delta}g^{\beta\delta} - \Gamma^{\alpha}_{\gamma\beta}\Gamma^{\gamma}_{\delta\alpha}g^{\beta\delta} + \Gamma^{\alpha}_{\beta\delta,\alpha}g^{\beta\delta} - \Gamma^{\alpha}_{\beta\alpha,\delta}g^{\beta\delta}.$$

contains four terms. These can be decomposed in the following way. Here [d-dimensional] denotes the corresponding term from d dimensions.

$$\begin{aligned}
\Gamma^{\alpha}_{\gamma\alpha}\Gamma^{\gamma}_{\beta\delta}g^{\beta\delta} &= -\tilde{d}^2(\partial\sigma)^2 + \tilde{d}\,{}^d\Gamma^{\rho}_{\mu\nu}g^{\mu\nu}\partial_\rho\sigma - \tilde{d}\,{}^d\Gamma^{\nu}_{\gamma\nu}g^{\mu\gamma}\partial_\mu\sigma + \quad [d\text{-dimensional}],\\
-\Gamma^{\alpha}_{\gamma\beta}\Gamma^{\gamma}_{\delta\alpha}g^{\beta\delta} &= \tilde{d}(\partial\sigma)^2 + \quad [d\text{-dimensional}],\\
\Gamma^{\alpha}_{\beta\delta,\alpha}g^{\beta\delta} &= -\tilde{d}g^{\nu\mu}{}_{,\nu}\partial_\mu\sigma - 2\tilde{d}(\partial\sigma)^2 - \tilde{d}\Box\sigma + \quad [d\text{-dimensional}],\\
-\Gamma^{\alpha}_{\beta\alpha,\delta}g^{\beta\delta} &= -\tilde{d}\Box\sigma + \quad [d\text{-dimensional}].
\end{aligned}$$

Thus we obtain for the RICCI curvature scalar

$${}^D R = {}^d R - 2\tilde{d}\Box\sigma - (\tilde{d} - \tilde{d}^2)(\partial\sigma)^2.$$

The second term with the D'ALEMBERT operator could by means of integration by parts rewritten into a term containing $(\partial\sigma)\cdot(\partial\Phi)$. These terms are canceled, if we define a new, d-dimensional dilaton

$$\phi = \Phi - \frac{\tilde{d}}{2}\sigma,$$

getting rid of the $\mathrm{e}^{2\sigma}$ factor in the action after compactification. The dilaton kinetic term then reads

$$4(\partial\Phi)^2 = 4(\partial\phi)^2 + 4\tilde{d}(\partial\phi)\cdot(\partial\sigma) + \tilde{d}^2(\partial\sigma)^2$$

and is, up to integration by parts inside the action integral with vanishing boundary terms, equivalent to

$$4(\partial\Phi)^2 \equiv 4(\partial\phi)^2 - 2\tilde{d}\Box\sigma + 2\tilde{d}\Gamma^{\nu}_{\mu\nu}g^{\mu\rho}\partial_\rho\sigma.$$

Finally we obtain

$${}^D R + 4(\partial\Phi)^2 \equiv {}^d R - \tilde{d}(\partial\sigma)^2 + 4(\partial\phi)^2. \tag{4.20}$$

After compactification the effective four dimensional action finally is given by

$$\begin{aligned}
S = \frac{L_2^2}{g_\mathrm{s}}\int \mathrm{d}^4x\,\mathrm{e}^{-2\phi}\sqrt{-g}\left({}^{(4)}R + 4\partial_\mu\phi\partial^\mu\phi - 4\partial_\mu\sigma\partial^\mu\sigma\right)\\
- \int \mathrm{d}^4x\,\mathrm{e}^{2\sigma-\phi}\sqrt{-g}\left(1 + \frac{1}{2}\partial_\mu T\partial^\mu T + \frac{m^2}{2}T^2\right),
\end{aligned} \tag{4.21}$$

where the quadratic action for the tachyon was obtained by expansion of the square root for small T and $\dot{T}$ as mentioned before.

Since we are interested in cosmological solutions, we make the ansatz of a FRIEDMANN ROBERTSON WALKER line element for the spacetime metric

$$\left(g_{\mu\nu}\right) = \mathrm{diag}\left(-1, a(t)^2, a(t)^2, a(t)^2\right). \tag{4.22}$$

Defining the HUBBLE parameter $H = \frac{\dot{a}}{a}$ in the usual way, we obtain the FRIEDMANN equations from the variation w. r. t. the components of the metric

$$3H^2 = 6H\dot{\phi} - 2\dot{\phi}^2 + 2\dot{\sigma}^2 + \frac{\mathrm{e}^{2\sigma+\phi}}{2\lambda^2}(1 + \frac{\dot{T}^2}{2} + \frac{m^2T^2}{2}), \tag{4.23a}$$

$$\dot{H} = -4H\dot{\phi} + 2\dot{\phi}^2 - 2\dot{\sigma}^2 - \frac{\mathrm{e}^{2\sigma+\phi}}{4\lambda^2}(3 + \frac{\dot{T}^2}{2} + \frac{3m^2T^2}{2}). \tag{4.23b}$$

By variation w. r. t. the dilaton ϕ and the modulus σ and the tachyon we obtain three further equations

$$\ddot{\phi} = -3H\dot{\phi} + 2\dot{\phi}^2 - \frac{\mathrm{e}^{2\sigma+\phi}}{4\lambda^2}(3 - \frac{\dot{T}^2}{2} + \frac{3m^2T^2}{2}), \tag{4.23c}$$

$$\ddot{\sigma} = -3H\dot{\sigma} + 2\dot{\phi}\dot{\sigma} - \frac{\mathrm{e}^{2\sigma+\phi}}{4\lambda^2}(1 - \frac{\dot{T}^2}{2} + \frac{3m_0^2T^2}{2}), \tag{4.23d}$$

$$\ddot{T} = -3H\dot{T} + \dot{T}\left(\dot{\phi} - 2\dot{\sigma}\right) - m^2T, \tag{4.23e}$$

where $m^2 = \frac{1}{2\alpha'}\left(-1 + \mathrm{e}^{-2\sigma}\right)$ and $m_0^2 = -\frac{1}{2\alpha'}$. We shall keep in mind that these equations are only applicable for small values of T and $\dot{T}$.

4.3.1. Asymptotic solutions

In the equations (4.23) the terms with the tachyon act effectively as a potential for dilaton ϕ and the modulus σ. But they come with an exponential $\mathrm{e}^{2\sigma+\phi}$ in front of them. If now σ is strongly negative we can neglect these terms in the equations of motion. Furthermore in this case m^2 is positive. We thus expect the 'tachyon' to be approximately $T = 0$ as long as we are in an expanding universe with $H > 0$.

Dilaton domination: Pre-big bang and Post-big bang solutions

If in addition we assume approximately constant modulus σ, we recover the standard pre-big bang scenario with the equations of motion

$$3H^2 = 6H\dot{\phi} - 2\dot{\phi}^2, \tag{4.24a}$$

$$\dot{H} = -4H\dot{\phi} + 2\dot{\phi}^2, \tag{4.24b}$$

$$\ddot{\phi} = -3H\dot{\phi} + 2\dot{\phi}^2. \tag{4.24c}$$

Taking a power law ansatz for the scale factor $a \propto t^n$ with HUBBLE parameter $H = \frac{n}{t}$, the constraint equation (4.24a) immediately suggest a dilaton of $\dot{\phi} = \frac{\phi_1}{t}$. Plugging in the ansatz we obtain after multiplying with t^2

$$
\begin{aligned}
3n^2 &= 6n\phi_1 - 2\phi_1^2, \\
-n &= -4n\phi_1 + 2\phi_1^2, \\
-\phi_1 &= -3n\phi_1 + 2\phi_1^2.
\end{aligned}
$$

These equations reduce to $\phi_1 = \frac{3n-1}{2}$ and $n^2 = \frac{1}{3}$. We thus obtain two solutions for $t > 0$:

1. an expanding solution with $a = a_0 t^{\frac{1}{\sqrt{3}}}$ and $\phi = \frac{\sqrt{3}-1}{2} \ln t + \phi_0$ and
2. a contracting solution with $a = a_0 t^{-\frac{1}{\sqrt{3}}}$ and $\phi = -\frac{\sqrt{3}+1}{2} \ln t + \phi_0$.

Both solutions are singular at $t = 0$ and decelerate. Since $t > 0$ for these solutions, the singularity is in the past and these solutions are 'post-big bang' solutions. Both of them are connected to each other with the aforementioned scale-factor duality transformation (2.45). Due to the symmetry of the field equations under time reflection $t \to -t$, we can extend these solutions to times $t < 0$ and obtain

1. an expanding solution with $a = a_0 |t|^{\frac{1}{\sqrt{3}}}$ and $\phi = \frac{\sqrt{3}-1}{2} \ln|t| + \phi_0$ and
2. a contracting solution with $a = a_0 |t|^{-\frac{1}{\sqrt{3}}}$ and $\phi = -\frac{\sqrt{3}+1}{2} \ln|t| + \phi_0$.

Again both solutions are singular, but now the singularity is in the future and they are 'pre-big bang' solutions. The rate of expansion or contraction is now accelerating.

Dilaton and modulus domination

We now drop the additional assumption of approximately constant modulus σ. The equations of motion read

$$
\begin{aligned}
3H^2 &= 6H\dot{\phi} - 2\dot{\phi}^2 + 2\dot{\sigma}^2, && (4.25a) \\
\dot{H} &= -4H\dot{\phi} + 2\dot{\phi}^2 - 2\dot{\sigma}^2, && (4.25b) \\
\ddot{\phi} &= -3H\dot{\phi} + 2\dot{\phi}^2. && (4.25c) \\
\ddot{\sigma} &= -3H\dot{\sigma} + 2\dot{\sigma}\dot{\phi}. && (4.25d)
\end{aligned}
$$

The similarity of the equations for ϕ and σ leads us to the ansatz

$$
a = a_0 t^n, \qquad \dot{\phi} = \frac{\phi_1}{t}, \qquad \dot{\sigma} = \frac{\sigma_1}{t}.
$$

Plugging in the ansatz we obtain after multiplying with t^2

$$\begin{aligned} 3n^2 &= 2\sigma_1^2 + 6n\phi_1 - 2\phi_1^2, \\ n &= 2\sigma_1^2 - 4n\phi_1 + 2\phi_1^2, \\ 0 &= \phi_1(1 - 3n + 2\phi_1), \\ 0 &= \sigma_1(1 - 3n + 2\phi_1). \end{aligned}$$

These equations reduce to $\phi_1 = \frac{3n-1}{2}$ and $\sigma_1^2 = \frac{1-3n^2}{4}$. We can now allow for arbitrary n as long as $|n| \leq \frac{1}{\sqrt{3}}$. We thus obtain a continous flock of solutions in the post-big bang regime $t > 0$:

$$a = a_0 t^n, \qquad \phi = \tfrac{3n-1}{2} \ln t + \phi_0, \qquad \sigma = \pm \tfrac{\sqrt{1-3n^2}}{2} \ln t + \sigma_0, \tag{4.26}$$

where for $-\frac{1}{\sqrt{3}} \leq n \leq 0$ we have a contracting solution and for $0 \leq n \leq \frac{1}{\sqrt{3}}$ the solution describes an expanding universe. Again these solutions are singular and accompanied by their time reflected variants for $t < 0$.

In order to resolve the singularity, the exponential terms would have to dominate equations (4.23c) and (4.23d). But since $H \propto \dot{\phi} \propto \dot{\sigma} \propto t^{-1}$ for the exponential term to be a dominant contribution to equations (4.23c) and (4.23d) we need to have

$$\mathrm{e}^{2\sigma+\phi} \propto t^{-2}, \qquad \text{for} \quad t \to 0. \tag{4.27}$$

But plugging in the expanding solution the dominance of the exponential term is equivalent to

$$2 < \frac{3n+1}{2} \pm \sqrt{1-3n^2}, \tag{4.28}$$

which is false for all n in the applicable regime $0 \leq n \leq \frac{1}{\sqrt{3}}$. We thus conclude that the inclusion of the matter terms in equations (4.23c) and (4.23d), which are suppressed by an exponential, cannot stop the post-big bang like inflationary approach to the singularity for $t \to 0$. In this approximation, the singularities are not resolved.

4.3.2. Full tachyon potential

Let us therefore keep the full tachyon potential in the action, while still expanding the square root into a power series. After doing the integration over the three coordinates that are not involved with the orbifold procedure we obtain

$$\begin{aligned} S_T &= -T_7 L_4^3 \int \mathrm{d}^4x\, \mathrm{e}^{-\Phi} \sqrt{-g} \mathrm{e}^{4\sigma} \int \mathrm{d}x^4\, V(\tilde{T}) \left(1 + \frac{1}{2}\partial_\mu \tilde{T} \partial^\mu \tilde{T} + \frac{1}{2}\partial_4 \tilde{T} \partial^4 \tilde{T}\right) \\ &= -T_7 L_4^3 \int \mathrm{d}^4x\, \mathrm{e}^{-\Phi} \sqrt{-g} \mathrm{e}^{4\sigma} \int_0^{\frac{L_4}{2}} \mathrm{d}x^4\, \mathrm{e}^{-\frac{1}{2\alpha'} \sin^2\left(\frac{2\pi x^4}{L_4}\right) T^2} \\ &\quad \cdot \left(1 + \sin^2\left(\frac{2\pi x^4}{L_4}\right) \partial_\mu T \partial^\mu T + \mathrm{e}^{-2\sigma} \left(\frac{2\pi}{L_4}\right)^2 \cos^2\left(\frac{2\pi x^4}{L_4}\right) T^2\right). \end{aligned} \tag{4.29}$$

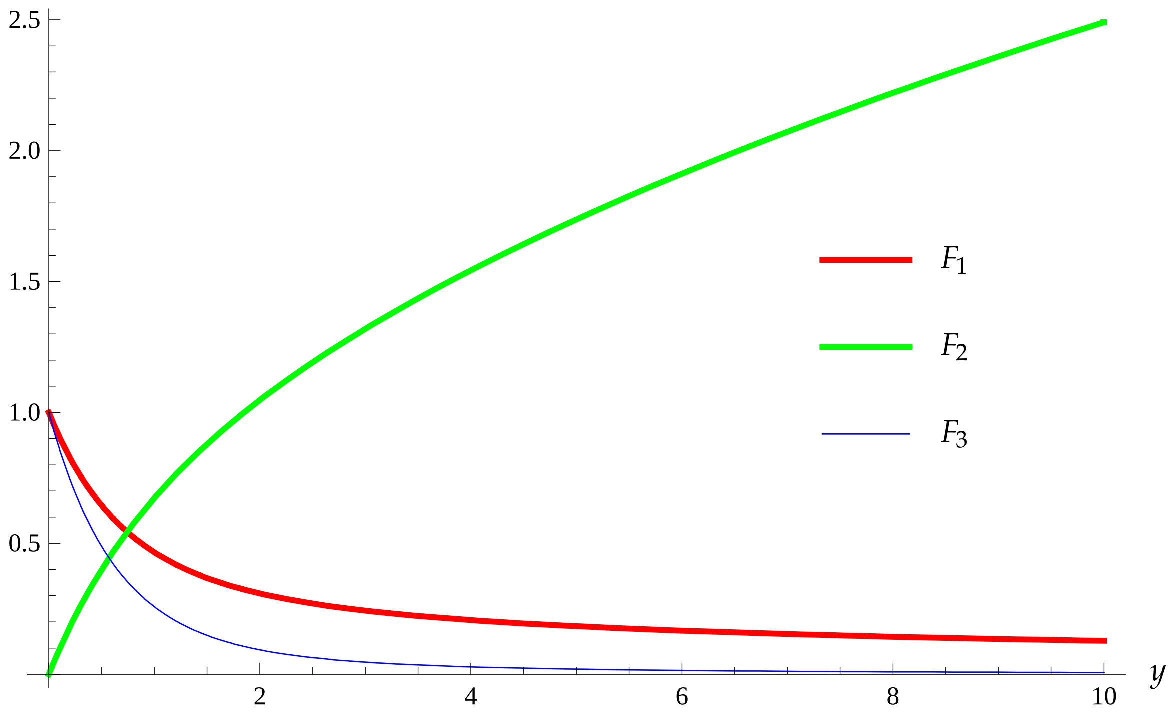


Figure 4.8.: $F_1(y) = e^{-y} I_0(y)$, $F_2(y) = y e^{-y} \left(I_0(y) + I_1(y) \right)$, and $F_3(y) = e^{-y} \left(I_0(y) - I_1(y) \right)$

The integration can be done with the help of the following identity of the modified BESSEL function I_0

$$\int_0^{\pi} e^{-2y \sin^2 t} \, dt = \pi e^{-y} I_0(y). \tag{4.30}$$

The remaining integrals containing $\sin^2$ and $\cos^2 = 1 - \sin^2$ can be expressed as derivatives of this relation

$$\frac{\partial}{\partial y} \int_0^{\pi} e^{-2y \sin^2 t} \, dt = -2 \int_0^{\pi} \sin^2 t e^{-2y \sin^2 t} \, dt = \pi e^{-y} \left(I_0'(y) - I_0(y) \right), \tag{4.31}$$

where $I_0'(y) = I_1(y)$. Using these relations, we obtain for the tachyon action (4.29)

$$S_T = -\frac{T_7 L_4^4}{2} \int d^4 x \, e^{2\sigma - \phi} \sqrt{-g} \left(F_1 \left(\tfrac{T^2}{4\alpha'} \right) + e^{-2\sigma} F_2 \left(\tfrac{T^2}{4\alpha'} \right) + \frac{1}{2} F_3 \left(\tfrac{T^2}{4\alpha'} \right) \partial_\mu T \partial^\mu T \right), \tag{4.32}$$

where the functions F_i are defined by

$$F_1(y) = e^{-y} I_0(y), \tag{4.33}$$

$$F_2(y) = y e^{-y} \left(I_0(y) + I_1(y) \right), \tag{4.34}$$

$$F_3(y) = e^{-y} \left(I_0(y) - I_1(y) \right), \tag{4.35}$$

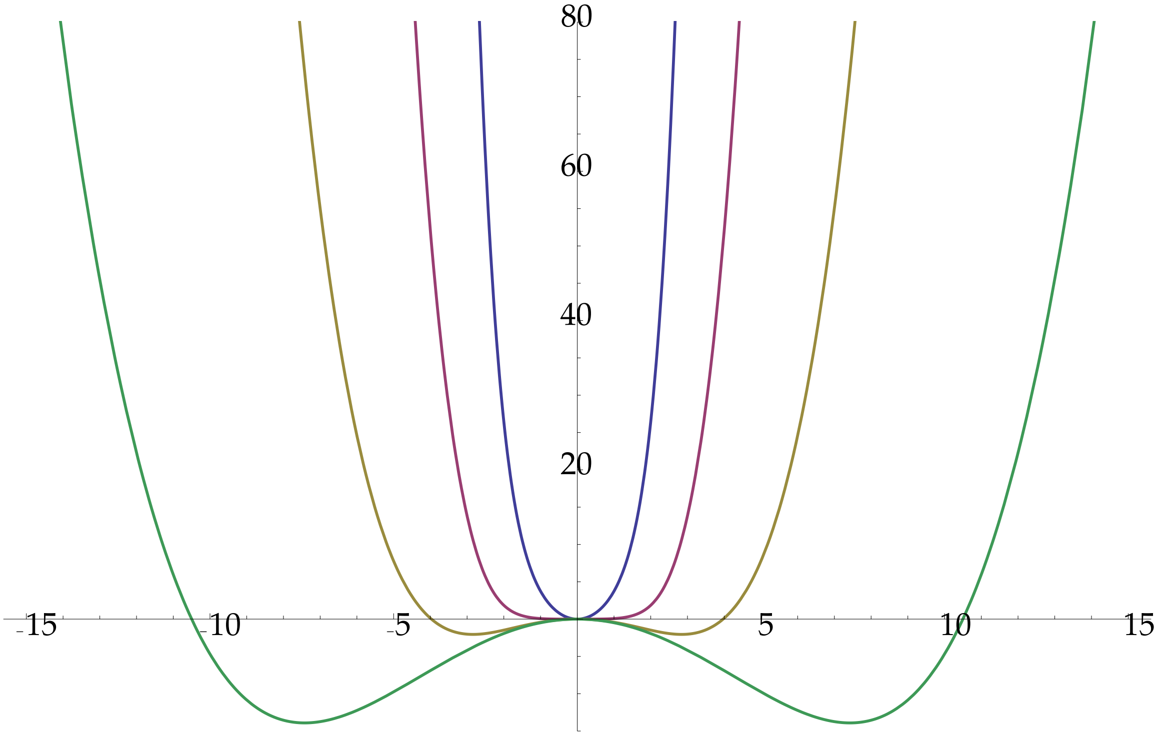


Figure 4.9.: Effective potential $V_{\text{eff}}(T)$ for modulus $\sigma = -1$, $\sigma = 0$, $\sigma = 1$, and $\sigma = 2$.

and sketched in figure 4.8. The field equations now read

$$3H^2 = 6H\dot{\phi} - 2\dot{\phi}^2 + 2\dot{\sigma}^2 + \frac{e^{2\sigma+\phi}}{2\lambda^2}\left(F_1 + e^{-2\sigma}F_2 + \frac{F_3}{2}\dot{T}^2\right), \tag{4.36a}$$

$$\dot{H} = -4H\dot{\phi} + 2\dot{\phi}^2 - 2\dot{\sigma}^2 - \frac{e^{2\sigma+\phi}}{4\lambda^2}\left(3F_1 + 3e^{-2\sigma}F_2 + \frac{F_3}{2}\dot{T}^2\right), \tag{4.36b}$$

$$\ddot{\phi} + 3H\dot{\phi} = 2\dot{\phi}^2 - \frac{e^{2\sigma+\phi}}{4\lambda^2}\left(3F_1 + 3e^{-2\sigma}F_2 - \frac{F_3}{2}\dot{T}^2\right), \tag{4.36c}$$

$$\ddot{\sigma} + 3H\dot{\sigma} = 2\dot{\sigma}\dot{\phi} - \frac{e^{2\sigma+\phi}}{4\lambda^2}\left(F_1 - \frac{F_3}{2}\dot{T}^2\right), \tag{4.36d}$$

$$\ddot{T} + 3H\dot{T} = \dot{T}\left(\dot{\phi} - 2\dot{\sigma} - \frac{1}{4\alpha'}\frac{F_3'}{F_3}T\dot{T}\right) - \frac{1}{4\alpha'}\frac{2}{F_3}\left(F_1' + e^{-2\sigma}F_2'\right)T. \tag{4.36e}$$

From the last equation, we can read off the effective potential, in which T is moving

$$V_{\text{eff}}(T) = \frac{1}{2\alpha'}\int_{T^*}^{T} d\tau\, \frac{F_1'(\frac{\tau^2}{4\alpha'}) + e^{-2\sigma}F_2'(\frac{\tau^2}{4\alpha'})}{F_3(\frac{\tau^2}{4\alpha'})}\tau. \tag{4.37}$$

Unfortunately a closed form of this integral is yet to be found. As long as $\sigma \leq 0$ the effective potential (4.37) has one minimum at $T = 0$. But if $\sigma > 0$ it has two absolute minima at $T = \pm T_0$ for some T_0. Numerical results of the integral are sketched for some values of σ in figure 4.9.

Solutions with approximately $T \equiv 0$

Solutions with a stable D-brane correspond to $T \equiv 0$ and describe classical universe if the dilaton $\Phi = \phi + 2\sigma$ is constant or at least changing only slowly. Plugging in $T \equiv 0$ into the equations of motion (4.36) and setting $\sigma = \frac{1}{2}(\Phi - \phi)$ we obtain

$$3H^2 = 6H\dot{\phi} - 2\dot{\phi}^2 + \frac{1}{2}\left(\dot{\Phi} - \dot{\phi}\right)^2 + \frac{e^{\Phi}}{2\lambda^2}, \tag{4.38a}$$

$$\dot{H} = -4H\dot{\phi} + 2\dot{\phi}^2 - \frac{1}{2}\left(\dot{\Phi} - \dot{\phi}\right)^2 - \frac{3e^{\Phi}}{4\lambda^2}, \tag{4.38b}$$

$$\ddot{\phi} + 3H\dot{\phi} = 2\dot{\phi}^2 - \frac{3e^{\Phi}}{4\lambda^2}, \tag{4.38c}$$

$$\ddot{\Phi} + 3H\dot{\Phi} = 2\dot{\phi}\dot{\Phi} - \frac{5e^{\Phi}}{4\lambda^2}. \tag{4.38d}$$

Assuming an asymptotically constant dilaton $\Phi = Bt^{-A} + C$ with $A > 0$ and taking a generic ansatz for $H = Dt^F$ and $\phi = Gt^J$ we obtain from (4.38d) that either $F = 1 + A$ or $J = 1 + A$ but in both cases the remaining equation (4.38b) is in contradiction with the constraint equation (4.38a), at least if C is non-vanishing.

We thus relax our requirement and allow for logarithmically growing dilaton $\Phi = -A\ln(Bt)$. Using again equation (4.38d) we obtain

$$0 = \frac{A}{t^2} + \frac{5(Bt)^{-}A}{4\lambda^2} - \frac{3AH}{t} + \frac{2A\phi'}{t}.$$

Assuming for now that $A < 2$ and thus dominates the $\frac{1}{t^2}$ term, we need to compensate the t^{-A}-term with either H or ϕ'. Both lead to a contradiction between equations (4.38b) and (4.38a) or to the assumption that $A < 2$. Analogously the case $A > 2$ does not fulfill (4.38a). In the case $A = 2$ the ansatz $\phi = Ct^D$ and $H = Ft^G$ in equation (4.38d) leads, since B is real, to the possibilities $G = -1$, $G = D - 1 > -1$ and $D = 0$, while the latter more precisely should be read as ϕ is logarithmically divergent. The latter two cases lead to a contradiction in (4.38a), while the first gives one analytic solution

$$\begin{aligned} \Phi &= -2\ln\frac{5t}{\sqrt{8}}, \\ \phi &= -\frac{6}{5}\ln t + \phi_0, \\ H &= -\frac{2}{5t}. \end{aligned} \tag{4.39}$$

This solution describes a decelerated contraction similar to the post-big bang scenario but with a $\frac{2}{5}$-power law for the scale factor instead of $\frac{1}{\sqrt{3}}$. The solution is singular with a $\frac{1}{t}$ pole in the HUBBLE parameter and a logarithmic divergence in the dilaton.

Solutions with large T

The other extremal case is the one of large T. In this case we cannot expand the square root in (4.29) anymore. But we can reorder the terms

$$\begin{aligned}\sqrt{1-\dot{\tilde{T}}^2+\mathrm{e}^{-2\sigma}\tilde{T}'^2} &= \sqrt{1-2\dot{T}^2\sin^2\left(\frac{2\pi x^4}{L_4}\right)+\frac{8\pi^2}{L_4^2}\mathrm{e}^{-2\sigma}T^2\cos^2\left(\frac{2\pi x^4}{L_4}\right)} \\ &= \sqrt{1+\frac{8\pi^2}{L_4^2}\mathrm{e}^{-2\sigma}T^2-\left(2\dot{T}^2+\frac{8\pi^2}{L_4^2}\mathrm{e}^{-2\sigma}T^2\right)\sin^2\left(\frac{2\pi x^4}{L_4}\right)} \\ &= \sqrt{1+\frac{8\pi^2}{L_4^2}\mathrm{e}^{-2\sigma}T^2}\sqrt{1-\frac{2\dot{T}^2+\frac{8\pi^2}{L_4^2}\mathrm{e}^{-2\sigma}T^2}{1+\frac{8\pi^2}{L_4^2}\mathrm{e}^{-2\sigma}T^2}\sin^2\left(\frac{2\pi x^4}{L_4}\right)}.\end{aligned}$$

We are especially interested in the potential part, i. e. the part without derivatives $\dot{T}$.

$$V_{\mathrm{eff}}(T) = \sqrt{1+4\mathrm{e}^{-2\sigma}y}\sqrt{1-\left(\frac{1}{1+\frac{4\mathrm{e}^{2\sigma}}{y}}\right)\sin^2(t)},$$

where we introduced $y=\frac{2\pi^2}{L_4^2}T^2$ and $t=\frac{2\pi x^4}{L_4}$. Using TAYLOR expansion we express the square root through a sum and note that within the action integral, we can obtain the n-th power of $\sin^2 t$ from the n-th derivative of $F_1(y)$ with the help of relation (4.31).

$$V_{\mathrm{eff}}(T) \to \sqrt{1+4\mathrm{e}^{-2\sigma}y}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2\left(\frac{1}{2}\right)_{(1-n)}n!}\left(\frac{1}{1+\frac{4\mathrm{e}^{2\sigma}}{y}}\right)^n\left(-\frac{1}{2}\right)^n\frac{\partial^n}{\partial y^n}F_1(y)\right),$$

where the prefactor within the sum includes the POCHHAMMER symbol $(a)_m = a(a+1)\cdots(a+m-1)$ and takes the values

$$\left(\frac{(-1)^n}{2\left(\frac{1}{2}\right)_{(1-n)}n!}\right)_{n\in\{0,1,2,\ldots\}} = \left(1,-\frac{1}{2},-\frac{1}{8},-\frac{1}{16},-\frac{5}{128},-\frac{7}{256},\ldots\right).$$

4.3.3. Discussion

The numerical integration of the differential equations are very unreliable. The tachyon T or its derivatives generically grow large. Especially if one could by fine tuning the parameters obtain a bouncing solution, this seems only possible with large T or $\dot{T}$. But in this regime the approximation applied to obtain the differential equations is not valid. One would have to use different approximations like we did in the case of the D9-brane in section 4.2.

4.4. Non-BPS D9-brane with orbifold

Let us therefore return to the D9-brane scenario of section 4.2 and introduce the orbifolding procedure here. But for simplicity we keep the remaining spacelike dimensions treated equal. We thus take the metric ansatz

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\boldsymbol{x}^2 + \mathrm{e}^{2\sigma(t)}(\mathrm{d}x^4)^2 + a(t)^2\mathrm{d}\boldsymbol{y}^2,$$

where $\boldsymbol{x}$ denote the spacelike coordinates of ordinary spacetime, x^4 is the coordinate of the torus T^6 which is orbifolded using the identification $x^4 \equiv -x^4$, and $\boldsymbol{y}$ are the five remaining coordinates in the torus T^6. We will use capital latin indices A and B running from 0 to 9 to label all spacetime coordinates.

We assume that the tachyon $T(t, x^4)$ has non-trivial dependence only on time t and the orbifolded coordinate x^4. After variation of the action we integrate over the x^4-direction and denote this by a bar on the respective components $\bar{\epsilon}$, $\bar{L}$, and $\bar{T}^4{}_4$ of the energy momentum tensor:

$$56H^2 + 16H\dot{\sigma} - 32H\dot{\Phi} + 4\dot{\Phi}^2 - 4\dot{\Phi}\dot{\sigma} = \frac{\mathrm{e}^{2\Phi}}{\lambda^2}\bar{\epsilon}, \tag{4.40a}$$

$$\ddot{\sigma} + \dot{\sigma}^2 - 2\ddot{\Phi} + 2\dot{\Phi}^2 - 2\dot{\sigma}\dot{\Phi} + 7H\dot{\sigma} - 14H\dot{\Phi} + 7\dot{H} + 28H^2 = \frac{\mathrm{e}^{2\Phi}}{2\lambda^2}\bar{L}, \tag{4.40b}$$

$$-2\ddot{\Phi} + 2\dot{\Phi}^2 - 16H\dot{\Phi} + 8\dot{H} + 36H^2 = \frac{\mathrm{e}^{2\Phi}}{2\lambda^2}\bar{T}^4{}_4, \tag{4.40c}$$

$$\ddot{\sigma} + \dot{\sigma}^2 - 2\ddot{\Phi} + 2\dot{\Phi}^2 - 2\dot{\sigma}\dot{\Phi} + 8H\dot{\sigma} - 16H\dot{\Phi} + 8\dot{H} + 36H^2 = \frac{\mathrm{e}^{2\Phi}}{4\lambda^2}\bar{L}, \tag{4.40d}$$

As usual it is useful to supplement these equations of motion by the conservation equation of the energy momentum tensor $\nabla_A T^A{}_B = -\partial_B\Phi L$, where we assume that the LAGRANGE has the usual form $L = \mathrm{e}^{-\Phi}V(T)K(X)$ with $X = g^{AB}(\partial_A T)(\partial_B T)$

$$\dot{\bar{\epsilon}} + \dot{\sigma}(\bar{\epsilon} - \bar{T}^4{}_4) + \dot{\Phi}\bar{L} + 8H(\bar{\epsilon} - \bar{L}) = 0. \tag{4.40e}$$

Let us now calculate the integrals of the energy momentum tensor averaged over the orbifolded dimension for the error function action (4.5). For the energy density $\bar{\epsilon}$ we use (4.8a) and assume that the tachyon $\tilde{T} = \sqrt{2}T(t)\sin\frac{2\pi x^4}{L_4}$ is in the lowest mode allowed after the orbifolding. We obtain

$$\begin{aligned}\bar{\epsilon} = \bar{T}^0{}_0 &= \frac{2}{L_4}\int_0^{\frac{L_4}{2}} \mathrm{d}x^4\,\epsilon \\ &= \mathrm{e}^{-\Phi}\mathrm{e}^{-\frac{T^2\mathrm{e}^{-2\sigma}}{2\alpha'}}F_1(z),\end{aligned}$$

where we used the modified BESSEL function as in (4.30), the shorthand (4.33) and introduced the variable

$$z = \frac{1}{4\alpha'}\left(T^2 - T^2 \mathrm{e}^{-2\sigma} - \dot{T}^2 2\alpha'\right).$$

For the LAGRANGE $\bar{L}$ we use the power series expansion of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!},$$

to obtain the expansion

$$\begin{aligned}
\mathrm{e}^{\Phi}\bar{L} &= -\frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n-1)} \int_0^{\pi} \mathrm{d}\xi \, \mathrm{e}^{-\frac{T^2}{2\alpha'}\sin^2\xi} \left(\frac{T^2 \mathrm{e}^{-2\sigma}}{2\alpha'} - \frac{T^2 \mathrm{e}^{-2\sigma} + \dot{T}^2 2\alpha'}{2\alpha'} \sin^2\xi \right)^n \\
&= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n-1)} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{T^2 \mathrm{e}^{-2\sigma}}{2\alpha'} \right)^{n-l} \left(\frac{T^2 \mathrm{e}^{-2\sigma} + \dot{T}^2 2\alpha'}{4\alpha'} \right)^l F_1^{(l)}(y),
\end{aligned}$$

where $y = \frac{T^2}{4\alpha'}$. As in the case of the D7-brane orbifold on page 73 we used that within the action integral we can obtain the l-th power of $\sin^2\xi$ from the l-th derivative of $F_1(y)$. In the case of large $\dot{T}^2$, i. e. $\dot{T}^2 \gg T^2$ and $\dot{T}^2 \gg T^2\mathrm{e}^{-2\sigma}$, the internal sum is dominated by the term for which $n = l$ and we approximately get

$$\mathrm{e}^{\Phi}\bar{L} \approx -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n-1)} \left(\frac{\dot{T}^2}{2} \right)^n F_1^{(n)}(y).$$

In the other extremal case with $\dot{T}^2 \ll T^2 \mathrm{e}^{-2\sigma}$ we get

$$\mathrm{e}^{\Phi}\bar{L} \approx -\frac{1}{2} \mathrm{e}^{-\frac{T^2}{2\alpha'}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n-1)} \left(\frac{T^2 \mathrm{e}^{-2\sigma}}{2} \right)^n F_1^{(n)}(y).$$

For the pressure component in the orbifolded direction $\bar{T}^4{}_4$ we apply equation (4.3) on the LAGRANGE $\bar{L}$ from the paragraph above and obtain

$$\begin{aligned}
\mathrm{e}^{\Phi}\bar{T}^4{}_4 &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{T^2 \mathrm{e}^{-2\sigma}}{2\alpha'} \right)^{n-l} \left(\frac{T^2 \mathrm{e}^{-2\sigma} + \dot{T}^2 2\alpha'}{4\alpha'} \right)^l F_1^{(l)}(y) \\
&\quad - \frac{\dot{T}^2}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n-1)} \sum_{l=0}^{n} \binom{n}{l} l \left(\frac{T^2 \mathrm{e}^{-2\sigma}}{2\alpha'} \right)^{n-l} \left(\frac{T^2 \mathrm{e}^{-2\sigma} + \dot{T}^2 2\alpha'}{4\alpha'} \right)^l F_1^{(l)}(y) \\
&= \mathrm{e}^{\Phi}\bar{\epsilon} - \frac{\dot{T}^2}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n-1)} \sum_{l=0}^{n} \binom{n}{l} l \left(\frac{T^2 \mathrm{e}^{-2\sigma}}{2\alpha'} \right)^{n-l} \left(\frac{T^2 \mathrm{e}^{-2\sigma} + \dot{T}^2 2\alpha'}{4\alpha'} \right)^l F_1^{(l)}(y).
\end{aligned}$$

4.4.1. Asymptotic solutions

In analogy with section 4.2.1 we can look for solutions, where the dilaton dominates the equations, i. e. $\dot{\Phi} \gg H$ and $\dot{\Phi} \gg \dot{\sigma}$. A linear combination of equations (4.40b) and (4.40d) gives in this limit

$$\ddot{\Phi} = \dot{\Phi}^2,$$

which is again solved by a logarithmic dilaton $\Phi = -\log(t - t_0)$. From equations (4.40a), (4.40b), and (4.40d) we can determine the equation of state parameter necessary for this solution to be $w = 0$. For the errorfunction Lagrangian this appears to be possible only if $T \gg 1$ and $\dot{T} \gg 1$ and thus $\bar{T}^4{}_4 = 0$. In this case, another linear combination of (4.40b) and (4.40d) gives

$$\dot{H} - 2\dot{\Phi}H + 8H^2 = 0,$$

which is asymptotically solved by $H = \frac{h}{(t-t_0)^2}$. Furthermore (4.40b) and (4.40c) imply

$$\ddot{\sigma} - 2\dot{\sigma}\dot{\Phi} - \dot{H} + 2H\dot{\Phi} \approx 0$$

and thus

$$\ddot{\sigma} + \frac{2\dot{\sigma}}{t_0 - t} \approx 0,$$

which is solved by $\sigma = \sigma_0 + \frac{\sigma_1}{t^2}$. For the tachyon T we obtain the asymptotic solution consistent with $w = 0$ for $t \to \infty$ to be

$$T \approx \mathrm{e}^{-t} + \left(\frac{t}{4} + \frac{1}{4}\log(t_0 - t)\right)\mathrm{e}^{t}.$$

4.4.2. Discussion

Allowing for arbitrary values of the tachyon T is necessary in order to understand the dynamics of the non-BPS brane system. Unfortunately the resulting system of differential equations contains rather complicated expressions for energy and pressure of the tachyon field. These contain double sums involving modified BESSEL functions, which cannot be solved efficiently during the numerical integration of the differential equations. Thus we were unable to obtain numerical solutions.

5. Conclusion

In this thesis we were concerned with the singularity problem of the general theory of relativity. After a short review of EINSTEIN's well established general relativity, and string theory as the best candidate for a fundamental theory we discussed two approaches on the singularity topic. On the one hand we obtained an educated guess for a non-singular theory in a bottom-up strategy. On the other hand we took the top-down perspective starting from string theory given as fundamental description of nature and analysed the resulting low energy effective theory.

In chapter 3 we applied the limiting curvature hypothesis on anisotropic cosmologies. By constructing a theory in which two carefully chosen curvature invariants are bounded, we obtain a modified theory of gravity. Analytic and numerical analysis suggest that in this theory all homogeneous and anisotropic spacetimes are non-singular. Singularities with small anisotropy are resolved and replaced by a DE SITTER phase which is completely analogous to the solutions obtained by BRANDENBERGER et al for isotropic spacetimes. There is circumstantial evidence for the existence of a global solution in the generic case, resolving the singularity with an approximately flat MINKOWSKI phase. This solution would interpolate between a contracting anisotropic universe and a universe that time-symmetrically expands anisotropically. While we are not able to proof the existence of such a solution globally at present, we think that a better understanding of this solution, if it exists, could have important applications in the resolution of cosmological singularities as an alternative to the so-called bounce solutions as they appear in pre-big-bang scenarios.

The application of the same mechanism on the space-like singularity inside a SCHWARZSCHILD black hole suggests that the surrounding of the singularity is replaced by a flat region connecting the black hole with a corresponding white hole solution in a causally disconnected spacetime region. Henceforth the global spacetime structure of the non-singular black hole is expected to be qualitatively comparable to the KERR solution's with its infinite series of black and white holes. We do not find evidence for the alternative scenario proposed in [7] where the black hole singularity is replaced by a region which is approximately DE SITTER spacetime and thus one would expect a global structure of one baby universe inside the black hole.

In chapter 4 we approached the singularity problem from the other side. Starting from type IIA super string theory we constructed a simple cosmological model. The full model consists of a non-BPS D-brane which is spacetime filling and wraps the remaining dimensions. The internal dimensions are compactified on a torus which is Z_2 orbifolded. This construction results in an interesting interplay between the size of the internal space given by some modulus field and the potential of the brane modes. We use a truncated action which was constructed in order to approximate the full string theory result for the dynamical creation and decay of non-BPS branes quite accurately. Thus we do not expect higher derivative corrections that could spoil our solutions.

Taking the lowest order effective action for metric, dilaton and an effective action for the open tachyonic mode, we obtained solutions with a bounce of the scale factor in the string frame. In EINSTEIN frame these solutions are not bouncing but expanding or contracting for all times. The bounce results from the positivity of the pressure of the tachyon field in our Lagrangian. Both curvature and time derivative of the dilaton remain small during our bounce so that the gravitational sector behaves entirely classical. Unfortunately there remain singularities in the curvature and the dilaton before or after the bounce. These asymptotic string frame curvature singularities can be resolved by the ad hoc addition of a potential, proportional to $R\mathrm{e}^{-\Phi}$. Such a term might result from α' corrections in the open string sector. Exact calculation of the corrections would be necessary in order to give a more precise picture of the effects resulting from the corrections. With our choice for the sign of the prefactor the gravitational coupling changes sign in the string frame at some time after or before the bounce. After transformation to the EINSTEIN frame, this turns into a bounce without violating the null energy condition.

While our phenomenological potential clearly stabilises the dilaton within the perturbative regime without the tachyon, the numerical analysis hints, that this is no longer the case once the tachyonic sector is included. The obvious question is then, whether a modified potential exists which stabilises the dilaton in our model. Furthermore it would be very interesting to see whether the string theory α' corrections result in such a modified potential.

We are still far from a solution of the singularity problem but both approaches give promising results that deserve further examination. It is quite reassuring that rather simple concepts can be used to investigate this issue. In this way it might be possible to build a solution step by step from small, well-understood pieces. Furthermore we feel that complementary approaches to the problem from different perspectives can be very fruitful.

A. A Toy model of brane tachyon dynamics

Let us look at a simplified toy model for the dynamics of the tachyon from a space filling brane wrapped on several extra dimensions that are toroidally compactified. We do not consider the back reaction of the fields on the geometry and assume a flat MINKOWSKI background metric $g_{\alpha\beta} = \eta_{\alpha\beta}$. Furthermore we assume a constant dilaton. The $(7+1)$-dimensional D-7-brane shall be spacetime filling and the remaining four dimensions shall wrap a torus with radii e^{σ_i}. This toy model shall only include the tachyon fields T_i resulting from the instability of this non-BPS-brane. For simplicity the last two spacelike dimensions shall be compactified on a 2-torus of fixed size and can thus essentially be ignored. Due to the KALUZA KLEIN mechanism the effective mass of each tachyon T_i is given by the corresponding modulus field

$$2\alpha' m_i^2 = -1 + \mathrm{e}^{-2\sigma_i}. \tag{A.1}$$

We simplify the setup further by the assumption of isotropy in the torus, i. e. $\sigma_i = \sigma$ for all i, and that the tachyon modes in the four internal directions are equal $T_i = T$. Our action will now contain kinetic terms for modulus and tachyon field and a mass term for the tachyon field with the modulus dependent mass:

$$S = \int \left(\mathrm{e}^{a\sigma} (\partial\sigma)^2 + g\mathrm{e}^{4\sigma} \left(-(\partial T)^2 - m^2 T^2 \right) \right) \mathrm{d}x^4. \tag{A.2}$$

Here we introduced a prefactor $\mathrm{e}^{a\sigma}$ for the modulus' kinetic term.

Variation of (A.2) w. r. t. σ and T yields the equations of motion

$$\ddot{T} + 4\dot{T}\dot{\sigma} + (\mathrm{e}^{-2\sigma} - 1)T = 0 \tag{A.3a}$$

$$\ddot{\sigma} + \frac{a}{2}\dot{\sigma}^2 - g\mathrm{e}^{(4-a)\sigma} \left(2\dot{T}^2 + (2 - \mathrm{e}^{-2\sigma})T^2 \right) = 0. \tag{A.3b}$$

We note that, by a rescaling of T with $\frac{1}{\sqrt{g}}$, we could remove the coupling from the equations.

For vanishing coupling g, we can immediately give the general solution of (A.3b)

$$\sigma = \sigma_0 + \frac{2}{a} \ln t. \tag{A.4}$$

The remaining equation (A.3a) for T is non-linear and contains the $\frac{2}{a}$-th power of t, which is integer only if $a \in \{1, 2\}$. For $a = 1$ one finds using MATHEMATICA an analytical solution in terms of hypergeometric functions but it is complex valued for all choices of the parameters. We thus introduce $\Sigma = \mathrm{e}^{\frac{a}{2}\sigma}$ and specialise to the case $a = 2$ in the following for reasons of convenience.

A.1. Equations of Motion

The equations of motion for Σ and T are

$$\ddot{T} = -4\frac{\dot{\Sigma}}{\Sigma}\dot{T} - \left(\Sigma^{-2} - 1\right) T \tag{A.5a}$$

$$\ddot{\Sigma} = g\Sigma\left(2\dot{T}^2\Sigma^2 + \left(2\Sigma^2 - 1\right) T^2\right) \tag{A.5b}$$

We remark that any physically viable solution is bound to $\Sigma > 0$ since it was defined as the exponential of the real field σ.

If $\Sigma > \frac{1}{\sqrt{2}}$ or T is small compared to $\dot{T}$ we immediately see from equation (A.5b) that Σ is a concave function. For large Σ equation (A.5a) becomes a harmonic oscillator equation with friction term $4\frac{\dot{\Sigma}}{\Sigma}$.

A.1.1. Asymptotic Solutions for $t \to 0$

We start to analyse the possible asymptotic solutions of (A.5) for small times t. Inspired by the solution with vanishing coupling (A.4) we seek solutions with a power law behaviour

$$\Sigma(t) = bt^\gamma + \mathcal{O}\left(t^{\gamma+\epsilon}\right), \tag{A.6}$$

where b and γ are arbitrary real constants and $\epsilon > 0$. Plugging this ansatz into equations (A.5) leaves in leading order:

$$\ddot{T} = -4\frac{\gamma}{t}\dot{T} - \left(\frac{t^{-2\gamma}}{b^2} - 1\right) T \tag{A.7a}$$

$$\frac{\gamma(\gamma-1)}{gb^2} = \dot{T}^2 t^{2\gamma+2} + T^2 2t^{2\gamma+2} - T^2 t^2 \tag{A.7b}$$

Depending on the value of γ there are different possibilities as solutions for T.

$\boxed{1 < \gamma}$ The reparametrisations $\bar{T} \coloneqq t^\lambda T$ and $\tau \coloneqq \left(\frac{k}{t}\right)^\delta$ gives (denoting derivatives with respect to τ by prime)

$$\begin{aligned} 0 = {} & \bar{T}'' + \frac{1}{\tau}\bar{T}'\left(\frac{1-4\gamma+\delta+2\lambda}{\delta}\right) \\ & + \bar{T}\left(\frac{1}{\tau^2}\left(\frac{\lambda - 4\gamma\lambda + \lambda^2}{\delta^2}\right) + \tau^{-\frac{2(1+\delta-\gamma)}{\delta}}\frac{k^{2(1-\gamma)}}{b^2\delta^2} - \frac{k^2}{\delta^2}\tau^{-2\left(1+\frac{1}{\delta}\right)}\right). \end{aligned} \tag{A.8}$$

Choosing $\lambda = 2\gamma - \frac{1}{2}$, $\delta = \gamma - 1$, and $k = (b(\gamma-1))^{-\frac{1}{\gamma-1}}$ we are left with essentially the modified BESSEL differential equation with $n = \frac{4\gamma-1}{2(\gamma-1)}$, where the last term is a small correction negligible against τ^2 for large τ. We thus get the asymptotical solutions for $t \to 0$

$$T = t^{-\lambda}\left(C_1 I_n\left(kt^{-\delta}\right) + C_2 K_n\left(kt^{-\delta}\right)\right). \tag{A.9}$$

The solutions (A.9) have the asymptotical behaviour of

$$T \sim \mathrm{e}^{\pm t^{1-\gamma}} t^{-\frac{3}{2}\gamma}, \tag{A.10}$$

where $+$ and $-$ refer to the I_n and K_n branch. The non-singular and the singular branch are incompatible to the second differential equation (A.7b) as the leading order term would require $\gamma = 1$ in both cases.

$\boxed{\gamma = 1}$ A similar construction as for $\gamma > 1$ with $\bar{T} := t^{\frac{3}{2}} T$ yields the exact modified BESSEL equation as long as $b \geq \frac{2}{3}$

$$t^2\ddot{\bar{T}} + t\dot{\bar{T}} - \left(t^2 + \frac{9}{4} - \frac{1}{b^2}\right)\bar{T} = 0. \tag{A.11}$$

Hence the general solution reads ($n = \sqrt{\frac{9}{4} - \frac{1}{b^2}}$)

$$T = t^{-\frac{3}{2}}\left(C_1 I_n(t) + C_2 K_n(t)\right), \tag{A.12}$$

with the asymptotical behaviour of $T \sim t^{\beta}$ with $\beta = -\frac{3}{2} \pm n$. This solution is compatible with the second equation

$$0 = 2b^2 t^{2\beta+3} + \left(\beta^2 b^2 - 1\right) t^{2\beta+1}, \tag{A.13}$$

if $2\beta + 3 \geq 0$, hence $\beta = -\frac{3}{2} + n$ and either $b = \pm\frac{1}{\beta}$ or $2\beta + 1 \geq 0$. While the first does not lead to a solution, the latter gives $b > \frac{2}{\sqrt{5}}$ a class of solutions.

$\boxed{0 < \gamma < 1}$ Equations (A.7) with the ansatz $T = c + t^{\alpha}$ are given by

$$\left(\alpha(\alpha-1) + 4\alpha\gamma\right) t^{\alpha-2} + \frac{c}{b^2} t^{-2\gamma} - c + t^{\alpha-2\gamma} - t^{\alpha} = 0 \tag{A.14a}$$

$$\begin{aligned} -\gamma(\gamma-1)t^{-2} - c^2 g - 2cgt^{\alpha} - gt^{2\alpha} + \alpha^2 b^2 g t^{2\alpha+2\gamma-2} \\ +2b^2c^2gt^{2\gamma} + 4b^2cgt^{\alpha+2\gamma} + 2b^2gt^{2\alpha+2\gamma} = 0 \end{aligned} \tag{A.14b}$$

$\alpha < 2(1-\gamma)$ In this case $\alpha - 2 < -2\gamma$ and hence the leading order is $t^{\alpha-2}$. The coefficient for this term vanishes for $\alpha = 1 - 4\gamma$. The only possibility in this range of γ for the most negative terms to cancel is $\gamma = \frac{1}{3}$ and thus $\alpha = -\frac{1}{3}$. Actual cancellation would require $gb^2 = -2$, which is impossible.

$\alpha = 2(1-\gamma)$ In this case is no way to cancel the t^{-2} pole.

$\alpha > 2(1-\gamma)$ We have to put $c = 0$, which leaves us with the two possibilities $\alpha = -1$ and $\alpha = -\gamma$ both outside this range of parameters.

$\boxed{\gamma = 0}$ i. e. $\Sigma = \Sigma_0 + Bt^\beta$ with $\beta > 0$ and $\Sigma_0 \neq 0$. Together with the ansatz $T = t^\alpha + Dt^\delta$ with $\alpha < \delta$ this gives in leading order of equation (A.5a)

$$\alpha(\alpha-1)t\alpha - 2 = -4\frac{4\alpha\beta B}{\Sigma_0}t^{\alpha+\beta-2} - \frac{1-\Sigma_0^2}{\Sigma_0^2}t\alpha \tag{A.15}$$

Since no term on the right hand side can compensate the left hand side, we have to require $\alpha \in \{0,1\}$. Equation (A.5b) reads

$$\frac{\beta(\beta-1)B}{g}t^{\beta-2} = \alpha^2\Sigma_0^3 t2\alpha - 2 + \left(\Sigma_0^3 - \Sigma_0\right)t^{2\alpha} \tag{A.16}$$

$\alpha = 1$ would require $\Sigma_0 = 0$ and is thus excluded but for $\alpha = 0$ we can for any $\beta > 0$ give $\Sigma_0 \neq 0$ such that the leading order coefficient vanishes.

$\boxed{\gamma < 0}$ The power law ansatz $T = c + dt^\alpha$ in (A.14b) shows, that cancellation is only possible for $\alpha > 1$ and $\gamma = -1$ or $0 < \alpha < 1$ and $\gamma = -\alpha$. Checking the leading order in (A.14a), we are left with the possibilities $\{\gamma = -1, \alpha = 2, c = -6d\}$ and $\{\gamma = -1, \alpha = 5, c = 0\}$.

Let us summarise the classes of solutions for small t. First there exists a class where both Σ and T are regular for small t. Second there is a class of asymptotical solutions where $\Sigma \sim t$ is linear in t and the tachyon $T \sim t^\beta$ behaves like a power law with some negative possibly fractional power $\beta \in (-\frac{3}{2}, 0)$. In the third class of asymptotical solutions for small times t the tachyon T is finite and $\Sigma \sim \frac{1}{t}$ has a pole.

A.1.2. Asymptotic solutions for $t \to \infty$

An exponentially growing ansatz for the tachyon

$$T \propto e^{\alpha t} \qquad \text{with } \alpha > 0 \tag{A.17}$$

or a divergent power law ansatz

$$T \propto t^\alpha \qquad \text{with } \alpha > 0 \tag{A.18}$$

plugged into equation (A.5a) yields an analytically solvable equation for Σ. Since this solution for Σ does not solve (A.5b), we can conclude for $t \to \infty$ the tachyon can neither grow exponentially nor with a power law.

Assuming asymptotically constant T, we immediately get $T = 0$. The other possibility $\Sigma = 1$ from equation (A.5a) results in $T = 0$ from (A.5b) as well. Since

$\Sigma = \text{const.}$ cannot be fulfilled with non-constant T, we are left with the two possibilities $T = \Sigma = 0$ and $T = 0, \Sigma \propto t$.

Therefore a natural ansatz for the asymptotical behaviour at large times is an exponentially decaying tachyon

$$T \propto \mathrm{e}^{-\alpha t} \qquad \text{with } \alpha > 0. \tag{A.19}$$

The equations (A.5a) and (A.5b) can now be solved in leading order by $\Sigma \propto t$.

A.1.3. Putting solutions together

We have to glue the asymptotical solution for large t where $\Sigma \sim t$ and $T \sim \mathrm{e}^{-t}$ to one of the two asymptotical solutions for small t. A solution which is singular in T lies outside the applicability of our discussion, since we would have to include higher order terms (i. e. a full DBI-action).

Starting at $t \gg 1$ with $\Sigma \sim t$ and $T \sim \mathrm{e}^{-t}$, we observe that

$$\ddot{\Sigma} = g\Sigma\left(\dot{T}^2\Sigma^2 + (2\Sigma^2 - 1)T^2\right) > 0 \tag{A.20}$$

$$\ddot{T} = -4\frac{\dot{\Sigma}}{\Sigma}\dot{T} - \left(\frac{1}{\Sigma^2} - 1\right)T > 0 \tag{A.21}$$

at least as long as we choose t large enough to have $\Sigma \geq \frac{1}{\sqrt{2}}$. To connect this to the asymptotical solution for small t which is singular in T, Σ has to turn around, i. e. $\ddot{\Sigma} < 0$. Thus at some point t_1 we have

$$\left(\frac{\dot{T}}{T}\right)^2 = \frac{1 - 2\Sigma^2}{\Sigma^2} \quad \text{at } t_1. \tag{A.22}$$

This requires $\Sigma < \frac{1}{\sqrt{2}}$ at t_1. Solving (A.22) for $\dot{T}$ we see that at t_1

$$\ddot{T} = -4\frac{T}{\Sigma^2}\left(\pm\sqrt{1 - 2\Sigma^2}|\dot{\Sigma}| + \left(1 - \Sigma^2\right)\right). \tag{A.23}$$

Where the ambiguity arises both from the square root in $\dot{T}$ and the sign of $\dot{\Sigma}$ which is up to now unknown. In the case of the $+$-sign in equation (A.23), $\ddot{T}$ is negative at t_1 and thus has to change sign again at some $t_2 < t_1$ in order to fit the asymptotical solution $T \sim t^{-\beta}$. But $\ddot{T}(t_2) = 0$ is impossible since $\Sigma < \frac{1}{\sqrt{2}} < 1$.

A.1.4. Numerical Solutions

Numerical integration of equations (A.5a) and (A.5b) starting at large times t and with initial data according to the just found asymptotical behaviour $T \propto \mathrm{e}^{-\alpha t}$ and

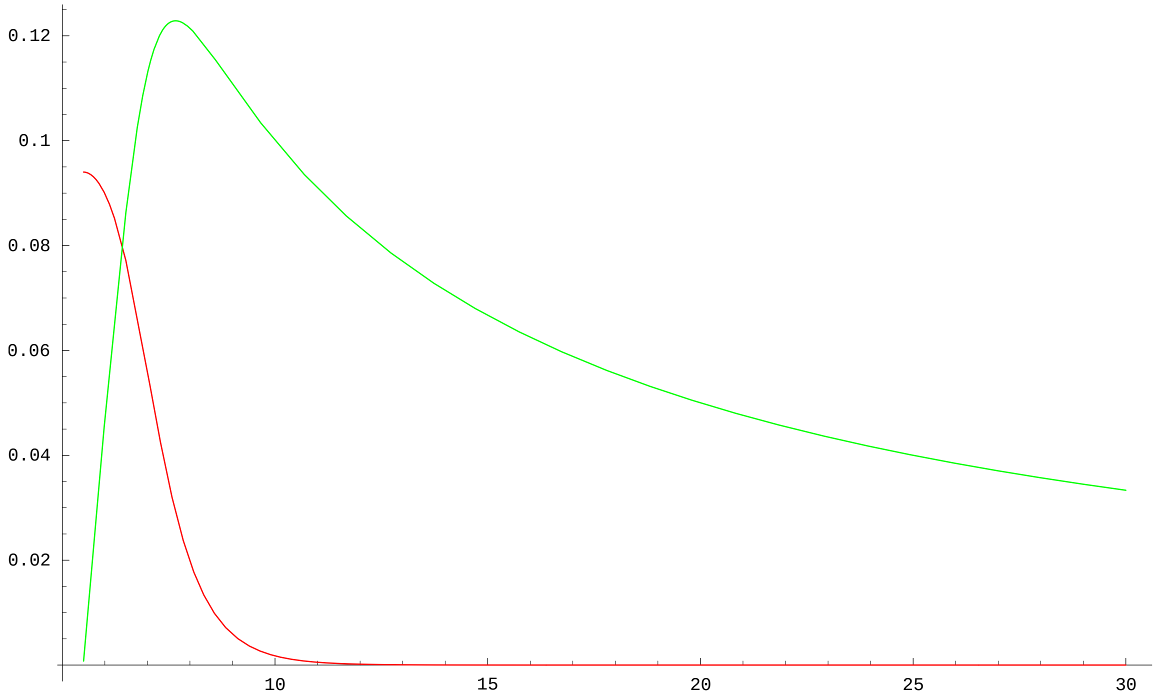


Figure A.1.: Numerical solution, starting at large t. The plot shows Σ^{-1} (green) and T (red).

$\Sigma \propto t$ results in a pole of Σ at small t coinciding with a stationary point of T. Indeed, the ansatz $T = \text{const.}$ plugged into (A.5b) gives

$$\ddot{\Sigma} = gT^2 \left(2\Sigma^3 - \Sigma\right). \tag{A.24}$$

In accordance with section A.1.1 equation (A.24) has a singular solution asymptotical to $\Sigma \propto \frac{1}{t}$ and two regular solutions. Hence in any linear combination a non-trivial part of the $\frac{1}{t}$-pole will dominate for small t.

A typical numerical solution is plotted in figure A.1. Here we integrated starting from some large t_1 and initial conditions corresponding to $T = \mathrm{e}^{-t}$ and $\Sigma = t$. T grows exponentially to smaller times and Σ falls off linearly. When finally Σ turns around and goes over into the $\frac{1}{t}$-pole T also reaches a turning point and ends in a maximal value at the time, when Σ is singular. Adjusting the initial values $\Sigma(t_1)$ and/or $\dot{\Sigma}(t_1)$ in a suitable way any positive value of T at the singular point at $t_0 \gg t_1$ can be achieved. Generally the larger we choose $\Sigma(t_1)$ and the smaller we choose $\dot{\Sigma}(t_1)$, the smaller $T(t_0)$ will be – but a negative $\dot{\Sigma}$ will result in $\Sigma = 0$ for some even larger $t > t_1$ and thus in a singularity of T (see figure A.2 for an example).

Starting integration at small t with finite values for Σ and T, we reach a similar singularity at some larger t (see e. g. figure A.3). Depending on the initial value of Σ we might observe an oscillatory behaviour in T (see e. g. figure A.4), since for small $\Sigma < 1$ T is not tachyonic but an ordinary massive particle with a standard

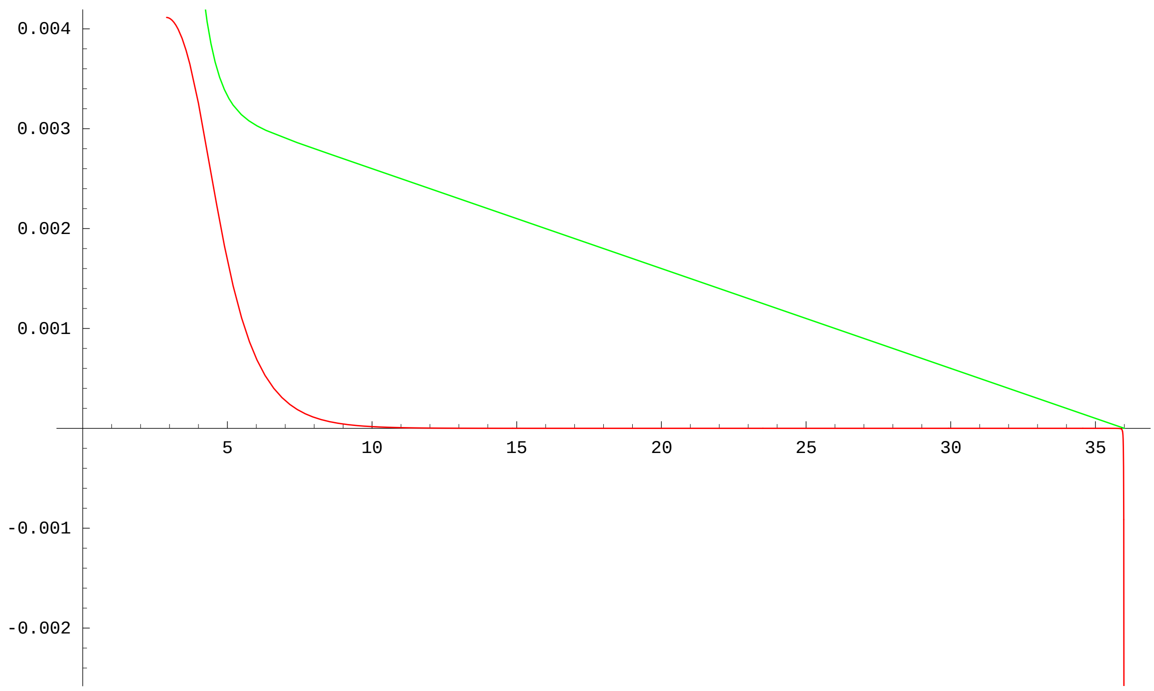


Figure A.2.: Numerical solution, starting at large t with negative $\dot{\Sigma}$ integrated in both directions. The plot shows Σ (downscaled, green) and T (red).

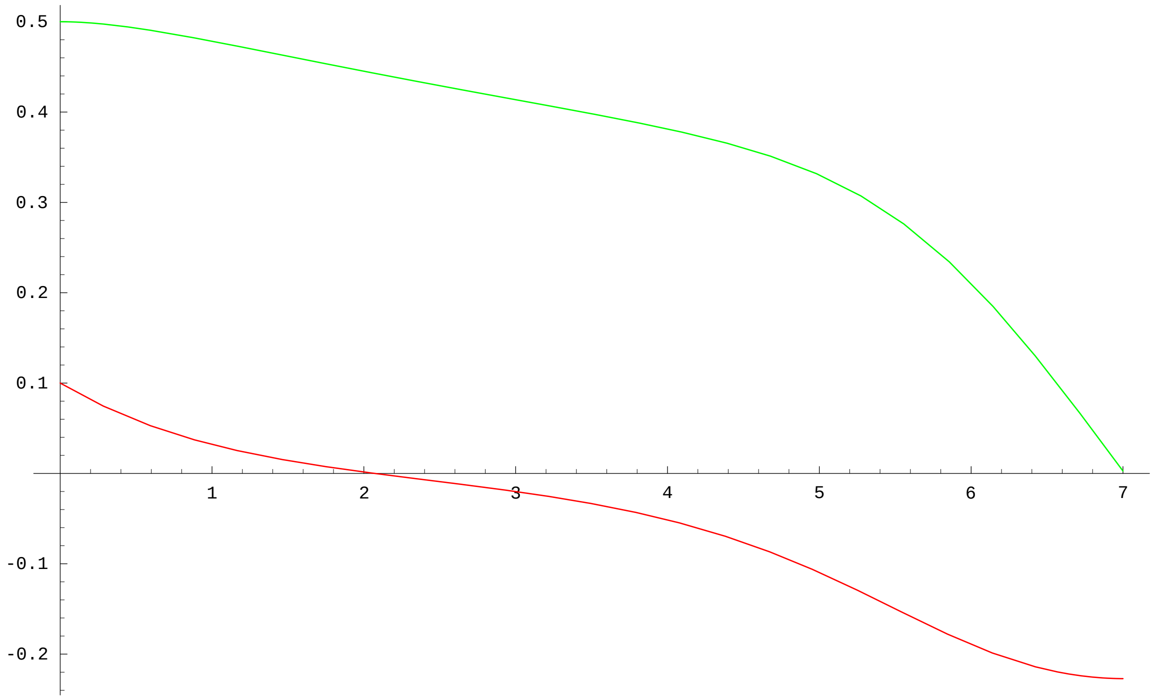


Figure A.3.: Numerical solution, starting at small t. The plot shows Σ^{-1} (green) and T (red).

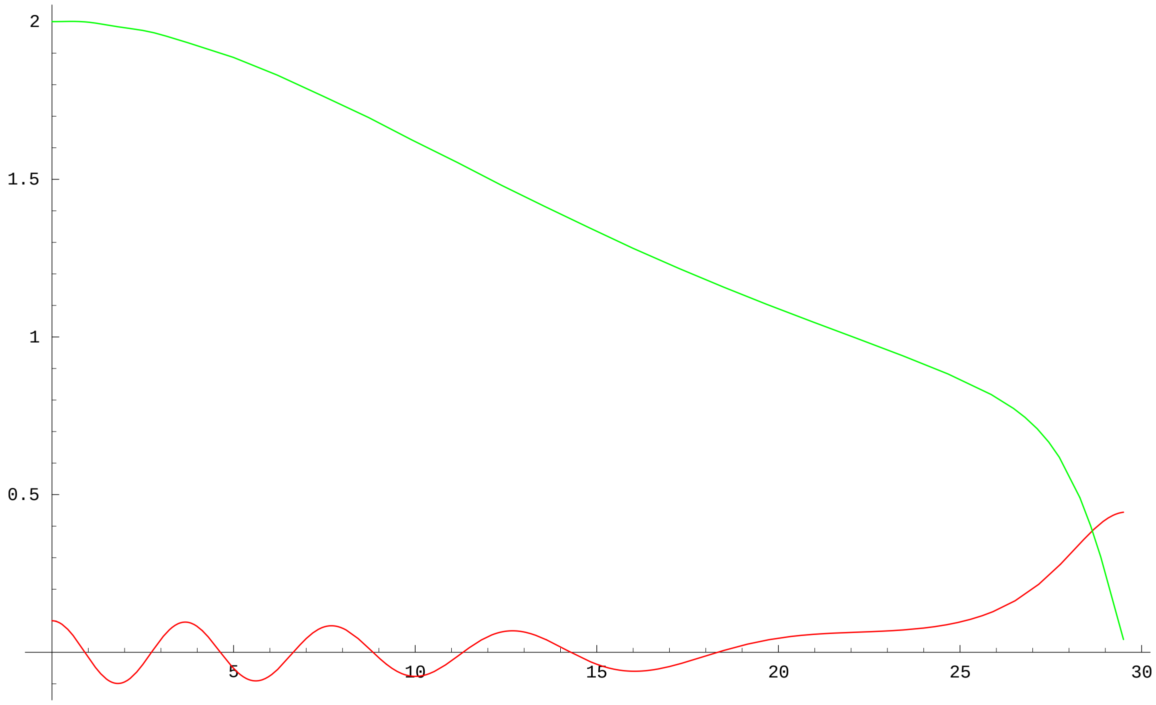


Figure A.4.: Numerical solution, starting at small t. The plot shows Σ^{-1} (green) and T (red).

wave equation.

One might hope to be able to express these two solutions in terms of a non-singular variable e. g. $\psi := \frac{1}{\Sigma}$ and then glue them together at the singular point. But this does not seem viable from a physical point of view (and it seems impossible anyway). Infinite Σ corresponds to infinite σ and thus a decompactified torus. Physically there is no sense in analytical continuation of a solution over a point in space infinitely far away in finite time.

A.2. Including Gravity

So far we ignored the backreaction of the fields onto the curvature of spacetime. To cure this we refine our ansatz such that the metric of the ordinary $(3+1)$-dimensional spacetime is given by a FRIEDMANN ROBERTSON WALKER cosmology with scale factor $a(t)$ and HUBBLE parameter $H(t)$. The equations of motion now read

$$\ddot{T} = -3H\dot{T} - 4\frac{\dot{\Sigma}}{\Sigma}\dot{T} - \left(\Sigma^{-2} - 1\right)T, \tag{A.25a}$$

$$\ddot{\Sigma} = -3H\dot{\Sigma} + g\Sigma\left(2\dot{T}^2\Sigma^2 + \left(2\Sigma^2 - 1\right)T^2\right), \tag{A.25b}$$

$$4\dot{H} = -6H^2 + \dot{\Sigma}^2 + g\Sigma^2\left(\Sigma^2\dot{T}^2 + T^2\left(\Sigma^2 - 1\right)\right), \tag{A.25c}$$

$$0 = 6\frac{H^2}{\Sigma^2} + \frac{\dot{\Sigma}^2}{\Sigma^2} + g\left(\Sigma^2\dot{T}^2 + T^2\left(\Sigma^2 - 1\right)\right). \tag{A.25d}$$

An exponentially decaying ansatz for the tachyon field $T \propto \mathrm{e}^{-t}$ at large t now induces an asymptotical solution, where the modulus $\Sigma \propto \frac{1}{t}$ tends to zero and the curvature $H = \frac{2}{3t}$ follows the evolution of a flat dust-filled universe. Unfortunately this solution is incompatible with the constraint equation (i. e. the 00-component of the field equations).

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