Subtle and Ineffable Tree Properties

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In memoriam

Friedrich Roesler

1945 – 2010

Zusammenfassung

In Anlehnung an die Baumeigenschaft geben wir kombinatorische Prinzipien an, die die Konzepte der sogenannten *subtle* und *ineffable* Kardinalzahlen so einfangen, dass diese auch für kleine Kardinalzahlen anwendbar sind. Auf diesen Prinzipien aufbauend entwickeln wir dann ein weiteres, das dies sogar für superkompakte Kardinalzahlen leistet.

Wir zeigen die Konsistenz dieser Prinzipien ausgehend von den jeweils entsprechenden großen Kardinalzahlen. Zudem zeigen wir die Äquikonsistenz für *subtle* und *ineffable*. Für Superkompaktheit beweisen wir durch das Fehlschlagen des Quadratprinzips, dass die besten derzeit bekannten unteren Schranken für Konsistenzstärke anwendbar sind.

Das Hauptresultat der Arbeit ist das Ergebnis, dass das Proper Forcing Axiom das der Superkompaktheit entsprechende Prinzip impliziert.

Summary

In the style of the tree property, we give combinatorial principles that capture the concepts of the so-called *subtle* and *ineffable* cardinals in such a way that they are also applicable to small cardinals. Building upon these principles we then develop a further one that even achieves this for supercompactness.

We show the consistency of these principles starting from the corresponding large cardinals. Furthermore we show the equiconsistency for *subtle* and *ineffable*. For supercompactness, utilizing the failure of square we prove that the best currently known lower bounds for consistency strength in general can be applied.

The main result of the thesis is the theorem that the Proper Forcing Axiom implies the principle corresponding to supercompactness.

Introduction

Foreword

In [Hau14, p. 131] Felix Hausdorff wrote that if there are inaccessible cardinals, “so ist die kleinste unter ihnen von einer so exorbitanten Größe, daß sie für die üblichen Zwecke der Mengenlehre kaum jemals in Betracht kommen wird.”[1] While he could not foresee the way set theory was heading and that large cardinals would once assume a central role in it, he is still right for most of other mathematics. This is not without reason, for the inaccessibility of a cardinal κ merely says that the set theoretic universe up to height κ already is a universe of set theory itself, so that all of usual mathematics fully lives within it.

So why should a mathematician not intrinsically interested in large cardinals care about them? The answer is that there are many questions about small cardinals like ω_2 that need large cardinals for their answer. But beyond that, for some large cardinal axioms there exist combinatorial principles that capture the crux of the axiom yet make sense even for small cardinals. They thus provide a framework for strong hypotheses without requiring any actual reference to large cardinals. Such canonical principles had previously only been known for the large cardinal properties Mahlo and weak compactness, giving the impression it was more an exception these large cardinals admitted such a definition.[2]

In the present work, we give such characterizations for subtle and ineffable cardinals and then extend them to supercompactness. Subtlety and ineffability are, by today’s standards, rather weak large cardinal concepts. They have also received relatively little attention lately, which we mainly attribute to the fact their definitions appeared to be conceptually different from other large cardinal axioms. We hope to counter this impression by demonstrating how ineffability is in many ways an evidently natural strengthening of weak compactness.

Supercompactness, on the other hand, is not only very strong, it is probably among the large cardinal axioms that currently receive most attention; one of the biggest open problems in set theory is to find an inner model for a supercompact cardinal. While it is not like the principle we isolated formed the puzzle stone that had been missing so far—such a stone is yet to be determined—it nonetheless is a new approach, and one might hope for it to be

[1] . . . , “then the least among them is of such exorbitant size that it will hardly ever come into consideration for the usual goals of set theory.”

[2] For weak compactness the corresponding canonical principle is the well known property that there exist no κ-Aronszajn trees. For Mahlo, it is the less well known principle that no *special* κ-Aronszajn trees exist, see [Tod87, (1.9)] or [Tod, Theorem 9.4].

of help by suggesting different strategies. Also, since we show the Proper Forcing Axiom PFA implies the supercompactness principle for ω_2, we can give some reassurance to the conjecture that the consistency strength of PFA is that of a supercompact cardinal and to the understanding that PFA makes ω_2 behave like a very large cardinal.

Overview

For a cardinal κ, let $\phi_0(\kappa)$ stand for the property that if $T \subset {}^{<\kappa}2$ is a downward closed[3] tree of height κ, then it has a cofinal branch. By König's Lemma, $\phi_0(\omega)$ holds. Thus it is not too surprising that $\phi_0(\omega)$ captures some of the large cardinal character of ω, and indeed $\phi_0(\kappa)$ is merely a slightly nonstandard way of saying κ is weakly compact for uncountable κ. So what is $\phi_0(\kappa)$ good for? Not much, in fact it one could argue it is a bad definition. For it *obscures* the fact that a cardinal κ is weakly compact iff[4] it is inaccessible and satisfies a combinatorial property κ-TP which is *not* restricted to inaccessibility. This κ-TP is, of course, the tree property on κ, or as it is more convenient for us, the property that any downward closed tree $T \subset {}^{<\kappa}2$ that is thin[5] has a cofinal branch.[6]

As Aronszajn trees exist in ZFC, the least uncountable cardinal κ for which κ-TP can hold is ω_2. By results of William Mitchell and Jack Silver [Mit73], ω_2-TP implies ω_2 is weakly compact in L and can be forced from a model of ZFC + "there exists a weakly compact cardinal." It therefore captures, in a very concrete way, the large cardinal character of ω_2 without requiring ω_2 to actually be inaccessible in V.

Now take κ to be subtle. Recall that a cardinal κ is subtle iff $\phi_1(\kappa)$ holds, where $\phi_1(\kappa)$ stands for the principle that for every club $C \subset \kappa$ and every sequence $\langle d_\alpha \mid \alpha < \kappa\rangle$ with $d_\alpha \subset \alpha$ for all $\alpha < \kappa$ there exist $\alpha, \beta \in C$ such that $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$. This $\phi_1(\kappa)$ suffers from the same problem as $\phi_0(\kappa)$ does, it already implies the inaccessibility of κ. However, having learned our lesson, we again require the sequence $\langle d_\alpha \mid \alpha < \kappa\rangle$ in the definition of $\phi_1(\kappa)$ to be thin, that is, for every $\delta < \kappa$ the set $\{d_\alpha \cap \delta \mid \alpha < \kappa\}$ is required to have cardinality less than κ. This yields a combinatorial principle κ-STP which relates to subtlety the same way κ-TP is related to weak compactness:

- A cardinal κ is subtle iff it is inaccessible and satisfies κ-STP.
- ω_2-STP can be forced from a model of ZFC + "there exists a subtle cardinal."
- ω_2-STP implies ω_2 is a subtle cardinal in L.

[3] By *downward closed* we mean that if $f \in T$ and $\alpha < \operatorname{dom} f$, then $f \upharpoonright \alpha \in T$. It feels more natural to restrict our attention to these trees, in particular in light of what we are aiming at.

[4] One can often read "iff" should be spelled out as "if and only if" in mathematical documents, the reason being it is not correct English and therefore bad style. Since it shortens the text, removes unnecessary redundancy, and thereby *aids* the reader, the author cannot help but to disagree with this point of view.

[5] We call a tree $T \subset {}^{<\kappa}2$ *thin* iff every of its levels has cardinality less than κ.

[6] See the next section as to why we consider this principle more natural.

It will not come as a surprise that the same can be done for ineffability. Still the reader will most likely not be entirely convinced of the importance of *thin*. We managed to replace the bad ϕ's by good TPs once more but still have not even left the realm of large cardinals compatible with $V = L$. Therefore it will hopefully be at least a little surprising that—naturally giving up the third point—the same can be done for supercompactness!

For $\lambda \geq \kappa$, κ is λ-ineffable iff for every sequence $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ such that $d_a \subset a$ for all $a \in \mathfrak{P}_\kappa\lambda$ there are a stationary $S \subset \mathfrak{P}_\kappa\lambda$ and $d \subset \lambda$ such that $d_a = d \cap a$ for all $a \in S$. We additionally require the sequence $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ to be thin in the sense that for a club $C \subset \mathfrak{P}_\kappa\lambda$ and every $c \in C$ we have $|\{d_a \cap c \mid c \subset a \in \mathfrak{P}_\kappa\lambda\}| < \kappa$ and call the resulting principle (κ, λ)-ITP. This yields κ is λ-ineffable iff κ is inaccessible and (κ, λ)-ITP holds. Menachem Magidor [Mag74] showed a cardinal κ is supercompact iff it is λ-ineffable for all $\lambda \geq \kappa$. Thus κ is supercompact iff it is inaccessible and (κ, λ)-ITP holds for all $\lambda \geq \kappa$. Since we can force (ω_2, λ)-ITP from a model of ZFC + "there exists a λ^{ω_1}-ineffable cardinal," we can produce a model in which ω_2 is, apart from the missing inaccessibility, "supercompact." This form of supercompactness implies the failure of a weak form of square, so that the best currently known lower bounds for consistency strength apply to it.

Let us now turn our attention to the forcing axiom PFA. As PFA implies ω_2-TP by work of James Baumgartner [Tod84b, Theorem 7.7], PFA preserves some large cardinal structure, the weak compactness, of ω_2. The word "preserve" is used as PFA is usually forced by collapsing a supercompact cardinal to ω_2 in a certain way, and one can consider PFA to "remember" that ω_2 used to be a very large cardinal. The existence of a supercompact cardinal is widely regarded as the correct consistency strength of PFA. We will provide what might be considered strong heuristic evidence that this is correct: We show PFA implies (ω_2, λ)-ITP for all $\lambda \geq \omega_2$. This, in the sense of the previous paragraph, says that PFA captures the "supercompactness" of ω_2.

The principle (κ, λ)-ITP might also prove useful for finding an inner model of a supercompact cardinal. While $\mathfrak{P}_\kappa\lambda$-combinatorics does not relativize as smoothly as ordinal combinatorics does, which is mainly due to the fact that being club is no longer absolute, the principle still seems to be a canonical candidate. One "only" needs to find an inner model that inherits (κ, λ)-ITP from V and in addition thinks κ is inaccessible.

In the course of developing a theory for κ-ITP and (κ, λ)-ITP, we introduce a weakening of *thin* called *slender* which comes from Saharon Shelah's approachability ideal. Its purpose is twofold. On the one hand, it pulls together different concepts like the tree property and the approachability property. On the other, it is the *correct* weakening of *thin* in the sense that most proofs that make use of *thin* actually use *slender*.

Simple Yet Powerful

At first glance, *thin* appears to be a standard concept. The tree property is well known, Aronszajn trees are one of the most prominent combinatorial objects in set theory. However, we shall try to argue *thin* has—quite surprisingly—been completely omitted from the literature during the last decades.

Historically, Nachman Aronszajn's result that ω_1-Aronszajn trees exist motivated the tree property, while weak compactness emerged as a generalization of the compactness theorem for first order predicate language. For the connection between these two, it was probably unfortunate that the definition of tree is of axiomatic nature. For despite the fact that for the various equivalences of weak compactness downward closed subtrees of ${}^{<\kappa}2$, let us call them *standard trees* for now, are the natural objects to consider, the focus remained on trees in general. However, unlike the set of standard trees, the class of trees of height κ also includes pathological trees with levels of size κ. These pathological cases misleadingly hide the fact that one can drop the thinness requirement by restricting oneself to standard trees. Without the understanding that *thin* is something that is automatically implied by inaccessibility but needs to be explicitly required for successor cardinals, it is less surprising that it was missed for subtlety and ineffability, properties that live on standard trees, and much more so for λ-ineffability.

So while there is a plausible argument for the omission of *thin*, it seems slightly astonishing that κ-STP has never been considered before. For a tree T of height κ, let us call an antichain $A \subset T$ a *club-antichain* iff there is a club $C \subset \kappa$ such that $A \cap T_\alpha \neq \emptyset$ for all $\alpha \in C$, where T_α denotes the αth level of T. We can thus rephrase κ-STP: If a κ-tree does not split at limit levels,[7] then it cannot have a club-antichain. Club-antichains have received some attention before, see [Tod84b, Remark 9.6 (iii)], but still κ-STP somehow remained unnoticed.

For one, *thin* enables us to see a much more natural connection between weak compactness, subtlety, and ineffability. But furthermore it gives rise to (κ, λ)-ITP for all $\lambda \geq \kappa$, apart from its strengthening (κ, λ)-ISP the only known combinatorial principle the author intuitively expects to have the consistency strength of a supercompact cardinal, even more so among the known consequences of PFA.

Annotation of Content

The thesis is divided into three chapters. Each chapter starts with a Preliminaries section which recalls definitions and standard lemmas and states nonstandard notation or technical lemmas.

Chapter 1 introduces the concepts fundamental to this work. Sections 1.2 and 1.3 define various ideals and what we will refer to as tree and forest properties.[8] The most important

[7] A tree T is said to *not split at limit levels* iff for every $t, t' \in T$ of limit height from $\{s \in T \mid s < t\} = \{s \in T \mid s < t'\}$ it follows that $t = t'$.

[8] The term *forest* was originally used by the author to address what should be called the downward closure of a $\mathfrak{P}_\kappa\lambda$-list in the present terminology, extending the meaning of *downward closure* to $\mathfrak{P}_\kappa\lambda$ in the obvious manner. The concept first appeared in [Jec73] and was called *binary mess* in there. However, the author felt

among these are probably the principles κ-ITP and (κ, λ)-ITP. Section 1.4 shows (κ, λ)-ITP implies the failure of a weak version of square on λ, which will be used in Section 2.4 to give lower bounds for the consistency strength of (κ, λ)-ITP. Section 1.5 gives results about the filter corresponding to (κ, λ)-ITP for inaccessible κ that will be needed in Chapter 2.

Chapter 2 deals with the consistency of the principles introduced in Chapter 1. In Section 2.2, a forcing construction originally developed by Mitchell is presented. It is then used in Section 2.3 to give upper bounds on the consistency strengths of our tree and forest properties. These are accompanied by the lower bounds established in Section 2.4.

Chapter 3 is mostly independent of Chapter 2. The main result of the thesis, that PFA implies ω_2-ITP (Section 3.2) and even (κ, λ)-ITP for all $\lambda \geq \omega_2$ (Section 3.3), is proved in it. The proof yields, as corollaries, several known implications of PFA.

Not every possible definition is fully exploited. One could consider principles κ-AITP and (κ, λ)-AITP by weakening "stationary" to "cofinal" in the definitions of κ-ITP and (κ, λ)-ITP respectively. These principle then naturally correspond to almost ineffability and λ-almost ineffability. In the one cardinal case, all the results generalize in a canonical way to κ-AITP. In the two cardinal case, for inaccessible κ the statements "(κ, λ)-AITP holds for all $\lambda \geq \kappa$" and "(κ, λ)-ITP holds for all $\lambda \geq \kappa$" are equivalent by [Car86], so that (κ, λ)-AITP appears to be rather fruitless.

A different aspect that has been neglected is (κ, λ)-TP, the generalization of the tree property to $\mathfrak{P}_\kappa\lambda$-combinatorics and thus a weakening of (κ, λ)-ITP: If $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ is a thin $\mathfrak{P}_\kappa\lambda$-list[9] and $E \subset \mathfrak{P}_\kappa\lambda$ is cofinal, then there exists $b \subset \lambda$ such that for every $a \in \mathfrak{P}_\kappa\lambda$ there exists $e \in E$ such that $a \subset e$ and $d_e \cap a = b \cap a$.[10] In [Jec73], this property is considered for inaccessible κ and shown to be the correct concept for strong compactness, that is, an inaccessible cardinal κ is strongly compact iff (κ, λ)-TP holds for all $\lambda \geq \kappa$. We omitted (κ, λ)-TP since we could have only derived corollaries for it from (κ, λ)-ITP.

As indicated earlier, throughout the thesis a weakening of *thin* called *slender* is being worked with. It proved to be a more natural replacement for *thin*. For the proof of ω_2-ITP and (ω_2, λ)-ITP for all $\lambda \geq \omega_2$ under PFA, a detour was taken. It did not, however, apply to the *slender* principles. Section 3.4 shows this detour was unnecessary and that *slender* can indeed fully replace *thin* for all our results.

It would have been possible to completely remove *thin* or at least special arguments concerning it. But as such a removal would have betrayed the results' history, we decided against any such omission. All results are thus developed for *thin* and *slender* in parallel. Still the reader is asked to bear in mind that some results are actually redundant, in particular what we will refer to as the *thin τ-approximation property* and its associated theorems.

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the word forest was more suggestive. While *forests* were removed from this thesis in an overall attempt to simplify the content by eliminating unnecessary definitions, the author thinks the name "forest properties" is still the catchiest way to describe the two cardinal combinatorial principles under consideration.

[9] See Definitions 1.3.1 and 1.3.2.

[10] This would be "every (κ, λ)-forest has a cofinal branch" in the abandoned terminology. It is called "every binary (κ, λ)-mess is solvable" for inaccessible κ in [Jec73].

Notation

As they say, the notation used is mostly standard. Ord denotes the class of all ordinals. For $A \subset \mathrm{Ord}$, $\operatorname{Lim} A$ denotes the class of limit points of A. Lim stands for Lim Ord. If a is a set of ordinals, $\operatorname{otp} a$ denotes the order type of a. For a regular cardinal δ, $\operatorname{cof} \delta$ denotes the class of all ordinals of cofinality δ, and $\operatorname{cof}(< \delta)$ denotes those of cofinality less than δ.

For forcings, we follow the *less is more, more or less* rule and write $p < q$ to mean p is stronger than q. A forcing $\mathbb{P}$ is *δ-closed* (*δ-directed closed*) iff it is closed under descending sequences (directed sets) of conditions of size less than δ. It is *δ-distributive* iff the intersection of fewer than δ many dense open sets is dense open. Names either carry a dot above them or are canonical names for elements of V, so that we can confuse sets in the ground model with their names. If $\mathbb{P}_\kappa$ is an iteration of forcings, then for $p \in \mathbb{P}_\kappa$ we let $\operatorname{supp} p$ denote the support of p.

The phrases *for large enough θ* and *for sufficiently large θ* will be used for saying that there exists a θ' such that the sentence's proposition holds for all $\theta \geq \theta'$.

The text requires knowledge of several standard definitions and notations of set theory. Chapter 2 expects the reader to be acquainted with iterated forcing, and for Chapter 3 it is helpful to be familiar with forcing axioms. To anyone meeting these conditions, the text should be readily accessible. In the inaccessible case, a standard textbook like [Jec03] should suffice for the necessary background theory. For further reading on large cardinals, [KM78] is recommended to the reader as a start.

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I wish to use this possibility to heartily thank my parents for their support and reassurance during all the time of my studies.

1 Tree and Forest Properties

1.1 Preliminaries

The following definition comprises most of what the reader needs to know about $\mathfrak{P}_\kappa\lambda$-combinatorics. It is possible that the reader is used to a slightly different definition of δ-closed. Apart from this, the only thing in Definition 1.1.1 which is not standard is the notation $\mathfrak{P}'_\kappa X$.

1.1.1 Definition. Let κ be a regular uncountable cardinal and X such that $\kappa \leq |X|$. Set

$$\mathfrak{P}_\kappa X := \{x \subset X \mid |x| < \kappa\}.$$

$C \subset \mathfrak{P}_\kappa X$ is called *δ-closed*, iff for any sequence $\langle x_\nu \mid \nu < \eta\rangle$ with $\eta < \delta$ such that $x_\nu \subset x_{\nu'}$ and $x_\nu \in C$ for all $\nu < \nu' < \eta$ it follows that $\sup_{\nu<\eta} x_\nu := \bigcup\{x_\nu \mid \nu < \eta\} \in C$. $C \subset \mathfrak{P}_\kappa X$ is called *closed* iff it is κ-closed. C is called *cofinal* iff for every $y \in \mathfrak{P}_\kappa X$ there exists $x \in C$ such that $y \subset x$. C is called *δ-club* iff C is δ-closed and cofinal. C is called *club* iff it is κ-club. $S \subset \mathfrak{P}_\kappa\lambda$ is called *stationary* iff $C \cap S \neq \emptyset$ for every club $C \subset \mathfrak{P}_\kappa\lambda$.

If $\kappa \subset X$, then

$$\mathfrak{P}'_\kappa X := \{x \in \mathfrak{P}_\kappa X \mid \kappa \cap x \in \mathrm{Ord},\ \langle x, \in\rangle \prec \langle X, \in\rangle\}$$

is club. For $x \in \mathfrak{P}'_\kappa X$ we set $\kappa_x := \kappa \cap x$.

For $f : \mathfrak{P}_\omega X \to \mathfrak{P}_\kappa X$ let $\mathrm{Cl}_f := \{x \in \mathfrak{P}_\kappa X \mid \forall z \in \mathfrak{P}_\omega x\ f(z) \subset x\}$. Cl_f is club, and it is well known that for any club $C \subset \mathfrak{P}_\kappa X$ there is an $f : \mathfrak{P}_\omega X \to \mathfrak{P}_\kappa X$ such that $\mathrm{Cl}_f \subset C$, see [Men75].

If $X \subset X'$, $R \subset \mathfrak{P}_\kappa X$, $U \subset \mathfrak{P}_\kappa X'$, then

$$U \restriction X := \{u \cap X \mid u \in U\} \subset \mathfrak{P}_\kappa X$$

and

$$R^{X'} := \{x' \in \mathfrak{P}_\kappa X' \mid x' \cap X \in R\} \subset \mathfrak{P}_\kappa X'.$$

It is easily seen that both operations preserve the property of containing a club and thus also stationarity. ⌟

1.2 Tree Properties

In this section, κ is assumed to be a regular uncountable cardinal. We first give three definitions that proved to be notationally useful but do not carry much weight beyond that.

1.2.1 Definition. A sequence $\langle d_\alpha \mid \alpha < \kappa\rangle$ is called *κ-list* iff $d_\alpha \subset \alpha$ for all $\alpha < \kappa$. ⌟

1.2.2 Definition. For $A \subset \{f \mid \exists\alpha < \mathrm{Ord}\ f : \alpha \to 2\}$ we call the tree of its initial segments

$$\mathrm{dc}\, A := \{f \upharpoonright \alpha \mid f \in A,\ \alpha < \mathrm{Ord}\}$$

its *downward closure*, ordered by inclusion. ⌟

1.2.3 Definition. Let $\langle d_\alpha \mid \alpha < \kappa\rangle$ be a κ-list. For $\alpha < \kappa$ define $t_\alpha : \alpha \to 2$ by $t_\alpha(\beta) = 1 \ :\leftrightarrow \beta \in d_\alpha$. We call $\langle t_\alpha \mid \alpha < \kappa\rangle$ the *characteristic functions* of $\langle d_\alpha \mid \alpha < \kappa\rangle$ and $\mathrm{dc}\{t_\alpha \mid \alpha < \kappa\}$ its *corresponding tree*. ⌟

The next definition is simple yet essential. While the property *thin* was of original interest to this work, the property *slender* gained attention as it became apparent it is the correct concept for the results in Chapter 2. It was key to understanding how to correctly attack the problems in Chapter 3. It comes from Shelah's approachability ideal but does not seem to have received attention as a more general concept before.

1.2.4 Definition. Let $\langle d_\alpha \mid \alpha < \kappa\rangle$ be a κ-list.

1. $\langle d_\alpha \mid \alpha < \kappa\rangle$ is called *thin* iff $|\{d_\alpha \cap \delta \mid \alpha < \kappa\}| < \kappa$ for all $\delta < \kappa$.
2. $\langle d_\alpha \mid \alpha < \kappa\rangle$ is called *slender* iff there exists a club $C \subset \kappa$ such that for every $\gamma \in C$ and every $\delta < \gamma$ there is $\beta < \gamma$ such that $d_\gamma \cap \delta = d_\beta \cap \delta$. ⌟

While *thin* is rather intuitive, *slender* might first appear to be not. But consider the characteristic functions $\langle t_\alpha \mid \alpha < \kappa\rangle$ of a κ-list $\langle d_\alpha \mid \alpha < \kappa\rangle$. Then $\langle d_\alpha \mid \alpha < \kappa\rangle$ is slender iff for club many $\gamma < \kappa$ the function t_γ is a branch in the tree $\mathrm{dc}\{t_\alpha \mid \alpha < \gamma\}$. In particular, *slender* is, as the following proposition shows, a weakening of *thin*.

1.2.5 Proposition. *Let $\langle d_\alpha \mid \alpha < \kappa\rangle$ be a κ-list. If $\langle d_\alpha \mid \alpha < \kappa\rangle$ is thin, then it is slender.*

Proof. Suppose to the contrary there is a stationary $S \subset \kappa$ such that

$$\forall\gamma \in S\ \exists\delta_\gamma < \gamma\ \forall\beta < \gamma\ d_\gamma \cap \delta_\gamma \neq d_\beta \cap \delta_\gamma.$$

We may assume $\delta_\gamma = \delta$ for some $\delta < \kappa$ and all $\gamma \in S$. But then in particular for distinct $\gamma, \gamma' \in S$ we have $d_\gamma \cap \delta \neq d_{\gamma'} \cap \delta$, so that $|\{d_\alpha \cap \delta \mid \alpha < \kappa\}| \geq |S| = \kappa$, contradicting $\langle d_\alpha \mid \alpha < \kappa\rangle$ being thin. □

For the following definition, the wording was chosen as to reflect the connection to the already existing terminology. If one is not familiar with these concepts, one might feel a little taken aback.

1.2.6 Definition. Let $A \subset \kappa$ and let $\langle d_\alpha \mid \alpha < \kappa \rangle$ be a κ-list.

1. $\langle d_\alpha \mid \alpha < \kappa \rangle$ is called A-*approachable*, iff there exists a club $C \subset \kappa$ such that $\sup d_\alpha = \alpha$ and $\operatorname{otp} d_\alpha = \operatorname{cf} \alpha < \alpha$ for all $\alpha \in A \cap C$.
2. $\langle d_\alpha \mid \alpha \in A \rangle$ is called A-*unsubtle* iff there exists a club $C \subset \kappa$ such that $d_\alpha \neq d_\beta \cap \alpha$ for all $\alpha, \beta \in A \cap C$ with $\alpha < \beta$.
3. $\langle d_\alpha \mid \alpha \in A \rangle$ is called A-*effable* iff for every $S \subset A$ which is stationary in κ there exist $\alpha, \beta \in S$ such that $\alpha \leq \beta$ and $d_\alpha \neq d_\beta \cap \alpha$.

$\langle d_\alpha \mid \alpha < \kappa \rangle$ is called *approachable*, *unsubtle*, or *effable* iff it is κ-approachable, κ-unsubtle, or κ-effable respectively. ⌟

Combining Definitions 1.2.4 and 1.2.6, it is natural to define the following ideals.

1.2.7 Definition. We let

$$\begin{aligned}
I_{\mathrm{AT}}[\kappa] &:= \{A \subset \kappa \mid \text{there exists a thin } A\text{-approachable } \kappa\text{-list}\},\\
I_{\mathrm{ST}}[\kappa] &:= \{A \subset \kappa \mid \text{there exists a thin } A\text{-unsubtle } \kappa\text{-list}\},\\
I_{\mathrm{IT}}[\kappa] &:= \{A \subset \kappa \mid \text{there exists a thin } A\text{-effable } \kappa\text{-list}\},\\
I_{\mathrm{AS}}[\kappa] &:= \{A \subset \kappa \mid \text{there exists a slender } A\text{-approachable } \kappa\text{-list}\},\\
I_{\mathrm{SS}}[\kappa] &:= \{A \subset \kappa \mid \text{there exists a slender } A\text{-unsubtle } \kappa\text{-list}\},\\
I_{\mathrm{IS}}[\kappa] &:= \{A \subset \kappa \mid \text{there exists a slender } A\text{-effable } \kappa\text{-list}\}.
\end{aligned}$$

By $F_{\mathrm{AT}}[\kappa]$, $F_{\mathrm{ST}}[\kappa]$, $F_{\mathrm{IT}}[\kappa]$, $F_{\mathrm{AS}}[\kappa]$, $F_{\mathrm{SS}}[\kappa]$, $F_{\mathrm{IS}}[\kappa]$ we denote the filters associated to $I_{\mathrm{AT}}[\kappa]$, $I_{\mathrm{ST}}[\kappa]$, $I_{\mathrm{IT}}[\kappa]$, $I_{\mathrm{AS}}[\kappa]$, $I_{\mathrm{SS}}[\kappa]$, $I_{\mathrm{IS}}[\kappa]$ respectively.

Furthermore we write κ-STP for $\kappa \notin I_{\mathrm{ST}}[\kappa]$, κ-ITP for $\kappa \notin I_{\mathrm{IT}}[\kappa]$, κ-SSP for $\kappa \notin I_{\mathrm{SS}}[\kappa]$, and κ-ISP for $\kappa \notin I_{\mathrm{IS}}[\kappa]$. ⌟

1.2.8 Proposition. *It holds that*

$$I_{\mathrm{AT}}[\kappa] \subset I_{\mathrm{ST}}[\kappa]$$

and

$$I_{\mathrm{AS}}[\kappa] \subset I_{\mathrm{SS}}[\kappa].$$

Proof. Let $\langle d_\alpha \mid \alpha < \kappa \rangle$ be an A-approachable κ-list, witnessed by a club $C \subset \mathrm{Lim}$. We define

$$\tilde{d}_\alpha := (\{\zeta + 1 \mid \zeta \in d_\alpha\} \cup \{\operatorname{cf} \alpha\}) - \operatorname{cf} \alpha$$

if $\operatorname{cf} \alpha < \alpha$ and set $\tilde{d}_\alpha := \emptyset$ otherwise.

Then $\langle \tilde{d}_\alpha \mid \alpha < \kappa \rangle$ is A-unsubtle, for let $\alpha, \alpha' \in A \cap C$, $\alpha < \alpha'$. If $\operatorname{cf} \alpha = \operatorname{cf} \alpha'$, then $\operatorname{otp}(\tilde{d}_{\alpha'} \cap \alpha) < \operatorname{cf} \alpha' = \operatorname{cf} \alpha = \operatorname{otp} \tilde{d}_\alpha$. If $\operatorname{cf} \alpha < \operatorname{cf} \alpha'$, then $\operatorname{cf} \alpha \in \tilde{d}_\alpha$ but $\operatorname{cf} \alpha \notin \tilde{d}_{\alpha'} \cap \alpha$, and if $\operatorname{cf} \alpha > \operatorname{cf} \alpha'$, then $\operatorname{cf} \alpha' \in \tilde{d}_{\alpha'} \cap \alpha$ but $\operatorname{cf} \alpha' \notin \tilde{d}_\alpha$.

If $\langle d_\alpha \mid \alpha < \kappa\rangle$ is thin, then so is $\langle \tilde{d}_\alpha \mid \alpha < \kappa\rangle$. For if $\delta < \kappa$ is a limit ordinal, then

$$\begin{aligned}\{\tilde{d}_\alpha \cap \delta \mid \alpha < \kappa\} &= \{((\{\zeta+1 \mid \zeta \in d_\alpha\} \cup \{\operatorname{cf}\alpha\}) - \operatorname{cf}\alpha) \cap \delta \mid \alpha < \kappa\} \\ &\subset \{(\{\zeta+1 \mid \zeta \in d_\alpha \cap \delta\} \cup \{\beta\}) - \beta \mid \alpha < \kappa,\ \beta < \delta\} \cup \{\emptyset\},\end{aligned}$$

which has cardinality $< \kappa$.

If $\langle d_\alpha \mid \alpha < \kappa\rangle$ is slender, without loss of generality witnessed by C, let $f : \kappa \times \kappa \to \kappa$ be bijective such that $f(\alpha,\alpha) = \alpha$ for club many $\alpha < \kappa$. We may assume $C \subset \{\alpha < \kappa \mid f''(\alpha \times \alpha) = \alpha,\ f(\alpha,\alpha) = \alpha\}$. Let

$$\bar{d}_\alpha := ((\{\zeta+1 \mid \zeta \in d_{\alpha_0}\} \cup \{\operatorname{cf}\alpha_1\}) - \operatorname{cf}\alpha_1) \cap \alpha,$$

where $\alpha = f(\alpha_0,\alpha_1)$. Then $\bar{d}_\gamma = \tilde{d}_\gamma$ for $\gamma \in C$. By assumption, if $\gamma \in C$ and $\delta < \gamma$, then there is a $\beta < \gamma$ such that $d_\gamma \cap \delta = d_\beta \cap \delta$. If $\operatorname{cf}\gamma = \gamma$, then $\bar{d}_\gamma = \emptyset$, so assume $\operatorname{cf}\gamma < \gamma$. Then for $\bar{\beta} := f(\beta, \operatorname{cf}\gamma) < \gamma$ we have

$$\bar{d}_{\bar{\beta}} = (\{\zeta+1 \mid \zeta \in d_\beta\} \cup \{\operatorname{cf}\gamma\}) - \operatorname{cf}\gamma,$$

so $\bar{d}_\gamma \cap \delta = \bar{d}_{\bar{\beta}} \cap \delta$. Thus $\langle \bar{d}_\alpha \mid \alpha < \kappa\rangle$ is a slender A-unsubtle κ-list. □

1.2.9 Proposition. *$I_{\mathrm{AS}}[\kappa]$ is the approachability ideal on κ, that is, if*

$$\begin{aligned}I[\kappa] := \{A \subset \kappa \mid\ & \text{there is a sequence } \langle a_\alpha \mid \alpha < \kappa\rangle \text{ of bounded subsets of } \kappa \text{ and a club } C \subset \kappa \\ & \text{such that } \forall \gamma \in A \cap C\ \exists b_\gamma \subset \gamma \text{ with } \operatorname{otp} b_\gamma = \operatorname{cf}\gamma < \gamma \text{ and } \sup b_\gamma = \gamma \\ & \text{such that } \forall \delta < \gamma\ \exists \beta < \gamma\ b_\gamma \cap \delta = a_\beta\},\end{aligned}$$

then $I_{\mathrm{AS}}[\kappa] = I[\kappa]$.

Proof. For the "$\subset$" part of the proof, let $\langle d_\alpha \mid \alpha < \kappa\rangle$ be an A-approachable slender κ-list, and let the club C witness it. Let $f : \kappa\times\kappa \to \kappa$ be a bijection such that $f(\alpha,\alpha) = \alpha$ for club many $\alpha < \kappa$. So we can assume $C \subset \{\alpha < \kappa \mid f(\alpha,\alpha) = \alpha,\ f''(\alpha \times \alpha) = \alpha\}$. Let $a_\alpha := d_{\alpha_0} \cap \alpha_1$, where $\alpha = f(\alpha_0,\alpha_1)$. Let $b_\gamma := a_\gamma = d_\gamma$ for $\gamma \in A \cap C$. Then if $\delta < \gamma \in A \cap C$, there is a $\beta < \gamma$ such that $d_\gamma \cap \delta = d_\beta \cap \delta$, so for $\tilde{\beta} := f(\beta,\delta) < \gamma$ we have $b_\gamma \cap \delta = d_\gamma \cap \delta = d_\beta \cap \delta = a_{\tilde{\beta}}$.

For the other direction, let $\langle a_\alpha \mid \alpha < \kappa\rangle$, a club $C \subset \kappa$ and sets $b_\gamma \subset \gamma$ for $\gamma \in A \cap C$ witness $A \in I[\kappa]$. Define $a'_{\alpha+1} := a_\alpha$, $a'_\gamma := b_\gamma$ for $\gamma \in A \cap C$, and $a'_\gamma := \emptyset$ in all other cases. Let $\langle U_\nu \mid \nu < \kappa\rangle$ be a partition of $\kappa - \mathrm{Lim}$ such that U_ν is unbounded in κ for all $\nu < \kappa$. We may assume $C \subset \Delta_{\nu<\kappa} \mathrm{Lim}\, U_\nu$. Set $d_\gamma := a'_\gamma$ if $\gamma \in \mathrm{Lim} \cap \kappa$, and for successor $\alpha < \kappa$ let $\nu < \kappa$ be such that $\alpha \in U_\nu$ and set $d_\alpha := a'_\nu \cap \alpha$. Then $\langle d_\alpha \mid \alpha < \kappa\rangle$ is an A-approachable slender κ-list. It is obviously A-approachable. To see it is slender, let $\delta < \gamma \in C$. Then there is $\beta < \gamma$ such that $b_\gamma \cap \delta = a_\beta = a'_{\beta+1}$. As $\gamma \in \mathrm{Lim}\, U_{\beta+1}$, there is a $\tilde{\beta} \in [\delta,\gamma)$ with $\tilde{\beta} \in U_{\beta+1}$. But then $d_\gamma \cap \delta = b_\gamma \cap \delta = a'_{\beta+1} \cap \delta = d_{\tilde{\beta}} \cap \delta$. □

$I_{\mathrm{AT}}[\kappa]$ was first considered by Mitchell in [Mit04].

By definition, all ideals from Definition 1.2.7 contain the nonstationary ideal on κ. Obviously, if a κ-list is A-unsubtle, then it is A-effable. Therefore $I_{\mathrm{AT}}[\kappa] \subset I_{\mathrm{ST}}[\kappa] \subset I_{\mathrm{IT}}[\kappa]$ and $I_{\mathrm{AS}}[\kappa] \subset I_{\mathrm{SS}}[\kappa] \subset I_{\mathrm{IS}}[\kappa]$. Furthermore, by Proposition 1.2.5, it is immediate that $I_{\mathrm{AT}}[\kappa] \subset I_{\mathrm{AS}}[\kappa]$, $I_{\mathrm{ST}}[\kappa] \subset I_{\mathrm{SS}}[\kappa]$, and $I_{\mathrm{IT}}[\kappa] \subset I_{\mathrm{IS}}[\kappa]$.

We will not actually consider the ideals $I_{\mathrm{AT}}[\kappa]$ and $I_{\mathrm{AS}}[\kappa]$ in great detail. The reason they are included here is that they canonically fit in and that some results are most naturally formulated for them.

The next two definitions are independently due to Kenneth Kunen and Ronald Jensen.

1.2.10 Definition. κ is called *subtle* iff there is no unsubtle κ-list. ⌟

1.2.11 Definition. κ is called *ineffable* iff there is no effable κ-list. ⌟

Note that, since for inaccessible κ every κ-list is thin, κ is subtle iff κ is inaccessible and κ-STP holds, and κ is ineffable iff κ is inaccessible and κ-ITP holds.

1.2.12 Proposition. *$I_{\mathrm{AT}}[\kappa]$, $I_{\mathrm{ST}}[\kappa]$, $I_{\mathrm{IT}}[\kappa]$, $I_{\mathrm{AS}}[\kappa]$, $I_{\mathrm{SS}}[\kappa]$, $I_{\mathrm{IS}}[\kappa]$ are normal ideals on κ.*

Proof. Let I be one of the ideals. Let $g : D \to \kappa$ be regressive with $D \notin I$ and assume to the contrary that $A_\gamma := g^{-1\prime\prime}\{\gamma\} \in I$ for every $\gamma < \kappa$. Without loss we may assume every element of D is indecomposable. For every $\gamma < \kappa$ let $\langle d^\gamma_\alpha \mid \alpha < \kappa\rangle$ be a κ-list witnessing $A_\gamma \in I$, and let C^γ be a club that witnesses that $\langle d^\gamma_\alpha \mid \alpha < \kappa\rangle$ is A_γ-approachable, A_γ-unsubtle, or A_γ-effable and, if appropriate, slender. Set $C := \Delta_{\gamma<\kappa} C^\gamma$. Then $A_\gamma \cap C \subset A_\gamma \cap C^\gamma$ as $A_\gamma \subset \kappa - (\gamma + 1)$.

In the thin case, for $\alpha \in D$ let $\gamma := g(\alpha)$ and set

$$e_\alpha := \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d^\gamma_\alpha\},$$

and for $\alpha \in \kappa - D$ let $e_\alpha := \emptyset$. Then $e_\alpha \subset \alpha$ as every $\alpha \in D$ is indecomposable. Also $\langle e_\alpha \mid \alpha < \kappa\rangle$ is thin. For take $\delta < \kappa$. Then

$$\begin{aligned}\{e_\alpha \cap \delta \mid \alpha < \kappa\} &= \bigcup_{\gamma<\kappa} \{e_\alpha \cap \delta \mid \alpha \in A_\gamma\} \cup \{\emptyset\}\\ &= \bigcup_{\gamma<\delta} \{e_\alpha \cap \delta \mid \alpha \in A_\gamma\} \cup \{\emptyset\}.\end{aligned}$$

But for fixed γ we have

$$\begin{aligned}|\{e_\alpha \cap \delta \mid \alpha \in A_\gamma\}| &= |\{(\{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d^\gamma_\alpha\}) \cap \delta \mid \alpha \in A_\gamma\}|\\ &\leq |\{d^\gamma_\alpha \cap \delta \mid \alpha \in A_\gamma\}|\\ &< \kappa\end{aligned}$$

by the thinness of $\langle d^\gamma_\alpha \mid \alpha < \kappa\rangle$, so $|\{e_\alpha \cap \delta \mid \alpha < \kappa\}| < \kappa$ as well.

In the slender case, as in the proof of Proposition 1.2.9 let $f : \kappa \times \kappa \to \kappa$ bijective be such that $\alpha \leq f(\alpha,\beta)$ for all $\alpha,\beta < \kappa$. $C' := \{\alpha < \kappa \mid f''(\alpha \times \alpha) \subset \alpha\}$ is club, and we may assume $C \subset C'$. Furthermore let $\langle U_\nu \mid \nu < \kappa\rangle$ be a partition of $\kappa - \mathrm{Lim}$ such that U_ν is unbounded in κ for all $\nu < \kappa$. We can assume $C \subset \Delta_{\nu<\kappa} \mathrm{Lim}\, U_\nu$. For $\alpha \in C \cap D$ let

$$e_\alpha := \{g(\alpha)\} \cup \{g(\alpha) + 1 + \eta \mid \eta \in d_\alpha^{g(\alpha)}\},$$

which is a subset of α by the indecomposability of α. Define

$$e_{\alpha+1} := (\{\nu\} \cup \{\nu + 1 + \eta \mid \eta \in d_{\alpha_1}^{\nu}\}) \cap \alpha,$$

where $\alpha = f(\alpha_0, \alpha_1)$ and $\nu < \kappa$ is such that $\alpha_0 + 1 \in U_\nu$. In all other cases, let $e_\alpha := \emptyset$. Then $\langle e_\alpha \mid \alpha < \kappa\rangle$ is slender, for take $\alpha \in C$ and $\delta < \alpha$. If $\alpha \notin D$, then $e_\alpha \cap \delta = \emptyset = e_0 \cap \delta$, so let $\alpha \in D$. Let $\gamma := g(\alpha)$. Then $\alpha \in C^\gamma$, so by the slenderness of $\langle d_\alpha^\gamma \mid \alpha < \kappa\rangle$ there is $\beta < \alpha$ such that $d_\alpha^\gamma \cap \delta = d_\beta^\gamma \cap \delta$. Since $\alpha \in \mathrm{Lim}\, U_\gamma$, there is an $\alpha_0 \in [\delta, \alpha)$ such that $\alpha_0 + 1 \in U_\gamma$. Let $\tilde{\beta} := f(\alpha_0,\beta) < \alpha$. Then $\tilde{\beta} \geq \delta$ and thus

$$e_\alpha \cap \delta = (\{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\alpha^\gamma\}) \cap \delta = (\{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\beta^\gamma\}) \cap \delta = e_{\tilde{\beta}+1} \cap \delta.$$

In both cases we arrived at an either thin or slender κ-list $\langle e_\alpha \mid \alpha < \kappa\rangle$ such that for every $\alpha \in C \cap D$

$$e_\alpha = \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\alpha^\gamma\} \subset [\gamma, \kappa),$$

where $\gamma := g(\alpha)$. We will now produce a contradiction to $D \notin I$ in each of the three possible cases.

If $I = I_{\mathrm{AT}}[\kappa]$ or $I = I_{\mathrm{AS}}[\kappa]$, then for every $\alpha \in C \cap D$ and $\gamma := g(\alpha)$ it holds that $\sup e_\delta = \sup d_\delta^\gamma$ and $\mathrm{otp}\, e_\delta = 1 + \mathrm{otp}\, d_\delta^\gamma = \mathrm{otp}\, d_\delta^\gamma$, so $\langle e_\alpha \mid \alpha < \kappa\rangle$ is D-approachable.

If $I = I_{\mathrm{ST}}[\kappa]$ or $I = I_{\mathrm{SS}}[\kappa]$, let $\alpha,\beta \in C \cap D$, $\alpha < \beta$. If $\gamma := g(\alpha) = g(\beta)$, then $e_\beta \cap \alpha = \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\beta^\gamma\} \cap \alpha = \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\beta^\gamma \cap \alpha\}$ because α is indecomposable. But $\langle d_\alpha^\gamma \mid \alpha < \kappa\rangle$ is A^γ-unsubtle, witnessed by C, so $e_\beta \cap \alpha = \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\beta^\gamma \cap \alpha\} \neq \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\alpha^\gamma\} = e_\alpha$. If $g(\alpha) \neq g(\beta)$, then for $\gamma := \min\{g(\alpha), g(\beta)\}$ we have $\gamma < \alpha,\beta$ and either $\gamma \in e_\alpha - (e_\beta \cap \alpha)$ or $\gamma \in (e_\beta \cap \alpha) - e_\alpha$, so again $e_\alpha \neq e_\beta \cap \alpha$. Therefore $\langle e_\alpha \mid \alpha < \kappa\rangle$ is D-unsubtle.

If $I = I_{\mathrm{IT}}[\kappa]$ or $I = I_{\mathrm{IS}}[\kappa]$, take an arbitrary stationary $S \subset C \cap D$. As g is regressive, we can assume $S \subset A_\gamma$ for some $\gamma < \kappa$. Since $\langle d_\alpha^\gamma \mid \alpha < \kappa\rangle$ is A_γ-effable, there are $\alpha,\beta \in S$ such that $\alpha < \beta$ and $d_\alpha^\gamma \neq d_\beta^\gamma \cap \alpha$. But then

$$\begin{aligned} e_\alpha &= \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\alpha^\gamma\} \\ &\neq \{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\beta^\gamma \cap \alpha\} \\ &= (\{\gamma\} \cup \{\gamma + 1 + \eta \mid \eta \in d_\beta^\gamma\}) \cap \alpha \\ &= e_\beta \cap \alpha. \end{aligned}$$

This means $\langle e_\alpha \mid \alpha < \kappa\rangle$ is D-effable. □

The following four propositions are all more or less folklore in the sense they are standard knowledge for $I_{\mathrm{AS}}[\kappa]$.

1.2.13 Proposition. *If κ is inaccessible, then*

$$\{\delta < \kappa \mid \delta \textit{ inaccessible}\} \in F_{\mathrm{AT}}[\kappa].$$

Proof. Because of the inaccessibility of κ the set $\{\delta < \kappa \mid \delta \text{ strong limit}\}$ is club in κ. But obviously $\{\delta < \kappa \mid \delta \text{ singular}\} \in I_{\mathrm{AT}}[\kappa]$. □

1.2.14 Proposition. $\operatorname{cof}\omega \cap \kappa \in I_{\mathrm{AT}}[\kappa]$.

Proof. For $\alpha \in \operatorname{cof}\omega \cap \kappa$, let $d_\alpha \subset \alpha$ be cofinal of order type ω, otherwise let $d_\alpha := \emptyset$. Then $\langle d_\alpha \mid \alpha < \kappa \rangle$ is thin and $(\operatorname{cof}\omega \cap \kappa)$-approachable. □

1.2.15 Proposition. *If λ is regular, then* $\operatorname{cof}(<\lambda) \cap \lambda^+ \in I_{\mathrm{AT}}[\lambda^+]$.

Proof. By [She91, Lemma 4.4] there exist $\langle S_i \mid i < \lambda \rangle$, $\langle C^i_\delta \mid \delta \in S_i \rangle$ such that $\bigcup_{i<\lambda} S_i = \operatorname{cof}(<\lambda) \cap \lambda^+$ and for every $i < \lambda$, $\alpha \in S_i$, and $\eta < \alpha$

(i) $C^i_\alpha \subset \alpha \cap \operatorname{cof}(<\lambda)$, $|C^i_\alpha| < \lambda$, C^i_α is closed, and C^i_α unbounded in α if $\alpha \in \mathrm{Lim}$,

(ii) $\eta \in C^i_\alpha \rightarrow \eta \in S_i \ \wedge\ C^i_\eta = C^i_\alpha \cap \eta$.

For $\delta < \lambda$ let $D_\delta \subset \delta$ be cofinal of order type $\operatorname{cf}\delta$. Let

$$C^i_{\alpha,\delta} := \{\gamma \in C^i_\alpha \mid \operatorname{otp}(C^i_\alpha \cap \gamma) \in D_\delta\},$$

and

$$T_\beta := \{C^i_{\alpha,\delta} \cup \{\eta\} \mid i < \lambda,\ \delta < \lambda,\ \eta < \beta,\ \alpha \leq \beta\},$$

for $\beta < \lambda^+$, so that $|T_\beta| \leq \lambda$.

For $\alpha \in \operatorname{cof}(<\lambda) \cap \lambda^+$, let $d_\alpha := C^i_{\alpha,\operatorname{otp} C^i_\alpha}$, where $i < \lambda$ is such that $\alpha \in S_i$. For $\alpha \in \operatorname{cof}\lambda \cap \lambda^+$, let $d_\alpha := \emptyset$. Then $\langle d_\alpha \mid \alpha < \lambda^+ \rangle$ is $(\operatorname{cof}(<\lambda) \cap \lambda^+)$-approachable as for $\alpha \in \operatorname{cof}(<\lambda) \cap \lambda^+$ the set d_α is cofinal in α and $\operatorname{otp} d_\alpha = \operatorname{otp} D_{\operatorname{otp} C^i_\alpha} = \operatorname{cf}\operatorname{otp} C^i_\alpha = \operatorname{cf}\alpha$. If $\beta < \alpha$ and $d_\alpha \cap \beta \neq \emptyset$, then for $\eta := \sup(C^i_\alpha \cap \beta)$ we have $\eta \in C^i_\alpha \cap S_i$. If $\eta = \beta$, this means $C^i_\alpha \cap \beta = C^i_\alpha \cap \eta = C^i_\eta$. If $\eta < \beta$, we have $C^i_\alpha \cap \beta = C^i_\alpha \cap \eta \cup \{\eta\} = C^i_\eta \cup \{\eta\}$. Therefore

$$\begin{aligned} d_\alpha \cap \beta - \{\eta\} &= \{\gamma \in C^i_\alpha \cap \beta - \{\eta\} \mid \operatorname{otp}(C^i_\alpha \cap \gamma) \in D_{\operatorname{otp} C^i_\alpha}\} \\ &= \{\gamma \in C^i_\eta \mid \operatorname{otp}(C^i_\eta \cap \gamma) \in D_{\operatorname{otp} C^i_\alpha}\} \\ &= C^i_{\eta,\operatorname{otp} C^i_\alpha}, \end{aligned}$$

so $d_\alpha \cap \beta \in T_\beta$. This shows $\langle d_\alpha \mid \alpha < \lambda^+ \rangle$ is thin. □

1.2.16 Proposition. *If $\square^*_\lambda$ holds, then $\lambda^+ \in I_{\mathrm{AT}}[\lambda^+]$.*

Proof. By Proposition 1.2.14 it suffices to show the set of all ordinals below λ^+ of uncountable cofinality is in $I_{\mathrm{AT}}[\lambda^+]$. Let $\langle \mathscr{C}_\alpha \mid \alpha \in \mathrm{Lim} \cap \lambda^+ \rangle$ be a $\square^*_\lambda$ sequence, and let $\mathscr{C}'_\alpha := \{\mathrm{Lim}\, C \mid C \in \mathscr{C}_\alpha\}$. Then for $\alpha < \kappa^+$ with $\mathrm{cf}\,\alpha \geq \omega_1$ every $C \in \mathscr{C}'_\alpha$ is club in α, and if $\beta \in C$, then $C \cap \beta \in \mathscr{C}'_\beta$.

Now as in the proof of Proposition 1.2.15 let $D_\delta \subset \delta$ be cofinal of order type $\mathrm{cf}\,\delta$ for $\delta \leq \lambda$. For $C \in \mathscr{C}'_\alpha$ let $C_\delta := \{\gamma \in C \mid \mathrm{otp}(C \cap \gamma) \in D_\delta\}$. For every $\alpha < \kappa^+$ of uncountable cofinality, let $C \in \mathscr{C}'_\alpha$ and set $d_\alpha := C_{\mathrm{otp}\, C}$. Set $d_\alpha := \emptyset$ otherwise. Then $\langle d_\alpha \mid \alpha < \kappa^+ \rangle$ is a thin $(\mathrm{cof}(\geq \omega_1) \cap \kappa^+)$-approachable κ^+-list. □

1.3 Forest Properties

For the whole section κ and λ are assumed to be cardinals, $\kappa \leq \lambda$, and κ is regular and uncountable. Our goal is to generalize the concepts of Section 1.2 to $\mathfrak{P}_\kappa\lambda$. The reader will notice that we could develop the theory more generally for H_λ or an arbitrary set X instead of λ. However, it appeared sensible to simplify the concepts by only considering λ.

1.3.1 Definition. $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ is called *$\mathfrak{P}_\kappa\lambda$-list* iff $d_a \subset a$ for all $a \in \mathfrak{P}_\kappa\lambda$. ⌟

1.3.2 Definition. Let $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ be a $\mathfrak{P}_\kappa\lambda$-list.

1. $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ is called *thin* iff there is a club $C \subset \mathfrak{P}_\kappa\lambda$ such that $|\{d_a \cap c \mid c \subset a \in \mathfrak{P}_\kappa\lambda\}| < \kappa$ for every $c \in C$.
2. $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ is called *slender* iff for every sufficiently large θ there is a club $C \subset \mathfrak{P}_\kappa H_\theta$ such that for all $M \in C$ and all $b \in M \cap \mathfrak{P}_\kappa\lambda$

$$d_{M\cap\lambda} \cap b \in M.$$ ⌟

The definition of *thin* introduces a club as this results in a more natural definition. We do not claim that the generalization of *slender* is obvious. However, applied by any standard it is the correct one, as we will see later.

1.3.3 Proposition. *Let $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ be a $\mathfrak{P}_\kappa\lambda$-list. If $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ is thin, then it is slender.*

Proof. Let $C \subset \mathfrak{P}_\kappa\lambda$ be a club that witnesses $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ is thin. Define $g : C \to \mathfrak{P}_\kappa H_\theta$ by

$$g(c) := \{d_a \cap c \mid c \subset a \in \mathfrak{P}_\kappa\lambda\}.$$

Let $\bar{C} := \{M \in C^{H_\theta} \mid \forall b \in M \cap \mathfrak{P}_\kappa\lambda\ \exists c \in M \cap C\ b \subset c,\ \forall c \in M \cap C\ g(c) \subset M\}$. Then $\bar{C}$ is club. Let $M \in \bar{C}$ and $b \in M \cap \mathfrak{P}_\kappa\lambda$. Then there is $c \in M \cap C$ such that $b \subset c$, so

$$d_{M\cap\lambda} \cap b = d_{M\cap\lambda} \cap c \cap b \in M$$

as $d_{M\cap\lambda} \cap c \in g(c) \subset M$. Therefore $\bar{C}$ witnesses $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ is slender. □

Looking at Definition 1.2.6, *approachable* seems to not generalize to $\mathfrak{P}_\kappa\lambda$. But neither does *unsubtle*. For if one takes the obvious generalization, then by [Men75, Lemma 1.12] κ is λ-subtle for all $\lambda \geq \kappa$ iff κ is subtle.

1.3.4 Definition. Let $A \subset \mathfrak{P}_\kappa\lambda$ and let $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ be a $\mathfrak{P}_\kappa\lambda$-list. $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is called *A-effable* iff for every $S \subset A$ which is stationary in $\mathfrak{P}_\kappa\lambda$ there are $a, b \in S$ such that $a \subset b$ and $d_a \neq d_b \cap a$. $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is called *effable* iff it is $\mathfrak{P}_\kappa\lambda$-effable. ⌟

1.3.5 Proposition. *A $\mathfrak{P}_\kappa\lambda$-list $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is A-effable iff for every $S \subset A$ which is stationary in $\mathfrak{P}_\kappa\lambda$ there are $a, b \in S$ such that $d_a \cap (a \cap b) \neq d_b \cap (a \cap b)$.*

Proof. Suppose that $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is not A-effable, that is, there exists a stationary $S \subset A$ such that for all $a, b \in S$, if $a \subset b$, then $d_a = d_b \cap a$. Let $a, b \in S$ be such that $d_a \cap (a \cap b) \neq d_b \cap (a \cap b)$. Let $c \in S$ be such that $a \cup b \subset c$. But then $d_a \cap (a \cap b) = d_c \cap (a \cap b) = d_b \cap (a \cap b)$, a contradiction. □

1.3.6 Definition. We let

$$I_{\mathrm{IT}}[\kappa, \lambda] := \{A \subset \mathfrak{P}_\kappa\lambda \mid \text{there exists a thin } A\text{-effable } \mathfrak{P}_\kappa\lambda\text{-list}\},$$
$$I_{\mathrm{IS}}[\kappa, \lambda] := \{A \subset \mathfrak{P}_\kappa\lambda \mid \text{there exists a slender } A\text{-effable } \mathfrak{P}_\kappa\lambda\text{-list}\}.$$

By $F_{\mathrm{IT}}[\kappa, \lambda]$ we denote the filter associated to $I_{\mathrm{IT}}[\kappa, \lambda]$, and by $F_{\mathrm{IS}}[\kappa, \lambda]$ the filter associated to $I_{\mathrm{IS}}[\kappa, \lambda]$.

We write (κ, λ)-ITP for $\mathfrak{P}_\kappa\lambda \notin I_{\mathrm{IT}}[\kappa, \lambda]$ and (κ, λ)-ISP for $\mathfrak{P}_\kappa\lambda \notin I_{\mathrm{IS}}[\kappa, \lambda]$. ⌟

By Proposition 1.3.3, every thin $\mathfrak{P}_\kappa\lambda$-list is slender, so $I_{\mathrm{IT}}[\kappa, \lambda] \subset I_{\mathrm{IS}}[\kappa, \lambda]$.

1.3.7 Proposition. *$I_{\mathrm{IT}}[\kappa, \lambda]$ and $I_{\mathrm{IS}}[\kappa, \lambda]$ are normal ideals on $\mathfrak{P}_\kappa\lambda$.*

Proof. Suppose $D \subset \mathfrak{P}_\kappa\lambda$ and $g : D \to \lambda$ is regressive. Set $A_\gamma := g^{-1\prime\prime}\{\gamma\}$. Let $f : \lambda \times \lambda \to \lambda$ be bijective, and define $f_{\alpha_1} : \lambda \to \lambda$ by $f_{\alpha_1}(\alpha_0) := f(\alpha_0, \alpha_1)$. We show that if $A_\gamma \in I_{\mathrm{IT}}[\kappa, \lambda]$ for all $\gamma < \lambda$, then $D \in I_{\mathrm{IT}}[\kappa, \lambda]$, and that if $A_\gamma \in I_{\mathrm{IS}}[\kappa, \lambda]$ for all $\gamma < \lambda$, then $D \in I_{\mathrm{IS}}[\kappa, \lambda]$.

In the thin case, that is, if $A_\gamma \in I_{\mathrm{IT}}[\kappa, \lambda]$ for all $\gamma < \lambda$, let $\langle d_a^\gamma \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ be a thin A_γ-effable $\mathfrak{P}_\kappa\lambda$-list for $\gamma < \lambda$. Let $C^\gamma \subset \mathfrak{P}_\kappa\lambda$ be a club witnessing $\langle d_a^\gamma \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is thin. Set $C := \Delta_{\gamma<\lambda} C^\gamma$. We may assume that for all $a \in C$ and all $\alpha_0, \alpha_1 < \lambda$

$$f(\alpha_0, \alpha_1) \in a \leftrightarrow \alpha_0, \alpha_1 \in a. \tag{1.1}$$

For $a \in C \cap D$ set

$$d_a := f''_{g(a)} d_a^{g(a)},$$

and set $d_a := \emptyset$ for $a \in \mathfrak{P}_\kappa\lambda - (C \cap D)$. If $c \in C$ and $a \in C \cap D$ are such that $c \subset a$ and $g(a) \notin c$, then

$$d_a \cap c = \emptyset. \tag{1.2}$$

For if $g(a) \notin c$, then by (1.1) we have $d_a \cap c = f''_{g(a)} d_a^{g(a)} \cap c \subset \operatorname{rng} f_{g(a)} \cap c = \emptyset$. Thus for fixed $c \in C$ we have

$$\begin{aligned}\{d_a \cap c \mid c \subset a \in C \cap D\} \cup \{\emptyset\} &= \{d_a \cap c \mid g(a) \in c,\ c \subset a \in C \cap D\} \cup \{\emptyset\} \\ &\subset \{f''_\gamma d_a^\gamma \cap c \mid \gamma \in c,\ c \subset a \in C \cap A_\gamma\} \cup \{\emptyset\}.\end{aligned}$$

For $\gamma \in c$ we have $c \in C^\gamma$ and thus, as C^γ witnesses $\langle d_a^\gamma \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is thin,

$$|\{d_a^\gamma \cap c \mid c \subset a \in C \cap A_\gamma\}| < \kappa.$$

Therefore

$$\begin{aligned}|\{d_a \cap c \mid c \subset a \in \mathfrak{P}_\kappa\lambda\}| &\leq |\{f''_\gamma d_a^\gamma \cap c \mid \gamma \in c,\ c \subset a \in C \cap A_\gamma\}| \\ &= |\{f''_\gamma (d_a^\gamma \cap f_\gamma^{-1}{}'' c) \mid \gamma \in c,\ c \subset a \in C \cap A_\gamma\}| \\ &= |\{f''_\gamma (d_a^\gamma \cap c) \mid \gamma \in c,\ c \subset a \in C \cap A_\gamma\}| \\ &< \kappa,\end{aligned}$$

which shows $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is thin.

If $A_\gamma \in I_{\mathrm{IS}}[\kappa, \lambda]$ for all $\gamma < \lambda$, let $\langle d_a^\gamma \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ be a slender A_γ-effable $\mathfrak{P}_\kappa\lambda$-list for $\gamma < \lambda$. Let $C^\gamma \subset \mathfrak{P}'_\kappa H_\theta$ be a club witnessing $\langle d_a^\gamma \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is slender, where θ is some large enough cardinal. Set $C := \Delta_{\gamma<\lambda} C^\gamma$. We can again assume that for all $M \in C$ and $\alpha_0, \alpha_1 < \lambda$

$$f(\alpha_0, \alpha_1) \in M \leftrightarrow \alpha_0, \alpha_1 \in M.$$

In addition, we may require that

$$\langle M, \in, f \restriction (M \times M)\rangle \prec \langle H_\theta, \in, f\rangle \tag{1.3}$$

for every $M \in C$. As above we define

$$d_a := f''_{g(a)} d_a^{g(a)}$$

for $a \in (C \restriction \lambda) \cap D$ and let $d_a := \emptyset$ otherwise. By the same argument that led to (1.2), we have

$$d_a \cap b = \emptyset \tag{1.4}$$

if $b \in \mathfrak{P}_\kappa\lambda$, $a \in (C \restriction \lambda) \cap D$, $b \subset a$, and $g(a) \notin b$. To show $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is slender, let $M \in C$ and $b \in M \cap \mathfrak{P}_\kappa\lambda$. Set $a := M \cap \lambda$. If $M \notin D$, then $d_a \cap b \subset d_a = \emptyset \in M$, so assume $M \in D$. Then

$$d_a \cap b = f''_{g(a)} d_a^{g(a)} \cap b = f''_{g(a)} (d_a^{g(a)} \cap f_{g(a)}^{-1}{}'' b).$$

If $g(a) \notin b$, then by (1.4) $d_a \cap b = \emptyset \in M$, so suppose $g(a) \in b$. Then $f_{g(a)}^{-1}{}'' b = b$, so by the slenderness of $\langle d_{\tilde a}^{g(a)} \mid \tilde a \in \mathfrak{P}_\kappa\lambda\rangle$ we have $d_a^{g(a)} \cap f_{g(a)}^{-1}{}'' b \in M$. Thus, as $g(a) \in b \subset M$, by (1.3)

$$d_a \cap b = f''_{g(a)} (d_a^{g(a)} \cap f_{g(a)}^{-1}{}'' b) \in M.$$

In both cases we arrived at a $\mathfrak{P}_\kappa\lambda$-list $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ such that for a club $C \subset \mathfrak{P}_\kappa\lambda$ that is closed under f and f^{-1} we have

$$d_a = f''_{g(a)} d_a^{g(a)}$$

for every $a \in C \cap D$, and $d_a = \emptyset$ for $a \in \mathfrak{P}_\kappa\lambda - (C \cap D)$. Suppose that $D \notin I_{\mathrm{IT}}[\kappa, \lambda]$ for the thin case or $D \notin I_{\mathrm{IS}}[\kappa, \lambda]$ for the slender case. Then there are $S \subset C \cap D$ stationary in $\mathfrak{P}_\kappa\lambda$ and $d \subset \lambda$ such that $d_a = d \cap a$ for all $a \in S$. Since g is regressive we may assume $S \subset A_\gamma$ for some $\gamma < \lambda$. But then for $\tilde{d} := f_\gamma^{-1''} d$ and $a \in S$ it holds that

$$d_a^\gamma = f_\gamma^{-1''} f_\gamma'' d_a^\gamma = f_\gamma^{-1''} d_a = f_\gamma^{-1''}(d \cap a) = f_\gamma^{-1''} d \cap f_\gamma^{-1''} a = \tilde{d} \cap a,$$

contradicting $\langle d_a^\gamma \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ being effable. □

1.3.8 Lemma. *Suppose $\lambda < \lambda'$. Then*

$$I_{\mathrm{IT}}[\kappa, \lambda] \subset \{A' \restriction \lambda \mid A' \in I_{\mathrm{IT}}[\kappa, \lambda']\}$$

and

$$I_{\mathrm{IS}}[\kappa, \lambda] \subset \{A' \restriction \lambda \mid A' \in I_{\mathrm{IS}}[\kappa, \lambda']\}.$$

Proof. Suppose $A \subset \mathfrak{P}_\kappa\lambda$ and $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is an A-effable list. Let $A' := A^{\lambda'}$. Define $d'_{a'} := d_{a' \cap \lambda}$ for $a' \in \mathfrak{P}_\kappa\lambda'$.

Suppose $S' \subset A'$ is stationary. Then $S := S' \restriction \lambda \subset A$ is stationary in $\mathfrak{P}_\kappa\lambda$. Thus there are $a, b \in S$ such that $d_a \cap (a \cap b) \neq d_b \cap (a \cap b)$. Let $a', b' \in S'$ be such that $a = a' \cap \lambda$, $b = b' \cap \lambda$. Then $d_{a'} \cap (a' \cap b') = d_a \cap (a \cap b) \neq d_b \cap (a \cap b) = d_{b'} \cap (a' \cap b')$. Therefore $\langle d'_{a'} \mid a' \in \mathfrak{P}_\kappa\lambda'\rangle$ is A'-effable by Proposition 1.3.5.

If $C \subset \mathfrak{P}_\kappa\lambda$ witnesses $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is thin, then for $C' := C^{\lambda'}$ and $c' \in C'$ we have

$$\begin{aligned} |\{d'_{a'} \cap c' \mid c' \subset a' \in \mathfrak{P}_\kappa\lambda'\}| &= |\{d_{a' \cap \lambda} \cap c' \mid c' \subset a' \in \mathfrak{P}_\kappa\lambda'\}| \\ &= |\{d_a \cap (c' \cap \lambda) \mid c' \cap \lambda \subset a \in \mathfrak{P}_\kappa\lambda\}| \\ &< \kappa, \end{aligned}$$

because $c' \cap \lambda \in C$. So $\langle d'_{a'} \mid a' \in \mathfrak{P}_\kappa\lambda'\rangle$ is thin.

If $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is slender and $C \subset \mathfrak{P}_\kappa H_\theta$ is a club witnessing the slenderness, then obviously C also witnesses $\langle d'_{a'} \mid a' \in \mathfrak{P}_\kappa\lambda'\rangle$ is slender. □

1.3.9 Proposition. *Suppose $\lambda \leq \lambda'$. Then (κ, λ')-*ITP *implies (κ, λ)-*ITP*, and (κ, λ')-*ISP *implies (κ, λ)-*ISP.

Proof. If (κ, λ')-ITP holds, then $\mathfrak{P}_\kappa\lambda' \notin I_{\mathrm{IT}}[\kappa, \lambda']$, so by Lemma 1.3.8 $\mathfrak{P}_\kappa\lambda \notin I_{\mathrm{IT}}[\kappa, \lambda]$, whence (κ, λ)-ITP. The proof is the same for (κ, λ)-ISP. □

1.3.10 Proposition. *It holds that*

$$I_{\mathrm{IT}}[\kappa] = \{A \cap \kappa \mid A \in I_{\mathrm{IT}}[\kappa, \kappa]\}$$

and

$$I_{\mathrm{IS}}[\kappa] = \{A \cap \kappa \mid A \in I_{\mathrm{IS}}[\kappa, \kappa]\}.$$

Proof. Since κ is club in $\mathfrak{P}_\kappa\kappa$, the only thing nontrivial is to show both definitions of slender coincide. So let $C \subset \kappa$ be a club witnessing $\langle d_\alpha \mid \alpha < \kappa\rangle$ is a slender κ-list. For large enough θ let

$$\tilde{C} := \{M \in \mathfrak{P}'_\kappa H_\theta \mid \kappa_M \in C,\ \forall\alpha < \kappa_M\ d_\alpha \in M\}.$$

Then $\tilde{C}$ is club in $\mathfrak{P}_\kappa H_\theta$. If $M \in \tilde{C}$ and $b \in M \cap \mathfrak{P}_\kappa\lambda$, then $\sup b < \kappa_M$. Since $\kappa_M \in C$, there is $\beta < \kappa_M$ such that $d_{\kappa_M} \cap \sup b = d_\beta \cap \sup b$, and thus $d_{M\cap\kappa} \cap b = d_{\kappa_M} \cap \sup b \cap b = d_\beta \cap \sup b \cap b = d_\beta \cap b \in M$. Therefore $\tilde{C}$ witnesses $\langle d_a \mid a \in \mathfrak{P}_\kappa\kappa\rangle$, where $d_a := \emptyset$ if $a \notin \kappa$, is a slender $\mathfrak{P}_\kappa\kappa$-list.

For the other direction, let $\langle d_a \mid a \in \mathfrak{P}_\kappa\kappa\rangle$ be a slender $\mathfrak{P}_\kappa\kappa$-list. We show there is a slender κ-list $\langle d'_\alpha \mid \alpha < \kappa\rangle$ such that $d_\alpha = d'_\alpha$ for club many $\alpha < \kappa$.

Let θ be large enough, and let $\bar{M} \prec H_\theta$ with $d_\beta \cap \alpha \in \bar{M}$ for all $\alpha \leq \beta < \kappa$ and $|\bar{M}| = \kappa$. Let $C \subset \mathfrak{P}_\kappa H_\theta$ be a club witnessing $\langle d_a \mid a \in \mathfrak{P}_\kappa\kappa\rangle$ is slender. Let $<_{\bar{M}}$ be a well-order of $\bar{M}$ of order type κ, and recursively define $g : \kappa \to \bar{M}$ by

$$g(\alpha) := <_{\bar{M}}\text{-}\min\{x \in \bar{M} - g''\alpha \mid x \in \mathfrak{P}_\kappa\kappa \to x \subset \alpha\}.$$

g is surjective, and for all $\alpha < \kappa$, if $g(\alpha) \in \mathfrak{P}_\kappa\kappa$, then $g(\alpha) \subset \alpha$. Define $h : \kappa \to \bar{M}$ by

$$h(\alpha) := \begin{cases} g(\beta) & \text{if } \alpha = \beta + 1 \text{ for some } \beta < \kappa, \\ d_\alpha & \text{otherwise.} \end{cases}$$

Then h is surjective, for every $\alpha < \kappa$

$$h(\alpha) \in \mathfrak{P}_\kappa\kappa \to h(\alpha) \subset \alpha, \tag{1.5}$$

and

$$h(\delta) = d_\delta \tag{1.6}$$

for all limit ordinals $\delta < \kappa$. Define

$$d'_\alpha := h(\alpha) \cap \alpha$$

for $\alpha < \kappa$. Then by (1.5) and (1.6) we have $d'_\delta = h(\delta) \cap \delta = d_\delta$ for all limit ordinals $\delta < \kappa$.

Pick a continuous $\in$-chain $\langle M_\nu \mid \nu < \kappa\rangle$ such that $\langle M_\nu, \in, g \restriction M_\nu\rangle \prec \langle \bar{M}, \in, g\rangle$, $\kappa \in M_\nu$, and $M_\nu \in C \restriction \bar{M}$ for all $\nu < \kappa$. Then $\tilde{C} := \{\kappa_{M_\nu} \mid \nu < \kappa\}$ is club in κ. If $\gamma \in \operatorname{Lim} \tilde{C}$, say $\gamma = \kappa_{M_{\nu^*}}$, and $\beta < \gamma$, then

$$d'_\gamma \cap \beta = d_\gamma \cap \beta \in M_{\nu^*}$$

by the slenderness of $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$. As $M_{\nu^*} = \bigcup\{M_\nu \mid \nu < \nu^*\}$, we have $d'_\gamma \cap \beta \in M_\nu$ for some $\nu < \nu^*$. Thus there exists an $\alpha < \kappa_{M_\nu}$ with $g(\alpha) = d'_\gamma \cap \beta$. Therefore

$$d'_{\alpha+1} = h(\alpha + 1) \cap (\alpha + 1) = g(\alpha) = d'_\gamma \cap \beta$$

by (1.5) and the definition of h, proving $\langle d'_\alpha \mid \alpha < \kappa\rangle$ is a slender κ-list. □

1.3.11 Proposition. *κ-ITP holds iff (κ, κ)-ITP holds, and κ-ISP holds iff (κ, κ)-ISP holds.*

Proof. Follows immediately from Proposition 1.3.10. □

Proposition 1.3.11 can be seen as affirmation that the two cardinal principles of this section correctly generalize those from Section 1.2.

The next definition is due to Thomas Jech [Jec73].

1.3.12 Definition. κ is called *λ-ineffable* iff there is no effable $\mathfrak{P}_\kappa\lambda$-list. ⌟

Again κ is λ-ineffable iff κ is inaccessible and (κ, λ)-ITP holds. It follows from Proposition 1.3.9 that if $\kappa \leq \lambda \leq \lambda'$ and κ is λ'-ineffable, then it is λ-ineffable.

The next two propositions are due to Magidor [Mag74]. The first is also independently due to Jech [Jec73].

1.3.13 Proposition. *If κ is λ-supercompact, then κ is λ-ineffable.*

1.3.14 Proposition. *If κ is $|V_\lambda|$-ineffable, then κ is δ-supercompact for all $\delta < \lambda$.*

Thus by Propositions 1.3.13 and 1.3.14 a cardinal κ is supercompact iff κ is inaccessible and (κ, λ)-ITP holds for all $\lambda \geq \kappa$.

The following proposition is the two cardinal analog of Proposition 1.2.14.

1.3.15 Proposition. *Suppose* $\operatorname{cf}\lambda \geq \kappa$. *Then*

$$\{a \in \mathfrak{P}_\kappa\lambda \mid \operatorname{Lim} a \cap \operatorname{cof}\omega \subset a\} \in F_{\mathrm{IT}}[\kappa, \lambda].$$

Proof. Let $A := \{a \in \mathfrak{P}_\kappa\lambda \mid \exists\eta_a \in \operatorname{Lim} a - a\ \ \operatorname{cf}\eta_a = \omega\}$ and for $a \in A$ let η_a be a witness. For $\delta \in \operatorname{cof}\omega \cap \lambda$ let $\langle d^\delta_\nu \mid \nu < \tau_\delta\rangle$ be an enumeration of $\{d \subset \delta \mid \operatorname{otp} d = \omega,\ \sup d = \delta\}$. For $a \in \mathfrak{P}_\kappa\lambda$ and $\delta \in \operatorname{Lim} a \cap \operatorname{cof}\omega$ let

$$\nu^\delta_a := \min\{\nu < \tau_\delta \mid \sup(d^\delta_\nu \cap a) = \delta\}.$$

For $a \in A$ set

$$d_a := d^{\eta_a}_{\nu^{\eta_a}_a} \cap a,$$

and for $a \in \mathfrak{P}_\kappa\lambda - A$ let $d_a := \emptyset$.

Then $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is A-effable, for suppose there were a cofinal $U \subset A$ and a $d \subset \lambda$ such that $d_a = d \cap a$ for all $a \in U$. Let $a \in U$. Since $\operatorname{cf}\lambda \geq \kappa$ there exists $b \in U$ such that $a \cup \operatorname{Lim} a \subset b$. But then $\operatorname{otp}(d_b \cap a) < \omega$, contradicting $d_b \cap a = d_a$.

$\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is also thin, for let $a \in \mathfrak{P}_\kappa\lambda$. Let

$$B_a := \{d^\delta_{\nu^\delta_a} \cap a \mid \delta \in \operatorname{Lim} a \cap \operatorname{cof}\omega\} \cup \mathfrak{P}_\omega a.$$

Then $|B_a| < \kappa$. Let $b \in A$ with $a \subset b$, and suppose $d_b \cap a \notin \mathfrak{P}_\omega a$. Since $a \subset b$, we have $\nu^\delta_b \leq \nu^\delta_a$ for all $\delta \in \operatorname{Lim} a \cap \operatorname{cof}\omega$. Because $|d_b \cap a| = \omega$ we also have that $d^{\eta_b}_{\nu^{\eta_b}_b} \cap a = d_b \cap a$ is unbounded in η_b. Therefore $\nu^{\eta_b}_a \leq \nu^{\eta_b}_b$, so that $\nu^{\eta_b}_a = \nu^{\eta_b}_b$. But this means $d_b \cap a = d^{\eta_b}_{\nu^{\eta_b}_a} \cap a \in B_a$. □

1.4 The Failure of Square

We define a variant of the weak square principle that is natural for our application. It is a “threaded” version of Ernest Schimmerling’s weak square principle and is only defined up to a given cofinality μ.

1.4.1 Definition. A sequence $\langle \mathscr{C}_\alpha \mid \alpha \in \operatorname{Lim} \cap \operatorname{cof}(<\mu) \cap \lambda\rangle$ is called a $\square^\mu(\kappa, \lambda)$-*sequence* iff it satisfies the following properties.

(i) $0 < |\mathscr{C}_\alpha| < \kappa$ for all $\alpha \in \operatorname{Lim} \cap \operatorname{cof}(<\mu) \cap \lambda$,

(ii) $C \subset \alpha$ is club for all $\alpha \in \operatorname{Lim} \cap \operatorname{cof}(<\mu) \cap \lambda$ and $C \in \mathscr{C}_\alpha$,

(iii) $C \cap \beta \in \mathscr{C}_\beta$ for all $\alpha \in \operatorname{Lim} \cap \operatorname{cof}(<\mu) \cap \lambda$, $C \in \mathscr{C}_\alpha$ and $\beta \in \operatorname{Lim} C$,

(iv) there is no club $D \subset \lambda$ such that $D \cap \delta \in \mathscr{C}_\delta$ for all $\delta \in \operatorname{Lim} D \cap \operatorname{cof}(<\mu) \cap \lambda$.

We say that $\square^\mu(\kappa, \lambda)$ holds iff there exists a $\square^\mu(\kappa, \lambda)$-sequence. $\square(\kappa, \lambda)$ stands for $\square^\lambda(\kappa, \lambda)$. ⌟

Note that $\square_{\tau,<\kappa}$ implies $\square(\kappa, \tau^+)$ and that $\square(\lambda)$ is $\square(2, \lambda)$. Also observe that $\square^\mu(\kappa, \lambda) \to \square^{\mu'}(\kappa', \lambda)$ for $\mu' \leq \mu$ and $\kappa \leq \kappa'$.

1.4.2 Theorem. *Suppose* $\operatorname{cf}\lambda \geq \kappa$ *and* $\square^\kappa(\kappa, \lambda)$ *holds. Then* $\neg(\kappa, \lambda)$-ITP.

Proof. Let $A := \{a \in \mathfrak{P}_\kappa\lambda \mid \operatorname{Lim} a \cap \operatorname{cof}\omega \subset a\}$. By Proposition 1.3.15, $A \in F_{\mathrm{IT}}[\kappa, \lambda]$. So it remains to show $A \in I_{\mathrm{IT}}[\kappa, \lambda]$. We may assume $\sup a \notin a$ for all $a \in A$. For $\gamma \in \operatorname{Lim} \cap \operatorname{cof}(<\kappa) \cap \lambda$ let $C_\gamma \in \mathscr{C}_\gamma$, and set $d_a := C_{\sup a} \cap a$ for $a \in A$, otherwise $d_a := \emptyset$. Then, since $\operatorname{Lim} a \cap \operatorname{cof}\omega \subset a$,

$$\sup d_a = \sup a \tag{1.7}$$

for every $a \in A$.

$\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is thin, for let $a \in \mathfrak{P}_\kappa\lambda$. Set

$$B_a := \{(C \cap a) \cup h \mid \exists\eta \in \operatorname{Lim} a\ C \in \mathscr{C}_\eta,\ h \in \mathfrak{P}_\omega a\} \cup \mathfrak{P}_\omega a.$$

Then $|B_a| < \kappa$. Let $b \in A$, $a \subset b$, and suppose $d_b \cap a \notin \mathfrak{P}_\omega a$. Let $\eta := \max \operatorname{Lim}(d_b \cap a)$. Then $\eta \in \operatorname{Lim} C_{\sup b}$, so there is a $C \in \mathscr{C}_\eta$ such that $d_b \cap \eta = C_{\sup b} \cap b \cap \eta = C \cap b$, so $d_b \cap a \cap \eta = C \cap a$. Since $|d_b \cap a - \eta| < \omega$, this means $d_b \cap a = (C \cap a) \cup (d_b \cap a - \eta) \in B_a$.

$\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is also A-effable. For suppose there were a cofinal $U \subset A$ and $d \subset \lambda$ such that $d_a = d \cap a$ for all $a \in U$. Then d is unbounded in λ by (1.7). Let $\delta \in \operatorname{Lim} d \cap \operatorname{cof}(<\kappa) \cap \lambda$. We will show $d \cap \delta \in \mathscr{C}_\delta$, which contradicts the fact that $\langle \mathscr{C}_\alpha \mid \alpha \in \operatorname{Lim} \cap \operatorname{cof}(<\kappa) \cap \lambda\rangle$ is a $\square^\kappa(\kappa, \lambda)$-sequence, thus finishing the proof. For every $a \in U$ such that $\delta \in \operatorname{Lim}(d \cap a)$ we have $C_{\sup a} \cap a = d_a = d \cap a$, and thus $\delta \in \operatorname{Lim} C_{\sup a}$, so that there is a $C_a \in \mathscr{C}_\delta$ such that $d \cap a \cap \delta = C_a \cap a$. But since $|\mathscr{C}_\delta| < \kappa$, there is a cofinal $U' \subset \{a \in U \mid \delta \in \operatorname{Lim}(d \cap a)\}$ such that $C_a = C$ for some $C \in \mathscr{C}_\delta$ and all $a \in U'$. But then we have $d \cap \delta \cap a = C \cap a$ for all $a \in U'$, which means $d \cap \delta = C \in \mathscr{C}_\delta$. □

The following corollary is originally due to Robert Solovay [Sol74].

1.4.3 Corollary. *Suppose κ is δ-supercompact. Then $\neg\square^\kappa(\kappa, \lambda)$ for all $\kappa \le \lambda \le \delta$ with* $\operatorname{cf}\lambda \ge \kappa$. *In particular, if κ is supercompact, then $\neg\square(\lambda)$ for all $\lambda \ge \kappa$ with* $\operatorname{cf}\lambda \ge \kappa$.

Proof. This follows directly from Proposition 1.3.13 and Theorem 1.4.2. □

1.5 Forest Properties for Inaccessible κ

When κ is inaccessible, the filter $F_{\mathrm{IT}}[\kappa, \lambda]$ has some additional helpful properties. These will be used in Section 2.3.

1.5.1 Proposition. *Let κ be inaccessible. Then*

$$\{a \in \mathfrak{P}'_\kappa\lambda \mid \kappa_a \textit{ inaccessible}\} \in F_{\mathrm{IT}}[\kappa, \lambda].$$

Proof. Let $X := \{\alpha < \kappa \mid \alpha \text{ inaccessible}\}$. By Proposition 1.2.13 $X \in F_{\mathrm{AT}}[\kappa] \subset F_{\mathrm{IT}}[\kappa]$. By Proposition 1.3.10 and Lemma 1.3.8 $F_{\mathrm{IT}}[\kappa] \subset F_{\mathrm{IT}}[\kappa, \kappa] \subset \{A \upharpoonright \kappa \mid A \in F_{\mathrm{IT}}[\kappa, \lambda]\}$, so there is an $X' \in F_{\mathrm{IT}}[\kappa, \lambda]$ such that $X = X' \upharpoonright \kappa$. But if $a \in \mathfrak{P}'_\kappa\lambda \cap X'$, then κ_a is inaccessible. □

1.5.2 Proposition. *Let κ be inaccessible. Let $g : \mathfrak{P}_\kappa\lambda \to \mathfrak{P}_\kappa\lambda$. Then*

$$\{a \in \mathfrak{P}'_\kappa\lambda \mid \forall z \in \mathfrak{P}_{\kappa_a} a\ g(z) \subset a\} \in F_{\mathrm{IT}}[\kappa, \lambda].$$

Proof. Suppose not. Then

$$B := \{a \in \mathfrak{P}'_\kappa\lambda \mid \exists z_a \in \mathfrak{P}_{\kappa_a}a\ g(z_a) \not\subset a\} \notin I_{\mathrm{IT}}[\kappa, \lambda].$$

So let $S \subset B$ be stationary and $z \subset \lambda$ be such that $z_a = z \cap a$ for all $a \in S$. For all $a \in S$ we have $\mu_a := |z_a| < \kappa_a$, so there are a stationary $S' \subset S$ and $\mu < \kappa$ such that $\mu_a = \mu$ for all $a \in S'$.

Suppose $|z| > \mu$. Then there is $y \subset z$ such that $|y| = \mu^+ < \kappa$. But $S'' := \{a \in S' \mid y \subset a\}$ is stationary and for every $a \in S''$ we have $z_a = z \cap a \supset y \cap a = y$, which implies $\mu = \mu_a = |z_a| \geq |y| = \mu^+$, a contradiction.

Since S' is cofinal, there is an $a \in S'$ such that $z \cup g(z) \subset a$. But then $z_a = z \cap a = z$ and $g(z_a) = g(z) \subset a$, so that $a \notin B$, contradicting $S' \subset B$. □

The next two Propositions are due to Chris Johnson. In [Joh90], he proves the stronger result that they hold for the so-called λ-Shelah property, a weakening of λ-ineffability that was introduced by Donna Carr in [Car86]. Note that the second one is a simple corollary of the first.

1.5.3 Proposition. *Suppose κ is inaccessible and $\lambda > \kappa$ is a successor cardinal. Then there is $A \in F_{\mathrm{IT}}[\kappa, \lambda]$ such that $\langle \sup a \mid a \in A\rangle$ is injective.*

1.5.4 Proposition. *Suppose κ is λ-ineffable and* cf $\lambda \geq \kappa$*. Then $\lambda^{<\kappa} = \lambda$.*

2 Consistency Results

2.1 Preliminaries

In this section, let μ be a regular uncountable cardinal.

Since we only need internal approachability of length ω, the following definition only covers the special case.

2.1.1 Definition. Let

$$\mathrm{IA}(\omega) := \{x \mid \exists\langle x_n \mid n < \omega\rangle\, (\bigcup\{x_n \mid n < \omega\} = x \ \wedge\ \forall n < \omega\, \langle x_i \mid i < n\rangle \in x)\}. \qquad \lrcorner$$

2.1.2 Proposition. *Let θ be regular and large enough. Then $\mathfrak{P}'_\mu H_\theta \cap \mathrm{IA}(\omega)$ is ω_1-club in $\mathfrak{P}_\mu H_\theta$.*

Proof. $\mathfrak{P}'_\mu H_\theta \cap \mathrm{IA}(\omega)$ is cofinal, for let $y \in \mathfrak{P}_\mu H_\theta$. Let $x_0 \in \mathfrak{P}'_\mu H_\theta$ such that $y \subset x_0$, and for $n < \omega$ let $x_{n+1} \in \mathfrak{P}'_\mu H_\theta$ be such that $x_n \cup \{\langle x_i \mid i < n\rangle\} \subset x_{n+1}$. Let $x := \bigcup\{x_n \mid n < \omega\}$. Then $y \subset x \in \mathfrak{P}'_\mu H_\theta \cap \mathrm{IA}(\omega)$.

To see that it is ω_1-closed, let $\langle x^k \mid k < \omega\rangle$ be a sequence in $\mathfrak{P}'_\mu H_\theta \cap \mathrm{IA}(\omega)$, and let $x := \bigcup\{x^k \mid k < \omega\}$. For $k < \omega$ let $\langle x^k_n \mid n < \omega\rangle$ be such that $x^k = \bigcup\{x^k_n \mid n < \omega\}$ and $\langle x^k_i \mid i < n\rangle \in x^k$ for all $n < \omega$. Let $x_n := \bigcup\{x^j_i \mid i, j < n\}$. Then $\langle x_n \mid n < \omega\rangle$ witnesses that $x \in \mathrm{IA}(\omega)$. □

We now give several definitions, propositions, and lemmas that are standard in the sense they have appeared in the literature in one form or another or are of simple technical nature. The exact terminology might be different from other texts or new, however.

2.1.3 Definition. Let $\mathbb{P}$ be a forcing. $q \in \mathbb{P}$ is called $(M, \mathbb{P})$-generic iff for every dense $D \subset \mathbb{P}$ with $D \in M$ the set $D \cap M$ is predense below q. ⌟

2.1.4 Definition. Let $\mathbb{P}$ be a forcing, E a class. $\mathbb{P}$ is *μ-proper for E* iff for all large enough regular θ there is a club $C \subset \mathfrak{P}_\mu H_\theta$ such that

$$C \cap E \subset \{M \in \mathfrak{P}_\mu H_\theta \mid \forall p \in \mathbb{P} \cap M\, \exists q \leq p\, (q \text{ is } (M, \mathbb{P})\text{-generic})\}.$$

$\mathbb{P}$ is *μ-proper* iff $\mathbb{P}$ is μ-proper for V. $\mathbb{P}$ is *proper* iff it is ω_1-proper. ⌟

Note that by Proposition 2.1.2, a forcing $\mathbb{P}$ is proper iff it is ω_1-proper for $\mathrm{IA}(\omega)$.

2.1.5 Proposition. *Let $\mathbb{P}$ be a forcing. If $\mathbb{P}$ is μ-cc, then it is μ-proper.*

Proof. Let θ be regular and large enough and let $M \in \mathfrak{P}'_\mu H_\theta$. Then any $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic because if $D \in M$ is dense in $\mathbb{P}$, then there is a maximal antichain $A \subset D$ such that $A \in M$. Since $\mathbb{P}$ is μ-cc, this means $|A| < \mu$ and thus $A \subset M$. Therefore $D \cap M \supset A \cap M = A$ is predense below q. □

2.1.6 Definition. Let $\mathbb{P}$ be a forcing. $\mathbb{P}$ satisfies the *μ-covering property* iff for V-generic $G \subset \mathbb{P}$ the class $\mathfrak{P}^V_\mu V$ is cofinal in $\mathfrak{P}^{V[G]}_\mu V$, that is, for every $x \in V[G]$ with $x \subset V$ and $|x| < \mu$ there is a $z \in \mathfrak{P}^V_\mu V$ such that $x \subset z$. ⌟

If a forcing $\mathbb{P}$ satisfies the μ-covering property, then in particular μ remains regular in $V^{\mathbb{P}}$.

2.1.7 Lemma. *Let $\mathbb{P}$ be a forcing, θ regular and large enough, and $M \prec H_\theta$ such that $\mathbb{P} \in M$. Suppose q is $(M, \mathbb{P})$-generic. Then*

$$q \Vdash M[\dot{G}] \cap V = M.$$

Proof. Let $G \subset \mathbb{P}$ be V-generic with $q \in G$. Let $\dot{x} \in M$ be such that $\dot{x}^G \in V$. Define

$$D := \{p \in \mathbb{P} \mid \exists y\, p \Vdash \dot{x} = y \ \vee \ p \Vdash \dot{x} \notin V\}.$$

Then $D \in M$ is dense, so $D \cap M$ is predense below q. Thus there exists a $p \in D \cap M \cap G$ and some y such that $p \Vdash \dot{x} = y$. But then y is definable from p and $\dot{x}$ in M, so $\dot{x}^G = y \in M$. □

2.1.8 Proposition. *Let $\mathbb{P}$ be a forcing. If $\mathbb{P}$ is μ-proper for $\mathrm{IA}(\omega)$, then $\mathbb{P}$ satisfies the μ-covering property.*

Proof. Let θ be regular and large enough, $p \in \mathbb{P}$ and $\dot{x} \in V^{\mathbb{P}}$ such that $p \Vdash \dot{x} \in \mathfrak{P}_\mu V$. By Proposition 2.1.2, there is an $M \in \mathfrak{P}'_\mu H_\theta \cap \mathrm{IA}(\omega)$ with $\dot{x}, p \in M$. Since $\mathbb{P}$ is μ-proper for $\mathrm{IA}(\omega)$, there is an $(M, \mathbb{P})$-generic $q \leq p$. Then $q \Vdash \dot{x} \in \mathfrak{P}_\mu(M[\dot{G}])$. By Lemma 2.1.7 $q \Vdash M[\dot{G}] \cap \mu = M \cap \mu < \mu$, so $q \Vdash \dot{x} \subset M[\dot{G}] \cap V = M$. □

2.1.9 Proposition. *Let $\mathbb{P}$ be a forcing. If $\mathbb{P}$ is μ-cc, then $\mathbb{P}$ satisfies the μ-covering property.*

Proof. This follows immediately from Propositions 2.1.5 and 2.1.8. □

2.1.10 Definition. Let $\mathbb{P}$ be a forcing. $\mathbb{P}$ satisfies the *thin μ-approximation property* iff the following holds. Suppose $\lambda \geq \mu$, $C \subset \mathfrak{P}_\mu \lambda$ club, and $T_z \in \mathfrak{P}_\mu(\mathfrak{P}z)$ for every $z \in C$. Then for V-generic $G \subset \mathbb{P}$ and $x \in V[G]$, $x \subset \lambda$, it holds that if $x \cap z \in T_z$ for all $z \in C$, then $x \in V$. ⌟

2.1.11 Definition. Let $\mathbb{P}$ be a forcing. $\mathbb{P}$ satisfies the *μ-approximation property* iff for V-generic $G \subset \mathbb{P}$ and $x \in V[G]$, $x \subset V$, it holds that if $x \cap z \in V$ for all $z \in \mathfrak{P}^V_\mu V$, then $x \in V$. ⌟

The following proposition is due to Silver, see [Kun80, chap. VIII, Lemma 3.4].

2.1.12 Proposition. *Let $\mathbb{P}$ be a forcing. If there is $\chi < \mu$ such that $2^\chi \geq \mu$ and $\mathbb{P}$ is χ^+-closed, then $\mathbb{P}$ satisfies the thin μ-approximation property.*

Proof. We may assume χ is minimal such that $2^\chi \geq \mu$.

Suppose to the contrary that for some λ and a club $C \subset \mathfrak{P}_\mu\lambda$, for each $z \in C$ there is $T_z \in \mathfrak{P}_\mu(\mathfrak{P}z)$ such that for some V-generic $G \subset \mathbb{P}$ there is an $x \in V[G]$, $x \subset \lambda$, such that $x \cap z \in T_z$ for all $z \in C$, but $x \notin V$. Let $\bar{p} \in G$ be such that $\bar{p} \Vdash \forall z \in C\ x \cap z \in T_z$ and $\bar{p} \Vdash x \notin V$.

Let $\dot{x}$ be a name for x and work in V. For $z \in C$ the set

$$D_z := \{p \in \mathbb{P} \mid \exists t \in T_z\ p \Vdash \dot{x} \cap z = t\}$$

is dense below $\bar{p}$. On the other hand, for every $q \leq \bar{p}$ there is $z(q) \in C$ such that for all $t \in T_{z(q)}$

$$q \nVdash \dot{x} \cap z(q) = t.$$

For otherwise for $\tilde{x} = \bigcup\{t \mid \exists z \in C\ q \Vdash \dot{x} \cap z = t\}$ we would have $q \Vdash \dot{x} = \tilde{x} \in V$, contradicting $q \leq \bar{p}$.

2.1.12.1 Claim. *For $q \leq \bar{p}$ and $z \in C$ with $z(q) \subset z$ there are $q_0, q_1 \leq q$ and $t_0, t_1 \in T_z$, $t_0 \neq t_1$, such that*

$$q_i \Vdash \dot{x} \cap z = t_i$$

for $i \in \{0, 1\}$.

Proof. Since D_z is dense below $\bar{p}$, there is $q_0 \leq q$ and t_0 such that $q_0 \Vdash \dot{x} \cap z = t_0$. As $q \nVdash \dot{x} \cap z = t_0$, there is an $r \leq q$ with $r \Vdash \dot{x} \cap z \neq t_0$. If $q_1 \leq r$ is in D_z, then for some $t_1 \neq t_0$ we have $q_1 \Vdash \dot{x} \cap z = t_1$, so q_0 and q_1 are as wanted. ⊣

For $s \in {}^{\leq\chi}2$ and $\delta \leq \chi$ we define $p_s \in \mathbb{P}$, t_s, and $z_\delta \in C$ such that the following holds.

(i) $\langle z_\delta \mid \delta \leq \chi \rangle$ is $\subset$-increasing and continuous,

(ii) if $s, s' \in {}^{\leq\chi}2$ and s is an initial segment of s', then $p_{s'} \leq p_s$,

(iii) $t_s \in T_{z_{\operatorname{dom} s}}$,

(iv) $t_{s\frown 0} \neq t_{s\frown 1}$ if $s \in {}^{<\chi}2$,

(v) $p_s \Vdash \dot{x} \cap z_{\operatorname{dom} s} = t_s$.

To see this can be done, let $\delta \leq \chi$ and suppose p_s, t_s and z_α have been defined for all $\alpha < \delta$ and $s \in {}^{<\delta}2$.

If $\delta = \beta + 1$, then let $z_\delta \in C$ such that $z_\delta \supset \bigcup\{z(p_s) \mid s \in {}^{<\delta}2\} \cup z_\beta$, which exists as $2^{<\delta} \leq 2^{<\chi} < \mu$ by assumption. By Claim 2.1.12.1, for every $s \in {}^{\beta}2$ there are $p_{s\frown 0}, p_{s\frown 1} \leq p_s$ and $t_{s\frown 0}, t_{s\frown 1} \in T_{z_\delta}$ such that (iv) and (v) are satisfied.

If δ is a limit ordinal, let $z_\delta := \bigcup\{z_\alpha \mid \alpha < \delta\}$, and for every $s \in {}^\delta 2$ let $\tilde{p}_s \in \mathbb{P}$ be such that $\tilde{p}_s \leq p_{s\restriction\alpha}$ for all $\alpha < \delta$, which exists because $\mathbb{P}$ is χ^+-closed. Let $p_s \in D_{z_\delta}$ with $p_s \leq \tilde{p}_s$. Then $p_s \Vdash \dot{x} \cap z_\delta = t_s$ for some $t_s \in T_{z_\delta}$.

If $s, s' \in {}^\chi 2$, $s \neq s'$, then $t_s \neq t_{s'}$. For let $\alpha < \chi$ be minimal such that $s \restriction (\alpha+1) \neq s' \restriction (\alpha+1)$. Then $p_s \Vdash t_s \cap z_{\alpha+1} = \dot{x} \cap z_\chi \cap z_{\alpha+1} = \dot{x} \cap z_{\alpha+1}$. But $p_{s\restriction(\alpha+1)} \Vdash \dot{x} \cap z_{\alpha+1} = t_{s\restriction(\alpha+1)}$, so $t_s \cap z_{\alpha+1} = t_{s\restriction(\alpha+1)}$. Likewise $t_{s'} \cap z_{\alpha+1} = t_{s'\restriction(\alpha+1)}$. By (iv) $t_{s\restriction(\alpha+1)} \neq t_{s'\restriction(\alpha+1)}$, so that $t_s \neq t_{s'}$. This is a contradiction because $\{t_s \mid s \in {}^\chi 2\} \subset T_{z_\chi}$, $|\{t_s \mid s \in {}^\chi 2\}| = 2^\chi \geq \mu$, and $|T_{z_\chi}| < \mu$. □

2.1.13 Proposition. *Let κ be regular uncountable, $\lambda \geq \kappa$, $\mathbb{P}$ κ-cc, $p \in \mathbb{P}$, and $\dot{C} \in V^{\mathbb{P}}$ such that $p \Vdash \dot{C} \subset \mathfrak{P}_\kappa\lambda$ club. Then there is a club $D \subset \mathfrak{P}_\kappa\lambda$ such that $p \Vdash D \subset \dot{C}$.*

Proof. Let $\dot{f}$ be such that $p \Vdash$ "$\dot{f} : \mathfrak{P}_\omega\lambda \to \mathfrak{P}_\kappa\lambda \wedge \mathrm{Cl}_{\dot{f}} \subset \dot{C}$". Define $g : \mathfrak{P}_\omega\lambda \to \mathfrak{P}_\kappa\lambda$ by

$$g(e) := \bigcup\{x \in \mathfrak{P}_\kappa\lambda \mid \exists q \leq p \; q \Vdash \dot{f}(e) = x\}.$$

Then $|g(e)| < \kappa$ by the regularity of κ and because $\mathbb{P}$ is κ-cc. For $e \in \mathfrak{P}_\omega\lambda$ we have $p \Vdash \dot{f}(e) \subset g(e)$. This means that for $x \in \mathrm{Cl}_g$ and $z \in \mathfrak{P}_\omega x$ we have $p \Vdash \dot{f}(z) \subset g(z) \subset x$. Hence $p \Vdash \mathrm{Cl}_g^V \subset \mathrm{Cl}_{\dot{f}} \subset \dot{C}$. □

2.1.14 Lemma. *Let $\kappa > \omega$ be regular, $\mathbb{P}_\kappa$ be the direct limit of an iteration $\langle \mathbb{P}_\nu \mid \nu < \kappa\rangle$. Suppose $\mathbb{P}_\kappa$ is κ-cc. Let $p \in \mathbb{P}_\kappa$ and $\dot{x} \in V^{\mathbb{P}_\kappa}$ such that $p \Vdash \dot{x} \in \mathfrak{P}_\kappa V$. Then there is $\rho < \kappa$ such that $p \Vdash \dot{x} \in V[\dot{G}_\rho]$.*

Proof. For $c \in V$ let $A_c \subset \mathbb{P}_\kappa$ be a maximal antichain below p that decides "$c \in \dot{x}$." Because $\mathbb{P}_\kappa$ satisfies the κ-covering property by Proposition 2.1.9, there is an $\varepsilon < \kappa$ and an $i : \varepsilon \to V$ such that $p \Vdash \dot{x} \subset \operatorname{rng} i$. Set

$$\dot{x}' := \{\langle \check{\ }(i(\nu)), a\rangle \mid \nu < \varepsilon,\; a \in A_{i(\nu)},\; a \Vdash i(\nu) \in \dot{x}\}.$$

Then $p \Vdash \dot{x} = \dot{x}'$, for let $G \subset \mathbb{P}_\kappa$ be V-generic with $p \in G$. Then

$$\begin{aligned} \delta \in \dot{x}^G &\leftrightarrow \exists a \in A_\delta \cap G \; a \Vdash \delta \in \dot{x} \\ &\leftrightarrow \exists \nu < \varepsilon \; \exists a \in A_{i(\nu)} \cap G \; (i(\nu) = \delta \wedge a \Vdash i(\nu) \in \dot{x}) \\ &\leftrightarrow \exists a \in G \; \langle \check{\delta}, a\rangle \in \dot{x}' \\ &\leftrightarrow \delta \in \dot{x}'^G. \end{aligned}$$

Since $\mathbb{P}_\kappa$ is direct limit and κ-cc, by the regularity of κ we have

$$\rho := \sup_{\nu<\varepsilon} \bigcup\{\operatorname{supp} a \mid a \in A_{i(\nu)}\} < \kappa.$$

But by definition $\dot{x}' \in V^{\mathbb{P}_\rho}$. □

2.1.15 Lemma. *Suppose η is regular uncountable and $\mathbb{P}_\eta$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta\rangle$. Also suppose $\{\nu < \eta \mid \mathbb{P}_\nu$ direct limit of $\langle \mathbb{P}_{\nu'} \mid \nu' < \nu\rangle\}$ is stationary in η. If $\mathbb{P}_\nu$ is η-cc for every $\nu < \eta$, then $\mathbb{P}_\eta$ is η-cc.*

The proof of Lemma 2.1.15 can be found in [Bau83, Theorem 2.2] or [Jec03, Theorem 16.30].

2.2 Forcing Constructions

We now describe a forcing construction that is originally due to Mitchell [Mit73]. The presentation follows [Kru08]. The reader should note that we use the convention that conditions are only defined on their support. We still write $p(\gamma) = \mathbf{1}$ to indicate $\gamma \notin \operatorname{supp} p$ though.

Let $\mathbb{C}$ denote the forcing for adding a Cohen real, and let $\operatorname{Coll}(\delta, \gamma)$ denote the forcing for collapsing γ onto δ. Suppose $C \subset \operatorname{Lim}$ is a set and τ is a regular uncountable cardinal. Let $\mathcal{L} := \{\alpha + 1 \mid \alpha \in C\}$ and $\zeta := \sup\{\alpha + 1 \mid \alpha \in \mathcal{L}\}$. We define an iterated forcing $\langle \mathbb{P}_\nu(C, \tau), \dot{\mathbb{Q}}_\gamma(C, \tau) \mid \nu \leq \zeta,\ \gamma < \zeta \rangle$ by the following four conditions. For the sake of brevity, let $\mathbb{P}_\nu := \mathbb{P}_\nu(C, \tau)$ and $\dot{\mathbb{Q}}_\gamma := \dot{\mathbb{Q}}_\gamma(C, \tau)$.

1. If $\gamma \in C$, then $\Vdash_\gamma \dot{\mathbb{Q}}_\gamma = \mathbb{C}$,
2. if $\gamma \in \mathcal{L}$, then $\Vdash_\gamma \dot{\mathbb{Q}}_\gamma = \operatorname{Coll}(\tau, \gamma)$,
3. if $\gamma \in \zeta - (C \cup \mathcal{L})$, then $\Vdash_\gamma \dot{\mathbb{Q}}_\gamma = \{\mathbf{1}\}$,
4. for $\nu \leq \zeta$ and $p \in \mathbb{P}_\nu$ it holds that
   - $|\operatorname{supp} p \cap C| < \omega$,
   - $|\operatorname{supp} p \cap \mathcal{L}| < \tau$.

We define an ordering $\leq^* \subset \leq$ on $\mathbb{P}_\nu$ by

$$p \leq^* q :\leftrightarrow (p \leq q \ \wedge\ \forall \gamma \in C \cap \nu\ p \upharpoonright \gamma \Vdash_\gamma p(\gamma) = q(\gamma)),$$

and write $\mathbb{P}^*_\nu$ for $\langle \mathbb{P}_\nu, \leq^* \rangle$. Furthermore let

$$\mathbb{P}'_\nu := \{p \in \mathbb{P}_\nu \mid \forall \alpha \in C \cap \nu\ \exists x\ p(\alpha) = \check{x}\}.$$

2.2.1 Lemma. *For $\nu \leq \zeta$, $\mathbb{P}^*_\nu$ is τ-closed.*

Proof. Let $\langle p_\alpha \mid \alpha < \delta \rangle$ be a decreasing sequence of conditions in $\mathbb{P}^*_\nu$ for some $\delta < \tau$. Let

$$S := \bigcup \{\operatorname{supp} p_\alpha \cap \mathcal{L} \mid \alpha < \delta\}.$$

Then $|S| < \tau$.

We recursively define p. So suppose $p \upharpoonright \gamma$ is given, $p \upharpoonright \gamma \leq^* p_\alpha \upharpoonright \gamma$ for all $\alpha < \delta$. If $\gamma \notin S$, set $p(\gamma) := p_0(\gamma)$. If $\gamma \in S$, then because $\Vdash_\gamma$ "$\dot{\mathbb{Q}}_\gamma = \operatorname{Coll}(\tau, \gamma)$ is τ-closed" and $p \upharpoonright \gamma \leq^* p_\alpha \upharpoonright \gamma$ for $\alpha < \delta$ there is an $\dot{r}$ such that $\Vdash_\gamma \dot{r} \in \dot{\mathbb{Q}}_\gamma$ and

$$p \upharpoonright \gamma \Vdash_\gamma \forall \alpha < \delta\ \dot{r} \leq p_\alpha(\gamma).$$

Set $p(\gamma)$ to be this $\dot{r}$. □

2.2.2 Lemma. *For $\nu \leq \zeta$, $\mathbb{P}'_\nu$ is dense in $\mathbb{P}_\nu$.*

Proof. Let $\gamma < \nu$ and suppose the statement is true for all $\beta < \gamma$. Let $p \in \mathbb{P}_\gamma$.

Suppose γ is a limit ordinal. Then there is $\beta < \gamma$ such that $\operatorname{supp} p \cap C \subset \beta$. Hence there is $\tilde{p} \in \mathbb{P}_\beta$ with $\tilde{p} \leq p \upharpoonright \beta$. If we set $q := \tilde{p} \cup p \upharpoonright [\beta, \gamma)$, then $q \leq p$ and $q \in \mathbb{P}'_\gamma$.

Suppose $\gamma = \beta + 1$ with $\beta \in C$. Since $\Vdash_\beta p(\beta) \in \mathbb{C}$, there are an x and $\bar{p} \in \mathbb{P}_\beta$ such that $\bar{p} \leq p \upharpoonright \beta$ and $\bar{p} \Vdash_\beta p(\beta) = x$. Let $\tilde{p} \in \mathbb{P}'_\beta$ such that $\tilde{p} \leq \bar{p}$. Then for $q := \tilde{p} \frown \langle \check{x} \rangle$ we have $q \leq p$ and $q \in \mathbb{P}'_\gamma$. □

The next lemma is very strong in that it directly implies important features of our forcing iteration $\langle \mathbb{P}_\nu \mid \nu \leq \zeta \rangle$.

2.2.3 Lemma. *Let $\nu \leq \zeta$. Suppose $M \in \mathfrak{P}_\tau H_\theta \cap \mathrm{IA}(\omega)$ for some large enough regular θ is such that $\mathbb{P}_\nu, C \in M$. Let $p \in \mathbb{P}'_\nu \cap M$ and suppose $\dot{X} \in V^{\mathbb{P}_\nu} \cap M$ is such that $p \Vdash_\nu \dot{X} \subset V \wedge \dot{X} \notin V$. Then there is $q \in \mathbb{P}_\nu$ such that $q \leq^* p$ and for every $r \leq q$ and $D \in M$ that is dense open in $\mathbb{P}_\nu$ there are $y_0, y_1 \in D \cap M$ and $x \in M$ with*

(i) y_0 *and* y_1 *are compatible with* r*,*

(ii) $y_0 \Vdash_\nu x \in \dot{X}$*,*

(iii) $y_1 \Vdash_\nu x \notin \dot{X}$*.*

Proof. There is a $q \in \mathbb{P}_\nu$ such that if $D \in M$ is dense in $\mathbb{P}^*_\nu$, then

$$\exists d \in D \cap M \; q \leq^* d \leq^* p. \tag{2.1}$$

To see this, let $\langle M_n \mid n < \omega \rangle$ witness $M \in \mathrm{IA}(\omega)$. Let $\mathcal{D}_n := \{D \in M_n \mid D \subset \mathbb{P}^*_\nu \text{ dense}\} \in M$. We recursively define a $\leq^*$-decreasing sequence $\langle q_n \mid n < \omega \rangle$ such that $q_n \in M$ and $q_n \leq^* p$ for all $n < \omega$. Suppose q_m has been defined for all $m < n$. For $D \in \mathcal{D}_n$, let $\bar{D} := \{r \in \mathbb{P}^*_\nu \mid \exists d \in D \; r \leq^* d\}$. Then $\bar{D}$ is dense open. The forcing $\mathbb{P}^*_\nu$ is τ-closed and thus in particular τ-distributive. Since $|\mathcal{D}_n| < \tau$, this means $F := \bigcap\{\bar{D} \mid D \in \mathcal{D}_n\}$ is dense open. Note that $F \in M$. So we can take $q_n \in F \cap M$ such that $q_n \leq^* p$ and $q_n \leq^* q_m$ for all $m < n$. If q is a lower bound for the sequence $\langle q_n \mid n < \omega \rangle$, then it satisfies (2.1).

We now show that this q satisfies the claim of the lemma. So let $r \leq q$, without loss $r \in \mathbb{P}'_\nu$, and let $D \in M$ be dense open in $\mathbb{P}_\nu$. Because $r \in \mathbb{P}'_\nu$, $r \upharpoonright (M \cap C) \in M$. Define $E \subset \mathbb{P}^*_\nu$ by $e \in E$ iff e and p are $\leq^*$-incompatible or $e \leq^* p$ and there are $y_0, y_1 \in D \cap \mathbb{P}'_\nu$ such that

(a) $\forall \gamma \in M \cap C \; \Vdash_\gamma y_0(\gamma), y_1(\gamma) \leq r(\gamma)$,

(b) $y_0 \upharpoonright \mathcal{L} = y_1 \upharpoonright \mathcal{L} = e \upharpoonright \mathcal{L}$,

(c) $\exists x \, (y_0 \Vdash_\nu x \in \dot{X} \wedge y_1 \Vdash_\nu x \notin \dot{X})$.

Then $E \in M$.

Suppose E is dense in $\mathbb{P}^*_\nu$. Then by (2.1) there is an $e \in E \cap M$ such that $q \leq^* e \leq^* p$. Since $e \leq^* p$ there are $y_0, y_1, x \in M$ that satisfy (a), (b), (c). Thus y_0 satisfies (ii) and y_1 satisfies (iii). To prove (i), suppose y_i and r are incompatible for either $i = 0$ or $i = 1$. Let γ be minimal such that $r \upharpoonright \gamma \not\Vdash_\gamma$ "$r(\gamma)$ and $y_i(\gamma)$ are compatible". If $\gamma \in C$, then $\gamma \in \operatorname{supp} y_i \cap C \subset M$, so (a) gives a contradiction. Thus $\gamma \in \mathcal{L}$. But because of $r \leq q \leq e$ and (b) we have $r \upharpoonright \gamma \Vdash_\gamma r(\gamma) \leq e(\gamma) = y_i(\gamma)$, which again is a contradiction.

So it remains to show that E is dense in $\mathbb{P}^*_\nu$. Let $s \in \mathbb{P}^*_\nu$ be $\leq^*$-compatible with p, and let $t \leq^* p, s$. Define

$$w := r \upharpoonright (M \cap C) \cup t \upharpoonright \mathcal{L}.$$

2.2.3.1 Claim. *It holds that $w \leq t$.*

Proof. Let $\gamma < \nu$ and suppose $w \upharpoonright \gamma \leq t \upharpoonright \gamma$.

- 1st case: $\gamma \in M \cap C$.

  We have $r \leq q \leq p$, so $r \upharpoonright \gamma \Vdash_\gamma r(\gamma) \leq p(\gamma)$. Since $r, p \in \mathbb{P}'_\nu$, $r(\gamma)$ and $p(\gamma)$ are canonical names and therefore $\Vdash_\gamma r(\gamma) \leq p(\gamma)$, so $\Vdash_\gamma w(\gamma) = r(\gamma) \leq p(\gamma)$. Also $t \leq^* p$, so $t \upharpoonright \gamma \Vdash_\gamma t(\gamma) = p(\gamma)$ and thus $w \upharpoonright \gamma \Vdash_\gamma w(\gamma) \leq t(\gamma)$.

- 2nd case: $\gamma \in \mathcal{L}$.

  Then $w \upharpoonright \gamma \Vdash_\gamma w(\gamma) = t(\gamma)$.

- 3rd case: $\gamma \in \nu - ((M \cap C) \cup \mathcal{L})$.

  Because $p \in M$ we have $\operatorname{supp} t \cap C = \operatorname{supp} p \cap C \subset M \cap C$, so $\gamma \notin \operatorname{supp} t$ and thus $w \upharpoonright \gamma \Vdash_\gamma w(\gamma) = \mathbf{1} = t(\gamma)$. ⊣

Let $z \in D$, $z \leq w$. Then $z \Vdash_\nu$ "$\dot{X} \subset V \ \wedge \ \dot{X} \notin V$" because $z \leq p$. Thus there exist an x and $\tilde{z}_0, \tilde{z}_1 \leq z$ such that $\tilde{z}_0 \Vdash_\nu x \in \dot{X}$ and $\tilde{z}_1 \Vdash_\nu x \notin \dot{X}$.

2.2.3.2 Claim. *There are $z_0, z_1 \in \mathbb{P}'_\nu$ such that $z_0 \leq \tilde{z}_0$, $z_1 \leq \tilde{z}_1$ and*

$$\forall \gamma \in \mathcal{L} \ \Vdash_\gamma z_0(\gamma), z_1(\gamma) \leq t(\gamma).$$

Proof. We can assume $\tilde{z}_0, \tilde{z}_1 \in \mathbb{P}'_\nu$. For $i \in \{0, 1\}$ we define z_i as follows. For $\gamma \in \nu - \mathcal{L}$, simply let $z_i(\gamma) := \tilde{z}_i(\gamma)$. For $\gamma \in \mathcal{L}$, let $z_i(\gamma)$ be such that $\tilde{z}_i \upharpoonright \gamma \Vdash_\gamma \tilde{z}_i(\gamma) = z_i(\gamma)$ and for all $u \in \mathbb{P}_\gamma$ that are incompatible with $\tilde{z}_i \upharpoonright \gamma$ we have $u \Vdash_\gamma z_i(\gamma) = t(\gamma)$. ⊣

Let z_0, z_1 be as in Claim 2.2.3.2.

2.2.3.3 Claim. *There are $y_0, y_1 \in D \cap \mathbb{P}'_\nu$ which satisfy (a) and (c). Furthermore for $\gamma \in \mathcal{L}$ $\Vdash_\gamma y_0(\gamma) = y_1(\gamma) \in \{z_0(\gamma), z_1(\gamma)\}$.*

Proof. For $\gamma < \min C$ set $y_0(\gamma) := y_1(\gamma) := \mathbf{1}$. Choose incompatible $y_0(\min C), y_1(\min C) \in \mathbb{C}$ such that $y_0(\min C) \leq z_0(\min C)$ and $y_1(\min C) \leq z_1(\min C)$.

Now suppose $y_0 \upharpoonright \gamma$ and $y_1 \upharpoonright \gamma$ have been defined. If $\gamma \notin \mathcal{L}$, let $y_0(\gamma) := z_0(\gamma)$ and $y_1(\gamma) := z_1(\gamma)$. For $\gamma \in \mathcal{L}$, let $\dot{a}$ be such that $y_0 \upharpoonright \gamma \Vdash_\gamma \dot{a} = z_0(\gamma)$ and for any $u \in \mathbb{P}_\gamma$ that is incompatible with $y_0 \upharpoonright \gamma$ it holds that $u \Vdash_\gamma \dot{a} = z_1(\gamma)$. Let $y_0(\gamma) := y_1(\gamma) := \dot{a}$.

Then $y_0 \leq z_0$, $y_1 \leq z_1$, $y_0 \upharpoonright \mathcal{L} = y_1 \upharpoonright \mathcal{L}$, and for any $\gamma \in C$

$$\Vdash_\gamma y_i(\gamma) = z_i(\gamma) \leq w(\gamma) = r(\gamma)$$

for $i \in \{0, 1\}$. ⊣

Let

$$e := t \upharpoonright C \cup y_0 \upharpoonright \mathcal{L}.$$

Then $e \leq^* t$, for if $\gamma \in C$, then $\Vdash_\gamma e(\gamma) = t(\gamma)$, and if $\gamma \in \mathcal{L}$, then by Claims 2.2.3.2 and 2.2.3.3 $\Vdash_\gamma \exists i \in \{0, 1\}\ e(\gamma) = y_0(\gamma) = z_i(\gamma) \leq t(\gamma)$.

So $e \leq^* t \leq^* s$. As e, y_0, y_1 satisfy (b), we have shown that E is dense. □

2.2.4 Proposition. *For $\nu \leq \zeta$, $\mathbb{P}_\nu$ is μ-proper for* IA(ω) *for every $\mu \in [\omega_1, \tau]$.*

Proof. Suppose $M \in \mathfrak{P}_\mu H_\theta \cap \mathrm{IA}(\omega)$ for some large enough regular θ with $\mathbb{P}_\nu, C \in M$ and $p \in \mathbb{P}_\nu \cap M$. Let $p' \leq p$ be such that $p' \in \mathbb{P}'_\nu \cap M$. Let $\dot{X} := \dot{G}$. Then by Lemma 2.2.3 there is $q \leq^* p'$ which is $(M, \mathbb{P}_\nu)$-generic. □

2.2.5 Proposition. *For $\nu \leq \zeta$, $\mathbb{P}_\nu$ satisfies the ω_1-approximation property.*

Proof. Suppose to the contrary that there is $p \in \mathbb{P}_\nu$ and $\dot{X} \in V^{\mathbb{P}_\nu}$ such that

$$p \Vdash_\nu \dot{X} \subset V \ \wedge\ \dot{X} \notin V \ \wedge\ \forall A \in \mathfrak{P}^V_{\omega_1} V\ \dot{X} \cap A \in V.$$

We can assume $p \in \mathbb{P}'_\nu$. Let θ be regular and large enough and, by Proposition 2.1.2, let $M \in \mathfrak{P}_{\omega_1} H_\theta \cap \mathrm{IA}(\omega)$ be such that $\dot{X}, \mathbb{P}_\nu, C, p \in M$. Let $q \leq^* p$ be as in Lemma 2.2.3. As $M \in \mathfrak{P}^V_{\omega_1} V$, there are $r \leq q$ and Y such that

$$r \Vdash_\nu \dot{X} \cap M = Y.$$

Thus, for $D = \mathbb{P}_\nu$, there are $y_0, y_1 \in M$ which are both compatible with r and $x \in M$ such that $y_0 \Vdash_\nu x \in \dot{X}$ and $y_1 \Vdash_\nu x \notin \dot{X}$. So there are $s_0 \leq y_0, r$ and $s_1 \leq y_1, r$. But then $s_0 \Vdash_\nu x \in Y$ and $s_1 \Vdash_\nu x \notin Y$, a contradiction. □

2.2.6 Theorem. *Let τ be a regular uncountable cardinal and $\gamma \geq \tau$ be a cardinal. Then the forcing $\mathbb{C} * \mathrm{Coll}(\tau, \gamma)$ is μ-proper for* IA(ω) *for every $\mu \in [\omega_1, \tau]$ and satisfies the ω_1-approximation property.*

Proof. Let $C := \{\gamma\}$. Then $\zeta = \gamma + 2$, and $\mathbb{P}_\zeta(C, \tau)$ is—up to a trivial isomorphism—the forcing $\mathbb{C} * \mathrm{Coll}(\tau, \gamma)$. Therefore it is μ-proper for $\mathrm{IA}(\omega)$ for every $\mu \in [\omega_1, \tau]$ and satisfies the ω_1-approximation property by Propositions 2.2.4 and 2.2.5. $\square$

Theorem 2.2.6 will be used in Chapter 3. While $\mathrm{Coll}(\omega_1, \gamma)$ only satisfies the thin ω_1-approximation property under PFA, $\mathbb{C} * \mathrm{Coll}(\omega_1, \gamma)$ satisfies the ω_1-approximation property and thus allows us to treat slenderness as well.

2.2.7 Proposition. *Let $\eta \in (\tau, \zeta]$ be inaccessible. Then $\mathbb{P}_\eta$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta \rangle$ and is η-cc.*

Proof. $\mathbb{P}_\nu$ is the direct limit of $\langle \mathbb{P}_{\nu'} \mid \nu' < \nu \rangle$ for every $\nu \leq \eta$ with $\mathrm{cf}\, \nu \geq \tau$. Since $|\mathbb{P}_\nu| < \eta$ for $\nu < \eta$, every $\mathbb{P}_\nu$ is η-cc. Thus $\mathbb{P}_\eta$ is η-cc by Lemma 2.1.15. $\square$

2.2.8 Theorem. *Let κ be inaccessible, $\tau < \kappa$ be regular and uncountable. Then there exists an iteration $\langle \mathbb{P}_\nu \mid \nu \leq \kappa \rangle$ such that $\Vdash_\kappa \kappa = \tau^+$ and for $\eta = 0$ and every inaccessible $\eta \leq \kappa$*

(i) $\mathbb{P}_\eta$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta \rangle$ and η-cc,

(ii) if $\mathbb{P}_\kappa = \mathbb{P}_\eta * \dot{\mathbb{Q}}$, then $\Vdash_\eta \dot{\mathbb{Q}}$ satisfies the ω_1-approximation property,

(iii) for every $\nu < \eta$, $\mathbb{P}_\nu$ is definable in H_η from the parameters τ and ν,

(iv) $\mathbb{P}_\eta$ is μ-proper for $\mathrm{IA}(\omega)$ *for every $\mu \in [\omega_1, \tau]$.*

Proof. Let $C := \{\gamma < \kappa \mid \gamma \text{ regular}\}$. Then $\zeta = \sup C = \kappa$.

Proposition 2.2.7 implies (i), (iii) follows from the definition of $\mathbb{P}_\kappa$, and (iv) follows from Proposition 2.2.4.

Furthermore $\Vdash_\kappa \kappa = \tau^+$, for if $\xi < \kappa$, then for some regular $\nu \geq \xi$ we have $\Vdash_{\nu+1} \dot{\mathbb{Q}}_{\nu+1} = \mathrm{Coll}(\tau, \nu + 1)$, so $\Vdash_\kappa |\xi| \leq \tau$ and thus $\Vdash_\kappa \kappa \leq \tau^+$. Since $\mathbb{P}_\kappa$ is κ-cc, κ is not collapsed and so $\Vdash_\kappa \kappa \geq \tau^+$.

To verify (ii), let $G_\eta \subset \mathbb{P}_\eta$ be V-generic and work in $V[G_\eta]$. But then $\dot{\mathbb{Q}}^{G_\eta} = \mathbb{P}_\kappa(C - \eta, \tau)$. Thus it satisfies the ω_1-approximation property by Proposition 2.2.5. $\square$

Theorem 2.2.8 is the basis for Section 2.3. Theorems 2.3.1 and 2.3.3 should be read with it in mind.

2.3 Preservation Theorems and Upper Bounds

The following theorem is the logical continuation of Mitchell's result that the tree property is preserved by the forcing from Section 2.2. As with the original result, it is more natural to formulate it for slenderness. It is probably helpful to the reader to understand Theorem 2.3.1 prior to Theorem 2.3.3 as it features the same structure and ideas and is less clouded by technicalities and additional problems that arise in the two cardinal case.

2.3.1 Theorem. *Let κ be a cardinal, $\tau < \kappa$ regular and uncountable, and $\langle \mathbb{P}_\nu \mid \nu \leq \kappa \rangle$ be an iteration such that for all inaccessible $\eta \leq \kappa$*

(i) $\mathbb{P}_\eta$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta \rangle$ and η-cc,

(ii) if $\mathbb{P}_\kappa = \mathbb{P}_\eta * \dot{\mathbb{Q}}$, then $\Vdash_\eta \dot{\mathbb{Q}}$ satisfies the ω_1-approximation property,

(iii) for every $\nu < \eta$, $\mathbb{P}_\nu$ is definable in H_η from the parameters τ and ν.

If κ is subtle, then $\Vdash_\kappa \kappa$-SSP, and if κ is ineffable, then $\Vdash_\kappa \kappa$-ISP.

Proof. Let $G \subset \mathbb{P}$ be V-generic and work in $V[G]$. Let $\langle d_\alpha \mid \alpha < \kappa \rangle$ be a slender κ-list and let $C' \subset \kappa - \tau$ be a club that witnesses the slenderness.

2.3.1.1 Claim. *There is a club $C \subset C'$ such that for all $\gamma \in C$*

$$\{d_\alpha \mid \alpha < \gamma\} \subset V[G_\gamma].$$

Proof. Let

$$g(\alpha) := \min\{\rho < \kappa \mid d_\alpha \in V[G_\rho]\}.$$

Then $g(\alpha) < \kappa$ for every $\alpha < \kappa$ by Lemma 2.1.14. Thus there is a club $C \subset C'$ such that $g''\gamma \subset \gamma$ for all $\gamma \in C$. ⊣

Let C be as in Claim 2.3.1.1. We may assume $C \in V$ by Proposition 2.1.13. Let

$$E := \{\eta \in C \mid \eta \text{ inaccessible in } V\}.$$

2.3.1.2 Claim. *If $\eta \in E$, then $d_\eta \in V[G_\eta]$.*

Proof. Let $\mathbb{P}_\kappa = \mathbb{P}_\eta * \dot{\mathbb{Q}}$. If $z \in \mathfrak{P}_{\omega_1}^{V[G_\eta]}\eta$, then $\delta := \sup z < \eta$, so as $\eta \in C'$ there is a $\beta < \eta$ such that $d_\eta \cap z = d_\eta \cap \delta \cap z = d_\beta \cap \delta \cap z = d_\beta \cap z \in V[G_\eta]$. Thus, since by (ii) $V[G_\eta] \models$ "$\dot{\mathbb{Q}}^{G_\eta}$ satisfies the ω_1-approximation property", we have $d_\eta \in V[G_\eta]$. ⊣

If $\eta < \kappa$ is inaccessible in V, by (i) and (iii) $\mathbb{P}_\eta \subset H_\eta^V$. By Claim 2.3.1.2, for $\eta \in E$ there is a $\mathbb{P}_\eta$-name $\dot{d}_\eta$ for d_η. Define, in V,

$$D_\eta := \{\langle p, \alpha, n \rangle \mid p \in \mathbb{P}_\eta,\ \alpha < \eta,\ (n = 0 \wedge p \Vdash_\eta \alpha \notin \dot{d}_\eta) \vee (n = 1 \wedge p \Vdash_\eta \alpha \in \dot{d}_\eta)\}$$

for $\eta \in E$. Then $\langle d_\eta \mid \eta \in E \rangle \in V$ and $D_\eta \subset H_\eta^V$.

Let

$$i : H_\kappa^V \to \kappa$$

be a bijection in V such that $i''H_\eta^V = \eta$ for all $\eta < \kappa$ inaccessible in V.

2.3.1.3 Claim. *If $\eta, \eta' \in E$, $\eta < \eta'$, and $i''D_\eta = i''D_{\eta'} \cap \eta$, then $d_\eta = d_{\eta'} \cap \eta$.*

Proof. If $\alpha \in d_\eta$, then there is a $p \in G$ such that $p \Vdash_\kappa \alpha \in \dot{d}_\eta$. Since $\dot{d}_\eta \in V^{\mathbb{P}_\eta}$, also $p \restriction \eta \Vdash_\eta \alpha \in \dot{d}_\eta$. Therefore $\langle p \restriction \eta, \alpha, 1 \rangle \in D_\eta = D_{\eta'} \cap H^V_\eta$, so $p \restriction \eta \Vdash_{\eta'} \alpha \in \dot{d}_{\eta'}$ and thus $\alpha \in d_{\eta'}$.

By the same argument, if $\alpha < \eta$ and $\alpha \notin d_\eta$, then $\alpha \notin d_{\eta'}$. Therefore $d_\eta = d_{\eta'} \cap \eta$. ⊣

By Proposition 1.2.13, $E \in F^V_{\mathrm{AT}}[\kappa]$. So if κ is subtle in V, then there are $\eta, \eta' \in E$, $\eta < \eta'$, with $i''D_\eta = i''D_{\eta'} \cap \eta$. Thus $d_\eta = d_{\eta'} \cap \eta$ by Claim 2.3.1.3, which means $\langle d_\alpha \mid \alpha < \kappa \rangle$ is not unsubtle.

If κ is ineffable in V, then in V there are a stationary $S \subset E$ and a $D \subset \kappa$ such that $i''D_\eta = D \cap \eta$ for all $\eta \in S$. S remains stationary in $V[G]$ by Proposition 2.1.13, so $\langle d_\alpha \mid \alpha < \kappa \rangle$ is not effable by Claim 2.3.1.3. □

2.3.2 Theorem. *If the theory* ZFC + *"there is a subtle cardinal" is consistent, then the theory* ZFC + ω_2-SSP *is consistent. If the theory* ZFC + *"there is an ineffable cardinal" is consistent, then the theory* ZFC + ω_2-ISP *is consistent.*

Proof. Taking $\tau = \omega_1$, this follows immediately from Theorems 2.2.8 and 2.3.1. □

The next theorem is the two cardinal version of Theorem 2.3.1. The two cardinal version of slenderness now pays off as the consideration of (κ, λ)-ISP greatly clarified its proof. In fact the proof of Theorem 2.3.3 was the main motivation for the generalization of slenderness to $\mathfrak{P}_\kappa\lambda$.

2.3.3 Theorem. *Let κ, λ be cardinals, τ regular uncountable, $\tau < \kappa \leq \lambda$, and $\langle \mathbb{P}_\nu \mid \nu \leq \kappa \rangle$ be an iteration such that for all inaccessible $\eta \leq \kappa$*

(i) $\mathbb{P}_\eta$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta \rangle$ and η-cc,

(ii) if $\mathbb{P}_\kappa = \mathbb{P}_\eta * \dot{\mathbb{Q}}$, then $\Vdash_\eta \dot{\mathbb{Q}}$ satisfies the ω_1-approximation property,

(iii) for every $\nu < \eta$, $\mathbb{P}_\nu$ is definable in H_η from the parameters τ and ν,

(iv) $\mathbb{P}_\eta$ satisfies the ω_1-covering property.

*Suppose κ is $\lambda^{<\kappa}$-ineffable. Then $\Vdash_\kappa (\kappa, \lambda)$-*ISP.

Proof. By Proposition 1.3.11, the case $\lambda = \kappa$ is already covered by Theorem 2.3.1, so we may assume $\lambda > \kappa$. Let $G \subset \mathbb{P}_\kappa$ be V-generic and work in $V[G]$. Let $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ be a slender $\mathfrak{P}_\kappa\lambda$-list, and let $C' \subset \mathfrak{P}_\kappa H_\theta$ be a club witnessing the slenderness of $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ for some large enough θ.

2.3.3.1 Claim. *There is a club $C \subset C'$ such that for all $M \in C$*

$$\mathfrak{P}_\kappa\lambda \cap M \subset V[G_{\kappa_M}].$$

Proof. For $x \in \mathfrak{P}_\kappa\lambda$ by Lemma 2.1.14 there is $\rho_x < \kappa$ such that $x \in V[G_{\rho_x}]$. Thus

$$C := \{M \in C' \mid \forall x \in \mathfrak{P}_\kappa\lambda \cap M\ \rho_x \in M\}$$

is as wanted. ⊣

Let C be as in Claim 2.3.3.1. Let $\sigma := (\lambda^{<\kappa})^V$. Let $\bar{M} \in V$ be such that $\bar{M} \prec H_\theta^V$, $\lambda \cup \mathfrak{P}_\kappa^V\lambda \subset \bar{M}$, $|\bar{M}|^V = \sigma$. Let $C_0 := C \upharpoonright \bar{M}$. By Proposition 2.1.13, there is a $C_1 \in V$ such that $C_1 \subset C_0$ and $V \models C_1 \subset \mathfrak{P}_\kappa\bar{M}$ club. Let

$$E := \{M \in C_1 \mid \kappa_M \text{ inaccessible in } V,\ \mathfrak{P}_{\omega_1}^V(M \cap \lambda) \subset M\}.$$

2.3.3.2 Claim. *If $M \in E$, then $d_{M\cap\lambda} \in V[G_{\kappa_M}]$.*

Proof. Let $z \in \mathfrak{P}_{\omega_1}^{V[G_{\kappa_M}]}(M \cap \lambda)$. $\mathbb{P}_{\kappa_M}$ satisfies the ω_1-covering property by (iv), so there is $b \in \mathfrak{P}_{\omega_1}^V(M \cap \lambda)$ such that $z \subset b$. Let $M' \in C$ be such that $M = M' \cap \bar{M}$. Then $b \in M \subset M'$. Therefore, by the slenderness of $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$, $d_{M\cap\lambda} \cap b = d_{M'\cap\lambda} \cap b \in \mathfrak{P}_\kappa\lambda \cap M' \subset V[G_{\kappa_{M'}}] = V[G_{\kappa_M}]$ and thus

$$d_{M\cap\lambda} \cap z = d_{M\cap\lambda} \cap b \cap z \in V[G_{\kappa_M}].$$

Let $\mathbb{P}_\kappa = \mathbb{P}_{\kappa_M} * \dot{\mathbb{Q}}$. Then $\dot{\mathbb{Q}}^{G_{\kappa_M}}$ satisfies the ω_1-approximation property by (ii), so $d_{M\cap\lambda} \in V[G_{\kappa_M}]$. ⊣

For $M \in E$ we have $\mathbb{P}_{\kappa_M} \subset M$ by (i) and (iii). By Claim 2.3.3.2 $d_{M\cap\lambda} \in V[G_{\kappa_M}]$, so there is $\dot{d}_M \in V^{\mathbb{P}_{\kappa_M}}$ such that $\dot{d}_M^{G_{\kappa_M}} = d_{M\cap\lambda}$. Let

$$D_M := \{\langle p, \alpha, n\rangle \mid p \in \mathbb{P}_{\kappa_M},\ \alpha \in M \cap \lambda,\ (n = 0 \wedge p \Vdash_{\kappa_M} \alpha \notin \dot{d}_M) \vee (n = 1 \wedge p \Vdash_{\kappa_M} \alpha \in \dot{d}_M)\}.$$

Then $\langle D_M \mid M \in E\rangle \in V$ and $D_M \subset M$.

Work in V. Let $f : \bar{M} \to \sigma$ be a bijection such that $f \upharpoonright \kappa = \mathrm{id} \upharpoonright \kappa$. By Propositions 1.5.1 and 1.5.2

$$F := \{m \in \mathfrak{P}_\kappa'\sigma \mid \kappa_m \text{ inaccessible},\ \mathfrak{P}_{\omega_1}(m \cap f''\lambda) \subset m\} \in F_{\mathrm{IT}}[\kappa, \sigma].$$

As κ is σ-ineffable, there exist a stationary $S' \subset \mathfrak{P}_\kappa\sigma$ and $d' \subset \sigma$ such that $f''D_{f^{-1}{}''m} = d' \cap m$ for all $m \in S'$ such that $f^{-1}{}''m \in E$. But $E = \{f^{-1}{}''m \mid m \in F\} \cap C_1$, so for $S := \{f^{-1}{}''m \mid m \in S' \cap F\} \cap C_1$ and for $D := f^{-1}{}''d'$ we have

$$D_M = D \cap M$$

for all $M \in S$.

Back in $V[G]$, let $T := S \upharpoonright \lambda$ and

$$d := \{\alpha < \lambda \mid \exists p \in G\ \langle p, \alpha, 1\rangle \in D\}.$$

2.3.3.3 Claim. *If $a \in T$, then $d_a = d \cap a$.*

Proof. If $a \in T$, then $a = M \cap \lambda$ for some $M \in S$. But then for $\alpha \in a$, if $\alpha \in d_a = d_{M\cap\lambda} = \dot{d}_M^{G_{\kappa_M}}$, then there is $p \in G_{\kappa_M}$ such that $p \Vdash_{\kappa_M} \alpha \in \dot{d}_M$. Thus $\langle p, \alpha, 1\rangle \in D_M = D \cap M$, so that $\alpha \in d$ by the definition of d.

By the same argument, if $\alpha \notin d_a$, then $\alpha \notin d$. ⊣

T is stationary in V, so by Proposition 2.1.13 it is also stationary in $V[G]$. Therefore by Claim 2.3.3.3, $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is not effable. □

If $\operatorname{cf}\lambda \geq \kappa$, the condition that κ is $\lambda^{<\kappa}$-ineffable in Theorem 2.3.3 can be weakened to λ-ineffable by Proposition 1.5.4.

2.3.4 Theorem. *If the theory* ZFC + *"there exists a supercompact cardinal" is consistent, then the theory* ZFC + *"(ω_2, λ)-*ISP *holds for every $\lambda \geq \omega_2$" is consistent.*

Proof. This follows immediately from Proposition 1.3.13 and Theorems 2.2.8 and 2.3.3. □

In Theorems 2.3.2 and 2.3.4, ω_2 only serves as the minimal cardinal for which the theorems hold true. One can of course take successors of larger regular cardinals instead. However, for simplicity the forcing described in Section 2.2 was defined only for adding Cohen subsets of ω, so that it blows up the continuum. This is not actually necessary. For example, starting from an ineffable cardinal and GCH, one could also force to get a model of "$2^\omega = \omega_1$" + "$2^{\omega_1} = 2^{\omega_2} = \omega_3$" + ω_3-ISP. The reader is referred to [Kru08] for a more thorough treatment of the degrees of freedom one has when defining the underlying forcing.

The ideals in the generic extension behave well with respect to those in the ground model, as shows the next theorem.

2.3.5 Theorem. *Let κ be inaccessible, $\lambda \geq \kappa$, and $\langle \mathbb{P}_\nu \mid \nu \leq \kappa\rangle$ be an iteration such that*

(i) $\mathbb{P}_\kappa$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta\rangle$ and κ-cc,

(ii) $\mathbb{P}_\kappa$ satisfies the ω_1-approximation property,

(iii) for every $\nu < \kappa$, $|\mathbb{P}_\nu| < \kappa$.

Let $G \subset \mathbb{P}_\kappa$ be V-generic. Then

$$I_{\mathrm{IT}}^{V}[\kappa, \lambda] \subset I_{\mathrm{IT}}^{V[G]}[\kappa, \lambda] \tag{2.2}$$

and

$$\mathfrak{P}_\kappa^{V[G]}\lambda - \mathfrak{P}_\kappa^{V}\lambda \in I_{\mathrm{IT}}^{V[G]}[\kappa, \lambda], \tag{2.3}$$

which furthermore implies

$$F_{\mathrm{IT}}^{V}[\kappa, \lambda] \subset F_{\mathrm{IT}}^{V[G]}[\kappa, \lambda]. \tag{2.4}$$

*So in particular, if $V[G] \models (\kappa, \lambda)$-*ITP*, then $V \models (\kappa, \lambda)$-*ITP.

Proof. Work in $V[G]$.

To prove (2.2), let $A \in I_{\mathrm{IT}}^V[\kappa,\lambda]$, and let $\langle d_a \mid a \in \mathfrak{P}_\kappa^V\lambda\rangle \in V$ be A-effable in V.

Then $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ is thin, where $d_a := \emptyset$ for $a \notin V$. For let $y \in \mathfrak{P}_\kappa\lambda$. By (i) and Lemma 2.1.14 there is $\rho < \kappa$, such that $y \in V[G_\rho]$. But since $V[G_\rho] \models$ "κ inaccessible" by (iii), we have $V[G_\rho] \models |\mathfrak{P}y| < \kappa$. This means that $|\{d_a \cap y \mid a \in A\}| \leq |\mathfrak{P}^{V[G_\rho]}y| < \kappa$.

Suppose $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda\rangle$ were not A-effable. Let $S \subset A$ be stationary and $d \subset \lambda$ such that $d_x = d \cap x$ for all $x \in S$. Suppose $d \notin V$. Then, by (ii), there is a $z \in \mathfrak{P}_{\omega_1}^V\lambda$ such that $d \cap z \notin V$. But for $x \in S$ with $z \subset x$ we have $d \cap z = d \cap x \cap z = d_x \cap z \in V$. Therefore $d \in V$, and $S \subset \bar{S} := \{x \in \mathfrak{P}_\kappa^V\lambda \mid d_x = d \cap x\} \in V$. Since $\langle d_a \mid a \in \mathfrak{P}_\kappa^V\lambda\rangle \in V$ is A-effable in V, $\bar{S}$ is not stationary in V. So there exists $C \in V$, $C \subset \mathfrak{P}_\kappa^V\lambda$ club in V such that $C \cap \bar{S} = \emptyset$. Let $f : \mathfrak{P}_\omega\lambda \to \mathfrak{P}_\kappa\lambda$ be in V such that $\mathrm{Cl}_f^V \subset C$. But then, by the stationarity of S, there is an $x \in S$ such that $x \in \mathrm{Cl}_f$, so that $x \in C \cap \bar{S}$, a contradiction.

For the proof of (2.3), let $B := \mathfrak{P}_\kappa\lambda - \mathfrak{P}_\kappa^V\lambda$. For $x \in B$ let $a_x \in \mathfrak{P}_{\omega_1}^V\lambda$ be such that $x \cap a_x \notin V$, which exists by (ii). Put $d_x := a_x \cap x$. For $x \in \mathfrak{P}_\kappa\lambda - B$, let $d_x := \emptyset$.

$\langle d_x \mid x \in \mathfrak{P}_\kappa\lambda\rangle$ is thin, for let $y \in \mathfrak{P}_\kappa\lambda$ and, by (i) and Lemma 2.1.14, let $\rho < \kappa$ be such that $y \in V[G_\rho]$. Then for $x \in \mathfrak{P}_\kappa\lambda$ with $y \subset x$ we have $d_x \cap y = a_x \cap y \in \mathfrak{P}_{\omega_1}^{V[G_\rho]}y$ and $|\mathfrak{P}_{\omega_1}^{V[G_\rho]}y| < \kappa$ because κ is inaccessible in $V[G_\rho]$ by (iii).

Suppose $\langle d_x \mid x \in \mathfrak{P}_\kappa\lambda\rangle$ were not B-effable. Then there are $d \subset \lambda$ and $U \subset B$ be such that U is cofinal and $d_x = d \cap x$ for all $x \in U$. Define a $\subset$-increasing sequence $\langle x_\alpha \mid \alpha < \omega_2\rangle$ with $x_\alpha \in U$ for all $\alpha < \omega_2$ and a sequence $\langle e_\alpha \mid \alpha < \omega_2\rangle$ such that $x_\alpha \subset e_\alpha$ and $e_\alpha \in \mathfrak{P}_\kappa^V\lambda$ for all $\alpha < \omega_2$ as follows. Let $\beta < \omega_2$ and suppose $\langle x_\alpha \mid \alpha < \beta\rangle$ and $\langle e_\alpha \mid \alpha < \beta\rangle$ have been defined. Let $x_\beta \in U$ be such that $\bigcup_{\alpha<\beta}(x_\alpha \cup a_\alpha \cup e_\alpha) \subset x_\beta$, and let $e_\beta \in \mathfrak{P}_\kappa^V\lambda$ be such that $x_\beta \subset e_\beta$, which exists since by (i) and Proposition 2.1.9 $\mathbb{P}_\kappa$ satisfies the κ-covering property.

Then $\langle d_{x_\alpha} \mid \alpha < \omega_2\rangle$ is $\subset$-increasing as $d_{x_\alpha} = d \cap x_\alpha$ for all $\alpha < \omega_2$, and since $|d_{x_\alpha}| < \omega_1$ for all $\alpha < \omega_2$, there is $\gamma < \omega_2$ such that $d_{x_\alpha} = d_{x_{\alpha'}}$ for all $\alpha, \alpha' \in [\gamma, \omega_2)$. But then $a_{x_{\gamma+1}} \cap e_\gamma \subset a_{x_{\gamma+1}} \cap x_{\gamma+1} = d_{x_{\gamma+1}} = d_{x_\gamma} \subset e_\gamma$ and $d_{x_{\gamma+1}} \subset a_{x_{\gamma+1}}$, so that $d_{x_\gamma} = a_{x_{\gamma+1}} \cap e_\gamma \in V$, a contradiction.

To see (2.4), let $A \in F_{\mathrm{IT}}^V[\kappa,\lambda]$. Then $\mathfrak{P}_\kappa^V\lambda - A \in I_{\mathrm{IT}}^V[\kappa,\lambda]$, so, by (2.2), $\mathfrak{P}_\kappa^V\lambda - A \in I_{\mathrm{IT}}^{V[G]}[\kappa,\lambda]$. Thus, by (2.3), $\mathfrak{P}_\kappa^{V[G]}\lambda - A = (\mathfrak{P}_\kappa^{V[G]}\lambda - \mathfrak{P}_\kappa^V\lambda) \cup (\mathfrak{P}_\kappa^V\lambda - A) \in I_{\mathrm{IT}}^{V[G]}[\kappa,\lambda]$, which means $A \in F_{\mathrm{IT}}^{V[G]}[\kappa,\lambda]$. □

Note that by [Git85, Theorem 1.1] the set $\mathfrak{P}_\kappa^{V[G]}\lambda - \mathfrak{P}_\kappa^V\lambda$ in (2.3) of Theorem 2.3.5 is stationary for $\lambda \geq \kappa^+$ if the forcing $\mathbb{P}_\kappa$ adds a real, as is the case with our forcing from Theorem 2.2.8. Also note that the proof of (2.2) only required the κ-approximation property. It was the proof of (2.3) that required the τ-approximation property for a $\tau < \kappa$, and we stated it for $\tau = \omega_1$.

The next theorem was a natural byproduct of the author's—in retrospect pointless—endeavor to show PFA does not imply ω_2-STP.

2.3.6 Theorem. *Let $\mathbb{P}$ be a κ-closed forcing. If κ-*STP *holds, then* $\Vdash \kappa$-STP. *If κ-*SSP *holds, then* $\Vdash \kappa$-SSP.

Proof. Suppose $\langle \dot{d}_\alpha \mid \alpha < \kappa \rangle$ is such that $\Vdash$ "$\langle \dot{d}_\alpha \mid \alpha < \kappa \rangle$ is a thin unsubtle κ-list" or $\Vdash$ "$\langle \dot{d}_\alpha \mid \alpha < \kappa \rangle$ is a slender unsubtle κ-list". By Lemma 2.1.13 we may assume there is a club $C \in V$ such that $\Vdash d_\gamma \neq d_{\gamma'} \cap \gamma$ and, in the slender case, $\Vdash \forall \delta < \gamma\, \exists \beta < \gamma\, \dot{d}_\gamma \cap \delta = \dot{d}_\beta \cap \delta$ for all $\gamma, \gamma' \in C$ with $\gamma < \gamma'$.

For $\alpha < \kappa$ and $\beta < \alpha$ let $D^\alpha_\beta := \{p \in \mathbb{P} \mid p \Vdash \beta \in \dot{d}_\alpha \;\vee\; p \Vdash \beta \notin \dot{d}_\alpha\}$. D^α_β is dense open, so $D^\alpha := \bigcap\{D^\alpha_\beta \mid \beta < \alpha\} \neq \emptyset$ because $\mathbb{P}$ is κ-distributive.

We inductively define a decreasing sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ and a sequence $\langle h_\alpha \mid \alpha < \kappa \rangle$. In the thin case we furthermore define $\langle \varepsilon_\alpha \mid \alpha < \kappa \rangle$, and in the slender case $\langle \beta^\alpha_\delta \mid \delta < \alpha < \kappa \rangle$, such that the following holds.

(i) $p_\alpha \in D^\alpha$,

(ii) $p_\alpha \Vdash h_\alpha = \dot{d}_\alpha$,

(iii) $\varepsilon_\alpha < \kappa$ and $p_\alpha \Vdash \{\dot{d}_\nu \cap \alpha \mid \nu < \kappa\} = \{\dot{d}_\nu \cap \alpha \mid \nu < \varepsilon_\alpha\}$,

(iv) if $\delta < \alpha \in C$, then $\beta^\alpha_\delta < \alpha$ and $p_\alpha \Vdash \dot{d}_\alpha \cap \delta = \dot{d}_{\beta^\alpha_\delta} \cap \delta$.

Suppose the sequences were defined for all $\alpha' < \alpha$. Let $r \in D^\alpha$ be such that $r \leq p_{\alpha'}$ for all $\alpha' < \alpha$, which exists since $\mathbb{P}$ is κ-closed, and let h_α be such that $r \Vdash \dot{d}_\alpha = h_\alpha$. In the thin case, since $r \Vdash \exists \varepsilon < \kappa\, \{\dot{d}_\nu \cap \alpha \mid \nu < \kappa\} = \{\dot{d}_\nu \cap \alpha \mid \nu < \varepsilon\}$, there are $p_\alpha \leq r$ and $\varepsilon_\alpha < \kappa$ such that (iii) holds. In the slender case, we can define a decreasing sequence $\langle r_\delta \mid \delta < \alpha \rangle$ along with $\langle \beta^\alpha_\delta \mid \delta < \alpha \rangle$ such that $r_0 \leq r$ and $r_\delta \Vdash \dot{d}_\alpha \cap \delta = \dot{d}_{\beta^\alpha_\delta} \cap \delta$. Again this can be done as $\mathbb{P}$ is κ-closed. If $p_\alpha \leq r_\delta$ for all $\delta < \alpha$, then (iv) is fulfilled.

In the thin case, $\langle h_\alpha \mid \alpha < \kappa \rangle$ is thin, for suppose not. Then there are $\alpha < \nu < \kappa$ such that $h_\nu \cap \alpha \notin \{h_\nu \cap \alpha \mid \nu < \varepsilon_\alpha\}$. But for $\eta := \max\{\nu, \varepsilon_\alpha\}$ we get $p_\eta \Vdash h_\nu \cap \alpha = \dot{d}_\nu \cap \alpha \in \{\dot{d}_\nu \cap \alpha \mid \nu < \kappa\} = \{\dot{d}_\nu \cap \alpha \mid \nu < \varepsilon_\alpha\} = \{h_\nu \cap \alpha \mid \nu < \varepsilon_\alpha\}$, a contradiction.

In the slender case, $\langle h_\alpha \mid \alpha < \kappa \rangle$ is slender. For let $\delta < \alpha \in C$. Then $p_\alpha \Vdash h_\alpha \cap \delta = \dot{d}_\alpha \cap \delta = \dot{d}_{\beta^\alpha_\delta} \cap \delta = h_{\beta^\alpha_\delta} \cap \delta$.

Furthermore $\langle h_\alpha \mid \alpha < \kappa \rangle$ is unsubtle, for if $\alpha, \alpha' \in C$, $\alpha < \alpha'$, then $p_{\alpha'} \Vdash h_\alpha = \dot{d}_\alpha \neq \dot{d}_{\alpha'} \cap \alpha = h_{\alpha'} \cap \alpha$. □

Theorem 2.3.6 is in some sense unique to subtlety. As mentioned in the introduction, in analogy to ineffability one can define a principle κ-AITP by weakening "stationary" to "unbounded" in the definition of κ-ITP. Then Theorem 2.3.1 shows κ-AITP can be forced from an almost ineffable cardinal. Let κ-AITP$'$ and κ-ITP$'$ be the restrictions of κ-AITP and κ-ITP to κ-lists whose corresponding trees have at most κ many cofinal branches. Then it is not hard to see that κ-AITP$'$ and κ-ITP$'$ are in fact equivalent.[11]

[11] If T is such a tree and $\langle b_\nu \mid \nu < \kappa \rangle$ enumerates its cofinal branches, define a partial function $f : T \to \kappa$ such that $t \in b_{f(t)}$ and $f(t) < \mathrm{ht}(t)$ whenever possible. Then κ-AITP$'$ implies there is a stationary $S \subset \kappa$ such that $T \upharpoonright S \subset \operatorname{dom} f$, which in turn implies κ-ITP$'$ holds for T.

Suppose now we start in a model with exactly one almost ineffable cardinal κ and one inaccessible λ above it. We force to get ω_2-AITP, preserving the inaccessibility of λ. If we now Lévy-collapse λ to ω_3, then in the extension any ω_2-tree has at most ω_2 many cofinal branches, making ω_2-AITP′ and ω_2-AITP as well as ω_2-ITP′ and ω_2-ITP equivalent. So if ω_2-AITP were preserved, then ω_2-ITP would hold in the extension. But Theorem 2.4.3 will show that ω_2-ITP has consistency strength of an ineffable cardinal, which is strictly above that of an almost ineffable cardinal and an inaccessible above it. This shows it is consistent that Lévy-collapsing an inaccessible cardinal λ to ω_3 destroys ω_2-AITP.

Theorem 3.2.4 will show ω_2-ITP′ is consistent relative to a Σ_1^2-indescribable cardinal, so by the same argument the ω_2-closed forcing that Lévy-collapses an inaccessible to ω_3 and thus forces the nonexistence of ω_2-Kurepa trees can destroy ω_2-ITP′. On the other hand, as PFA is preserved by ω_2-closed forcings and implies ω_2-ITP, in models of PFA it holds that ω_2-ITP is preserved by ω_2-closed forcings.

2.4 Lower Bounds

This section deals with lower bounds for the consistency strength of our combinatorial principles. We first consider the one cardinal variants, showing that Theorem 2.3.2 was best possible.

2.4.1 Theorem. *Suppose κ is regular and uncountable. If κ-STP holds, then $L \models \kappa$ is subtle.*

Proof. First note that $L \models \kappa$ is inaccessible. For if κ is a limit cardinal, then it is a regular limit cardinal in L and thus inaccessible by GCH in L. So assume $\kappa = \lambda^+$. Then by Proposition 1.2.16, $\square_\lambda^*$ fails, and it is well known that this implies κ is inaccessible in L, see [Mit73].

Let $\langle d_\alpha \mid \alpha < \kappa \rangle \in L$ be a κ-list and $C \in L$ be a club in κ. Then $\{d_\alpha \cap \beta \mid \alpha \leq \kappa\} \subset \mathfrak{P}^L\beta$. But $L \models 2^\beta < \kappa$ since κ is strong limit in L, so $|\mathfrak{P}^L\beta| < \kappa$, which means $\langle d_\alpha \mid \alpha < \kappa \rangle$ is thin in V. Thus there are $\alpha, \beta \in C$, $\alpha < \beta$, such that $d_\alpha = d_\beta \cap \alpha$. □

The next lemma is usually only given in its weaker version where κ is required to be weakly compact.

2.4.2 Lemma. *Suppose κ is regular uncountable and the tree property holds for κ. Let $A \subset \kappa$. If $A \cap \alpha \in L$ for all $\alpha < \kappa$, then $A \in L$.*

Proof. Let $\delta := \kappa + \omega$.

2.4.2.1 Claim. *There exists a nonprincipal κ-complete ultrafilter on $\mathfrak{P}^{L[A]}\kappa \cap L_\delta[A]$.*

Proof. Work in $L[A]$. Let $\langle A^i_\alpha \mid \alpha < \kappa,\ i < 2\rangle$ be an enumeration of $\mathfrak{P}^{L[A]}\kappa \cap L_\delta[A]$ such that $A^0_\alpha = \kappa - A^1_\alpha$ for all $\alpha < \kappa$. Define a tree

$$T := \{f \in {}^{<\kappa}2 \cap L[A] \mid |\bigcap\{A^{f(\alpha)}_\alpha \mid \alpha < \operatorname{dom} f\}| = \kappa\},$$

ordered by inclusion. By [Mit73], κ is inaccessible, so every level of T has cardinality less than κ. To see T has height κ, let $\nu < \kappa$. For $\beta < \kappa$, choose $f_\beta \in {}^\nu 2$ such that $\beta \in A^{f_\beta(\alpha)}_\alpha$ for all $\alpha < \nu$. Then, as $|{}^\nu 2| < \kappa$ by the inaccessibility of κ, there is an $f \in {}^\nu 2$ such that $|\{\beta < \kappa \mid f_\beta = f\}| = \kappa$. Hence $f \in T_\nu$.

Now work in V. By assumption, T has a cofinal branch $b \in {}^\kappa 2$. But then $U := \{A^{b(\alpha)}_\alpha \mid \alpha < \kappa\}$ is as wanted. ⊣

Let U be as in Claim 2.4.2.1. Let M be the transitive collapse of the internal ultrapower of $L_\delta[A]$ by U, and let $j : L_\delta[A] \to M$ be the corresponding embedding. Then j has critical point κ. As $L_\delta[A] \models V = L[A]$, we have $M \models V = L[j(A)]$, so $M = L_\gamma[j(A)]$ for some limit ordinal $\gamma \geq \delta$. It holds that $L_\delta[A] \models \forall\alpha < \kappa\, A \cap \alpha \in L$, so $L_\gamma[j(A)] \models \forall\alpha < j(\kappa)\ j(A) \cap \alpha \in L$, so in particular $L_\gamma[j(A)] \models A = j(A) \cap \kappa \in L$. Therefore really $A \in L$. □

2.4.3 Theorem. *Suppose κ is regular and uncountable. If κ-ITP holds, then $L \models \kappa$ is ineffable.*

Proof. By Theorem 2.4.1, κ is inaccessible in L.

Let $\langle d_\alpha \mid \alpha < \kappa\rangle \in L$ be a κ-list. As in the proof of Theorem 2.4.1, it follows that $\langle d_\alpha \mid \alpha < \kappa\rangle$ is thin in V. Thus by κ-ITP there is a $d \subset \kappa$ such that $d_\alpha = d \cap \alpha$ for stationarily many $\alpha < \kappa$. This also means $d \cap \gamma \in L$ for all $\gamma < \kappa$. Therefore $d \in L$ by Lemma 2.4.2. Since $\{\alpha < \kappa \mid d_\alpha = d \cap \alpha\} \in L$ is also stationary in L, the proof is finished. □

Since as remarked above κ is supercompact iff it is inaccessible and (κ, λ)-ITP holds for all $\lambda \geq \kappa$, (κ, λ)-ITP appears to be the correct concept for supercompactness for small cardinals. The best known lower bounds for its consistency are derived from the failure of square. The following proposition is due to Jensen, Schimmerling, Ralf Schindler, and John Steel [JSSS09].[12]

2.4.4 Proposition. *Suppose $\lambda \geq \omega_3$ is regular such that $\eta^\omega < \lambda$ for all $\eta < \lambda$. If $\neg\square(\lambda)$ and $\neg\square_\lambda$, then there exists an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

2.4.5 Theorem. *The consistency of* ZFC + *"there is a κ^+-ineffable cardinal κ" implies the consistency of* ZFC + *"there is a proper class of strong cardinals and a proper class of Woodin cardinals."*

[12] The result is actually even stronger, they show there exists a sharp for a proper class model of the cardinals in Proposition 2.4.4.

Proof. If κ is κ^+-ineffable, then it is inaccessible and thus $\eta^\omega < \kappa$ for all $\eta < \kappa$. By Proposition 1.3.9, (κ, κ)-ITP holds. By Theorem 1.4.2, (κ, κ)-ITP and (κ, κ^+)-ITP imply $\neg\square(\kappa)$ and $\neg\square(\kappa^+)$, so by Proposition 2.4.4 there is an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals. □

2.4.6 Theorem. *Suppose κ is regular uncountable and $\lambda \geq \omega_3$ is such that* $\operatorname{cf} \lambda \geq \kappa$ *and $\eta^\omega < \lambda$ for all $\eta < \lambda$. If (κ, λ^+)-ITP holds, then there exists an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

Proof. This follows again from Proposition 1.3.9, Theorem 1.4.2, and Proposition 2.4.4. □

3 Implications under PFA

3.1 Preliminaries

3.1.1 Definition. Let $\mathbb{P}$ be a forcing. By $\mathsf{MA}(\mathbb{P})$ we denote the following principle: If D_α is dense in $\mathbb{P}$ for $\alpha < \omega_1$, then there exists a filter $G \subset \mathbb{P}$ such that $D_\alpha \cap G \neq \emptyset$ for all $\alpha < \omega_1$.

$\mathsf{PFA}(\delta)$ stands for the principle that $\mathsf{MA}(\mathbb{P})$ holds for every proper forcing $\mathbb{P}$ with $|\mathbb{P}| \leq \delta$. PFA holds iff $\mathsf{PFA}(\delta)$ holds for every δ. ⌟

The following proposition is due to Hugh Woodin [Woo99, Proof of Theorem 2.53], where it is shown for what is commonly referred to as weakly stationary. It provides a means to "applying PFA coherently stationarily often."

Recall that $G \subset \mathbb{P}$ is said to be *M-generic* iff G is a filter on $\mathbb{P}$ and $G \cap D \cap M \neq \emptyset$ for all $D \in M$ that are dense in $\mathbb{P}$.

3.1.2 Proposition. *Let $\mathbb{P}$ be a forcing such that $\mathsf{MA}(\mathbb{P})$ holds, and let θ be sufficiently large. Then*

$$\{M \in \mathfrak{P}_{\omega_2} H_\theta \mid \exists G \subset \mathbb{P}\ G \text{ is } M\text{-generic}\}$$

is stationary in $\mathfrak{P}_{\omega_2} H_\theta$.

Proof. Pick a club $C \subset \mathfrak{P}_{\omega_2} H_\theta$ and let $f : \mathfrak{P}_\omega H_\theta \to \mathfrak{P}_{\omega_2} H_\theta$ be such that $\mathrm{Cl}_f \subset C$.

3.1.2.1 Claim. *There is $\dot{h} \in V^{\mathbb{P}}$ such that*

(i) $\Vdash \dot{h} : \omega_1 \to H_\theta^V$ *injective,*

(ii) $\Vdash \operatorname{rng} \dot{h} \in \mathrm{Cl}_f$,

(iii) $\Vdash \forall D \in \operatorname{rng} \dot{h}\ (D \text{ is dense in } \mathbb{P} \to \dot{G} \cap D \cap \operatorname{rng} \dot{h} \neq \emptyset)$.

Proof. Let $H \subset \mathbb{P}$ be V-generic and work in $V[H]$. Let $M_0 \in \mathfrak{P}_{\omega_2} H_\theta^V \cap \mathrm{Cl}_f$. Suppose M_i has been defined. Let $M_{i+1} \in \mathfrak{P}_{\omega_2} H_\theta^V \cap \mathrm{Cl}_f$ be such that $M_i \subset M_{i+1}$ and $G \cap D \cap M_{i+1} \neq \emptyset$ for all $D \in M_i$ that are dense subsets of $\mathbb{P}$. Let $M := \bigcup_{i<\omega} M_i$ and $h : \omega_1 \to M$ be bijective. Then $h : \omega_1 \to H_\theta^V$ is injective, $\operatorname{rng} h \in \mathrm{Cl}_f$, and if $D \in \operatorname{rng} h$ is dense in $\mathbb{P}$, then $D \in M_i$ for some $i < \omega$, so $G \cap D \cap M \supset G \cap D \cap M_{i+1} \neq \emptyset$. Therefore if $\dot{h}$ is a name for such h, then $\dot{h}$ is as wanted. ⊣

Let $\dot h$ be as in Claim 3.1.2.1. For $\alpha < \omega_1$ define

$$D_\alpha := \{p \in \mathbb{P} \mid \exists x \in H_\theta \; p \Vdash \dot h(\alpha) = x\},$$

and for $a \in \mathfrak{P}_\omega \omega_1$ let

$$D_a := \bigcap\{D_\alpha \mid \alpha \in a\}.$$

Then D_α and D_a are dense open. Furthermore let

$$D'_\alpha := \{p \in D_\alpha \mid \forall q \in \mathbb{P} \; (p \Vdash \text{“}q = \dot h(\alpha) \in \dot G\text{”} \rightarrow p \leq q)\}.$$

D'_α is open, for otherwise there would be a $p_0 \in D_\alpha$ and $q \in \mathbb{P}$ such that $p_0 \Vdash q = \dot h(\alpha) \in \dot G$ and $\forall p \leq p_0 \; p \nleq q$, so that p_0 and q are incompatible, a contradiction.

For $x \in \mathfrak{P}_{\omega_2} H_\theta$ let $i_x : \omega_1 \rightarrow x$ be surjective, and for $a \in \mathfrak{P}_\omega \omega_1$ and $\beta < \omega_1$ let

$$E^\beta_a := \{p \in D_a \mid \exists \gamma < \omega_1 \; p \Vdash i_{f(\dot h'' a)}(\beta) = \dot h(\gamma)\}.$$

Then E^β_a is dense open in $\mathbb{P}$ for every $a \in \mathfrak{P}_\omega \omega_1$ and $\beta < \omega_1$. For suppose that for some $a \in \mathfrak{P}_\omega \omega_1$ and $\beta < \omega_1$ E^β_a were not dense, that is, there is a $p \in \mathbb{P}$ such that

$$\forall q \leq p \; \forall \gamma < \omega_1 \; q \nVdash i_{f(\dot h'' a)}(\beta) = \dot h(\gamma).$$

Then $p \Vdash \forall \gamma < \omega_1 \; i_{f(\dot h'' a)}(\beta) \neq \dot h(\gamma)$, so $p \Vdash i_{f(\dot h'' a)}(\beta) \in f(\dot h'' a) - \operatorname{rng} \dot h$, contradicting $\Vdash \operatorname{rng} \dot h \in \mathrm{Cl}_f$.

Finally for $\delta < \omega_1$ let

$$F_\delta := \{p \in D_\delta \mid p \Vdash \text{“}\dot h(\delta) \text{ is dense in } \mathbb{P}\text{”} \rightarrow \exists \beta < \omega_1 \; p \Vdash \dot h(\beta) \in \dot h(\delta) \cap \dot G\}.$$

F_δ is open dense in $\mathbb{P}$ for every $\delta < \omega_1$, for otherwise there were a $p \in \mathbb{P}$ such that $p \Vdash$ “$\dot h(\delta)$ is dense in $\mathbb{P}$” and $\forall q \leq p \; \forall \beta < \omega_1 \; q \nVdash \dot h(\beta) \in \dot h(\delta) \cap \dot G$, so that $p \Vdash \operatorname{rng} \dot h \cap \dot h(\delta) \cap \dot G = \emptyset$, contradicting (iii) of Claim 3.1.2.1.

By $\mathsf{MA}(\mathbb{P})$ there is a filter $G \subset \mathbb{P}$ that has nonempty intersection with D'_α, E^β_a, and F_δ for all $\alpha, \beta, \delta < \omega_1$ and $a \in \mathfrak{P}_\omega \omega_1$. Set $h := \dot h^G$.

3.1.2.2 Claim. *It holds that* $\operatorname{rng} h \in \mathrm{Cl}_f$.

Proof. Let $b \in \mathfrak{P}_\omega \operatorname{rng} h$ and take $z \in f(b)$. Let $a \in \mathfrak{P}_\omega \omega_1$ be such that $b = h'' a$ and $\beta < \omega_1$ such that $i_{f(b)}(\beta) = z$. Let $p \in E^\beta_a \cap G$. Then there is a $\gamma < \omega_1$ with $p \Vdash i_{f(b)}(\beta) = \dot h(\gamma)$. Let $p' \leq p$ be such that $p' \in D_\gamma \cap G$. Then $p' \Vdash z = i_{f(b)}(\beta) = \dot h(\gamma) = h(\gamma)$, so $z \in \operatorname{rng} h$. ⊣

3.1.2.3 Claim. *If* $D \in \operatorname{rng} h$ *is dense in* $\mathbb{P}$*, then* $G \cap D \cap \operatorname{rng} h \neq \emptyset$.

Proof. Let δ be such that $h(\delta) = D$. Let $p \in F_\delta \cap G$. Then $p \Vdash$ “$\dot h(\delta) = D$ is dense in $\mathbb{P}$”, so there is $\beta < \omega_1$ with $p \Vdash \dot h(\beta) \in D \cap \dot G$. Let $p' \leq p$ and $q \in \mathbb{P}$ be such that $p' \in D'_\beta \cap G$ and $p' \Vdash q = \dot h(\beta)$. Then $p' \leq q$, so $q \in G \cap D \cap \operatorname{rng} h$. ⊣

By Claims 3.1.2.2 and 3.1.2.3, $\operatorname{rng} h \in C$ is such that if $D \in \operatorname{rng} h$ is dense in $\mathbb{P}$, then $G \cap D \cap \operatorname{rng} h \neq \emptyset$, finishing the proof. □

3.1.3 Lemma. *Let $\mathbb{P}$ be a forcing, $\mathbb{Q} \subset \mathbb{P}$ dense. If $\dot{x} \in V^{\mathbb{P}}$, then there is $\dot{x}' \in V^{\mathbb{Q}}$ such that*

$$\Vdash_{\mathbb{P}} \dot{x} = \dot{x}'.$$

Proof. The proof is by induction over $\dot{x}$. Let

$$\dot{x}' := \{\langle \dot{y}', q\rangle \mid q \in \mathbb{Q},\ \exists p \in \mathbb{P}\ (q \leq p \wedge \langle \dot{y}, p\rangle \in \dot{x})\},$$

where $\dot{y}' \in V^{\mathbb{Q}}$ is such that $\Vdash_{\mathbb{P}} \dot{y} = \dot{y}'$, which exists by the induction hypothesis. Then $\Vdash_{\mathbb{P}} \dot{x} = \dot{x}'$, for let $G \subset \mathbb{P}$ be V-generic. Obviously $\dot{x}'^G \subset \dot{x}^G$, so for the other direction let $y \in \dot{x}^G$. Then there is a $p \in G$ and $\dot{y} \in V^{\mathbb{P}}$ such that $\langle \dot{y}, p\rangle \in \dot{x}$ and $\dot{y}^G = y$. The set $\{q \in \mathbb{Q} \mid q \leq p\}$ is dense below p, so there is $q \in \mathbb{Q} \cap G$ with $q \leq p$. Therefore $\langle \dot{y}', q\rangle \in \dot{x}'$ and thus $y = \dot{y}^G = \dot{y}'^G \in \dot{x}'^G$. □

3.1.4 Lemma. *Let $\mathbb{P}$ be a proper forcing. If $\mathbb{R} \subset \mathbb{P}$ is dense, then $\mathbb{R}$ is proper.*

Proof. Let $M \in \mathfrak{P}'_{\omega_1} H_\theta$ and $p \in M \cap \mathbb{R}$. As $\mathbb{P}$ is proper, there is an $(M, \mathbb{P})$-generic $\tilde{p} \leq p$. Let $\bar{p} \leq \tilde{p}$, $\bar{p} \in \mathbb{R}$. Then $\bar{p}$ is $(M, \mathbb{R})$-generic, for let $D \in M$ be dense in $\mathbb{R}$. Then D is dense in $\mathbb{P}$, so by the $(M, \mathbb{P})$-genericity of $\bar{p}$ the set $D \cap M$ is predense below $\bar{p}$ in $\mathbb{P}$. But then $D \cap M$ is also predense below $\bar{p}$ in $\mathbb{R}$. □

3.1.5 Lemma. *Let $\mathbb{P}, \dot{\mathbb{Q}}$, and δ be such that $\mathbb{P}$ is proper, $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is proper, $|\mathbb{P}| \leq \delta$, and $\Vdash_{\mathbb{P}} |\dot{\mathbb{Q}}| \leq \delta$. Then there is a dense $\mathbb{R} \subset \mathbb{P} * \dot{\mathbb{Q}}$ with $|\mathbb{R}| \leq \delta$.*

Proof. Since $\Vdash_{\mathbb{P}} |\dot{\mathbb{Q}}| \leq \delta$ there is $\dot{f} \in V^{\mathbb{P}}$ such that $\Vdash_{\mathbb{P}} \dot{f} : \delta \to \dot{\mathbb{Q}}$ surjective. Let $g : \delta \to V^{\mathbb{P}}$ be such that $\Vdash_{\mathbb{P}} \dot{f}(\alpha) = g(\alpha)$ for all $\alpha < \delta$.

Let $\mathbb{R} := \mathbb{P} \times \operatorname{rng} g$, ordered by the restriction of the ordering of $\mathbb{P} * \dot{\mathbb{Q}}$, that is $\langle p, g(\alpha)\rangle \leq \langle p', g(\alpha')\rangle$ iff $p \leq p'$ and $p \Vdash_{\mathbb{P}} g(\alpha) \leq g(\alpha')$. Then $|\mathbb{R}| \leq \delta$. $\mathbb{R}$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. For let $\langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}}$. Then $p \Vdash_{\mathbb{P}} \exists \alpha < \delta\ \dot{q} = \dot{f}(\alpha)$, so for some $\bar{p} \leq p$ and $\alpha < \delta$ we have $\bar{p} \Vdash_{\mathbb{P}} \dot{q} = \dot{f}(\alpha) = g(\alpha)$. This means $\langle \bar{p}, g(\alpha)\rangle \leq \langle p, \dot{q}\rangle$. □

3.1.6 Lemma. *Let $\mathbb{P}$ be proper, $|\mathbb{P}| \leq \delta$, and $\dot{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is ccc". If D_α, $\alpha < \delta$, are dense open subsets of $\mathbb{P} * \dot{\mathbb{Q}}$, then there is a proper $\mathbb{R} \subset \mathbb{P} * \dot{\mathbb{Q}}$ with $|\mathbb{R}| \leq \delta$ such that $D_\alpha \cap \mathbb{R}$ is dense open in $\mathbb{R}$ for all $\alpha < \delta$.*

Proof. For $\alpha < \delta$ let

$$\dot{E}_\alpha := \{\langle \dot{q}, p\rangle \mid \langle p, \dot{q}\rangle \in D_\alpha\}.$$

3.1.6.1 Claim. *For $\alpha < \delta$*

$$\Vdash_{\mathbb{P}} \dot{E}_\alpha \text{ is dense in } \dot{\mathbb{Q}}.$$

Proof. Let $p \in \mathbb{P}$ and $\dot q \in V^{\mathbb{P}}$ with $p \Vdash_{\mathbb{P}} \dot q \in \dot{\mathbb{Q}}$. Let $\langle \tilde p, \dot{\tilde q}\rangle \in D_\alpha$ such that $\langle \tilde p, \dot{\tilde q}\rangle \leq \langle p, \dot q\rangle$. Then $\langle \dot{\tilde q}, \tilde p\rangle \in \dot E_\alpha$ and thus $\tilde p \Vdash_{\mathbb{P}}$ "$\dot{\tilde q} \leq \dot q \wedge \dot{\tilde q} \in \dot E_\alpha$". ⊣

3.1.6.2 Claim. *There is $\dot{\mathbb{Q}}'$ such that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}' \subset \dot{\mathbb{Q}}$, $\Vdash_{\mathbb{P}} |\dot{\mathbb{Q}}'| \leq \delta$, $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}'$ is ccc", and $\Vdash_{\mathbb{P}}$ "$\dot E_\alpha \cap \dot{\mathbb{Q}}'$ is dense in $\dot{\mathbb{Q}}'$" for every $\alpha < \delta$.*

Proof. Let $G \subset \mathbb{P}$ be V-generic and work in $V[G]$. Set $\mathbb{Q} := \dot{\mathbb{Q}}^G$. For compatible $q, q' \in \mathbb{Q}$ let $c(q,q')$ be such that $c(q,q') \leq q, q'$, and for $\alpha < \delta$ and $q \in \mathbb{Q}$ let $f_\alpha(q) \in \dot E^G_\alpha$ be such that $f_\alpha(q) \leq q$, which exists since $\dot E^G_\alpha$ is dense in $\mathbb{Q}$ by Claim 3.1.6.1. Let $\mathbb{Q}' \in \mathfrak{P}_{\delta^+}\mathbb{Q}$ be closed under c and f_α for all $\alpha < \delta$.

Then any antichain in $\mathbb{Q}'$ is an antichain in $\mathbb{Q}$, so $\mathbb{Q}'$ is ccc, and if $\dot{\mathbb{Q}}'$ is a name for $\mathbb{Q}'$, then it is as wanted. ⊣

Let $\dot{\mathbb{Q}}'$ be as in Claim 3.1.6.2. We apply Lemma 3.1.5 to $\mathbb{P} * \dot{\mathbb{Q}}'$ and get a dense $\mathbb{R} \subset \mathbb{P} * \dot{\mathbb{Q}}'$ which satisfies $|\mathbb{R}| \leq \delta$. By Lemma 3.1.4, $\mathbb{R}$ is proper.

So it remains to show $D_\alpha \cap \mathbb{R}$ is dense open in $\mathbb{R}$ for all $\alpha < \delta$. Let $\langle p, \dot q\rangle \in \mathbb{R}$. As $\Vdash_{\mathbb{P}}$ "$\dot E_\alpha \cap \dot{\mathbb{Q}}'$ is dense in $\dot{\mathbb{Q}}'$", there is $\dot q' \in V^{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}} \dot q' \in \dot E_\alpha \cap \dot{\mathbb{Q}}' \wedge \dot q' \leq \dot q$.

3.1.6.3 Claim. *There is $\tilde p \leq p$ and $\langle \dot q_0, p_0\rangle \in \dot E_\alpha$ such that $\tilde p \leq p_0$ and $\tilde p \Vdash_{\mathbb{P}} \dot q_0 = \dot q'$.*

Proof. Suppose not, so for all $\tilde p \leq p$ and all $\langle \dot q_0, p_0\rangle \in \dot E_\alpha$ it holds that $\tilde p \leq p_0 \to \tilde p \nVdash_{\mathbb{P}} \dot q_0 = \dot q'$. Let $G \subset \mathbb{P}$ be V-generic such that $p \in G$ and work in $V[G]$. Then $\dot q^G \in \dot E^G_\alpha$, so there is $\langle \dot q_0, p_0\rangle \in \dot E_\alpha$ with $p_0 \in G$ and $\dot q^G_0 = \dot q^G$. Let $p_1 \in G$ such that $p_1 \leq p_0, p$. Then $\tilde p \nVdash_{\mathbb{P}} \dot q_0 = \dot q'$ for all $\tilde p \leq p_1$, so $p_1 \Vdash_{\mathbb{P}} \dot q_0 \neq \dot q'$, a contradiction. ⊣

Let $\tilde p \leq p$ and $\langle \dot q_0, p_0\rangle \in \dot E_\alpha$ be as in Claim 3.1.6.3. Then $\langle p_0, \dot q_0\rangle \in D_\alpha$, so $\langle \tilde p, \dot q_0\rangle \in D_\alpha$. Furthermore $\tilde p \Vdash_{\mathbb{P}} \dot q_0 = \dot q' \in \dot{\mathbb{Q}}'$, so $\langle \tilde p, \dot q_0\rangle \in \mathbb{P} * \dot{\mathbb{Q}}'$. So for $\langle \bar p, \dot{\bar q}\rangle \in \mathbb{R}$ with $\langle \bar p, \dot{\bar q}\rangle \leq \langle \tilde p, \dot q_0\rangle$ we have $\langle \bar p, \dot{\bar q}\rangle \in D_\alpha \cap \mathbb{R}$ and $\langle \bar p, \dot{\bar q}\rangle \leq \langle p, \dot q\rangle$. □

3.1.7 Proposition (PFA(2^ω)). *Suppose $\mathbb{P} = \mathbb{D} * \mathrm{Coll}(\omega_1, \omega_2) * \dot{\mathbb{Q}}$ is such that $\mathbb{D}$ is proper, $|\mathbb{D}| \leq 2^\omega$, $\Vdash_{\mathbb{D}} 2^\omega = \omega_2 = \omega_2^V$, and $\Vdash_{\mathbb{D}*\mathrm{Coll}(\omega_1,\omega_2)}$ "$\dot{\mathbb{Q}}$ is ccc". Then* MA($\mathbb{P}$).

Proof. For $\alpha < \omega_1$, let $D_\alpha \subset \mathbb{D} * \mathrm{Coll}(\omega_1, \omega_2) * \mathbb{Q}$ be dense. Since

$$\Vdash_{\mathbb{D}} |\mathrm{Coll}(\omega_1, \omega_2)| = \omega_2^\omega = 2^\omega = \omega_2^V,$$

by Lemma 3.1.5 there is a dense $\mathbb{R} \subset \mathbb{D} * \mathrm{Coll}(\omega_1, \omega_2)$ such that $|\mathbb{R}| \leq 2^\omega$. Then $\mathbb{R}$ is proper by Lemma 3.1.4. By Lemma 3.1.3 there is $\dot{\mathbb{Q}}' \in V^{\mathbb{R}}$ such that

$$\Vdash_{\mathbb{D}*\mathrm{Coll}(\omega_1,\omega_2)} \dot{\mathbb{Q}} = \dot{\mathbb{Q}}'.$$

Let $D'_\alpha := \{p \in \mathbb{D} * \mathrm{Coll}(\omega_1, \omega_2) * \dot{\mathbb{Q}}' \mid \exists q \in D_\alpha\ p \leq q\}$. Then D'_α is dense open in $\mathbb{D} * \mathrm{Coll}(\omega_1, \omega_2) * \dot{\mathbb{Q}}'$ for $\alpha < \omega_1$. Therefore by Lemma 3.1.6 there is a proper $\bar{\mathbb{R}} \subset \mathbb{R} * \dot{\mathbb{Q}}'$ with $|\bar{\mathbb{R}}| \leq 2^\omega$ and $D'_\alpha \cap \bar{\mathbb{R}}$ dense open in $\bar{\mathbb{R}}$ for all $\alpha < \omega_1$. By PFA(2^ω) there is a filter $\bar G \subset \bar{\mathbb{R}}$ such that $G \cap D'_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Let $G := \{q \in \mathbb{P} \mid \exists p \in G\ p \leq q\}$. Then G is a filter on $\mathbb{P}$ that has nonempty intersection with every D_α for $\alpha < \omega_1$, for if $p \in D'_\alpha \cap \bar G$, then there is $q \in D_\alpha$ with $p \leq q$, so that also $q \in G$. □

3.1.8 Proposition. *If T is a tree without uncountable branches, then there exists a ccc forcing $\mathbb{Q}$ that specializes T, that is, it holds that*

$$V^{\mathbb{Q}} \models \exists s : T \to \omega\ \forall t, t' \in T\ (t <_T t' \to s(t') \neq s(t)).$$

The proof of Lemma 3.1.8 can be found in [She98, chap. III, Theorem 5.4] or [Jec03, Theorem 16.17].

3.1.9 Proposition (PFA). *Let $\mathbb{P}$ be an ω_2-directed closed forcing. Then $V^{\mathbb{P}} \models$ PFA.*

Proof. Let $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ be a name for a proper forcing and let $\langle \dot{D}_\alpha \mid \alpha < \omega_1 \rangle$ be names for dense subsets of $\dot{\mathbb{Q}}$. Let

$$E_\alpha := \{\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \mid p \Vdash \dot{q} \in \dot{D}_\alpha\}.$$

Then the sets E_α are dense in $\mathbb{P} * \dot{\mathbb{Q}}$.

By PFA there exists a filter $G = G_1 * \dot{G}_2 \subset \mathbb{P} * \dot{\mathbb{Q}}$ such that $G \cap E_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. For $\alpha < \omega_1$ let $\langle p_\alpha, \dot{q}_\alpha \rangle \in G$ be such that $p_\alpha \Vdash \dot{q}_\alpha \in \dot{D}_\alpha$. Then $\{p_\alpha \mid \alpha < \omega_1\}$ is directed because G_1 is a filter. As $\mathbb{P}$ is ω_2-directed closed, there exists a $p \in \mathbb{P}$ with $p \leq p_\alpha$ for all $\alpha < \omega_1$. But $p \Vdash \forall \alpha < \omega_1\ \dot{q}_\alpha \in \dot{G}_2 \cap \dot{D}_\alpha$, so $p \Vdash \forall \alpha < \omega_1\ \dot{G}_2 \cap \dot{D}_\alpha \neq \emptyset$. □

Using a technique of Robert Beaudoin [Bea91], in [KY04] it is shown that Proposition 3.1.9 actually only requires the forcing $\mathbb{P}$ to be ω_2-closed.

3.2 Tree Properties

The goal of this section is to prove Theorem 3.2.5, which says that PFA implies ω_2-ITP. We have to face one major obstacle. The usual applications of PFA use forcings of the form $\mathrm{Coll}(\omega_1, \omega_2) *$ ccc or $\mathbb{C} * \mathrm{Coll}(\omega_1, \omega_2) *$ ccc, where $\mathbb{C}$ denotes the forcing for adding a Cohen real, which by Proposition 3.1.7 actually only require PFA(2^ω). However, by [NS08] PFA(2^ω) is consistent relative to the existence of a Σ^2_1-indescribable cardinal, which has consistency strength below that of a subtle cardinal! So in view of Theorem 2.4.1 we cannot hope to prove ω_2-ITP or even ω_2-STP using such a construction.

3.2.1 Definition. Let T be a tree and B be a set of cofinal branches of T. A function $g : B \to T$ is called *Baumgartner function* iff g is injective and for all $b, b' \in B$ it holds that

(i) $g(b) \in b$,

(ii) $g(b) < g(b') \to g(b') \notin b$. ⌟

The following lemma is due to Baumgartner, see [Bau84].

3.2.2 Lemma. *Let T be a tree and B be a set cofinal branches of T. Suppose $\kappa := \mathrm{ht}(T)$ is regular and $|B| \leq \kappa$. Then there is a Baumgartner function $g : B \to T$.*

Proof. Let $\langle b_\alpha \mid \alpha < \mu \rangle$ enumerate B, with $\mu \leq \kappa$. Recursively define g by

$$g(b_\alpha) := \min(b_\alpha - \bigcup\{b_\beta \mid \beta < \alpha\}).$$

This can be done since κ is regular. Suppose $g(b_\alpha) < g(b_{\alpha'})$ for some $\alpha, \alpha' < \mu$. Then $g(b_{\alpha'}) \in b_{\alpha'}$, so $g(b_\alpha) \in b_{\alpha'}$, so $\alpha < \alpha'$ and thus $g(b_{\alpha'}) \notin b_\alpha$. □

Recall that a tree T is said to *not split at limit levels* iff for all $t, t' \in T$ such that $\mathrm{ht}\, t = \mathrm{ht}\, t' \in \mathrm{Lim}$ and $\{s \in T \mid s < t\} = \{s \in T \mid s < t'\}$ it follows that $t = t'$.

3.2.3 Lemma. *Let T be a tree that does not split at limit levels and suppose B is a set of cofinal branches of T. Suppose $g : B \to T$ is a Baumgartner function. Suppose $\langle \alpha_\nu \mid \nu < \omega_1 \rangle$ is continuous and increasing. Let $\alpha := \sup_{\nu<\omega_1} \alpha_\nu$ and $t \in T_\alpha$. Suppose that for all $\nu < \omega_1$ there is $b_\nu \in B$ such that $g(b_\nu) < t \restriction \alpha_\nu \in b_\nu$. Then there is an $s < t$ such that $t \in g^{-1}(s)$.*

Proof. Let $h(\nu) := \mathrm{ht}(g(b_\nu)) < \alpha_\nu$ for $\nu < \omega_1$.

3.2.3.1 Claim. *There is an $\eta < \alpha$ such that $h^{-1\prime\prime}\{\eta\}$ is unbounded in ω_1.*

Proof. We first observe that for $\nu, \nu' < \omega_1$

$$h(\nu) < h(\nu') \to h(\nu) < \alpha_\nu < h(\nu') < \alpha_{\nu'}. \tag{3.1}$$

For if $h(\nu) < h(\nu')$, then $g(b_{\nu'}) \notin b_\nu$. But $t \restriction \alpha_\nu \in b_\nu$, so $\alpha_\nu < h(\nu')$. From (3.1) and the monotonicity of $\langle \alpha_\nu \mid \nu < \omega_1 \rangle$ we also get

$$\nu \leq \nu' \to h(\nu) \leq h(\nu'). \tag{3.2}$$

Now suppose $h^{-1\prime\prime}\{\eta\}$ is bounded in ω_1 for all $\eta < \alpha$. Let $\nu_0 := 0$. Suppose ν_i has been defined. Let ν_{i+1} be such that $\nu_{i+1} > \sup h^{-1\prime\prime}\{h(\nu_i)\}$. Then $\nu_i < \nu_{i+1}$ and $h(\nu_i) \neq h(\nu_{i+1})$, so by (3.2) $h(\nu_i) < h(\nu_{i+1})$. Set $\nu^* := \sup_{i<\omega} \nu_i$. Then $h(\nu_i) < h(\nu^*)$ for all $i < \omega$, so $\alpha_{\nu_i} < h(\nu^*)$ by (3.1). But this implies $\alpha_{\nu^*} = \sup_{i<\omega} \alpha_{\nu_i} \leq h(\nu^*)$, a contradiction. ⊣

By Claim 3.2.3.1 there is $\eta < \alpha$ such that $U := h^{-1\prime\prime}\{\eta\}$ is unbounded in ω_1. This means that for all $\nu \in U$ $t \restriction \alpha_\nu \in b_\nu = g^{-1}(g(b_\nu)) = g^{-1}(t \restriction \eta)$. But then $t \in g^{-1}(t \restriction \eta)$ since T does not split at limit levels. □

PFA can be seen as some sort of a reflection principle. One takes a structure of size ω_2, collapses its size to ω_1, and then uses a forcing to fix some of its properties. In the ground model, one can then apply PFA in such a way that these properties must already hold for a substructure of size ω_1.

However, it is not apparent what should be reflected when one tries to prove ω_2-ITP or just ω_2-STP from PFA. The solution to this problem becomes clearer when one concentrates on ω_2-ISP rather than ω_2-ITP. For slenderness is, roughly spoken, just the property that, when we consider the corresponding tree to be growing along the ordinals, new branches will be added only very rarely. Thus it seems natural to try reflecting the tree's cofinal branches—they all already exist and thus never need to be added. The two previous lemmas provide the framework for such a reflection.[13]

The problem that opens up now is that one does not know much about the set of cofinal branches of the corresponding tree. Therefore we just assume there are at most ω_2 many.[14]

3.2.4 Theorem (PFA(2^ω))**.** *Let $\langle d_\alpha \mid \alpha < \omega_2\rangle$ be a slender ω_2-list. If its corresponding tree has at most ω_2 many cofinal branches, then $\langle d_\alpha \mid \alpha < \omega_2\rangle$ is not effable.*

Proof. Let $T := \mathrm{dc}\{t_\alpha \mid \alpha < \omega_2\}$, where $\langle t_\alpha \mid \alpha < \omega_2\rangle$ are the characteristic functions of $\langle d_\alpha \mid \alpha < \omega_2\rangle$, and let $B := \{b \subset T \mid b \text{ cofinal branch}\}$. By Lemma 3.2.2, there is a Baumgartner function $g : B \to T$. Define

$$T^0 := \{t \in T \mid \exists b \in B\ g(b) < t \in b\},$$
$$T^1 := T - T^0.$$

Then T^1 has no cofinal branch, for if $b \subset T^1$ were one, then one would have $t \leq g(b) \in b$ for all $t \in b$.

Let $\mathbb{P} := \mathbb{C} * \mathrm{Coll}(\omega_1, \omega_2)$. Let $\dot{c} \in V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models$ "$\dot{c} : \omega_1 \to \omega_2^V$ is continuous and cofinal". By Theorem 2.2.6, $\mathbb{P}$ has the ω_1-approximation property, so $V^{\mathbb{P}} \models$ "$(T \restriction \operatorname{rng} \dot{c}) \cap T^1$ has no cofinal branches". Therefore by Proposition 3.1.8 there is a $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ such that $V^{\mathbb{P}} \models$ "$\dot{\mathbb{Q}}$ is ccc and specializes $(T \restriction \operatorname{rng} \dot{c}) \cap T^1$". Let $\dot{f} \in V^{\mathbb{P}*\dot{\mathbb{Q}}}$ be a name for the specialization map.

Let θ be large enough.

3.2.4.1 Claim. *Let $M \in \mathfrak{P}'_{\omega_2} H_\theta$ be such that $\langle d_\alpha \mid \alpha < \omega_2\rangle, T^0, T^1, \mathbb{P} * \dot{\mathbb{Q}}, \dot{f}, \dot{c} \in M$. Suppose there is an M-generic filter $G \subset \mathbb{P} * \dot{\mathbb{Q}}$. Let $\delta_M := M \cap \omega_2$. Then there is an $s \in M$ such that*

$$t_{\delta_M} \in g^{-1}(s).$$

Proof. For $\alpha < \delta_M$ and $\nu < \omega_1$ define

$$D_\alpha := \{p \in \mathbb{P} * \dot{\mathbb{Q}} \mid \exists \nu < \omega_1\ p \Vdash \dot{c}(\nu) \geq \alpha\}$$

[13]This is the line of thought by which the author found a solution to the problem. It is not obvious how the reflection must work in detail, and the author only arrived at the current solution after an odyssey of different attempts that eventually led to what had already been done by Baumgartner.

[14]Section 3.4 will show there is a much better solution.

and

$$E_\nu := \{p \in \mathbb{P} * \dot{\mathbb{Q}} \mid \exists \alpha < \omega_2 \; p \Vdash \dot{c}(\nu) = \alpha\}.$$

Then $D_\alpha, E_\nu \in M$ are dense open in $\mathbb{P}$. Therefore $c := \dot{c}^G : \omega_1 \to \delta_M$ is continuous and cofinal. For $\alpha < \delta_M$ and $\gamma < \omega_1$ let

$$F_\alpha^\gamma := \{p \in \mathbb{P} * \dot{\mathbb{Q}} \mid \exists n < \omega \; p \Vdash \text{“}t_\alpha \restriction \dot{c}(\gamma) \in T^1 \to \dot{f}(t_\alpha \restriction \dot{c}(\gamma)) = n\text{”}\}.$$

$F_\alpha^\gamma \in M$ is also dense open in $\mathbb{P} * \dot{\mathbb{Q}}$. Therefore for $f := \dot{f}^G$ we have

$$f : (\mathrm{dc}\{t_\alpha \mid \alpha < \delta_M\} \restriction \mathrm{rng}\, c) \cap T^1 \to \omega$$

is a specialization map. As $M \models$ “$\langle d_\alpha \mid \alpha < \omega_2\rangle$ is slender”, there is a club $C \in M$ that witnesses the slenderness of $\langle d_\alpha \mid \alpha < \omega_2\rangle$. This means $\delta_M \in C$ so that t_{δ_M} is a branch through the tree $\mathrm{dc}\{t_\alpha \mid \alpha < \delta_M\}$. Thus there is an $\alpha < \delta_M$ such that $t_{\delta_M} \restriction \beta \in T^0$ for all $\beta \in \mathrm{rng}\, c - \alpha$.

For $\beta \in \mathrm{rng}\, c - \alpha$ let b_β be such that $g(b_\beta) < t_{\delta_M} \restriction \beta \in b_\beta$. Then, letting $\langle \alpha_\nu \mid \nu < \omega_1\rangle$ enumerate $\mathrm{rng}\, c - \alpha$, by Lemma 3.2.3 there is an $s < t_{\delta_M}$ with $t_{\delta_M} \in g^{-1}(s)$. Then also $s \in M$. ⊣

By Proposition 3.1.7, $\mathsf{MA}(\mathbb{P} * \dot{\mathbb{Q}})$ holds. Thus by Proposition 3.1.2

$$S := \{M \in \mathfrak{P}'_{\omega_2} H_\theta \mid \langle d_\alpha \mid \alpha < \omega_2\rangle, T^0, T^1, \mathbb{P} * \dot{\mathbb{Q}}, \dot{f}, \dot{c} \in M,\; \exists G \subset \mathbb{P}\; G \text{ is } M\text{-generic}\}$$

is stationary in $\mathfrak{P}_{\omega_2} H_\theta$. By Claim 3.2.4.1, for every $M \in S$ there is an $s_M \in M$ such that $t_{M\cap\omega_2} \in g^{-1}(s_M)$. Thus there are a stationary $S' \subset S$ and an $s \in T$ such that $s_M = s$ for all $M \in S'$. But then

$$t_{M\cap\omega_2} \in g^{-1}(s)$$

for all $M \in S$, which implies

$$d_\alpha = d_\beta \cap \alpha$$

for $\alpha < \beta$ and $\alpha, \beta \in \{M \cap \omega_2 \mid M \in S'\} =: \tilde{S}$. $\tilde{S}$ is stationary in ω_2, and $\langle d_\alpha \mid \alpha < \omega_2\rangle$ is therefore not effable. □

Our original goal to prove ω_2-ITP from PFA is now just one collapse away.

3.2.5 Theorem (PFA). *ω_2-ITP holds.*

Proof. Let $\langle d_\alpha \mid \alpha < \omega_2\rangle$ be a thin list. Let $\langle t_\alpha \mid \alpha < \omega_2\rangle$ be the list of its characteristic functions, and let $T := \mathrm{dc}\{t_\alpha \mid \alpha < \omega_2\}$. Set $\lambda := |\{b \subset T \mid b \text{ cofinal branch}\}|$. If $\lambda > \omega_2$, let $\mathbb{P} := \mathrm{Coll}(\omega_2, \lambda)$, otherwise $\mathbb{P} := \{\mathbf{1}\}$.

$\mathbb{P}$ is ω_2-directed closed, so by Proposition 3.1.9 $V^{\mathbb{P}} \models \mathsf{PFA}$. As $2^{\omega_1} = \omega_2$, by Proposition 2.1.12 $\mathbb{P}$ satisfies the thin ω_2-approximation property and hence does not add any new branches through T, so $V^{\mathbb{P}} \models$ "T has at most ω_2 many cofinal branches". Therefore we can apply Theorem 3.2.4 and get $V^{\mathbb{P}} \models$ "$\exists d \subset \omega_2\ \{\alpha < \omega_2 \mid d_\alpha = d \cap \alpha\}$ stationary". But this d corresponds to a cofinal branch in T and thus is already in V. So

$$S := \{\alpha < \omega_2 \mid d_\alpha = d \cap \alpha\}$$

is in V as well and stationary in $V^{\mathbb{P}}$. But then S is also stationary in V and therefore $\langle d_\alpha \mid \alpha < \omega_2\rangle$ not effable. □

3.2.6 Corollary (PFA). *ω_2 is ineffable in L.*

Proof. This follows from Theorems 3.2.5 and 2.4.3. □

The fact that there can be no ω_2-Aronszajn trees under PFA is originally due to Baumgartner, see [Tod84b, Theorem 7.7].

3.2.7 Corollary (PFA(2^ω)). *Suppose $\langle d_\alpha \mid \alpha < \omega_2\rangle$ is a slender ω_2-list. Then its corresponding tree has a cofinal branch. In particular there is no ω_2-Aronszajn tree.*

Proof. This follows immediately from Theorem 3.2.4. □

The next corollary is originally independently due to Matthew Foreman and Stevo Todorčević, see [KY04].

3.2.8 Corollary (PFA(2^ω)). *The approachability property fails for ω_1, that is $\omega_2 \notin I_{\mathrm{AS}}[\omega_2]$.*

Proof. The proof is actually trivial, but unfortunately some cosmetics are required as our definition of an approachable slender ω_2-list allows, for example, an ω_2-Kurepa tree to hide in some nonstationary part of it. Suppose $\langle d_\alpha \mid \alpha < \omega_2\rangle$ is an approachable slender ω_2-list, its approachability witnessed by a club C. Let $\langle t_\alpha \mid \alpha < \omega_2\rangle$ be its characteristic functions. For $\alpha \in \omega_2 - C$, let

$$g(\alpha) := \sup\{\delta + 1 \mid \delta \leq \alpha,\ \exists \gamma \in C\ d_\gamma \cap \delta = d_\alpha \cap \delta\},$$

and let

$$d'_\alpha := \begin{cases} d_\alpha & \text{if } \alpha \in C, \\ d_\alpha \cap g(\alpha) & \text{if } \alpha \notin C \end{cases}$$

for $\alpha < \omega_2$. Then $\langle d'_\alpha \mid \alpha < \omega_2\rangle$ is still approachable and slender. Let $\langle t'_\alpha \mid \alpha < \omega_2\rangle$ be its characteristic functions and T' be its corresponding tree.

3.2.8.1 Claim. *If b is a branch through T', then $b^{-1\prime\prime}\{1\}$ is bounded ω_2.*

Proof. Suppose b is a branch through T' such that $b^{-1\prime\prime}\{1\}$ is unbounded in ω_2. For every $\delta < \omega_2$ there is an $\alpha_\delta < \omega_2$ such that $b \restriction \delta = t'_{\alpha_\delta} \restriction \delta$.

We may assume $\alpha_\delta \in C$ for all $\delta < \omega_2$, for take $\delta < \omega_2$. Then there is $\delta' \geq \delta$ such that $b(\delta') = 1$. Since $b \restriction (\delta' + 1) = t'_{\alpha_{\delta'+1}} \restriction (\delta' + 1)$, this means $t'_{\alpha_{\delta'+1}}(\delta') = 1$, so that either

$\alpha_{\delta'+1} \in C$ or $g(\alpha_{\delta'+1}) \geq \delta' + 1$. In the first case we can simply choose α_δ to be $\alpha_{\delta'+1}$, so suppose the second case holds. Then, by the definition of g, there is a $\gamma \in C$ such that $d_\gamma \cap \delta' = d_{\alpha_{\delta'+1}} \cap \delta'$, so $t'_{\alpha_{\delta'+1}} \upharpoonright \delta' = t_{\alpha_{\delta'+1}} \upharpoonright \delta' = t_\gamma \upharpoonright \delta' = t'_\gamma \upharpoonright \delta'$. So we can choose α_δ to be this γ.

Now if $b^{-1\prime\prime}\{1\}$ is unbounded in ω_2, pick $\delta < \omega_2$ such that $\operatorname{otp}(b^{-1\prime\prime}\{1\} \cap \delta) > \omega_1$. Then $\operatorname{otp} d_{\alpha_\delta} = \operatorname{otp} t_{\alpha_\delta}^{-1\prime\prime}\{1\} = \operatorname{otp} t_{\alpha_\delta}^{\prime\, -1\prime\prime}\{1\} = \operatorname{otp}(b^{-1\prime\prime}\{1\} \cap \delta) > \omega_1$, contradicting the approachability of $\langle d_\alpha \mid \alpha < \omega_2\rangle$ since $\alpha_\delta \in C$. ⊣

By Claim 3.2.8.1, there are at most $2^{<\omega_2} = \omega_2$ many branches through T'. Thus by Theorem 3.2.4 there are a stationary $S \subset C$ and $d \subset \omega_2$ such that $d_\alpha = d'_\alpha = d \cap \alpha$ for all $\alpha \in S$, which contradicts the approachability of $\langle d_\alpha \mid \alpha < \omega_2\rangle$. □

3.3 Forest Properties

Using another collapse, we can extend the results of Section 3.2 to $\mathfrak{P}_\kappa\lambda$. It is slightly ironic that the proof of Theorem 3.2.4 required so much work although its conclusion is weak measured by its consistency strength, while concluding now to theorems of much higher consistency strength is almost straightforward.

3.3.1 Theorem (PFA). (ω_2, λ)-ITP *holds for all* $\lambda \geq \omega_2$.

Proof. Let $\langle d_a \mid a \in \mathfrak{P}_{\omega_2}\lambda\rangle$ be a thin list, and let $E \subset \mathfrak{P}_{\omega_2}\lambda$ be club witnessing it is thin. For $z \in E$ let $T_z := \{d_a \cap z \mid z \subset a \in \mathfrak{P}_{\omega_2}\lambda\} \in \mathfrak{P}_{\omega_2}(\mathfrak{P}z)$.

Let $\mathbb{P} := \operatorname{Coll}(\omega_2, \lambda)$ and $G \subset \mathbb{P}$ be V-generic.

3.3.1.1 Claim. *In* $V[G]$, *there is a* $\subset$-*increasing continuous sequence* $\langle X_\alpha \mid \alpha < \omega_2\rangle$ *cofinal in* $\mathfrak{P}^V_{\omega_2}\lambda$ *with* $X_\alpha \in E$ *for all* $\alpha < \omega_2$.

Proof. Let $f : \omega_2 \to \lambda$ be bijective. Recursively define $\langle X_\alpha \mid \alpha < \omega_2\rangle$ as follows. Suppose $\delta < \omega_2$ and X_α has been defined for all $\alpha < \delta$. If $\delta = \beta + 1$, take $X_\delta \in E$ such that $X_\beta \cup \{f(\beta)\} \subset X_\delta$. If δ is a limit ordinal, then, as $\mathbb{P}$ is ω_2-distributive, $\langle X_\alpha \mid \alpha < \delta\rangle \in V$, so $X_\delta := \bigcup\{X_\alpha \mid \alpha < \delta\} \in E$. ⊣

Now work in $V[G]$ and let $\langle X_\alpha \mid \alpha < \omega_2\rangle$ be as in Claim 3.3.1.1. We may assume $\emptyset \in E$ and $X_0 = \emptyset$, as well as $|X_{\alpha+1} - X_\alpha| = \omega_1$ for all $\alpha < \omega_2$. For $\alpha < \omega_2$, let $s_\alpha : [\omega_1 \cdot \alpha, \omega_1 \cdot (\alpha + 1)) \to X_{\alpha+1} - X_\alpha$ be bijective. Set $s := \bigcup\{s_\alpha \mid \alpha < \omega_2\}$. Then $s : \omega_2 \to \lambda$ is bijective, $C := \{\gamma < \omega_2 \mid \forall\alpha < \gamma\ \omega_1 \cdot (\alpha + 1) < \gamma\}$ is club, and if $\gamma \in C$, then

$$s''\gamma = s''\bigcup\{[\omega_1 \cdot \alpha, \omega_1 \cdot (\alpha + 1)) \mid \alpha < \gamma\} = \bigcup\{\operatorname{rng} s_\alpha \mid \alpha < \gamma\} = X_\gamma. \tag{3.3}$$

For $\alpha < \omega_2$, define $\tilde{d}_\alpha := s^{-1\prime\prime} d_{X_\alpha}$ if $\alpha \in C$, $\tilde{d}_\alpha := \emptyset$ otherwise. Then for $\alpha \in C$ by (3.3) $\tilde{d}_\alpha \subset \alpha$. Furthermore $\langle \tilde{d}_\alpha \mid \alpha < \omega_2\rangle$ is thin, for if $\alpha, \delta \in C$, then

$$\tilde{d}_\alpha \cap \delta = s^{-1\prime\prime} d_{X_\alpha} \cap \delta = s^{-1\prime\prime}(d_{X_\alpha} \cap s''\delta) = s^{-1\prime\prime}(d_{X_\alpha} \cap X_\delta) \in \{s^{-1\prime\prime} y \mid y \in T_{X_\delta}\}.$$

By Proposition 3.1.9, PFA holds in $V[G]$, so by Theorem 3.2.5 $\langle \tilde{d}_\alpha \mid \alpha < \omega_2 \rangle$ is not effable. Thus there are a stationary $\tilde{S} \subset C$ and $\tilde{d} \subset \omega_2$ such that $\tilde{d}_\alpha = \tilde{d} \cap \alpha$ for all $\alpha \in \tilde{S}$. Let $d := s''\tilde{d}$ and

$$S := \{a \in \mathfrak{P}^V_{\omega_2}\lambda \mid d_a = d \cap a\}.$$

3.3.1.2 Claim. *If $D \in V$ is such that $D \subset \mathfrak{P}^V_{\omega_2}\lambda$ is club in V, then $D \cap S \neq \emptyset$.*

Proof. We first show $R := \{\alpha \in C \mid X_\alpha \in D\}$ is club in ω_2. To see it is unbounded, take $\alpha_0 < \omega_2$ and recursively define sequences $\langle \alpha_n \mid n < \omega \rangle$ and $\langle l_n \mid n < \omega \rangle$ such that $l_n \in D$ for all $n < \omega$ and $X_{\alpha_0} \subset l_0 \subset X_{\alpha_1} \subset l_1 \subset \ldots$. Let $\alpha := \sup_{n<\omega} \alpha_n$. Then by the ω_2-distributivity of $\mathbb{P}$ it follows that $\langle l_n \mid n < \omega \rangle \in V$, so $X_\alpha = \bigcup \{l_n \mid n < \omega\} \in D$ and thus $\alpha \in R$. R is closed, for if $\delta < \omega_2$ and $\langle \alpha_\nu \mid \nu < \delta \rangle$ is an increasing sequence with supremum α such that $\alpha_\nu \in R$ for all $\nu < \delta$, then $\langle X_{\alpha_\nu} \mid \nu < \delta \rangle \in V$ and thus $X_\alpha \in D$, so $\alpha \in R$.

Now since R is club, there exists $\alpha \in R \cap \tilde{S}$. But then $d_{X_\alpha} = s''\tilde{d}_\alpha = s''(\tilde{d} \cap \alpha) = s''\tilde{d} \cap s''\alpha = d \cap X_\alpha$, so $X_\alpha \in D \cap S$. ⊣

It remains to show $d \in V$ because then $S \in V$ and S is stationary in V by Claim 3.3.1.2. Let $z \in E$. Choose $a \in S$ with $z \subset a$. Then $d \cap z = d \cap a \cap z = d_a \cap z \in T_z$. By Proposition 2.1.12, $\mathbb{P}$ satisfies the thin ω_2-approximation property, so that $d \in V$ as wanted. □

A closer look at the proof of Theorem 3.3.1 will reveal we actually did not require the full strength of thin. It is sufficient for the $\mathfrak{P}_\kappa\lambda$-list $\langle d_a \mid a \in \mathfrak{P}_\kappa\lambda \rangle$ to satisfy the weaker requirement that for a club $C \subset \mathfrak{P}_\kappa\lambda$ and every $c \in C$ there is a $z_c \in \mathfrak{P}_\kappa\lambda$ such that $|\{d_a \cap c \mid z_c \subset a \in \mathfrak{P}_\kappa\lambda\}| < \kappa$ for all $c \in C$. Since this definition is equivalent in the one cardinal case, it might be a reasonable modification of thin, possibly even a more natural generalization.

The failure of weak square under PFA is originally due to Todorčević and Magidor, see [Tod84a] and [Sch95, Theorem 6.3].

3.3.2 Corollary (PFA). *Suppose* cf $\lambda \geq \omega_2$. *Then* $\neg\square^{\omega_2}(\omega_2, \lambda)$.

Proof. This follows from Theorems 1.4.2 and 3.3.1. □

If we weaken Theorem 3.2.4 by replacing "slender" with "thin," then in its proof we can replace $\mathbb{P}$ by $\mathrm{Coll}(\omega_1, \omega_2)$ as we only need the thin ω_1-approximation property. In [Kön07], Bernhard König introduces a weakening PFA(Γ_Σ) of PFA which is sufficient for forcings of the form $\mathrm{Coll}(\omega_1, \omega_2) * \mathrm{ccc}$ and $\mathrm{Coll}(\omega_2, \lambda)$ but consistent with $\omega_2 \in I_{\mathrm{AS}}[\omega_2]$. Thus in any model of PFA(Γ_Σ) + "$\omega_2 \in I_{\mathrm{AS}}[\omega_2]$" Theorem 3.2.4 holds for thin ω_2-lists, and thus also Theorem 3.3.1. Therefore it is consistent that we have (ω_2, λ)-ITP holds for all $\lambda \geq \omega_2$ but, via Corollary 3.2.8, $\neg\omega_2$-ISP.

3.4 Slenderness Revisited

While Chapters 1 and 2 provided a homogeneous picture of *slender*, suggesting it is a more natural replacement for *thin*, Chapter 3 so far left much about *slender* in the unclear. In this section, we are going to close this gap.

The main problem of our previous attempt to prove ω_2-ISP from PFA was the limitation on the number of the branches of the corresponding tree. Reading a draft of this thesis, Matteo Viale made an important observation; a small modification of the proof of Theorem 3.2.4 solves the problem. One just needs to reverse two steps of the proof: *First* collapse everything necessary to ω_1, *then* apply Lemma 3.2.2 to find the Baumgartner function g. Theorem 3.4.1 should thus be seen as his contribution.

3.4.1 Theorem (PFA). *ω_2-ISP holds.*

Proof. Let $\langle d_\alpha \mid \alpha < \omega_2 \rangle$ be a slender ω_2-list, and let $T := \mathrm{dc}\{t_\alpha \mid \alpha < \omega_2\}$, where $\langle t_\alpha \mid \alpha < \omega_2 \rangle$ are the characteristic functions of $\langle d_\alpha \mid \alpha < \omega_2 \rangle$. Let $B := \{b \subset T \mid b \text{ cofinal branch}\}$.

Define $\mathbb{P} := \mathbb{C} * \mathrm{Coll}(\omega_1, \max\{|B|, \omega_2\})$. Let $\dot{c} \in V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models$ "$\dot{c} : \omega_1 \to \omega_2^V$ is continuous and cofinal". As $\mathbb{P}$ satisfies the ω_1-approximation property by Theorem 2.2.6, we have $V^{\mathbb{P}} \models B = \{b \subset T \mid b \text{ cofinal branch}\}$. Thus by Lemma 3.2.2, there is $\dot{g} \in V^{\mathbb{P}}$ such that $V^{\mathbb{P}} \models$ "$\dot{g} : B \to T \upharpoonright \mathrm{rng}\, \dot{c}$ is a Baumgartner function".

Let $\dot{T}^0, \dot{T}^1 \in V^{\mathbb{P}}$ be such that

$$V^{\mathbb{P}} \models \dot{T}^0 = \{t \in T \upharpoonright \mathrm{rng}\, \dot{c} \mid \exists b \in B\ \dot{g}(b) < t \in b\},$$
$$V^{\mathbb{P}} \models \dot{T}^1 = T \upharpoonright \mathrm{rng}\, \dot{c} - \dot{T}^0.$$

Then again $V^{\mathbb{P}} \models$ "$\dot{T}^1$ has no cofinal branch". Therefore there is a $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ such that $V^{\mathbb{P}} \models$ "$\dot{\mathbb{Q}}$ is ccc and specializes $\dot{T}^1$". Let $\dot{f} \in V^{\mathbb{P} * \dot{\mathbb{Q}}}$ be a name for the specialization map.

Let θ be large enough.

3.4.1.1 Claim. *Let $M \in \mathfrak{P}'_{\omega_2} H_\theta$ be such that $\langle d_\alpha \mid \alpha < \omega_2 \rangle, T, \dot{c}, \dot{g}, \dot{T}^0, \dot{T}^1, \mathbb{P} * \dot{\mathbb{Q}}, \dot{f} \in M$. Suppose there is an M-generic filter $G \subset \mathbb{P} * \dot{\mathbb{Q}}$. Let $\delta_M := M \cap \omega_2$. Then there is a $b_M \in B \cap M$ such that*

$$t_{\delta_M} \in b_M.$$

Proof. Again $c := \dot{c}^G : \omega_1 \to \delta_M$ is continuous and cofinal. For $b \in B \cap M$ let

$$D_b := \{p \in \mathbb{P} * \dot{\mathbb{Q}} \mid \exists t \in T\ p \Vdash \dot{g}(b) = t\}.$$

Then $D_b \in M$ is dense, so $g := \dot{g}^G \upharpoonright M : B \cap M \to (T \upharpoonright \mathrm{rng}\, c) \cap M = \mathrm{dc}\{t_\alpha \mid \alpha < \delta_M\} \upharpoonright \mathrm{rng}\, c$ is a Baumgartner function. Let $T^0 := (\dot{T}^0)^G \cap M$ and $T^1 := (\dot{T}^1)^G \cap M$. Again for $f := \dot{f}^G \upharpoonright M$ we have $f : T^1 \to \omega$ is a specialization map. Also t_{δ_M} is a branch in $\mathrm{dc}\{t_\alpha \mid \alpha < \delta_M\}$ as

$M \models$ "$\langle d_\alpha \mid \alpha < \omega_2 \rangle$ is slender". Thus there is an $\alpha < \delta_M$ such that $t_{\delta_M} \upharpoonright \beta \in T^0$ for all $\beta \in \operatorname{rng} c - \alpha$.

For $\beta \in \operatorname{rng} c - \alpha$

$$E_\beta := \{p \in \mathbb{P} * \dot{\mathbb{Q}} \mid \exists b \in B \; p \Vdash \text{“}t_{\delta_M} \upharpoonright \beta \in \dot{T}^0 \rightarrow \dot{g}(b) < t_{\delta_M} \upharpoonright \beta \in b\text{”}\}$$

is dense. Also $E_\beta \in M$ as $t_{\delta_M} \upharpoonright \beta \in M$ because t_{δ_M} is a branch in $\operatorname{dc}\{t_\alpha \mid \alpha < \delta_M\}$. Hence for every $\beta \in \operatorname{rng} c - \alpha$ there is $b_\beta \in B \cap M$ such that $g(b_\beta) < t_{\delta_M} \upharpoonright \beta \in b_\beta$.

Thus we can apply Lemma 3.2.3 and get an $s < t_{\delta_M}$ such that $t_{\delta_M} \in g^{-1}(s)$. So $b_M := g^{-1}(s) \in B \cap M$ is as wanted. ⊣

The proof is now finished exactly as the proof of Theorem 3.2.4. □

Note that Theorem 3.4.1 implies Corollary 3.2.8 immediately. It does require the stronger assumption of PFA, not just PFA(2^ω), though.

Using the same technique, we will now show PFA implies (ω_2, λ)-ISP for all $\lambda \geq \omega_2$. In the proofs of Theorems 3.2.4 and 3.4.1, we used the forcing $\mathbb{C} * \operatorname{Coll}(\omega_1, \omega_2)$, which satisfies the ω_1-approximation property by Theorem 2.2.6, for adding a club of order type ω_1 through ω_2. Here we need something adapted to the $\mathfrak{P}_{\omega_2} H_\theta$ situation. The following theorem provides us with the necessary tool. It is due to John Krueger [Kru08].[15]

3.4.2 Theorem. *Suppose $E \subset \mathfrak{P}_{\omega_2} H_\theta$ is club, where θ is sufficiently large. Let $\dot{\mathbb{P}}(E) \in V^{\mathbb{C}}$ be such that*

$$V^{\mathbb{C}} \models \dot{\mathbb{P}}(E) = \{\langle e_\alpha \mid \alpha \leq \gamma \rangle \mid \gamma < \omega_1,\ \forall \alpha \leq \gamma\ e_\alpha \in E,\ \langle e_\alpha \mid \alpha \leq \gamma \rangle \text{ is } \subset\text{-increasing and continuous}\},$$

ordered by end extension. Then $\mathbb{C} * \dot{\mathbb{P}}(E)$ is proper, satisfies the ω_1-approximation property, and there is $\dot{c} \in V^{\mathbb{C} * \dot{\mathbb{P}}(E)}$ such that

$$V^{\mathbb{C} * \dot{\mathbb{P}}(E)} \models \dot{c} : \omega_1 \rightarrow E \text{ is continuous and cofinal.}$$

In the proof of the next theorem, we use Theorem 3.4.2 to shoot a club of order type ω_1 through the club witnessing the slenderness of a $\mathfrak{P}_{\omega_2}\lambda$-list. This basically results in a slender tree in the extension, and we can treat it the same way we did in the proof of Theorem 3.4.1.

3.4.3 Theorem (PFA). (ω_2, λ)-ISP *holds for all $\lambda \geq \omega_2$.*

[15] It should be noted the theorem does not literally exist in [Kru08]. Proposition 2.2 of it shows, together with the rest of the paper, the forcing $\mathbb{C} * \dot{\mathbb{P}}(\mathfrak{P}^V_{\omega_1} H^V_\theta)$ for shooting a club of order type ω_1 through $\mathfrak{P}_{\omega_1} H_\theta$ satisfies the ω_1-approximation property. However, the proof is easily adapted to what we claim, and since the modifications are all straightforward and trivial, it is not reasonable to repeat the argument.

Proof. Let $\langle d_a \mid a \in \mathfrak{P}_{\omega_2}\lambda\rangle$ be a slender $\mathfrak{P}_{\omega_2}\lambda$-list, and let $E \subset \mathfrak{P}'_{\omega_2}H_\theta$ be a club witnessing its slenderness for some large enough θ. Define $B := {}^{\lambda}2$. Let $\mathbb{P} := \mathbb{C} * \dot{\mathbb{P}}(E)$ be as in Theorem 3.4.2.

Work in $V^{\mathbb{P}}$. As $\mathbb{P}$ satisfies the ω_1-approximation property, we have $B = \{h \in {}^{\lambda}2 \mid \forall a \in \mathfrak{P}^V_{\omega_1}\lambda\ h \restriction a \in V\}$. Let $\dot{c} : \omega_1 \to E$ be continuous and cofinal. Define

$$\dot{T} := \bigcup\{{}^{\dot{c}(\alpha)}2 \cap V \mid \alpha < \omega_1\},$$

ordered by inclusion. Then $\dot{T}$ is a tree of height ω_1 and B the set of its cofinal branches. Since $|B| = \omega_1$, we can apply Lemma 3.2.2 and get a Baumgartner function $\dot{g} : B \to \dot{T}$. Let

$$\dot{T}^0 := \{t \in \dot{T} \mid \exists b \in B\ \dot{g}(b) < t \in b\},$$
$$\dot{T}^1 := \dot{T} - \dot{T}^0.$$

As $\dot{T}^1$ does not have cofinal branches, there is a ccc forcing $\dot{\mathbb{Q}}$ that specializes $\dot{T}^1$ with a specialization map $\dot{f}$.

Now work in V. Let $\bar{\theta}$ be large enough and set $\bar{E} := E^{H_{\bar{\theta}}}$.

3.4.3.1 Claim. *Let $\bar{M} \in \bar{E}$ be such that $B, E, \dot{c}, \dot{T}, \dot{g}, \dot{T}^0, \dot{T}^1, \mathbb{P} * \dot{\mathbb{Q}}, \dot{f} \in \bar{M}$. Suppose there is an $\bar{M}$-generic filter $G \subset \mathbb{P} * \dot{\mathbb{Q}}$. Then there is $h_{\bar{M}} \in B \cap \bar{M}$ such that*

$$d_{\bar{M}\cap\lambda} = h_{\bar{M}}^{-1\prime\prime}\{1\} \cap \bar{M} \cap \lambda.$$

Proof. By the usual density argument, $c := \dot{c}^G : \omega_1 \to E \cap \bar{M}$ is continuous and cofinal. We let $g := \dot{g}^G \restriction \bar{M}$, $T := \dot{T}^G \cap \bar{M}$, $T^0 := (\dot{T}^0)^G \cap \bar{M}$, $T^1 := (\dot{T}^1)^G \cap \bar{M}$, and $f := \dot{f}^G \restriction \bar{M}$. Then $g : B \cap \bar{M} \to T$ is a Baumgartner function, and $f : T^1 \to \omega$ is a specialization map.

Let $t : \bar{M} \cap \lambda \to 2$ be the characteristic function of $d_{\bar{M}\cap\lambda}$. We show t is a branch through T. So let $\beta < \omega_1$. Since $\bar{M} \cap H_\theta \in E$ and $c(\beta) \cap \lambda \in \bar{M} \cap H_\theta \cap \mathfrak{P}_{\omega_2}\lambda$, we have $d_{\bar{M}\cap\lambda} \cap c(\beta) \in \bar{M} \cap H_\theta$ by the slenderness of $\langle d_a \mid a \in \mathfrak{P}_{\omega_2}\lambda\rangle$. Thus $t \restriction c(\beta) \in \bigcup\{{}^{c(\alpha)}2 \cap \bar{M} \mid \alpha < \omega_1\} = T$.

As f shows T^1 is special, there is an $\alpha < \omega_1$ such that $t \restriction c(\beta) \in T^0$ for all $\beta \in \operatorname{rng} c - \alpha$, so by Lemma 3.2.3 we have $t \in g^{-1}(s) =: h_{\bar{M}}$ for some $s \in T$. Hence $d_{\bar{M}\cap\lambda} = h_{\bar{M}}^{-1\prime\prime}\{1\} \cap \bar{M} \cap \lambda$. ⊣

By Proposition 3.1.2, there are stationarily many $\bar{M} \in \bar{E}$ that satisfy the premise of Claim 3.4.3.1, and we may assume $h_{\bar{M}} = h$ for some h and all those $\bar{M}$. This shows $\langle d_a \mid a \in \mathfrak{P}_{\omega_2}\lambda\rangle$ is not effable. □

Conclusion

Closing Words

To sum up the course of the thesis, one could say we started with *thin* and arrived at *slender*. While *thin* provided the basis for this research, viewing back one cannot help but feel *slender* has risen as its heir. It might not be as intuitive as *thin*, but the way it naturally generalizes the approachability ideal and finds applications in proofs makes it feel like a fruitful concept which deserves further investigation. Although (κ, λ)-ITP does not imply (κ, λ)-ISP, they coincide in the natural models we considered. Hence we believe *thin* should only be seen as an intermediate concept that was necessary to find *slender*, not unlike PFA, which can be seen as a technical precursor to the natural forcing axiom MM. Like PFA however, *thin* still deserves attention as it is easier to grasp and a sufficient replacement for *slender* in many applications.

One of the main advantages of MM over PFA had always been the knowledge about the framework of strong reflection principles MM implies. Therefore one might hope for (ω_2, λ)-ITP or (ω_2, λ)-ISP to form a similar unified framework that can be utilized under PFA. Corollaries 3.2.7, 3.2.8, and 3.3.2 showed at least some of the known standard consequences of PFA can be derived from this framework.

Open Questions

Finally we present some open questions and conjectures. Some were left open from the beginning, others were grown back by the mathematical Hydra when one was answered.

Looking at how Proposition 1.3.15 generalized Proposition 1.2.14, it seems reasonable to expect Proposition 1.2.15 to generalize in the following way.

Conjecture 1. *If τ is regular and $\kappa = \tau^+$, then*

$$\{a \in \mathfrak{P}_\kappa \lambda \mid \operatorname{Lim} a \cap \operatorname{cof}(< \tau) \subset a\} \in F_{\mathrm{IT}}[\kappa, \lambda].$$

It might be possible to find a variant of the existence of partial squares that enables one to prove Conjecture 1.

Proposition 1.5.3 gives rise to the next question.

Question 2. *Suppose κ is regular uncountable and $\lambda \geq \kappa$ is such that* $\operatorname{cf} \lambda \geq \kappa$. *Is there an* $A \in F_{\mathrm{IT}}[\kappa, \lambda]$ *such that* $\langle \sup a \mid a \in A \rangle$ *is injective?*

A positive answer to Question 2 would entail the following generalization of Proposition 1.5.4, even with the requirement $2^{<\kappa} = \kappa$ replaced by $2^{<\kappa} \leq \lambda$.

Conjecture 3. *Suppose κ is regular uncountable such that* $2^{<\kappa} = \kappa$. *If* $\operatorname{cf} \lambda \geq \kappa$ *and* (κ, λ)-ITP *holds, then*

$$\lambda^{<\kappa} = \lambda.$$

Note that Conjecture 3 would imply a different proof of the theorem due to Viale that PFA implies the Singular Cardinal Hypothesis, confer [Via06].

Mitchell showed in [Mit09] it is consistent that $I_{\mathrm{AS}}[\omega_2]$ is the nonstationary ideal on the ordinals of uncountable cofinality.

Question 4. *Is it consistent that* $I_{\mathrm{IS}}[\omega_2]$ *does not contain any stationary subset of* $\operatorname{cof} \omega_1 \cap \omega_2$?

Since in any such model by an argument due to Shelah it holds that $2^{\omega} \geq \omega_3$, a positive answer to Question 4 would also imply a positive answer to Question 5.

Question 5. *Is it consistent that* ω_2-ITP *holds and* $2^{\omega} \geq \omega_3$?

Question 6. *Suppose* (ω_2, λ)-ISP *holds for all* $\lambda \geq \omega_2$. *Does this imply* $2^{\omega} = \omega_2$?

The last two questions are motivated by [MS96], which shows that if κ is the singular limit of strongly compact cardinals, then κ^+-TP holds.

Question 7. *Suppose κ is the singular limit of supercompact cardinals. Does this imply* κ^+-ITP?

From this, the authors establish the consistency of $\omega_{\omega+1}$-TP using some very large cardinal assumptions.

Question 8. *Is* $\omega_{\omega+1}$-ITP *consistent?*

Bibliography

[Bau83] J. E. Baumgartner, *Iterated forcing*, Surveys in set theory, London Math. Soc. Lecture Note Ser., vol. 87, Cambridge Univ. Press, Cambridge, 1983, pp. 1–59. MR 823775, Zbl 0524.03040

[Bau84] ______, *Applications of the proper forcing axiom*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 913–959. MR 776640, Zbl 0556.03040

[Bea91] R. E. Beaudoin, *The proper forcing axiom and stationary set reflection*, Pacific J. Math. **149** (1991), no. 1, 13–24. MR 1099782, Zbl 0687.03034

[Car86] D. M. Carr, *The structure of ineffability properties of* $P_\kappa\lambda$, Acta Math. Hungar. **47** (1986), no. 3-4, 325–332. MR 854519, Zbl 0612.03023

[Git85] M. Gitik, *Nonsplitting subset of* $\mathscr{P}_\kappa(\kappa^+)$, J. Symbolic Logic **50** (1985), no. 4, 881–894 (1986). MR 820120, Zbl 0601.03021

[Hau14] F. Hausdorff, *Grundzüge der Mengenlehre*, Veit, Leipzig, 1914 (German), Mit 53 Figuren im Text. MR 31025, Zbl 45.0123.01

[Jec03] T. Jech, *Set theory*, Springer Monographs in Mathematics, Springer, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513, Zbl 1007.03002

[Jec73] ______, *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic **5** (1972/73), 165–198. MR 325397, Zbl 0262.02062

[Joh90] C. A. Johnson, *Some partition relations for ideals on* $P_\kappa\lambda$, Acta Math. Hungar. **56** (1990), no. 3-4, 269–282. MR 1111312, Zbl 0733.03039

[JSSS09] R. Jensen, E. Schimmerling, R. Schindler, and J. Steel, *Stacking mice*, J. Symbolic Logic **74** (2009), no. 1, 315–335. MR 2499432, Zbl 1161.03031

[KM78] A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory*, Higher set theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977), Lecture Notes in Math., vol. 669, Springer, Berlin, 1978, pp. 99–275. MR 520190, Zbl 0381.03038

[Kön07] B. König, *Forcing indestructibility of set-theoretic axioms*, J. Symbolic Logic **72** (2007), no. 1, 349–360. MR 2298486, Zbl 1128.03043

[Kru08] J. Krueger, *A general Mitchell style iteration*, MLQ Math. Log. Q. **54** (2008), no. 6, 641–651. MR 2472470, Zbl 1160.03033

[Kun80] K. Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, Amsterdam, 1980, An introduction to independence proofs. MR 597342, Zbl 0443.03021

[KY04] B. König and Y. Yoshinobu, *Fragments of Martin's maximum in generic extensions*, MLQ Math. Log. Q. **50** (2004), no. 3, 297–302. MR 2050172, Zbl 1047.03039

[Mag74] M. Magidor, *Combinatorial characterization of supercompact cardinals*, Proc. Amer. Math. Soc. **42** (1974), 279–285. MR 327518, Zbl 0279.02050

[Men75] T. K. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic **7** (1974/75), 327–359. MR 357121, Zbl 0299.02084

[Mit04] W. J. Mitchell, *A weak variation of Shelah's* $I[\omega_2]$, J. Symbolic Logic **69** (2004), no. 1, 94–100. MR 2039349, Zbl 1073.03028

[Mit09] ______, $I[\omega_2]$ *can be the nonstationary ideal on* $\mathrm{Cof}(\omega_1)$, Trans. Amer. Math. Soc. **361** (2009), no. 2, 561–601. MR 2452816

[Mit73] ______, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic **5** (1972/73), 21–46. MR 313057, Zbl 0255.02069

[MS96] M. Magidor and S. Shelah, *The tree property at successors of singular cardinals*, Arch. Math. Logic **35** (1996), no. 5-6, 385–404. MR 1420265, Zbl 0874.03060, arXiv:math.LO/9501220

[NS08] I. Neeman and E. Schimmerling, *Hierarchies of forcing axioms. I*, J. Symbolic Logic **73** (2008), no. 1, 343–362. MR 2387946, Zbl 1154.03032

[Sch95] E. Schimmerling, *Combinatorial principles in the core model for one Woodin cardinal*, Ann. Pure Appl. Logic **74** (1995), no. 2, 153–201. MR 1342358, Zbl 0834.03018

[She91] S. Shelah, *Reflecting stationary sets and successors of singular cardinals*, Arch. Math. Logic **31** (1991), no. 1, 25–53. MR 1126352, Zbl 0742.03017

[She98] ______, *Proper and improper forcing*, second ed., Perspectives in Mathematical Logic, Springer, Berlin, 1998. MR 1623206, Zbl 0889.03041

[Sol74] R. M. Solovay, *Strongly compact cardinals and the GCH*, Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971) (Providence, R.I.), Amer. Math. Soc., 1974, pp. 365–372. MR 379200, Zbl 0317.02083

[Tod] S. Todorčević, *Coherent sequences*, Handbook of Set Theory, North-Holland, Amsterdam, forthcoming.

[Tod84a] ———, *A note on the proper forcing axiom*, Axiomatic set theory (Boulder, Colo., 1983), Contemp. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1984, pp. 209–218. MR 763902, Zbl 0564.03038

[Tod84b] ———, *Trees and linearly ordered sets*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 235–293. MR 776625, Zbl 0557.54021

[Tod87] ———, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), no. 3-4, 261–294. MR 908147, Zbl 0658.03028

[Via06] M. Viale, *The proper forcing axiom and the singular cardinal hypothesis*, J. Symbolic Logic **71** (2006), no. 2, 473–479. MR 2225888, Zbl 1098.03053

[Woo99] W. H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, de Gruyter Series in Logic and its Applications, vol. 1, Walter de Gruyter & Co., Berlin, 1999. MR 1713438, Zbl 1159.03326

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