Self-organized critical phenomena. Forest fire and sandpile models

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Zusammenfassung

Das Konzept selbstorganisierter Kritizität wurde eingeführt um das Auftreten fraktaler Strukturen in verschiedenen natürlichen Phänomenen besser zu verstehen. Selbstorganisierter Kritizität liegt die Idee zugrunde, dass eine interne Dynamik ein System zu einem stationären Zustand führt, der sich durch Wechselbeziehungen für die Potenzgesetze in Zeit und Ort gelten charakterisiert. Wir untersuchen die zwei bekanntesten Modelle, die eingeführt wurden, um das Phänomen selbstorganisierter Kritizität zu studieren.

Das erste der Modelle ist das Waldbrandmodell. In einem Waldbrandmodell ist ein jeder Knoten eines Graphen entweder frei oder durch einen Baum belegt. Freie Knoten werden entsprechend unabhängiger Poisson Prozesse mit Rate eins belegt. Unabhängig davon tritt an einem jeden Knoten Entzündung (durch Blitzschlag) auf, entsprechend unabhängiger Poisson Prozesse mit Rate $\lambda > 0$. Wenn sich ein Knoten entzündet, so wird die gesamte Zusammenhangskomponente belegter Knoten des entzündeten Knotens unverzüglich frei.

Es ist bekannt, dass im unendlichen Volumen Waldbrandprozesse zu einer jeden Blitzschlagrate $\lambda > 0$ existieren. Der Existenzbeweis ist ziemlich abstrakt und impliziert nicht die Eindeutigkeit. Des Weiteren beantwortet die Konstruktion nicht die Frage, ob ein Waldbrandprozess auf einem Graphen G mit unendlichem Volumen messbar bezüglich seiner treibenden Poisson Prozesse ist. Motiviert durch diese Fragen zeigen wir die fast sichere Konvergenz einer Folge von Waldbrandprozessen auf endlichen, gegen den Graphen G wachsender Teilgraphen bezüglich ihrer treibenden Poisson Prozesse. Der Beweis ist ziemlich allgemeingültig und umfasst alle Graphen mit beschränktem Verzweigungsgrad, alle positiven Blitzschlagraten $\lambda > 0$, und eine relativ große Klasse von Anfangsbedingungen. Einer der Hauptbestandteile des Beweises ist eine Abschätzung des Abfalls der Größenverteilung der Zusammenhangskomponenten in einem Waldbrandmodell. Für $\gamma > 0$ betrachten wir die Wahrscheinlichkeit, dass die Zusammenhangskomponente eines Knoten x zu einer Zeit $t \geq \gamma$ größer als m ist, bedingt auf die Konfiguration einiger weiterer Zusammenhangskomponenten zur Zeit t. Wir zeigen, dass diese bedingte Wahrscheinlichkeit im Limes m gegen unendlich gegen Null konvergiert; uniform in der Wahl des Knoten x, der Zeit $t \geq \gamma$ und der Konfiguration der weiteren Zusammenhangskomponenten auf die wir bedingen. Als Konsequenz der fast sicheren Konvergenz erhalten wir die Messbarkeit und Eindeutigkeit bezüglich der treibenden Poisson Prozesse, und die Markov Eigenschaft.

Das zweite untersuchte Modell ist das Abelsche Sandstapelmodell. Es sei Λ eine endliche Teilmenge des zweidimensionalen Gitters $\mathbb{Z}^2$. Wir betrachten das folgende Sandstapelmodell auf Λ: Ein jeder Knoten in Λ enthält einen Sandstapel mit einer Höhe von ein bis vier Sandkörnern. Zu diskreten Zeitpunkten wählen wir zufällig einen Knoten $v \in \Lambda$ und fügen ein Sandkorn zu dem Knoten v hinzu. Falls nach dem Hinzufügen des Sandkorns die Höhe des Stapels bei v echt größer als vier ist, so fällt der Stapel zusammen. Das heißt, vier Sandkörner verlassen den Knoten v, und ein jeder Nachbar mit Abstand eins von v erhält eines dieser Sandkörner. Falls es nach dem Zusammenfallen des Stapels bei v weitere Stapel mit eine Höhe strikt größer als vier gibt, so lassen wir diese zusammenfallen, bis wir eine Konfiguration erhalten in der alle Stapel eine Höhe

zwischen eins und vier haben.

Wir untersuchen den Skalenlimes des Feldes von Seiten mit Höhe eins in solch einem Sandstapelmodell. Genauer gesagt, wir identifizieren den Skalenlimes der Kovarianz davon Höhe eins bei zwei makroskopisch voneinander entfernten Knoten zu haben. Wir zeigen, dass dieser Skalenlimes konform kovariant ist. Darüber hinaus zeigen wir einen zentralen Grenzwertsatz für das Feld der Knoten mit Höhe. Unsere Resultate basieren auf einer Darstellung der gemeinsamen Intensitäten der Indikatorfunktionen die Höhe eins anzeigen, welche einer blockdeterminantalen Struktur ähnlich ist.

Abstract

The concept of self-organized criticality was proposed as an explanation for the occurrence of fractal structures in diverse natural phenomena. Roughly speaking the idea behind self-organized criticality is that a dynamic drives a system towards a stationary state that is characterized by power law correlations in space and time. We study two of the most famous models that were introduced as models exhibiting self-organized criticality.

The first of them is the forest fire model. In a forest fire model each site (vertex) of a graph is either vacant or occupied by a tree. Vacant sites get occupied according to independent rate 1 Poisson processes. Independently, at each sites ignition (by lightning) occurs according to independent Poisson processes that have rate $\lambda > 0$. When a site is ignited its whole cluster of occupied sites becomes vacant instantaneously.

It is known that infinite volume forest fire processes exist for all ignition rates $\lambda > 0$. The proof of existence is rather abstract, and does not imply uniqueness. Nor does the construction answer the question whether infinite volume forest fire processes are measurable with respect to their driving Poisson processes. Motivated by these questions, we show the almost sure infinite volume convergence for forest fire models with respect to their driving Poisson processes. Our proof is quite general and covers all graphs with bounded vertex, all positive ignition rates $\lambda > 0$, and a quite large set of initial configurations. One of the main ingredients of the proof is an estimate for the decay of the cluster size distribution in a forest fire model. For $\gamma > 0$, we study the probability that the cluster at site x and time $t \geq \gamma$ is larger than m, conditioned on the configuration of some further clusters at time t. We show that as m tend to infinity, this conditional probability decays to zero. The convergence is uniform in the choice of the site x, the time t, and the configuration of the further clusters we condition on. Being a consequence of almost sure infinite volume convergence, we obtain uniqueness and measurability with respect to the driving Poisson processes, and the Markov property.

The second model in focus is the Abelian sandpile model. Let Λ be a finite subset of the two-dimensional integer lattice. We consider the following sandpile model on Λ: each vertex in Λ contains a sandpile with a height between one and four sand grains. At discrete times, we choose a site $v \in \Lambda$ randomly and add a sand grain at the site v. If after adding the sand grain the height at the site v is strictly larger than four, then the site topples. That is, four sand grains leave the site v, and each distance-one-neighbour of v gets one of these grains. If after toppling the site v there are other sites with a height strictly larger than four, we continue by toppling these sites until we obtain a configuration where all sites have a height between one and four.

We study the scaling limit for the height one field in such a sandpile model. More precisely, we identify the scaling limit for the covariance of having height one at two macroscopically distant sites. We show that this scaling limit is conformally covariant. Furthermore, we show a central limit theorem for the sandpile height one field. Our results are based on a representation of the height one joint intensities that is close to a block-determinantal structure.

Contents

Chapter 1

Introduction

1.1 Self-organized criticality

Many systems like a collection of electrons, a pile of sand grains, a bucket of fluid, or an ecosystem consist of many components that have some internal mechanism of interaction. Additionally to these internal interactions, there may be some driving external forces like a magnetic field, or a lightning that hits and ignites a tree in a forest. Driven by its external forces and its internal interactions, such a system will evolve in time. What happens? Does the behaviour depend crucially on the details of the system, or is there some simplifying mechanism that produces a typical behaviour shared by a large class of systems?

In [26] Mandelbrot discovered that many naturally occurring objects like mountain ranges, river networks, or coastlines are best described as fractals. Fractal structures frequently come along with correlation functions that show non-trivial power law behaviour.

Systems that exhibit correlations with power law decay over a wide range of length scales are said to have critical correlations. This is because correlations much larger than the length scale of interactions were first studied in equilibrium statistical mechanics in the neighbourhood of the critical phase transition. But, to observe such critical phenomena in equilibrium systems, one needs to fine-tune some physical parameters to specific critical values, something rather unlikely for a naturally occurring process.

In [3] P. Bak, C. Tang and K. Wiesenfeld argued that the dynamics which gives rise to the power law correlations seen in nature must not involve any fine tuning of parameters. It must be that some internal mechanism drives the system to a state that shows equilibrium critical phenomena. They coined the term ‘self-organized criticality’ to name such mechanisms. Phenomena in many fields of science have been claimed to exhibit self-organized criticality. It begun with sandpiles, earthquakes and forest fires. Next came electric breakdown, motion of magnetic flux lines in superconductors, water droplets on surfaces, dynamics of magnetic domains, and growing interfaces. Later on self-organized criticality models were applied to economics, and proposed as a way of understanding biological evolution. Various physical situations where the concept may apply are discussed in [16]. However, so far there does not exist a mathematical, nor a generally accepted definition of what self-organized criticality is.

In this work, we study two of the most famous models that were introduced as models exhibiting self-organized criticality: the forest fire process and the Abelian sandpile model. More precisely, we show almost sure infinite volume limit convergence for forest fire processes. And we study the scaling limit for the Abelian sandpile height one field.

1.2 Almost sure infinite volume convergence for forest fire processes

In a forest fire model on a graph $G = (V, E)$ each site (vertex) $x \in V$ has two possible states: either the site x is vacant, or occupied by a tree. The driving forces are two independent families of independent Poisson processes $(G_{t,x})_{t\geq 0}$, $x \in V$, and $(I_{t,x})_{t\geq 0}$, $x \in V$. We call $(G_{t,x})_{t\geq 0}$, $x \in V$, the growth processes. They have rate parameter 1. If one of them jumps the corresponding site gets occupied, respectively remains occupied. The processes $(I_{t,x})_{t\geq 0}$, $x \in V$, have rate parameter $\lambda > 0$. We call them the ignition processes. At the jump times of the ignition process at a site $x \in V$, the site x and its whole cluster burn down instantaneously. That is, the maximal nearest neighbour connected subset containing x and being occupied at each site gets vacant. Our forest fire model is a continuous time version of the Drossel-Schwabl forest fire model which has received much attention in the physics literature. See e.g. [9], [13] and [30].

The mathematical consideration of the forest fire process begun with [5]. Here J. van den Berg and A. A. Járai study the density of vacant sites and the cluster size distribution for forest fires on $\mathbb{Z}$. Likewise restricted to the case of forest fires on $\mathbb{Z}$, in [6] R. Brouwer and J. Pennanen show that there exists at least one stationary measure, and study the cluster size distribution in stationary state. For forest fires on $\mathbb{Z}^2$ with all sites vacant at time 0, the paper [4] discusses the behaviour near the 'critical time' t_c. Here t_c is defined by the relation $1 - \exp(-t_c) = p_c^2$, where p_c^d is the critical probability for site percolation on $\mathbb{Z}^d$, $d \in \mathbb{N}$. A percolation like assumption provided, it is shown that for fixed $t > t_c$, as simultaneously $\lambda \to 0$ and $m \to \infty$, the probability that some tree at distance smaller than m from 0 is burnt before time t does not converge to 1.

The existence of forest fire processes on $\mathbb{Z}^d$, $d \geq 2$, for all parameter $\lambda > 0$ is shown in [10]. A sequence of forest fire processes on finite sets $\Lambda \nearrow \mathbb{Z}^d$, the existence of weakly convergent subsequences, and Kolmogorov's Extension Theorem are used to define a process η on $\mathbb{Z}^d$. The key observation [[10], Lemma 18] to show that η satisfies the definition of a forest fire process is that the probability that a given non-empty cluster grows on its boundary before it gets hit by ignition is bounded by $2d/(2d+\lambda)$. However, due to the weak convergence, the construction does not imply the measurability of infinite volume forest fire processes with respect to their driving growth and ignition processes, nor their uniqueness. Closely related to this is the question whether the finite volume forest fire processes converge almost surely with respect to their driving growth and ignition processes.

In [11] subcritical site percolation is used to dominate the forest fire process and answer the latter questions affirmatively for the graph $\mathbb{Z}^d$ and $\lambda > (1-p_c^d)/p_c^d$. But [11] does not cover the case of major interest, where the parameter λ is small. Furthermore [11] restricts to forest fire processes with all sites vacant at initial time 0.

The main result of this work is the almost sure infinite volume limit convergence for forest fire processes with respect to their driving growth and ignition processes. Our result covers all ignition rates $\lambda > 0$, and all graphs where there exists a bound $d \in \mathbb{N}$ for the vertex degree. The only assumption we have to make concerns the initial configuration: let

$$P_{x,m} := P\left(|C_{0,x}| > m\right)$$

be the probability that the cluster at site $x \in V$ and initial time 0 is bigger than $m \in \mathbb{N}$. Our assumption is that there exists $m \in \mathbb{N}$ so that for all $x \in V$ the probability $P_{x,m}$ is smaller than some constant $D_{\lambda,d} < 1$, even if we additionally condition on the configuration of some further clusters. Here $D_{\lambda,d}$ is a constant that depends on the ignition rate λ and the bound for the vertex degree d. This assumption is satisfied for a quite general class of initial configurations. For example all configurations where the cluster size is bounded by a constant are covered. Another covered configuration is independent site percolation on $\mathbb{Z}^d$, as long as there are no infinite clusters.

Using the infinite volume convergence, we are able to answer our original question: we show that an infinite volume forest fire process is measurable with respect to its driving growth and ignition processes. Furthermore, we are able to show the Markov property for such forest fire processes.

In the course of the proof of almost sure convergence, we study the cluster size distribution. As a result, for all $\gamma > 0$ and all $\delta > 0$ we explicitly give a $m = m_{\gamma,\delta} \geq 0$ such that for all $t \geq \gamma$, all finite $B, D \subset V$, and all $x \in V \setminus D$

$$P\left(|C_{t,x}| > m \middle| \cup_{y \in B} C_{t,y} = D\right) < \delta.$$

Here we write $C_{t,x}$ for the cluster at site x and time t, and $|\cdot|$ to denote the cardinality.

Our results for the forest fire model are represented in Chapter 2. We start with a formal introduction to the forest fire model in Section 2.1. Our main results are stated in Section 2.2, and proven in Sections 2.3 - 2.6.

1.3 Scaling limit for the Abelian sandpile height one field

The Abelian sandpile model was introduced by P. Bak, C. Tang and K. Wiesenfeld [3, 2], and generalized by D. Dhar [8]. The model is defined on a finite lattice. We consider finite cut-out portions $\Lambda \subset \mathbb{Z}^2$ of the two-dimensional square lattice. Every site in $v \in \Lambda$ has a positive integer valued height variable η_v. We call η_v the height at the site $v \in \Lambda$. The system evolves discrete in time.

The dynamics of one time step is defined with respect to a toppling matrix. We explain the dynamics corresponding to the discrete Laplacian in Λ:

$$\Delta_\Lambda(v,w) = \begin{cases} 4 & \text{if } v = w; \\ -1 & \text{if } |v-w| = 1; \\ 0 & \text{otherwise,} \end{cases} \qquad \text{all } v,w \in \Lambda.$$

At each time step we pick a site $v \in \Lambda$ randomly, and increase its height by one. If the height of the site v became larger than $\Delta_\Lambda(v,v) = 4$, the site v is called unstable. It relaxes by toppling whereby four sand grains leave the site v, and each distance one neighbours gets one grain. If after toppling the site v there is any unstable site remaining, it is toppled, too. In case of toppling a site at the boundary of Λ, some grains are removed from the system. This process continues until all sites of Λ are stable, that is, have a height of at most four.

The mathematical study of the Abelian sandpile model was initiated by D. Dhar [8], who coined the name Abelian. He gave an characterization of the configurations that occur in the stationary state with positive probability. In [25] S.N. Majumdar and D. Dhar revealed a correspondence between such configurations and spanning trees. In [1] S.R. Athreya and A.A. Járai use this correspondence to study the infinite volume limit for the stationary distribution of Abelian sandpile models. They show the existence of the infinite volume limit in the weak sense. For a mathematical introduction to the Abelian sandpile model and further results see the review papers [14, 22, 27].

In [24] S.N. Majumdar and D. Dhar develop a powerful method to calculate the probability of specific subconfigurations in stationary state. In particular, they show that the covariance of having height one at two sites separated by distance r decays as r^{-4}. Their method has been extensively used and extended in the physics literature to support the conjecture that the scaling limit of the Abelian sandpile model can be described by a logarithmic conformal field theory (see e.g. [29, 23, 15] and [28]). Although the special case of the height one field seems to be well understood in the physics literature, we did not find any mathematical discussion of its scaling limit.

We study the scaling limit for the height one field of the two-dimensional Abelian sandpile model. Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a bounded connected domain with smooth boundary, and $U_\epsilon := U/\epsilon \cap \mathbb{Z}^2$, $\epsilon > 0$. We write μ_{U_ϵ} for the stationary distribution of the sandpile model on U_ϵ. Let $h_{U_\epsilon}(v)$ denote the indicator function of having height one at the site $v \in U_\epsilon$, and write $\mathbb{E}[h_{U_\epsilon}]$ to denote its expectation with respect to μ_{U_ϵ}. For every $u \in U$ and all $\epsilon > 0$ let $u_\epsilon \in U_\epsilon$ within $O(1)$ of u/ϵ.

Our first result concerns the covariance of having height one at two macroscopically distant sites: let $v, w \in U$, $v \neq w$, be two points in the interior of U. Then as $\epsilon \to 0$ the covariance of $h_{U_\epsilon}(v_\epsilon)$ and $h_{U_\epsilon}(w_\epsilon)$ rescaled by ϵ^{-4} tends to a finite limit $\mathrm{Cov}_U(v,w)$ which is conformally covariant with scale dimension 2. Here by conformal covariance with scale dimension 2 we mean that for any conformal isomorphism $f : U \mapsto U'$

$$\mathrm{Cov}_U(v,w) = |f'(v)|^2 \cdot |f'(w)|^2 \cdot \mathrm{Cov}_{U'}\left(f(v), f(w)\right).$$

More generally, we give an explicit and conformally covariant representation for the scaling limit of the rescaled and centred height one joint moments

$$\epsilon^{-2|V|}\mathbb{E}\Big[\prod_{v\in V}\big(h_{U_\epsilon}(v_\epsilon)-\mathbb{E}[h_{U_\epsilon}(v_\epsilon)]\big)\Big],$$

where $V\subset U$ is a set of finitely many points in the interior of U.

Furthermore, we show that the sandpile height one field converges to Gaussian white noise in the following sense. Let $n\geq 1$ and for all $1\leq i\leq n$ let $f_i : U\mapsto\mathbb{R}$ be a smooth function with support compactly contained in U. We integrate these test function over the centred height one field. For all $1\leq i\leq n$ let

$$f_i\diamond h_{U_\epsilon} := \frac{\epsilon}{\sqrt{\mathcal{V}}}\sum_{v\in U_\epsilon} f_i(\epsilon v)\cdot\big(h_{U_\epsilon}(v)-\mathbb{E}[h_{U_\epsilon}(v)]\big),$$

where $\mathcal{V}$ is a positive constant. We show that as ϵ tends to zero the random variables $f_i\diamond h_{U_\epsilon}$, $1\leq i\leq n$, converge in distribution to jointly normal random variables with mean zero and covariance matrix

$$\left(\int_U f_i(z)f_j(z)dz\right)_{1\leq i,j\leq n}.$$

The results are based on a representation of the height one joint intensities that is close to a block-determinantal structure.

We present our results for the Abelian sandpile model in Chapter 3. In Section 3.1 we start with an introduction to the model, and review some of its basic properties. Thereafter, we state our main results in Section 3.2, and prove them in Sections 3.3 - 3.6.

Chapter 2

Almost sure infinite volume convergence for forest fire processes

In this chapter we present our results for the forest fire model. We show for certain initial configurations the almost sure infinite volume convergence for forest fire models on graphs with bounded vertex degree with respect to their driving growth and ignition processes, for all positive ignition rates. Furthermore, we show that in a forest fire model the cluster size distribution decays uniformly in space and time.

In Section 2.1 we introduce the forest fire model. We state our main results in Section 2.2, and prove them subject to three key propositions in Section 2.3. The key propositions are proven in Sections 2.4 - 2.6.

2.1 The forest fire process

In this section we introduce the forest fire process. From now on let the graph $G = (V, E)$ be a connected graph with vertex set V, and E its set of undirected edges. Here by undirected we mean $\{x, y\} = \{y, x\}$ for all $\{x, y\} \in E$. We say that a graph $G = (V, E)$ is connected, if for all $x, y \in V$, $x \neq y$, there exists $\{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\} \subset E$ such that $v_1 = x$ and $v_n = y$. Furthermore, we suppose that the vertex degree of G is bounded. That is, we suppose the existence of $d \in \mathbb{N} := \{1, 2, \ldots\}$ so that every site $v \in V$ has at most d neighbours. Here we say that two sites $v, w \in V$ are neighbours if $\{v, w\} \in E$.

In a forest fire process on G every site $x \in V$ is either vacant or occupied by a tree. We write $\eta_{t,x} = 0$ to denote that the site $x \in V$ is vacant at time $t \geq 0$, and $\eta_{t,x} = 1$ if it is occupied. To describe the dynamics, we assign a pair of independent Poisson processes $(G_{t,x})_{t\geq 0}$ and $(I_{t,x})_{t\geq 0}$ to each site $x \in V$, independently of all other sites. The processes $(G_{t,x})_{t\geq 0, x\in V}$ have rate parameter 1 and are called the growth processes. If one of them jumps, there is a growth attempt at the corresponding site. The site gets occupied, respectively remains occupied if it has already been occupied. The processes $(I_{t,x})_{t\geq 0, x\in V}$ are called the ignition processes and have rate parameter $\lambda > 0$. At the jump times of the ignition process at a site $x \in V$, the site x and its whole cluster burn down instantaneously. Here by cluster at x we mean the maximal connected set of occupied sites containing the site x. For a formal definition of the term cluster, we introduce the term path first.

Definition 1 (Path)**.** Let $x \in V$, $S \subset V$ and $n \in \mathbb{N}$. A path of length n that connects the site x to a site in S is a vector $(p_i)_{1 \le i \le n} \in V^n$ that has the following properties:

(i) for all $1 < i \le n$ the sites p_{i-1} and p_i are neighbours, that is, $\{p_{i-1}, p_i\} \in E$;

(ii) the path consists of distinct sites, that is, $p_i \neq p_j$ for all $1 \le i \neq j \le n$;

(iii) the path starts at x and ends at a site in S, that is, $p_1 = x$ and $p_n \in S$.

We write $\mathrm{PATH}_n(x, S) \subset V^n$ to denote the set of paths of length n that connect the site x to a site in S, and define $\mathrm{PATH}(x, S) := \cup_{k \in \mathbb{N}} \mathrm{PATH}_k(x, S)$. For every path $\mathcal{P} \in \mathrm{PATH}(x, S)$, $\mathcal{P} = (p_i)_{1 \le i \le k}$, let $\mathcal{P}_V := \{p_i, 1 \le i \le k\}$.

Definition 2 (Cluster)**.** Let $x \in V$ and $t \ge 0$. We write

$$C_{t,x} := \left\{ y \in V \middle| \exists \mathcal{P} \in \mathrm{PATH}(x, y) \, \forall p \in \mathcal{P}_V : \eta_{t,p} = 1 \right\}$$

to denote the cluster at site x and time t.

We define the forest fire process formally. We write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition 3 (Forest fire process)**.** A forest fire process on G with parameter $\lambda > 0$ is a process $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{x \in V}$ with values in $(\{0,1\} \times \mathbb{N}_0 \times \mathbb{N}_0)^V$, $t \ge 0$, that has the following properties:

(a) the processes $(G_{t,x})_{t \ge 0}$ and $(I_{t,x})_{t \ge 0}$, $x \in V$, are independent Poisson processes with parameters 1 and λ, respectively. They are independent of the initial configuration $(\eta_{0,x})_{x \in V}$;

(b) for all $x \in V$ the process $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t \ge 0}$ is cádlág. That is, it is right continuous, and for all $t > 0$ the left limit $(\eta_{t-,x}, G_{t-,x}, I_{t-,x}) := \lim_{s \to t, s < t} (\eta_{s,x}, G_{s,x}, I_{s,x})$ exists;

(c) for all $x \in V$ and all $t > 0$

- if there is the growth of a tree at the site x at time t, then the site x is occupied at time t:

$$\left\{ G_{t-,x} < G_{t,x} \right\} \subset \{\eta_{t,x} = 1\}$$

- if the site x gets occupied at time t, then there must be the growth of a tree at the site x at time t:

$$\left\{ \eta_{t-,x} < \eta_{t,x} \right\} \subset \left\{ G_{t-,x} < G_{t,x} \right\}$$

- if the site x is hit by ignition at time t, then all sites of the cluster at x get vacant at time t:

$$\{I_{t^-,x} < I_{t,x}\} \subset \{\forall y \in C_{t^-,x} : \eta_{t,y} = 0\}$$

- if the site x gets vacant at time t, then the cluster at x is hit by ignition at time t:

$$\{\eta_{t^-,x} > \eta_{t,x}\} \subset \{\exists y \in C_{t^-,x} :\ I_{t^-,y} < I_{t,y}\}$$

Remark 1. For a more convenient notation, in the definition of a forest fire process we define the growth and ignition processes as follows. We require that

(i) the jump times of the growth and ignition processes are distinct. That is, there do not exist two growth processes, a growth and an ignition process, or two ignition processes that jump at the same time;

(ii) for all $x \in V$ it holds $\lim_{t\to\infty} G_{t,x} = \infty$ and $\lim_{t\to\infty} I_{t,x} = \infty$.

Remark 2. In [10] the existence of forest fire processes on $\mathbb{Z}^n$, $n \geq 2$, for all parameter $\lambda > 0$ is shown. The key observation [[10], Lemma 18] is the fact that the probability that a given non-empty cluster grows on its boundary before it gets hit by ignition is bounded by $2n/(2n + \lambda)$. This and therefore the existence result of [10] has a straightforward generalization to the case of arbitrary graphs with bounded vertex degree.

Let the process $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\geq 0, x\in V}$ be a forest fire process on G. To be able to state our results in the next section, we introduce some more notation. The main result is the almost sure infinite volume limit convergence for forest fire processes, where we consider forest fire processes on the following sequence of finite volume sub graphs of G.

Definition 4 (The subgraph G_n)**.** For all $x \in V$ and $n \in \mathbb{N}$ let

$$B_n(x) := \{z \in V | d(x,z) \leq n\}$$

denote the box with centre x and radius n, where

$$d(x,y) := \min\left\{n \in \mathbb{N}_0 \middle| \operatorname{PATH}_{n+1}(x,y) \neq \emptyset\right\}.$$

We write

$$G_n := \left(B_n, \left\{\{x,y\} \in E \middle| x,y \in B_n\right\}\right)$$

to denote the subgraph induced by $B_n := B_n(0)$, where $0 \in V$ is a distinguished site.

Definition 5 (The forest fire process on G_n)**.** For $n \in \mathbb{N}$ let $(\eta^{(n)}_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0, x\in B_n}$ denote the forest fire process on G_n with configuration $(\eta_{0,x})_{x\in B_n}$ at initial time 0. That is, the processes $\eta^{(n)}$, $n \in \mathbb{N}$, are living on the same probability space as the process η, and are driven by the same growth and ignition processes as η. For all $n \in \mathbb{N}$ the processes $\eta^{(n)}$ and η coincide on the set B_n at time 0.

One of the motivations to show infinite volume convergence is the question whether an infinite volume forest fire process is measurable with respect to its driving growth and ignition processes. We introduce the according σ-field.

Definition 6 (The σ-field $\mathcal{GI}_t$)**.** For all $t > 0$ and $B \subset V$ let

$$\mathcal{GI}_t(B) := \hat{\sigma}\big(\eta_{0,x}, G_{s,y}, I_{s,y} : 0 \leq s \leq t, x \in V, y \in B\big)$$

denote the completion of the σ-field generated by the initial configuration and the growth and ignition events that occur on B during the time interval $[0, t]$. For abbreviation let $\mathcal{GI}_t := \mathcal{GI}_t(V)$.

Remark 3. It is easy to see that a finite volume forest fire process is uniquely determined by its initial configuration and its driving growth and ignition processes (see e.g. [10] for the sketch of a recursive construction of finite volume forest fire processes). For all $n \in \mathbb{N}$, for all $t \geq 0$ the process $(\eta^{(n)}_{s,x})_{0 \leq s \leq t, x \in B_n}$ is measurable with respect to the σ-field $\mathcal{GI}_t$. Hence, for all $n \in \mathbb{N}$, for all $t \geq 0$ the process $(\eta^{(n)}_{s,x})_{0 \leq s \leq t, x \in B_n}$ is independent of the increments of the growth and ignition processes after time t.

As mentioned in the introduction, our results are restricted to a special class of initial configurations.

Definition 7 (Conditioned cluster size bound)**.** For all $s \geq 0$, $\delta > 0$ and $m \in \mathbb{N}$, we say that η has CCSB(s, δ, m), if the following holds: let $B, D \subset V$ finite and $x \in V \setminus D$. Then conditioned on the occurrence of $\cup_{y \in B} C_{s,y} = D$ the probability that the cluster at x is bigger than m at time s is smaller than or equal to δ. And almost surely the cluster at x is finite at time s. More formally, for all finite $B, D \subset V$, for all $x \in V \setminus D$

$$P\left(|C_{s,x}| > m, \cup_{y \in B} C_{s,y} = D\right) \leq \delta \cdot P\left(\cup_{y \in B} C_{s,y} = D\right)$$

and $P\left(|C_{s,x}| = \infty\right) = 0$, where $|\cdot|$ denotes the cardinality.

Remark 4. Our results restrict to the case where there exists $m \in \mathbb{N}$ so that the forest fire process has CCSB$(0, \lambda/(4d^2), m)$, where d is the bound for the vertex degree, and λ is the ignition rate. The following initial configurations are examples of such configurations:

(i) the empty initial configuration with all sites vacant at time 0;

(ii) every initial configuration $(\eta_{0,x})_{x \in V}$ where there exists $m \geq 0$ such that for all $x \in V$ the relation $P\left(|C_{0,x}| > m\right) = 0$ holds;

(iii) independent site percolation on $\mathbb{Z}^n$ with no infinite clusters, $n \geq 1$.

2.2 Statement of the main results

We are now ready to state the main results. Let $G = (V, E)$ be a connected graph with vertex degree bounded by $d \geq 2$.

Our central theorem states almost sure infinite volume limit convergence for the sequence of forest fire processes on the sub graphs G_n, $n \in \mathbb{N}$.

Theorem 1 (Almost sure infinite volume convergence). *Let $\lambda > 0$ and let the process $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\geq 0, x\in V}$ be a forest fire process on G with parameter λ that has* $\mathrm{CCSB}(0, \lambda/(4d^2), m)$ *for some $m \in \mathbb{N}$. For all $n \in \mathbb{N}$ let $\eta^{(n)}$ be the forest fire process on G_n coupled to η, as defined in Definition 5. Then the sequence of forest fire processes $(\eta^{(n)})_{n\in\mathbb{N}}$ converges to the forest fire process η uniformly on compact sets almost surely. That is, for all $t \geq 0$ and every finite $S \subset V$ it holds*

$$\lim_{n\to\infty} P\left(\sup_{l\geq n}\sup_{0\leq s\leq t}\sup_{x\in S}|\eta_{s,x} - \eta^{(l)}_{s,x}| > 0\right) = 0.$$

For the cluster size distribution we have the following result.

Theorem 2 (Uniform decay of the cluster size distribution). *Let the forest fire process $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\geq 0, x\in V}$ be as in Theorem 1. For every $\gamma > 0$ there exists a function $m_{\gamma,\lambda,d} :]0,1] \mapsto \mathbb{N}$ so that for all $\delta \in]0,1]$, for all $s \geq \gamma$ the forest fire process η has* $\mathrm{CCSB}(s, \delta, m_{\gamma,\lambda,d}(\delta))$.

The explicit formula for $m_{\gamma,\lambda,d}(\delta)$ is stated in Section 2.3, Proposition 3. As last result, we state the measurability and uniqueness of the forest fire process with respect to its driving growth and ignition processes, and the Markov property.

Theorem 3. *Let $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\geq 0, x\in V}$ be as in Theorem 1.*

(a) (Uniqueness) *Let $(\tilde\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\geq 0, x\in V}$ be a forest fire process on the same probability space as η that has the following properties: the forest fire process $\tilde\eta$ is driven by the same growth and ignition processes as the forest fire process η. Both forest fire processes have the same initial configuration, that is, $(\tilde\eta_{0,x})_{x\in V} = (\eta_{0,x})_{x\in V}$. Then with probability one both forest fire process are equal:*

$$P\big(\forall t \geq 0\ \forall x \in V : \tilde\eta_{t,x} = \eta_{t,x}\big) = 1$$

(b) (Measurability) *For all $t \geq 0$ the process $(\eta_{s,x})_{0\leq s\leq t, x\in V}$ is $\mathcal{GI}_t$-measurable;*

(c) (Markov Property) *For all $t \geq 0$ and all $t' \geq t$, for all $A \in \sigma\left(\eta_{t',x} : x \in V\right)$ we have*

$$P\left(\eta_{t',x} \in A \middle| (\eta_{s,x})_{0\leq s\leq t, x\in V}\right) = P\left(\eta_{t',x} \in A \middle| (\eta_{t,x})_{x\in V}\right)$$

almost surely.

2.3 Key propositions and proof of the main results

Throughout the remainder of this chapter let $G = (V,E)$ be a connected graph with vertex degree bounded by $d \geq 2$. Let $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\geq 0, x\in V}$ be a forest fire process on G with parameter $\lambda > 0$ that has $\mathrm{CCSB}(0, \lambda/(4d^2), m)$ for some $m \in \mathbb{N}$. For all $n \in \mathbb{N}$ we write $\eta^{(n)}$ to denote the (finite volume) forest fire processes on G_n coupled to η, as defined in Definition 5.

2.3.1 Organization of the proof of Theorems 1 - 3

We organized the presentation of the proof of Theorems 1 - 3 into three key propositions. We state them in Section 2.3.3, and prove Theorems 1 - 3 subject to them in Section 2.3.4. The three key propositions are independently proven in Sections 2.4, 2.5 and 2.6.

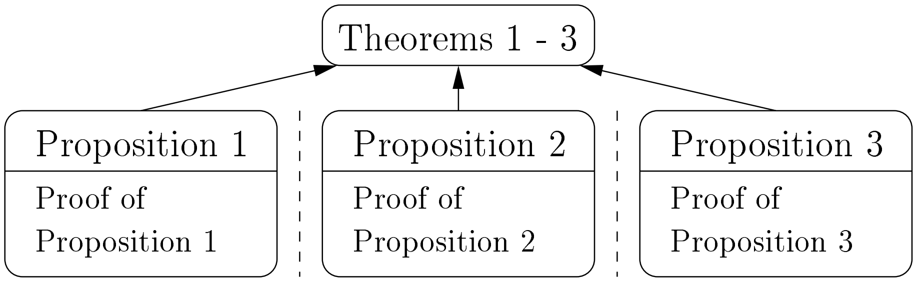

Figure 2.1: Proof of Theorems 1 - 3

Prior to giving the key propositions, we introduce an auxiliary process.

2.3.2 The blur process

We introduce an auxiliary process in this section. Given $S \subset V$ finite, $t \geq 0$ and the state of the forest fire process on the set S at time t, we have the following goal. After time t we want to keep track of a set of sites whose state can be determined without considering the growth and ignition processes that occur outside the set S.

Informally speaking, at time $s \geq t$ we mark those sites whose status might depend on the growth and ignition jumps that occurred outside the set S during $[t, s]$. We call these sites (t, S)-blurred at time s, where we proceed as follows. If the cluster at a given site $x \in S$ is not a subset of the set S at time t, then the site x is (t, S)-blurred at time t: it might be that shortly after time t the cluster at x gets hit by an ignition that occurs outside the set S. At time $s > t$ an occupied site $x \in S$ gets (t, S)-blurred, if its cluster gets connected to a site $y \in S$ that has already been (t, S)-blurred: it might be that the site y is occupied and thus the site x might get vacant due to an ignition that hits the cluster at y.

Definition 8 (Boundary)**.** For $S \subset V$ let

$$\partial S := \big\{ x \in V \setminus S \big| \exists y \in S : \{x, y\} \in E \big\}$$

denote the boundary of S, and $\overline{S} := S \cup \partial S$.

The next definition formalizes the blur process. Here with slight abuse of notation, we write $\overline{C_{t,x}} := x$ in case of $\eta_{t,x} = 0$.

Definition 9 (Blur process)**.** Let $t \geq 0$ and $S \subset V$ finite. The (t, S)-blur process is a right continuous process $\big(\beta^{t,S}_{s,x}\big)_{x \in \overline{S}}$ with values in $\{0, 2\}^{\overline{S}}$, $s \geq t$, that has the following properties: for all $x \in \overline{S}$

(a) the site x is (t,S)-blurred at time t, if and only if the cluster at x is connected to the boundary of S:
$$\beta^{t,S}_{t,x} = \begin{cases} 2 & \text{if } \overline{C_{t,x}} \cap \partial S \neq \emptyset, \\ 0 & \text{else;} \end{cases}$$

(b) for all $s \geq t$, $\beta^{t,S}_{s,x} = 2$ implies $\beta^{t,S}_{s',x} = 2$ for all $s' \geq s$. That is, a (t,S)-blurred site remains (t,S)-blurred forever;

(c) for all $s > t$ the site x is (t,S)-blurred at time s, if and only if the set $\overline{C_{s,x}}$ contains a site that has been (t,S)-blurred before time s:
$$\{\beta^{t,S}_{s,x} = 2\} = \left\{\exists z \in \overline{C_{s,x}} \cap \overline{S} : \beta^{t,S}_{s^-,z} = 2\right\}$$

Remark 5. The proof of existence of the blur process is part of the proof of Lemma 1 in Section 2.4.

2.3.3 Key propositions

In this section we state the three key proposition our proof of Theorems 1 - 3 is based on.

The first proposition concerns the blur process.

Proposition 1. *Let $t \geq 0$, $t' > t$, $m \in \mathbb{N}$, $x \in B_m$ and $l \geq m$. Suppose that the forest fire processes $\eta^{(k)}$, $k \geq l$, and the forest fire process η coincide on the set B_m at time t, but differ at the site x within the time interval $[t,t']$. Then the site x is (t,B_m)-blurred at time t':*
$$\left\{\sup_{k\geq l}\sup_{y\in B_m}|\eta_{t,y} - \eta^{(k)}_{t,y}| = 0, \sup_{k\geq l}\sup_{t\leq s'\leq t'}|\eta_{s',x} - \eta^{(k)}_{s',x}| > 0\right\} \subset \left\{\beta^{t,B_m}_{t',x} = 2\right\}.$$

The second proposition considers the probability that a given site is (t,B_n)-blurred at time $t+\epsilon$.

Proposition 2. *For all $m \in \mathbb{N}$ there exists $\epsilon_m > 0$ with the following property. Let $t \geq 0$ and suppose that the forest fire process has* CCSB$(t, \lambda/(4d^2), m)$*. Furthermore, suppose that we have almost sure infinite volume convergence at time t, that is,*
$$\lim_{n\to\infty} P\big(\sup_{l\geq n}|\eta_{t,y} - \eta^{(l)}_{t,y}| > 0\big) = 0 \qquad \textit{all } y \in V.$$

Then for all $x \in V$, as n tends to infinity the probability that the site x is (t,B_n)-blurred at time $t+\epsilon_m$ tends to zero:
$$\lim_{n\to\infty} P\left(\beta^{t,B_n}_{t+\epsilon_m,x} = 2\right) = 0$$

The assumption of almost sure convergence at time t in Proposition 2 has the following reason: Remark 3 implies that the configuration of the (finite volume) forest fire processes $\eta^{(n)}$, $n \in \mathbb{N}$, at time t is independent of the increments of the growth and ignition after time t. The almost sure convergence at time t allows us to carry this fact over to the forest fire process η.

The next proposition states that almost sure infinite volume limit convergence at time t implies that the assumption on the cluster size distribution in Proposition 2 holds at time t. Here we need infinite volume limit convergence, since for technical reasons we are going to study the cluster size distribution for finite volume forest fire processes first.

Definition 10 (The bound $m_{\gamma,\lambda,d}(\delta)$)**.** For all $\gamma > 0$ and all $\delta \in]0,1]$ let

$$m_{\gamma,\lambda,d}(\delta) := \left(\left\lceil\left(\frac{8d^3(N_{\gamma,\lambda,d}(\delta)-1)}{2\delta\lambda}+1\right)\right\rceil \vee (M_{\gamma,\lambda,d}(\delta)\vee d)\right)^{N_{\gamma,\lambda,d}(\delta)},$$

where

$$N_{\gamma,\lambda,d}(\delta) := \left\lceil\frac{\ln(2)-\ln(\delta)}{\ln(d+\lambda)-\ln(d)}\right\rceil,$$

$$\epsilon_{\gamma,\lambda,d}(\delta) := \left\{-\ln\left(1-\frac{\delta}{8d(N_{\gamma,\lambda,d}(\delta)-1)}\right)\right\}\wedge\gamma$$

and

$$M_{\gamma,\lambda,d}(\delta) := \left\lceil\frac{\ln(8)-\ln(7\delta)}{\lambda\epsilon_{\gamma,\lambda,d}(\delta)}\right\rceil.$$

Here for all $s, s' \in \mathbb{R}$ we write $\lceil s\rceil := \min\{z \in \mathbb{Z} | z \geq s\}$, $s \vee s' := \max\{s, s'\}$ and $s \wedge s' := \min\{s, s'\}$.

Proposition 3. *Let $\gamma > 0$, $t \geq \gamma$ and suppose we have almost sure infinite volume convergence at time t, that is,*

$$\lim_{n\to\infty} P\big(\sup_{l\geq n}|\eta_{t,y}-\eta^{(l)}_{t,y}| > 0\big) = 0 \qquad \textit{all } y \in V.$$

Then for all $\delta \in]0,1]$ the forest fire process η has $\mathrm{CCSB}(t,\delta,m_{\gamma,\lambda,d}(\delta))$.

We sketch the proof of Theorem 1 subject to Propositions 1, 2 and 3. For all $n \in \mathbb{N}$, the forest fire processes η and $\eta^{(n)}$ coincide on the set B_n at time 0. By assumption there exists a $m_0 \in \mathbb{N}$ such that the forest fire process η has $\mathrm{CCSB}(0,\lambda/(4d^2),m_0)$. Therefore Proposition 2 yields the existence of $\gamma > 0$ so that $\lim_{n\to\infty} P\big(\beta^{0,B_n}_{\gamma,x} = 2\big) = 0$ for all $x \in V$. Along with Proposition 1 this implies infinite volume limit convergence up to time γ. Thus Proposition 3 shows that there exists $m \in \mathbb{N}$ so that the forest fire process has $\mathrm{CCSB}(\gamma,\lambda/(4d^2),m)$. Along with Proposition 2 we obtain the existence of $\epsilon \geq 0$ such that $\lim_{n\to\infty} P(\beta^{\gamma,B_n}_{\gamma+\epsilon,x} = 2) = 0$ for all $x \in V$. Combining this, the infinite volume limit convergence up to time γ and Proposition 1, we obtain infinite volume limit convergence up to time $\gamma+\epsilon$. Hence, the assumption of Proposition 3 is satisfied at time $\gamma+\epsilon$. Going on recursively, we obtain Theorem 1.

2.3.4 Proof of the main results

We proof Theorems 1 - 3 subject to Propositions 1 - 3 in this section.

Proof of Theorem 1 subject to Propositions 1 - 3. For all $t \geq 0$, all $n \in \mathbb{N}$ and all $S \subset B_n$ let

$$\mathcal{E}(n,t,S) := \left\{ \sup_{l \geq n} \sup_{0 \leq s' \leq t} \sup_{x \in S} |\eta_{s',x} - \eta^{(l)}_{s',x}| > 0 \right\}.$$

There exists $m \in \mathbb{N}$ such that the forest fire process has CCSB$(0, \lambda/(4d^2), m)$. For all $n \in \mathbb{N}$ the configuration of the forest fire processes η and $\eta^{(n)}$ coincide on the set B_n at time 0. Therefore Proposition 2 provides the existence of $\gamma > 0$ such that for all $x \in V$

$$\lim_{n\to\infty} P\left(\beta^{0,B_n}_{\gamma,x} = 2\right) = 0.$$

Thus applying Proposition 1 we obtain for all finite $S \subset V$

$$\begin{aligned}\lim_{n\to\infty} P\left(\mathcal{E}(n,\gamma,S)\right) &\leq \lim_{n\to\infty} \sum_{x\in S} P\left(\mathcal{E}(n,\gamma,x)\right) \\ &\leq \lim_{n\to\infty} \sum_{x\in S} P\left(\beta^{0,B_n}_{\gamma,x} = 2\right) = 0.\end{aligned}$$

That is,

$$\lim_{n\to\infty} P\left(\sup_{l\geq n} \sup_{0\leq s\leq t} \sup_{x\in S} |\eta_{s,x} - \eta^{(l)}_{s,x}| > 0 \right) = 0 \tag{2.1}$$

holds for $t = \gamma$ for all finite $S \subset V$.

To conclude the theorem we show the existence of $\epsilon > 0$ so that if (2.1) holds at time $t = s$, $s \geq \gamma$, for all finite $S \subset V$, then (2.1) holds at time $t = s + \epsilon$ for all finite $S \subset V$. In particular, the choice of ϵ does not depend on the choice of $s \geq \gamma$.

Let $s \geq \gamma$ and suppose that (2.1) holds for $t = s$ for all finite $S \subset V$. Then from Proposition 3 there exists $m_\gamma \in \mathbb{N}$ so that the forest fire process has CCSB$(s, \frac{\lambda}{4d^2}, m_\gamma)$. Thus Proposition 2 provides the existence of $\epsilon > 0$ so that for all $x \in V$

$$\lim_{n\to\infty} P\left(\beta^{s,B_n}_{s+\epsilon,x} = 2\right) = 0. \tag{2.2}$$

Note that the choice of ϵ does not depend on the choice of $s \geq \gamma$.

Let $x \in V$ and $m \in \mathbb{N}$ such that $x \in B_m$, and let $n \geq m$. On the complement of $\mathcal{E}(n,t,B_m)$ it holds

$$\sup_{0\leq s\leq s'} \sup_{l\geq n} \sup_{y\in B_m} |\eta_{s',y} - \eta^{(l)}_{s',y}| = 0.$$

This implies

$$\begin{aligned}&\mathcal{E}(n,s+\epsilon,x)\setminus\mathcal{E}(n,s,B_m)\\&\qquad\subset\left\{\sup_{l\geq n}\sup_{y\in B_m}|\eta_{s,y}-\eta^{(l)}_{s,y}|=0,\sup_{l\geq n}\sup_{s\leq s'\leq s+\epsilon}|\eta_{s',x}-\eta^{(l)}_{s',x}|>0\right\}.\end{aligned}$$

Proposition 1 yields

$$P\left(\mathcal{E}(n,s+\epsilon,x)\right)\leq P\left(\mathcal{E}(n,s,B_m)\right)+P\left(\beta^{s,B_m}_{s+\epsilon,x}=2\right). \tag{2.3}$$

Let $\delta>0$. From (2.2) there exists $M\in\mathbb{N}$ so that $P(\beta^{s,B_M}_{s+\epsilon,x}=2)<\delta/2$. From our induction hypothesis there exists $N_0=N_0(M)\in\mathbb{N}$ so that $P\left(\mathcal{E}(N_0,s,B_M)\right)<\delta/2$. With this choice of N_0 and M, (2.3) and the monotonicity of the event $\mathcal{E}\left(N,s+\epsilon,x\right)$ imply $P\left(\mathcal{E}\left(N,s+\epsilon,x\right)\right)<\delta$ for all $N\geq N_0$. This shows (2.1) for $t=s+\epsilon$ for all finite $S\subset V$.

We conclude the theorem. In the first step, we showed that there exists $\gamma>0$ such that (2.1) holds for $t=\gamma$ for all finite $S\subset V$. In the second step, we show the existence of $\epsilon>0$ so that if (2.1) holds at time $t=s$ for all finite $S\subset V$, then (2.1) holds at time $t=s+\epsilon$ for all finite $S\subset V$. Hence, Theorem 1 follows by induction. □

Proof of Theorem 2 subject to Theorem 1 and Proposition 3. Theorem 2 follows directly from Theorem 1 and Proposition 3. □

Proof of Theorem 3 subject to Theorem 1 and 2. We use Theorem 1 and 2 to conclude the theorem.

(a) Let $\tilde\eta$ be as in Theorem 3, let $t\geq 0$ and $S\subset V$ finite. Then from the almost sure convergence proven in Theorem 1

$$\lim_{n\to\infty}P\left(\sup_{l\geq n}\sup_{0\leq s\leq t}\sup_{x\in S}|\eta_{s,x}-\eta^{(l)}_{s,x}|>0\right)=0$$

and

$$\lim_{n\to\infty}P\left(\sup_{l\geq n}\sup_{0\leq s\leq t}\sup_{x\in S}|\tilde\eta_{s,x}-\eta^{(l)}_{s,x}|>0\right)=0.$$

This implies

$$P\big(\exists s\in[0,t]\ \exists x\in S:\tilde\eta_{s,x}\neq\eta_{s,x}\big)=0.$$

The assertion follows.

(b) Let $t\geq 0$. As mentioned in Remark 3, the processes $(\eta^{(n)}_{s,x})_{0\leq s\leq t,x\in B_n}$, $n\in\mathbb{N}$, are measurable with respect to $\mathcal{GI}_t$. Therefore Theorem 1 implies that for all finite $S\subset V$ the process $(\eta_{s,x})_{0\leq s\leq t,x\in S}$ is $\mathcal{GI}_t$-measurable. The $\mathcal{GI}_t$-measurability of $(\eta_{s,x})_{0\leq s\leq t,x\in V}$ follows.

(c) Let $t \geq 0$ and $t' \geq t$. For all $s \geq 0$ and $x \in V$ let $\tilde{G}_{s,x} := G_{t+s,x} - G_{t,x}$ and $\tilde{I}_{s,x} := I_{t+s,x} - I_{t,x}$. From (a) the process $(\eta_{s,x})_{s\in[0,t],x\in V}$ is $\mathcal{GI}_t$-measurable. The increments of the growth and ignition processes after time t are independent of the σ-field $\mathcal{GI}_t$. This implies for all $B \in \sigma(\tilde{G}_{s,x}, \tilde{I}_{s,x} : s \in [0, t'-t], x \in V)$ and all $C \in \sigma(\eta_{t,x} : x \in V)$

$$P\left(B \cap C \middle| \mathcal{GI}_t\right) = 1_C \cdot P\left(B \middle| \mathcal{GI}_t\right) = 1_C \cdot P\left(B \middle| (\eta_{t,x})_{x\in V}\right) = P\left(B \cap C \middle| (\eta_{t,x})_{x\in V}\right)$$

almost surely. It follows for all $A \in \mathcal{GI}_{t,t'}$

$$P\left(A \middle| \mathcal{GI}_t\right) = P\left(A \middle| (\eta_{t,x})_{x\in V}\right) \tag{2.4}$$

almost surely, where

$$\mathcal{GI}_{t,t'} := \hat{\sigma}\left(\eta_{t,x}, \tilde{G}_{s,x}, \tilde{I}_{s,x} : s \in [0, t'-t], x \in V\right).$$

For all $s \geq 0$ and all $x \in V$ let $\tilde{\eta}_{s,x} := \eta_{t+s,x}$, and note that $(\tilde{\eta}_{s,x}, \tilde{G}_{s,x}, \tilde{I}_{s,x})_{s\geq 0, x\in V}$ satisfies the definition of a forest fire process on G with parameter λ and initial configuration $(\eta_{t,x})_{x\in V}$. Theorem 2 implies the existence of $m \in \mathbb{N}$ such that the forest fire process η has CCSB$(t, \lambda/(4d^2), m)$. Hence, the forest fire process $\tilde{\eta}$ has CCSB$(0, \lambda/(4d^2), m)$, and part (a) implies that $(\eta_{t',x})_{x\in V} = (\tilde{\eta}_{t'-t,x})_{x\in V}$ is measurable with respect to $\mathcal{GI}_{t,t'}$. That is, (2.4) holds for all $A \in \sigma\left(\eta_{t',x} : x \in V\right)$. Applying the $\mathcal{GI}_t$-measurability of $(\eta_{s,x})_{s\in[0,t],x\in V}$ again, we obtain

$$P\left(A \middle| (\eta_{s,x})_{s\in[0,t],x\in V}\right) = P\left(P\left(A \middle| \mathcal{GI}_t\right) \middle| (\eta_{s,x})_{s\in[0,t],x\in V}\right) = P\left(A \middle| (\eta_{t,x})_{x\in V}\right) \quad \text{a.s.}$$

for all $A \in \sigma\left(\eta_{t',x} : x \in V\right)$.

□

2.4 Proof of Proposition 1

We proof Proposition 1 in this section. First we prove the following version of it.

Lemma 1. *For all $t \geq 0$ and all finite $S \subset V$ the (t, S)-blur process exists and has the following property: let $x \in S$, $t' \geq t$ and $n \geq 1$ such that $S \subset B_n$. Suppose that the processes η and $\eta^{(n)}$ coincide on the set S at time t, and that the site x is not (t, S)-blurred at time t'. Then they coincide at the site x at time t':*

$$\left\{\forall y \in S : \eta_{t,y} = \eta^{(n)}_{t,y}, \beta^{t,S}_{t',x} = 0\right\} \subset \left\{\eta_{t',x} = \eta^{(n)}_{t',x}\right\}$$

Proof. Let t, t', S and n as in the lemma. The proof of the lemma is based on a recursive construction of the blur process. The set of sites that are not (t, S)-blurred at time $t =: \tau_0$ is

$$B(0) := \left\{ y \in \overline{S} \middle| \overline{C_{t,y}} \cap \partial S = \emptyset \right\}.$$

Recursively for all $i \in \mathbb{N}$ let

$$B(i) := B(i-1) \setminus \cup_{z \in V(i-1)} C_{\tau_i, z} \qquad (B(i) := \emptyset \text{ if } B(i-1) = \emptyset),$$

where

$$\tau_i := \min \left\{ t > \tau_{i-1} \middle| \exists y \in V(i-1) : G_{t,y} > G_{\tau_{i-1}, y} \right\} \qquad (\min\{\emptyset\} := \infty)$$

is the first time after τ_{i-1} at which one of the growth processes on

$$V(i-1) := \left\{ y \in B(i-1) \middle| \partial y \not\subset B(i-1) \right\}$$

jumps.

By induction on $i \in \mathbb{N}_0$ we show the following:

(i) the (t, S)-blur process is well defined up to time τ_i, and $B(i)$ is the set of those sites in $\overline{S}$ that are not (t, S)-blurred at time τ_i;

(ii) at time τ_i the entire set $V(i)$ is vacant with respect to η, that is, $\eta_{\tau_i, z} = 0$ for all $z \in V(i)$;

(iii) suppose that the processes η and $\eta^{(n)}$ coincide on the set $B(-1) := S$ at time τ_0. Then they coincide on $B(i-1)$ throughout $[\tau_0, \tau_i]$.

We begin with the induction. The sites in $B(0)$ are those sites in $\overline{S}$ that are not (t, S)-blurred at time τ_0. Hence, $\overline{C_{\tau_0, x}} \cap \partial S = \emptyset$ for all $x \in V(0)$. Every site $x \in V_0$ has a neighbour y so that $y \in B(0)$, that is, $\overline{C_{\tau_0, y}} \cap \partial S \neq \emptyset$. It follows that at time τ_0 the entire set $V(0)$ is vacant with respect to η.

In the induction step $k \to k+1$ suppose (i), (ii) and (iii) for $i = k$. The set $V(k)$ satisfies $\partial\left(B(k) \setminus V(k)\right) \subset V(k)$. The set $V(k)$ is vacant at time τ_k, and the definition of the time τ_{k+1} implies that the set $V(k)$ remains vacant throughout $[\tau_k, \tau_{k+1}[$. That is, $\overline{C_{s,y}} \subset B(k)$ for all $s \in [\tau_k, \tau_{k+1}[$ and all $y \in B(k)$. Therefore, during the time interval $[\tau_k, \tau_{k+1}[$ none of the sites in $B(k)$ can get vacant due to an ignition that occurs outside the set $B(k)$. Hence, throughout $[\tau_k, \tau_{k+1}[$ the configuration of η on $B(k)$ is uniquely determined by its configuration on $B(k)$ at time τ_k and the finitely many growth and ignition jumps that occur on $B(k)$ within $[\tau_k, \tau_{k+1}[$. At time τ_{k+1} a site $z \in V(k)$ gets occupied. Thus, since growth and ignition jumps occur at distinct times, all sites except for z remain unchanged at time τ_{k+1}. This implies that the configuration on $B(k)$ at time τ_{k+1} and the growth and ignition jumps that occur on $B(k)$ within $[\tau_k, \tau_{k+1}]$ determine the configuration of η on $B(k)$ at time τ_{k+1}. Since the processes η and $\eta^{(n)}$ are adapted to the same family of growth and ignition processes, this implies the following: if η and

$\eta^{(n)}$ coincide on the set $B(k)$ at time τ_k, then they coincide on the set $B(k)$ throughout $[\tau_k, \tau_{k+1}]$. Along with $B(k) \subset B(k-1)$ we conclude (iii) for $i = k+1$, where we use that (iii) holds for $i = k$.

The sites in $B(k)$ are those sites in $\overline{S}$ that are not (t, S)-blurred at time τ_k. For all $s \in [\tau_k, \tau_{k+1}[$ and $y \in B(k)$ we have $\overline{C_{s,y}} \subset B(k)$. Thus none of the sites of $B(k)$ get (t, S)-blurred within $[\tau_k, \tau_{k+1}[$. Let $y \in B(k)$ and suppose that y is (t, S)-blurred at time τ_{k+1}. Then at time τ_{k+1} the cluster at y is connected to a site that has been (t, S)-blurred before. That is, $\overline{C_{\tau_{k+1},y}} \not\subset B(k)$ holds. Therefore, $\partial (B(k) \setminus V(k)) \subset V(k)$ implies the existence of $z \in V(k)$ so that $y \in C_{\tau_{k+1},z}$. Conversely, every site $y \in B(k)$ that satisfies $y \in C_{\tau_{k+1},z}$ for some $z \in V(k)$ is (t, S)-blurred at time τ_{k+1}. This shows (i) for $i = k + 1$.

Let $y \in V(k + 1)$. Then either $y \in V(k)$ and $C_{\tau_{k+1},y} = \emptyset$ or $y \in \partial C_{\tau_{k+1},z}$ for some $z \in V(k)$. In both cases, the site y is vacant at time τ_{k+1}. We obtain (ii) for $i = k + 1$.

We conclude the lemma: let $x \in S$ and suppose that the site x is not (t, S)-blurred at time t', and that the processes η and $\eta^{(n)}$ coincide on the set S at time t. Then $\lim_{i\to\infty} \tau_i = \infty$ implies the existence of $i \in \mathbb{N}_0$ such that $t' \in [\tau_i, \tau_{i+1}[$. Since x is not (t, S)-blurred at time t', we have $x \in B(i)$. From (iii) the processes η and $\eta^{(n)}$ coincide on the set $B(i)$ throughout $[\tau_0, \tau_{i+1}]$. Hence, they coincide at the site x at time t'. □

Proof of Proposition 1. Let $t \geq 0$, $m \in \mathbb{N}$, $k \geq m$ and $x \in B_m$. Lemma 1 implies the following for all $s \geq t$. Suppose that η and $\eta^{(k)}$ coincide on B_m at time t, but differ at the site x at time s. Then the site x is (t, B_m)-blurred at time s. This implies for all $t' > t$

$$\begin{aligned} &\left\{\forall y \in B_m : \eta_{t,y} = \eta^{(k)}_{t,y}, \exists s \in [t, t'] : \eta_{s,x} \neq \eta^{(k)}_{s,x}\right\} \\ &\subset \left\{\exists s \in [t, t'] : \beta^{t,B_m}_{s,x} = 2\right\} \subset \left\{\beta^{t,B_m}_{t',x} = 2\right\}. \end{aligned}$$

Here to get the second relation, we use that a (t, B_m)-blurred site remains (t, B_m)-blurred forever. Proposition 1 follows. □

2.5 Proof of Proposition 2

2.5.1 Organization of the proof of Proposition 2

Let $t \geq 0$ and $x \in V$. To show Proposition 2 we have to show the existence of $\epsilon > 0$ such that

$$\lim_{n\to\infty} P\left(\beta^{t,B_n}_{t+\epsilon,x} = 2\right) = 0.$$

But due to the strong dependence of $\beta^{t,B_n}_{t+\epsilon,x} = 2$ on the configuration of η on B_n at time t, it seems to be difficult to estimate the probability directly. Therefore, we estimate the probability that the set of blurred sites reaches the site x from k clusters away. More formally, we study $\beta^{t,\mathrm{C}_{t,x}(k)}_{t+\epsilon,x} = 2$, where $\mathrm{C}_{t,x}(k)$ is defined as follows.

Definition 11 (The set $\mathrm{C}_{t,x}(k)$)**.** For all $t \geq 0$ and all $x \in V$ we define

$$\mathrm{C}_{t,x}(1) := \overline{C_{t,x}},$$

and recursively for all $k \geq 2$:

$$\mathrm{C}_{t,x}(k) := \mathrm{C}_{t,x}(k-1) \cup \bigcup_{y \in \partial \mathrm{C}_{t,x}(k-1)} \overline{C_{t,y}}$$

Here we write $\overline{C_{t,x}} := x$ in case of $\eta_{t,x} = 0$.

To conclude Proposition 2 we are going to use the following Proposition.

Proposition 4. *Let $m \in \mathbb{N}$, $t \geq 0$, and $\epsilon > 0$ so that $P(G_{\epsilon,0} > 0) < 1/(4md^2)$. Suppose that the forest fire process has* CCSB$(t, \lambda/(4d^2), m)$*, and almost sure infinite volume convergence at time t:*

$$\lim_{n\to\infty} P\big(\sup_{l \geq n} |\eta_{t,z} - \eta^{(l)}_{t,z}| > 0\big) = 0 \quad \textit{all} \quad z \in V. \tag{2.5}$$

Then for all $x \in V$ and all $k \in \mathbb{N}$

$$P\left(\beta^{t,\mathrm{C}_{t,x}(k)}_{t+\epsilon,x} = 2, |\,\mathrm{C}_{t,x}(k)| < \infty\right) \leq \left(\frac{3}{4}\right)^{k-1}.$$

To prove Proposition 2 we are going to proceed as follows. We start in Section 2.5.2 (General properties for the blur process) by showing some general properties for the blur process. In Section 2.5.3 (Growth and ignition estimates for the blur process) we use the growth and ignition processes to estimate some events described in terms of the blur process. Thereafter, in Section 2.5.4 (Estimates for the proof of Proposition 4) we use the results of Section 2.5.3 to derive the two key estimates the proof of Proposition 4 is based on. We use them and results of Section 2.5.3 to prove Proposition 4 in Section 2.5.5 (Proof of Proposition 4). Finally, in Section 2.5.6 (Proof of Proposition 2) we use Proposition 4 and results from Section 2.5.3 to conclude Proposition 2. Figure 2.2 illustrates the way the single parts of the proof of Proposition 2 depend on each other.

2.5.2 General properties for the blur process

In this section we show two general properties for the blur process. Throughout this section let $t \geq 0$ and $S \subset V$ finite. We start with a definition.

Definition 12 (The times $\beta^{t,S}_x$ and τ_n)**.** For all $x \in \overline{S}$ let

$$\beta^{t,S}_x := \min\left\{s \geq t \middle| \beta^{t,S}_{s,x} = 2\right\}$$

be the first time at which the site x is (t,S)-blurred. We write $\tau_0 := t$ and for all $1 \leq n \leq |S|$

$$\tau_n := \min\left\{\beta^{t,S}_x, x \in S \middle| \beta^{t,S}_x > \tau_{n-1}\right\} \quad (\min\{\emptyset\} := \infty)$$

to denote the nth time at which a new site gets (t,S)-blurred.

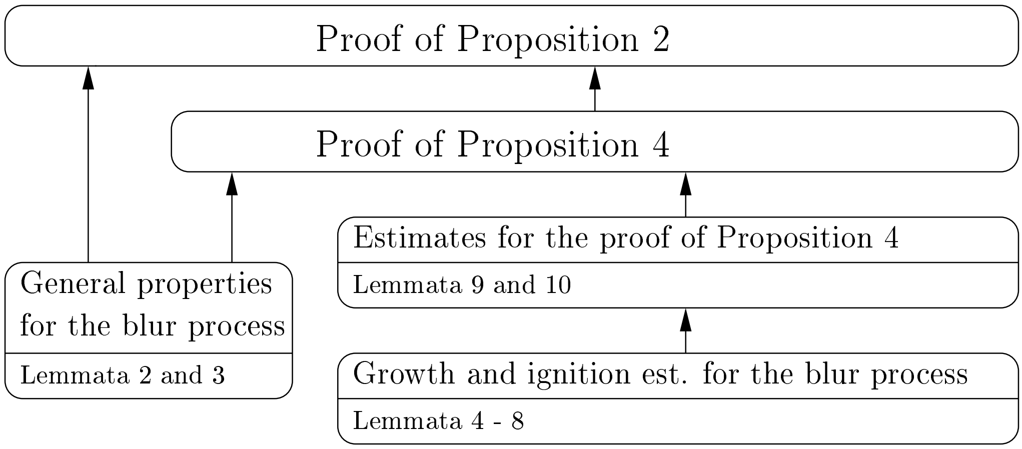

Figure 2.2: Proof of Proposition 2

Let $R \subset S$. We compare the (t, R)- and the (t, S)-blur process. If a site $x \in \overline{R}$ is (t, S)-blurred at some time $s \geq t$, then until time s the set of (t, S)-blurred grew from the boundary of S to the site x. Since R is a subset of S, it is likely to assume that then at time s the set of (t, R)-blurred contains the site x, too. This implies that if a site $x \in \overline{R}$ is (t, S)-blurred then it is (t, R)-blurred, too. The next lemma states this kind of monotonicity formally.

Lemma 2 (Monotonicity of the blur process)**.** *Let $R \subset S$, $x \in \overline{R}$. Then $\beta_x^{t,R} \leq \beta_x^{t,S}$.*

Proof. Let R as in the Lemma. By induction on $0 \leq i \leq |S|$ we show the following for all $x \in \overline{R}$. If the site x is (t, S)-blurred at time τ_i, then it is (t, R)-blurred. More formally, we show that $\beta_x^{t,S} \leq \tau_i$ implies $\beta_x^{t,R} \leq \tau_i$.

Let $x \in \overline{R}$ and suppose that the site x is (t, S) blurred at time τ_0. Then at time t the cluster at x is connected to the boundary of S. Or more formally, we have $\overline{C_{t,x}} \cap \partial S \neq \emptyset$. Hence, $R \subset S$ and $x \in \overline{R}$ imply $\overline{C_{t,x}} \cap \partial R \neq \emptyset$. It follows $\beta_x^{t,R} \leq \tau_0$.

In the induction step $i \to i + 1$, we suppose that $\beta_x^{t,S} \leq \tau_i$ implies $\beta_x^{t,R} \leq \tau_i$ for all $x \in \overline{R}$. Let $x \in \overline{R}$ and suppose that the site x is (t, S)-blurred at time τ_{i+1}. Then at time τ_{i+1} the cluster at x is connected to a site $y \in \overline{S}$ that is (t, S)-blurred at time τ_i. In case of $y \in \overline{R}$ the induction hypothesis implies that the site y is (t, R)-blurred at time τ_i. Otherwise if $y \notin \overline{R}$, then $x \in R$ implies that at time τ_{i+1} the cluster at x contains a site $z \in \partial R$. Such a site is (t, R)-blurred at time τ_0, and hence at time τ_i. In both cases at time τ_{i+1} the cluster at x is connected to a site that is (t, R)-blurred at time τ_i. That is, the site x is (t, R)-blurred at time τ_{i+1}.

We conclude the lemma. Let $x \in \overline{R}$. If $\beta_x^{t,S} < \infty$, there exists $0 \leq i \leq |S|$ so that $\beta_x^{t,S} = \tau_i$. We showed that $\beta_x^{t,S} = \tau_i$ implies $\beta_x^{t,R} \leq \tau_i$. It follows $\beta_x^{t,R} \leq \beta_x^{t,S}$. □

The definition of the blur process implies that if a site $x \in S$ is (t, S)-blurred, then there exists a path of (t, S)-blurred sites that connects the site x to the boundary of S. We study such paths in the next lemma.

Definition 13 (BlurPath)**.** For $t' \geq t$, $x \in \overline{S}$ and $W \subset V$ let

$$\mathrm{BP}_{t'}^{t,S}(x,W) := \left\{ \begin{array}{c} \exists (p_i)_{1\leq i\leq n} \in \mathrm{PATH}(x,\partial S) : (p_i)_{1\leq i\leq n} \in (\overline{S}\setminus W)^n, \\ \forall 1 \leq i < n : \beta_{p_{i+1}}^{t,S} \leq \beta_{p_i}^{t,S} \leq t' \end{array} \right\}$$

denote the existence of a path $(p_i)_{1\leq i\leq n} \in \mathrm{PATH}(x,\partial S)$ that connects the site x to the boundary of S with the following properties:

(i) the path consists of sites in $\overline{S}\setminus W$ only;

(ii) for all $1 \leq i < n$ the site p_i does not get (t,S)-blurred before the site p_{i+1}. The site p_1 is (t,S)-blurred at time t'.

Lemma 3 (Existence of a BlurPath)**.** *For all $t' \geq t$ and all $x \in \overline{S}$*

$$\left\{\beta_x^{t,S} \leq t'\right\} \subset \mathrm{BP}_{t'}^{t,S}(x,\emptyset).$$

That is, if s site $x \in \overline{S}$ is (t,S)-blurred at time $t' \geq t$, then $\mathrm{BP}_{t'}^{t,S}(x,\emptyset)$ occurs.

Proof. We show by induction on $0 \leq i \leq |S|$ for all $z \in \overline{S}$

$$\left\{\beta_z^{t,S} \leq \tau_i\right\} \subset \mathrm{BP}_{\tau_i}^{t,S}(z,\emptyset).$$

To begin the induction, let $i = 0$ and $y \in \overline{S}$. Suppose that the site y is (t,S)-blurred at time $\tau_0 = t$. Then the cluster at y is connected to the boundary of S and all sites of the cluster at y are (t,S)-blurred at time τ_0. This implies the existence of a path showing that $\mathrm{BP}_{\tau_0}^{t,S}(y,\emptyset)$ occurs.

Let $1 \leq i \leq |S|$. As induction hypothesis suppose for all $z \in \overline{S}$

$$\left\{\beta_z^{t,S} \leq \tau_{i-1}\right\} \subset \mathrm{BP}_{\tau_{i-1}}^{t,S}(z,\emptyset).$$

Let $y \in \overline{S}$ and suppose that y is (t,S)-blurred at time τ_i. Then at time τ_i the entire set $C_{\tau_i,y} \cap \overline{S}$ is (t,S)-blurred, and there exists a site in $\overline{C_{\tau_i,y}} \cap \overline{S}$ that has already been (t,S)-blurred at time τ_{i-1}. Thus there exists a path $(p_l)_{1\leq l\leq j} \in \mathrm{PATH}(x,\overline{S})$ with the following properties:

(i) for all $1 \leq l < j$, the site $p_l \in \overline{S}$ gets blurred at time τ_i, that is, $\beta_{p_l}^{t,S} = \tau_i$.

(ii) the site $p_j \in \overline{S}$ is (t,S)-blurred at time τ_{i-1}, that is, the induction hypothesis implies the occurrence of $\mathrm{BP}_{\tau_{i-1}}^{t,S}(p_j,\emptyset)$.

Let $(p'_l)_{0\leq l\leq m} \in \mathrm{PATH}(p_j,\partial S)$ be a path showing that $\mathrm{BP}_{\tau_{i-1}}^{t,S}(p_j,\emptyset)$ occurs. Then $p'_l \in \overline{S}$ for all $0 \leq l \leq m$, and $\beta_{p_{l+1}}^{t,S} \leq \beta_{p_l}^{t,S} \leq \tau_{i-1}$ for all $0 \leq l < m$. We concatenate the disjoint paths $(p_l)_{1\leq l<j}$ and $(p'_l)_{0\leq l\leq m}$: for all $1 \leq l \leq j+m$ let

$$\tilde{p}_l := \begin{cases} p_l & \text{if } 1 \leq l < j; \\ p'_{l-j} & \text{otherwise.} \end{cases}$$

Then $\tilde{p}_l \in \overline{S}$ for all $1 \leq l \leq j+m$, and $\beta^{t,S}_{\tilde{p}_{l+1}} \leq \beta^{t,S}_{\tilde{p}_l} \leq \tau_i$ for all $1 \leq l < j+m$. That is, $(\tilde{p}_l)_{1 \leq l < j+m}$ shows that $\mathrm{BP}^{t,S}_{\tau_i}(y, \emptyset)$ occurs.

We conclude the lemma: let $t' \geq t$, $x \in \overline{S}$, and suppose that the site x is (t,S)-blurred at time t'. Then there exists $0 \leq i \leq |S|$ so that $\beta^{t,S}_x \leq \tau_i \leq t'$. This implies the occurrence of $\mathrm{BP}^{t,S}_{\tau_i}(x, \emptyset)$, and hence the occurrence of $\mathrm{BP}^{t,S}_{t'}(x, \emptyset)$. □

2.5.3 Growth and ignition estimates for the blur process

In this section we show that some events in terms of the blur process imply the occurrence of some growth and ignition events. Later on in Section 2.5.4 (Estimates for the proof of Proposition 4) we use these growth and ignition events to derive the key estimates we need to prove Proposition 4.

Throughout this section let $t \geq 0$ and $S \subset V$ finite. As in the definition of the blur process, we write $\overline{C_{s,x}} := x$ in case of $\eta_{s,x} = 0$, and $\overline{C_{s^-,x}} := x$ in case of $\eta_{s^-,x} = 0$, all $s > 0$ and $x \in S$.

Lemma 4. *Let $x \in S$ and suppose $\beta^{t,S}_x > t$. Then there is the growth of a tree on the set S at time $\beta^{t,S}_x$, and the site x is occupied at time $\beta^{t,S}_x$.*

Proof. Let $x \in S$ and suppose $b := \beta^{t,S}_x > t$. Then the set $\overline{C_{b,x}}$ contains a site that has been (t,S)-blurred before time b. That is, there exists $y \in \overline{C_{b,x}}$ so that $\beta^{t,S}_y < b$. Since the site x is not (t,S)-blurred before time b, for all $0 \leq s < b$ the set $\overline{C_{s,x}}$ does not contain a site that has been (t,S)-blurred before time s. This implies $\overline{C_{b^-,x}} \subset S$, and $\beta^{t,S}_z \geq b$ all $z \in \overline{C_{b^-,x}}$. That is, there exists $w \in \overline{C_{b,x}}$ so that $w \notin \overline{C_{b^-,x}}$. It follows that the site x is occupied at time b, and that at time b the cluster at x grows on its boundary. Using $\overline{C_{b^-,x}} \subset S$, we conclude that there is the growth of a tree on the set S at time b. □

Definition 14 (The time σ^t_x)**.** For all $x \in V$ we define

$$\sigma^t_x := \min\left\{s \geq t \middle| \eta_{s,x} = 0\right\}$$

to be the first time $s \geq t$ the site x is vacant.

Definition 15 (The event $G_{t,t',F}$)**.** For all $t' > t$ and all $F \subset V$ we write

$$G_{t,t',F} := \left\{\forall y \in F : G_{t,y} < G_{t',y}\right\}$$

to describe the event that at each site of the set F there occurs the growth of a tree in between time t and t'.

Lemma 5. *For all $x \in S$ the set $\left\{\beta^{t,S}_x = \sigma^t_x\right\}$ is empty.*

Proof. The definition of the blur process implies that if a site $x \in S$ is vacant at time t, then it is not (t,S)-blurred at time t. That is, the relation $\sigma^t_x = t$ implies $\beta^{t,S}_x > t$.

In case of $\beta^{t,S}_x > t$ from Lemma 4 there is the growth of tree on the set S at time $\beta^{t,S}_x$. Conversely, the relation $\sigma^t_x > t$ implies that the site x gets vacant at time σ^t_x, that is, that there occurs an ignition at time σ^t_x. It follows $\beta^{t,S}_x \neq \sigma^t_x$ since growth and ignition jumps occur at distinct times. □

Lemma 6 (Site vacant first, then whole cluster vacant first). *Let $x \in S$. If after time t the site x has been vacant before it gets (t,S)-blurred, then all sites of the set $\overline{C_{t,x}}$ have been vacant before they get (t,S)-blurred:*

$$\left\{\sigma_x^t < \beta_x^{t,S}\right\} = \left\{\forall y \in \overline{C_{t,x}} : y \in S, \sigma_y^t < \beta_y^{t,S}\right\}$$

Proof. The relation $\sigma_x^t < \beta_x^{t,S}$ implies that the site x is not (t,S)-blurred at time t. That is, the relation $\overline{C_{t,x}} \subset S$ holds. The definition of the blur process implies that if a site $y \in S$ is vacant at time t, then it is not (t,S)-blurred at time t. We obtain

$$\left\{\sigma_x^t < \beta_x^{t,S}\right\} \subset \left\{\overline{C_{t,x}} \subset S\right\} \subset \left\{\forall y \in \partial C_{t,x} : y \in S, \sigma_y^t < \beta_y^{t,S}\right\}.$$

The time σ_x^t is the first time $s \geq t$ the site $x \in S$ is vacant. In particular, none of the sites of the set $C_{t,x}$ get vacant during the time interval $[t, \sigma_x^t[$. Otherwise the entire cluster at x would get vacant. We conclude $C_{s,x} \supseteq C_{t,x}$ for all $s \in [t, \sigma_x^t[$. Thus, if there would exists $y \in C_{t,x}$ so that $\beta_y^{t,S} < \sigma_x^t$, then $\beta_x^{t,S} \leq \sigma_x^t$ would hold. This shows that $\beta_x^{t,S} > \sigma_x^t$ implies $\beta_y^{t,S} \geq \sigma_x^t$ for all $y \in C_{t,x}$. At time σ_x^t the cluster at x is hit by ignition and all its sites get vacant. Therefore, we have $\sigma_y^t = \sigma_x^t$ for all $y \in C_{t,x}$. We obtain

$$\begin{aligned}\left\{\sigma_x^t < \beta_x^{t,S}\right\} &\subset \left\{\forall y \in C_{t,x} : y \in S, \sigma_y^t = \sigma_x^t \leq \beta_y^{t,S}\right\} \\ &\subset \left\{\forall y \in C_{t,x} : y \in S, \sigma_y^t < \beta_y^{t,S}\right\},\end{aligned}$$

where we use Lemma 5 to conclude the second equality. □

Lemma 7 (Occurrence of a growth). *Let $x \in S$ and $t' > t$. Suppose that after time t the site x is vacant before it gets (t,S)-blurred, and that the site x is (t,S)-blurred at time t'. Then there must occur the growth of a tree at the site x in between time t and t':*

$$\left\{\sigma_x^t < \beta_x^{t,S} \leq t'\right\} \subset G_{t,t',x}$$

Proof. Let $x \in S$, $t' > t$, and suppose $\sigma_x^t < \beta_x^{t,S} \leq t'$. The relation $\sigma_x^t < \beta_x^{t,S}$ implies $\beta_x^{t,S} > t$. Thus from Lemma 4 the site $x \in S$ is occupied at time $\beta_x^{t,S}$. The site x is vacant at time σ_x^t, and we have $t \leq \sigma_x^t < \beta_x^{t,S} \leq t'$. Hence, there must occur the growth of a tree in between time t and t'. □

Definition 16 (The event GBI). For $R \subset V$ we write

$$\begin{aligned}&\mathrm{GrowthBeforeIgnition}_t(R,S) := \mathrm{GBI}_t(R,S) \\ &:= \{\exists x \in R\, \exists s > t\, \forall y \in S : I_{t,y} = I_{s,y}, G_{t,s,x}\}\end{aligned}$$

to describe the event that after time t there occurs the growth of a tree on the set R before the set S gets hit by ignition. For all $t' > t$ and all $x \in V$, we write

$$\mathrm{GBI}_{t,t'}(x,S) := \mathrm{GBI}_t(x,S) \cap G_{t,t',x}$$

if we additionally require that such a growth occurs at the site x until time t'.

Lemma 8 (Occurrence of GBI). *Let $t' > t$ and $x, y \in S$. Suppose that the site y is vacant at time t, and that y is (t, S)-blurred at time t' and before time σ_x^t. Then after time t there must have been the growth of a tree at the site y before time t', and before the set $C_{t,x}$ has been hit by ignition:*

$$\left\{\beta_y^{t,S} < \sigma_x^t, \beta_y^{t,S} \le t', \eta_{t,y} = 0\right\} \subset \mathrm{GBI}_{t,t'}(y, C_{t,x})$$

Proof. Let $t' > t$, let $x, y \in S$, and suppose $\beta_y^{t,S} < \sigma_x^t$. Then the site x is occupied throughout $[t, \beta_y^{t,S}]$. That is, the set $C_{t,x}$ does not get hit by ignition within $[t, \beta_y^{t,S}]$. We obtain

$$\left\{\beta_y^{t,S} < \sigma_x^t\right\} \subset \left\{\forall z \in C_{t,x} : I_{t,z} = I_{\beta_y^{t,S}, z}\right\}.$$

Suppose that the site y is vacant at time t. Then Lemma 5 implies $\beta_y^{t,S} > t$. Along with Lemma 4 this shows that the site y is occupied at time $\beta_y^{t,S}$. Since y is vacant at time t, this implies the growth of a tree at y in between time t and $\beta_y^{t,S}$. We obtain

$$\begin{aligned}&\left\{\beta_y^{t,S} < \sigma_x^t, \beta_y^{t,S} \le t', \eta_{t,y} = 0\right\}\\ &\subset \left\{\forall z \in C_{t,x} : I_{t,z} = I_{\beta_y^{t,S}, z}, G_{t, \beta_y^{t,S}, y}, \beta_y^{t,S} \le t'\right\} \subset \mathrm{GBI}_{t,t'}(y, C_{t,x}).\end{aligned}$$

□

2.5.4 Estimates for the proof of Proposition 4

We derive the two estimates the proof of Proposition 4 is based on. Throughout this section let m, ϵ, t, x and k as in Proposition 4, and suppose $\mathrm{CCSB}(t, \lambda/(4d^2), m)$ and (2.5). Then from $\mathrm{CCSB}(t, \lambda/(4d^2), m)$ except on a null set there does not exist an infinite cluster at time t. For a more convenient notation, we restrict the forest fire process to the complement of the latter null set throughout this section. For abbreviation, let $\mathcal{C} := \mathrm{C}_{t,x}(k)$, and write $\{s < \beta_z^{t,\mathcal{C}}\}$ to denote $\{z \in \mathcal{C}, s < \beta_z^{t,\mathcal{C}}\}$ all $z \in V$ and $s \ge 0$

The estimates the proof of Proposition 4 is based on are quite technical. To motivate them, we give a rough sketch of the proof of Proposition 4 first. From Lemma 3 (Existence of a BlurPath) we have

$$P\left(\beta_{t+\epsilon,x}^{t,\mathcal{C}} = 2, |\mathcal{C}| < \infty\right) = P\left(\mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(x, \emptyset)\right).$$

To estimate the right hand site, we successively split up the event $\mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(x, \emptyset)$: let $\mathcal{P}$ be a path showing the occurrence of the latter event. The definition of the set $\mathcal{C}$ implies that on its way from x to the boundary of $\mathcal{C}$ the path $\mathcal{P}$ intersects at least k (possibly empty) neighbouring clusters. Starting at the cluster at x we jump from one of the latter clusters to the next one. After $i - 1$ jumps we reach a cluster at some site $y \in \mathrm{C}_{t,x}(i)$. We distinguish whether y gets $(t, \mathcal{C})$-blurred before or after time σ_y^t:

- ($\beta_y^{t,\mathcal{C}} < \sigma_y^t$ implies GBI) We suppose that y gets $(t,\mathcal{C})$-blurred before time σ_y^t, that is, $\beta_y^{t,\mathcal{C}} < \sigma_y^t$. Then let $z \in \partial C_{t,y}$ be the site at which the path $\mathcal{P}$ leaves the set $C_{t,y}$ the last time. We apply the lemmata from Section 2.5.3 to conclude the occurrence of $\mathrm{GBI}_{t,t+\epsilon}(z, C_{t,y})$. In the next step we jump to the next (possibly empty) cluster the path $\mathcal{P}$ passes on its way from z to the boundary. Our choice of z to be the last site at which the path $\mathcal{P}$ leaves the set $C_{t,y}$ the last time has the following reason. Each time we arrive at a cluster, we derive the occurrence of an event described by increments of the growth and ignition processes on or next to the cluster. To estimate their probabilities we want them to be independent. The event $\mathrm{GBI}_{t,t+\epsilon}(z, C_{t,y})$ depends on the increments of the growth and ignition events that occur on the set $z \cup C_{t,y}$ after time t. Thus after visiting the cluster at y we do not want to re-enter the cluster at y again;

- ($\beta_y^{t,\mathcal{C}} > \sigma_y^t$ implies growths along a sub path) Suppose that y gets $(t,\mathcal{C})$-blurred after time σ_y^t, that is, suppose $\beta_y^{t,\mathcal{C}} > \sigma_y^t$. We choose $z \in \partial C_{t,y}$ to be the site at which the path $\mathcal{P}$ leaves the set $C_{t,y}$ the first time. We write P to denote the sites of the sub path of $\mathcal{P}$ that connects the sites y and z. Using the lemmata from Section 2.5.3, we conclude the occurrence of $G_{t,t+\epsilon,P}$. In the next step we jump to the next (possibly empty) cluster the path $\mathcal{P}$ passes on its way from z to the boundary. In difference from the latter case, it might be that on its way from z to the boundary the path $\mathcal{P}$ re-enters the set cluster at y. The event $G_{t,t+\epsilon,P}$ depends on the increments of the growth processes on the set P. Thus to assure independence we have to take care that we do not use these increments in one of the later steps again.

Altogether after $i-1$ jumps we have gained the following information. Upon our travel we derived an event described by increments of the growth and ignition processes on a finite set $B \subset V$. And we know the configuration of the visited clusters, that is, we know $\cup_{w\in B} C_{t,w}$. Furthermore, we are aware of the remaining tail of $\mathcal{P}$ connecting a site $y \in \mathrm{C}_{t,x}(i)$ to the boundary of $\mathcal{C}$. For a formal statement of the latter, we introduce the event $\mathrm{SPREAD}\,(y,i,B)$. Here the occurrence of the set B and a further event $\mathcal{E}(B)$ has technical reasons and is due to the possibility of re-entering a cluster a second time.

Definition 17 (The event SPREAD)**.** Let $1 \le i \le k$, $B \subset V$ finite and $y \in V$. We write

$$\mathrm{SPREAD}\,(y,i,B) := \left\{\mathcal{E}(B), y \in \mathrm{C}_{t,x}(i), \mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(y,B)\right\}.$$

Here

$$\mathcal{E}(B) := \bigcap_{z\in B} \left(\{\sigma_z^t < \beta_z^{t,\mathcal{C}}\} \cup \{C_{t,z} \subset B\}\right).$$

denotes the event that for all $z \in B$ the cluster at z at time t is a subset of B, or it holds $\sigma_z^t < \beta_z^{t,\mathcal{C}}$.

Definition 18 (The set C_y)**.** For all $y \in V$ let

$$C_y := \{S \subset V \big| y \in S, |S| < \infty, S \text{ is connected}\}$$

denote the possible configurations of the cluster at y, provided it is non-empty.

Definition 19 (The σ-field $\mathcal{INCR}$)**.** We write

$$\mathcal{INCR}_t(B) := \sigma\left(G_{t+s,y} - G_{t,y}, I_{t+s,y} - I_{t,y} : s \geq 0, y \in B\right)$$

to denote the σ-field generated by the increments of the growth and ignition processes on the set $B \subset V$ after time $t \geq 0$.

The next lemma formalizes the distinction we sketched above. We are going to use the lemma to successively split up the probability $P\big(\mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(x,\emptyset)\big) = P\big(\mathrm{SPREAD}\,(x,1,\emptyset)$.

Lemma 9. *Let $B, D \subset V$ finite, $GI_t(B) \in \mathcal{INCR}_t(B)$, $y \in V$ and $1 \leq i < k$. Then the probability*

$$P\left(GI_t(B), \cup_{w\in B} C_{t,w} = D, \mathrm{SPREAD}\,(y,i,B)\right) \tag{2.6}$$

is smaller than or equal to the sum of

$$1_{\{y\notin D\}} \sum_{\substack{C\in C_y \\ C\cap B=\emptyset}} \sum_{\substack{z\in\partial C \\ z\notin B}} \sum_{y'\in\partial z} P\begin{pmatrix} GI_t(B,z,C), C_{t,y} = C, \cup_{w\in B} C_{t,w} = D, \\ \mathrm{SPREAD}\,(y', i+1, B\cup C\cup z) \end{pmatrix}$$

and

$$\sum_{C\in C_y} \sum_{z\in\partial C} \sum_{n\in\mathbb{N}} \sum_{\substack{\mathcal{P}\in \mathrm{PATH}_n(y,z) \\ \mathcal{P}_V\cap B=\emptyset}} \sum_{y'\in\partial z} P\begin{pmatrix} GI_t(B,\mathcal{P}_V), C_{t,y} = C, \cup_{w\in B} C_{t,w} = D, \\ \mathrm{SPREAD}\,(y', i+1, B\cup \mathcal{P}_V) \end{pmatrix}$$
$$+1_{\{y\notin B\}} \sum_{y'\in\partial y} P\begin{pmatrix} GI_t(B,y), \eta_{t,y} = 0, \cup_{w\in B} C_{t,w} = D, \\ \mathrm{SPREAD}\,(y', i+1, B\cup y) \end{pmatrix}.$$

Here we write $GI_t(B,z,C)$ to denote the intersection of the independent events $GI_t(B)$ and $\mathrm{GBI}_{t,t+\epsilon}(z,C)$, and $GI_t(B,\mathcal{P}_V)$ for the intersection of the independent events $GI_t(B)$ and $G_{t,t+\epsilon,\mathcal{P}_V}$.

Proof. Let B, D, y and i as in the lemma. If $y \in B$, then $\mathrm{SPREAD}\,(y,i,B)$ is the empty set. Thus it suffices to study the case $y \notin B$. From Lemma 5 the set $\{\sigma_y^t = \beta_y^{t,\mathcal{C}}\}$ is empty. If $\beta_y^{t,\mathcal{C}} < \sigma_y^t$, we proceed as sketched in ($\beta_y^{t,\mathcal{C}} > \sigma_y^t$ implies GBI) and show

$$\begin{aligned} &\left\{\mathrm{SPREAD}\,(y,i,B)\,, \beta_y^{t,\mathcal{C}} < \sigma_y^t\right\} \\ &\subset \bigcup_{C\in C_y} \bigcup_{\substack{z\in\partial C \\ z\notin B}} \bigcup_{y'\in\partial z} \left\{ \begin{array}{c} C_{t,y} = C, \mathrm{GBI}_{t,t+\epsilon}(z,C), \\ \mathrm{SPREAD}\,(y', i+1, B\cup C\cup z) \end{array} \right\}. \end{aligned} \tag{2.7}$$

Otherwise, if $\beta_y^{t,\mathcal{C}} > \sigma_y^t$, the method described in ($\beta_y^{t,\mathcal{C}} > \sigma_y^t$ implies growths along a sub path) yields

$$\begin{aligned}&\left\{\mathrm{SPREAD}\left(y,i,B\right),\beta_y^{t,\mathcal{C}} > \sigma_y^t\right\}\\&\subset \bigcup_{C\in C_y}\bigcup_{z\in\partial C}\bigcup_{\substack{\mathcal{P}\in\mathrm{PATH}(y,z)\\ \mathcal{P}_V\cap B=\emptyset}}\bigcup_{y'\in\partial z}\left\{\begin{array}{c} C_{t,y}=C, G_{t,t+\epsilon,\mathcal{P}_V},\\ \mathrm{SPREAD}\left(y',i+1,B\cup\mathcal{P}_V\right)\end{array}\right\}\\&\qquad\cup\bigcup_{y'\in\partial y}\left\{\begin{array}{c}\eta_{t,y}=0, G_{t,t+\epsilon,y},\\ \mathrm{SPREAD}\left(y',i+1,B\cup y\right)\end{array}\right\}.\end{aligned}\tag{2.8}$$

In the summation in the lemma, the independence of the events $\mathrm{GBI}_{t,t+\epsilon}(z,C)$ and $GB_t(B)$ arises from the disjointness of the sets $C\cup z$ and B. To obtain this disjointness, we are going to show

$$\left\{\cup_{w\in B}C_{t,w}=D,\mathcal{E}\left(B\right),\beta_y^{t,\mathcal{C}}<\sigma_y^t\right\}\subset\{C_{t,y}\cap B=\emptyset, y\notin D\}.\tag{2.9}$$

Note that the relation (2.7), (2.8) and (2.9) imply the lemma. It remains to prove these relations.

Proof of (2.7). We proceed as in ($\beta_y^{t,\mathcal{C}} > \sigma_y^t$ implies GBI). The relation $\beta_y^{t,\mathcal{C}} < \sigma_y^t$ implies $\eta_{t,y}=1$, and there does not exist infinite clusters at time t. We condition on the shape of the cluster at y at time t, and obtain

$$\begin{aligned}&\left\{\beta_y^{t,\mathcal{C}}<\sigma_y^t,\mathrm{SPREAD}\left(y,i,B\right)\right\}\\&=\bigcup_{C\in C_y}\left\{C_{t,y}=C,\beta_y^{t,\mathcal{C}}<\sigma_y^t,\mathrm{SPREAD}\left(y,i,B\right)\right\}.\end{aligned}\tag{2.10}$$

The occurrence of $\mathrm{SPREAD}\left(y,i,B\right)$ implies $y\in\mathrm{C}_{t,x}(i)$ and $\mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(y,B)$. That is, there exists a path $(p_l)_{1\le l\le n}\in\mathrm{PATH}(y,\partial\mathcal{C})$ so that

$$(p_l)_{1\le l\le n}\in\left(\overline{\mathcal{C}}\setminus B\right)^n;\tag{2.11}$$

and

$$\beta_{p_{l+1}}^{t,S}\le\beta_{p_l}^{t,S}\le t+\epsilon\qquad\text{all } 1\le l<n.\tag{2.12}$$

Let $C\in C_y$ and suppose $C_{t,y}=C$. Starting at the site $y=p_1$, there exists a site in $z\in\overline{C}$ at which the path $(p_l)_{1\le l\le n}$ leaves the set $\overline{C}$ the last time. More precisely, let

$$j:=\max\left\{1\le l\le n|p_l\in\overline{C}\right\}$$

and $z:=p_j$. We show $z\in\partial C$, $z\notin B$, and $j<n$. By definition $\partial\mathcal{C}\cap\mathrm{C}_{t,x}(i)=\emptyset$, and $y\in\mathrm{C}_{t,x}(i)$ implies $\overline{C}\subset\mathrm{C}_{t,x}(i)$. Therefore, we have $\partial\mathcal{C}\cap\overline{C}=\emptyset$. Along with $p_n\in\partial\mathcal{C}$ it follows $j<n$ and $z\in\partial C$. From (2.11) we conclude $z\notin B$.

Let $y' := p_{j+1}$. Then (2.12) implies $\beta_z^{t,\mathcal{C}} \le \beta_y^{t,\mathcal{C}} \le t+\epsilon$, and $y' \in \partial z$ and $z \in \mathrm{C}_{t,x}(i)$ imply $y' \in \mathrm{C}_{t,x}(i+1)$. Furthermore, $(p_l)_{j+1\le l\le n} \in \mathrm{PATH}(y', \partial\mathcal{C})$, and (2.11) and our choice of j imply

$$(p_l)_{j+1\le l\le n} \in \left(\overline{\mathcal{C}} \setminus (B \cup C \cup z)\right)^{n-j}.$$

Together with (2.12) it follows that the path $(p_l)_{j+1\le l\le n}$ shows that $\mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(y', B\cup C\cup z)$ occurs. Altogether, we obtain

$$\begin{aligned}&\left\{C_{t,y} = C, \beta_y^{t,\mathcal{C}} < \sigma_y^t, \mathrm{SPREAD}\,(y,i,B)\right\}\\ &\subset \bigcup_{\substack{z\in\partial C\\ z\notin B}} \bigcup_{y'\in\partial z} \left\{\begin{array}{c} C_{t,y} = C, \beta_y^{t,\mathcal{C}} < \sigma_y^t, \beta_z^{t,\mathcal{C}} \le \beta_y^{t,\mathcal{C}} \le t+\epsilon, \\ y' \in \mathrm{C}_{t,x}(i+1), \mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(y', B\cup C\cup z), \mathcal{E}(B)\end{array}\right\}.\end{aligned} \tag{2.13}$$

Let $z \in \partial C$ and suppose $C_{t,y} = C$. Then the site z is vacant at time t. It follows

$$\begin{aligned}&\left\{C_{t,y} = C, \beta_y^{t,\mathcal{C}} < \sigma_y^t, \beta_z^{t,\mathcal{C}} \le \beta_y^{t,\mathcal{C}} \le t+\epsilon, \mathcal{E}(B)\right\}\\ &\subset \left\{C_{t,y} = C, \beta_z^{t,\mathcal{C}} < \sigma_y^t, \beta_z^{t,\mathcal{C}} \le t+\epsilon, \eta_{t,z} = 0\right\}\end{aligned}$$

Application of Lemma 8 (Occurrence of GBI) provides

$$\begin{aligned}&\left\{C_{t,y} = C, \beta_z^{t,\mathcal{C}} < \sigma_y^t, \beta_z^{t,\mathcal{C}} \le t+\epsilon, \eta_{t,z} = 0\right\}\\ &\subset \left\{C_{t,y} = C, \mathrm{GBI}_{t,t+\epsilon}(z,C), \beta_z^{t,\mathcal{C}} \le t+\epsilon, \eta_{t,z} = 0\right\}.\end{aligned}$$

If the site z is vacant at time t, then $\sigma_z^t = 0$. From Lemma 5 the set $\{\sigma_z^t = \beta_z^{t,\mathcal{C}}\}$ is empty. It follows

$$\{C_{t,y} = C, \beta_z^{t,\mathcal{C}} \le t+\epsilon, \eta_{t,z} = 0\} \subset \{C_{t,y} = C, \sigma_z^t < \beta_z^{t,\mathcal{C}}\} \subset \mathcal{E}(C\cup z).$$

Altogether, this shows

$$\begin{aligned}&\left\{C_{t,y} = C, \beta_y^{t,\mathcal{C}} < \sigma_y^t, \beta_z^{t,\mathcal{C}} \le \beta_y^{t,\mathcal{C}} \le t+\epsilon, \mathcal{E}(B)\right\}\\ &\subset \{C_{t,y} = C, \mathrm{GBI}_{t,t+\epsilon}(z,C), \mathcal{E}(C\cup z)\}.\end{aligned}$$

Combining with (2.13) we obtain

$$\begin{aligned}&\left\{C_{t,y} = C, \beta_y^{t,\mathcal{C}} < \sigma_y^t, \mathrm{SPREAD}\,(y,i,B)\right\}\\ &\subset \bigcup_{\substack{z\in\partial C\\ z\notin B}} \bigcup_{y'\in\partial z} \left\{\ C_{t,y} = C, \mathrm{GBI}_{t,t+\epsilon}(z,C), \mathrm{SPREAD}\,(y', i+1, B\cup C\cup z)\ \right\}.\end{aligned}$$

Along with (2.10), this concludes the prove of (2.7).

Proof of (2.8). The proof of (2.8) is the formal analogon to ($\beta_y^{t,\mathcal{C}} > \sigma_y^t$ implies growths along a sub path). We consider the case $\eta_{t,y} = 1$ first. Then

$$\begin{aligned}&\left\{\eta_{t,y} = 1, \beta_y^{t,\mathcal{C}} > \sigma_y^t, \mathrm{SPREAD}\,(y,i,B)\right\}\\ &= \bigcup_{C\in C_y} \left\{C_{t,x} = C, \beta_y^{t,\mathcal{C}} > \sigma_y^t, \mathrm{SPREAD}\,(y,i,B)\right\}.\end{aligned} \tag{2.14}$$

Let $C \in C_y$ and suppose $\{C_{t,x} = C, \text{SPREAD}\,(y, i, B)\}$. Let $(p_l)_{1 \le l \le n} \in \text{PATH}(y, \partial \mathcal{C})$ be a path showing the occurrence of $\text{BP}^{t,\mathcal{C}}_{t+\epsilon}(y, B)$. Then the path $(p_l)_{1\le l\le n}$ satisfies (2.11) and (2.12). In the proof of (2.7) we picked the site at which the path $(p_l)_{1\le l\le n}$ leaves the set $\overline{C}$ the last time, and showed that such a site is an element of ∂C. Hence

$$j := \min\{1 \le l \le n | p_l \in \partial C\}$$

is well defined, and $z := p_j$ is the first site at which the path $(p_l)_{1\le l\le n}$ intersects the set ∂C. We write $\mathcal{P} := (p_l)_{1\le l\le j}$, and note that our choice of j implies $z \in \partial C$ and $\mathcal{P}_V \subset \overline{C}$. From (2.11) and (2.12) it follows $\mathcal{P}_V \subset \overline{C} \setminus B$, and $\beta^{t,\mathcal{C}}_w \le t + \epsilon$ for all $w \in \mathcal{P}_V$. As in the proof of (2.7) we obtain $j < n$, $y' := p_{j+1} \in \mathrm{C}_{t,x}(i + 1)$, and that $(p_l)_{j+1\le l\le n}$ is path showing the occurrence of $\text{BP}^{t,\mathcal{C}}_{t+\epsilon}(y', B \cup \mathcal{P}_V)$. Altogether, we have

$$\begin{aligned}&\left\{C_{t,y} = C, \beta^{t,\mathcal{C}}_y > \sigma^t_y, \text{SPREAD}\,(y, i, B)\right\}\\ &\subset \bigcup_{z\in\partial C} \bigcup_{\substack{\mathcal{P}\in\text{PATH}(y,z)\\ \mathcal{P}_V\subset\overline{C}\setminus B}} \bigcup_{y'\in\partial z} \left\{\begin{array}{c} C_{t,y} = C, \beta^{t,\mathcal{C}}_y > \sigma^t_y, \forall w \in \mathcal{P}_V : \beta^{t,\mathcal{C}}_w \le t + \epsilon,\\ y' \in \mathrm{C}_{t,x}(i + 1), \text{BP}^{t,\mathcal{C}}_{t+\epsilon}(y', B \cup \mathcal{P}_V), \mathcal{E}(B)\end{array}\right\}.\end{aligned} \tag{2.15}$$

Suppose that the site y gets $(t, \mathcal{C})$-blurred after time σ^t_y. Then from Lemma 6 (Site vacant first, then whole cluster vacant first) all sites $w \in \overline{C}$ get $(t, \mathcal{C})$-blurred after time σ^t_w. By Lemma 7 (Occurrence of a growth) if a site $w \in \mathcal{C}$ gets $(t, \mathcal{C})$-blurred in between time σ^t_w and $t+\epsilon$, then there must have been the growth of a tree at the site x in between time t and $t + \epsilon$. Therefore we have for all $A \subset \overline{C}$

$$\begin{aligned}&\left\{C_{t,y} = C, \beta^{t,\mathcal{C}}_y > \sigma^t_y, \forall w \in A : \beta^{t,\mathcal{C}}_w \le t + \epsilon\right\}\\ &\subset \left\{\forall w \in A : \sigma^t_w < \beta^{t,\mathcal{C}}_w \le t + \epsilon, \right\} \subset \left\{G_{t,t+\epsilon,A}, \mathcal{E}(A)\right\}.\end{aligned}$$

Combing with (2.14) and (2.15) it follows

$$\begin{aligned}&\left\{\eta_{t,y} = 1, \beta^{t,\mathcal{C}}_y > \sigma^t_y, \text{SPREAD}\,(y, i, B)\right\}\\ &\subset \bigcup_{C\in C_y} \bigcup_{z\in\partial C} \bigcup_{\substack{\mathcal{P}\in\text{PATH}(y,z)\\ \mathcal{P}_V\cap B=\emptyset}} \bigcup_{y'\in\partial z} \left\{\begin{array}{c} C_{t,y} = C, G_{t,t',\mathcal{P}_V},\\ \text{SPREAD}\,(y', i + 1, B \cup \mathcal{P}_V)\end{array}\right\}.\end{aligned}$$

In case of $\eta_{t,y} = 0$ the same arguments as above provide

$$\begin{aligned}&\left\{\eta_{t,y} = 0, \beta^{t,\mathcal{C}}_y > \sigma^t_y, \text{SPREAD}\,(y, i, B)\right\}\\ &\subset \bigcup_{y'\in\partial y} \left\{\begin{array}{c} \eta_{t,y} = 0, G_{t,t+\epsilon,y},\\ \text{SPREAD}\,(y', i + 1, B \cup y)\end{array}\right\}.\end{aligned}$$

This concludes the proof of (2.8).

Proof of (2.9). The relation (2.9) is

$$\left\{\cup_{w\in B} C_{t,w} = D, \mathcal{E}\,(B)\,, \beta^{t,\mathcal{C}}_y < \sigma^t_y\right\} \subset \{C_{t,y} \cap B = \emptyset, y \notin D\}.$$

We start the proof of (2.9) by showing that the set

$$\left\{\mathcal{E}\left(B\right),\beta_y^{t,\mathcal{C}}<\sigma_y^t,C_{t,y}\cap B\neq\emptyset\right\}$$

is empty. Suppose that $C_{t,y}\cap B\neq\emptyset$. Then there exists a site $z\in B$ such that $y\in C_{t,z}$ holds. From Lemma 6 (Site vacant first, then whole cluster vacant first) if z would get $(t,\mathcal{C})$-blurred after time σ_z^t, the site y would get $(t,\mathcal{C})$-blurred after time σ_y^t. Thus we obtain

$$\left\{\mathcal{E}\left(B\right),\beta_y^{t,\mathcal{C}}<\sigma_y^t,C_{t,y}\cap B\neq\emptyset\right\}\subset\left\{\mathcal{E}\left(B\right),\exists z\in B:\beta_z^{t,\mathcal{C}}<\sigma_z^t,y\in C_{t,z}\right\}.$$

The event $\mathcal{E}(B)$ implies that for all $z\in B$ it holds $\beta_z^{t,\mathcal{C}}>\sigma_z^t$, or $C_{t,z}\subset B$. We conclude

$$\left\{\mathcal{E}\left(B\right),\exists z\in B:\beta_z^{t,\mathcal{C}}<\sigma_z^t,y\in C_{t,z}\right\}\subset\left\{\exists z\in B:C_{t,z}\subset B,y\in C_{t,z}\right\}=\emptyset,$$

where we use $y\notin B$.

We conclude (2.9). Since the set $\left\{\mathcal{E}\left(B\right),\beta_y^{t,\mathcal{C}}<\sigma_y^t,C_{t,y}\cap B\neq\emptyset\right\}$ is empty, we have

$$\left\{\cup_{w\in B}C_{t,w}=D,\mathcal{E}\left(B\right),\beta_y^{t,\mathcal{C}}<\sigma_y^t\right\}\subset\left\{\cup_{w\in B}C_{t,w}=D,C_{t,y}\cap B=\emptyset\right\}.$$

Suppose that $\cup_{w\in B}C_{t,w}=D$ occurs. In case of $y\in D$ there exists a $z\in B$ such that $z\in C_{t,y}$. In particular, $\cup_{w\in B}C_{t,w}=D$ and $y\in D$ imply $C_{t,y}\cap B\neq\emptyset$. Hence

$$\left\{\cup_{w\in B}C_{t,w}=D,C_{t,y}\cap B=\emptyset\right\}\subset\left\{y\notin D,C_{t,y}\cap B=\emptyset\right\}.$$

□

Lemma 9 enables us to successively split up the probability $P\big(\mathrm{BP}_{t+\epsilon}^{t,\mathcal{C}}(x,\emptyset)\big)$. In the next lemma we estimate the sums we obtain by doing so.

Lemma 10. *Let B, D, $GI_t(B)$, and y as in Lemma 9. Then the sum of*

$$1_{\{y\notin D\}}\sum_{\substack{C\in C_y\\ C\cap B=\emptyset}}\sum_{\substack{z\in\partial C\\ z\notin B}}\sum_{y'\in\partial z}P\left(GI_t(B,z,C),C_{t,y}=C,\cup_{w\in B}C_{t,w}=D\right)$$

and

$$\sum_{C\in C_y}\sum_{z\in\partial C}\sum_{n\in\mathbb{N}}\sum_{\substack{\mathcal{P}\in\mathrm{PATH}_n(y,z)\\ \mathcal{P}_V\cap B=\emptyset}}\sum_{y'\in\partial z}P\left(GI_t(B,\mathcal{P}_V),C_{t,y}=C,\cup_{w\in B}C_{t,w}=D\right)$$
$$+1_{\{y\notin B\}}\sum_{y'\in\partial y}P\left(GI_t(B,y),\eta_{t,y}=0,\cup_{w\in B}C_{t,w}=D\right)$$

is smaller than or equal to

$$\frac{3}{4}\cdot P\left(GI_t(B),\cup_{w\in B}C_{t,w}=D\right).$$

Proof. Let B, D, $GI_t(B)$ and y as in the lemma. Let $C \in C_y$, $K \subset V$ such that $K \cap B = \emptyset$ and $GI_t(K) \in \mathcal{INCR}_t(K)$. The occurrence of the event $\{C_{t,y} = C, \cup_{w\in B} C_{t,w} = D\}$ is determined by the configuration of the forest fire process η on the finite set $\overline{C} \cup \overline{D} \cup B$. Hence, the almost sure infinite volume convergence at time t, (2.5), implies

$$\begin{aligned} &P\left(GI_t(B), GI_t(K), C_{t,y} = C, \cup_{w\in B} C_{t,w} = D\right) \\ &= \lim_{n\to\infty} P\left(GI_t(B), GI_t(K), C_{t,y}^{(n)} = C, \cup_{w\in B} C_{t,w}^{(n)} = D\right) \end{aligned}$$

Let $n \in \mathbb{N}$ so that $\overline{C} \cup \overline{D} \cup B \subset B_n$. From Remark 3 the configuration of the finite volume forest fire process $\eta^{(n)}$ at time t is independent of the increments of the growth and ignition processes after time t. Furthermore, the disjointness of K and B implies that the increments of the growth and ignition processes on K and B are independent. Hence,

$$\begin{aligned} &P\left(GI_t(B), GI_t(K), C_{t,y}^{(n)} = C, \cup_{w\in B} C_{t,w}^{(n)} = D\right) \\ &= P\left(GI_t(B)\right) \cdot P\left(GI_t(K)\right) \cdot P\left(C_{t,y}^{(n)} = C, \cup_{w\in B} C_{t,w}^{(n)} = D\right), \end{aligned}$$

and using (2.5) again it follows

$$\begin{aligned} &P\left(GI_t(B), GI_t(K), C_{t,y} = C, \cup_{w\in B} C_{t,w} = D\right) \\ &= P\left(GI_t(B)\right) \cdot P\left(GI_t(K)\right) \cdot P\left(C_{t,y} = C, \cup_{w\in B} C_{t,w} = D\right). \end{aligned}$$

Therefore, to show the lemma it suffices to show

$$\begin{aligned} 1_{\{y\notin D\}} \sum_{C\in C_y} \sum_{z\in\partial C} P\left(\mathrm{GBI}_{t,t+\epsilon}(z,C)\right) \cdot P\left(C_{t,y} = C, \cup_{w\in B} C_{t,w} = D\right) \\ \leq \frac{1}{4d} \cdot P\left(\cup_{w\in B} C_{t,w} = D\right) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \sum_{C\in C_y \cup\{\emptyset\}} \sum_{z\in\partial C} \sum_{n\in\mathbb{N}} \sum_{\mathcal{P}\in \mathrm{PATH}_n(y,z)} P\left(G_{t,t+\epsilon,\mathcal{P}_V}\right) \cdot P\left(C_{t,y} = C, \cup_{w\in B} C_{t,w} = D\right) \\ \leq \frac{2}{4d} \cdot P\left(\cup_{w\in B} C_{t,w} = D\right). \end{aligned} \tag{2.17}$$

Here in the summation in (2.17) we write $\partial C := y$ in case of $C = \emptyset$.

We show (2.17) first. The vertex degree of the graph G is bounded by d. That is, for all $n \in \mathbb{N}$ there exist at most d^{n-1} different paths with origin y and length n. Therefore, the choice of ϵ provides for all $C \in C_y \cup \{\emptyset\}$

$$\begin{aligned} \sum_{z\in\partial C} \sum_{n\in\mathbb{N}} \sum_{\mathcal{P}\in \mathrm{PATH}_n(y,z)} P\left(G_{t,t+\epsilon,\mathcal{P}_V}\right) &< \sum_{n\in\mathbb{N}} \sum_{\mathcal{P}\in \mathrm{PATH}_n(y,\partial C)} P\left(G_{\epsilon,0} > 0\right)^n \\ &< \sum_{n\in\mathbb{N}} \frac{d^{n-1}}{(4md^2)^n} < \frac{1}{4d}. \end{aligned}$$

It follows (2.17). To prove (2.16), we distinguish whether at time t the cluster at y is bigger than m, or not. Note that since the vertex degree is bounded by d, $|\partial S| \leq d|S|$ for all finite, non-empty $S \subset V$. Thus for $C \in C_y$ such that $|C| \leq m$,

$$\sum_{z \in \partial C} P\left(\mathrm{GBI}_{t,t+\epsilon}(z, C)\right) \leq \sum_{z \in \partial C} P\left(G_{\epsilon,0} > 0\right) \leq \frac{md}{4md^2} = \frac{1}{4d}. \tag{2.18}$$

For $C \in C_y$ satisfying $|C| > m$, we obtain

$$\sum_{z \in \partial C} P\left(\mathrm{GBI}_{t,t+\epsilon}(z, C)\right) \leq \sum_{z \in \partial C} P\left(\mathrm{GBI}_{t}(z, C)\right) = \frac{|\partial C|}{1 + \lambda|C|} \leq \frac{d}{\lambda}. \tag{2.19}$$

We assumed $\mathrm{CCSB}(t, \lambda/(4d^2), m)$, that is, in particular

$$1_{\{y \notin D\}} \cdot P\left(|C_{t,y}| > m, \cup_{w \in B} C_{t,w} = D\right) \leq \frac{\lambda}{4d^2} \cdot P\left(\cup_{w \in B} C_{t,w} = D\right).$$

Combining with (2.18) and (2.19), it follows

$$\begin{aligned} &1_{\{y \notin D\}} \sum_{C \in C_y} \sum_{z \in \partial C} P\left(\mathrm{GBI}_{t,t+\epsilon}(z, C)\right) \cdot P\left(C_{t,y} = C, \cup_{w \in B} C_{t,w} = D\right) \\ &\leq \left(\frac{1}{4d} + \frac{d}{\lambda} \cdot \frac{\lambda}{4d^2}\right) \cdot P\left(\cup_{w \in B} C_{t,w} = D\right) = \frac{2}{4d} \cdot P\left(\cup_{w \in B} C_{t,w} = D\right). \end{aligned}$$

This shows (2.16). □

2.5.5 Proof of Proposition 4

We prove Proposition 4 in this section.

Proof of Proposition 4. Let $m \in \mathbb{N}$, $t \geq 0$, and $\epsilon > 0$ as in Proposition 4 and suppose $\mathrm{CCSB}(t, \lambda/(4d^2), m)$ and (2.5). From $\mathrm{CCSB}(t, \lambda/(4d^2), m)$ except on a null set there does not exist an infinite cluster at time t. We restrict the forest fire process to the complement of the latter null set. We have to show that for all $x \in V$ and all $k \in \mathbb{N}$

$$P\left(\beta_{t+\epsilon,x}^{t,\mathrm{C}_{t,x}(k)} = 2\right) \leq \left(\frac{3}{4}\right)^{k-1}.$$

Let $x \in V$ and $k \in \mathbb{N}$. Lemma 3 (Existence of a BlurPath) implies

$$P\left(\beta_{t+\epsilon,x}^{t,\mathrm{C}_{t,x}(k)} = 2\right) = P\left(\mathrm{SPREAD}\left(x, 1, \emptyset\right)\right).$$

Applying Lemma 9 successively $k-1$ times and thereafter using Lemma 10 to estimate the derived sum yields the desired result. □

2.5.6 Proof of Proposition 2

In this section we give the proof of Proposition 2.

Proof of Proposition 2. Let $t \geq 0$, $x \in V$ and $m \in \mathbb{N}$, and choose $\epsilon > 0$ such that $P(G_{\epsilon,0} > 0) < 1/(4md^2)$. Suppose $\mathrm{CCSB}(t, \lambda/(4d^2), m)$ and (2.5). Let $M \in \mathbb{N}$ such that $x \in B_M$. Lemma 2 (Monotonicity of the blur process) implies for all $n \geq M$

$$\left\{\beta_{t+\epsilon,x}^{t,B_{n+1}} = 2\right\} \subset \left\{\beta_{t+\epsilon,x}^{t,B_n} = 2\right\}.$$

That is, to show $\lim_{n\to\infty} P\left(\beta_{t+\epsilon,x}^{t,B_n} = 2\right) = 0$, it suffices to show that the set

$$E := \bigcap_{n \geq M} \left\{\beta_{t+\epsilon,x}^{t,B_n} = 2\right\}$$

is a null set. Let $k \in \mathbb{N}$. From $\mathrm{CCSB}(t, \lambda/(4md^2), m)$, with probability one there does not exist an infinite cluster at time t. Hence, except on a null set $\mathcal{N}$, the set $\mathrm{C}_{t,x}(k)$ is finite. Along with Lemma 2 (Monotonicity of the blur process) we get

$$E \setminus \mathcal{N} \subset \left\{\exists n \geq M : \mathrm{C}_{t,x}(k) \subset B_n, \beta_{t+\epsilon,x}^{t,B_n} = 2\right\} \subset \left\{\beta_{t+\epsilon,x}^{t,\mathrm{C}_{t,x}(k)} = 2, |\,\mathrm{C}_{t,x}(k)| < \infty\right\}.$$

Application of Proposition 4 yields

$$P(E) \leq P\left(\beta_{t+\epsilon,x}^{t,\mathrm{C}_{t,x}(k)} = 2, |\,\mathrm{C}_{t,x}(k)| < \infty\right) \leq \left(\frac{3}{4}\right)^k.$$

This shows $P(E) \leq (3/4)^k$ all $k \in \mathbb{N}$, that is, $P(E) = 0$. The proposition follows. □

2.6 Proof of Proposition 3

2.6.1 Organization of the proof of Proposition 3

Our proof of Proposition 3 is based on the following finite volume version of Theorem 2.

Proposition 5 (Theorem 2 restricted to finite volume). *Let $\gamma > 0$ and suppose that the graph $G = (V, E)$ is finite volume. Let $m_{\gamma,\lambda,d}$ as in Proposition 3. Then for all $\delta \in]0, 1]$, for all $s \geq \gamma$ the forest fire process η has* $\mathrm{CCSB}(s, \delta, m_{d,\lambda,\gamma}(\delta))$.

Due to its length, we split the proof of Proposition 5 into several lemmata. We sketch the underlying intuition first.

Let $t \geq 0$ and $x \in V$. Our goal is to choose $m > 0$ such that $|C_{t,x}| > m$ has small probability. We distinguish three major cases, and estimate their probabilities separately.

- (Site vacant before) Suppose that the site x is vacant within $[t - \epsilon, t]$ for some $\epsilon \in]0, \gamma]$. From $|C_{t,x}| > m$ the site x is occupied at time t. Thus there must occur the growth of a tree at x in between time $t - \epsilon$ and t. We can choose $\epsilon \in]0, \gamma]$ such that the latter has small probability;

- (Site occupied before, cluster large) If the site x is occupied throughout $[t-\epsilon, t]$, then the cluster at x must not get hit by ignition in between time $t-\epsilon$ and t. An event with small probability, provided the size of the cluster at x at time $t-\epsilon$ is larger than some $M = M(\epsilon) > 0$;
- (Site occupied before, cluster small) The last case is the case where the size of the cluster at x at time $t-\epsilon$ is smaller than or equal to M, and x is occupied throughout $[t-\epsilon, t]$. More formally, we have to show that we can choose m such that

  $$\{|C_{t-\epsilon,x}| \leq M, |C_{t,x}| > m, \forall s \in [t-\epsilon, t] : \eta_{s,x} = 1\}$$

  has small probability. Since the site x is occupied throughout $[t-\epsilon, t]$, the cluster at x must not get hit by ignition during $[t-\epsilon, t]$. First we can choose N such that the probability that the cluster at x grows more than N times without getting hit by ignition is small. To grow from size M to size $m >> M$ within less than N growth steps, at least at one of the at most N growth steps the cluster at x has to get connected to a comparatively big cluster. We are going to show that we can choose m so that this event has small probability.

That is, to estimate the probability of $|C_{t,x}| > m$ we use events described by the increments of the growth and ignition processes after time $t-\epsilon$. But, to obtain the full statement of Proposition 5, we have to condition on the occurrence of $\cup_{t\in B} C_{t,y} = D$ additionally. This event obviously depends on the growth and ignition jumps that occur in between time $t-\epsilon$ and t. To handle this, we introduce the domain of dependence, a minimal space time region DOD $\subset V \times [0, t]$ with the property that the occurrence of $\cup_{t\in B} C_{t,y} = D$ is measurable with respect to the initial configuration and the growth and ignition jumps that occur on DOD. The consideration of the domain of dependence and the thereby required measurability with respect to the growth and ignition processes explain why we restrict Theorem 2 to finite volume first.

The further organization is as follows. In Section 2.6.2 we introduce the domain of dependence. In the first part of the section (Basic properties of the domain of dependence) we show the most important properties of the domain of dependence. In particular, we show that the domain of dependence is measurable with respect to the initial configuration and the growth and ignition jumps that occur on it. In the second part (Working with the domain of dependence) of Section 2.6.2 we use this property to estimate the probabilities of some growth and ignition events, uniformly in the condition $\cup_{t\in B} C_{t,y} = D$. Thereafter, in Section 2.6.3 we use the results of Section 2.6.2 to formalize the three cases sketched above. Finally, we show Proposition 5 in Section 2.6.4, and conclude Proposition 3 in Section 2.6.5. Figure 2.3 illustrates the way the single parts of the proof of Proposition 3 depend on each other.

2.6.2 The domain of dependence

In this section we introduce the domain of dependence. We show some of its basic properties, and how to work with it. All lemmata of this section restrict to the case

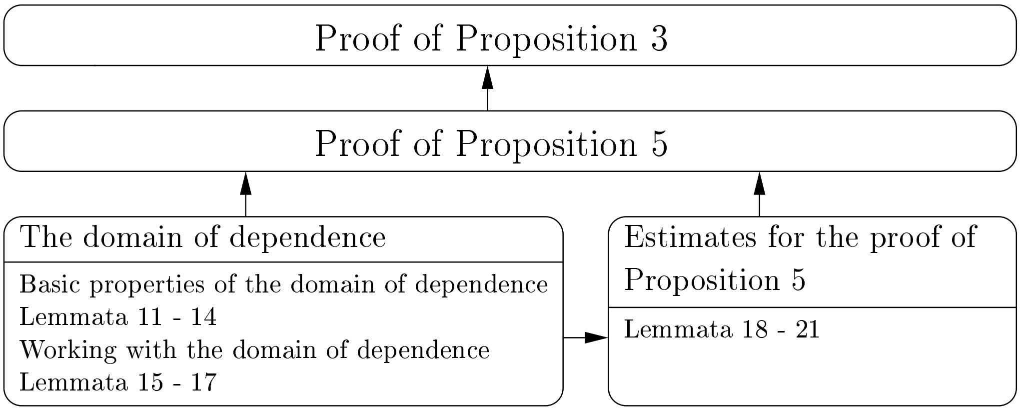

Figure 2.3: Proof of Proposition 3

where the graph $G = (V, E)$ is finite volume.

We explain the domain of dependence less formally first. Here we restrict to the special case $C_{t_0,y} = C$ for some $y \in V$, $C \in \mathcal{C}_y$ and $t_0 > 0$. Let $\mathcal{D}_0 := C$ and $0 < \epsilon \leq t_0$. If $C_{t_0,y} = C$, then $\mathcal{D}'_0 := \partial C$ is vacant at time t_0. Going back in time there is a smallest time $\delta(1) \leq t_0$ such that the set $\mathcal{D}'_0$ has been vacant throughout $[\delta(1), t_0]$. This boundary of vacant sites blocks fires caused by ignitions outside the set $\mathcal{D}_0$. We are going to use this to show that $C_{t_0,y} = C$ is determined with respect to the configuration of the forest fire process on $\overline{\mathcal{D}_0}$ at time $\delta(1)$ and the growth and ignition jumps that occur on $\overline{\mathcal{D}_0} \times [\delta(1), t_0]$. In the next step, we choose a sufficiently large set $\mathcal{D}_1 \supset \mathcal{D}_0$ such that $\mathcal{D}'_1 := \partial\mathcal{D}_1$ is vacant throughout $[t_1, \delta(1)]$ for some $t_1 < \delta(1)$. We define $\delta(2)$ to be the smallest time such that the set $\mathcal{D}'_1$ is vacant throughout the entire time interval $[\delta(2), \delta(1)]$. We proceed as in the previous step, and go on recursively until we reach time $t_0 - \epsilon$.

From now on throughout the remainder of this section let $B, D_0 \subset V$, $\epsilon > 0$ and $t_0 \geq \epsilon$. To handle the general case, we note that $\{\cup_{y \in B} C_{t_0,y} = D_0\} \neq \emptyset$ implies

$$\{\cup_{y \in B} C_{t_0,y} = D_0\} = \{\forall y \in D_0 : \eta_{t_0,y} = 1, \forall z \in D'_0 : \eta_{t_0,z} = 0\}. \tag{2.20}$$

Here $W' := \partial W \cup (B \setminus W)$ for $W \subset V$.

For abbreviation, for all $W \subset V$, $S \subset [0, \infty[$ and $a \in \{0, 1\}$ let

$$\{\eta_{S,W} = a\} := \{\forall (s, w) \in S \times W : \eta_{s,w} = a\}$$

denote the event that all sites of the set W are vacant ($a = 0$), respectively occupied ($a = 1$) throughout the entire time set S. In case of $S \times W = \emptyset$, we define $\{\eta_{S,W} = a\}$ to denote the sure event.

Definition 20 (The domain of dependence)**.** Let $\delta(0) := t_0$, $\mathcal{D}_0 := D_0$, $\mathcal{D}'_0 := D'_0$, and suppose $\eta_{t_0,D'_0} = 0$. Recursively for all $0 \leq i \leq |V|$ let

$$\delta(i+1) := \min\left\{ s \in [t_0 - \epsilon, \delta(i)] \middle| \eta_{[s,\delta(i)],\mathcal{D}'_i} = 0 \right\},$$

be the smallest time $s \in [t_0 - \epsilon, \delta(i)]$ such that all sites of the set $\mathcal{D}'_i$ are vacant throughout the time interval $[s, \delta(i)]$. Here

$$\mathcal{D}_{i+1} := \begin{cases} \mathcal{D}_i \cup \bigcup_{x \in \mathcal{D}'_i} C_{\delta(i+1)^-,x} & \text{if } \delta(i+1) > t_0 - \epsilon; \\ \mathcal{D}_i & \text{if } \delta(i+1) = t_0 - \epsilon \end{cases}$$

denotes the union of $\mathcal{D}_i$ and the cluster that gets vacant at time $\delta(i+1)$. The domain of dependence is

$$\text{DOD} := V \times [0, \delta(|V|+1)] \cup \bigcup_{0 \le i \le |V|} \left\{ \mathcal{D}_i \cup \mathcal{D}'_i \right\} \times [0, \delta(i)].$$

We write

$$\delta_x := \min \left\{ 0 \le i \le |V| + 1 \middle| (x, \delta(i)) \in \text{DOD} \right\}$$

for the last time the site $x \in V$ is part of the domain of dependence.

Basic properties of the domain of dependence. We now show some important properties of the domain of dependence.

Lemma 11 (On the shape of the domain of dependence)**.** *In case of* $\eta_{t_0, D'_0} = 0$*:*

(i) the domain of dependence is well defined;

(ii) for all $1 \le i \le |V|$ *the sets* $\mathcal{D}'_i$ *and* $\mathcal{D}_i \setminus \mathcal{D}_{i-1}$ *are vacant at time* $\delta(i)$*;*

(iii) $\delta(|V|+1) = t_0 - \epsilon$*;*

(iv) if a site $x \in V \setminus D_0$ *is part of the domain of dependence after time* $t_0 - \epsilon$*, then it is vacant at time* δ_x*:*

$$\left\{ \eta_{t_0, D'_0} = 0, \delta_x > t_0 - \epsilon \right\} \subset \left\{ \eta_{t_0, D'_0} = 0, \eta_{\delta_x, x} = 0 \right\}$$

Proof. Suppose $\eta_{t_0, D'_0} = 0$.

Proof of (i) and (ii). We prove (i) and (ii) by induction on $0 \le i \le n+1$. From $\eta_{t_0, D'_0} = 0$, the set $\mathcal{D}'_0$ is vacant at time $\delta(0)$.

In the induction step $i - 1 \to i$, suppose that $\delta(i-1)$ is well defined and that $\mathcal{D}'_{i-1}$ is vacant at time $\delta(i-1)$. Then $\delta(i)$ is well defined, and $\mathcal{D}'_{i-1}$ is vacant at time $\delta(i)$.

In case of $\delta(i) = t_0 - \epsilon$, we have $\mathcal{D}_i = \mathcal{D}_{i-1}$ and $\mathcal{D}'_i = \mathcal{D}'_{i-1}$, and thus the sets $\mathcal{D}'_i$ and $\mathcal{D}_i \setminus \mathcal{D}_{i-1}$ are vacant at time $\delta(i)$. Suppose $\delta(i) > t_0 - \epsilon$, and let $y \in \partial \mathcal{D}_i$. Then either $y \in \partial C_{\delta(i)^-,x}$ for some $x \in \mathcal{D}'_{i-1}$, or $y \in \mathcal{D}'_{i-1}$ and $C_{\delta(i)^-,x} = \emptyset$. Both cases imply $\eta_{\delta(i)^-,y} = 0$. From $\delta(i) > t_0 - \epsilon$ a site of $\mathcal{D}'_{i-1}$ gets vacant at time $\delta(i)$. That is, there occurs an ignition on V at time $\delta(i)$. Thus, since growth and ignition jumps occur at distinct times, the site y remains vacant at time $\delta(i)$. Hence, the set $\partial \mathcal{D}_i$ is vacant at time $\delta(i)$. Furthermore, since $\mathcal{D}'_{i-1}$ is vacant at time $\delta(i)$, it follows that $\mathcal{D}_i \setminus \mathcal{D}_{i-1} = \cup_{x\in\mathcal{D}'_{i-1}} C_{\delta(i)^-,x}$ is vacant at time $\delta(i)$. Finally, $B \setminus \mathcal{D}_{i-1} \subset \mathcal{D}'_{i-1}$ and $\mathcal{D}_{i-1} \subset \mathcal{D}_i$ imply $B \setminus \mathcal{D}_i \subset \mathcal{D}'_{i-1}$. Thus, since $\mathcal{D}'_{i-1}$ is vacant at time $\delta(i)$, the set $B \setminus \mathcal{D}_i$ is vacant at time $\delta(i)$.

Proof of (iii). Let $1 \leq i \leq |V|$ and suppose $\delta(i) > t_0 - \epsilon$. Then a site of $\mathcal{D}'_{i-1}$ gets vacant at time $\delta(i)$. Thus, the set $\mathcal{D}_i \setminus \mathcal{D}_{i-1} = \cup_{x\in\mathcal{D}'_{i-1}} C_{\delta(i)^-,x}$ is non-empty. We conclude $|\mathcal{D}_i| \geq |\mathcal{D}_{i-1}| + 1$. Therefore, $\delta(|V|) > t_0 - \epsilon$ implies $|\mathcal{D}_{|V|}| \geq |V|$, and hence $\mathcal{D}_{|V|} = V$ and $\mathcal{D}'_{|V|} = \emptyset$. It follows $\delta(|V| + 1) = t_0 - \epsilon$.

Proof of (iv). Let $x \in V \setminus D_0$ and suppose $\delta_x > t_0 - \epsilon$. From $\delta(|V| + 1) = t_0 - \epsilon$ there exists $0 \leq i \leq |V|$ such that $\delta_x = \delta(i)$.

In case of $\delta_x = \delta(0)$, the choice of $x \in V \setminus D_0$ implies $x \in D'_0$. By assumption, the set D'_0 is vacant at time $\delta(0)$. Let $1 \leq i \leq |V|$ and suppose $\delta_x = \delta(i)$. If $x \in \mathcal{D}_{i-1}$ would hold, we would have $\delta_x \geq \delta(i-1)$. This shows $x \in \mathcal{D}'_i \cup (\mathcal{D}_i \setminus \mathcal{D}_{i-1})$. From part (ii) the sets $\mathcal{D}_i \setminus \mathcal{D}_{i-1}$ and $\mathcal{D}'_i$ are vacant at time $\delta(i)$. It follows that the site x is vacant at time $\delta(i)$. □

The next lemma states that the domain of dependence is self-determined in the following sense: the configuration of the domain of dependence is measurable with respect to the initial configuration and the growth and ignition jumps hat occur on it. We define the according σ-field.

Definition 21 (The σ-field $\mathcal{DOD}$)**.** For all $1 \leq i \leq |V|$ let $t_i \in [t_0 - \epsilon, t_{i-1}]$ and $D_i \subset V$. Let

$$\mathcal{DOD}(t_i, D_i : 1 \leq i \leq |V|) := \sigma\Big(\mathcal{GI}_{t_0-\epsilon}(V) \cup \bigcup_{0\leq i\leq|V|} \mathcal{GI}_{t_i}(D_i \cup D'_i)\Big),$$

be the σ-field generated by the initial configuration and the growth and ignition jumps that occur within the space time set

$$V \times [0, t_0 - \epsilon] \cup \bigcup_{0\leq i\leq|V|} \{D_i \cup D'_i\} \times [0, t_i].$$

Lemma 12 (The domain of dependence is self-determined)**.** *For all* $1 \leq i \leq |V|$ *let* $t_i \in [t_0 - \epsilon, t_{i-1}]$ *and* $D_i \subset V$ *such that* $D_i \supset D_{i-1}$. *Then the event*

$$\Big\{\eta_{t_0,D_0} = 1, \eta_{t_0,D'_0} = 0, \forall 1 \leq i \leq |V| : \delta(i) \leq t_i, \mathcal{D}_i = D_i\Big\}$$

is measurable with respect to the σ*-field* $\mathcal{DOD}(t_i, D_i : 1 \leq i \leq |V|)$.

Proof. For all $1 \leq i \leq |V|$ let $t_i \in [t_0 - \epsilon, t_{i-1}]$ and $D_i \subset V$ such that $D_i \supset D_{i-1}$. For abbreviation, we write $\mathcal{DOD} := \mathcal{DOD}(t_i, D_i : 1 \leq i \leq |V|)$. Without less of generality we can assume $D_0 \neq V$, since otherwise $\mathcal{DOD} = \mathcal{GI}_{t_0}(V)$ would imply the lemma (the graph G is finite volume, and hence the configuration of the forest fire process on $V \times [0, t_0]$ is $\mathcal{GI}_{t_0}(V)$-measurable). Let

$$j := \max\big\{0 \leq i \leq |V| \big| D_i \neq V, t_i \neq t_0 - \epsilon\big\}$$

and $\tau_{j+1} := t_{j+1}$, where $t_{|V|+1} := t_0 - \epsilon$. For all $0 \le i \le j$ let

$$\tau_i := \max\left\{s \in [\tau_{i+1}, t_i] \big| \eta_{[\tau_{i+1}, s[, D'_i} = 0\right\}$$

be the last time $s \in [\tau_{i+1}, t_i]$ such that all sites of D'_i are vacant during $[\tau_{i+1}, s[$.

The proof consists of the following steps. We use an induction on $j+1 \ge k \ge 0$ to show that the time τ_k and the configuration of the forest fire process on the space time set

$$M(k) := V \times [0, \tau_{j+1}] \cup \bigcup_{k \le i \le j} \{D_i \cup D'_i\} \times [0, \tau_i]$$

are $\mathcal{DOD}$-measurable. Thereafter, by induction on $0 \le k \le j+1$ we show that the event $\mathcal{E}_k \cap \{\tau_0 = t_0\}$ is $\mathcal{DOD}$-measurable, where

$$\mathcal{E}_k := \left\{\eta_{t_0, D_0} = 1, \eta_{t_0, D'_0} = 0, \forall 1 \le i \le k : \delta(i) \le t_i, \mathcal{D}_i = D_i\right\}.$$

Finally, we obtain the $\mathcal{DOD}$-measurability of $\mathcal{E}_{|V|} \cap \{\tau_0 = t_0\}$, and conclude the lemma by showing $\mathcal{E}_{|V|} = \mathcal{E}_{|V|} \cap \{\tau_0 = t_0\}$.

We start with the proof of the $\mathcal{DOD}$-measurability of the configuration on the sets $M(k)$, $j+1 \ge k \ge 0$.

Our choice of τ_{j+1} implies that at least one of the relations $D_{j+1} = V$ and $\tau_{j+1} = t_0 - \epsilon$ holds. Hence, we have $\mathcal{GI}_{\tau_{j+1}}(V) \subset \mathcal{DOD}$. Since G is finite volume, this implies the $\mathcal{DOD}$-measurability of the configuration of the forest fire process on $M(j+1)$.

In the induction step $k \to k-1$, suppose that τ_k and the configuration on $M(k)$ are $\mathcal{DOD}$-measurable. Then $\{D_{k-1} \cup D'_{k-1}\} \subset \{D_k \cup D'_k\}$ implies that the configuration on $D_{k-1} \cup D'_{k-1}$ at time τ_k is $\mathcal{DOD}$-measurable. If one of the sites of D'_{k-1} is occupied at time τ_k, then $\tau_{k-1} = \tau_k$ and $M(k-1)$ equals $M(k)$. We suppose that the entire set D'_{k-1} is vacant at time τ_k. Then τ_{k-1} is the minimum of t_{k-1} and the first time after τ_k at which a growth on D'_{k-1} occurs. That is, the time τ_{k-1} is $\mathcal{DOD}$-measurable, and the entire set D'_{k-1} is vacant throughout $[\tau_k, \tau_{k-1}[$. Furthermore, along with $\partial D_{k-1} \subset D'_{k-1}$ this implies that throughout $[\tau_k, \tau_{k-1}[$ the configuration on D_{k-1} does not depend on the growth and ignition jumps that occur outside the set D_{k-1}. In other words, on the time-space set $D_{k-1} \times [\tau_k, \tau_{k-1}[$ the forest fire process η evolves as a forest fire process on D_{k-1} conditioned on having configuration $(\eta_{\tau_k, y})_{y \in D_{k-1}}$ at time τ_k. Therefore, the configuration on $\{D_{k-1} \cup D'_{k-1}\} \times [\tau_k, \tau_{k-1}[$ is $\mathcal{DOD}$-measurable. At time τ_{k-1} either a site of D'_{k-1} gets occupied, or all sites of D'_{k-1} remain vacant. In case of the former, the configuration on D_{k-1} remains unchanged at time τ_{k-1}, since growth and ignition jumps occur at distinct times. In case of the latter, the set D'_{k-1} is vacant throughout the entire time interval $[\tau_k, \tau_{k-1}]$. In both cases it follows that the configuration on $\{D_{k-1} \cup D'_{k-1}\} \times [\tau_k, \tau_{k-1}]$ is $\mathcal{DOD}$-measurable. Along with the induction hypothesis, we obtain the $\mathcal{DOD}$-measurability of the configuration on $M(k-1)$.

We go on by showing that the events $\mathcal{E}_k \cap \{\tau_0 = t_0\}$, $0 \le k \le j+1$, are measurable with respect to $\mathcal{DOD}$.

From the last step, the time τ_0 and the configuration of the forest fire process on $M(0)$ are $\mathcal{DOD}$-measurable. Thus, restricted to $\{\tau_0 = t_0\}$ the configuration of the forest fire process on $\{D_0 \cup D'_0\} \times [0, \delta(0)]$ is $\mathcal{DOD}$-measurable. In particular, this shows the $\mathcal{DOD}$-measurability of the event $\mathcal{E}_0 \cap \{\tau_0 = t_0\}$. We note $\mathcal{E}_0 \cap \{\tau_0 = t_0\} \subset \{\delta(0) \leq \tau_0\}$.

In the induction step $k \to k+1$ our induction hypothesis is as follows. We suppose that $\mathcal{E}_k \cap \{\tau_0 = t_0\}$ is $\mathcal{DOD}$-measurable, and that $\mathcal{E}_k \cap \{\tau_0 = t_0\} \subset \{\delta(k) \leq \tau_k\}$. Furthermore, we suppose that restricted to $\mathcal{E}_k \cap \{\tau_0 = t_0\}$ the time $\delta(k)$ and the configuration of the forest fire process on $\{D_k \cup D'_k\} \times [0, \delta(k)]$ are $\mathcal{DOD}$-measurable.

Suppose the occurrence of $\mathcal{E}_k \cap \{\tau_0 = t_0\}$. Then $\mathcal{D}'_k = D'_k$, from Lemma 11 (On the shape of the domain of dependence) the set D'_k is vacant at time $\delta(k)$, and the time $\delta(k+1)$ is the smallest time $s \in [t_0 - \epsilon, \delta(k)]$ such that all sites of the set D'_k are vacant throughout the time interval $[s, \delta(k)]$. Therefore, the induction hypothesis implies the $\mathcal{DOD}$-measurability of $\delta(k+1)$. Moreover, the occurrence of $\mathcal{E}_k \cap \{\tau_0 = t_0\}$ implies $\delta(k) \leq \tau_k$, and the set D'_k is vacant throughout $[\tau_{k+1}, \tau_k[$. It follows that the set $\mathcal{D}'_k$ is vacant throughout $[\tau_{k+1} \wedge \delta(k), \delta(k)]$. Along with $t_0 - \epsilon \leq \tau_{k+1} \leq t_{k+1}$, we conclude $\mathcal{E}_{k+1} \cap \{\tau_0 = t_0\} \subset \{\delta(k+1) \leq \tau_{k+1} \leq t_{k+1}\}$. In the previous step we showed that the configuration of the forest fire process on $M(0)$, and therefore the configuration on $\{D_{k+1} \cup D'_{k+1}\} \times [0, \tau_{k+1}]$ is $\mathcal{DOD}$-measurable. The time $\delta(k+1)$ is $\mathcal{DOD}$-measurable, and satisfies $\delta(k+1) \leq \tau_{k+1}$. It follows that the configuration of the forest fire process on $\{D_{k+1} \cup D'_{k+1}\} \times [0, \delta_{k+1}]$ is $\mathcal{DOD}$-measurable. In particular, this shows the $\mathcal{DOD}$-measurability of $\mathcal{E}_{k+1} \cap \{\tau_0 = t_0\} = \mathcal{E}_k \cap \{\tau_0 = t_0, \mathcal{D}_{k+1} = D_{k+1}\}$.

We conclude the $\mathcal{DOD}$-measurability of $\mathcal{E}_{|V|} \cap \{\tau_0 = t_0\}$. If the event $\mathcal{E}_{j+1} \cap \{\tau_0 = t_0\}$ occurs, then $\mathcal{D}_{j+1} = D_{j+1}$ and $\delta(j+1) \leq t_{j+1}$. Our choice of j implies that at least one of the relations $D_{j+1} = V$ and $t_{j+1} = t_0 - \epsilon$ holds. Thus from the definition of the domain of dependence, it holds $\mathcal{D}_l = D_{j+1}$ and $\delta(l) = t_0 - \epsilon$ for all $j+1 < l \leq |V|$. Hence,

$$\begin{aligned}\mathcal{E}_{|V|} \cap \{\tau_0 = t_0\} &= \mathcal{E}_{j+1} \cap \{\tau_0 = t_0\} \cap \{\forall j+1 < l \leq |V| : \delta(l) \leq t_l, \mathcal{D}_l = D_l\} \\ &= \mathcal{E}_{j+1} \cap \{\tau_0 = t_0\} \cap \{\forall j+1 < l \leq |V| : D_l = D_{j+1}\}.\end{aligned}$$

That is, the $\mathcal{DOD}$-measurability of $\mathcal{E}_{j+1} \cap \{\tau_0 = t_0\}$ implies the $\mathcal{DOD}$-measurability of $\mathcal{E}_{|V|} \cap \{\tau_0 = t_0\}$.

To conclude $\mathcal{E}_{|V|} = \mathcal{E}_{|V|} \cap \{\tau_0 = t_0\}$, we show $\mathcal{E}_{|V|} \subset \{\delta(i) \leq \tau_i\}$ by induction on $j+1 \geq i \geq 0$. If $j < |V|$, the occurrence of $\mathcal{E}_{|V|}$ implies $\delta(j+1) \leq t_{j+1} = \tau_{j+1}$. Otherwise in case of $j = |V|$, Lemma 11 (On the shape of the domain of dependence) yields $\delta(|V|+1) = t_0 - \epsilon = \tau_{|V|+1}$.

In the induction step $i+1 \to i$ suppose $\mathcal{E}_{|V|} \subset \{\delta(i+1) \leq \tau_{i+1}\}$. If $\delta(i) \leq \tau_{i+1}$, then $\tau_{i+1} \leq \tau_i$ provides $\delta(i) \leq \tau_i$. Assume $\tau_{i+1} < \delta(i)$ and the occurrence of $\mathcal{E}_{|V|}$. Then $\delta(i) \leq t_i$, the set D'_i is vacant throughout $[\delta(i+1), \delta(i)]$, and our induction hypothesis implies $\delta(i+1) \leq \tau_{i+1} < \delta(i)$. Thus, the set D'_i is vacant throughout $[\tau_{i+1}, \delta(i)]$. The time τ_i is the last time $s \in [\tau_{i+1}, t_i]$ such that all sites of D'_i are vacant during $[\tau_{i+1}, s[$. We conclude $\delta(i) \leq \tau_i$, where we use $\delta(i) \leq t_i$.

The induction yields $\mathcal{E}_{|V|} = \mathcal{E}_{|V|} \cap \{\delta(0) \leq \tau_0\}$. Along with $t_0 = \delta(0)$ and $\tau(0) \leq t_0$, we obtain $\mathcal{E}_{|V|} = \mathcal{E}_{|V|} \cap \{\tau_0 = t_0\}$. This concludes the proof of the lemma, since we showed the $\mathcal{DOD}$-measurability of $\mathcal{E}_{|V|} \cap \{\tau_0 = t_0\}$. □

To prove Proposition 5, we estimate the probability that some growth and ignition events occur within time $t_0 - \epsilon$ and t_0. Intuitively spoken, to handle the fact that we are conditioning on the configuration on $D_0 \cup D_0'$ at time t_0, we use the domain of dependence as follows. From Lemma 12 (The domain of dependence is self-determined) the occurrence of $\{\eta_{t_0,D_0} = 1, \eta_{t_0,D_0'} = 0\}$ is measurable with respect to the growth and ignition jumps that occur within the domain of dependence. Therefore, the occurrence of $\{\eta_{t_0,D_0} = 1, \eta_{t_0,D_0'} = 0\}$ is independent from growth and ignition jumps that occur on a space time set that is distinct from the domain of dependence. The next two lemmata provide us with tools to show that at after a given time a given site is not part of the domain of dependence.

Lemma 13 (All time occupied, then not part of DOD)**.** *Suppose $\eta_{t_0,D_0'} = 0$ and let $s \in [t_0 - \epsilon, t_0]$. If a site $x \in V \setminus D_0$ is occupied throughout the entire time interval $[s, t_0]$, then x is not part of the domain of dependence after time s:*

$$\left\{\eta_{t_0,D_0'} = 0, \eta_{[s,t_0],x} = 1\right\} \subset \left\{\eta_{t_0,D_0'} = 0, \delta_x \leq s\right\}$$

Proof. Suppose $\eta_{t_0,D_0'} = 0$ and let $x \in V \setminus D_0$. Lemma 11 (On the shape of the domain of dependence) states that if $\delta(x) > t_0 - \epsilon$, then the site x is vacant at time δ_x. We obtain for all $s \in [t_0 - \epsilon, t_0]$

$$\begin{aligned}\left\{\eta_{t_0,D_0'} = 0, \delta_x > s\right\} &= \left\{\eta_{t_0,D_0'} = 0, \delta_x > s, \eta_{\delta_x,x} = 0\right\} \\ &\subset \left\{\eta_{t_0,D_0'} = 0, \exists s' \in [s, t_0] : \eta_{s',x} = 0\right\}.\end{aligned}$$

This shows the assertion. □

Lemma 14 (Site not part of DOD, then whole cluster not part of DOD)**.** *Let $x \in V$ and $s \geq t_0 - \epsilon$. Suppose $\eta_{t_0,D_0'} = 0$, and that the site x is not part of the domain of dependence after time s. Then none of the sites of $C_{s,x}$ are part of the domain of dependence after time s:*

$$\left\{\eta_{t_0,D_0'} = 0, \delta_x \leq s\right\} \subset \left\{\eta_{t_0,D_0'} = 0, \forall y \in C_{s,x} : \delta_y \leq s\right\} \tag{2.21}$$

Furthermore, for all $z \in \partial C_{s,x}$ there does not grow a tree at the site z in between time s and $\delta_z \vee s$:

$$\left\{\eta_{t_0,D_0'} = 0, \delta_x \leq s\right\} \subset \left\{\eta_{t_0,D_0'} = 0, \forall z \in \partial C_{s,x} : G_{s,z} = G_{\delta_z \vee s,z}\right\} \tag{2.22}$$

Proof. Let $s \geq t_0 - \epsilon$ and suppose $\eta_{t_0,D_0'} = 0$. For all $y \in V$ it holds $\delta_y \leq t_0$. That is, $s \geq t_0$ implies (2.21) and (2.22). Therefore, we assume $s \in [t_0 - \epsilon, t_0[$.

Lemma 11 (On the shape of the domain of dependence) states $\delta(|V| + 1) = t_0 - \epsilon$. Hence, $s \in [t_0 - \epsilon, t_0[$ yields the existence of $0 \leq i \leq |V|$ so that $s \in [\delta(i + 1), \delta(i)[$.

Let $0 \le i \le |V|$ and $x \in V$. In the next step, we show that if $s \in [\delta(i+1), \delta(i)[$ and $\delta_x \le s$, then $C_{s,x} \cap \{\mathcal{D}_i \cup \mathcal{D}'_i\} = \emptyset$. More formally, we show

$$\begin{aligned}&\left\{\eta_{t_0,D'_0} = 0, s \in [\delta(i+1), \delta(i)[, \delta_x \le s\right\} \\ &\subset \left\{\eta_{t_0,D'_0} = 0, s \in [\delta(i+1), \delta(i)[, C_{s,x} \cap \left\{\mathcal{D}_i \cup \mathcal{D}'_i\right\} = \emptyset\right\}. \end{aligned} \tag{2.23}$$

Suppose $s \in [\delta(i+1), \delta(i)[$ and $\delta_x \le s$. For all $y \in \mathcal{D}_i \cup \mathcal{D}'_i$ it holds $\delta_y \ge \delta(i) > s$. It follows $x \notin \mathcal{D}_i \cup \mathcal{D}'_i$. The entire set $\mathcal{D}'_i$ is vacant throughout $[\delta(i+1), \delta(i)[$, and therefore at time s. A cluster does not contain any vacant sites. Along with $\partial \mathcal{D}_i \subset \mathcal{D}'_i$ and $x \notin \mathcal{D}_i$ it follows $C_{s,x} \cap \{\mathcal{D}_i \cup \mathcal{D}'_i\} = \emptyset$.

For all $y \in V$, by definition $y \notin \mathcal{D}_i \cup \mathcal{D}'_i$ implies $\delta_y \le \delta(i+1)$. It follows

$$\begin{aligned}&\left\{\eta_{t_0,D'_0} = 0, s \in [\delta(i+1), \delta(i)[, y \notin \mathcal{D}_i \cup \mathcal{D}'_i\right\} \\ &= \left\{\eta_{t_0,D'_0} = 0, s \in [\delta(i+1), \delta(i)[, \delta_y \le \delta(i+1)\right\} \subset \left\{\eta_{t_0,D'_0} = 0, \delta_y \le s\right\}.\end{aligned}$$

Along with (2.23) this concludes the proof of (2.21).

To show (2.22), we first show for all $y \in V$ that $s \in [\delta(i+1), \delta(i)[$ and $y \notin \mathcal{D}_i$ imply $G_{s,y} = G_{\delta_y \vee s, y}$:

$$\left\{\eta_{t_0,D'_0} = 0, s \in [\delta(i+1), \delta(i)[, y \notin \mathcal{D}_i\right\} \subset \left\{\eta_{t_0,D'_0} = 0, G_{s,y} = G_{\delta_y \vee s, y}\right\} \tag{2.24}$$

Let $y \in V \setminus \mathcal{D}_i$ and suppose $s \in [\delta(i+1), \delta(i)[$. Then either $y \in V \setminus \{\mathcal{D}_i \cup \mathcal{D}'_i\}$, or $y \in \mathcal{D}'_i$.

First we suppose $y \notin \mathcal{D}_i \cup \mathcal{D}'_i$. Then $\delta_y \le \delta(i+1)$. Hence, $s \in [\delta(i+1), \delta(i)[$ implies $\delta_y \vee s = s$ and $G_{s,y} = G_{\delta_y \vee s, y}$.

Now we suppose $y \in \mathcal{D}'_i$. Then there exists $0 \le j \le i$ so that $\delta(j) = \delta_y$. Let $i \ge l \ge j$. Then $y \in \mathcal{D}_l \cup \mathcal{D}'_l$, and by definition $\mathcal{D}_l \cap \mathcal{D}'_i = \emptyset$. Hence, $y \in \mathcal{D}'_i$ implies $y \in \mathcal{D}'_l$. The set $\mathcal{D}'_l$ is vacant throughout $[\delta(l+1), \delta(l)]$, and hence the site y is. It follows that the site y is vacant throughout $[\delta(i+1), \delta_y]$. Thus, $s \in [\delta(i+1), \delta(i)[$ implies $G_{s,y} = G_{\delta_y \vee s, y}$.

By definition $\partial \mathcal{D}_i \subset \mathcal{D}'_i$. Therefore, $C_{s,x} \cap \{\mathcal{D}_i \cup \mathcal{D}'_i\} = \emptyset$ implies $\partial C_{s,x} \cap \mathcal{D}_i = \emptyset$. Along with (2.24) this yields

$$\begin{aligned}&\left\{\eta_{t_0,D'_0} = 0, s \in [\delta(i+1), \delta(i)[, C_{s,x} \cap \left\{\mathcal{D}_i \cup \mathcal{D}'_i\right\} = \emptyset\right\} \\ &\subset \left\{\eta_{t_0,D'_0} = 0, s \in [\delta(i+1), \delta(i)[, \partial C_{s,x} \cap \mathcal{D}_i = \emptyset\right\} \\ &\subset \left\{\eta_{t_0,D'_0} = 0, \forall z \in \partial C_{s,x} : G_{s,z} = G_{\delta_z \vee s, z}\right\}.\end{aligned}$$

Combining with (2.23), we obtain (2.22). □

Working with the domain of dependence. We now show how to use the domain of dependence to estimate the probabilities of some key events, uniformly in the condition $\{\eta_{t_0,D_0} = 1, \eta_{t_0,D'_0} = 0\}$.

We consider the following situation first. Suppose that the site $x \in V$ is occupied throughout $[t_0-\epsilon, t_0]$. Then the cluster at x must not get hit by ignition during $[t_0-\epsilon, t_0]$. The configuration of the (finite volume) forest fire process up to time $t_0-\epsilon$ is independent of the increments of the growth and ignition processes after time $t_0 - \epsilon$. We obtain for $C \in C_x$

$$\begin{aligned} P\left(\eta_{[t_0-\epsilon,t_0],x} = 1, C_{t_0-\epsilon,x} = C\right) &\le P\left(\forall y \in C : I_{t_0-\epsilon,y} = I_{t_0,y}, C_{t_0-\epsilon,x} = C\right) \\ &= P\left(I_{\epsilon,0} = 0\right)^{|C|} \cdot P\left(C_{t_0-\epsilon,x} = C\right). \end{aligned}$$

The next lemma states that the same relation holds, even if we condition on the occurrence of $\{\eta_{t_0,D_0} = 1, \eta_{t_0,D_0'} = 0\}$.

Lemma 15 (All time occupied, then no ignition)**.** *Let $x \in V \setminus D_0$ and $C \in C_x$. Then*

$$\begin{aligned} &P\left(\eta_{[t_0-\epsilon,t_0],x} = 1, C_{t_0-\epsilon,x} = C, \eta_{t_0,D_0} = 1, \eta_{t_0,D_0'} = 0\right) \\ &\le P\left(I_{\epsilon,0} = 0\right)^{|C|} \cdot P\left(C_{t_0-\epsilon,x} = C, \eta_{t_0,D_0} = 1, \eta_{t_0,D_0'} = 0\right). \end{aligned}$$

Proof. Suppose that $\eta_{t_0,D_0'} = 0$, that the site $x \in V \setminus D_0$ is occupied during the time interval $[t_0 - \epsilon, t_0]$, and that $C_{t_0-\epsilon,x} = C$ for some $C \in C_x$. Then from Lemma 13 (All time occupied, then not part of DOD) the site x is not part of the domain of dependence after time $t_0-\epsilon$. Therefore, Lemma 14 (Site not part of DOD, then whole cluster not part of DOD) yields that the whole cluster at x is not part of the domain of dependence after time $t_0 - \epsilon$. It follows that the sets C and $\mathcal{D}_{|V|} \cup \mathcal{D}'_{|V|}$ are disjoint. Furthermore, if the site x is occupied throughout $[t_0 - \epsilon, t_0]$, there must not occur an ignition on $C_{t_0-\epsilon,x} = C$ within $[t_0 - \epsilon, t_0]$. Formally, we obtain

$$\begin{aligned} &P\left(\eta_{[t_0-\epsilon,t_0],x} = 1, \mathcal{C}\right) \\ &\le P\left(\forall y \in C : I_{t_0-\epsilon,y} = I_{t_0,y}, C \cap (\mathcal{D}_{|V|} \cup \mathcal{D}'_{|V|}) = \emptyset, \mathcal{C}\right), \end{aligned}$$

where $\mathcal{C} := \{C_{t_0-\epsilon,x} = C, \eta_{t_0,D_0} = 1, \eta_{t_0,D_0'} = 0\}$. Conditioning on the shape of the domain of dependence yields

$$\begin{aligned} &P\left(\forall y \in C : I_{t_0-\epsilon,y} = I_{t_0,y}, C \cap (\mathcal{D}_{|V|} \cup \mathcal{D}'_{|V|}) = \emptyset, \mathcal{C}\right) \\ &= \sum_{(D_i)_{1\le i\le |V|} \in S} P\left(\forall y \in C : I_{t_0-\epsilon,y} = I_{t_0,y}, \forall 1 \le i \le |V| : \mathcal{D}_i = D_i, \mathcal{C}\right), \end{aligned}$$

where

$$\mathcal{S} := \left\{ (D_i)_{1\le i\le |V|} \middle| \begin{array}{c} \forall 1 \le i \le |V| : D_{i-1} \subset D_i \subset V, \\ C \cap (D_{|V|} \cup D'_{|V|}) = \emptyset \end{array} \right\}.$$

For $(D_i)_{1\le i\le |V|} \in \mathcal{S}$, from Lemma 12 (The domain of dependence is self-determined)

$$\{\mathcal{C}, \forall 1 \le i \le |V| : \mathcal{D}_i = D_i\} \in \sigma\left(\mathcal{GI}_{t_0-\epsilon}(V) \cup \mathcal{GI}_{t_0}(D_{|V|} \cup D'_{|V|})\right).$$

Hence, the disjointness of $D_{|V|} \cup D'_{|V|}$ and C implies

$$\begin{aligned}&P\left(\forall y \in C : I_{t_0-\epsilon,y} = I_{t_0,y}, \forall 1 \leq i \leq |V| : \mathcal{D}_i = D_i, \mathcal{C}\right)\\ &\leq P\left(I_{\epsilon,0} = 0\right)^{|C|} \cdot P\left(\forall 1 \leq i \leq |V| : \mathcal{D}_i = D_i, \mathcal{C}\right),\end{aligned}$$

where we use $\{\forall y \in C : I_{t_0-\epsilon,y} = I_{t_0,y}\} \in \mathcal{INCR}_{t_0-\epsilon}(C)$. Summing up again we obtain

$$\begin{aligned}&\sum_{(D_i)_{1\leq i\leq |V|}\in S} P\left(I_{\epsilon,0} = 0\right)^{|C|} \cdot P\left(\forall 1 \leq i \leq |V| : \mathcal{D}_i = D_i, \mathcal{C}\right)\\ &\leq P\left(I_{\epsilon,0} = 0\right)^{|C|} \cdot P\left(\mathcal{C}\right).\end{aligned}$$

□

Let $\tau \geq t_0 - \epsilon$ be a finite $\mathcal{GI}.(V)$-stopping time. It follows easily (see e.g. [10]) that the increments of the growth and ignition processes after time τ are independent of the configuration of the (finite volume) forest fire process η until time τ. Furthermore, these increments are distributed as after time 0. This implies for $x \in V$, $C \in C_x$ and $C' \subset \partial C$

$$\begin{aligned}P\left(\mathrm{GBI}_\tau(C', C), C_{\tau,x} = C\right) &= P\left(\mathrm{GBI}_0(C', C)\right) \cdot P\left(C_{\tau,x} = C\right)\\ &= \frac{|C'|}{|C'| + \lambda|C|} \cdot P\left(C_{\tau,x} = C\right).\end{aligned}$$

In the next lemma we extend this observation to the case where we condition on the occurrence of $\{\eta_{t_0,D_0} = 1, \eta_{t_0,D'_0} = 0\}$ additionally.

Lemma 16 (Estimate GBI). *Let $x \in V$, $C \in C_x$, $C' \subset \partial C$ and*

$$E \in \mathcal{GI}_\tau := \sigma\left(A \middle| \forall s \geq 0 : \{\tau \leq s\} \cap A \in \mathcal{GI}_s\right).$$

Then

$$\begin{aligned}&P\Big(\mathrm{GBI}_\tau(C', C), C_{\tau,x} = C, \delta_x \leq \tau, E, \eta_{t_0,D_0} = 1, \eta_{t_0,D'_0} = 0\Big)\\ &\leq \frac{|C'|}{|C'| + \lambda|C|} \cdot P\Big(C_{\tau,x} = C, E, \eta_{t_0,D_0} = 1, \eta_{t_0,D'_0} = 0\Big)\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}&P\Big(\mathrm{GBI}_\tau(C', C), C_{\tau,x} = C, \delta_x \leq t_0 - \epsilon, E, \eta_{t_0,D_0} = 1, \eta_{t_0,D'_0} = 0\Big)\\ &\leq \frac{|C'|}{|C'| + \lambda|C|} \cdot P\Big(C_{\tau,x} = C, \delta_x \leq t_0 - \epsilon, E, \eta_{t_0,D_0} = 1, \eta_{t_0,D'_0} = 0\Big).\end{aligned} \tag{2.26}$$

In the course of the proof of this and the next lemma, we are going to approximate the times τ and $\delta(i)$, $1 \leq i \leq |V|$, by discrete times. Prior to the proof of Lemma 16, we introduce the notation we are going to use.

Let $n \in \mathbb{N}$ and for all $1 \leq i \leq |V|$ let

$$\tau^{(n)} := \frac{1}{n} \min \{k \in \mathbb{N}_0 | k \geq \tau \cdot n\}$$

and

$$\delta^{(n)}(i) := \frac{1}{n} \min \{k \in \mathbb{N}_0 | k \geq \delta(i) \cdot n\}.$$

Let $\delta^{(n)}(0) = \delta(0)$, $\delta^{(n)}(|V|+1) = \delta(|V|+1)$. For all $y \in V$ the time δ_y is the last time the site y is part of the domain of dependence. We approximate theses times, too. For all $y \in V$ let $i_y \in \{0, \dots, |V|+1\}$ so that $\delta_y = \delta(i_y)$, and write $\delta_y^{(n)} := \delta^{(n)}(i_y)$.

To approximate the shape of the domain of dependence, let $\mathcal{S}_n$ be the set of those $s = (k, d(1), \dots, d(|V|), D_1, \dots, D_{|V|}) \in ((1/n)\mathbb{N}_0)^{|V|+1} \times V^{|V|}$ such that

(i) $k \geq t_0 - \epsilon$ and $d(1) < t_0 + 1/n$;

(ii) for all $1 \leq i \leq |V|$, $D_{i-1} \subset D_i$ and $d(i) \geq d(i+1)$ $(d(|V|+1) := t_0 - \epsilon)$;

For $s = (k, d(1), \dots, d(|V|), D_1, \dots, D_{|V|}) \in S_n$ we write

$$DOD(s) := V \times [0, d(|V|+1)] \cup \bigcup_{0 \leq i \leq |V|} \{D_i \cup D_i'\} \times [0, d(i)],$$

where $d(0) := t_0$. For all $z \in V$ we choose $z_s \in \{0, \dots, |V|+1\}$ so that $d(z_s)$ is the last time at which the site z is part of the set $DOD(s)$:

$$z_s := \min \left\{ 0 \leq i \leq |V|+1 \middle| (z, d(i)) \in DOD(s) \right\}$$

Proof of Lemma 16. Let x, C, C' and E as in the lemma. Suppose $\eta_{t_0, D_0'} = 0$, $C_{\tau,x} = C$ and $\delta_x \leq \tau$. Lemma 14 (Site not part of DOD, then whole cluster not part of DOD) implies $\delta_y \leq \tau$ all $y \in C$, and $G_{\tau,z} = G_{\delta_z \vee \tau, z}$ all $z \in C'$. Therefore, we have

$$\begin{aligned} &P\left(\mathrm{GBI}_\tau(C', C), C_{\tau,x} = C, \delta_x \leq \tau, E, \mathcal{C}\right) \\ &= P\begin{pmatrix} \mathrm{GBI}_\tau(C', C), \forall z \in C' : G_{\tau,z} = G_{\delta_z \vee \tau, z}, \\ C_{\tau,x} = C, \forall y \in C : \delta_y \leq \tau, E, \mathcal{C} \end{pmatrix}, \end{aligned} \tag{2.27}$$

where $\mathcal{C} := \left\{\eta_{t_0, D_0} = 1, \eta_{t_0, D_0'} = 0\right\}$. Let

$$\mathrm{GBI}(\tau, \delta.) := \left\{ \exists z \in C' \, \exists s > (\tau \vee \delta_z) \, \forall y \in C : G_{\tau \vee \delta_z, s, z}, I_{\tau,y} = I_{s,y} \right\},$$

and note that

$$\mathrm{GBI}_\tau(C', C) \cap \{\forall z \in C' : G_{\tau,z} = G_{\delta_z \vee \tau, z}\} \subset \mathrm{GBI}(\tau, \delta.).$$

Hence, (2.27) transforms into

$$\begin{aligned} &P\left(\mathrm{GBI}_\tau(C', C), C_{\tau,x} = C, \delta_x \leq \tau, E, \mathcal{C}\right) \\ &= P\left(\mathrm{GBI}(\tau, \delta.), C_{\tau,x} = C, \forall y \in C : \delta_y \leq \tau, E, \mathcal{C}\right). \end{aligned}$$

The finitely many growth and ignition processes on $C \cup C'$ are right continuous and integer valued. Therefore

$$\begin{aligned} &P\left(\mathrm{GBI}(\tau, \delta.), C_{\tau,x} = C, \forall y \in C : \delta_y \leq \tau, E, \mathcal{C}\right) \\ &= \lim_{n\to\infty} P\left(\mathrm{GBI}(\tau^{(n)}, \delta.^{(n)}), C_{\tau,x} = C, \forall y \in C : \delta_y \leq \tau, E, \mathcal{C}\right). \end{aligned}$$

Let $n \in \mathbb{N}$. Suppose that $\delta_y \leq \tau$ all $y \in C$. Then the set $C \times]\tau, \infty[$ and the domain of dependence are disjoint. This motivates the choice of $\mathcal{S}_{n,C}$ to be the set of those $s = (k, d(1), \ldots, d(|V|), D_1, \ldots, D_{|V|}) \in \mathcal{S}_n$ so that the sets $C \times]k, \infty[$ and $DOD(s)$ are disjoint. We obtain

$$\begin{aligned} &P\left(\mathrm{GBI}(\tau^{(n)}, \delta.^{(n)}), C_{\tau,x} = C, \forall y \in C : \delta_y \leq \tau, E, \mathcal{C}\right) \\ &\leq \sum_{s \in \mathcal{S}_{n,C}} P\big(\mathrm{GBI}(s), \mathrm{DOD}(s), E(s)\big), \end{aligned}$$

where we write for $s = (k, d(1), \ldots, d(|V|), D_1, \ldots, D_{|V|})$

$$\begin{aligned} E(s) &:= \left\{C_{\tau,x} = C, \tau^{(n)} = k, E\right\}, \\ \mathrm{DOD}(s) &:= \left\{\mathcal{C}, \forall 1 \leq i \leq |V| : \delta^{(n)}(i) = d(i), \mathcal{D}_i = D_i\right\}, \end{aligned}$$

and

$$\mathrm{GBI}(s) := \left\{\exists z \in C' \, \exists s > (k \vee d(z_s)) \, \forall y \in C : G_{(k\vee d(z_s)),s,z}, I_{k,y} = I_{s,y}\right\}.$$

Let $s \in \mathcal{S}_{n,C}$, $s = (k, d(1), \ldots, d(|V|), D_1, \ldots, D_{|V|})$. Lemma 12 (The domain of dependence is self-determined) and $E(s) \in \mathcal{GI}_k(V)$ imply

$$\{E(s), \mathrm{DOD}(s)\} \in \sigma\left(\mathcal{GI}_k(V) \cup \bigcup_{0\leq i\leq |V|} \mathcal{GI}_{d(i)}(D_i \cup D_i')\right).$$

The event $\mathrm{GBI}(s)$ is measurable with respect to the σ-field

$$\sigma\left(\mathcal{INCR}_k(C) \cup \bigcup_{z\in C'} \mathcal{INCR}_{(k\vee d(z_s))}(z)\right).$$

The sets $C \times]k, \infty[$ and $DOD(s)$ are disjoint. By our choice of z_s the same holds for all $z \in C'$ for the sets $z \times]k \vee d(z_s), \infty[$ and $DOD(s)$. This implies independence of the latter two σ-fields. In particular, we get

$$\begin{aligned} P\big(\mathrm{GBI}(s), \mathrm{DOD}(s), E(s)\big) &= P\big(\mathrm{GBI}(s)\big) \cdot P\big(\mathrm{DOD}(s), E(s)\big) \\ &\leq \frac{|C'|}{|C'| + \lambda|C|} \cdot P\big(\mathrm{DOD}(s), E(s)\big), \end{aligned}$$

where we use

$$P\big(\text{GBI}(s)\big) \le P\big(\text{GBI}_k(C', C)\big) = \frac{|C'|}{|C'| + \lambda|C|}.$$

Summing up again we obtain

$$\sum_{s\in\mathcal{S}_{n,C}} P\big(\text{GBI}(s), \text{DOD}(s), E(s)\big) \le \frac{|C'|}{|C'| + \lambda|C|} \cdot P\big(C_{\tau,x} = C, E, \mathcal{C}\big).$$

This shows (2.25).

We note that $\delta_x \le t_0 - \epsilon$ if and only if $x \notin \mathcal{D}_{|V|} \cup \mathcal{D}'_{|V|}$. Thus, to show (2.26) we refine $S_{n,C}$ so that for all $(k, d(1), \dots, d(|V|), D_1, \dots, D_{|V|}) \in \mathcal{S}_{n,C}$ we have $x \notin D_{|V|} \cup D'_{|V|}$. Then summing up yields

$$\begin{aligned}\sum_{s\in\mathcal{S}_{n,C}} P\big(\text{DOD}(s), E(s)\big) &\le P\left(C_{\tau,x} = C, x \notin \mathcal{D}_{|V|} \cup \mathcal{D}'_{|V|}, E, \mathcal{C}\right)\\ &= P\big(C_{\tau,x} = C, \delta_x \le t_0 - \epsilon, E, \mathcal{C}\big).\end{aligned}$$

We obtain (2.26). □

We now consider the following situation. Suppose $\tau \le t_0$ and that the site $x \in V$ is vacant within $[\tau, t_0]$, but occupied at time t_0. Then there must be the growth of a tree at the site $x \in V$ in between time τ and t_0. From $t_0 - \epsilon \le \tau \le t_0$ this implies the growth of a tree in between time τ and $\tau + \epsilon$. We obtain for $E \in \mathcal{GI}_\tau$

$$\begin{aligned}&P\left(\tau \le t_0, \exists s \in [\tau, t_0] : \eta_{s,x} = 0, \eta_{t_0,x} = 1, E\right)\\ &\le P\left(G_{\tau,\tau+\epsilon,x}, E\right) = P\left(G_{\epsilon,0} > 0\right) \cdot P\left(E\right).\end{aligned}$$

For $W \subset V$ and $s \ge 0$ let

$$\mathcal{GI}_s^W := \sigma(\mathcal{GI}_s \cup \mathcal{INCR}_0(W))$$

be the σ-field generated by the initial configuration and the growth and ignition jumps that occur within the space time set $\{V \times [0, s]\} \cup \{W \times [0, \infty[\}$. We write

$$\mathcal{GI}_\tau^W := \sigma\left(A \middle| \forall s \ge 0 : \{\tau \le s\} \cap A \in \mathcal{GI}_s^W\right).$$

Lemma 17 (Vacant then occupied implies growth)**.** *Let $W \subset V$, $x \in V \setminus (D_0 \cup W)$ and $E \in \mathcal{GI}_\tau^W$. Then*

$$\begin{aligned}&P\left(\tau \le t_0, \exists s \in [\tau, t_0] : \eta_{s,x} = 0, \eta_{t_0,x} = 1, E, \eta_{t_0,D'_0} = 0, \eta_{t_0,D_0} = 1\right)\\ &\le P(G_{\epsilon,0} > 0) \cdot P\left(E, \eta_{t_0,D'_0} = 0, \eta_{t_0,D_0} = 1\right).\end{aligned}$$

Proof. Let W, x and E as in the lemma. Suppose that $\tau \leq t_0$. In case of $\delta_x > \tau$ we have $\delta_x > t_0 - \epsilon$, and Lemma 11 (On the shape of the domain of dependence) implies $\eta_{\delta_x,x} = 0$. That is, if the site x is vacant within $[\tau, t_0]$, then the site x is vacant within $[\tau \vee \delta_x, t_0]$. Suppose that the site x is vacant within $[\tau \vee \delta_x, t_0]$, but occupied at time t_0. Then there must occur the growth of a tree at x in between time $\tau \vee \delta_x$ and t_0. From $t_0 - \epsilon \leq \tau \vee \delta_x \leq t_0$ this implies the growth of a tree in between time $\tau \vee \delta_x$ and $(\tau \vee \delta_x) + \epsilon$. Formally, we have

$$\begin{aligned} &P\Big(\tau \leq t_0, \exists s \in [\tau, t_0] : \eta_{s,x} = 0, \eta_{t_0,x} = 1, E, \eta_{t_0,D_0'} = 0, \eta_{t_0,D_0} = 1\Big) \\ &\leq P\Big(G_{\tau\vee\delta_x,(\tau\vee\delta_x)+\epsilon,x}, E, \eta_{t_0,D_0'} = 0, \eta_{t_0,D_0} = 1\Big). \end{aligned}$$

The growth process at the site x takes values in $\mathbb{N}_0$ and is right continuous. This implies

$$\begin{aligned} &P\Big(G_{\tau\vee\delta_x,\tau\vee\delta_x+\epsilon,x}, E, \eta_{t_0,D_0'} = 0, \eta_{t_0,D_0} = 1\Big) \\ &= \lim_{n\to\infty} P\Big(G_{\tau^{(n)}\vee\delta_x^{(n)},\tau^{(n)}\vee\delta_x^{(n)}+\epsilon,x}, E, \eta_{t_0,D_0'} = 0, \eta_{t_0,D_0} = 1\Big). \end{aligned}$$

Let $n \in \mathbb{N}$. For $s = (k, d(1), \ldots, d(|V|), D_1, \ldots, D_{|V|}) \in \mathcal{S}_n$ let $E(s) := \{\tau^{(n)} = k, E\}$, $\mathrm{GR}(s) := G_{k\vee d(x_s),k\vee d(x_s)+\epsilon,x}$, and $\mathrm{DOD}(s)$ be defined as in the proof of Lemma 16. Then it holds

$$\begin{aligned} &P\Big(G_{\tau^{(n)}\vee\delta_x^{(n)},\tau^{(n)}\vee\delta_x^{(n)}+\epsilon,x}, E, \eta_{t_0,D_0'} = 0, \eta_{t_0,D_0} = 1\Big) \\ &= \sum_{s\in\mathcal{S}_n} P\big(GR(s), \mathrm{DOD}(s), E(s)\big). \end{aligned}$$

Let $s = (k, d(1), \ldots, d(|V|), D_1, \ldots, D_{|V|}) \in \mathcal{S}_n$. Lemma 12 (The domain of dependence is self-determined) along with $E(s) \in \mathcal{GI}_k^W(V)$ implies

$$\{E(s), \mathrm{DOD}(s)\} \in \sigma\Bigg(\mathcal{GI}_k^W \cup \bigcup_{0\leq i\leq|V|} \mathcal{GI}_{d(i)}(D_i \cup D_i')\Bigg).$$

The event $\mathrm{GR}(s)$ is measurable with respect to the σ-field $\mathcal{INCR}_{k\vee d(x_s)}(x)$. The sets $x \times]k \vee d(x_s), \infty[$ and $DOD(s)$ are disjoint, and $x \notin W$ provides the same for the sets $x \times]k \vee d(x_s), \infty[$ and $\{V \times [0, k]\} \cup \{W \times [0, \infty[\}$. This implies independence of the latter two σ-fields. We obtain

$$P\big(\,\mathrm{GR}(s), \mathrm{DOD}(s), E(s)\big) = P\big(G_{\epsilon,0} > 0\big) \cdot P\big(\,\mathrm{DOD}(s), E(s)\big).$$

Summing up again we get

$$\begin{aligned} &\sum_{s\in\mathcal{S}_n} P\big(\,\mathrm{GR}(s), \mathrm{DOD}(s), E(s)\big) \\ &= P\big(G_{\epsilon,0} > 0\big) \cdot P\Big(E, \eta_{t_0,D_0'} = 0, \eta_{t_0,D_0} = 1\Big). \end{aligned}$$

This concludes the proof of the lemma. □

2.6.3 Estimates for the proof of Proposition 5

In Section 2.6.1 we sketched the three cases (Site vacant before), (Site occupied before, cluster large) and (Site occupied before, cluster small), the proof of Proposition 5 is based on. In this section we estimate these three cases formally . All lemmata of this section restrict to the case where the graph $G = (V, E)$ is finite volume. Throughout this section let $\epsilon > 0$, $t \geq \epsilon$, $B, D \subset V$, $x \in V \setminus D$ and $m \geq 1$.

We formalize the distinction between the cases first.

Lemma 18 (Case distinction)**.** *It holds*

$$\begin{aligned} &P\left(|C_{t,x}| > m, \cup_{y\in B} C_{t,y} = D\right) \\ &= \sum_{C_0 \in C_x} P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right) \\ &\quad + P\left(\exists s \in [t-\epsilon, t] : \eta_{s,x} = 0, |C_{t,x}| > m, \cup_{y\in B} C_{t,y} = D\right). \end{aligned} \tag{2.28}$$

Proof. Either the site x is occupied throughout $[t-\epsilon, t]$, or not. In case of the former, the site x is occupied at time $t - \epsilon$. Thus conditioning on the shape of the cluster at x at time $t - \epsilon$ yields the lemma. □

We estimate the summands of the right hand site of (2.28). We start with the analogon to (Site vacant before).

Lemma 19 (Site vacant before)**.** *We have*

$$\begin{aligned} &P\left(\exists s \in [t-\epsilon, t] : \eta_{s,x} = 0, |C_{t,x}| > m, \cup_{y\in B} C_{t,y} = D\right) \\ &\leq P\left(G_{\epsilon,0} > 0\right) \cdot P\left(\cup_{y\in B} C_{t,y} = D\right). \end{aligned}$$

Proof. We suppose $\{\cup_{y\in B} C_{t,y} = D\} \neq \emptyset$, since otherwise the assertion is obvious. If the cluster at x at time t is bigger than m, then the site x is occupied at time t. Along with (2.20) this shows

$$\begin{aligned} &P\left(\exists s \in [t-\epsilon, t] : \eta_{s,x} = 0, |C_{t,x}| > m, \cup_{y\in B} C_{t,y} = D\right) \\ &\leq P\left(\exists s \in [t-\epsilon, t] : \eta_{s,x} = 0, \eta_{t,x} = 1, \eta_{t,D'} = 0, \eta_{t,D} = 1\right), \end{aligned}$$

where $D' := \partial D \cup (B \setminus D)$. Application of Lemma 17 (Vacant then occupied implies growth) provides

$$\begin{aligned} &P\left(\exists s \in [t-\epsilon, t] : \eta_{s,x} = 0, \eta_{t,x} = 1, \eta_{t,D'} = 0, \eta_{t,D} = 1\right) \\ &\leq P(G_{\epsilon,0} > 0) \cdot P\left(\cup_{y\in B} C_{t,y} = D\right), \end{aligned}$$

where we use (2.20) again. □

Let $C_0 \in C_x$. Depending on the size of C_0, we proceed as described in (Site occupied before, cluster large), respectively (Site occupied before, cluster small) to estimate the probability

$$P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right).$$

We formalize (Site occupied before, cluster large) first. For abbreviation we write

$$\mathcal{C} := \left\{C_{t-\epsilon,x} = C_0, \eta_{t,D} = 1, \eta_{t,D'} = 0\right\},$$

where $D' := \partial D \cup \{D \setminus B\}$.

Lemma 20 (Site occupied before, cluster large). *It holds*

$$\begin{aligned} &P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right) \\ &\leq P\left(I_{\epsilon,0} = 0\right)^{|C_0|} \cdot P\left(C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right). \end{aligned}$$

Proof. As in Lemma 19, (2.20) implies that it suffices to show

$$P\left(\eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) \leq P\left(I_{\epsilon,0} = 0\right)^{|C_0|} \cdot P\left(\mathcal{C}\right).$$

This is the statement of Lemma 15 (All time occupied, then no ignition). □

We are going to use Lemma 20 if the size of C_0 is sufficiently large (compared to ϵ). It remains to consider (Site occupied before, cluster small).

Lemma 21 (Site occupied before, cluster small). *Let $N \in \mathbb{N}$ and suppose $\sqrt[N]{m} \geq |C_0| \vee d$. Then*

$$\begin{aligned} &P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right) \\ &\leq \left(C^N + D(N,m)\right) \cdot P\left(C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right). \end{aligned}$$

Here $C := d/(d+\lambda)$ bounds the probability that after a given time the cluster at x grows before it gets hit by ignition. And

$$D(N,m) := (N-1) \cdot \left(P(G_{\epsilon,0} > 0) \cdot d + \frac{d^3}{\lambda \cdot (\sqrt[N]{m} - 1)}\right)$$

derives as a bound for the probability that during $[t-\epsilon, t]$ the cluster at x gets bigger than m within less than N growth steps.

Proof. Let $N \in \mathbb{N}$ and suppose $\sqrt[N]{m} \geq |C_0| \vee d$. We show

$$P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, \mathcal{C}\right) \leq \left(C^N + D(N,m)\right) \cdot P\left(\mathcal{C}\right). \tag{2.29}$$

First we distinguish whether the cluster at x grew more than N times in between time $t-\epsilon$ and t, or not. To do so, let $\tau_0 := t-\epsilon$ and recursively for all $n \in \mathbb{N}$, let τ_n be the first time after τ_{n-1} at which either the cluster at x is hit by ignition, or at which there occurs the growth of a tree next to it. Formally, we define

$$\tau_n := \min\left\{s > \tau_{n-1} | A(\tau_{n-1}, s)\right\},$$

where

$$A(\tau_{n-1}, s) := \left\{\exists y \in \partial C_{\tau_{n-1},x} : G_{\tau_{n-1},s,y}\right\} \cup \left\{\exists y \in C_{\tau_{n-1},x} : I_{\tau_{n-1},s,y}\right\}.$$

Here with slight abuse of notation we write $\partial C_{\tau_{n-1},x} = x$ if $\eta_{\tau_{n-1},x} = 0$. That is, if the site x is vacant at time τ_{n-1}, then the time τ_n is the first time after τ_{n-1} at which there occurs the growth of a tree at the site x. We distinguish whether $\tau_N \leq t$, and obtain

$$\begin{aligned}
&P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, \mathcal{C}\right) \\
&\leq P\left(\tau_N \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) + P\left(\tau_N > t, \eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, \mathcal{C}\right).
\end{aligned}$$

Hence, (2.29) follows if we show

$$P\left(\tau_N \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) \leq C^N \cdot P\left(\mathcal{C}\right) \tag{2.30}$$

and

$$P\left(\tau_N > t, \eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, \mathcal{C}\right) \leq D(N,m) \cdot P\left(\mathcal{C}\right). \tag{2.31}$$

We first prove (2.30), and then (2.31).

Proof of (2.30). To prove (2.30) we proceed as follows. Suppose $\tau_N \leq t$, and that the site x is occupied throughout $[t-\epsilon, t]$. Then the site x is occupied at time τ_n all $1 \leq n \leq N$. We show that this implies the following for all $1 \leq n \leq N$: after time τ_{n-1} the cluster at x grows before it gets hit by ignition, that is, the event $\mathrm{GBI}_{\tau_{n-1}}(\partial C_{\tau_{n-1},x}, C_{\tau_{n-1},x})$ holds. Then we use Lemma 16 (Estimate GBI) to estimate the probability of the latter event.

Suppose $\eta_{t,D'} = 0$, and that the site x is occupied throughout $[t-\epsilon, t]$. Lemma 13 (All time occupied, then not part of DOD) implies $\delta_x \leq t - \epsilon$, where we use $x \in V \setminus D$. If $\tau_N \leq t$ and $\eta_{[t-\epsilon,t],x} = 1$, then for all $0 \leq i \leq N$ the site x is occupied at time τ_i. We conclude

$$P\left(\tau_N \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) \leq P\left(\delta_x \leq t-\epsilon, \forall 0 \leq i \leq N : \eta_{\tau_i,x} = 1, \mathcal{C}\right). \tag{2.32}$$

Let $0 < n \leq N$ and suppose that the site x is occupied at time τ_{n-1} and at time τ_n. Then there is the growth of a tree on $\partial C_{\tau_{n-1},x}$ at time τ_n: otherwise the cluster at x would be hit by ignition at time τ_n, and hence the site x would be vacant. That is, after time τ_{n-1} there occurs the growth of a tree on $\partial C_{\tau_{n-1},x}$, before $C_{\tau_{n-1},x}$ gets hit by ignition. Formally, we have

$$\begin{aligned}
&P\left(\delta_x \leq t-\epsilon, \forall 0 \leq i \leq n : \eta_{\tau_i,x} = 1, \mathcal{C}\right) \\
&= P\left(\mathrm{GBI}_{\tau_{n-1}}(\partial C_{\tau_{n-1},x}, C_{\tau_{n-1},x}), \delta_x \leq t-\epsilon, \forall 0 \leq i \leq n-1 : \eta_{\tau_i,x} = 1, \mathcal{C}\right).
\end{aligned}$$

We condition on the shape of the cluster at x and time τ_{n-1}, apply Lemma 16 (Estimate GBI), and obtain

$$\begin{aligned}
&P\left(\mathrm{GBI}_{\tau_{n-1}}(\partial C_{\tau_{n-1},x}, C_{\tau_{n-1},x}), \delta_x \leq t-\epsilon, \forall 0 \leq i \leq n-1 : \eta_{\tau_i,x} = 1, \mathcal{C}\right) \\
&\leq \sum_{C_n \in C_x} \frac{|\partial C_n|}{|\partial C_n| + \lambda |C_n|} \cdot P\begin{pmatrix} C_{\tau_{n-1},x} = C_n, \delta_x \leq t-\epsilon, \\ \forall 0 \leq i \leq n-1 : \eta_{\tau_i,x} = 1, \mathcal{C} \end{pmatrix}.
\end{aligned}$$

For all $C_n \in C_x$, we have $|\partial C_n| \leq d \cdot |C_n|$, and therefore

$$\frac{|\partial C_n|}{|\partial C_n| + \lambda |C_n|} = 1 - \frac{\lambda |C_n|}{|\partial C_n| + \lambda |C_n|} \leq 1 - \frac{\lambda}{d+\lambda} = \frac{d}{d+\lambda}.$$

It follows

$$\begin{aligned}
&P\left(\delta_x \leq t-\epsilon, \forall 0 \leq i \leq n : \eta_{\tau_i,x} = 1, \mathcal{C}\right) \\
&\leq \frac{d}{d+\lambda} \cdot P\left(\delta_x \leq t-\epsilon, \forall 0 \leq i \leq n-1 : \eta_{\tau_i,x} = 1, \mathcal{C}\right).
\end{aligned}$$

Using this to successively estimate the right hand site of (2.32) provides (2.30).

Proof of (2.31). The intuition underlying the proof of (2.31) is the following. Suppose that the cluster at x is large (larger than some $m \in \mathbb{N}$) at time t, but comparatively small (smaller than $\sqrt[N]{m}$) at time $t-\epsilon$. Furthermore suppose that the cluster at x grows less than N times within $t-\epsilon$ and t. Then at least at one of the at most $N-1$ growth steps the cluster at x must grow a comparatively large amount of sites. We are going to show that this event has small probability.

In the first step we distinguish at which growth step the cluster at x grows an amount of sites that is comparatively large enough. Suppose $C_{t-\epsilon,x} = C_0$ and $|C_{t,x}| > m$. Then our choice of C_0 provides $|C_{\tau_0,x}| \leq \sqrt[N]{m}$. Hence, the cluster at x must grow at least one time in between time τ_0 and t, that is, we have $\tau_1 \leq t$. We distinct on the occurrence of $|C_{\tau_1,x}| \leq (\sqrt[N]{m})^2$, and obtain

$$\begin{aligned}
&P\left(\tau_N > t, \eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, \mathcal{C}\right) \\
&\leq P\left(|C_{\tau_0,x}| \leq \sqrt[N]{m}, |C_{\tau_1,x}| > \left(\sqrt[N]{m}\right)^2, \tau_1 \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) \\
&+ P\left(|C_{\tau_1,x}| \leq \left(\sqrt[N]{m}\right)^2, |C_{t,x}| > m, \tau_1 < t < \tau_N, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right).
\end{aligned}$$

We apply the same argument to the cluster at x at time τ_1, and obtain for the second summand

$$\begin{aligned}
&P\left(|C_{\tau_1,x}| \leq \left(\sqrt[N]{m}\right)^2, |C_{t,x}| > m, \tau_1 < t < \tau_N, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) \\
&\leq P\left(|C_{\tau_1,x}| \leq \left(\sqrt[N]{m}\right)^2, |C_{\tau_2,x}| > \left(\sqrt[N]{m}\right)^3, \tau_2 \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) \\
&+ P\left(|C_{\tau_2,x}| \leq \left(\sqrt[N]{m}\right)^3, |C_{t,x}| > m, \tau_2 < t < \tau_N, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right).
\end{aligned}$$

Going on iteratively it follows

$$\begin{aligned}
&P\left(\tau_N > t, \eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, \mathcal{C}\right) \\
&\leq \sum_{n=1}^{N-1} P\left(\begin{array}{c} |C_{\tau_{n-1},x}| \leq \left(\sqrt[N]{m}\right)^n, |C_{\tau_n,x}| > \left(\sqrt[N]{m}\right)^{n+1}, \\ \tau_n \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C} \end{array} \right).
\end{aligned}$$

Here, to see that the iteration ends after $N-1$ steps, we use that $\tau_{N-1} \leq t < \tau_N$ and $|C_{t,x}| > m$ imply $|C_{\tau_{N-1},x}| > m$.

If at some growth step the cluster at x grows a comparatively large amount of sites, then at this growth step the cluster at x must get connected to a comparatively large cluster. We now estimate the minimal size for such a cluster to be comparatively large enough. Let $1 \leq n \leq N-1$, and suppose $|C_{\tau_{n-1},x}| \leq (\sqrt[N]{m})^n$ and $|C_{\tau_n,x}| > (\sqrt[N]{m})^{n+1}$. Then there occurs the growth of a tree on $\partial C_{\tau_{n-1},x}$ at time τ_n. Furthermore, $\tau_n \leq t$ and $\eta_{[t-\epsilon,t],x} = 1$ imply $\eta_{\tau_{n-1},x} = 1$, and hence $C_{\tau_{n-1},x} = C_n$ for some $C_n \in C_x$. It follows

$$\begin{aligned} &P\left(|C_{\tau_{n-1},x}| \leq (\sqrt[N]{m})^n, |C_{\tau_n,x}| > (\sqrt[N]{m})^{n+1}, \tau_n \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right) \\ &\leq \sum_{\substack{C_n \in C_x \\ |C_n| \leq (\sqrt[N]{m})^n}} \sum_{y \in \partial C_n} P\left(\begin{array}{c} C_{\tau_{n-1},x} = C_n, |C_{\tau_n,x}| > (\sqrt[N]{m})^{n+1}, \\ G_{\tau_{n-1},\tau_n,y}, \tau_n \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C} \end{array}\right). \end{aligned}$$

Let $C_n \in C_x$ such that $|C_n| \leq (\sqrt[N]{m})^n$, let $y \in \partial C_n$ and suppose that

$$\left\{C_{\tau_{n-1},x} = C_n, |C_{\tau_n,x}| > (\sqrt[N]{m})^{n+1}, G_{\tau_{n-1},\tau_n,y}, \tau_n \leq t, \eta_{[t-\epsilon,t],x} = 1, \mathcal{C}\right\}$$

occurs. Then throughout $[\tau_{n-1}, \tau_n[$ the cluster at x equals C_n, and at time τ_n the cluster at x grows at the site y. Growth and ignition jumps occur at distinct times. Hence, no site gets vacant at time τ_n, and the site y is the only site that gets occupied at time τ_n. It follows that at time τ_n the cluster at x is the union of C_n, the site y and the clusters that contained a neighbour of y:

$$C_{\tau_n,x} = C_n \cup \{y\} \cup \bigcup_{z \in \partial y \setminus \overline{C_n}} C_{\tau_n^-,z} \tag{2.33}$$

Along with $|C_n| \leq (\sqrt[N]{m})^n$ this yields

$$|C_{\tau_n,x}| \leq (\sqrt[N]{m})^n + 1 + (d-1) \cdot \max_{z \in \partial y \setminus \overline{C_n}} |C_{\tau_n^-,z}|.$$

Hence, $|C_{\tau_n,x}| > (\sqrt[N]{m})^{n+1}$ implies the existence of $z \in \partial y \setminus \overline{C_n}$ so that

$$|C_{\tau_n^-,z}| \geq \frac{(\sqrt[N]{m})^{n+1} - (\sqrt[N]{m})^n - 1}{d-1} \geq \frac{(\sqrt[N]{m})^{n+1} - (\sqrt[N]{m})^n}{d} =: M,$$

where the second inequality is due to $\sqrt[N]{m} \geq d$.

That is, there exists $z \in \partial y \setminus \overline{C_n}$ so that $|C_{\tau_n^-,z}| \geq M$. To be able to use Lemmata 17 (Vacant then occupied implies growth) and 16 (Estimate GBI) later on, we show $z \notin D$.

Let $z \in \partial y \setminus \overline{C_n}$ and suppose $|C_{\tau_n^-,z}| \geq M$. From (2.33) we have $z \in C_{\tau_n,x}$. Hence, $\eta_{[\tau_n,t],x} = 1$ implies $\eta_{[\tau_n,t],z} = 1$. Lemma 13 (All time occupied, then not part of DOD) and $x \in V \setminus D$ imply $\delta_x \leq \tau_n$. Along with Lemma 14 (Site not part of DOD, then whole cluster not part of DOD) it follows $\delta_z \leq \tau_n$. In case of $\tau_n < t$, this implies $\delta_z < t$ and therefore $z \notin D$. Otherwise in case of $\tau_n = t$, the relations $\eta_{D',t} = 0$, $\partial D \subset D'$ and $x \in V \setminus D$ imply $C_{t,x} \cap D = \emptyset$, and in particular $z \notin D$.

Altogether, we obtain

$$\begin{aligned}&P\left(C_{\tau_{n-1},x}=C_n,|C_{\tau_n,x}|>\left(\sqrt[N]{m}\right)^{n+1},G_{\tau_{n-1},\tau_n,y},\tau_n\leq t,\eta_{[t-\epsilon,t],x}=1,\mathcal{C}\right)\\&\leq\sum_{z\in\partial y\setminus\{\overline{C_n}\cup D\}}P\left(\begin{array}{c}C_{\tau_{n-1},x}=C_n,|C_{\tau_n^-,z}|\geq M,\\G_{\tau_{n-1},\tau_n,y},\tau_n\leq t,\eta_{[\tau_n,t],z}=1,\mathcal{C}\end{array}\right).\end{aligned}$$

Let $z\in\partial y\setminus\{\overline{C_n}\cup D\}$ and

$$\sigma_z:=\min\left\{s\geq\tau_{n-1}\big||C_{s,z}|\geq M\right\}\wedge\tau_n$$

be the minimum of τ_n and the first time $s\geq\tau_{n-1}$ at which the cluster at z is bigger than or equal to M. Either the site z is occupied throughout the entire time interval $[\tau_{n-1},\tau_n]$, or not. Furthermore, $|C_{\tau_n^-,z}|\geq M$ implies $\sigma_z<\tau_n$. It follows

$$\begin{aligned}&P\left(C_{\tau_{n-1},x}=C_n,|C_{\tau_n^-,z}|\geq M,G_{\tau_{n-1},\tau_n,y},\tau_n\leq t,\eta_{[\tau_n,t],z}=1,\mathcal{C}\right)\\&\leq P_0(n,C_n,y,z)+P_1(n,C_n,y,z),\end{aligned}$$

where

$$P_0(n,C_n,y,z):=P\left(\begin{array}{c}C_{\tau_{n-1},x}=C_n,G_{\tau_{n-1},\tau_n,y},\tau_{n-1}\leq t,\\\exists s\in[\tau_{n-1},t]:\eta_{s,z}=0,\eta_{t,z}=1,\mathcal{C}\end{array}\right)$$

and

$$P_1(n,C_n,y,z):=P\left(C_{\tau_{n-1},x}=C_n,\sigma_z<\tau_n\leq t,G_{\tau_{n-1},\tau_n,y},\eta_{[\sigma_z,t],z}=1,\mathcal{C}\right).$$

Altogether, we obtain

$$\begin{aligned}&P\left(\tau_N>t,\eta_{[t-\epsilon,t],x}=1,|C_{t,x}|>m,\mathcal{C}\right)\\&\leq\sum_{n=1}^{N-1}\sum_{\substack{C_n\in C_x\\|C_n|\leq\left(\sqrt[N]{m}\right)^n}}\sum_{y\in\partial C_n}\sum_{z\in\partial y\setminus\{\overline{C_n}\cup D\}}\Big(P_0(n,C_n,y,z)+P_1(n,C_n,y,z)\Big).\end{aligned}$$

That is, to prove (2.31) it suffices to show for all $1\leq n\leq N-1$

$$\sum_{\substack{C_n\in C_x\\|C_n|\leq\left(\sqrt[N]{m}\right)^n}}\sum_{y\in\partial C_n}\sum_{z\in\partial y\setminus\{\overline{C_n}\cup D\}}P_0(n,C_n,y,z)\leq d\cdot P(G_{\epsilon,0}>0)\cdot P\left(\mathcal{C}\right)\tag{2.34}$$

and

$$\sum_{\substack{C_n\in C_x\\|C_n|\leq\left(\sqrt[N]{m}\right)^n}}\sum_{y\in\partial C_n}\sum_{z\in\partial y\setminus\{\overline{C_n}\cup D\}}P_1(n,C_n,y,z)\leq\frac{d^3}{\lambda\left(\sqrt[N]{m}-1\right)}\cdot P\left(\mathcal{C}\right).\tag{2.35}$$

The proof of (2.34) is based on Lemma 17 (Vacant then occupied implies growth), and to prove (2.35) we are going to use 16 (Estimate GBI).

We start we the proof of (2.34). Let $1 \leq n \leq N-1$ and $C_n \in C_x$ such that $|C_n| \leq (\sqrt[N]{m})^n$. Let $y \in \partial C_n$, $z \in \partial y \setminus \{\overline{C_n} \cup D\}$ and suppose $C_{\tau_{n-1},x} = C_n$. Then τ_n is the first time after τ_{n-1} at which either there occurs a growth on ∂C_n, or there occurs an ignition on C_n. Therefore, we have

$$\begin{aligned}&\left\{C_{\tau_{n-1},x} = C_n, G_{\tau_{n-1},\tau_n,y}\right\}\\ &= \left\{\begin{array}{c} C_{\tau_{n-1},x} = C_n, \exists s > \tau_{n-1} : G_{\tau_{n-1},s,y}, \\ \forall w \in \partial C_n \setminus \{y\} : G_{\tau_{n-1},w} = G_{s,w}, \forall v \in C_n : I_{\tau_{n-1},w} = I_{s,w} \end{array}\right\}.\end{aligned}$$

That is, $\left\{C_{\tau_0,x} = C_0, C_{\tau_{n-1},x} = C_n, G_{\tau_{n-1},\tau_n,y}\right\} \in \mathcal{GI}_{\tau_{n-1}}^{\overline{C_n}}$. Hence, Lemma 17 (Vacant then occupied implies growth) implies

$$P_0(n, C_n, y, z) \leq P(G_{\epsilon,0} > 0) \cdot P\left(C_{\tau_{n-1},x} = C_n, G_{\tau_{n-1},\tau_n,y}, \mathcal{C}\right). \tag{2.36}$$

For $w \in \partial C_n$ if $\left\{C_{\tau_{n-1},x} = C_n, G_{\tau_{n-1},\tau_n,w}\right\}$ occurs, then the growth process at the site w jumps at time τ_n. Growth jumps occur at distinct times. It follows that the events $\left\{C_{\tau_{n-1},x} = C_n, G_{\tau_{n-1},\tau_n,w}\right\}$, $w \in \partial C_n$, are disjoint. We obtain

$$\begin{aligned}&\sum_{\substack{C_n \in C_x \\ |C_n| \leq (\sqrt[N]{m})^n}} \sum_{y \in \partial C_n} \sum_{z \in \partial y} P(G_{\epsilon,0} > 0) \cdot P\left(C_{\tau_{n-1},x} = C_n, G_{\tau_{n-1},\tau_n,y}, \mathcal{C}\right)\\ &\leq d \cdot P(G_{\epsilon,0} > 0) \cdot P\left(\mathcal{C}\right).\end{aligned}$$

This shows (2.34). For the proof of (2.35) suppose that

$$\left\{C_{\tau_{n-1},x} = C_n, \sigma_z < \tau_n \leq t, G_{\tau_{n-1},\tau_n,y}, \eta_{[\sigma_z,t],z} = 1, \mathcal{C}\right\}$$

occurs. Then the growth process at the site y jumps at time τ_n. It follows $G_{\sigma_z,\tau_n,y}$. Furthermore, $\eta_{[\sigma_z,t],z} = 1$ implies that the cluster at the site z does not get hit by ignition within $[\sigma_z, \tau_n]$. That is, $I_{\sigma_z,u} = I_{\tau_n,u}$ for all $u \in C_{\sigma_z,z}$. Hence, $\mathrm{GBI}_{\sigma_z}(y, C_{\sigma_z,z})$ occurs. Since the site $z \in V \setminus D$ is occupied throughout $[\sigma_z, t]$, Lemma 13 (All time occupied, then not part of DOD) implies $\delta_z \leq \sigma_z$. Formally, we have

$$P_1(n, C_n, y, z) \leq P\left(C_{\tau_{n-1},x} = C_n, \mathrm{GBI}_{\sigma_z}(y, C_{\sigma_z,z}), \delta_z \leq \sigma_z < \tau_n, \mathcal{C}\right).$$

The relation $\sigma_z < \tau_n$ implies $|C_{\sigma_z,z}| > M$. From $y \in \partial C_n$ and $\sigma_z < \tau_n$ the site y is vacant at time σ_z. Along with $z \in \partial y$ it follows $y \in \partial C_{\sigma_z,z}$. From Lemma 16 (Estimate

GBI) we obtain

$$\begin{aligned}
&P\left(C_{\tau_{n-1},x} = C_n, \mathrm{GBI}_{\sigma_z}(y, C_{\sigma_z,z}), \delta_z \le \sigma_z < \tau_n, \mathcal{C}\right) \\
&\le \sum_{\substack{C \in C_z \\ y \in \partial C, |C| > M}} P\left(C_{\tau_{n-1},x} = C_n, C_{\sigma_z,z} = C, \mathrm{GBI}_{\sigma_z}(y, C), \delta_z \le \sigma_z, \mathcal{C}\right) \\
&\le \sum_{\substack{C \in C_z \\ |C| > M}} \frac{1}{1+\lambda|C|} \cdot P\left(C_{\tau_{n-1},x} = C_n, C_{\sigma_z,y} = C, \mathcal{C}\right) \\
&\le \frac{1}{\lambda M} \cdot P\left(C_{\tau_{n-1},x} = C_n, \mathcal{C}\right).
\end{aligned}$$

Summing up we get

$$\sum_{\substack{C_n \in C_x \\ |C_n| \le (\sqrt[N]{m})^n}} \sum_{y \in \partial C_n} \sum_{z \in \partial y} \frac{1}{\lambda M} \cdot P\left(C_{\tau_{n-1},x} = C_n, \mathcal{C}\right) \le \frac{d^2 \, (\sqrt[N]{m})^n}{\lambda M} \cdot P\left(\mathcal{C}\right),$$

where we use $|\partial C_n| \le d \cdot |C_n|$ all $C_n \in C_x$. Inserting the definition of M provides

$$\frac{d^2 \, (\sqrt[N]{m})^n}{\lambda M} = \frac{d^3 \, (\sqrt[N]{m})^n}{\lambda \left((\sqrt[N]{m})^{n+1} - (\sqrt[N]{m})^n \right)} = \frac{d^3}{\lambda \, (\sqrt[N]{m} - 1)}.$$

This shows (2.35). □

2.6.4 Proof of Proposition 5

In this section we use Lemmata 18 - 21 to show Proposition 5.

Proof of Proposition 5. Let $\gamma > 0$ and suppose that the graph $G = (V, E)$ is finite volume. Let $\delta > 0$. We write $N := N_{\gamma,\lambda,d}(\delta)$, $\epsilon := \epsilon_{\gamma,\lambda,d}(\delta)$, $m_0 := M_{\gamma,\lambda,d}(\delta)$ and $m := m_{\gamma,\lambda,d}(\delta)$ with the notation from Definition 10.

Let $t \ge \gamma$, $B \subset V$, $D \subset V$ and $x \in V \setminus D$. Lemma 18 (Case distinction) implies

$$\begin{aligned}
&P\left(|C_{t,x}| > m, \cup_{y \in B} C_{t,y} = D\right) \\
&= \sum_{C_0 \in C_x} P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y \in B} C_{t,y} = D\right) \\
&\quad + P\left(\exists s \in [t-\epsilon, t] : \eta_{s,x} = 0, |C_{t,x}| > m, \cup_{y \in B} C_{t,y} = D\right).
\end{aligned}$$

We use Lemmata 19 - 21 to estimate the right hand site of this equation. Lemma 19 (Site vacant before) provides

$$\begin{aligned}
&P\left(\exists s \in [t-\epsilon, t] : \eta_{s,x} = 0, |C_{t,x}| > m, \cup_{y \in B} C_{t,y} = D\right) \\
&\le \frac{\delta}{8} \cdot P\left(\cup_{y \in B} C_{t,y} = D\right).
\end{aligned}$$

Let $C_0 \in C_x$. In case of $|C_0| \geq m_0$ we have

$$P(I_{\epsilon,0} = 0)^{|C_0|} \leq P(I_{\epsilon,0} = 0)^{m_0} = \left(e^{-\lambda\epsilon}\right)^{m_0} \leq \frac{7\delta}{8},$$

and along with Lemma 20 (Site occupied before, cluster large) it follows

$$P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right) \\ \leq \frac{7\delta}{8} \cdot P\left(C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right).$$

Suppose that $|C_0| < m_0$. Then $\sqrt[N]{m} \geq |C_0| \vee d$. Thus Lemma 21 (Site occupied before, cluster small) provides

$$P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right) \\ \leq \left(C^N + D(N,m)\right) \cdot P\left(C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right),$$

where $C := d/(d+\lambda)$ and

$$D(N,m) := (N-1)\left(P(G_{\epsilon,0} > 0)\cdot d + \frac{d^3}{\lambda\left(\sqrt[N]{m}-1\right)}\right).$$

Note that N, ϵ and m are chosen such that $C^N \leq \delta/2$,

$$P(G_{\epsilon,0} > 0) = 1 - e^{-\epsilon} \leq \frac{\delta}{8d(N-1)}$$

and

$$\frac{d^3}{\lambda\left(\sqrt[N]{m}-1\right)} \leq \frac{2\delta}{8(N-1)}.$$

It follows

$$C^N + D(N,m) \leq \frac{\delta}{2} + (N-1)\cdot\left(\frac{d\delta}{8d(N-1)} + \frac{2\delta}{8(N-1)}\right) = \frac{7\delta}{8}.$$

Altogether, we get

$$\sum_{C_0\in C_x} P\left(\eta_{[t-\epsilon,t],x} = 1, |C_{t,x}| > m, C_{t-\epsilon,x} = C_0, \cup_{y\in B} C_{t,y} = D\right) \\ \leq \frac{7\delta}{8}\cdot P\left(\cup_{y\in B} C_{t,y} = D\right).$$

This shows

$$P\left(|C_{t,x}| > m, \cup_{y\in B} C_{t,y} = D\right) \leq \delta \cdot P\left(\cup_{y\in B} C_{t,y} = D\right).$$

□

2.6.5 Proof of Proposition 3

Finally, we show Proposition 3.

Proof of Proposition 3. Let $\gamma > 0$, $\delta > 0$ and $m := m_{\gamma,\lambda,d}(\delta)$. Let $t \geq \gamma$ and suppose almost sure infinite volume convergence at time t:

$$\lim_{n\to\infty} P\big(\sup_{l\geq n} |\eta_{t,y} - \eta_{t,y}^{(l)}| > 0\big) = 0 \qquad \text{all } y \in V. \tag{2.37}$$

Let $B, D \subset V$ finite, and $x \in V \setminus D$. The occurrence of the event

$$\{|C_{t,x}| > m, \cup_{y\in B} C_{t,y} = D\}$$

is determined by the status of the forest fire process η on the finite set $B_{2m}(x) \cup B \cup \overline{D}$ at time t. Therefore, the almost sure infinite volume convergence at time t, (2.37), implies

$$\begin{aligned} &P\left(|C_{t,x}| > m, \cup_{y\in B} C_{t,y} = D\right) = \lim_{n\to\infty} P\left(|C_{t,x}^{(n)}| > m, \cup_{y\in B} C_{t,y}^{(n)} = D\right) \\ &\leq \delta \cdot \lim_{n\to\infty} P\left(\cup_{y\in B} C_{t,y}^{(n)} = D\right) = \delta \cdot P\left(\cup_{y\in B} C_{t,y} = D\right), \end{aligned}$$

where we use that the finite volume forest fire processes $(\eta_{t,y}^{(n)})_{t\geq 0, y\in B_n}$, $n \geq 1$, have $\mathrm{CCSB}(t, \delta, m)$ (Proposition 5).

This shows $P\big(|C_{t,x}| > m_{\gamma,\lambda,d}(\delta')\big) \leq \delta'$ all $\delta' > 0$. It follows $P\left(|C_{t,x}| = \infty\right) = 0$. □

Chapter 3

Scaling limit for the Abelian sandpile height one field

In this chapter we study the scaling limit for the height one field of the two-dimensional Abelian sandpile model. We identify the scaling limit for the covariance of having height one at two macroscopically distant sites, and show that it is conformally covariant. Furthermore, we show a central limit theorem for the sandpile height one field. The results are based on a representation of the height one joint intensities that is close to a block-determinantal structure. In Section 3.1 we start with an introduction to the model, and review some of its basic properties. Thereafter, in Section 3.2 we state our main results, and prove them in Sections 3.3 - 3.6.

3.1 The Abelian sandpile model

This section introduces the Abelian sandpile model, based on the works [14] and [27].

3.1.1 The model

Let Λ be a finite subset of $\mathbb{Z}^2$. The sandpile model on Λ is defined with respect to a toppling matrix.

Definition 22 (Toppling matrices)**.** A matrix $\Delta \in \mathbb{Z}^{\Lambda\times\Lambda}$ is a toppling matrix on Λ, if it satisfies the following conditions:

(i) for all $v, w \in \Lambda$, $v \neq w$, $\Delta(v,w) = \Delta(w,v) \leq 0$;

(ii) for all $v \in \Lambda$, $\Delta(v,v) \geq 1$;

(iii) for all $v \in \Lambda$, $\sum_{w\in\Lambda} \Delta(v,w) \geq 0$;

(iv) for all $v_1 \in \Lambda$ there exists $n \in \mathbb{N} := \{1,2,\ldots\}$, and $v_i \in \Lambda$, $2 \leq i \leq n$, such that $\sum_{w\in\Lambda} \Delta(v_n,w) > 0$ and $\Delta(v_{i-1},v_i) < 0$ for all $1 < i \leq n$.

The forth condition in the definition of a toppling matrix is fundamental to have a well defined toppling rule later on.

Let Δ be a toppling matrix on Λ. In a sandpile model on Λ every site $v \in \Lambda$ has a height $\eta_v \in \mathbb{N}$.

Definition 23 (Stable configurations)**.** Let $(\eta_v)_{v\in\Lambda} \in \mathbb{N}^\Lambda$ be a height configuration. A site $v \in \Lambda$ is called stable with respect to the toppling matrix Δ, if $\eta_v \leq \Delta(v,v)$. We write $\Omega_\Delta := \prod_{v\in\Lambda}\{1,\dots,\Delta(v,v)\}$ to denote the set of height configurations that are stable with respect to Δ. A site that is not stable is called unstable.

The dynamics of the sandpile model corresponding to the toppling matrix Δ is as follows. Let $(\eta_v)_{v\in\Lambda} \in \Omega_\Delta$ be a stable height configuration. We choose a site $v \in \Lambda$ and increase the height at v by one. If the site v became unstable, that is, in case of $\eta_v + 1 > \Delta(v,v)$ we topple v according to Δ. For all $w \in \Lambda$ we decrease the height at w by $\Delta(v,w)$. It might be that from toppling the site v, one or more sites $w \neq v$ became unstable. Then we continue by toppling all unstable sites until we obtain a stable configuration. For a formal statement, we introduce the addition operator $a_{v,\Delta}$.

Definition 24 (The addition operator)**.** Let $\{v_1,\dots,v_{|V|}\}$ be an enumeration of the set V. We define the toppling transformation $\mathcal{T}_\Delta : \mathbb{N}^\Lambda \mapsto \Omega_\Delta$ by

$$\mathcal{T}_\Delta(\eta) := \lim_{N\to\infty}\left(\prod_{i=1}^{|V|} T_{v_i}(\,\cdot\,)\right)^N(\eta). \tag{3.1}$$

Here for all $\eta = (\eta_w)_{w\in\Lambda} \in \mathbb{N}^\Lambda$, for all $w \in \Lambda$

$$(T_v(\eta))_w := \begin{cases} \eta_w - \Delta(v,w) & \text{if } \eta(v) > \Delta(v,v); \\ \eta_w & \text{otherwise.} \end{cases}$$

For all $v \in \Lambda$ we define the addition operator $a_{v,\Delta} : \Omega_\Delta \mapsto \Omega_\Delta$ by

$$a_{v,\Delta}(\eta) := \mathcal{T}_\Delta(\eta^v),$$

where $(\eta^v)_w := \eta_w + 1_{\{v=w\}}$, $w \in \Lambda$.

The addition operator is well defined (see e.g. [27]):

Remark 6. As a consequence of the forth condition in Definition 22, the limit in (3.1) exists. And for all $\eta \in \Omega_\Delta$ the configuration $a_{v,\Delta}(\eta)$ is independent of the chosen enumeration of V. This is the famous 'Abelian' property.

The Abelian sandpile model is defined as a Markov chain.

Definition 25 (The Abelian sandpile model)**.** The Abelian sandpile model corresponding to the toppling matrix Δ is a discrete time Markov chain $\{\zeta_n : n \geq 0\}$ on Ω_Δ with the following transition operator: given a configuration in Ω_Δ, we pick a site $v \in \Lambda$ according to the uniform distribution on Λ, and apply the addition operator $a_{v,\Delta}$ to the configuration. We write $P_{\Delta,\eta}$ to denote the Markov measure of the chain starting from $\eta \in \Omega_\Delta$.

Definition 26 (Recurrent configurations)**.** We call a configuration $\eta \in \Omega_\Delta$ recurrent with respect to Δ, if

$$P_{\Delta,\eta}\big(\zeta_n = \eta \text{ for infinitely many } n\big) = 1.$$

We write $\mathcal{R}_\Delta := \{\eta \in \Omega_\Delta | \eta \text{ is recurrent}\}$ for the set of recurrent configurations.

We recall two results discovered by D. Dhar and S.N. Majumdar in [8] and [24] (see [27] for an alternative proof).

Remark 7. The number of recurrent configurations satisfies

$$|\mathcal{R}_\Delta| = \det(\Delta).$$

There exists a unique measure μ_Δ that is invariant with respect to the Abelian sandpile model corresponding to Δ. It is the uniform measure on the set of recurrent configurations $\mathcal{R}_\Delta$.

This work focuses on the Abelian sandpile height one field.

Definition 27 (The height one indicator function)**.** For all $v \in \Lambda$ let $h_\Delta(v) : \Omega_\Delta \mapsto \{0,1\}$ denote the indicator function of having height one at the site v.

In our notation of the height one indicator function we use the index Δ to denote the measure we take expectations with respect to: $\mathbb{E}\big[h_\Delta(v)\big]$ means the expectation of $h_\Delta(v)$ with respect to the measure μ_Δ.

Our main results concern the sandpile model corresponding to the discrete Laplacian with open boundary conditions.

Definition 28 (The discrete Laplacian Δ_Λ)**.** For all $v, w \in \Lambda$ let

$$\Delta_\Lambda(v,w) := \begin{cases} 4 & \text{if } v = w; \\ -1 & \text{if } |v - w| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

We note that Δ_Λ is a toppling matrix on Λ, and write Ω_Λ, $\mathcal{R}_\Lambda$, μ_Λ and $h_\Lambda(\,\cdot\,)$ to denote Ω_{Δ_Λ}, $\mathcal{R}_{\Delta_\Lambda}$, μ_{Δ_Λ} and $h_{\Delta_\Lambda}(\,\cdot\,)$.

3.1.2 The thermodynamic limit

Let $\Lambda_n := [-n,n]^2 \cap \mathbb{Z}^2$. In [1] S.R. Athreya and A.A. Járai show that as $n \to \infty$ the measures μ_{Λ_n} weakly converge to a translation invariant measure μ_0 on $\Omega_0 := \{1,2,3,4\}^{\mathbb{Z}^2}$.

Lemma 22 ([14], Theorem 4.1)**.** *The limit $\mu_0 = \lim_{n\to\infty} \mu_{\Lambda_n}$ exists in the sense of weak convergence. μ_0 is translation invariant.*

For all $v \in \mathbb{Z}^2$ let $h_0(v) : \Omega_0 \mapsto \{0,1\}$ denote the indicator function of having height one at the site v.

3.2 Statement of the main results

From now on throughout the remainder let $U \subset \mathbb{C} = \mathbb{R}^2$ be a bounded connected domain with smooth boundary. We write $\mathcal{C}_c^\infty(U)$ to denote the set of smooth functions $f : U \mapsto \mathbb{R}$ with support compactly contained in U. For all $\epsilon > 0$ let $U_\epsilon := U/\epsilon \cap \mathbb{Z}^2$. For every $u \in U$ let $\epsilon_u > 0$ so that for all $\epsilon \in]0, \epsilon_u]$ there exists $u_\epsilon \in U_\epsilon$ such that $|u/\epsilon - u_\epsilon| \leq 2$. Our first result concerns the scaling limit for the covariance of having height one at two macroscopically distant points.

3.2.1 Conformal scaling for the height one joint moments

Let v, w be two distinct points in the interior of U, and $\mathrm{Cov}\left(h_{U_\epsilon}(v_\epsilon), h_{U_\epsilon}(w_\epsilon)\right)$ be the covariance of having height one at the sites v_ϵ and w_ϵ in a sandpile model on U_ϵ. Then rescaled by ϵ^{-4} this covariance converges to a finite limit which is conformally covariant with scale dimension 2. More formally, we have for the height one joint moments of a finite set of points in the interior of U:

Theorem 4 (Conformal scaling for the height one joint moments)**.** *Let $V \subset U$ be a set of finitely many points in the interior of U. Then as $\epsilon \to 0$ the rescaled joint moment*

$$\epsilon^{-2|V|}\mathbb{E}\left[\prod_{v\in V}\left(h_{U_\epsilon}(v_\epsilon) - \mathbb{E}[h_{U_\epsilon}(v_\epsilon)]\right)\right]$$

tends to a finite limit $E_U\left(v : v \in V\right)$ which is conformally covariant with scale dimension 2.

By conformal covariance with scale dimension 2 we mean that for any conformal isomorphism $f : U \mapsto U'$

$$E_U\left(v : v \in V\right) = E_{U'}\left(f(v) : v \in V\right) \cdot \prod_{v\in V} |f'(v)|^2.$$

To obtain Theorem 4 we derive an explicit representation for $E_U\left(v : v \in V\right)$. The formula is given in Section 3.2.3.

Our next result concerns the scaling limit for the height one field itself.

3.2.2 Scaling limit for the height one field

In the scaling limit the Abelian sandpile height one field converges to Gaussian white noise in the following sense.

Theorem 5 (Scaling limit for the height one field)**.** *Let $n \geq 1$ and for all $1 \leq i \leq n$ let $f_i \in \mathcal{C}_c^\infty(U)$. Then as $\epsilon \to 0$ the random variables*

$$f_i \diamond h_{U_\epsilon} := \frac{\epsilon}{\sqrt{\mathcal{V}}}\sum_{v\in U_\epsilon} f_i(\epsilon v)\cdot\left(h_{U_\epsilon}(v) - \mathbb{E}[h_{U_\epsilon}(v)]\right), \quad 1 \leq i \leq n$$

converge in distribution to jointly normal random variables with mean zero and covariance matrix

$$\left(\int_U f_i(z) f_j(z) dz\right)_{1\leq i,j\leq n}.$$

Here $\mathcal{V}$ denotes

$$\mathcal{V} := \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathbb{V}\Big[\sum_{v\in\Lambda_n} h_0(v)\Big]$$

and satisfies $0 < \mathcal{V} = \sum_{v\in\mathbb{Z}^2} \mathrm{Cov}(h_0(0), h_0(v)) < \infty$.

Note that we use two different scalings in Theorem 4 and Theorem 5. This has the following reason. In a sandpile model the covariance of having height one at two points with distance r decays as r^{-4}. That is, for two distinct points v, w in the interior of U, as ϵ tends to zero $\mathrm{Cov}\left(h_{U_\epsilon}(v_\epsilon), h_{U_\epsilon}(w_\epsilon)\right)$ decays as $(\epsilon/|v-w|)^4$. Hence, in Theorem 4 we have to rescale the covariance by ϵ^{-4}. Conversely, the variance of $f_i \diamond h_{U_\epsilon}$ is a sum including all the covariance terms $\mathrm{Cov}\left(h_{U_\epsilon}(u_\epsilon), h_{U_\epsilon}(z_\epsilon)\right)$ where $u_\epsilon, z_\epsilon \in U_\epsilon$ are microscopically close to each other. These covariance terms are $O(1)$. Thus, to obtain a finite variance in Theorem 5, we have to rescale the covariance by ϵ^{-2}. As a consequence, the limit in Theorem 5 ignores the way the fluctuations of the height one variables are spatially coupled.

3.2.3 Scaling limit for the height one joint cumulants

We now give the representation for $E_U(v : v \in V)$. The explicit formula is given in terms of the scaling limits for the height one joint cumulants.

Definition 29 (Cumulants)**.** Let X be a random variable with all moments finite. We define the cumulants $\kappa_n(X)$, $n \in \mathbb{N}$, to be the Taylor coefficients of the logarithm of the characteristic function:

$$\log \mathbb{E}\left[\exp(itX)\right] = \sum_{n=1}^{\infty} \kappa_n(X) \frac{(it)^n}{n!}$$

Given a finite family $(X_v)_{v\in V}$ of random variables with all moments finite, we write $\kappa\left(X_v : v \in V\right)$ to denote the joint cumulant of $(X_v)_{v\in V}$. That is,

$$\mathbb{E}\Big[\prod_{v\in V} X_v\Big] = \sum_{\Pi\in\Pi(V)} \prod_{B\in\Pi} \kappa\left(X_v : v \in B\right), \tag{3.2}$$

where

$$\Pi(V) := \left\{ \{A_1, \cdots, A_n\} \,\middle|\, \begin{array}{c} n\in\mathbb{N}, \forall 1 \leq i \neq j \leq n : \emptyset \neq A_i \subset V, \\ A_i \cap A_j = \emptyset, \cup_{l=1}^n A_l = V \end{array} \right\}$$

denotes the set of partitions of V.

Remark 8. Let $(X_v)_{v\in V}$ as in Definition 29. We note that (3.2) uniquely defines the joint cumulants: suppose $|V| = 1$, that is, $V = \{v\}$ for some v. Then (3.2) implies $\kappa(X_v) = \mathbb{E}[X_v]$. In case of $|V| = 2$, $V = \{v, w\}$, then $\kappa(X_v) = \mathbb{E}[X_v]$, $\kappa(X_w) = \mathbb{E}[X_w]$ and (3.2) imply

$$\kappa(X_v, X_w) = \mathbb{E}[X_v \cdot X_w] - \mathbb{E}[X_w] \cdot \mathbb{E}[X_v].$$

Proceeding by an induction based on (3.2), we obtain the assertion.

Remark 9. Let X be a random variable with all moments finite, and write $X_i := X$ for all $i \in \mathbb{N}$. Faá die Bruno's Formula provides the following relation between cumulants and moments (see e.g. [20]):

$$\mathbb{E}(X^n) = \sum_{\Pi\in\Pi(\{1,\dots,n\})} \prod_{B\in\Pi} \kappa_{|B|}(X), \qquad n \in \mathbb{N}$$

Hence, using an induction on $n \in \mathbb{N}$ it follows $\kappa(X_i : 1 \le i \le n) = \kappa_n(X)$. Finally, we note that joint cumulants are multilinear.

To express the scaling limit for the height one joint cumulants, we use the continuous Green's function. We write ∂_x and ∂_y to denote the derivative in direction of the real, respectively imaginary axis. For a function $f : U^k \mapsto \mathbb{R}$ let $\partial_x^{(i)} f$ denote the ∂_x-derivative of f as a function of the ith variable, provided it exists. Similarly we define $\partial_y^{(i)}$ and write $\Delta := \big(\partial_x^{(1)}\big)^2 + \big(\partial_y^{(1)}\big)^2$ to denote the continuous Laplacian in $\mathbb{C}$.

Definition 30 (The continuous Green's function)**.** Let g_U denote the continuous Green's function on U. That is, g_U is the real valued function satisfying $-\Delta g_U(\,\cdot, w) = \delta_w$ on $U \times U$ in the sense of distributions, and which is zero when v is on the boundary of U.

To state the explicit formula for $E_U\,(v : v \in V)$, we need one more definition.

Definition 31 (Cycles)**.** Let V be a finite set. We write

$$S(V) := \{\phi : V \mapsto V \big| \phi \text{ bijective}\}$$

to denote the set permutations of V, and

$$S_{\text{cycl}}(V) := \big\{\sigma \in S(V) \big| \forall \emptyset \ne P \subsetneq V : \sigma(P) \ne P\big\}$$

for the full cycles of V. Here $\sigma(P) := \cup_{p\in P}\{\sigma(p)\}$ is the image of P under σ.

Theorem 6 (Scaling limit for the height one joint cumulants)**.** *Let V as in Theorem 4, and suppose $|V| \ge 2$. Then as $\epsilon \to 0$ the rescaled joint cumulant $\epsilon^{-2|V|}\kappa\big(h_{U_\epsilon}(v_\epsilon) : v \in V\big)$ converges to*

$$\kappa_U(v : v \in V) := -C^{|V|} \sum_{\sigma\in S_{\text{cycl}}(V)} \sum_{(k^v)_{v\in V}\in\{x,y\}^V} \prod_{v\in V} \partial^{(1)}_{k^v} \partial^{(2)}_{k^{\sigma(v)}} g_U\,(v, \sigma(v))\,.$$

Here $C := (2/\pi) - (4/\pi^2) = \pi \cdot \mathbb{E}\,[h_0(0)]$.

Remark 10. Let V as in Theorem 4, and write $\kappa_U(v) := 0$ for $v \in V$. Then Theorem 6 along with (3.2) yields

$$E_U(v : v \in V) := \lim_{\epsilon \to 0} \epsilon^{-2|V|} \mathbb{E}\Big[\prod_{v \in V} \big(h_{U_\epsilon}(v_\epsilon) - \mathbb{E}[h_{U_\epsilon}(v_\epsilon)]\big)\Big]$$
$$= \sum_{\Pi \in \Pi(V)} \prod_{B \in \Pi} \kappa_U(v : v \in B).$$

The further organization of this chapter is as follows. Our results are based on an expression for the height one joint cumulants in terms of differences of discrete Green's functions. In Section 3.3 (The height one field in finite volume) we use a correspondence of sandpile models and spanning trees, and the matrix tree theorem to derive this expression. Thereafter, we use the theory of harmonic functions to study the asymptotics of the Green's function differences in Section 3.4 (Green's function asymptotics). We combine results of Sections 3.3 and 3.4 to conclude Theorems 4 and 6 in Section 3.5 (Scaling limit for the height one joint cumulants). Independently of the proof of Theorems 4 and 6, using the method of moments and results of Sections 3.3 and 3.4 we prove Theorem 5 in Section 3.6 (Scaling limit for the sandpile height one field).

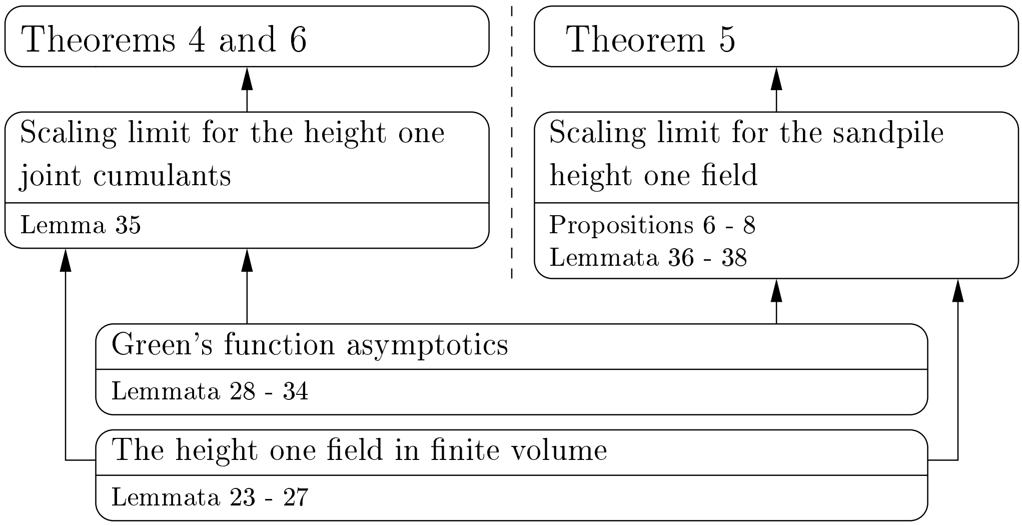

Figure 3.1: Proof of Theorems 4 - 6

3.3 The height one field in finite volume

In this section we study the height one field for finite $\Lambda \subset \mathbb{Z}^2$. In the first part of this section we are going to recall a characterization of recurrent configurations. Thereafter, we use the burning test to calculate height one probabilities for the sandpile model corresponding to the discrete Laplacian Δ_Λ. Finally, in the last part of this section we combinatory decompose the height one probabilities to derive expressions for the height one joint cumulants.

We write $\overline{\Lambda} := \Lambda \cup \partial\Lambda$, where

$$\partial\Lambda := \{v \in \mathbb{Z}^2 \setminus \Lambda | \exists w \in \Lambda : |v - w| = 1\}$$

denotes the set of those sites in the complement of Λ that have a distance-one-neighbour in Λ.

3.3.1 The burning test

We recall a characterization of recurrent configurations which was first discovered by D. Dhar and S. N. Majumdar in [8] and [24]. Let Δ be a toppling matrix on Λ. Then a configuration $\eta = (\eta_v)_{v\in\Lambda} \in \Omega_\Delta$ is recurrent with respect to Δ, if and only if it passes the following burning test. For all $V \subset \Lambda$ and all $v \in V$ we say that the site v is burnable in V, if

$$\eta_v > - \sum_{w \in V\setminus\{v\}} \Delta(v, w).$$

In the first step of the burning test, burn the set V_1 of those sites $v \in \Lambda$ that are burnable in Λ. Iterate this procedure with $\Lambda_1 := \Lambda\setminus V_1$ and burn the sites $v \in \Lambda_1$ that are burnable in Λ_1, and so on. If and only if at the end all sites are burned, the configuration passes the burning test, that is, is recurrent with respect to Δ.

More formally, a configuration $\eta \in \Omega_\Delta$ is recurrent with respect to Δ, if and only if it is Λ-burnable as follows.

Definition 32 (The burning test)**.** Let $C \subset \Lambda$ and $\eta \in \prod_{v\in C}\{1, \ldots, \Delta(v, v)\}$. Then η is C-burnable with respect to Δ, if there exists a bijection $v : \{1, \ldots, |C|\} \mapsto C$ as follows: for every $1 \le j \le |C|$ the site $v(j)$ is burnable in $C_j := C \setminus \{v(i), 1 \le i \le j - 1\}$, that is,

$$\eta_{v(j)} > - \sum_{w\in C_j\setminus\{v(j)\}} \Delta(v(j), w).$$

We now consider the sandpile model that corresponds to the discrete Laplacian Δ_Λ. The burning test implies the following Lemma, which appears in a similar version in [21].

Lemma 23. *Let $v \in \Lambda$ and $C \subset \Lambda\setminus\{v\}$ such that $\mathcal{D}_v := \{v\pm1, v\pm i, v\pm1\pm i\} \subset C$. Fix an arbitrary configuration $\sigma_C = (\sigma_C(w))_{w\in C} \in \Omega_C := \{1, 2, 3, 4\}^C$ so that $\mu_\Lambda(\eta_C = \sigma_C) > 0$, and $\sigma_C(w) = 4$ for all $w \in \mathcal{D}_v$. Here $\eta_C := (\eta_w)_{w\in C}$. Then for all $A \subset \Omega_{\{v\}}$, and all events $B \subset \Omega_{\Lambda\setminus\{v\}}$ that depend on the configuration on $v^c := \Lambda \setminus \{v\}$ only,*

$$\mu_\Lambda(\eta_v \in A, \eta_{v^c} \in B|\eta_C = \sigma_C) = \frac{|A|}{4} \cdot \mu_\Lambda(\eta_{v^c} \in B|\eta_C = \sigma_C).$$

Proof. Let v, C and σ_C as in the lemma, and let $\eta \in \Omega_\Lambda$ so that $\eta_C = \sigma_C$. To burn the site v, we have to burn a neighbour of v first. All sites of $\mathcal{D}_v$ have maximal height four. That is, if we are able to burn a neighbour of v, we can burn the entire set $\mathcal{D}_v$ without burning any further site of $\Lambda \setminus \mathcal{D}_v$. After burning all neighbours of v, we are able to burn the site v independently of its height. That is, η belongs to $\mathcal{R}_\Lambda$, if and only if η_{v^c} is v^c-burnable with respect to Δ_Λ. It follows

$$\begin{aligned}\mu_\Lambda(\eta_v \in A, \eta_{v^c} \in B | \eta_C = \sigma_C) &= \frac{\sum_{\eta \in \mathcal{R}_\Lambda} 1_{\{\eta_v \in A\}} \cdot 1_{\{\eta_{v^c} \in B, \eta_C = \sigma_C\}}}{\sum_{\eta \in \mathcal{R}_\Lambda} 1_{\{\eta_C = \sigma_C\}}} \\ &= \frac{\sum_{\eta_v \in \Omega_v} 1_{\{\eta_v \in A\}} \cdot \sum_{\tilde{\eta} \in \mathcal{R}^v_\Lambda} 1_{\{\tilde{\eta}_{v^c} \in B, \tilde{\eta}_C = \sigma_C\}}}{\sum_{\eta \in \mathcal{R}_\Lambda} 1_{\{\eta_C = \sigma_C\}}} \\ &= \frac{|A|}{|\Omega_v|} \cdot \mu_\Lambda(\eta_{v^c} \in B | \eta_C = \sigma_C),\end{aligned}$$

where $\mathcal{R}^v_\Lambda \subset \Omega_{v^c}$ denotes the set of sub configurations that are v^c-burnable with respect to Δ_Λ. □

3.3.2 Height one probabilities

We use the burning test to calculate height one probabilities for the sandpile model corresponding to the discrete Laplacian Δ_Λ. We start with a characterization for the height one field.

We glue the sites (vertices) of $\partial\Lambda$ together to be one site ν, and write $\Lambda^\nu := \Lambda \cup \{\nu\}$. That is, $|\nu - \nu| = 0$, and for all $v \in \Lambda$, $|\nu - v| = |v - \nu| = \min_{w \in \partial\Lambda} |v - w|$. To denote the set of paths that connect two sites $v, w \in \Lambda^\nu$, we write

$$\mathrm{PATH}(v, w) := \left\{ \{v_i, 1 \le i \le n\} \subset \Lambda^\nu \,\middle|\, \begin{array}{c} n \in \mathbb{N}, v_1 = v, v_n = w, \\ \forall 1 < i \le n : |v_{i-1} - v_i| = 1 \end{array} \right\}.$$

Lemma 24 (Characterization of the height one field)**.** *The probability of having height one at each site of a set $V \subset \Lambda$, that is,*

$$\mathbb{E}\Big[\prod_{v \in V} h_\Lambda(v) \Big]$$

is non-zero if and only if

(i) the set V does not contain any neighbours: $|v - w| \ne 1$ for all $v, w \in V$;

(ii) for every site $v \in \Lambda \setminus V$ there exists a path $P \in \mathrm{PATH}(v, \nu)$ so that P and V are disjoint.

Proof. Suppose we have height one at two neighbours $v, w \in \Lambda$, $|v - w| = 1$. Then from the burning test, to burn the site v, we have to burn all neighbours of v first, in particular the site w. Conversely, to burn the site w, we have to burn the site v first.

Hence, a configuration that has height one on v and w is not burnable. That is, such a configuration is not recurrent and occurs with probability zero.

Let $V \subset \Lambda$ so that $|v-w| \neq 1$ for all $v, w \in V$. Suppose that there exists a site $v \in \Lambda \setminus V$ with the property that for all $P \in \mathrm{PATH}(v, \nu)$ we have $P \cap V \neq \emptyset$. Then $W := \{w \in \Lambda | \exists P \in \mathrm{PATH}(v, w) : P \cap V = \emptyset\}$ satisfies $\partial W \subset V$. We suppose that all sites of the set ∂W have height one. Then to burn a site of ∂W, we have to burn a site of W first. But to burn a site of W, we have to burn a site of ∂W first. It follows that every configuration that has height one on ∂W is not Λ-burnable with respect to Δ_Λ, that is, occurs with probability zero. In particular, every configuration that has height one on V occurs with probability zero.

We suppose that for all $v \in \Lambda \setminus V$ there exists $P_v \in \mathrm{PATH}(v, \nu)$ so that $P_v \cap V = \emptyset$. Then the configuration η that has height four on $\Lambda \setminus V$ and height one on V is Λ-burnable with respect to Δ_Λ as follows. First we burn every site $w \in \Lambda \setminus V$ using the path P_w that connects w to the site ν. The set V does not contain any neighbours. Thus after burning the set $\Lambda \setminus V$, we can burn the entire set V. That is, η is Λ-burnable with respect to Δ_Λ and occurs with probability $1/|\mathcal{R}_\Lambda|$. □

In [24] S. N. Majumdar and D. Dhar use determinantal formulas to express the probabilities of certain height configurations in stationary state. We use their method to obtain an explicit expression for height one probabilities. The representation is in terms of differences of the Green's function on Λ.

Definition 33 (The Green's function on Λ)**.** We define $G_\Lambda \in \mathbb{R}^{\Lambda \times \Lambda}$ through $G_\Lambda := \Delta_\Lambda^{-1}$, and call G_Λ the Green's function on Λ.

Definition 34 (The difference operators)**.** Let $V \subset \mathbb{Z}^2$, and for all $a + ib \in \mathbb{Z}^2$ let $V_{a+ib} := \{v \in V | v + a + ib \in V\}$. We define the difference operators

$$\begin{aligned} \partial_x^{(1)} &: \mathbb{C}^V \times \mathbb{C}^V \mapsto \mathbb{C}^{V_1} \times \mathbb{C}^V, & \partial_x^{(1)} f(v,w) &:= f(v+1,w) - f(v,w), \\ \partial_{-x}^{(1)} &: \mathbb{C}^V \times \mathbb{C}^V \mapsto \mathbb{C}^{V_{-1}} \times \mathbb{C}^V, & \partial_{-x}^{(1)} f(v,w) &:= f(v-1,w) - f(v,w), \end{aligned}$$

and

$$\begin{aligned} \partial_y^{(1)} &: \mathbb{C}^V \times \mathbb{C}^V \mapsto \mathbb{C}^{V_i} \times \mathbb{C}^V, & \partial_y^{(1)} f(v,w) &:= f(v+i,w) - f(v,w), \\ \partial_{-y}^{(1)} &: \mathbb{C}^V \times \mathbb{C}^V \mapsto \mathbb{C}^{V_{-i}} \times \mathbb{C}^V, & \partial_{-y}^{(1)} f(v,w) &:= f(v-i,w) - f(v,w). \end{aligned}$$

Similarly, we define $\partial_x^{(2)}$, $\partial_y^{(2)}$, $\partial_{-x}^{(2)}$ and $\partial_{-y}^{(2)}$ with respect to the second variable.

Lemma 25 (Height one probabilities)**.** *Let $V \subset \Lambda$ such that for all $v, v' \in V$, $|v-v'| \neq 1$ and $\partial v \subset \Lambda$. Then the probability of having height one at each site of V satisfies*

$$\mathbb{E}\left[\prod_{v \in V} h_\Lambda(v)\right] = \det\left(1_{\{v=v'\}} - K_\Lambda(v,v')\right)_{v,v' \in V},$$

where

$$K_\Lambda(v,v') := \begin{pmatrix} \partial_x^{(1)}\partial_x^{(2)}G_\Lambda(v,v') & \partial_x^{(1)}\partial_{-x}^{(2)}G_\Lambda(v,v') & \partial_x^{(1)}\partial_y^{(2)}G_\Lambda(v,v') \\ \partial_{-x}^{(1)}\partial_x^{(2)}G_\Lambda(v,v') & \partial_{-x}^{(1)}\partial_{-x}^{(2)}G_\Lambda(v,v') & \partial_{-x}^{(1)}\partial_y^{(2)}G_\Lambda(v,v') \\ \partial_y^{(1)}\partial_x^{(2)}G_\Lambda(v,v') & \partial_y^{(1)}\partial_{-x}^{(2)}G_\Lambda(v,v') & \partial_y^{(1)}\partial_y^{(2)}G_\Lambda(v,v') \end{pmatrix},$$

and $1_{\{v=v'\}}$ *denotes the product of the identity matrix and the indicator function of* $\{v=v'\}$.

Proof. Let V as in the lemma. We call $E := \{\{v,w\}\subset\Lambda^\nu \,\big|\, |v-w|=1\}$ the set of edges. For all edges $\{v,w\}\in E$ let the weight $x_{\Delta_\Lambda}(\{v,w\})$ induced by Δ_Λ be

$$x_{\Delta_\Lambda}(\{v,w\}) := \begin{cases} -\Delta_\Lambda(v,w) & \text{if } v,w\in\Lambda; \\ \sum_{z'\in\Lambda}\Delta_\Lambda(z,z') & \text{if } \{v,w\}=\{z,\nu\} \text{ for a } z\in\Lambda. \end{cases}$$

We modify the weights induced by Δ_Λ as follows. For every $v\in V$ we decrease the weight of the three edges connecting the site v to its neighbours in $\mathcal{N}_i(v) := \{v\pm 1, v+i\}$ by one. That is, we modify the toppling matrix by setting

$$\Delta_G := \Delta_\Lambda + \sum_{v\in V} B_v,$$

where

$$B_v(u,w) := \begin{cases} -3 & \text{if } v=u=w; \\ -1 & \text{if } u=w\in\mathcal{N}_i(v); \\ 1 & \text{if } \{u,w\}=\{v,v'\} \text{ for a } v'\in\mathcal{N}_i(v); \\ 0 & \text{otherwise.} \end{cases}$$

Here we note that from $|v-v'|\neq 1$ for $v,v'\in V$, we do not decrease the weight of the same edge two times. Thus for all $v\in V$ and all $v'\in\mathcal{N}_i(v)$ the weight induced by Δ_G satisfies $x_{\Delta_G}(\{v,v'\}) = -\Delta_G(v,v') = 0$.

In the first step we show

$$\mathbb{E}\Big[\prod_{v\in V} h_\Lambda(v)\Big] = \frac{\det(\Delta_G)}{\det(\Delta_\Lambda)} = \det\Big(1 + G_\Lambda\cdot\sum_{v\in V} B_v\Big), \tag{3.3}$$

where $\Delta_\Lambda^{-1} = G_\Lambda$ implies the second equality.

Suppose that there exists $v_1\in\Lambda\setminus V$ such that for all $P\in\mathrm{PATH}(v_1,\nu)$ we have $P\cap V\neq\emptyset$. Then from Lemma 24 (Characterization of the height one field) the left hand side of (3.3) equals zero. To show that the same holds for $\det(\Delta_G)$, we use the matrix tree theorem. From the matrix tree theorem (see e.g. [32]), $\det(\Delta_G)$ is the Δ_G-weighted number of spanning trees of $\Lambda\cup\nu$:

$$\det(\Delta_G) = \sum_{T\in\mathcal{T}}\prod_{\{v,w\}\in T} x_{\Delta_G}(\{v,w\})$$

Here $\mathcal{T}$ is the set of spanning trees of $\Lambda \cup \nu$, were a spanning tree is viewed as a subset of E. Let $T \in \mathcal{T}$ be spanning tree of $\Lambda \cup \nu$, and $B_T = \big\{\{v_i, v_{i+1}\}, 1 \le i \le n\big\} \subset T$, $v_i \ne v_j$ for all $1 \le i \ne j \le n+1$, be its branch that connects the site v_1 to the site $v_{n+1} = \nu$. We note that $P_B := \{v_i, 1 \le i \le n+1\}$ satisfies $P_B \in \mathrm{PATH}(v_1, \nu)$. Hence, our choice of v_1 implies $P_B \cap V = \emptyset$. That is, there exists $2 \le j \le n$ so that $v_j \in V$. Along with $v_{j-1} \ne v_{j+1}$ it follows that the branch B, and hence the spanning tree T contains an edge $\{v, v'\}$ that connects a site $v \in V$ to a site $v' \in \mathcal{N}_i(v)$. Such an edge has Δ_G-weight $x_{\Delta_G}(\{v, v'\}) = 0$. We conclude $\det(\Delta_G) = 0$.

Suppose that for every $v \in \Lambda \setminus V$ there exists $P \in \mathrm{PATH}(v, \nu)$ so that P and V are disjoint. Then Δ_G is a toppling matrix, and the set of configurations that are recurrent with respect to Δ_G satisfies $|\mathcal{R}_G| = \det(\Delta_G)$. We write $\phi : \mathbb{Z}^\Lambda \mapsto \mathbb{Z}^\Lambda$ for the map that is defined by successively for all $v \in V$ decreasing the height by one at all sites of $\mathcal{N}_i(v)$. More formally, for all $\eta = (\eta_w)_{w \in \Lambda} \in \mathbb{Z}^\Lambda$ let

$$\phi(\eta) := \left(\eta_w - \sum_{v \in V} 1_{\{w \in \mathcal{N}_i(v)\}}\right)_{w \in \Lambda}.$$

From the burning test, in a recurrent configuration a site with height k has less than k neighbours with height one. This implies $\phi(\eta) \in \Omega_{\Delta_G}$ for $\eta \in \mathcal{R}_{\Lambda,V}$, where $\mathcal{R}_{\Lambda,V}$ denotes the set of configurations in $\mathcal{R}_\Lambda$ that have height one on V. Furthermore, it is easy to see that each sequence that burns a configuration $\eta \in \mathcal{R}_{\Lambda,V}$ with respect to Δ_Λ, burns $\phi(\eta)$ with respect Δ_G. Hence, $\phi(\eta) \in \mathcal{R}_G$ for $\eta \in \mathcal{R}_{\Lambda,V}$. Along with similar considerations for ϕ^{-1}, it follows that ϕ defines a one-to-one mapping of $\mathcal{R}_{\Lambda,V}$ onto $\mathcal{R}_G$. Therefore, we have

$$\mathbb{E}\Big[\prod_{v \in V} h_\Lambda(v)\Big] = \frac{|\mathcal{R}_{\Lambda,V}|}{|\mathcal{R}_\Lambda|} = \frac{|\mathcal{R}_G|}{|\mathcal{R}_\Lambda|} = \frac{\det(\Delta_G)}{\det(\Delta_\Lambda)}.$$

This concludes the proof of (3.3).

For a shorter notation, we suppress the dependence on Λ and write $G(v,w)$ instead of $G_\Lambda(v,w)$ in the following. Using elementary row and column operations (see Remark 11 below) it follows

$$\det\left(1 + G_\Lambda \cdot \sum_{v \in V} B_v\right) = \det\left(1_{\{v=v'\}} + G_{v,v'} \cdot B\right)_{v,v' \in V}, \qquad (3.4)$$

where

$$B := \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

and $G_{v,v'}$ denotes

$$\begin{pmatrix} G(v,v') & G(v,v'+1) & G(v,v'-1) & G(v,v'+i) \\ G(v+1,v') & G(v+1,v'+1) & G(v+1,v'-1) & G(v+1,v'+i) \\ G(v-1,v') & G(v-1,v'+1) & G(v-1,v'-1) & G(v-1,v'+i) \\ G(v+i,v') & G(v+i,v'+1) & G(v+i,v'-1) & G(v+i,v'+i) \end{pmatrix}.$$

In the next step, adding the second, third and forth column to the first column, and subtracting the first row from the second third and forth row in each block, we obtain

$$\det\left(1_{\{v=v'\}} + G_{v,v'} \cdot B\right)_{v,v'\in V} = \det\left(1_{\{v=v'\}} - \tilde{G}_{v,v'} \cdot \tilde{B}\right)_{v,v'\in V}.$$

Here $\tilde{G}_{v,v'}$ denotes

$$\begin{pmatrix} G(v,v') & G(v,v'+1) & G(v,v'-1) & G(v,v'+i) \\ \partial_x^{(1)} G(v,v') & \partial_x^{(1)} G(v,v'+1) & \partial_x^{(1)} G(v,v'-1) & \partial_x^{(1)} G(v,v'+i) \\ \partial_{-x}^{(1)} G(v,v') & \partial_{-x}^{(1)} G(v,v'+1) & \partial_{-x}^{(1)} G(v,v'-1) & \partial_{-x}^{(1)} G(v,v'+i) \\ \partial_y^{(1)} G(v,v') & \partial_y^{(1)} G(v,v'+1) & \partial_y^{(1)} G(v,v'-1) & \partial_y^{(1)} G(v,v'+i) \end{pmatrix}$$

and

$$\tilde{B} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, the product of $\tilde{G}_{v,v'}$ and $\tilde{B}$ equals

$$\begin{pmatrix} 0 & \partial_x^{(2)} G(v,v') & \partial_{-x}^{(2)} G(v,v') & \partial_y^{(2)} G(v,v') \\ 0 & \partial_x^{(1)}\partial_x^{(2)} G(v,v') & \partial_x^{(1)}\partial_{-x}^{(2)} G(v,v') & \partial_x^{(1)}\partial_y^{(2)} G(v,v') \\ 0 & \partial_{-x}^{(1)}\partial_x^{(2)} G(v,v') & \partial_{-x}^{(1)}\partial_{-x}^{(2)} G(v,v') & \partial_{-x}^{(1)}\partial_y^{(2)} G(v,v') \\ 0 & \partial_y^{(1)}\partial_x^{(2)} G(v,v') & \partial_y^{(1)}\partial_{-x}^{(2)} G(v,v') & \partial_y^{(1)}\partial_y^{(2)} G(v,v') \end{pmatrix}.$$

This shows

$$\det\left(1_{\{v=v'\}} - \tilde{G}_{v,v'} \cdot \tilde{B}\right)_{v,v'\in V} = \det\left(1_{\{v=v'\}} - K_\Lambda(v,v')\right)_{v,v'\in V}.$$

Along with (3.3) and (3.4) this concludes the proof of the lemma. □

We sketch the row and column operations underlying (3.4).

Remark 11. Let $V, W \subset \Lambda$ so that $V \cup W = \Lambda$. Let $(G_{v,w})_{v,w\in\Lambda} \in \mathbb{R}^{\Lambda\times\Lambda}$, $(A_{v,w})_{v,w\in V} \in \mathbb{R}^{V\times V}$ and $(B_{v,w})_{v,w\in W} \in \mathbb{R}^{W\times W}$. For $U, U' \subset \Lambda$ we abbreviate $G_{U,U'} := (G_{v,w})_{v\in U, w\in U'}$. We write $X := V \setminus W$, $Y := V \cap W$, $Z := W \setminus V$ and

$$E := \begin{pmatrix} G_{X,X} & G_{X,Y} & G_{X,Z} & G_{X,Y} \\ G_{Y,X} & G_{Y,Y} & G_{Y,Z} & G_{Y,Y} \\ G_{Z,X} & G_{Z,Y} & G_{Z,Z} & G_{Z,Y} \\ G_{Y,X} & G_{Y,Y} & G_{Y,Z} & G_{Y,Y} \end{pmatrix} \cdot \begin{pmatrix} A_{X,X} & A_{X,Y} & 0 & 0 \\ A_{Y,X} & A_{Y,Y} & 0 & 0 \\ 0 & 0 & B_{Z,Z} & B_{Z,Y} \\ 0 & 0 & B_{Y,Z} & B_{Y,Y} \end{pmatrix}.$$

Elementary row and column operations that leave the unit matrix unchanged transform E into

$$\tilde{E} := \begin{pmatrix} G_{X,X} & G_{X,Y} & G_{X,Z} & G_{X,Y} \\ G_{Y,X} & G_{Y,Y} & G_{Y,Z} & G_{Y,Y} \\ G_{Z,X} & G_{Z,Y} & G_{Z,Z} & G_{Z,Y} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} A_{X,X} & A_{X,Y} & 0 & 0 \\ A_{Y,X} & A_{Y,Y} & 0 & 0 \\ 0 & B_{Z,Y} & B_{Z,Z} & B_{Z,Y} \\ 0 & B_{Y,Y} & B_{Y,Z} & B_{Y,Y} \end{pmatrix}.$$

The matrix $\tilde{E}$ equals

$$\begin{pmatrix} G_{X,X} & G_{X,Y} & G_{X,Z} & 0 \\ G_{Y,X} & G_{Y,Y} & G_{Y,Z} & 0 \\ G_{Z,X} & G_{Z,Y} & G_{Z,Z} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} A_{X,X} & A_{X,Y} & 0 & 0 \\ A_{Y,X} & A_{Y,Y}+B_{Y,Y} & B_{Y,Z} & B_{Y,Y} \\ 0 & B_{Z,Y} & B_{Z,Z} & B_{Z,Y} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows $\det(1+E) = \det(1+F)$, where

$$F := \begin{pmatrix} G_{X,X} & G_{X,Y} & G_{X,Z} \\ G_{Y,X} & G_{Y,Y} & G_{Y,Z} \\ G_{Z,X} & G_{Z,Y} & G_{Z,Z} \end{pmatrix} \cdot \begin{pmatrix} A_{X,X} & A_{X,Y} & 0 \\ A_{Y,X} & A_{Y,Y}+B_{Y,Y} & B_{Y,Z} \\ 0 & B_{Z,Y} & B_{Z,Z} \end{pmatrix}.$$

Remark 12. The idea to consider the modified matrix Δ_G to calculate the probabilities of specific sub configurations is due to S. N. Majumdar and D. Dhar [24]. G. Piroux and P. Ruelle extended their method in [28].

3.3.3 Height one joint cumulants

We now combinatory decompose the height one joint moments into the height one joint cumulants.

Our presentation for the height one joint moments has a block-determinantal structure. The block indexed by the sites $v, w \in \Lambda$ is the three by three matrix

$$\left(1_{\{v^i=w^j\}} - \partial_i^{(1)}\partial_j^{(2)} G_\Lambda(v,w)\right)_{i,j\in\{x,-x,y\}}.$$

For notational reasons for every $v \in \Lambda$ let v^x, v^{-x} and v^y denote three distinguishable copies of v, and write

$$\left(k_\Lambda\left(v^i, w^j\right)\right)_{i,j\in\{x,-x,y\}} := \left(1_{\{v^i=w^j\}} - \partial_i^{(1)}\partial_j^{(2)} G_\Lambda(v,w)\right)_{i,j\in\{x,-x,y\}}.$$

Here x, $-x$ and y are simple indexes. For $V \subset \Lambda$ let $V^{\overline{x}y} := \bigcup_{v\in V}\{v^x, v^{-x}, v^y\}$ and write

$$S_{\text{cycl}}^{\overline{x}y}(V) := \left\{\sigma \in S(V^{\overline{x}y}) \middle| \forall \emptyset \neq P \subsetneq V : \sigma\left(P^{\overline{x}y}\right) \neq P^{\overline{x}y}\right\}$$

for the set of permutations of $V^{\overline{x}y}$ that do not operate as a permutation on $P^{\overline{x}y}$ for a proper non-empty subset P of V. In our definition of the sets $V^{\overline{x}y}$ and $S_{\text{cycl}}^{\overline{x}y}(V)$ the index $\overline{x}y$ denotes that they are defined with respect to the three copies v^x, v^{-x} and v^y each $v \in V$.

During the proof of Theorem 4 we are going to introduce two distinguishable copies (v,x) and (v,y) for every site $v \in \Lambda$. Again x and y will be simple indexes. As equivalents to the sets $V^{\overline{x}y}$ and $S_{\text{cycl}}^{\overline{x}y}(V)$ we are going to define V^{xy} and $S_{\text{cycl}}^{xy}(V)$, where the index xy denotes that we are in the situation of two copies (v,x) and (v,y) each $v \in V$.

We start with the height one joint cumulants for sets $V \subset \Lambda$ where all sites have a distance greater than one.

Lemma 26 (Height one joint cumulants)**.** *Let $V \subset \Lambda$ as in Lemma 25. Then the joint cumulant $\kappa\left(h_\Lambda(v) : v \in V\right)$ satisfies*

$$\kappa\left(h_\Lambda(v) : v \in V\right) = \sum_{\sigma \in S^{\overline{x}y}_{\mathrm{cycl}}(V)} \mathrm{sign}(\sigma) \prod_{v \in V^{\overline{x}y}} k_\Lambda\left(v, \sigma(v)\right). \tag{3.5}$$

Proof. Let V as in the Lemma 25. The proof of the lemma is based on the representation of the height one joint moments shown in Lemma 25 (Height one probabilities), and the decomposition of joint moments into joint cumulants (3.2).

For $P \subset V$ and $\sigma \in S(V^{\overline{x}y})$ we write $\sigma_P : P^{\overline{x}y} \mapsto V^{\overline{x}y}$ to denote σ restricted to $P^{\overline{x}y}$. For $\Pi \in \Pi(V)$ let

$$S^{\overline{x}y}_\Pi(V) := \left\{\sigma \in S(V^{\overline{x}y}) \middle| \forall P \in \Pi : \sigma_P \in S^{\overline{x}y}_{\mathrm{cycl}}(P)\right\}.$$

Then

$$S(V^{\overline{x}y}) = \sum_{\Pi \in \Pi(V)} S^{\overline{x}y}_\Pi(V),$$

where $\sum$ denotes the disjoint union. Hence, Lemma 25 (Height one probabilities) implies

$$\begin{aligned}
\mathbb{E}\left[\prod_{v \in V} h_\Lambda(v)\right] &= \sum_{\sigma \in S(V^{\overline{x}y})} \mathrm{sign}(\sigma) \prod_{v \in V^{\overline{x}y}} k_\Lambda\left(v, \sigma(v)\right) \\
&= \sum_{\Pi \in \Pi(V)} \sum_{\sigma \in S^{\overline{x}y}_\Pi(V)} \mathrm{sign}(\sigma) \prod_{v \in V^{\overline{x}y}} k_\Lambda\left(v, \sigma(v)\right) \\
&= \sum_{\Pi \in \Pi(V)} \prod_{P \in \Pi} \sum_{\sigma \in S^{\overline{x}y}_{\mathrm{cycl}}(P)} \mathrm{sign}(\sigma) \prod_{v \in P^{\overline{x}y}} k_\Lambda\left(v, \sigma(v)\right).
\end{aligned} \tag{3.6}$$

Using this, an induction on $|V|$ and (3.2) yields the lemma. First suppose $|V| = 1$, that is, $V = \{v\}$ for some $v \in \Lambda$. Then from (3.2) it holds $\mathbb{E}[h_\Lambda(v)] = \kappa(h_\Lambda(v))$. Hence, (3.6) implies (3.5). In the induction step $n \to n+1$ suppose that (3.5) holds for all $V \subset \Lambda$ so that $|V| \leq n$. Let $V \subset \Lambda$ so that $|V| = n+1$. Then from (3.2)

$$\kappa\left(X_v : v \in V\right) = \mathbb{E}\left[\prod_{v \in V} X_v\right] - \sum_{\substack{\Pi \in \Pi(V) \\ \Pi \neq \{V\}}} \prod_{P \in \Pi} \kappa\left(X_v : v \in P\right). \tag{3.7}$$

For all $\Pi \in \Pi(V)$ so that $\Pi \neq \{V\}$, for all $B \in \Pi$ it holds $|B| \leq n$. Thus, we can use the induction hypothesis to express the cumulants that occur on the right hand site of (3.7). Comparing with (3.6) yields (3.5). □

In the next lemma we consider the height one joint cumulants that are not covered by Lemma 26. For all $n \in \mathbb{N}$ we write $\langle n \rangle := \{i, 1 \leq i \leq n\}$.

Lemma 27 (Height one joint cumulants including neighbours)**.** *Let $n \in \mathbb{N}$, $n \geq 2$, and for all $1 \leq i \leq n$ let $v_i \in \Lambda$. For $B \subset \langle n \rangle$ we write $\kappa\left(B\right) := \kappa\left(h_\Lambda(v_i) : i \in B\right)$.*

(i) If $v_2 \in \partial v_1$*, then*

$$\kappa(\langle n\rangle) = -\sum_{P\subset\langle n\rangle\setminus\{1,2\}} \kappa(P\cup\{1\})\cdot\kappa(\langle n\rangle\setminus\{P\cup\{1\}\}).$$

(ii) If $v_1 = v_2$*, then*

$$\kappa(\langle n\rangle) = \kappa(\langle n\rangle\setminus\{1\}) - \sum_{P\subset\langle n\rangle\setminus\{1,2\}} \kappa(P\cup\{1\})\cdot\kappa(\langle n\rangle\setminus\{P\cup\{1\}\}).$$

Here $\sum_{P\subset\langle n\rangle\setminus\{1,2\}}$ *denotes the sum over all* $P\subset\langle n\rangle\setminus\{1,2\}$ *including the empty set.*

Proof. The proof of the lemma is based on the decomposition of joint moments into joint cumulants (3.2), and Lemma 24 (Characterization of the height one field).

Let n and v_i, $1\le i\le n$, as in the lemma. For $P\subset\langle n\rangle\setminus\{1,2\}$ let $\Pi_P(\langle n\rangle)$ denote the set of those $\Pi\in\Pi(\langle n\rangle)$ such that either $\{1,2\}\cup P\in\Pi$, or there exists $P'\subset P$ so that $\{1\}\cup P'\in\Pi$ and $\{2\}\cup P\setminus P'\in\Pi$. Then $\Pi(\langle n\rangle)=\sum_{P\subset\langle n\rangle\setminus\{1,2\}}\Pi_P(\langle n\rangle)$. Hence, (3.2) implies

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^n h_\Lambda(v_i)\Big] &= \sum_{P\subset\langle n\rangle\setminus\{1,2\}}\ \sum_{\Pi\in\Pi_P(\langle n\rangle)}\ \prod_{B\in\Pi}\kappa(B)\\ &= K(\langle n\rangle\setminus\{1,2\}) + \sum_{P\subsetneq\langle n\rangle\setminus\{1,2\}}\ \sum_{\Pi\in\Pi(\langle n\rangle\setminus(\{1,2\}\cup P))}\ \prod_{B\in\Pi}\kappa(B)\cdot K(P),\end{aligned} \tag{3.8}$$

where $K(P) := \kappa(\{1,2\}\cup P) + \sum_{P'\subset P}\kappa(\{1\}\cup P')\cdot\kappa(\{2\}\cup P\setminus P')$.

Proof of (i). Suppose $v_2\in\partial v_1$. Then Lemma 24 (Characterization of the height one field) gives $\mathbb{E}[\prod_{i=1}^n h_\Lambda(v_i)] = 0$. Along with (3.8) it follows

$$0 = K(\langle n\rangle\setminus\{1,2\}) + \sum_{P\subsetneq\langle n\rangle\setminus\{1,2\}}\ \sum_{\Pi\in\Pi(\langle n\rangle\setminus(\{1,2\}\cup P))}\ \prod_{B\in\Pi}\kappa(B)\cdot K(P),$$

and an induction on $n\ge 2$ yields $K(\langle n\rangle\setminus\{1,2\})=0$. This is the first assertion.

Proof of (ii). Suppose $v_2 = v_1$. We show

$$K(\langle n\rangle\setminus\{1,2\}) = \kappa(\langle n\rangle\setminus\{1\})$$

by induction on $n\ge 2$. Note that $v_1 = v_2$ implies $h(v_1)\cdot h(v_2) = h(v_2)$, and hence $\mathbb{E}[\prod_{i=1}^n h_\Lambda(v_i)] = \mathbb{E}[\prod_{i=2}^n h_\Lambda(v_i)]$.

For $n=2$ from (3.8) we have

$$K(\langle 2\rangle\setminus\{1,2\}) = \mathbb{E}[h_\Lambda(v_1)\cdot h_\Lambda(v_2)] = \mathbb{E}[h_\Lambda(v_2)] = \kappa(h_\Lambda(v_2)) = \kappa(\langle 2\rangle\setminus\{1\}).$$

In the induction step $n-1 \to n$, we suppose that $K(\langle k\rangle \setminus \{1,2\}) = \kappa(\langle k\rangle \setminus \{1\})$ for all $2 \le k \le n-1$. Then $K(P) = \kappa(P \cup \{2\})$ for $P \subsetneq \langle n\rangle \setminus \{1,2\}$, and (3.8) yields

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{i=1}^{n} h_\Lambda(v_i)\Big] - K(\langle n\rangle \setminus \{1,2\}) + \kappa(\langle n\rangle \setminus \{1\}) \\
&= \sum_{P \subsetneq \langle n\rangle\setminus\{1,2\}} \sum_{\Pi\in\Pi(\langle n\rangle\setminus(\{1,2\}\cup P))} \prod_{B\in\Pi} \kappa(B)\cdot\kappa(P\cup\{2\}) + \kappa(\langle n\rangle \setminus \{1\}) \\
&= \sum_{\Pi\in\Pi(\langle n\rangle\setminus\{1\})} \prod_{B\in\Pi} \kappa(B) = \mathbb{E}\Big[\prod_{i=2}^{n} h_\Lambda(v_i)\Big].
\end{aligned}$$

Along with $\mathbb{E}[\prod_{i=1}^n h_\Lambda(v_i)] = \mathbb{E}[\prod_{i=2}^n h_\Lambda(v_i)]$ this concludes the induction step. □

3.4 Green's function asymptotics

In this section we study the Green's function differences that occur in our expression for the joint cumulants. We restrict our representation to the $\partial_y^{(1)}\partial_x^{(2)}$-difference. The same proofs yield similar results for the $\partial_x^{(1)}\partial_y^{(2)}$-, the $\partial_x^{(1)}\partial_x^{(2)}$- and the $\partial_y^{(1)}\partial_y^{(2)}$-difference. First we introduce the Classical Green's function on $\mathbb{Z}^2$, and recall it's asymptotic behaviour. We compare the $\partial_y^{(1)}\partial_x^{(2)}$-difference of the Classical Green's function and of the Green's function on U_ϵ, and use the theory of harmonic functions to estimate the difference. Thereafter, we use the derived results to study the convergence behaviour of the $\partial_y^{(1)}\partial_x^{(2)}$-difference of the Green's function on U_ϵ in the limit $\epsilon \to 0$.

Definition 35 (The discrete Laplacian Δ_0)**.** We write

$$\Delta_0 := \partial_{-x}^{(1)}\partial_x^{(1)} + \partial_{-y}^{(1)}\partial_y^{(1)}$$

to denote the discrete Laplacian in $\mathbb{Z}^2$.

Remark 13. The Green's function $G_\Lambda : \Lambda\times\Lambda \mapsto \mathbb{R}$ naturally extends to a function on $\overline{\Lambda}\times\overline{\Lambda}$ by setting $G_\Lambda(v,w) := 0$ all $v,w\in\overline{\Lambda}$ so that $\{v,w\}\cap\partial\Lambda \neq \emptyset$. Then for all $v,w\in\Lambda$ it holds $\Delta_0 G_\Lambda(v,w) = 1_{\{v=w\}}$. That is, G_Λ is $1/4$ times the Green function of simple random walk in Λ, killed on exit from Λ.

Definition 36 (The Classical Green's function G_0)**.** For all $v,w\in\mathbb{Z}^2$ let

$$G_0(v,w) := -(1/4)a(w-v),$$

where a denotes the potential kernel of simple random walk on the plane. We call G_0 the Classical Green's function on the plane, and note $\Delta_0 G_0(v,w) = 1_{\{v=w\}}$ all $v,w\in\mathbb{Z}^2$ ([19], Section 1.6).

In [12] Y. Fukai and K. Uchiyama prove an asymptotic expansion for the potential kernel a of simple random walk on the plane.

Lemma 28 ([12], Remark 2). *For $v=(v_x,v_y)\in\mathbb{Z}^2$ as $|v|\to\infty$*

$$G_0(0,v)=-\frac{1}{2\pi}\log|v|+C_1+C_2\cdot\frac{v_x^2v_y^2}{|v|^6}+\frac{C_3}{|v|^2}+O\left(|v|^{-3}\right)$$

for some constants C_1, C_2 and C_3. Here we say for $f,g:\mathbb{Z}^2\mapsto\mathbb{R}$ that $g(v)=O(h(v))$ as $|v|\to\infty$, if there exist $C>0$ and $R>0$ so that $|v|>R$ implies $|g(v)|\le C\cdot h(v)$.

Lemma 28 implies the following asymptotic expansion for the Green's function differences.

Lemma 29 (Asymptotic expansion for the Green's function differences). *As $|v|\to\infty$*

$$\partial_x^{(2)}G_0(0,v)=-\mathrm{Re}\,\frac{1}{2\pi v}+O\left(|v|^{-2}\right)$$

and

$$\partial_y^{(1)}\partial_x^{(2)}G_0(0,v)=\mathrm{Im}\,\frac{1}{2\pi v^2}+O\left(|v|^{-3}\right).$$

Proof. Lemma 28 yields

$$\begin{aligned}\partial_y^{(1)}\partial_x^{(2)}G_0(0,v)&=G_0(0,v+1-i)-G_0(0,v+1)-G_0(0,v-i)+G_0(0,0)\\&=\frac{1}{2\pi}A+C_2\cdot B+C_3\cdot C+O\left(|v|^{-3}\right),\end{aligned}$$

where we write

$$\begin{aligned}A:=&-\log|v+1-i|+\log|v+1|+\log|v-i|-\log|v|,\\B:=&\frac{(v_x+1)^2(v_y-1)^2}{|v+1-i|^6}-\frac{(v_x+1)^2v_y^2}{|v+1|^6}-\frac{v_x^2(v_y-1)^2}{|v-i|^6}+\frac{v_x^2v_y^2}{|v|^6}\\=&O\left(|v|^{-3}\right)\end{aligned}$$

and

$$C:=\frac{1}{|v+1-i|^2}-\frac{1}{|v+1|^2}-\frac{1}{|v-i|^2}+\frac{1}{|v|^2}=O\left(|v|^{-3}\right).$$

Using $\log(1+z)=z+O(|z|^2)$, we get

$$\begin{aligned}A&=\mathrm{Re}\,\log\left(\frac{(v+1)(v-i)}{v(v+1-i)}\right)=-\mathrm{Re}\frac{i}{v(v+1-i)}+O\left(|v|^{-4}\right)\\&=\mathrm{Im}\,\frac{1}{v^2}+O\left(|v|^{-3}\right).\end{aligned}$$

This shows the second statement. Similarly, the first statement follows from Lemma 28, and $\log|v|-\log|v+1|=-\mathrm{Re}\,(1/v)+O\left(|v|^{-2}\right)$. □

Definition 37 (Harmonic functions)**.** We say that a function $f : \overline{C} \mapsto \mathbb{R}$ is harmonic on $C \subset \mathbb{Z}^2$, if $\Delta_0 f(v) = 0$ for all $v \in C$.

To estimate harmonic functions we have the following lemma.

Lemma 30 ([19], Theorem 1.7.1)**.** *There exists a constant $C > 0$ with the following property: for all $n \in \mathbb{N}$ if a function $f : \overline{C}_n \mapsto \mathbb{R}$ is harmonic on $C_n := \{z \in \mathbb{Z}^2 \big| |z| < n\}$, then*

$$|\partial_y f(0)| \leq \max\left\{|f(v)|, v \in \overline{C_n}\right\} \cdot \frac{C}{n}.$$

We estimate the Green's function differences in the next lemma.

Lemma 31 (Estimates for the Green's function differences)**.** *Let $D \subset U$ so that the distance of D and ∂U is nonvanishing, that is,* $\operatorname{dist}(D, \partial U) := \inf_{(x,y)\in D\times\partial U} |x - y| > 0$. *Then there exist $c_D > 0$ and $\epsilon_D > 0$ as follows. For all $\epsilon \in]0, \epsilon_D]$, restricted to $D_\epsilon \times D_\epsilon$ the difference of the $\partial_y^{(1)}\partial_x^{(2)}$-difference quotients of the Classical Green's function and the Green's function on U_ϵ is bounded by c_D. More formally, for all $\epsilon \in]0, \epsilon_D]$, for all $v, w \in D_\epsilon$*

$$\left|\partial_y^{(1)}\partial_x^{(2)} G_{U_\epsilon}(v, w) - \partial_y^{(1)}\partial_x^{(2)} G_0(v, w)\right| \leq c_D \cdot \epsilon^2.$$

And for all $\epsilon \in]0, \epsilon_D]$, for all $v, w \in D_\epsilon$ the $\partial_y^{(1)}\partial_x^{(2)}$-difference of the Green's function on U_ϵ satisfies

$$\left|\partial_y^{(1)}\partial_x^{(2)} G_{U_\epsilon}(v, w)\right| \leq c_D \cdot \begin{cases} \frac{1}{|w-v|^2} & \text{if } v \neq w; \\ 1 & \text{if } v = w. \end{cases}$$

Proof. Let D as in the lemma. For all $\epsilon > 0$ and all $v \in \overline{U_\epsilon}$, $w \in U_\epsilon$ we write

$$H_\epsilon(v, w) := \partial_x^{(2)} G_0(v, w) - \partial_x^{(2)} G_{U_\epsilon}(v, w).$$

Here as in Remark 13 we write $G_{U_\epsilon}(v, w) := 0$ in case of $\{v, w\} \cap \partial U_\epsilon \neq \emptyset$. For all $(v, w) \in \partial U_\epsilon \times U_\epsilon$ we have $H_\epsilon(v, w) = \partial_x^{(2)} G_0(v, w)$. Thus Lemma 29 (Asymptotic expansion for the Green's function differences) and $\operatorname{dist}(D, \partial U) > 0$ imply the existence of $\tilde{c}_D > 0$ and $\tilde{\epsilon}_D > 0$ so that for all $\epsilon \in]0, \tilde{\epsilon}_D]$, for all $(v, w) \in \partial U_\epsilon \times D_\epsilon$

$$|H_\epsilon(v, w)| \leq \tilde{c}_D \cdot \epsilon. \tag{3.9}$$

For all $w \in D_\epsilon$ the function $H_\epsilon(\,\cdot\,, w) : \overline{U_\epsilon} \mapsto \mathbb{R}$ is harmonic on U_ϵ. Therefore, the maximum principle for harmonic functions implies that (3.9) holds for all $\epsilon \in]0, \tilde{\epsilon}_D]$, for all $(v, w) \in \overline{U_\epsilon} \times D_\epsilon$. Using Lemma 30 and $\operatorname{dist}(D, \partial U) > 0$, we obtain the first assertion. As a consequence there exist $\hat{c}_D > 0$ and $\hat{\epsilon}_D > 0$ with the property that for all $\epsilon \in]0, \hat{\epsilon}_D]$, for all $v, w \in D_\epsilon$

$$\left|\partial_y^{(1)}\partial_x^{(2)} G_{U_\epsilon}(v, w)\right| \leq \left|\partial_y^{(1)}\partial_x^{(2)} G_0(v, w)\right| + \hat{c}_D \cdot \epsilon^2.$$

Therefore, Lemma 29 and the boundedness of D imply the second statement. □

The next Lemma is well known for differences of the Green's function that restrict to one variable (see e.g. [7] §3, or compare with [17] Lemma 17). However, in the literature we did not see a proof that directly extends to the case of differences with respect to both variables of the Green's function.

Lemma 32 (Convergence of the Green's function differences)**.** *Let v and w be points in the interior of U, $v \neq w$. Then as ϵ tends to zero the second difference quotient $(1/\epsilon^2)\partial_y^{(1)}\partial_x^{(2)}G_{U_\epsilon}(v_\epsilon, w_\epsilon)$ converges to $\partial_y^{(1)}\partial_x^{(2)}g_U(v,w)$.*

To identify the $\partial_x^{(2)}$-derivative of the continuous Green's function g_U, we show the following lemma first.

Lemma 33. *Let w be a point in the interior of U. Then $\partial_x^{(2)}g_U(z,w)$ as a function of z is continuous up to the boundary of U, where it vanishes.*

Proof. Let w be a point in the interior of U and choose $R > 0$ such that $\overline{D_R} \subset U$, where $D_R := \{z \in \mathbb{C} \big| |w - z| \leq R\}$. Then as a function of v, $g_U(v,w)$ is harmonic on $U \setminus D_R$. Let $(B_t)_{t\geq 0}$ be a two-dimensional Brownian motion. We write P_v and $\mathbb{E}_v$ for probabilities and expectations, if $(B_t)_{t\geq 0}$ is started in $B_0 = v \in U \setminus D_R$. Then for all $v \in U \setminus D_R$ (see e.g. [18])

$$g_U(v,w) = \mathbb{E}_v\left[g(B_{\tau^c}, w)\right],$$

where

$$\tau^c := \inf\left\{t > 0 : B_t \notin U \setminus D_R\right\}.$$

That is, for $v \in U \setminus D_R$

$$\begin{aligned}\partial_x^{(2)} g_U(v,w) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(g_U(v,w+\epsilon) - g_U(v,w)\right)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}_v\big[g(B_{\tau^c}, w+\epsilon) - g(B_{\tau^c}, w)\big].\end{aligned}$$

The continuous Green's function $g_U(z,w)$ vanishes for z on the boundary of U. We obtain for all $v \in U \setminus D_R$

$$\left|\partial_x^{(2)} g_U(v,w)\right| \leq C \cdot P_v\left(B_{\tau^c} \in \partial D_R\right),$$

where

$$C := \sup_{z\in\partial D_R}\left|\partial_x^{(2)} g_U(z,w)\right| < \infty.$$

It follows for z on the boundary of U

$$\lim_{v\to z}\left|\partial_x^{(2)} g_U(v,w)\right| \leq C \cdot \lim_{v\to z} P\left(B_{\tau^c} \in \partial D_R\right) = 0.$$

□

Definition 38 (The Landau symbols O and o)**.** Let $g, h :]0, \infty[\mapsto \mathbb{R}$ be two functions. We write $g(\epsilon) = O(h(\epsilon))$ if there exist constants $C > 0$ and $\epsilon_0 > 0$ so that $|h(\epsilon)| \leq C \cdot g(\epsilon)$ for all $\epsilon \in]0, \epsilon_0]$. If we wish to imply that the constants may depend on some further quantity α, we write $O_\alpha(g(\epsilon))$. Similarly, we write $h(\epsilon) = o(g(\epsilon))$ in case of $\lim_{\epsilon \to 0} h(\epsilon)/g(\epsilon) = 0$.

Proof of Lemma 32. Let v and w be two points in the interior of U, $v \neq w$. For all $\epsilon > 0$, for all $z \in \overline{U}_\epsilon$ let

$$\mathcal{G}_\epsilon(z) := (1/\epsilon) \partial_x^{(2)} G_{U_\epsilon}(z, w_\epsilon)$$

and

$$\mathcal{G}_0(z) := (1/\epsilon) \partial_x^{(2)} G_0(z, w_\epsilon).$$

Let $\mathcal{H}_\epsilon : \overline{U_\epsilon} \mapsto \mathbb{R}$ be the harmonic function with the same boundary values which the function $f(z) := \mathrm{Re}\ (2\pi(w - \epsilon z))^{-1}$ assumes for $z \in \partial U_\epsilon$. The function $\mathcal{G}_\epsilon$ is zero on the boundary, and ϵw_ϵ is within $O_w(\epsilon)$ of w. Along with Lemma 29 (Asymptotic expansion for the Green's function differences) it follows for all $z \in \partial U_\epsilon$

$$\begin{aligned} \mathcal{G}_\epsilon(z) - \mathcal{G}_0(z) - \mathcal{H}_\epsilon(z) &= \mathrm{Re}\, \frac{1}{2\pi(\epsilon w_\epsilon - \epsilon z)} - \mathrm{Re}\, \frac{1}{2\pi(w - \epsilon z)} + O_w(\epsilon) \\ &= O_w(\epsilon). \end{aligned}$$

Therefore, the maximum principle for harmonic functions implies

$$\sup_{z \in \overline{U}_\epsilon} \left| \mathcal{G}_\epsilon(z) - \mathcal{G}_0(z) - \mathcal{H}_\epsilon(z) \right| = O_w(\epsilon),$$

and Lemma 30 yields

$$\frac{1}{\epsilon} \partial_y \big(\mathcal{G}_\epsilon(v_\epsilon) - \mathcal{G}_0(v_\epsilon) - \mathcal{H}_\epsilon(v_\epsilon) \big) = O_{w,v}(\epsilon).$$

Let $h : \overline{U} \to \mathbb{R}$ be the harmonic function with the same boundary values which the function $\tilde{f}(z) := \mathrm{Re}\ (2\pi(w - z))^{-1}$ assumes for $z \in \partial U$. Let $\tilde{\mathcal{H}}_\epsilon : \epsilon U_\epsilon \mapsto \mathbb{R}$ be defined by $\tilde{\mathcal{H}}_\epsilon(v) := \mathcal{H}_\epsilon(v/\epsilon)$. In [7] it is shown that as $\epsilon \to 0$ the function $\tilde{\mathcal{H}}_\epsilon$ converges to the function h, and that for any region lying entirely within U the difference quotients of $\tilde{\mathcal{H}}_\epsilon$ tend uniformly towards the corresponding partial derivatives of h. In particular, as ϵ tends to zero $\tilde{\mathcal{H}}_\epsilon(\epsilon v_\epsilon) = \mathcal{H}_\epsilon(v_\epsilon)$ tends to $h(v)$, and $(1/\epsilon)\partial_y \tilde{\mathcal{H}}_\epsilon(\epsilon v_\epsilon) = (1/\epsilon)\partial_y \mathcal{H}_\epsilon(v_\epsilon)$ tends to $\partial_y h(v)$. Along with Lemma 29 (Asymptotic expansion for the Green's function differences) this shows that as ϵ tends to zero $\mathcal{G}_\epsilon(v_\epsilon)$ tends to

$$-\mathrm{Re}\, \frac{1}{2\pi(w - v)} + h(v) =: \tilde{g}(v),$$

and $(1/\epsilon)\mathcal{G}_\epsilon(v_\epsilon) = (1/\epsilon^2) \partial_y^{(1)} \partial_x^{(2)} G_{U_\epsilon}(v_\epsilon, w_\epsilon)$ tends to

$$\mathrm{Im}\, \frac{1}{2\pi(w - v)^2} + \partial_y h(v) = \partial_y \tilde{g}(v).$$

The function $\tilde{g}$ has boundary values zero and a single 'pole' of residue $(2\pi)^{-1}$ at w. Along with Lemma 33 it follows $\tilde{g}(v) = \partial_x^{(2)} g_U(v,w)$. We conclude $\partial_y \tilde{g}(v) = \partial_y^{(1)}\partial_x^{(2)} g_U(v,w)$. □

We have one more estimate for the Green's function differences.

Lemma 34 (Convergence of the Green's function differences on the diagonal)**.** *For v in the interior of U*

$$\lim_{\epsilon\to 0} \partial_x^{(1)}\partial_x^{(2)} G_{U_\epsilon}(v_\epsilon, v_\epsilon) = \lim_{\epsilon\to 0} \partial_y^{(1)}\partial_y^{(2)} G_{U_\epsilon}(v_\epsilon, v_\epsilon) = \frac{1}{2},$$
$$\lim_{\epsilon\to 0} \partial_y^{(1)}\partial_x^{(2)} G_{U_\epsilon}(v_\epsilon, v_\epsilon) = \lim_{\epsilon\to 0} \partial_y^{(1)}\partial_{-x}^{(2)} G_{U_\epsilon}(v_\epsilon, v_\epsilon) = \frac{1}{2} - \frac{1}{\pi}$$

and

$$\lim_{\epsilon\to 0} \partial_x^{(1)}\partial_{-x}^{(2)} G_{U_\epsilon}(v_\epsilon, v_\epsilon) = -\frac{1}{2} + \frac{2}{\pi}.$$

Proof. Let $v \in U$. From Lemma 31 (Estimates for the Green's function differences) we have

$$\left|\partial_y^{(1)}\partial_x^{(2)} G_{U_\epsilon}(v_\epsilon, v_\epsilon) - \partial_y^{(1)}\partial_x^{(2)} G_0(v_\epsilon, v_\epsilon)\right| = O_v(\epsilon^2).$$

Explicit values for the potential kernel a of simple random walk on the plane are known [[31], page 148] and yield

$$\partial_y^{(1)}\partial_x^{(2)} G_0(v_\epsilon, v_\epsilon) = -\frac{1}{4}\big(a(1-i) - a(-i) - a(1) + a(0)\big) = \frac{1}{2} - \frac{1}{\pi}.$$

This shows $\lim_{\epsilon\to 0} \partial_y^{(1)}\partial_x^{(2)} G_{U_\epsilon}(v_\epsilon, v_\epsilon) = 1/2 - 1/\pi$. The other relations follow along the same lines. □

3.5 Scaling limit for the height one joint cumulants

We show Theorem 4 and Theorem 6 in this section. We proceed as follows. Let V be defined as in Theorem 6, and write $V_\epsilon := \cup_{v\in V}\{v_\epsilon\}$. In Lemma 6 we derived an expression of the height one joint cumulant $\kappa(h_{U_\epsilon}(v) : v \in V_\epsilon)$ in terms of the matrices

$$\left(k_{U_\epsilon}(v^i, w^j)\right)_{i,j\in\{x,-x,y\}} := \left(1_{\{v^i = w^j\}} - \partial_i^{(1)}\partial_j^{(2)} G_{U_\epsilon}(v,w)\right)_{i,j\in\{x,-x,y\}} \quad v, w \in V_\epsilon.$$

In Lemmata 34 (Convergence of the Green's function differences on the diagonal) and 32 (Convergence of the Green's function differences) we studied the limit $\epsilon \to 0$ of the second differences of the Green's function on U_ϵ. In the next lemma, Lemma 35, we combine these results to express $\lim_{\epsilon\to 0}\epsilon^{-2|V|}\kappa(h_{U_\epsilon}(v) : v\in V_\epsilon)$ in terms of the 2×2 matrices $\left(A_U\left((v,i),(w,j)\right)\right)_{i,j\in\{x,y\}}$, $v,w\in V$, given by

$$A_U\left((v,i),(w,j)\right) := \begin{cases} 1 & \text{if } v=w \text{ and } i=j,\\ 0 & \text{if } v=w \text{ and } i\neq j,\\ -\partial_i^{(1)}\partial_j^{(2)} g_U(v,w) & \text{if } v\neq w \text{ and } i=j,\\ \partial_i^{(1)}\partial_j^{(2)} g_U(v,w) & \text{otherwise.}\end{cases}$$

Here (v,x) and (v,y) denote two distinguishable copies of the site $v\in V$. Thereafter, we use this expression to show Theorems 4 and 6.

To state Lemma 35 we have to introduce some more notation. For all $P\subset V$ let $P^{xy} := \bigcup_{v\in P}\{(v,x),(v,y)\}$. Let

$$S^{xy}_{\text{cycl}}(V) := \left\{\sigma\in S(V^{xy})\middle|\forall\emptyset\neq P\subsetneq V : \sigma\left(P^{xy}\right)\neq P^{xy}\right\}$$

denote the set of permutations of V^{xy} that do not operate as a permutation on P^{xy} for a proper non-empty subset P of V. We write

$$S^{xy}_1(V) := \left\{\sigma\in S^{xy}_{\text{cycl}}(V)\middle|\forall v\in V : |\sigma(v^{xy})\cap v^{xy}| = 1\right\},$$

to denote the permutations in $S^{xy}_{\text{cycl}}(V)$ where for every $v\in V$ the set v^{xy} and the image of v^{xy} have exactly one point in common. In the definition of P^{xy} and $S^{xy}_{\text{cycl}}(V)$ the index xy denotes the correspondence to the case of two copies (v,x) and (v,y) each $v\in V$.

Lemma 35. *As $\epsilon\to 0$ the rescaled joint cumulant $\epsilon^{-2|V|}\kappa(h_{U_\epsilon}(v) : v\in V_\epsilon)$ tends to*

$$\left(\frac{2}{\pi}-\frac{4}{\pi^2}\right)^{|V|}\sum_{\sigma\in S^{xy}_1(V)}\text{sign}(\sigma)\prod_{v\in V^{xy}} A_U\left(v,\sigma(v)\right).$$

Proof. Lemma 26 (Height one joint cumulants) gives

$$\kappa\left(h_{U_\epsilon}(v) : v\in V_\epsilon\right) = \sum_{\sigma\in S^{\overline{x}y}_{\text{cycl}}(V_\epsilon)}\text{sign}(\sigma)\prod_{v\in V_\epsilon^{\overline{x}y}} k_{U_\epsilon}\left(v,\sigma(v)\right).$$

Let $v,w\in V$ so that $v\neq w$, and let $i,j\in\{x,-x,y\}$. Lemma 34 (Convergence of the Green's function differences on the diagonal) gives explicit values for $\lim_{\epsilon\to 0} k_{U_\epsilon}(v^i_\epsilon,v^j_\epsilon)$. From Lemma 32 (Convergence of the Green's function differences) as ϵ tends to zero $(1/\epsilon^2)k_{U_\epsilon}(v^i_\epsilon,w^j_\epsilon)$ tends to $-\partial_i^{(1)}\partial_j^{(2)}g_U(v,w)$. For all $\sigma\in S^{\overline{x}y}_{\text{cycl}}(V_\epsilon)$ and all $v\in V_\epsilon$ the sets $v^{\overline{x}y}$ and $\sigma(v^{\overline{x}y})$ have at most two points in common. Let

$$S^{\overline{x}y}_2(V_\epsilon) := \left\{\sigma\in S^{\overline{x}y}_{\text{cycl}}(V_\epsilon)\middle|\forall v\in V_\epsilon : |\sigma(v^{\overline{x}y})\cap v^{\overline{x}y}| = 2\right\},$$

be the set of those permutations in $S^{\overline{x}y}_{\text{cycl}}(V)$ where for every $v\in V_\epsilon$ the set $v^{\overline{x}y}$ and the image of $v^{\overline{x}y}$ have exactly two points in common. Then

$$C(\sigma) := \left|\left\{v^i\in V_\epsilon^{\overline{x}y}\middle|\sigma(v^i)\notin v^{\overline{x}y}\right\}\right|$$

satisfies $C(\sigma) = |V|$ for $\sigma \in S_2^{\overline{x}y}(V_\epsilon)$, and $C(\sigma) > |V|$ for $\sigma \in S_{\text{cycl}}^{\overline{x}y}(V_\epsilon) \setminus S_2^{\overline{x}y}(V_\epsilon)$. It follows

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon^{-2|V|}\kappa\left(h_{U_\epsilon}(v) : v \in V_\epsilon\right) \\
&= \lim_{\epsilon\to 0} \sum_{\sigma\in S_2^{\overline{x}y}(V_\epsilon)} \operatorname{sign}(\sigma) \prod_{\substack{v^i\in V_\epsilon^{\overline{x}y}\\ \sigma(v^i)\in v^{\overline{x}y}}} k_{U_\epsilon}\left(v^i,\sigma(v^i)\right) \prod_{\substack{v^i\in V_\epsilon^{\overline{x}y}\\ \sigma(v^i)\notin v^{\overline{x}y}}} \frac{k_{U_\epsilon}\left(v^i,\sigma(v^i)\right)}{\epsilon^2} \\
&= \sum_{\sigma\in S_2^{\overline{x}y}(V)} \operatorname{sign}(\sigma) \prod_{v\in V^{\overline{x}y}} k_U\left(v,\sigma(v)\right),
\end{aligned}$$

where $\left(k_U(v^i,v^j)\right)_{i,j\in\{x,-x,y\}} =: k_U^{\overline{x}y}(v,w)$ is given by

$$k_U^{\overline{x}y}(v,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}-\frac{2}{\pi} & \frac{1}{\pi}-\frac{1}{2} \\ \frac{1}{2}-\frac{2}{\pi} & \frac{1}{2} & \frac{1}{\pi}-\frac{1}{2} \\ \frac{1}{\pi}-\frac{1}{2} & \frac{1}{\pi}-\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

respectively for $v \neq w$

$$k_U^{\overline{x}y}(v,w) = \begin{pmatrix} -\partial_x^{(1)}\partial_x^{(2)}g_U(v,w) & \partial_x^{(1)}\partial_x^{(2)}g_U(v,w) & -\partial_x^{(1)}\partial_y^{(2)}g_U(v,w) \\ \partial_x^{(1)}\partial_x^{(2)}g_U(v,w) & -\partial_x^{(1)}\partial_x^{(2)}g_U(v,w) & \partial_x^{(1)}\partial_y^{(2)}g_U(v,w) \\ -\partial_y^{(1)}\partial_x^{(2)}g_U(v,w) & \partial_y^{(1)}\partial_x^{(2)}g_U(v,w) & -\partial_y^{(1)}\partial_y^{(2)}g_U(v,w) \end{pmatrix}.$$

Adding the second row to the first row, and the second column to the first column transforms $k_U^{\overline{x}y}(v,v)$ into

$$\overline{k}_U^{\overline{x}y}(v,v) := \begin{pmatrix} 2-\frac{4}{\pi} & 1-\frac{2}{\pi} & \frac{2}{\pi}-1 \\ 1-\frac{2}{\pi} & \frac{1}{2} & \frac{1}{\pi}-\frac{1}{2} \\ \frac{2}{\pi}-1 & \frac{1}{\pi}-\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

respectively $k_U^{\overline{x}y}(v,w)$ for $v \neq w$ into

$$\overline{k}_U^{\overline{x}y}(v,w) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\partial_x^{(1)}\partial_x^{(2)}g_U(v,w) & \partial_x^{(1)}\partial_y^{(2)}g_U(v,w) \\ 0 & \partial_y^{(1)}\partial_x^{(2)}g_U(v,w) & -\partial_y^{(1)}\partial_y^{(2)}g_U(v,w) \end{pmatrix}.$$

Another row and column operation that leaves $\overline{k}_U^{\overline{x}y}(v,w)$ unchanged in case of $v \neq w$, transforms $\overline{k}_U^{\overline{x}y}(v,v)$ into

$$\begin{pmatrix} 2-\frac{4}{\pi} & 0 & 0 \\ 0 & \frac{1}{\pi} & 0 \\ 0 & 0 & \frac{1}{\pi} \end{pmatrix}.$$

This implies the representation stated in the lemma. □

We now use Lemma 35 to show Theorem 6.

Proof of Theorem 6. In Lemma 35 we showed that as $\epsilon \to 0$ the rescaled joint cumulant $\epsilon^{-2|V|}\kappa(h_{U_\epsilon}(v) : v \in V_\epsilon)$ tends to

$$\left(\frac{2}{\pi} - \frac{4}{\pi^2}\right)^{|V|} \sum_{\sigma \in S_1^{xy}(V)} \operatorname{sign}(\sigma) \prod_{v \in V^{xy}} A_U\left(v, \sigma(v)\right).$$

The assertion of Theorem 6 is basically a transformation of this representation. Let $\sigma \in S_1^{xy}(V)$ such that

$$\prod_{v \in V^{xy}} A_U\left(v, \sigma(v)\right) \neq 0. \tag{3.10}$$

For all $v \in V$ it holds $A_U((v,x),(v,y)) = A_U((v,y),(v,x)) = 0$ and $|\sigma(v^{xy}) \cap v^{xy}| = 1$. Hence, (3.10) implies the existence of $(h^v)_{v\in V} \in \{x,y\}^V$ so that $\sigma((v,h^v)) = (v,h^v)$ for all $v \in V$. We write $k^v := \{x,y\} \setminus h^v$ and conclude

$$\sigma \in S(V,(k^v)_{v\in V}) := \left\{\sigma \in S_1^{xy}(V) \middle| \forall v \in V : \sigma((v,h^v)) = (v,h^v)\right\}.$$

Along with $A_U\big((v,x),(v,x)\big) = A_U\big((v,y),(v,y)\big) = 1$ all $v \in V$, this implies

$$\begin{aligned}
&\sum_{\sigma \in S_1^{xy}(V)} \operatorname{sign}(\sigma) \prod_{v \in V^{xy}} A_U\left(v, \sigma(v)\right) \\
&= \sum_{(k^v)_{v\in V} \in \{x,y\}^V} \sum_{\sigma \in S(V,(k^v)_{v\in V})} \operatorname{sign}(\sigma) \prod_{v\in V} A_U\big((v,k^v), \sigma((v,k^v))\big).
\end{aligned} \tag{3.11}$$

Let $(k^v)_{v\in V} \in \{x,y\}^V$. To further simplify (3.11) we construct a one-to-one mapping of $S(V,(k^v)_{v\in V})$ onto $S_{\text{cycl}}(V)$. For all $\sigma \in S(V,(k^v)_{v\in V})$ and $v \in V$ there exists a unique $v_\sigma \in V$ so that $\sigma((v,k^v)) = (v_\sigma, k^{v_\sigma})$. We define $\phi : S(V,(k^v)_{v\in V}) \mapsto S(V)$ by $\phi(\sigma)(v) := v_\sigma$ for $v \in V$, and note that ϕ is injective. Furthermore, if there would exist $\sigma \in S(V,(k^v)_{v\in V})$ and $\emptyset \neq P \subsetneq V$ so that $\phi(\sigma)(P) = P$, then $\sigma(P^{xy}) = P^{xy}$ would hold. Thus, $S(V,(k^v)_{v\in V}) \subset S_{\text{cycl}}^{xy}(V)$ implies $\phi : S(V,(k^v)_{v\in V}) \mapsto S_{\text{cycl}}(V)$. The inverse of ϕ, $\phi^{-1} : S_{\text{cycl}}(V) \mapsto S(V,(k^v)_{v\in V})$ exists and is given by $\phi^{-1}(\sigma)((v,k^v)) := (\sigma(v), k^{\sigma(v)})$ and $\phi^{-1}(\sigma)((v,h^v)) := (v,h^v)$, $v \in V$. In particular, ϕ is a one-to-one mapping of $S(V,(k^v)_{v\in V})$ onto $S_{\text{cycl}}(V)$. We obtain

$$\begin{aligned}
&\sum_{\sigma \in S(V,(k^v)_{v\in V})} \operatorname{sign}(\sigma) \prod_{v\in V} A_U\big((v,k^v), \sigma((v,k^v))\big) \\
&= (-1)^{|V|-1} \sum_{\sigma \in S_{\text{cycl}}(V)} \prod_{v\in V} A_U\big((v,k^v), (\sigma(v), k^{\sigma(v)})\big) \\
&= -\sum_{\sigma \in S_{\text{cycl}}(V)} \prod_{v\in V} \partial_{k^v}^{(1)} \partial_{k^{\sigma(v)}}^{(2)} g_U\left(v, \sigma(v)\right),
\end{aligned}$$

where we use $\operatorname{sign}(\sigma) = \operatorname{sign}(\phi(\sigma)) = (-1)^{|V|-1}$ for $\sigma \in S(V,(k^v)_{v\in V})$. This concludes the proof of the theorem. □

We proceed with the proof of Theorem 4.

Proof of Theorem 4. Let $f : U \mapsto U'$ be a conformal isomorphism, and let $u, v, w \in V$ such that $u \neq v \neq w$. The continuous Green's function is conformally invariant, that is, satisfies $g_U(u,v) = g_{U'}(f(u), f(v)) = (g_{U'} \circ F)(u,v)$, where $F(u,v) := (f(u), f(v))$. Therefore, we obtain for $k^u, k^w \in \{x, y\}$

$$
\begin{aligned}
&\sum_{k^v \in \{x,y\}} \partial^{(1)}_{k^u} \partial^{(2)}_{k^v} g_U\left(u,v\right) \cdot \partial^{(1)}_{k^v} \partial^{(2)}_{k^w} g_U\left(v,w\right) \\
&= \sum_{k^v \in \{x,y\}} \partial^{(1)}_{k^u} \partial^{(2)}_{k^v} \left(g_{U'} \circ F\right)(u,v) \cdot \partial^{(1)}_{k^v} \partial^{(2)}_{k^w} \left(g_{U'} \circ F\right)(v,w) \\
&= |f'(v)|^2 \cdot \sum_{k^v \in \{x,y\}} \partial^{(1)}_{k^u} \Big((\partial^{(2)}_{k^v} g_{U'}) \circ F \Big)(u,v) \cdot \partial^{(2)}_{k^w} \Big((\partial^{(1)}_{k^v} g_{U'}) \circ F \Big)(v,w)\,,
\end{aligned}
$$

where we use that f satisfies the Cauchy-Riemann equations. It follows for $\sigma \in S(V)$

$$
\begin{aligned}
&\sum_{(k^v)_{v\in V} \in \{x,y\}^V} \prod_{v \in V} \partial^{(1)}_{k^v} \partial^{(2)}_{k^{\sigma(v)}} g_U\left(v, \sigma(v)\right) \\
&= \left(\prod_{v\in V} |f'(v)|^2 \right) \cdot \sum_{(k^v)_{v\in V} \in \{x,y\}^V} \prod_{v \in V} \Big((\partial^{(1)}_{k^v} \partial^{(2)}_{k^{\sigma(v)}} g_{U'}) \circ F \Big) \big(v, \sigma(v)\big).
\end{aligned}
$$

We conclude

$$
\kappa_U(v : v \in V) = \left(\prod_{v\in V} |f'(v)|^2 \right) \cdot \kappa_{U'}(f(v) : v \in V).
$$

Along with Theorem 6 and Remark 10 this shows Theorem 4. □

3.6 Scaling limit for the sandpile height one field

We are going to use the method of moments to prove Theorem 5. That is, we show that the cumulants of the test integrals from Theorem 5 converge to the cumulants of a normal distribution.

Due to its length, we split the proof of Theorem 5 into the following propositions. Since the random variables considered in Theorem 5 contain the term $1/\sqrt{\mathcal{V}}$, we have to assure well definedness first:

Proposition 6 ($\mathcal{V}$ is well defined)**.** *Let $\mathcal{V}$ be defined as in Theorem 5. Then $\mathcal{V}$ is well defined and satisfies*

$$
0 < \mathcal{V} = \sum_{v \in \mathbb{Z}^2} \mathrm{Cov}(h_0(0), h_0(v)) < \infty.
$$

Our proof of Theorem 5 is based on the convergence of the cumulants. For all $n \geq 3$ the nth cumulant of a normal distributed random variable is zero.

Proposition 7 (Higher cumulants vanish). *Let $f \in \mathcal{C}_c^\infty(U)$ and $n \geq 3$. Then as $\epsilon \to 0$ the nth cumulant of $f \diamond h_{U_\epsilon}$ tends to zero.*

To identify the covariance matrix we have the following proposition.

Proposition 8 (The covariance matrix). *Let $f, g \in \mathcal{C}_c^\infty(U)$. Then as $\epsilon \to 0$ the covariance of $f \diamond h_{U_\epsilon}$ and $g \diamond h_{U_\epsilon}$ converges to $\int_U f(z)g(z)dz$.*

Our proofs of Propositions 6, 7 and 8 are based on the same estimate for the height one joint cumulants.

3.6.1 Estimate for the height one joint cumulants

In this section we derive the key estimate the proofs of Propositions 6 - 8 are based on.

We begin with a short motivation of the estimate. Let $f \in \mathcal{C}_c^\infty(U)$, $n \geq 2$ and $D := \mathrm{supp}(f) := \{x \in U | f(x) \neq 0\}$. The nth cumulant of $f \diamond h_{U_\epsilon}$ satisfies

$$\kappa_n\left(f \diamond h_{U_\epsilon}\right) = \left(\frac{\epsilon}{\sqrt{\mathcal{V}}}\right)^n \cdot \sum_{v_1,\ldots,v_n \in D_\epsilon} \left(\prod_{i=1}^n f(\epsilon v_i)\right) \cdot \kappa\left(h_{U_\epsilon}(v_i) : 1 \leq i \leq n\right).$$

Therefore, we estimate

$$\epsilon^n \cdot \sum_{\substack{v_1,\ldots,v_n \in D_\epsilon \\ |v_i - v_j| > 1 \text{ for } i \neq j}} \left|\kappa\left(h_{U_\epsilon}(v_i) : 1 \leq i \leq n\right)\right|. \tag{3.12}$$

First we consider the joint cumulants.

Lemma 36 (Estimate for the joint cumulants). *Let $n \geq 2$ and $D \subset U$ such that $\mathrm{dist}(D, \partial U) > 0$. Then there exist $c_{D,n} > 0$ and $\epsilon_D > 0$ with the following property. Let $\epsilon \in]0, \epsilon_D]$, and $V \subset D_\epsilon$ so that $|V| = n$ and $|v - w| \neq 1$ all $v, w \in V$. Then it holds*

$$\left|\kappa\left(h_{U_\epsilon}(v) : v \in V\right)\right| \leq c_{D,n} \cdot \sum_{\sigma \in S_{\mathrm{cycl}}(V)} \prod_{v \in V} \frac{1}{|v - \sigma(v)|^2}.$$

Proof. Let $D \subset U$ so that $\mathrm{dist}(D, \partial U) > 0$, and let $n \geq 2$. Lemma 26 (Height one joint cumulants) along with Lemma 31 (Estimates for the Green's function differences) imply the existence of $c_D > 0$ and $\epsilon_D > 0$ with the following property. Let $\epsilon \in]0, \epsilon_D]$ and $V \subset D_\epsilon$ so that $|V| = n$ and $|v - w| \neq 1$ all $v, w \in V$. For all $i, j \in \{x, -x, y\}$ and all $u, w \in V$ let

$$\tilde{k}\left(u^i, w^j\right) := \begin{cases} \frac{1}{|u-w|^2} & \text{if } v \neq w; \\ 1 & \text{if } v = w. \end{cases}$$

Then

$$\left|\kappa\left(h_{U_\epsilon}(v) : v \in V\right)\right| \leq (c_D)^{3n} \cdot \sum_{\sigma \in S_{\mathrm{cycl}}^{\overline{x}y}(V)} \prod_{v \in V^{\overline{x}y}} \tilde{k}\left(v, \sigma(v)\right). \tag{3.13}$$

To estimate the right hand side of (3.13) we show that for every $\sigma \in S_{\text{cycl}}^{\overline{x}y}(V)$ there exists a $\sigma' \in S_{\text{cycl}}(V)$ satisfying

$$\prod_{v \in V^{\overline{x}y}} \tilde{k}\,(v, \sigma(v)) \leq 4^{18n^3} \cdot \prod_{v \in V} \frac{1}{|v - \sigma'(v)|^2}. \tag{3.14}$$

Let $\sigma \in S_{\text{cycl}}^{\overline{x}y}(V)$. To show (3.14) we consider the orbits of σ. For all $v \in V^{\overline{x}y}$ let $\text{orbit}_\sigma(v) := \cup_{l \geq 0}\{\sigma^l(v)\}$ be the orbit of σ at v. Let $\text{ORBIT}_\sigma(v)$ be the orbit of σ at v projected onto V, that is, let $\text{ORBIT}_\sigma(v)$ be the minimal subset of V such that $\text{orbit}_\sigma(v) \subset \{\text{ORBIT}_\sigma(v)\}^{\overline{x}y}$. We proceed as follows. We choose a minimal set $P \subset V^{\overline{x}y}$ so that the sets $\text{ORBIT}_\sigma(p)$, $p \in P$, cover the set V in an appropriate way. Then we show that for every $p \in P$ the permutation $\sigma \in S_{\text{cycl}}^{\overline{x}y}(V)$ induces a permutation $\sigma_p \in S_{\text{cycl}}(\text{ORBIT}_\sigma(p))$. Finally, we use the permutations σ_p, $p \in P$, to construct a permutation $\sigma' \in S_{\text{cycl}}(V)$ satisfying (3.14).

First we define what we mean by a minimal set $P \subset V^{\overline{x}y}$ so that the sets $\text{ORBIT}_\sigma(p)$, $p \in P$, cover the set V in an appropriate way: we say that $P \subset V^{\overline{x}y}$ is an appropriate orbit-covering of V, if

(i) for all $p \in P$ the set $\text{ORBIT}_\sigma(p)$ is non-trivial: $|\text{ORBIT}_\sigma(p)| \geq 2$;

(ii) the set P does not contain two elements of the same orbit: for all $p, p' \in P$, $p \neq p'$ implies $\text{orbit}_\sigma(p) \cap \text{orbit}_\sigma(p') = \emptyset$;

(iii) for all $\emptyset \neq P' \subsetneq P$ it holds $(\cup_{p \in P'} \text{ORBIT}_\sigma(p)) \cap (\cup_{p \in P \setminus P'} \text{ORBIT}_\sigma(p)) \neq \emptyset$;

(iv) the sets $\text{ORBIT}_\sigma(p)$, $p \in P$, cover V. That is, it holds $\cup_{p \in P} \text{ORBIT}_\sigma(p) = V$.

We show the existence of an appropriate orbit-covering of V. We note that $\sigma \in S_{\text{cycl}}^{\overline{x}y}(V)$ implies the existence of $p \in V^{\overline{x}y}$ such that $|\text{ORBIT}_\sigma(p)| \geq 2$. If $\text{ORBIT}_\sigma(p) = V$, then $P := \{p\}$ is an appropriate orbit-covering of V. Otherwise, we successively extend P until we reach an appropriate orbit-covering of V, where we proceed as follows. Let $\emptyset \neq P \subset V^{\overline{x}y}$, and suppose $P_\sigma := \cup_{p \in P} \text{ORBIT}_\sigma(p) \neq V$. Then there exists $p' \in P_\sigma^{\overline{x}y} \setminus P$ with the property that $\text{ORBIT}_\sigma(p') \not\subset P_\sigma$. Otherwise we would have $\sigma(P_\sigma^{\overline{x}y}) = P_\sigma^{\overline{x}y}$, in contradiction to $\sigma \in S_{\text{cycl}}^{\overline{x}y}(V)$. We note that such a p' satisfies $\text{ORBIT}_\sigma(p') \cap P_\sigma \neq \emptyset$, $|\text{ORBIT}_\sigma(p')| \geq 2$, and $\text{orbit}_\sigma(p') \cap \text{orbit}_\sigma(p) = \emptyset$ for all $p \in P$. Furthermore,

$$(\cup_{p \in P'} \text{ORBIT}_\sigma(p)) \cap (\cup_{p \in P \setminus P'} \text{ORBIT}_\sigma(p)) \neq \emptyset \quad \text{all} \quad \emptyset \neq P' \subsetneq P$$

and $\text{ORBIT}_\sigma(p') \cap P_\sigma \neq \emptyset$ imply

$$(\cup_{p \in P'} \text{ORBIT}_\sigma(p)) \cap (\cup_{p \in \{P \cup \{p'\}\} \setminus P'} \text{ORBIT}_\sigma(p)) \neq \emptyset \quad \text{all} \quad \emptyset \neq P' \subsetneq \{P \cup \{p'\}\}.$$

Thus $|P_\sigma| < |\text{ORBIT}_\sigma(p') \cup P_\sigma|$ assures that extending P finitely many times yields an appropriate orbit-covering of V.

Let $P \subset V^{\overline{x}y}$ be an appropriate orbit-covering of V. For all $p, p' \in P$ so that $p \neq p'$ it holds $\operatorname{orbit}_\sigma(p) \cap \operatorname{orbit}_\sigma(p') = \emptyset$. Hence, $0 < \tilde{k}(v, w) \leq 1$ all $v, w \in V^{\overline{x}y}$ implies

$$\prod_{v \in V^{\overline{x}y}} \tilde{k}\,(v, \sigma(v)) \leq \prod_{p \in P} \prod_{v \in \operatorname{orbit}_\sigma(p)} \tilde{k}\,(v, \sigma(v))\,. \tag{3.15}$$

To handle the right hand site of (3.15) let $p \in P$. We show that the permutation σ induces a permutation $\sigma_p \in S_{\text{cycl}}(\text{ORBIT}_\sigma(p))$ so that

$$\prod_{v \in \operatorname{orbit}_\sigma(p)} \tilde{k}\,(v, \sigma(v)) \leq 4^{3n^2} \cdot \prod_{v \in \text{ORBIT}_\sigma(p)} \frac{1}{|v - \sigma_p(v)|^2}. \tag{3.16}$$

Let $v_0 \in V$ such that $p \in v_0^{\overline{x}y}$. Let $\tau_0 = 0$, and recursively for all $1 \leq i < |\,\text{ORBIT}_\sigma(p)|$ let

$$\tau_i := \min\left\{k > \tau_{i-1} \middle| \sigma^k(p) \notin \{v_0^{\overline{x}y}, \ldots, v_{i-1}^{\overline{x}y}\}\right\},$$

and $v_i \in \text{ORBIT}_\sigma(p)$ so that $\sigma^{\tau_i}(p) \in v_i^{\overline{x}y}$. We write $v_{|\,\text{ORBIT}_\sigma(p)|} := v_0$, and define $\sigma_p \in S_{\text{cycl}}(\text{ORBIT}_\sigma(p))$ by $\sigma_p(v_i) := v_{i+1}$, $0 \leq i < |\,\text{ORBIT}_\sigma(p)|$. For $u, v, w \in \mathbb{Z}^2$ such that $u \neq v \neq w \neq u$ the triangle inequality $|u - w| \leq |u - v| + |v - w|$ yields

$$\frac{1}{|u - v|^2} \cdot \frac{1}{|v - w|^2} \leq 4 \cdot \frac{1}{|u - w|^2}.$$

Therefore, for all $0 \leq i < |\,\text{ORBIT}_\sigma(p)|$

$$\prod_{l=\tau_i}^{\tau_{i+1}-1} \tilde{k}\left(\sigma^l(p), \sigma^{l+1}(p)\right) \leq 4^{3n} \cdot \frac{1}{|v_i - v_{i+1}|^2},$$

where $\tau_{|\,\text{ORBIT}_\sigma(p)|} := |\operatorname{orbit}_\sigma(p)|$. That is,

$$\begin{aligned} \prod_{v \in \operatorname{orbit}_\sigma(p)} \tilde{k}\,(v, \sigma(v)) &= \prod_{i=0}^{|\,\text{ORBIT}_\sigma(p)|-1} \prod_{l=\tau_i}^{\tau_{i+1}-1} \tilde{k}\left(\sigma^l(p), \sigma^{l+1}(p)\right) \\ &\leq \left(4^{3n}\right)^n \cdot \prod_{i=0}^{|\,\text{ORBIT}_\sigma(p)|-1} \frac{1}{|v_i - v_{i+1}|^2} = 4^{3n^2} \cdot \prod_{v \in \text{ORBIT}_\sigma(p)} \frac{1}{|v - \sigma_p(v)|^2}. \end{aligned}$$

This shows (3.16). Along with (3.15) it follows

$$\prod_{v \in V^{\overline{x}y}} \tilde{k}\,(v, \sigma(v)) \leq 4^{9n^3} \cdot \prod_{p \in P} \prod_{v \in \text{ORBIT}_\sigma(p)} \frac{1}{|v - \sigma_p(v)|^2} \tag{3.17}$$

for some $\sigma_p \in S_{\text{cycl}}(T_\sigma(p))$, $p \in P$.

To conclude (3.14), we use the permutations σ_p, $p \in P$, to construct a permutation $\sigma' \in S_{\text{cycl}}(V)$ such that

$$\prod_{p\in P}\prod_{v\in \text{ORBIT}_\sigma(p)} \frac{1}{|v-\sigma_p(v)|^2} \le 4^{n^2}\cdot \prod_{v\in V}\frac{1}{|v-\sigma'(v)|^2}. \tag{3.18}$$

Here we proceed as follows. Similarly as in the previous step, we obtain the following. Let $A, B \subset V$ such that $A\cap B \neq \emptyset$, let $\sigma_A \in S_{\text{cycl}}(A)$ and let $\sigma_B \in S_{\text{cycl}}(A)$. Then there exists a $\sigma_C \in S_{\text{cycl}}(A\cup B)$ such that

$$\prod_{v\in A}\frac{1}{|v-\sigma_A(v)|^2}\cdot\prod_{v\in B}\frac{1}{|v-\sigma_B(v)|^2} \le 4^n\cdot\prod_{v\in A\cup B}\frac{1}{|v-\sigma_C(v)|^2}. \tag{3.19}$$

This enables us to estimate the right hand side of (3.17): let $\emptyset \neq P' \subset P$ so that $O_1 := \cup_{p\in P'}\text{ORBIT}_\sigma(p) \neq V$. Suppose that for all $p\in P$ either $O_1 \cap \text{ORBIT}_\sigma(p) = \emptyset$ or $\text{ORBIT}_\sigma(p) \subset O_1$ holds. Let $\tilde{P} := \{p \in P | \,\text{ORBIT}_\sigma(p) \subset O_1\}$ and note that $\tilde{P}$ satisfies $\cup_{p\in\tilde{P}}\text{ORBIT}_\sigma(p) = O_1 \neq V$. That is, there exists $\emptyset \neq \tilde{P} \subsetneq P$ such that $(\cup_{v\in\tilde{P}}\text{ORBIT}_\sigma(v)) \cap (\cup_{v\in P\setminus\tilde{P}}\text{ORBIT}_\sigma(v)) = \emptyset$, a contradiction since the set P is an appropriate orbit-covering of V. This shows $O_1 \cap \text{ORBIT}_\sigma(p) \neq \emptyset$ and $T_\sigma(p) \not\subset O_1$ for some $p \in P\setminus P'$. Along with (3.19) it follows that for every $\sigma_1 \in S_{\text{cycl}}(O_1)$ and every $\sigma_p \in S_{\text{cycl}}(S_\sigma(p))$ there exists a $\sigma_2 \in S_{\text{cycl}}(O_1 \cup S_\sigma(p))$ so that

$$\prod_{v\in O_1}\frac{1}{|v-\sigma_1(v)|^2}\cdot\prod_{v\in S_\sigma(p)}\frac{1}{|v-\sigma_p(v)|^2} \le 4^n\cdot\prod_{v\in O_1\cup S_\sigma(p)}\frac{1}{|v-\sigma_2(v)|^2}.$$

Using this successively to estimate the right hand side of (3.17), we obtain (3.18). Along with (3.17) it follows (3.14), and combining with (3.13) yields the lemma. □

Lemma 36 (Estimate for the joint cumulants) enables us to estimate (3.12).

Lemma 37 (Key estimate for the proof of Theorem 5)**.** *Let $D \subset U$, $\text{dist}(D, \partial U) > 0$. Then for all $n \geq 2$ it holds*

$$\epsilon^n \cdot \sum_{\substack{v_1,\dots,v_n\in D_\epsilon\\ |v_i-v_j|>1 \text{ for } i\neq j}} \left|\kappa\left(h_{U_\epsilon}(v_i) : 1 \le i \le n\right)\right| = O_{D,n}\left(\epsilon^{(n-2)/2}\right).$$

Proof. Let $D \subset U$ so that $\text{dist}(D,\partial U) > 0$, and let $n \geq 2$. From Lemma 36 (Estimate for the joint cumulants) it suffices to show

$$\epsilon^{(n+2)/2}\cdot\sum_{\substack{v_1,\dots,v_n\in D_\epsilon\\ v_i\neq v_j \text{ for } i\neq j}}\left(\prod_{i=1}^{n-1}\frac{1}{|v_i-v_{i+1}|^2}\right)\cdot\frac{1}{|v_n-v_1|^2} = O_{D,n}(1). \tag{3.20}$$

We do this by induction on $n \geq 2$.

For $n = 2$ it holds $\epsilon^2 |D_\epsilon| = O_D(1)$, and thus

$$\epsilon^2 \cdot \sum_{\substack{v_1, v_2 \in D_\epsilon \\ v_1 \neq v_2}} \frac{1}{|v_1 - v_2|^4} \leq \epsilon^2 \cdot \sum_{v_1 \in D_\epsilon} \sum_{\substack{z \in \mathbb{Z}^2 \\ z \neq 0}} \frac{1}{|z|^4} = O_D(1).$$

In the induction step $k-1 \to k$ suppose that (3.20) holds for $n = k-1$. Let v_1, v_{k-1} and $v_k \in D_\epsilon$ so that $v_0 \neq v_{k-1} \neq v_k \neq v_0$. In case of $|v_{k-1} - v_k| \leq |v_k - v_1|$ the triangle inequality $|v_{k-1} - v_1| \leq |v_{k-1} - v_k| + |v_k - v_1| \leq 2 \cdot |v_k - v_1|$ implies

$$\frac{1}{|v_{k-1} - v_k|^2} \cdot \frac{1}{|v_k - v_1|^2} \leq \frac{1}{|v_{k-1} - v_1|^2} \cdot \frac{4}{|v_{k-1} - v_k|^2}.$$

A similar relation holds in case of $|v_{k-1} - v_k| \geq |v_k - v_1|$. We obtain

$$\begin{aligned} &\epsilon^{(k+2)/2} \cdot \sum_{\substack{v_1, \cdots, v_k \in D_\epsilon \\ v_i \neq v_j \text{ for } i \neq j}} \left(\prod_{i=1}^{k-1} \frac{1}{|v_i - v_{i+1}|^2} \right) \cdot \frac{1}{|v_k - v_1|^2} \\ &\leq \epsilon^{(k+1)/2} \cdot \sum_{\substack{v_1, \cdots, v_{k-1} \in D_\epsilon \\ v_i \neq v_j \text{ for } i \neq j}} \left(\prod_{i=1}^{k-2} \frac{1}{|v_i - v_{i+1}|^2} \right) \cdot \frac{1}{|v_{k-1} - v_1|^2} \cdot C(v_1, v_{k-1}), \end{aligned} \tag{3.21}$$

where

$$C(v_1, v_{k-1}) := \sum_{v_k \in D_\epsilon \setminus \{v_1\}} \frac{4 \cdot \epsilon^{1/2}}{|v_k - v_1|^2} + \sum_{v_k \in D_\epsilon \setminus \{v_{k-1}\}} \frac{4 \cdot \epsilon^{1/2}}{|v_{k-1} - v_k|^2}.$$

For $v, w \in D_\epsilon$ we have $\epsilon \cdot |v - w| = O_D(1)$. Therefore, we obtain

$$C(v_1, v_{k-1}) = \sum_{z \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{|z|^{5/2}} \cdot O_D(1) = O_D(1).$$

Along with the induction hypothesis and (3.21) this concludes the induction step. □

A further consequence of Lemma 36 is the following lemma.

Lemma 38 (Estimate for the covariances)**.** *For all $\epsilon > 0$, for all $v \in D_\epsilon$ it holds*

$$\sum_{w \in D_\epsilon} |\kappa\left(h_{U_\epsilon}(v), h_{U_\epsilon}(w)\right)| = O_D(1),$$

and

$$\lim_{\epsilon \to 0} \left(\epsilon^2 \cdot \sum_{\substack{v, w \in D_\epsilon \\ |v - w| > 1/\sqrt{\epsilon}}} |\kappa(h_{U_\epsilon}(v), h_{U_\epsilon}(w))| \right) = 0.$$

Proof. Lemma 36 (Estimate for the joint cumulants) implies for all $\epsilon > 0$, for all $v, w \in D_\epsilon$, $|v-w| > 1$,

$$|\kappa(h_{U_\epsilon}(v), h_{U_\epsilon}(w))| = \frac{1}{|v-w|^4} \cdot O_D(1).$$

It follows

$$\sum_{w \in D_\epsilon} |\kappa\left(h_{U_\epsilon}(v), h_{U_\epsilon}(w)\right)| = O_D(1) + \sum_{\substack{z \in \mathbb{Z}^2 \\ |z|>1}} \frac{1}{|z|^4} \cdot O_D(1) = O_D(1),$$

and

$$\lim_{\epsilon \to 0} \left(\epsilon^2 \cdot \sum_{\substack{v,w \in D_\epsilon \\ |v-w|>1/\sqrt{\epsilon}}} |\kappa(h_{U_\epsilon}(v), h_{U_\epsilon}(w))| \right) = \lim_{\epsilon \to 0} \sum_{\substack{z \in \mathbb{Z}^2 \\ |z|>1/\sqrt{\epsilon}}} \frac{1}{|z|^4} \cdot O_D(1) = 0.$$

□

3.6.2 Proof of Propositions 6, 7 and 8

In preparation for the proof of Proposition 6 ($\mathcal{V}$ is well defined) we state some remarks on the height one covariance in infinite volume.

Remark 14. In Lemma 26 (Height one joint cumulants) we showed that for all $n \in \mathbb{N}$, for all $v, w \in \Lambda_n$ so that $|v-w| > 1$ it holds

$$\kappa\left(h_{\Lambda_n}(v), h_{\Lambda_n}(w)\right) = \sum_{\sigma \in S^{\overline{x}y}_{\text{cycl}}(\{v,w\})} \text{sign}(\sigma) \prod_{u \in \{v,w\}^{\overline{x}y}} k_{\Lambda_n}\left(u, \sigma(u)\right).$$

Here $k_{\Lambda_n}\left(u^i, z^j\right) = 1_{\{u^i = z^j\}} - \partial_i^{(1)} \partial_j^{(2)} G_{\Lambda_n}(u,z)$. Let $v, w \in \mathbb{Z}^2$, $v \neq w$. As in the proof of Lemma 31 (Estimates for the Green's function differences) it follows for all $u^i, z^j \in \{v,w\}^{\overline{x}y}$

$$\lim_{n \to \infty} \partial_i^{(1)} \partial_j^{(2)} G_{\Lambda_n}(u,z) = \partial_i^{(1)} \partial_j^{(2)} G_0(u,z),$$

and therefore

$$\lim_{n \to \infty} k_{\Lambda_n}\left(u^i, z^j\right) = 1_{\{u^i = z^j\}} - \partial_i^{(1)} \partial_j^{(2)} G_0(u,z) =: k_0\left(u^i, z^j\right).$$

Along with the weak convergence stated in Lemma 22 this shows

$$\kappa\left(h_0(v), h_0(w)\right) = \sum_{\sigma \in S^{\overline{x}y}_{\text{cycl}}(\{v,w\})} \text{sign}(\sigma) \prod_{u \in \{v,w\}^{\overline{x}y}} k_0\left(u, \sigma(u)\right). \tag{3.22}$$

Hence, the translation invariance of the Classical Green's function implies for $v, w \in \mathbb{Z}^2$

$$\kappa\left(h_0(v), h_0(w)\right) = \kappa\left(h_0(0), h_0(w-v)\right). \tag{3.23}$$

Lemma 29 (Asymptotic expansion for the Green's function differences) and (3.22) provide the existence of $c' > 0$ such that for all $v, w \in \mathbb{Z}^2$, $v \neq w$,

$$\left|\kappa(h_0(v), h_0(w))\right| \leq c' \cdot \sum_{\sigma \in S^{\overline{x}y}_{\text{cycl}}(\{v,w\})} \text{sign}(\sigma) \prod_{u \in \{v,w\}^{\overline{x}y}} \tilde{k}\left(u, \sigma(u)\right),$$

where $\tilde{k}$ is defined as in (3.13). We proceed as in our estimate for the right hand site of (3.13), and obtain the existence of $c > 0$ so that for all $v, w \in \mathbb{Z}^2$, $v \neq w$,

$$\left|\kappa(h_0(v), h_0(w))\right| \leq c \cdot \frac{1}{|v-w|^4}. \tag{3.24}$$

Let $D \subset U$ so that $\text{dist}(D, \partial U) > 0$. From Lemma 31 (Estimates for the Green's function differences) there exist $c_D > 0$ and $\epsilon_D > 0$ with the following property. Let $\epsilon \in]0, \epsilon_D]$ and $u^i, z^j \in D^{\overline{x}y}_\epsilon$. Then $|k_{U_\epsilon}\left(u^i, z^j\right)| \leq c_D$ and $\left|k_{U_\epsilon}\left(u^i, z^j\right) - k_0\left(u^i, z^j\right)\right| \leq c_D \cdot \epsilon^2$. Thus (3.22) and Lemma 26 (Height one joint cumulants) imply the existence of $c'_D > 0$ and $\epsilon'_D > 0$ so that for all $\epsilon \in]0, \epsilon'_D]$, for all $v, w \in D_\epsilon$

$$\left|\kappa(h_{U_\epsilon}(v), h_{U_\epsilon}(w)) - \kappa(h_0(v), h_0(w))\right| \leq c'_D \cdot \epsilon^2. \tag{3.25}$$

Next we prove Proposition 6 ($\mathcal{V}$ is well defined).

Proof of Proposition 6. For $B \subset \mathbb{Z}^2$ finite let $h_0(B) := \sum_{v \in B} h_0(v)$. For all $n \geq 1$ let

$$\mathcal{V}_n := \frac{1}{|\Lambda_n|}\mathbb{V}[h_0(\Lambda_n)] = \frac{1}{|\Lambda_n|} \sum_{v,w \in \Lambda_n} \kappa(h_0(0), h_0(v-w)).$$

To show Proposition 6 we have to show that

$$\mathcal{V} := \lim_{n \to \infty} \mathcal{V}_n$$

is well defined and satisfies $0 < \mathcal{V} = \sum_{v \in \mathbb{Z}^2} \kappa(h_0(0), h_0(v)) < \infty$. First we show well-definedness and finiteness. Thereafter, we use Lemma 23 to conclude $\mathcal{V} > 0$.

Note that (3.24) implies

$$\sum_{v \in \mathbb{Z}^2} \left|\kappa(h_0(0), h_0(v))\right| < \infty.$$

Hence, to show that $\mathcal{V} := \lim_{n\to\infty} \mathcal{V}_n$ is well defined and finite, it suffices to show

$$\lim_{n \to \infty} \mathcal{V}_n = \sum_{v \in \mathbb{Z}^2} \kappa(h_0(0), h_0(v)). \tag{3.26}$$

Using (3.24) we obtain

$$\lim_{n \to \infty} \sum_{v \in \mathbb{Z}^2 \setminus \Lambda_n} \left|\kappa(h_0(0), h_0(v))\right| = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{\substack{v,w\in\Lambda_n\\ v-w\notin\Lambda_n}} \big|\kappa(h_0(0), h_0(v-w))\big| \leq \lim_{n\to\infty} \sum_{z\in\mathbb{Z}^2\setminus\Lambda_n} \big|\kappa(h_0(0), h_0(z))\big| = 0.$$

Therefore, to conclude (3.26) it remains to show that as n tends to infinity

$$\begin{aligned} R_n := &\sum_{v\in\Lambda_n} \kappa(h_0(0), h_0(v)) - \frac{1}{|\Lambda_n|} \sum_{\substack{v,w\in\Lambda_n\\ v-w\in\Lambda_n}} \kappa(h_0(0), h_0(v-w)) \\ = &\frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} \big(|\Lambda_n| - |\{(u,w)\in\Lambda_n^2 | u-w=v\}|\big)\cdot \kappa(h_0(0), h_0(v)) \end{aligned}$$

tends to zero. For all $n\in\mathbb{N}$, for all $v\in\Lambda_n$ we have

$$|\Lambda_n| \geq \big|\, \{(u,w)\in\Lambda_n^2 | u-w=v\} \,\big| \geq |\Lambda_n| - 6n|v|.$$

It follows

$$|R_n| \leq \frac{6n}{|\Lambda_n|} \sum_{v\in\Lambda_n} |v|\cdot|\kappa(h_0(0), h_0(v))|.$$

Hence, (3.24) implies $\lim_{n\to\infty} R_n = 0$. It remains to show $\mathcal{V} > 0$.

To show $\mathcal{V} > 0$ we are going to use Lemma 23 as follows. Let $n \geq 1$ be odd and $\Lambda_n' := \Lambda_n \cap 2\mathbb{Z}^2$. We condition on the configuration on $\Lambda_n \setminus \Lambda_n'$, and consider those sites $v\in\Lambda_n'$ where all sites of $\mathcal{D}_v$ have height four. By Lemma 23 the conditioned distribution of the height variable at such a site is the uniform distribution on $\{1,2,3,4\}$. This enables us to estimate the conditional variance of $h_0(\Lambda_n)$.

Let $\sigma = (\sigma_w)_{w\in\Lambda_n\setminus\Lambda_n'} \in \Omega_{\Lambda_n\setminus\Lambda_n'}$ such that $\mu_0(\eta_{\Lambda_n\setminus\Lambda_n'} = \sigma) > 0$, and write

$$V_\sigma := \big\{v\in\Lambda_n' \big| \forall w\in\mathcal{D}_v : \sigma_w = 4\big\}.$$

From Lemma 23 for $k \geq n$

$$\begin{aligned} \mathbb{V}\big[h_{\Lambda_k}(\Lambda_n)\big|\eta_{\Lambda_n\setminus\Lambda_n'} = \sigma\big] &= \sum_{v\in V_\sigma} \mathbb{V}\big[h_v(v)\big] + \mathbb{V}\big[h_{\Lambda_k}(\Lambda_n\setminus V_\sigma)\big|\eta_{\Lambda_n\setminus\Lambda_n'} = \sigma\big] \\ &\geq \sum_{v\in V_\sigma} \mathbb{V}\big[h_v(v)\big] = \frac{3}{16}\cdot|V_\sigma|. \end{aligned}$$

Therefore, the weak convergence implies

$$\mathbb{V}\big[h_0(\Lambda_n)\big|\eta_{\Lambda_n\setminus\Lambda_n'} = \sigma\big] \geq \frac{3}{16}\cdot|V_\sigma|,$$

and the law of total variance yields

$$|\Lambda_n|\cdot\mathcal{V}_n \geq \mathbb{E}\big[\mathbb{V}\big(h_0(\Lambda_n)\big|\eta_{\Lambda_n\setminus\Lambda_n'}\big)\big] \geq \frac{3}{16}\cdot\sum_{v\in\Lambda_n'} \mathbb{E}\big[k_0(v)\big]. \tag{3.27}$$

Here $k_0(v)$ denotes the indicator function of $\{\forall w \in \mathcal{D}_v : \eta_w = 4\}$, and $\mathbb{E}\big[k_0(v)\big]$ its expectation with respect to μ_0.

Let $v \in \mathbb{Z}^2$. To estimate $\mathbb{E}[k_0(v)]$ we count recurrent configurations where all sites $w \in \mathcal{D}_v$ have maximal height four: let $k \in \mathbb{N}$ so that $v \in \Lambda_k$. For $\sigma = (\sigma_w)_{w\in\Lambda_k} \in \mathcal{R}_{\Lambda_k}$ let

$$\phi(\sigma) \in \mathcal{R}_{\Lambda_k,\mathcal{D}_v} := \{\sigma \in \mathcal{R}_{\Lambda_k} | \forall w \in \mathcal{D}_v : \sigma(w) = 4\}$$

be defined by

$$(\phi(\sigma))_w := \begin{cases} 4 & \text{if } w \in \mathcal{D}_v; \\ \sigma_w & \text{otherwise.} \end{cases}$$

Then $|\phi^{-1}(\sigma')| \leq 4^8$ for all $\sigma' \in \mathcal{R}_{\Lambda_k,\mathcal{D}_v}$. It follows

$$\mathbb{E}[k_{\Lambda_k}(v)] = \frac{|\mathcal{R}_{\Lambda_k,\mathcal{D}_v}|}{|\mathcal{R}_{\Lambda_k}|} \geq \frac{|\mathcal{R}_{\Lambda_k}|}{4^8 \cdot |\mathcal{R}_{\Lambda_k}|} \geq \frac{1}{4^8}.$$

In particular, this shows $\mathbb{E}[k_0(v)] = \lim_{k\to\infty} \mathbb{E}[k_{\Lambda_k}(v)] \geq 4^{-8}$. Along with (3.27) we get

$$\mathcal{V} = \lim_{n\to\infty} \mathcal{V}_n \geq \lim_{n\to\infty} \frac{3}{4^{10}} \cdot \frac{|\Lambda'_{2n+1}|}{|\Lambda_{2n+1}|} \geq \frac{1}{3 \cdot 4^{10}}.$$

□

The estimate proven in Lemma 37 (Key estimate for the proof of Theorem 5) enables us to show Proposition 7 (Higher cumulants vanish).

Proof of Proposition 7. Let $f \in \mathcal{C}_c^\infty(U)$, $D := \operatorname{supp}(f)$ and $n \geq 3$. The nth cumulant of $f \diamond h_{U_\epsilon}$ satisfies

$$\kappa_n\left(f \diamond h_{U_\epsilon}\right) = \left(\frac{\epsilon}{\sqrt{\mathcal{V}}}\right)^n \cdot \sum_{v_1,\ldots,v_n \in D_\epsilon} \left(\prod_{i=1}^n f(\epsilon v_i)\right) \cdot \kappa\left(h_{U_\epsilon}(v_i) : 1 \leq i \leq n\right).$$

Since f is bounded, to prove $\lim_{\epsilon\to 0} \kappa_n\left(f \diamond h_{U_\epsilon}\right) = 0$ it suffices to show

$$\epsilon^n \cdot \sum_{v_1,\ldots,v_n \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_i) : 1 \leq i \leq n\right)\right| = O_{D,n}(\sqrt{\epsilon}). \tag{3.28}$$

Lemma 37 (Key estimate for the proof of Theorem 5) implies

$$\epsilon^n \cdot \sum_{v_1,\ldots,v_n \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_i) : 1 \leq i \leq n\right)\right| = O_{D,n}\left(\sqrt{\epsilon}\right) + \sum_{e \in \{0,\pm 1,\pm i\}} B_{n,e} \cdot O_n(1),$$

where

$$B_{n,e} := \epsilon^n \cdot \sum_{\substack{v_1,\ldots,v_n \in D_\epsilon \\ v_1+e=v_2}} \left|\kappa\left(h_{U_\epsilon}(v_i) : 1 \leq i \leq n\right)\right|.$$

It remains to show $B_{n,e} = O_{D,n}(\sqrt{\epsilon})$ for all $e \in \{0, \pm 1, \pm i\}$. We do this by induction on $n \geq 3$. In the induction step $k-1 \to k$ suppose that for all $3 \leq n \leq k-1$ the relation (3.28) holds, respectively note that Lemma 37 (Key estimate for the proof of Theorem 5) provides

$$\begin{aligned}&\epsilon^2 \cdot \sum_{v_1, v_2 \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_1), h_{U_\epsilon}(v_2)\right)\right| \\ &= O_D(1) + \epsilon^2 \cdot \sum_{\substack{v_1, v_2 \in D_\epsilon \\ |v_1 - v_2| > 1}} \left|\kappa\left(h_{U_\epsilon}(v_1), h_{U_\epsilon}(v_2)\right)\right| = O_D(1). \end{aligned} \tag{3.29}$$

Lemma 27 (Height one joint cumulants including neighbours) gives for all $e \in \{0, \pm 1, \pm i\}$

$$\begin{aligned}B_{k,e} \leq &\epsilon^k \cdot \sum_{P \subset \langle k \rangle \setminus \{1,2\}} \sum_{\substack{v_1, \ldots, v_k \in D_\epsilon \\ v_1 + e = v_2}} \left|\kappa(h_{U_\epsilon}(v_i) : i \in P_1)\right| \cdot \left|\kappa(h_{U_\epsilon}(v_i) : i \in \langle k \rangle \setminus P_1)\right| \\ &+ \epsilon^k \cdot \sum_{v_2, \ldots, v_k \in D_\epsilon} \left|\kappa(h_{U_\epsilon}(v_i) : 2 \leq i \leq k))\right|, \end{aligned} \tag{3.30}$$

where $P_1 := P \cup \{1\}$. The induction hypothesis in case of $k > 3$, respectively (3.29) for $k = 3$ implies

$$\epsilon^k \cdot \sum_{v_2, \ldots, v_k \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_i) : 2 \leq i \leq k)\right)\right| = O_{D,k}(\epsilon).$$

To handle the other summands in (3.30), let $e \in \{0, \pm 1, \pm i\}$ and $P \subset \langle k \rangle \setminus \{1, 2\}$. First suppose that $P = \emptyset$. For all $v_1 \in D_\epsilon$ we have $|\kappa(h_{U_\epsilon}(v_1))| = \mathbb{E}[h_{U_\epsilon}(v_1)] \leq 1$. Therefore, the induction hypothesis, respectively (3.29) provides

$$\begin{aligned}&\epsilon^k \cdot \sum_{\substack{v_1, \ldots, v_k \in D_\epsilon \\ v_1 + e = v_2}} \left|\kappa\left(h_{U_\epsilon}(v_1)\right)\right| \cdot \left|\kappa\left(h_{U_\epsilon}(v_i) : 2 \leq i \leq k\right)\right| \\ \leq &\epsilon^k \cdot \sum_{v_2, \ldots, v_k \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_i) : 2 \leq i \leq k\right)\right| = O_{D,k}(\epsilon).\end{aligned}$$

The same estimate holds for $P = \langle k \rangle \setminus \{1, 2\}$. It remains to study the case $1 \leq |P| < k-2$. Suppose $1 = |P| < k - 2$, without loss of generality let $P = \{v_k\}$. Then

$$\begin{aligned}&\epsilon^k \cdot \sum_{\substack{v_1, \ldots, v_k \in D_\epsilon \\ v_1 + e = v_2}} \left|\kappa\left(h_{U_\epsilon}(v_1), h_{U_\epsilon}(v_k)\right)\right| \cdot \left|\kappa\left(h_{U_\epsilon}(v_i) : 2 \leq i \leq k-1\right)\right| \\ &= \epsilon^k \cdot \sum_{v_2, \ldots, v_{k-1} \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_i) : 2 \leq i \leq k-1\right)\right| \cdot \sum_{v_k \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_2 - e), h_{U_\epsilon}(v_k)\right)\right| \\ &= \epsilon^k \cdot \sum_{v_2, \ldots, v_{k-1} \in D_\epsilon} \left|\kappa\left(h_{U_\epsilon}(v_i) : 2 \leq i \leq k-1\right)\right| \cdot O_D(1) = O_{D,k}(\epsilon),\end{aligned}$$

where the second equality is due to first equation in Lemma 38 (Estimate for the covariances) , and the third equality follows from the induction hypothesis, respectively (3.29).

We suppose $1 < |P| < k-2$. Then $|P_1| \geq 3$ and $|\langle k\rangle \setminus P_1| \geq 2$. Hence, the induction hypothesis and (3.29) imply

$$\begin{aligned}
&\epsilon^k \cdot \sum_{\substack{v_1,\ldots,v_k \in D_\epsilon \\ v_1+e=v_2}} |\kappa(h_{U_\epsilon}(v_i) : i \in P_1)| \cdot |\kappa(h_{U_\epsilon}(v_i) : i \in \langle k\rangle \setminus P_1)| \\
&\leq \left(\epsilon^{|P_1|} \cdot \sum_{(v_i)_{i\in P_1} \in D_\epsilon^{P_1}} |\kappa_U(h_{U_\epsilon}(v_i) : i \in P_1)|\right) \cdot \left(\epsilon^{k-|P_1|} \cdot \sum_{(v_i)_{i\in\langle k\rangle\setminus P_1} \in D_\epsilon^{\langle k\rangle\setminus P_1}} |\kappa(h_{U_\epsilon}(v_i) : i \in \langle k\rangle \setminus P_1)|\right) \\
&= O_{D,|P_1|}(\sqrt{\epsilon}) \cdot O_{D,k-|P_1|}(1) = O_{D,k}(\sqrt{\epsilon}).
\end{aligned}$$

Altogether, this shows $B_{k,e} = O_{D,k}(\sqrt{\epsilon})$. □

We now prove Proposition 8 (The covariance matrix).

Proof of Proposition 8. Let $f, g \in \mathcal{C}_c^\infty(U)$ and $D := \text{supp}(f) \cup \text{supp}(g)$. Using the boundedness of f and g, and the second equation from Lemma 38 (Estimate for the covariances) we get

$$\mathcal{V} \cdot \kappa\left(f \diamond h_{U_\epsilon}, g \diamond h_{U_\epsilon}\right) = \epsilon^2 \cdot \sum_{\substack{v,w\in U_\epsilon \\ |v-w|\leq 1/\sqrt{\epsilon}}} f(\epsilon v) g(\epsilon w) \kappa(h_{U_\epsilon}(v), h_{U_\epsilon}(w)) + o(1).$$

From (3.25) there exist $c_D > 0$ and ϵ_D such that for all $\epsilon \in]0, \epsilon_D]$

$$\epsilon^2 \cdot \sum_{\substack{v,w\in D_\epsilon \\ |v-w|\leq 1/\sqrt{\epsilon}}} \left|\kappa(_{U_\epsilon}(v), h_{U_\epsilon}(w)) - \kappa(h_0(v), h_0(w))\right| \leq \epsilon^4 \cdot c_D \cdot d_\epsilon,$$

where $d_\epsilon := |D_\epsilon| \cdot |\{w \in \mathbb{Z}^2 : |w| \leq 1/\sqrt{\epsilon}\}|$. We note $\epsilon^4 \cdot d_\epsilon = o(1)$ and obtain

$$\mathcal{V} \cdot \kappa\left(f \diamond h_{U_\epsilon}, g \diamond h_{U_\epsilon}\right) = \epsilon^2 \cdot \sum_{\substack{v,w\in U_\epsilon \\ |v-w|\leq 1/\sqrt{\epsilon}}} f(\epsilon v) g(\epsilon w) \kappa(h_0(v), h_0(w)) + o(1).$$

Our choice of $g \in \mathcal{C}_c^\infty$ implies the existence of $C_g > 0$ so that for all $z_1, z_2 \in U$ it holds $|g(z_1) - g(z_2)| \leq C_g \cdot |z_1 - z_2|$. That is, for all $\epsilon > 0$, all $v, w \in U_\epsilon$ with $|v - w| \leq 1/\sqrt{\epsilon}$, we have $|g(\epsilon v) - g(\epsilon w)| \leq C_g \cdot \sqrt{\epsilon}$. Therefore, (3.24) implies

$$\epsilon^2 \cdot \sum_{\substack{v,w\in U_\epsilon \\ |v-w|\leq 1/\sqrt{\epsilon}}} |g(\epsilon v) - g(\epsilon w)| \cdot |\kappa(h_0(v), h_0(w))| = o(1).$$

It follows

$$\begin{aligned}\mathcal{V}\cdot\kappa\left(f\diamond h_{U_\epsilon},g\diamond h_{U_\epsilon}\right)&=\epsilon^2\cdot\sum_{\substack{v,w\in U_\epsilon\\|v-w|\le 1/\sqrt{\epsilon}}} f(\epsilon v)g(\epsilon v)\kappa(h_0(v),h_0(w))+o(1)\\&=\Big(\sum_{v\in\mathbb{Z}^2}\kappa(h_0(0),h_0(v))\Big)\cdot\int_U f(z)g(z)dz+o(1),\end{aligned}$$

where the second equality is due to (3.23) and (3.24). Along with (3.26) this concludes the proof of the proposition. □

3.6.3 Proof of Theorem 5

Finally, we show Theorem 5.

Proof of Theorem 5. Let $n\in\mathbb{N}$ and for all $1\le i\le n$ let $f_i\in\mathcal{C}_c^\infty(U)$ and $t_i\in\mathbb{R}$. In Proposition 6 ($\mathcal{V}$ is well defined) it is proven $0<\mathcal{V}=\sum_{v\in\mathbb{Z}^2}\mathrm{Cov}(h_0(0),h_0(v))<\infty$. Therefore, the family of random variables $f_i\diamond h_{U_\epsilon}$, $1\le i\le n$, is well defined. We write $f:=\sum_{i=1}^n t_if_i$, and note $\sum_{1\le i\le n}t_i\cdot(f_i\diamond h_{U_\epsilon})=f\diamond h_{U_\epsilon}$ and $f\in\mathcal{C}_c^\infty(U)$. From Propositions 8 (The covariance matrix) and 7 (Higher cumulants vanish) as ϵ tends to zero the cumulants of $f\diamond h_{U_\epsilon}$ converge to the cumulants of the normal distribution with mean zero and variance $\int_U f^2(z)dz$. This is equivalent to convergence of the moments which in turn implies convergence in distribution. From Proposition 8 for all $1\le i,j\le n$ as $\epsilon\to 0$ the covariance of $f_i\diamond h_{U_\epsilon}$ and $f_j\diamond h_{U_\epsilon}$ tends to $\int_U f_i(z)f_j(z)dz$. □

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