Pricing in new markets: an application to insurance and electricity products

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Zusammenfassung

In der vorliegenden Dissertation werden neueste Entwicklungen in der Modellierung von Versicherungs- und Elektrizitätsfinanzprodukten untersucht. Insbesondere konzentrieren wir uns auf zwei Kernprobleme. Zunächst betrachten wir die Bewertung von Katastrophenoptionen. Danach zeigen wir, dass ähnliche Techniken für die Bewertung europäischer Elektrizitätsoptionen angewendet werden können. Zu den Hauptergebnissen der Arbeit gehören realistische mathematische Modelle für Katastrophenschadenindizes und Elektrizitätsterminmärkte, die mit Hilfe von Fourier-Transformationstechniken die Herleitung analytischer Bewertungsformeln für europäische Optionen ermöglichen.

Katastrophenoptionen sind Finanzinstrumente für den Transfer von Versicherungsrisiken in den Kapitalmarkt. Sie basieren auf einem Index, der Versicherungsverluste durch Naturkatastrophen quantitativ erfasst. Im Rahmen dieser Arbeit betrachten wir nur standardisierte Optionskontrakte, d.h. börsennotierte Katastrophenderivate auf Grundlage eines Marktschadenindex, beispielsweise des PCS-Index des Property Claims Service, einer international anerkannten Marktautorität für Vermögensschäden in den USA. Ein Marktschadenindex spiegelt die angefallenen Schäden der Versicherungswirtschaft nach einer Naturkatastrophe wider. Zur realistischen Abbildung eines Marktschadenindex entwickeln wir ein mathematisches Modell, bei dem die Vorabschätzung der Schäden jeder Naturkatastrophe sofort mit einem positiven Martingal neu geschätzt wird, das ab einem zufälligen Zeitpunkt des Schadeneintritts beginnt. Der wesentliche Vorteil unseres Modells ist die Anwendbarkeit auf heavy-tailed-verteilte Schäden (die üblichen Verteilungen für die Modellierung von Katastrophenschäden sind heavy-tailed).

Ferner wird in dieser Arbeit ein Elektrizitätsmarktmodell entwickelt, bei dem wir Elektrizitätsterminpreise (Forwards, Futures) und Elektrizitätsspotpreise gleichzeitig modellieren. Andererseits haben wir in unserem Modell eine direkte Verbindung zwischen Elektrizitätsterminpreis- und Spotpreisprozessen. Der Terminpreis unterscheidet sich vom Spotpreis um einen stocha-

stischen positiven Faktor mit dem Endwert (terminal value) eins. Deswegen kann dieser Faktor als eine Nullkuponanleihe modelliert werden. Ein wichtiger Vorteil unseres Modells ist die Markov-Eigenschaft des Spotpreisprozesses, die für die Bewertung von pfadabhängigen Elektrizitätsoptionen (path-dependent options), wie zum Beispiel Swingoptionen, entscheidend ist. Insbesondere beinhaltet unser Elektrizitätsmarktmodell ein allgemein anerkanntes Modell, bei dem der Spotpreisprozess das Exponential eines Ornstein-Uhlenbeck-Prozesses ist.

Mit dieser Arbeit hoffen wir, zur Entwicklung quantitativer Instrumente beizutragen, die einen liquiden Handel und die Bewertung von Katastrophen- und Elektrizitätsoptionen unterstützen.

Abstract

In this thesis we consider recent developments in insurance and electricity financial products. In particular, we investigate the interplay between insurance and finance, and therein the problem of pricing *catastrophe insurance options written on a loss index* as well as *electricity products*.

Catastrophe insurance options are standardized exchange-traded financial securities based on an underlying index, e.g. a PCS index, that encompasses insurance losses due to natural catastrophes. The PCS index is provided by the Property Claim Services (PCS), a US independent industry authority which estimates catastrophic property damage. The advantages of the catastrophe options in comparison to other capital market insurance solutions are lower transaction costs relative to the reinsurance and minimal credit risk, because of the guarantee of the exchange.

The main results of the thesis are fairly realistic models for catastrophe loss indexes and electricity futures markets, where by employing Fourier transform techniques we are able to provide analytical pricing formulas for European type options traded in the markets.

For the catastrophe loss index we specify a model, where the initial estimate of each catastrophe loss is re-estimated immediately by a positive martingale starting from the random time of loss occurrence. Significant advantage of this methodology is that it can be applied to loss distributions with heavy tails – the appropriate tail behavior for catastrophe modeling. The case when the re-estimation factors are given by positive affine martingales is also discussed and a characterization of positive affine local martingales is provided.

For electricity futures markets we derive a model, where we can simultaneously model evolution of futures and spot prices. At the same time we have an explicit connection between electricity futures and spot price processes. Furthermore, an important achievement is that the spot price dynamics in

this model becomes multi-dimensional Markovian. The Markovian structure is crucial for pricing of path dependent electricity options.

Contents

Introduction

Overview

In the recent years a variety of new financial markets has been developed. In the early 90s, several countries started to liberalize their electricity markets by leaving the price determination to the market principles of supply and demand. Many countries have since reformed their power sector. One important consequence is the emergence of trade of electricity delivery contracts on exchanges, similar to the trade of shares. The new freedom achieved has the drawback of increased uncertainty about the price development. The most significant challenge for pricing of derivatives is the non-storability of electricity, which implies that traditional valuation methods for storable commodities are not adequate. New approaches are required to price even the simplest energy derivatives.

At the same time insurance firms have introduced a new class of financial instruments that transfer catastrophe risk to the capital markets. Over the past decades the rise in insured losses has exploded from USD 2.5 billions per year to an average value of the aggregated insurance losses of USD 30.4 billions per year, in prices of 2006 (see [53]). Table 1 gives a summary of the ten most expensive natural catastrophes for the last 20 years. In particular, the increasing risks point out that a single catastrophe could ruin the whole insurance market. Therefore, actuaries started to look for alternative possibilities to transfer catastrophic risk.[1]

[1] For a general overview of the capital market insurance solutions see [51] and [52].

Insured Loss (USD Billions)	Year	Event	Country
66.3	2005	Hurricane Katrina; floods, dams burst, damage to oil rigs	U.S., Gulf of Mexico, Bahamas, North Atlantic
23.0	1992	Hurricane Andrew; flooding	U.S., Bahamas
21.4	2001	Terrorist attack on World Trade Center, Pentagon and other buildings	U.S.
19.0	1994	Northridge earthquake	U.S.
13.7	2004	Hurricane Ivan; damage to oil rigs	U.S., Caribbean
13.0	2005	Hurricane Wilma; torrential rain, floods	U.S., Mexico, Jamaica, Haiti
10.4	2005	Hurricane Rita; floods, damage to oil rigs	U.S., Gulf of Mexico, Cuba
8.6	2004	Hurricane Charley	U.S., Caribbean
8.4	1991	Typhoon Mireille	Japan
7.4	1989	Hurricane Hugo	U.S., Puerto Rico

Table 1: Top 10 Insured Catastrophe Losses (Source: Swiss Re, Sigma Nr. 2/2007).

In this thesis we consider recent developments in insurance and electricity financial products. In particular, we focus on two main issues. First, we consider the problem of pricing catastrophe insurance derivatives written on a loss index. Then we show that similar techniques can be applied for pricing flow commodity options. We specify fairly realistic models for catastrophe loss indexes and for electricity futures markets, where we provide explicit pricing formulas for European options using Fourier transform methods.

The thesis is organized as follows. We continue the introduction with the discussion of aforementioned new markets and give an overview of the existing models. In particular, we explain how our approaches are related to the previous ones. The main part of the thesis is divided into two parts. In the first part we focus on pricing of catastrophe insurance derivatives only. This part is based on [3] and [4]. In Part II we consider the modeling and the pricing of electricity products. Each part of the thesis is self-contained and has its own outline. However, in Part II we use Fourier transform methods for pricing European options introduced in Part I.

Catastrophe insurance options

In order to securitize increasing catastrophe risks, insurance companies have tried to take advantage of the vast potential of capital markets by introducing exchange-traded catastrophe insurance options. Exchange-traded insurance instruments present several advantages with respect to reinsurance. For instance, they offer lower transaction costs because they are standardized. Furthermore, they include minimal credit risk because the obligations are guaranteed by the exchange. A comprehensive comparison of insurance securities is given in [51] and [52]. In particular, catastrophe options are standardized contracts based on an index of catastrophe losses, for example compiled by Property Claim Service (PCS), an internationally recognized market authority on property losses from catastrophes in the US.

The first index-based catastrophe derivatives were CAT futures, which were introduced by the Chicago Board of Trade (CBOT) in 1992. Some models for the index underlying the CAT futures can be found in [1] and [10]. However, due to the structure of these products, there was only little trading activity on CAT futures in the market. A second version of catastrophe insurance derivatives were PCS options based on the index compiled by PCS. For the description of PCS catastrophe insurance options [41], [50] or [51] can be consulted. On its peak, the total capacity created by this version of insurance options amounted to 89 millions USD per year. Trading in PCS options slowed down in 1999 because of market illiquidity and lack of qualified personnel (see e.g. [51]).

However, the record losses caused by the hurricanes Katrina, Rita and Wilma in 2005 have been a catalyst for creating new derivative instruments to trade catastrophe risks in capital markets. Since March 2007, the New York Mercantile Exchange (NYMEX) has begun trading of catastrophe futures and options again. These new contracts have been designed to bring the transparency and liquidity of the capital markets to the insurance sector. They have provided effective ways of protection against property catastrophe risk and have given the investors the opportunity to trade a new class of assets which has little or no correlation with other exchange traded asset classes. The NYMEX catastrophe options are settled against the Re-Ex loss index, which is created from the data supplied by PCS.

The structure of catastrophe options can be described as follows. The option is written on an index that evolves over two periods, the *loss period* and the *development period*. During the contract specific *loss period* $[0, T_1]$ the index measures catastrophic events that occur. In addition to the loss period, option users choose a *development period* $[T_1, T_2]$. During the *development period* damages of catastrophes occurred in the loss period are reestimated and continue to affect the index. The contract expires at the end of the chosen development period.

Since the introduction of catastrophe insurance derivatives in 1992, the pricing of these products has been a problem. The underlying loss index is not traded and hence the market becomes incomplete. It is then an open question how the pricing measure should be determined. The next challenge is that even for fairly simple models of the loss index the pricing problem is rather complicated.

To date, several approaches have been proposed to model a catastrophe index and to price catastrophe options written on it. In [41], [42] and [43], the underlying catastrophe index has been represented as a compound Poisson process with nonnegative jumps. However, no distinction between loss and reestimation periods has been made. In [9] and [40], the authors distinguish between loss and reestimation periods and model the index as an exponential Lévy process over each period. While technicalities for pricing purposes are simplified in this setting, the assumption of an exponential model for accumulated losses during the loss period seems to be quite unrealistic. For instance, it implies that later catastrophes are more severe than earlier ones, and that the index starts in a positive value (instead of starting at 0). Yet another model is proposed in [49] where immediate reestimation is assumed and modeled through individual reestimation factors for each catastrophe. However, no explicit pricing methods are obtained for this model.

In this thesis, we consider the distinction between loss and reestimation period as in [9] and [40], but propose two more realistic models for the loss index. To begin with, we assume in Chapter 1 that the index is described by a time-inhomogeneous compound Poisson process during the loss period, and that during the reestimation period the index is reestimated by a factor (common for all catastrophes) which is given as an exponential time inhomogeneous Lévy process. In this framework, we consider the problem of pricing European catastrophe options written on the index. Interpreting the option as a payoff on a two-dimensional asset, we are able to obtain analytical pricing formulas by employing Fourier transform techniques. To this end, we extend Fourier transform techniques for dampened payoff functions as introduced in [8] and [18] to the case of a general payoff depending on two factors. We conclude Chapter 1 by calculating explicitly the price of the most commonly traded catastrophe options in the market.

However, although the assumption of common reestimation factor is accepted among practitioners, it may be considered unrealistic because loss reestimation happens individually for each catastrophe and begins almost immediately after the catastrophic event. We resolve this problem in Chapter 2, where we offer an even more realistic model for the loss index that allows immediate loss reestimation. This approach includes the model proposed in [49] as a particular case. In Chapter 2 we assume that catastrophe occurrence is modeled by a Poisson process, and consider individual reestimation for each catastrophe where the initial estimate of every catastrophe loss is reestimated immediately by a positive martingale starting from the random time of loss occurrence. We then consider the pricing of catastrophe options written on the index. As in Chapter 1 we employ Fourier transform techniques in order to obtain option pricing formulas. To this end, we manage to simplify the calculation of the characteristic function of the index.

We mention in particular, that our approaches work for loss distributions with heavy tails, which is the appropriate tail behavior for catastrophe modeling. We then proceed to discuss the case when the reestimation factors are given by positive affine martingales. In this situation, we provide a characterization of positive affine (local) martingales. We explain our approaches more precisely in Part I. See also [3] and [4].

In our opinion the use of exchange traded insurance derivatives will play a crucial role in the securitization of increasing catastrophe risks in the future. For this purpose, one essential task is to develop quantitative tools that help to establish liquid trading of these instruments. We hope that this work contributes to this aim providing new insights into the pricing of catastrophe options.

Electricity pricing

In Part II of this thesis we consider the modeling of electricity markets. In the stochastic modeling of electricity markets, there are two main approaches in the literature (see e.g. [5], [26]). The first one starts with a stochastic model for the spot price and derives futures price dynamics from it by using the arbitrage theory. The second approach directly models the price dynamics of forward and futures contracts traded in electricity markets. We refer to [5] and [32] for an overview of the literature on electricity markets.

Spot price models have two major disadvantages. Since electricity is non-storable, the spot electricity price is not a tradeable asset. This implies that it is not easy to give a precise definition of spot prices in the electricity market (see [5], [32]). For the same reason the valuation methods for traded asset prices are not adequate. The second disadvantage is that the connection between the spot and futures prices is not straightforward (see [26]). The modeled dynamics of the entire futures curve can be rarely consistent with the actually observed curves. On the contrary, futures price models attempt to systematically describe changes of the entire curve.

However, futures price models, since they normally imply a very complex non-Markovian dynamics for the spot price, are not well suited for pricing of path dependent electricity products like, for example, swing options (see e.g. [25], [31] and [54]). Markovian property of the spot price is essential to solve the constrained stochastic optimal control problem of maximizing the expected profit of the path dependent options.

Another drawback of the aforementioned spot and futures models is the lack of flexibility to decouple spot and futures price evolution. By calibrating the futures price according to observed market data, it is no longer possible to control the spot price and vice versa. In [26] an approach is introduced which converts the flow commodity market into a money market. By a currency change correspondence is obtained between given electricity market and a market consisting of bonds and a risky asset. The significant benefit of this transformation is an additional source of randomness in the modeling of electricity prices. Namely, it is possible to calibrate the spot and futures processes independently including features of both electricity price processes. Furthermore, the approach of [26] allows to apply the full potential of the well-established interest rate theory for pricing electricity derivatives.

In this thesis we generalize the approach of [26] replacing, in the dynamics of the asset prices, the Brownian motion by a more general Lévy process taking into account the occurrence of spikes. Interest rate theory combined with change of numeraire techniques is used to find a new electricity spot price model with sufficiently flexible futures curve. In particular, our framework contains as a special case the commonly accepted model for electricity market, where the spot price process is an exponential of an Ornstein-Uhlenbeck process. In addition, we consider valuation of electricity products in this framework. Using Fourier transform techniques introduced in Part I, we provide analytical pricing formulas for European electricity options.

The valuable feature of our approach is that the dynamics becomes multidimensional Markovian (see Section 3.4). As mentioned above, the Markovian structure is significant to prove the dynamic programming principle needed to find viscosity solutions of Hamilton-Jacobi-Bellman equation associated with pricing of path dependent electricity products like swing options. See [54] for more details on dynamic programming principle and pricing of electricity derivatives known as tolling agreements. Note that the framework of [54] includes as a special case continuous time swing options previously studied in [39], [31] and [25]. We finish Part II with the derivation Hamilton-Jacobi-Bellman equation for the value function of a continuous time electricity swing option in our setting.

Part I

Pricing of catastrophe insurance options written on a loss index

Outline and main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We consider a financial market endowed with a *risk-free asset* with *deterministic* interest rate r_t, and the *possibility of trading catastrophe insurance options*, written on a loss index $L = (L_t)_{0 \le t \le T_2}$. In short, we define catastrophe insurance option as a European derivative written on the loss index L with maturity T_2 and payoff

$$h(L_{T_2}) > 0 \tag{1}$$

for a continuous payoff function $h : \mathbb{R} \mapsto \mathbb{R}_+$. Since we have assumed that the interest rate r is deterministic, without loss of generality, we can express the price process of the insurance derivative in discounted terms, i.e. we can set $r \equiv 0$.

Before we give the precise definition of the loss index process L in Chapters 1 and 2, let us recall the common structure of catastrophe insurance options following the description in [41], [9], [50], [40], and [51]. The catastrophe options are written on a loss index that evolves over two time periods, the *loss period* $[0, T_1]$ and the consecutive *development period* $[T_1, T_2]$. During the contract specific loss period the index measures catastrophic occurring events. After the loss period, option users can choose either a six-month or a twelve-month *development period* $[T_1, T_2]$, where the reestimates of catastrophe losses that occurred during the loss period continue to affect the index. The option contract matures at the end of the chosen development period T_2.

Here we consider two models for the loss index. Throughout Chapter 1 we assume that the reestimation begins at T_1 for all insurance claims that have occurred during the loss period. In reality the starting point of reestimation might differ from claim to claim. However, the approximation using one common starting point for reestimation is accepted among practitioners and can be found in the literature (see for example [9] and [40]). Technically, as we will see in Section 1.2, this assumption facilitates the derivation of explicit pricing formulas. The main results of Chapter 1 were recently published in [3].

In Chapter 2 we consider option pricing in a model with immediate reestimation of single loss occurrences. This more realistic model requires a more complex setting (see also [49] and [4]). Here we assume that at the time of catastrophe occurrence the reported losses are only estimates of the true losses, and these estimates are consecutively reestimated until the end T_2 of the development period. The loss index provides thus at any $t \in [0, T_2]$ an *estimation* of the accumulation of the final time (T_2) amounts of catastrophe losses that have occurred during the loss period. Let N_t, $t \in [0, T_1]$ denote the number of catastrophes up to time t, and U_i, $i = 1, ..., N_t$ the corresponding final amounts of the losses at time T_2 (which are unknown at time $0 \leq t < T_2$). Then the value L_t of the loss index can be expressed as

$$L_t = \sum_{i=1}^{N_{t \wedge T_1}} E\left[U_i \,|\, \mathcal{F}_t\right] , \qquad t \in [0, T_2] , \tag{2}$$

where the filtration $\{\mathcal{F}_t\,, t \in [0, T_2]\}$ represents the information available. If the number N_t of catastrophes is assumed to follow a Poisson process, then the structure of the index is a compound Poisson sum with martingales as summands. As we will see in Section 2.1.1, this model is more suitable for option pricing with heavy-tailed losses. We mention that Chapter 2 is based on [4].

The main results of Part I are analytical pricing formulas for catastrophe options traded in the market. To this end we employ Fourier transform techniques. In particular, we explicitly compute prices for call, put, and spread options, which are the typical instruments in the market. Furthermore, in Section 2.3 we discuss the case when the reestimation of losses are given by positive affine martingales and provide a characterization of positive affine (local) martingales.

More precisely, Part I is organized as follows. In Section 1.1 we specify our first model for the loss index. In Section 1.2 we introduce a class of structure preserving pricing measures, before we derive the price process of European style catastrophe options for the model introduced in Section 1.1 by using Fourier transform techniques. Finally, in Section 2.3 we compute explicitly the prices of the most common option types traded in the market. In particular, Section 1.4 is devoted to pricing in the case of heavy-tailed losses.

In Section 2.1 we present a more realistic and complicated model for the loss index. Further, in Sections 2.1.1–2.1.2 we consider the pricing of general European options in the model described in Section 2.1, before we explicitly compute prices for spread options in Section 2.2, which are the typical instruments in the market.

We conclude Part I with Section 2.3, where we discuss the special case of positive affine martingales as reestimation factors.

Chapter 1

Pricing of catastrophe options under assumption of common reestimation factor

1.1 Modeling of the loss index

Here, we model the loss index by the stochastic process $L = (L_t)_{0 \le t \le T_2}$ as follows:

i) For $t \in [0, T_1]$,

$$L_t = \sum_{j=1}^{N_t} Y_j \tag{1.1}$$

is a time inhomogeneous compound Poisson process, where

- N_t is a time inhomogeneous Poisson process with deterministic intensity $\lambda(t) > 0$,
- Y_j, $j = 1, 2, \ldots$, are positive i.i.d. random variables with distribution function G, independent of N_t.

Note that we allow for *seasonal behavior of loss occurrence* modeled by a time dependent intensity $\lambda(t)$.

ii) For $t \in [T_1, T_2]$

$$L_t = L_{T_1+u} = L_{T_1} Z_u, \quad u = t - T_1 \in [0, T_2 - T_1], \tag{1.2}$$

where Z_u is a process that represents the *reestimation factor* with

- $Z_0 = 1$ a.s.,
- $(L_t)_{t \le T_1}$ and $(Z_u)_{0 \le u \le T_2 - T_1}$ are independent.

We suppose that all investors in the market observe the past evolution of the loss index including the current value. Therefore, the *flow of information* is given by the filtration $(\mathcal{F}_t^0)_{0 \le t \le T_2}$ generated by the process L, which is of the form

- $\mathcal{F}_0^0 = \{\emptyset, \Omega\}$,
- $\mathcal{F}_t^0 := \sigma(L_u,\, u \le t) = \sigma(\sum_{j=1}^{N_u} Y_j,\, u \le t)$, for $t \in [0, T_1]$,
- $\mathcal{F}_t^0 := \sigma(L_u,\, u \le t) = \sigma(L_s,\, s \le T_1) \vee \sigma(Z_{u-T_1},\, T_1 < u \le t)$, for $t \in (T_1, T_2]$,
- $\mathcal{F}_{T_2}^0 \subseteq \mathcal{F}$.

We assume that the filtration $(\mathcal{F}_t^0)_{0 \le t \le T_2}$ is right-continuous. Let $(\mathcal{F}_t)_{0 \le t \le T_2}$ be the completion of the filtration $(\mathcal{F}_t^0)_{0 \le t \le T_2}$ with $\mathbb{P}$-null sets of $\mathcal{F}$.

It is reasonable to assume that the *reestimation is not biased* (see also [49]). Therefore, we suppose that $(Z_t)_{0 \le t \le T_2 - T_1}$ is a positive martingale with respect to the filtration $(\mathcal{F}_t)_{0 \le t \le T_2}$ of the form

$$Z_t = e^{X_t} \tag{1.3}$$

for a process $X = (X_t)_{0 \le t \le T_2 - T_1}$ such that $X_0 = 0$ a.s.. More precisely, in this section we assume that X_t is a *time inhomogeneous Lévy process.*

Definition 1.1.1. *An adapted stochastic process $(X_t)_{t \in [0,T]}$ with values in $\mathbb{R}$ is a* time inhomogeneous Lévy process *or a* process with independent increments and absolutely continuous characteristics*, if the following conditions hold:*

1. *X has independent increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t \leq T$.*

2. *For every $t \in [0,T]$, the law of X_t is characterized by the characteristic function*

$$E[e^{iuX_t}] = \exp\left\{\int_0^t \left(iub_s - \frac{1}{2}c_s u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iuxI_{\{|x|\leq 1\}})F_s(dx)\right) ds\right\}$$

with deterministic functions

$$\begin{aligned} b_\cdot &: [0,T] \to \mathbb{R}, \\ c_\cdot &: [0,T] \to \mathbb{R}^+, \\ F_\cdot &: [0,T] \to LM(\mathbb{R}), \end{aligned}$$

where $LM(\mathbb{R})$ is the family of Lévy measures $\nu(dx)$ on $\mathbb{R}$, i.e.

$$\int_{\mathbb{R}} (x^2 \wedge 1)\nu(dx) < \infty \quad \text{and} \quad \nu(\{0\}) = 0.$$

It is assumed that

$$\int_0^T \left(|b_s| + c_s + \int_{\mathbb{R}} (x^2 \wedge 1)F_s(dx)\right) ds < \infty.$$

The triplet $(b, c, F) := (b_s, c_s, F_s)_{s\in[0,T]}$ is called the characteristics *of X.*

Note that by Lemma 1.4 and Lemma 1.5 of [33] X is a semimartingale, and the semimartingale characteristics (B, C, ν) of X associated with the truncation function $h(x) = xI_{|x|\leq 1}$ are given by

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0,t] \times A) = \int_0^t \int_A F_s(dx)ds \tag{1.4}$$

for $A \in \mathcal{B}(\mathbb{R})$.
We assume the following exponential integrability condition.

(C1) There exists $\epsilon > 0$ such that for all $u \in [-(1+\epsilon), 1+\epsilon]$

$$E[e^{uX_t}] < \infty \quad \forall t \in [0,T].$$

By Lemma 1.6 of [33] this is equivalent to the following integrability condition on F_s:

(C1') There exists $\epsilon > 0$ such that for all $u \in [-(1+\epsilon), 1+\epsilon]$

$$\int_0^T \int_{\{|x|>1\}} e^{ux} F_s(dx)ds < \infty.$$

In particular, $E[Z_t] < \infty$ for all $t \in [0,T]$, if (C1) is in force. Furthermore we require the following condition on the characteristics

$$\text{(C2)} \qquad \int_0^t b_s ds + \frac{1}{2}\int_0^t c_s ds + \int_0^t \int_{\mathbb{R}} (e^x - 1 - h(x))F_s(dx)ds = 0,$$

which implies (see e.g. [18], Remark 3.1, and [29], Lemma 4.4) that $Z_t = e^{X_t}$ is a martingale. We note that (C2) also implies that the process

$$I_{\{x>1\}} e^x * \nu = \int_0^T \int_{\{x>1\}} e^{ux} F_s(dx)ds$$

has finite variation, or equivalently (by [27], Proposition 8.26) that $Z_t = e^{X_t}$ is a special semimartingale.

Further, as in [33] we obtain that X_t can be canonically represented as

$$X_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}} x(\mu(ds,dx) - F_s(dx)ds), \qquad (1.5)$$

where W_t is a standard Brownian motion and μ is the integer-valued random measure associated with the jumps of X_t.

Remark 1.1.2. By assuming time-inhomogeneous Lévy process to model Z_t, we allow for time dependent reestimation behavior. For example, one could imagine that the reestimation frequency is higher in the beginning than later on.

Another possible choice of the reestimation factor Z_t is a positive affine martingale. In Section 2.3 we give a characterization of this class of processes.

Example 1.1.3. In particular, our framework includes the case when

$$Z_t = e^{X_t} \quad \text{and} \quad X_t = \int_0^t \sigma(s) dV_s, \tag{1.6}$$

where $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}$ is a càglàd (left continuous with right limits) deterministic function with $\sigma \neq 0$ a.s., and $V = (V_t)_{t \geq 0}$ is a Lévy process. See for example [6], [12], or [48] for more details on Lévy processes.

In this case Z_t is not a martingale, as requested by the assumption that the reestimation is unbiased. However, by using the following lemma, we justify why we can directly consider the process Z_t of the form (1.6) as a model for the reestimation factor.

Lemma 1.1.4. *Consider the process* $Z_t = \exp\{\int_0^t \sigma(s) dV_s\}$ *defined in* (1.6) *such that*

$$E[Z_t] < \infty, \quad \forall t \geq 0. \tag{1.7}$$

Let (b, c, ν) *be the characteristic triplet of the Lévy process* V *and let* $\psi : \mathbb{R} \to \mathbb{C}$ *be the characteristic exponent of* V*, i.e.*

$$E[e^{iuV_t}] = e^{t\psi(u)}, \quad u \in \mathbb{R}.$$

Then

$$E[Z_t] = E\left[e^{\int_0^t \sigma(s) dV_s}\right] = e^{\int_0^t \psi(-i\sigma(u)) du} < \infty,$$

where

$$\psi(-i\sigma(u)) = \frac{1}{2} c\sigma^2(u) + b\sigma(u) + \int_{-\infty}^{\infty} \left(e^{\sigma(u)x} - 1 - ux 1_{|x| \leq 1}\right) \nu(dx). \tag{1.8}$$

Proof. By Theorem 25.17 in [49] we have that $\psi(-iu)$ is well-defined in $u \in \mathbb{R}$ if

$$E\left[e^{uV_t}\right] < \infty$$

for some $t > 0$ (or equivalently for every $t > 0$) and then

$$E\left[e^{uV_t}\right] = e^{t\psi(-iu)} < \infty, \tag{1.9}$$

where

$$\psi(-iu) = \frac{1}{2} cu^2 + bu + \int_{-\infty}^{\infty} \left(e^{ux} - 1 - ux 1_{|x| \leq 1}\right) \nu(dx).$$

Following the proof of Proposition 3.14 of [12], for $\sigma(t)$ we consider a piecewise constant left-continuous approximation $\sigma^{\Delta}:\mathbb{R}_+\mapsto\mathbb{R}$,

$$\sigma^{\Delta}(u)=\sum_{j=1}^{N}\sigma_j I_{(t_{j-1},t_j]}(u).$$

Then

$$\int_0^t \sigma^{\Delta}(u)dV_u=\sum_{i=1}^{N}\sigma_j(V_{t_j}-V_{t_{j-1}})$$

and by (1.7) and (1.9) we obtain

$$\begin{aligned} E[e^{\int_0^t \sigma^{\Delta}(u)dV_u}] &= \prod_{j=1}^{N} E[e^{\sigma_j(V_{t_j}-V_{t_{j-1}})}]=\prod_{j=1}^{N}E[e^{\sigma_j V_{t_j-t_{j-1}}}] \\ &= \prod_{j=1}^{N} e^{(t_j-t_{j-1})\psi(-i\sigma_j)}=e^{\int_0^t \psi(-i\sigma^{\Delta}(u))du}<\infty. \end{aligned} \tag{1.10}$$

Equality (1.10) can be extended to an arbitrary càglàd function σ.

□

By Lemma 1.1.4 we have that

$$Z_t\cdot e^{-\int_0^t \psi(-i\sigma(u))du}=e^{\int_0^t \sigma(u)dV_u-\int_0^t \psi(-i\sigma(u))du}$$

is a martingale. Since $e^{-\int_0^t \psi(-i\sigma(u))du}$ is deterministic, the presence of this deterministic multiplicative factor in the expression for Z_t will not play any role in the computation of Section 1.2. Hence, without loss of generality we can assume that the reestimation factor Z_t is of the form (1.6).

Now we consider the problem of pricing of insurance European derivatives with payoff depending on the value L_{T_2} of the loss index at maturity T_2.

1.2 Pricing of catastrophe options

1.2.1 Pricing measure

In the catastrophe insurance market the underlying index L is not traded. Hence the market is incomplete and there exist infinitely many equivalent martingale measures. If we can include in the capital market the presence of a reinsurance portfolio, then the reinsurance portfolio specifies a premium process p_t for the overall insured losses $(L_t)_{t\geq 0}$. The premium p_t defines the price at time t of the remaining risk $L_{T_2} - L_t$ (see also [14]). If the insurance market is liquid enough, we can consider p_t as the price of an asset. In this way the loss index could be approximated by an insurance portfolio. Consider now a contingent claim $H = h(L_{T_2})$ defined in (1). Recall that h is a nonnegative continuous deterministic function and that we consider the price processes of all derivatives in discounted terms. Therefore, in the absence of arbitrage, given an equivalent martingale measure $\mathbb{Q}$, the premium price and the price of an insurance derivatives that pays out $H = h(L_{T_2})$ at the maturity are given by

$$p_t^{\mathbb{Q}} = E^{\mathbb{Q}}\left[L_{T_2} - L_t \Big| \mathcal{F}_t\right]$$

and

$$\pi_t^{\mathbb{Q}} = E^{\mathbb{Q}}\left[h(L_{T_2}) \Big| \mathcal{F}_t\right], \tag{1.11}$$

respectively. The problem is now how to choose an equivalent martingale measure $\mathbb{Q}$.

We make here the usual assumption that under the pricing measure $\mathbb{Q}$ the index process is described by the same kind of process as under $\mathbb{P}$. This means that we assume that:

(A1) Z_t remains a positive martingale under $\mathbb{Q}$;

(A2) Before T_1, L_t remains a compound Poisson process, otherwise it would be possible to obtain information on the next catastrophe;

(A3) N, Z, Y_i remain mutually independent, otherwise under $\mathbb{Q}$ the reestimation would be influenced by the catastrophes previously occurred. This would also mean that the agent believes that different catastrophes are estimated differently.

In particular, we assume that the class of pricing measures is determined by Radon-Nykodym derivatives of the following form: Since hypothesis (A3) is in force and taking into account that the reestimation factor Z is already a martingale, we choose a measure with the density given by

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp&\left\{\sum_{j=1}^{N_{T_1}} \beta(Y_j) - \int_0^{T_1} \lambda_s ds E\left[e^{\beta(Y_1)} - 1\right]\right\}\\ \times \exp&\left\{\int_0^T \gamma(s) dW_s - \frac{1}{2}\int_0^T \gamma^2(s) ds\right\}\\ \times \exp&\left\{\int_0^T \ln\phi(s,x)\left(\mu(ds,dx) - F_s(dx)ds\right)\right.\\ &\left.- \int_0^T \int_{\mathbb{R}} (\phi(s,x) - 1 - \ln\phi(s,x)) F_s(dx) ds\right\} \qquad (1.12)\end{aligned}$$

for some Borel function β with $E\left[e^{\beta(Y_1)}\right] < \infty$ and positive deterministic integrands $\phi(t,x)$ and $\gamma(t)$ such that $E[\frac{d\mathbb{Q}}{d\mathbb{P}}] = 1$.

By Girsanov's Theorem for Brownian motion and random measures (see [27]) this class of pricing measures preserves the structure of our model. In particular, under the measure $\mathbb{Q}$ the process $L_t,\ t \in [0, T_1]$, is again a time inhomogeneous compound Poisson process with intensity

$$\lambda_t^{\mathbb{Q}} = \lambda_t E[e^{\beta(Y_1)}] \qquad (1.13)$$

and distribution function of jumps

$$dG^{\mathbb{Q}}(y) = \frac{e^{\beta(y)}}{E[e^{\beta(Y_1)}]} dG(y). \qquad (1.14)$$

Further, under $\mathbb{Q}$ the process X is again a time inhomogeneous Lévy process independent of $L_t,\ t \in [0, T_1]$, with characteristics $(b^{\mathbb{Q}}, c^{\mathbb{Q}}, F^{\mathbb{Q}})$ given by

$$\begin{aligned} b_t^{\mathbb{Q}} &= b_t - \gamma_t \sqrt{c_t},\\ c_t^{\mathbb{Q}} &= c_t,\\ F_t^{\mathbb{Q}}(dx) &= \phi(t,x) F_t(dx).\end{aligned}$$

In order to specify a pricing measure $\mathbb{Q}$, one possible method is now to calibrate β, ϕ and γ to observed market prices. For example, in [43] the pricing measure is calibrated on the prices of insurance portfolios (i.e. from the premiums) and the prices of catastrophe derivatives. Another approach to pick a pricing measure is chosen in [9], [40] and [49], where the choice of the pricing measure for catastrophe insurance options is motivated through an equilibrium argument between the premium price and the price of an insurance derivative written on the same catastrophe losses. In [9] and [40] the Esscher transform is used to compute the equivalent martingale measure, which is justified by looking at a representative investor maximizing her expected utility.

Here we do not discuss the problem of choosing β, ϕ and γ, but we assume to be given an equivalent martingale measure $\mathbb{Q}$ of the form (1.12) and proceed to the risk neutral pricing under $\mathbb{Q}$ of catastrophe options as described in the next section.

1.2.2 Pricing via Fourier transform techniques

Now, let us return to the price process $\pi_t^{\mathbb{Q}}$ given in (1.11). By (1.2) we can rewrite (1.11) as

$$\pi_t^{\mathbb{Q}} = E^{\mathbb{Q}}\left[h(L_{T_1} Z_{T_2-T_1})|\mathcal{F}_t\right] = E^{\mathbb{Q}}\left[h(L_{T_1} e^{X_{T_2-T_1}})|\mathcal{F}_t\right] .$$

Interpreting the claim as a payoff on two factors, we can rewrite the price process as

$$\pi_t^{\mathbb{Q}} = E^{\mathbb{Q}}\left[g(L_{T_1}, X_{T_2-T_1})|\mathcal{F}_t\right], \tag{1.15}$$

where $g : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is defined by

$$g(x_1, x_2) := h(x_1 e^{x_2}) \;\text{ for any } (x_1, x_2) \in \mathbb{R}^2. \tag{1.16}$$

In the following we will calculate the expected payoff in (1.15) by Fourier transform techniques. To this end we extend the approach of dampened payoffs on one dimensional assets of [45] (see also [8]) to general payoffs on two dimensional assets. We impose the following hypotheses:
Assume that

(H1) $I_1 := \{(\alpha, \beta) \in \mathbb{R}^2 | \int_{\mathbb{R}^2} e^{-\alpha x_1 - \beta x_2} g(x_1, x_2) dx_1 dx_2 < \infty\} \neq \emptyset$.

Let

$$I_2 := \{(\alpha, \beta) \in \mathbb{R}^2 | \int_{\mathbb{R}^2} e^{\alpha x_1 + \beta x_2} G^{\mathbb{Q}}_{(L_{T_1}, X_{T_2 - T_1})}(dx_1, dx_2) < \infty\},$$

where $G^{\mathbb{Q}}_{(L_{T_1}, X_{T_2 - T_1})}$ is the cumulative distribution function of $(L_{T_1}, X_{T_2 - T_1})$ under $\mathbb{Q}$. Assume that

(H2) $I_1 \cap I_2 \neq \emptyset$.

Note that, since by Assumption (A3), L_{T_1} and $X_{T_2 - T_1}$ remain independent under $\mathbb{Q}$, it follows that

$$I_2 = \{(\alpha, \beta) \in \mathbb{R}^2 | E^{\mathbb{Q}}[e^{\alpha L_{T_1}}] < \infty \text{ and } E^{\mathbb{Q}}[e^{\beta X_{T_2 - T_1}}] < \infty\}. \tag{1.17}$$

Now, the *dampened payoff function* is introduced as

$$f(x_1, x_2) = e^{-\alpha x_1 - \beta x_2} g(x_1, x_2) \quad \text{for } (\alpha, \beta) \in I_1 \cap I_2. \tag{1.18}$$

Note, that under Hypothesis (H1), we have that

$$f(\cdot) \in L^1(\mathbb{R}^2)$$

for $(\alpha, \beta) \in I_1 \cap I_2$. Hence the Fourier transform

$$\hat{f}(u_1, u_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x_1 u_1 + x_2 u_2)} f(x_1, x_2) dx_1 dx_2 \tag{1.19}$$

is well defined for every $u = (u_1, u_2) \in \mathbb{R}^2$. Assuming also

(H3) $\hat{f}(\cdot) \in L^1(\mathbb{R}^2)$,

we get by the Inversion Theorem (cf. [33], Section 8.2) that

$$f(x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(x_1 u_1 + x_2 u_2)} \hat{f}(u_1, u_2) du_1 du_2. \tag{1.20}$$

Remark 1.2.1. Note that the equality in (1.20) holds everywhere and not only almost everywhere because we have assumed a continuous payoff function h. If the probability distribution of L_{T_2} would have a Lebesgue density, an almost everywhere equality in (1.20) would have been sufficient for the following computations. However, since the loss index is driven by a compound Poisson process, the distribution of L_{T_2} has atoms and we need an everywhere equality to guarantee (1.21) below.

Returning to the valuation problem (1.15), we obtain that

$$\begin{aligned}
\pi_t^{\mathbb{Q}} &= E^{\mathbb{Q}}\left[g(L_{T_1}, X_{T_2-T_1})|\mathcal{F}_t\right] = E^{\mathbb{Q}}\left[e^{\alpha L_{T_1}+\beta X_{T_2-T_1}} f(L_{T_1}, X_{T_2-T_1})|\mathcal{F}_t\right] \\
&= \frac{1}{2\pi} E^{\mathbb{Q}}\Big[e^{\alpha L_{T_1}+\beta X_{T_2-T_1}} \int_{\mathbb{R}^2} e^{-i(u_1 L_{T_1}+u_2 X_{T_2-T_1})} \hat{f}(u_1,u_2)du_1 du_2 \Big|\mathcal{F}_t\Big] \quad (1.21) \\
&= \frac{1}{2\pi} E^{\mathbb{Q}}\Big[\int_{\mathbb{R}^2} e^{-i\{(u_1+i\alpha)L_{T_1}+(u_2+i\beta)X_{T_2-T_1}\}} \hat{f}(u_1,u_2)du_1 du_2 \Big|\mathcal{F}_t\Big] \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} E^{\mathbb{Q}}\Big[e^{-i\{(u_1+i\alpha)L_{T_1}+(u_2+i\beta)X_{T_2-T_1}\}}|\mathcal{F}_t\Big] \hat{f}(u_1,u_2)du_1 du_2 \quad (1.22) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} E^{\mathbb{Q}}\Big[e^{-i(u_1+i\alpha)L_{T_1}}|\mathcal{F}_t\Big] E^{\mathbb{Q}}\Big[e^{-i(u_2+i\beta)X_{T_2-T_1}}|\mathcal{F}_t\Big] \hat{f}(u_1,u_2)du_1 du_2, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.23)
\end{aligned}$$

where in the equality (1.22) we could apply Fubini's theorem, because Hypothesis (H3) holds. The last equation holds by the independence of L_{T_1} and $X_{T_2-T_1}$ and by (1.17).

Since L is a time inhomogeneous compound Poisson process until T_1 and X is a time inhomogeneous Lévy process independent of $L_t,\ t \in [0, T_1]$, we can explicitly compute the conditional expectations in (1.23) by using the known form of the conditional characteristic functions:

1. If $t < T_1$, we have

$$\begin{aligned}
& E^{\mathbb{Q}}\left[e^{-i(u_1+i\alpha)L_{T_1}}|\mathcal{F}_t\right] \\
&= e^{-i(u_1+i\alpha)L_t} E^{\mathbb{Q}}\left[e^{-i(u_1+i\alpha)(L_{T_1}-L_t)}\right] \\
&= e^{-i(u_1+i\alpha)L_t} \exp\{-\int_t^{T_1} \lambda_s^{\mathbb{Q}} ds \int_0^\infty (1-e^{-i(u_1+i\alpha)x}) G^{\mathbb{Q}}(dx)\} \\
&= e^{-\int_t^{T_1}\lambda_s^{\mathbb{Q}} ds} e^{-i(u_1+i\alpha)L_t} \exp\{\int_t^{T_1} \lambda_s^{\mathbb{Q}} ds \int_0^\infty e^{-i(u_1+i\alpha)x} G^{\mathbb{Q}}(dx)\},
\end{aligned}$$

and

$$\begin{aligned}
&E^{\mathbb{Q}}\left[e^{-i(u_2+i\beta)X_{T_2-T_1}}|\mathcal{F}_t\right] = E^{\mathbb{Q}}\left[e^{-i(u_2+i\beta)X_{T_2-T_1}}\right]\\
&= \exp\{\int_0^{T_2-T_1}\left(i(u_2+i\beta)b_s^{\mathbb{Q}} - \frac{1}{2}c_s^{\mathbb{Q}}(u_2+i\beta)^2\right)ds\}\\
&\times \exp\{\int_0^{T_2-T_1}\int_{\mathbb{R}}(e^{i(u_2+i\beta)x} - 1 - i(u_2+i\beta)xI_{\{|x|\leq 1\}})F_s^{\mathbb{Q}}(dx)ds\}\,.
\end{aligned}$$

2. If $t \in [T_1, T_2]$,

$$E^{\mathbb{Q}}\left[e^{-i(u_1+i\alpha)L_{T_1}}|\mathcal{F}_t\right] = e^{-i(u_1+i\alpha)L_{T_1}}\;;$$

and

$$\begin{aligned}
&E^{\mathbb{Q}}\left[e^{-i(u_2+i\beta)X_{T_2-T_1}}|\mathcal{F}_t\right]\\
&= e^{-i(u_2+i\beta)X_{t-T_1}}E^{\mathbb{Q}}\left[e^{-i(u_2+i\beta)(X_{T_2-T_1}-X_{t-T_1})}\right] \qquad (1.24)\\
&= e^{-i(u_2+i\beta)X_{t-T_1}}\exp\{\int_{t-T_1}^{T_2-T_1}\left(i(u_2+i\beta)b_s^{\mathbb{Q}} - \frac{1}{2}c_s^{\mathbb{Q}}(u_2+i\beta)^2\right)ds\\
&\times \exp\{\int_{t-T_1}^{T_2-T_1}\int_{\mathbb{R}}(e^{i(u_2+i\beta)x} - 1 - i(u_2+i\beta)xI_{\{|x|\leq 1\}})F_s^{\mathbb{Q}}(dx)\,ds\}.
\end{aligned}$$

Example 1.2.2. Let us return to Example 1.1.3, where X is the process defined in (1.6). In this case we can simplify the characteristic function in (1.24):

$$E^{\mathbb{Q}}\left[e^{-i(u_2+i\beta)(X_{T_2-T_1}-X_{t-T_1})}\right] = \exp\{\int_{t-T_1}^{T_2-T_1}\psi^{\mathbb{Q}}(-(u_2+i\beta)\sigma(s))ds\},$$

where $\psi^{\mathbb{Q}}$ is the time-independent characteristic exponent of the Lévy process V under $\mathbb{Q}$, i.e $E^{\mathbb{Q}}[e^{iuV_t}] = e^{t\psi^{\mathbb{Q}}(u)}$.

Hence, in order to calculate the price process $(\pi_t^{\mathbb{Q}})_{t\in[0,T_2]}$ the only remaining task is to compute the Fourier transform of the dampened payoff function f.

We summarize our results in the following

Theorem 1.2.3. *Under the Hypotheses (H1)-(H3), the price process $\pi_t^{\mathbb{Q}}$ of a catastrophe insurance option written on the loss index with maturity T_2 and payoff $h(L_{T_2}) > 0$ is given*

1. *for* $t \in [0, T_1]$ *by*

$$\pi_t^{\mathbb{Q}} = \frac{1}{2\pi} e^{-\int_t^{T_1} \lambda_s^{\mathbb{Q}} ds} \int_{\mathbb{R}^2} \hat{f}(u_1, u_2) e^{-i(u_1 + i\alpha) L_t} \exp\{\int_t^{T_1} \lambda_s^{\mathbb{Q}} ds \int_0^{\infty} e^{-i(u_1+i\alpha)x} G^{\mathbb{Q}}(dx)\}$$

$$\times \exp\{\int_0^{T_2 - T_1} \left(i(u_2 + i\beta) b_s^{\mathbb{Q}} - \frac{1}{2} c_s^{\mathbb{Q}} (u_2 + i\beta)^2 \right) ds\}$$

$$\times \exp\{\int_0^{T_2 - T_1} \int_{\mathbb{R}} \left(e^{i(u_2 + i\beta)x} - 1 - i(u_2 + i\beta) x I_{\{|x| \leq 1\}} \right) F_s^{\mathbb{Q}}(dx) ds\} du_1 du_2,$$

and

2. *for* $t > T_1$ *by*

$$\pi_t^{\mathbb{Q}} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(u_1, u_2) e^{-i(u_1 + i\alpha) L_{T_1}} e^{-i(u_2 + i\beta) X_{t - T_1}}$$

$$\times \exp\{\int_{t-T_1}^{T_2 - T_1} \left(i(u_2 + i\beta) b_s^{\mathbb{Q}} - \frac{1}{2} c_s^{\mathbb{Q}} (u_2 + i\beta)^2 \right) ds\}$$

$$\times \exp\{\int_{t-T_1}^{T_2 - T_1} \int_{\mathbb{R}} \left(e^{i(u_2 + i\beta)x} - 1 - i(u_2 + i\beta) x I_{\{|x| \leq 1\}} \right) F_s^{\mathbb{Q}}(dx) ds\} du_1 du_2.$$

Here f *is the dampened payoff as defined in* (1.18)*, and* $\hat{f}$ *is its Fourier transform* (1.19)*.*

Remark 1.2.4. In order to estimate $\pi_t^{\mathbb{Q}}$ numerically several methods are possible. One commonly used technique is the fast Fourier transform (FFT). In our case we need to apply FFT for a double integral which implies reduced speed of convergence. There exist various techniques to improve the convergence speed (see for example the “integration-along-cut” method suggested in [7]). However, speed becomes an issue only when one repeatedly needs to price a large number of options. For further discussion on this topic we refer to [12].

Remark 1.2.5. In this section we have chosen to model Z_t by a time inhomogeneous Lévy process. This class of processes is very rich and flexible to model a wide range of phenomena, and at the same time it is analytically very tractable. Note, however, that all the calculations go through exactly in the same way even for other choices of processes for Z_t, as long as the conditional characteristic function is known. See also Section 2.3 for another possible choice for Z.

1.3 Applications: call, put and spread catastrophe options

In this section we consider the most common catastrophe insurance options traded in the market: call, put, and spread options. By computing explicitly the Fourier transform corresponding to the payoff, we are able to provide pricing formulas for these options using Theorem 1.2.3.

Example 1.3.1 (Call option).

Consider the payoff function of a catastrophe call option in the form

$$h_{\text{call}}(x) = (x - K)^+ \tag{1.25}$$

for some strike price $K > 0$. Then the corresponding payoff on a two dimensional asset as introduced in (1.16) is

$$g_{\text{call}}(x_1, x_2) = (x_1 e^{x_2} - K)^+ I_{\{x_1>0\}} = (x_1 e^{x_2} - K) I_{\{x_1>0,\ x_2>\ln \frac{K}{x_1}\}},$$

and the dampened payoff function is

$$\begin{aligned} f_{\text{call}}(x_1, x_2) &= e^{-\alpha x_1 - \beta x_2} g_{\text{call}}(x_1, x_2) \\ &= e^{-\alpha x_1 - \beta x_2} (x_1 e^{x_2} - K) I_{\{x_1>0,\ x_2>\ln \frac{K}{x_1}\}}. \end{aligned} \tag{1.26}$$

Note that f_{call} belongs to $L^1(\mathbb{R}^2)$ for all $(\alpha, \beta) \in I_1 = (0, \infty) \times (1, \infty)$. For the Fourier transform $\hat{f}_{\text{call}}$ we obtain

$$\begin{aligned}
\hat{f}_{\text{call}}(u_1,u_2) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(x_1u_1+x_2u_2)} f_{\text{call}}(x_1,x_2)dx_1dx_2 \\
&= \frac{1}{2\pi}\int_0^\infty \int_{\ln\frac{K}{x_1}}^\infty e^{-(\alpha-iu_1)x_1-(\beta-iu_2)x_2}(x_1e^{x_2}-K)dx_2dx_1 \\
&= \frac{1}{2\pi}\Bigg[\int_0^\infty x_1e^{-(\alpha-iu_1)x_1}\int_{\ln\frac{K}{x_1}}^\infty e^{-(\beta-1-iu_2)x_2}dx_2dx_1 \\
&\quad -K\int_0^\infty e^{-(\alpha-iu_1)x_1}\int_{\ln\frac{K}{x_1}}^\infty e^{-(\beta-iu_2)x_2}dx_2dx_1\Bigg] \\
&= \frac{1}{2\pi}\Bigg[\frac{1}{\beta-1-iu_2}\int_0^\infty x_1e^{-(\alpha-iu_1)x_1}e^{-(\beta-1-iu_2)\ln K/x_1}dx_1 \\
&\quad -\frac{K}{\beta-iu_2}\int_0^\infty e^{-(\alpha-iu_1)x_1}e^{-(\beta-iu_2)\ln K/x_1}dx_1\Bigg] \\
&= \frac{1}{2\pi}\Bigg[\frac{1}{(\beta-1-iu_2)K^{(\beta-1-iu_2)}}\int_0^\infty x_1^{\beta-iu_2}e^{-(\alpha-iu_1)x_1}dx_1 \\
&\quad -\frac{1}{(\beta-iu_2)K^{(\beta-1-iu_2)}}\int_0^\infty x_1^{\beta-iu_2}e^{-(\alpha-iu_1)x_1}dx_1\Bigg] \\
&= \frac{1}{2\pi}\frac{\int_0^\infty x_1^{\beta-iu_2}e^{-(\alpha-iu_1)x_1}dx_1}{(\beta-1-iu_2)(\beta-iu_2)K^{(\beta-1-iu_2)}} \\
&= \frac{1}{2\pi}\frac{\Gamma(\beta+1-iu_2)}{(\beta-1-iu_2)(\beta-iu_2)(\alpha-iu_1)^{(\beta+1-iu_2)}K^{(\beta-1-iu_2)}},
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

To prove that the payoff function of a catastrophe call option (1.25) satisfies the conditions of Theorem 1.2.3, it remains to show that

$$\hat{f}_{\text{call}}(u_1,u_2) \in L^1(\mathbb{R}^2). \tag{1.27}$$

Note that to prove (1.27) it is sufficient to consider the asymptotics of $|\hat{f}_{\text{call}}(u_1,u_2)|$ for $|u_1|,|u_2| \to \infty$. In fact, since

$$\lim_{|u_2|\to\infty} |\Gamma(\beta+1-iu_2)|e^{\frac{\pi}{2}|u_2|}|u_2|^{-\beta-\frac{1}{2}} = \sqrt{2\pi} \tag{1.28}$$

(see 8.328.1 in [23]), we get

$$
\begin{aligned}
|\hat{f}_{\text{call}}(u_1,u_2)| &= \frac{1}{2\pi}\frac{1}{K^{\beta-1}|e^{-iu_2\ln K}|} \\
&\quad\times \frac{|\Gamma(\beta+1-iu_2)||e^{iu_2(\ln|\alpha-iu_1|-i\arctan\frac{u_1}{\alpha})}|}{|(\beta-1-iu_2)(\beta-iu_2)(\alpha-iu_1)^{(\beta+1)}|} \\
&= \frac{1}{2\pi}\frac{1}{K^{\beta-1}}\frac{|\Gamma(\beta+1-iu_2)|e^{u_2\arctan\frac{u_1}{\alpha}}}{|(\beta-1-iu_2)(\beta-iu_2)(\alpha-iu_1)^{(\beta+1)}|} \\
&\sim \frac{1}{2\pi}\frac{1}{K^{\beta-1}}\frac{\sqrt{2\pi}e^{-\frac{\pi}{2}|u_2|}|u_2|^{\beta+\frac{1}{2}}e^{u_2\arctan\frac{u_1}{\alpha}}}{|u_2|^2|u_1|^{\beta+1}} \\
&\sim \frac{1}{\sqrt{2\pi}}\frac{1}{K^{\beta-1}}\frac{e^{-\frac{\pi}{2}|u_2|}|u_2|^{\beta-\frac{3}{2}}e^{u_2\arctan\frac{u_1}{\alpha}}}{|u_1|^{\beta+1}},
\end{aligned}
\tag{1.29}
$$

where

$$
f_1(u_1,u_2)\sim f_2(u_1,u_2) \quad :\Leftrightarrow \quad \lim_{|u_1|,|u_2|\to\infty}\frac{|f_1(u_1,u_2)|}{|f_2(u_1,u_2)|}=1.
$$

Now we distinguish the following cases:

1. If $u_1u_2<0$, then (1.29) simplifies to

$$
\begin{aligned}
|\hat{f}_{\text{call}}(u_1,u_2)| &\sim \frac{1}{\sqrt{2\pi}}\frac{1}{K^{\beta-1}}\frac{e^{-\frac{\pi}{2}|u_2|}|u_2|^{\beta-\frac{3}{2}}e^{-|u_2\arctan\frac{u_1}{\alpha}|}}{|u_1|^{\beta+1}} \\
&\sim \frac{1}{\sqrt{2\pi}}\frac{1}{K^{\beta-1}}\frac{e^{-\pi|u_2|}|u_2|^{\beta-\frac{3}{2}}}{|u_1|^{\beta+1}},
\end{aligned}
\tag{1.30}
$$

   where the right hand side of (1.30) is integrable at infinity.

2. If $u_1u_2>0$, then (1.29) is equivalent to

$$
|\hat{f}_{\text{call}}(u_1,u_2)| \sim \frac{1}{\sqrt{2\pi}}\frac{1}{K^{\beta-1}}\frac{e^{-\frac{\pi}{2}|u_2|}|u_2|^{\beta-\frac{3}{2}}e^{|u_2|\arctan\frac{|u_1|}{\alpha}}}{|u_1|^{\beta+1}} \tag{1.31}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\frac{1}{K^{\beta-1}}\frac{e^{-\frac{\pi}{2}|u_2|}|u_2|^{\beta-\frac{3}{2}}e^{|u_2|(\frac{\pi}{2}-\arctan\frac{\alpha}{|u_1|})}}{|u_1|^{\beta+1}} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{K^{\beta-1}}\frac{|u_2|^{\beta-\frac{3}{2}}e^{-|u_2|\arctan\frac{\alpha}{|u_1|}}}{|u_1|^{\beta+1}} \\
&\sim \frac{1}{\sqrt{2\pi}}\frac{1}{K^{\beta-1}}\frac{|u_2|^{\beta-\frac{3}{2}}e^{-|u_2|\frac{\alpha}{|u_1|}}}{|u_1|^{\beta+1}}.
\end{aligned}
\tag{1.32}
$$

Since

$$\int_0^\infty \frac{u_2^{\beta-\frac{3}{2}} e^{-u_2 \frac{\alpha}{|u_1|}}}{|u_1|^{\beta+1}} du_2 = \alpha^{\frac{1}{2}-\beta}\Gamma(\beta - \frac{1}{2})|u_1|^{-\frac{3}{2}}$$

is integrable at infinity, the right hand side of (1.32) is integrable as $|u_1|, |u_2| \to \infty$.

We can thus apply Theorem 1.2.3 and obtain an explicit price for the call option.

Once we know the price for call options, pricing of catastrophe insurance put and spread options can be reduced to the pricing of call options with standard arguments as we show in the next examples.

Example 1.3.2 (Put option)**.** Let

$$h_{put}(x) = (K - x)^+$$

be the payoff of a catastrophe insurance put option. Then the payoffs of call and put options with the same strike K are related through the formula

$$h_{put}(x) = h_{call}(x) + K - L_{T_2}.$$

We can thus determine the price $\pi^{\mathbb{Q}}_{\text{put}}(t)$ of the put option by computing the price $\pi^{\mathbb{Q}}_{\text{call}}(t)$ of the call option and the following call-put parity:

$$\begin{aligned}\pi^{\mathbb{Q}}_{put}(t) &= \pi^{\mathbb{Q}}_{call}(t) + K - E^{\mathbb{Q}}[L_{T_2}|\mathcal{F}_t]\\ &= \pi^{\mathbb{Q}}_{call}(t) + K - E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1}|\mathcal{F}_t].\end{aligned}$$

For the conditional expectation $E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1}|\mathcal{F}_t]$ we get by independence of $(L_t)_{t\le T_1}$ and $(Z_{T_1+u})_{u\le T_2-T_1}$ that

1. if $t \le T_1$,

$$\begin{aligned}&E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1}|\mathcal{F}_t] = E^{\mathbb{Q}}[L_{T_1}|\mathcal{F}_t]E^{\mathbb{Q}}[Z_{T_2-T_1}|\mathcal{F}_t]\\ &= (L_t + E^{\mathbb{Q}}[L_{T_1} - L_t])E^{\mathbb{Q}}[e^{X_{T_2-T_1}}] = (L_t + E^{\mathbb{Q}}[Y_1]\int_t^{T_1} \lambda^{\mathbb{Q}}_s ds)\\ &\times \exp\{\int_0^{T_2-T_1} \left(b^{\mathbb{Q}}_s + \frac{1}{2}c^{\mathbb{Q}}_s + \int_{\mathbb{R}} (e^x - 1 + xI_{\{|x|\le 1\}})F^{\mathbb{Q}}_s(dx) \right) ds\};\end{aligned}$$

2. if $t \in [T_1, T_2]$,

$$\begin{aligned} & E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1}|\mathcal{F}_t] = E^{\mathbb{Q}}[L_{T_1} e^{X_{T_2-T_1}}|\mathcal{F}_t] \\ & = L_{T_1} Z_{t-T_1} E^{\mathbb{Q}}[\exp\{X_{T_2-T_1} - X_{t-T_1}\}] = L_{T_1} Z_{t-T_1} \\ & \times \exp\{\int_{t-T_1}^{T_2-T_1} \left(b_s^{\mathbb{Q}} + \frac{1}{2} c_s^{\mathbb{Q}} + \int_{\mathbb{R}} (e^x - 1 + x I_{\{|x| \le 1\}}) F_s^{\mathbb{Q}}(dx) \right) ds\}. \end{aligned}$$

Example 1.3.3 (Call and put spread options)**.** A call spread option is a capped call option which is a combination of buying a call option with strike price K_1, and selling at the same time a call option with the same maturity but with the strike price $K_2 > K_1$. This corresponds to a payoff function at maturity of the form

$$\begin{aligned} h_{spread}(x) &= (x - K_1)^+ - (x - K_2)^+ \\ &= \begin{cases} 0, & \textit{if } 0 \le x \le K_1; \\ x - K_1, & \textit{if } K_1 < x \le K_2; \\ K_2 - K_1, & \textit{if } x > K_2. \end{cases} \end{aligned}$$

The price of the catastrophe call spread option is thus the difference of the prices of the call options with strike prices K_1 and K_2 respectively.
Analogously we can calculate the price of a put spread catastrophe option using the results in Example 1.3.2.

Remark 1.3.4. Note that for the above computations the damping parameter α in (1.26) has to be strictly bigger than zero. By (H2) and (1.17) this implies that the distribution $G^{\mathbb{Q}}$ of the claim sizes $Y_i,\ i = 1, 2, \ldots,$ has to fulfill

$$\int_{\mathbb{R}_+} e^{\alpha x} G^{\mathbb{Q}}(dy) < \infty, \quad \text{for some } \alpha > 0\,. \tag{1.33}$$

Typical examples of the distributions satisfying (1.33) are the exponential, Gamma, and truncated normal distributions. An important class of distribution functions which also satisfy (1.33) is the class of convolution equivalent distribution functions $\mathcal{S}(\alpha)$ for $\alpha > 0$, which is convenient for the modeling

of the claim sizes. See [34] for the definition and properties, and [35] for an application of the convolution equivalent distributions. The generalized inverse Gaussian distribution is one of the most important examples of the convolution equivalent distributions.
On the other hand, distributions $G^{\mathbb{Q}}$ with heavy tails do not fulfill (1.33) (they would require $\alpha \leq 0$). However, because the class of heavy tailed distributions is very relevant for catastrophe claim size modeling, we will in the next subsection specify a framework, in which we can also price catastrophe options with heavy tailed claims.

1.4 Pricing with heavy-tailed losses

In order to treat heavy-tailed losses, i.e. to be able to take a damping parameter $\alpha = 0$ in (1.17), we make the assumption that the distribution function $G^{\mathbb{Q}}$ of Y_i, $i = 1, 2, \ldots$, has support on (ϵ, ∞) for some $\epsilon > 0$. In other words, we assume that if a catastrophe occurs then the corresponding loss amount is greater than some small $\epsilon > 0$. This assumption is obviously no serious restriction, especially in the light of the fact that PCS defines a catastrophe as a single incident or a series of related incidents (man-made or natural disasters) that causes insured property losses of at least \$25 million. Note that this implies

$$\{L_{T_1} > 0\} = \{L_{T_1} > \epsilon\}, \tag{1.34}$$

since L is a time inhomogeneous compound Poisson process until T_1.

In this framework we now want to apply the Fourier technique of Section 1.2.2 to price a catastrophe put option. To this end we first perform the following transformations. The price process of a catastrophe put option is given by

$$\pi_t^{\mathbb{Q}} = E^{\mathbb{Q}}\left[(K - L_{T_1} e^{X_{T_2 - T_1}})^+ | \mathcal{F}_t\right]. \tag{1.35}$$

Since L is a time inhomogeneous compound Poisson process until T_1 under

$\mathbb{Q}$, we can rewrite (1.35) as

$$\begin{aligned}\pi_t^{\mathbb{Q}} &= E^{\mathbb{Q}}\left[(K - L_{T_1}e^{X_{T_2-T_1}})^+ I_{\{N_{T_1}=0\}}|\mathcal{F}_t\right]\\ &\quad + E^{\mathbb{Q}}\left[(K - L_{T_1}e^{X_{T_2-T_1}})^+ I_{\{N_{T_1}>0\}}|\mathcal{F}_t\right]\\ &= K\mathbb{Q}(N_{T_1}=0|\mathcal{F}_t) + E^{\mathbb{Q}}\left[(K - L_{T_1}e^{X_{T_2-T_1}})^+ I_{\{L_{T_1}>0\}}|\mathcal{F}_t\right], \qquad (1.36)\end{aligned}$$

where we have used that $L_{T_1}I_{\{N_{T_1}=0\}} = 0$.

Let $\bar{L}_{T_1} := L_{T_1} - \epsilon$. Then by (1.34)

$$\{L_{T_1} > 0\} = \{L_{T_1} > \epsilon\} = \{\bar{L}_{T_1} + \epsilon > \epsilon\} = \{\bar{L}_{T_1} > 0\}.$$

Hence we obtain

$$\begin{aligned}&E^{\mathbb{Q}}\left[(K - L_{T_1}e^{X_{T_2-T_1}})^+ I_{\{L_{T_1}>0\}}|\mathcal{F}_t\right]\\ &= E^{\mathbb{Q}}\left[(K - (\bar{L}_{T_1}+\epsilon)e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1}>0\}}|\mathcal{F}_t\right]. \qquad (1.37)\end{aligned}$$

Define the pay off function g by

$$g(x_1, x_2) = (K - (x_1+\epsilon)e^{x_2})^+ I_{\{x_1>0\}}.$$

In order to apply the Fourier method of Theorem 1.2.3, we continuously extend g from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}^2$ as

$$\bar{g}(x_1, x_2) = (K - (|x_1|+\epsilon)e^{x_2})^+.$$

Then we have

$$\begin{aligned}E^{\mathbb{Q}}\left[\bar{g}(\bar{L}_{T_1}, X_{T_2-T_1})|\mathcal{F}_t\right] &= E^{\mathbb{Q}}\left[(K - (\bar{L}_{T_1}+\epsilon)e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1}>0\}}|\mathcal{F}_t\right]\\ &+ E^{\mathbb{Q}}\left[(K - (|\bar{L}_{T_1}|+\epsilon)e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1}\le 0\}}|\mathcal{F}_t\right]. \qquad (1.38)\end{aligned}$$

Since $\{\bar{L}_{T_1} \le 0\} = \{L_{T_1} = 0\} = \{\bar{L}_{T_1} = -\epsilon\}$, the second term on the right-hand side of (1.38) is

$$\begin{aligned}&E^{\mathbb{Q}}\left[(K - (|\bar{L}_{T_1}|+\epsilon)e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1}\le 0\}}|\mathcal{F}_t\right]\\ &= E^{\mathbb{Q}}\left[(K - 2\epsilon e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1}=-\epsilon\}}|\mathcal{F}_t\right]\\ &= E^{\mathbb{Q}}\left[(K - 2\epsilon e^{X_{T_2-T_1}})^+|\mathcal{F}_t\right]\mathbb{Q}(\bar{L}_{T_1}=-\epsilon|\mathcal{F}_t)\\ &= E^{\mathbb{Q}}\left[(K - 2\epsilon e^{X_{T_2-T_1}})^+|\mathcal{F}_t\right]\mathbb{Q}(L_{T_1}=0|\mathcal{F}_t)\\ &= E^{\mathbb{Q}}\left[(K - 2\epsilon e^{X_{T_2-T_1}})^+|\mathcal{F}_t\right]\mathbb{Q}(N_{T_1}=0|\mathcal{F}_t). \qquad (1.39)\end{aligned}$$

Together, equations (1.36)–(1.39) lead to the following expression for the price process of a put option.

Proposition 1.4.1. *Under Assumption* (1.34)*, the price process of a catastrophe put option is given by*

$$\pi_t^{\mathbb{Q}} = KP_t^0 + P_t^1 P_t^0 + P_t^2,$$

where

$$\begin{aligned}
P_t^0 &= e^{-\int_t^{T_1} \lambda^{\mathbb{Q}}(s)ds} I_{\{N_t=0\}}, \\
P_t^1 &= E^{\mathbb{Q}}\left[(K - 2\epsilon e^{X_{T_2-T_1}})^+ | \mathcal{F}_t\right], \\
P_t^2 &= E^{\mathbb{Q}}\left[(K - (|\bar{L}_{T_1}| + \epsilon)e^{X_{T_2-T_1}})^+ | \mathcal{F}_t\right] .
\end{aligned}$$

Proof. Given equations (1.36)–(1.39), it only remains to validate the expression for P^0. Since N_t is a time inhomogeneous Poisson process with deterministic intensity $\lambda^{\mathbb{Q}}(t) > 0$ under $\mathbb{Q}$, we have

$$\begin{aligned}
\mathbb{Q}(N_{T_1} = 0|\mathcal{F}_t) &= \mathbb{Q}((N_{T_1} - N_t) + N_t = 0|\mathcal{F}_t) \\
&= \mathbb{Q}((N_{T_1} - N_t) + n = 0|\mathcal{F}_t)|_{n=N_t} \\
&= e^{-\int_t^{T_1} \lambda^{\mathbb{Q}}(s)ds} I_{\{N_t=0\}}.
\end{aligned}$$

□

Note that P_t^1 is the price process of a regular put option written on a one dimensional asset that is given by an exponential Lévy process. This price can be obtained by Fourier transform techniques or any other favorite method. To use in one dimension the Fourier transform methods of Theorem 1.2.3, one computes that the dampened pay off

$$f_2(x_2) := (K - 2\epsilon e^{x_2})^+ e^{\beta x_2} \quad \text{for } \beta > 1$$

has the Fourier transform

$$\begin{aligned}
\hat{f}_2(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\ln \frac{K}{2\epsilon}} e^{iux_2} e^{\beta x_2} (K - 2\epsilon e^{x_2}) dx_2 \\
&= \frac{K}{\sqrt{2\pi}} \left(\frac{K}{2\epsilon}\right)^{\beta+iu} \frac{1}{(\beta+iu)(\beta+1+iu)} \in L^1(\mathbb{R}).
\end{aligned}$$

In order to calculate the last term P_t^2 of the put price process $\pi_t^{\mathbb{Q}}$ we can now use Theorem 1.2.3 with damping parameter $\alpha = 0$ (which then allows for heavy tailed loss distributions by Remark 1.3.4). For this purpose we check that Hypothesis (H1)–(H3) hold true. First we consider the dampened function

$$f_1(x_1, x_2) := e^{\beta x_2}\bar{g}(x_1, x_2) = e^{\beta x_2}(K - (|x_1| + \epsilon)e^{x_2})^+ \quad \text{for } \beta > 1.$$

Since $f_1 \in L^1(\mathbb{R}^2)$, we have $(0, -\beta) \in I_1$ for all $\beta > 1$. Hence Hypothesis (H1) is satisfied for $\beta > 1$ and $\alpha = 0$. We assume that $E^{\mathbb{Q}}[e^{\beta X_{T_2 - T_1}}] < \infty$ for some $\beta > 1$. Then by (1.17), we have $(0, \beta) \in I_2 \cap I_1$. Thus (H2) is also satisfied.

Remark 1.4.2. *Note that we can now admit heavy-tailed loss distributions, because we don't need to dampen in x_1 anymore, since $\alpha = 0$.*

To check (H3) we consider the Fourier transform of f_1:

$$\begin{aligned}
\hat{f}_1(u_1, u_2) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(x_1u_1 + x_2u_2)} f_1(x_1, x_2)dx_1dx_2 \\
&= \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} e^{i(x_1u_1 + x_2u_2)} e^{\beta x_2}(K - (|x_1| + \epsilon)e^{x_2}) I_{\{|x_1| \le Ke^{-x_2} - \epsilon,\, x_2 \le \ln\frac{K}{\epsilon}\}} dx_1dx_2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\ln\frac{K}{\epsilon}} \int_{-Ke^{-x_2}+\epsilon}^{Ke^{-x_2}-\epsilon} e^{i(u_1x_1 + u_2x_2)} e^{\beta x_2}(K - (|x_1| + \epsilon)e^{x_2})dx_1dx_2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\ln\frac{K}{\epsilon}} e^{iu_2x_2} e^{(\beta+1)x_2} \frac{1 - \cos(u_1(Ke^{-x_2} - \epsilon))}{u_1^2} dx_2.
\end{aligned}$$

Lemma 1.4.3. *There exists $C > 0$ such that*

$$|\hat{f}_1(u_1, u_2)|(1 + u_2^2u_1 + u_1^2 + u_2^2) \le C \quad \textit{for all } u_1, u_2 \in \mathbb{R}. \tag{1.40}$$

Proof. We prove Lemma 1.4.3 in four steps:

1. Since $f_1 \in L^1(\mathbb{R}^2)$, $\hat{f}_1$ is bounded, i.e. there exists $0 < C_1 < \infty$ such that
$$|\hat{f}_1(u_1, u_2)| \le C_1 \quad \text{for all } u_1, u_2 \in \mathbb{R}.$$

2. Then we have

$$|\hat{f}_1(u_1,u_2)|u_1^2 \le \frac{1}{2\pi}\int_{-\infty}^{\ln\frac{K}{\epsilon}} 2e^{(\beta+1)x_2}dx_2 = \frac{1}{\pi}\frac{1}{\beta+1}\left(\frac{K}{\epsilon}\right)^{\beta+1} =: C_2 < \infty.$$

3. Integrating by parts we obtain

$$\begin{aligned}
&|\hat{f}_1(u_1,u_2)|u_2^2 \\
&= \frac{1}{2\pi u_1^2}\left|\int_{-\infty}^{\ln\frac{K}{\epsilon}} \frac{\partial^2}{\partial x_2^2}(e^{iu_2x_2})\cdot e^{(\beta+1)x_2}\Big(1-\cos(u_1(Ke^{-x_2}-\epsilon))\Big)dx_2\right| \\
&= \frac{1}{2\pi u_1^2}\Big|\int_{-\infty}^{\ln\frac{K}{\epsilon}} \frac{\partial}{\partial x_2}(e^{iu_2x_2})\cdot e^{(\beta+1)x_2}\Big((\beta+1)\Big(1-\cos(u_1(Ke^{-x_2}-\epsilon))\Big) \\
&\quad -\sin(u_1(Ke^{-x_2}-\epsilon))u_1Ke^{-x_2}\Big)\,dx_2\Big| \\
&= \frac{1}{2\pi u_1^2}\Big|\int_{-\infty}^{\ln\frac{K}{\epsilon}} e^{iu_2x_2}\Big\{(\beta+1)^2e^{(\beta+1)x_2}\Big(1-\cos(u_1(Ke^{-x_2}-\epsilon))\Big) \\
&\quad -2(\beta+1)e^{\beta x_2}u_1K\sin(u_1(Ke^{-x_2}-\epsilon)) + e^{\beta x_2}u_1K\Big(\sin(u_1(Ke^{-x_2}-\epsilon)) \\
&\qquad +u_1Ke^{-x_2}\cos u_1(Ke^{-x_2}-\epsilon)\Big)\Big\}\,dx_2\Big|.
\end{aligned}$$

Substituting $s = s(x_2) := u_1(Ke^{-x_2}-\epsilon)$ we note that

$$\begin{aligned}
&|\hat{f}_1(u_1,u_2)|u_2^2 \le \frac{1}{2\pi u_1^2}\int_{-\infty}^{\ln\frac{K}{\epsilon}} \Big|(\beta+1)^2e^{(\beta+1)x_2}\Big(1-\cos s(x_2)\Big) \\
&\quad -(2\beta+1)e^{\beta x_2}u_1K\sin s(x_2) + u_1^2K^2e^{(\beta-1)x_2}\cos s(x_2)\Big|dx_2 \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\ln\frac{K}{\epsilon}} \Big((\beta+1)^2e^{(\beta+1)x_2}\frac{u_1^2(Ke^{-x_2}-\epsilon)^2}{2u_1^2} \\
&\quad +(2\beta+1)e^{\beta x_2}\left|\frac{\sin(u_1(Ke^{-x_2}-\epsilon))}{u_1}\right| + K^2e^{(\beta-1)x_2}|\cos s(x_2)|\Big)dx_2 \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\ln\frac{K}{\epsilon}} \Big((\beta+1)^2e^{(\beta+1)x_2}\frac{K^2e^{-2x_2}+\epsilon^2}{2} + (2\beta+1)e^{\beta x_2}|Ke^{-x_2}-\epsilon| \\
&\quad +K^2e^{(\beta-1)x_2}\Big)dx_2 =: C_3 < \infty.
\end{aligned}$$

4. Further we consider $|\hat{f}_1(u_1,u_2)|u_2^2u_1$. Since for $0<|u_1|<1$ we have

$$|\hat{f}_1(u_1,u_2)|u_2^2u_1 \le |\hat{f}_1(u_1,u_2)|u_2^2 \le C_3,$$

we can assume that $|u_1| > 1$. As above we get

$$\begin{aligned}
|\hat{f}_1(u_1,u_2)|u_2^2 u_1 &= \frac{1}{2\pi u_1}\Big|\int_{-\infty}^{\ln\frac{K}{\epsilon}} e^{iu_2x_2}\left\{(\beta+1)^2 e^{(\beta+1)x_2}(1-\cos s)\right.\\
&\quad \left. -2(\beta+1)e^{\beta x_2}u_1K\sin s + e^{\beta x_2}u_1K(\sin s + u_1Ke^{-x_2}\cos s)\right\}dx_2\Big|\\
&= \frac{1}{2\pi u_1}\Big|\int_{-\infty}^{\ln\frac{K}{\epsilon}} e^{iu_2x_2}\left\{(\beta+1)^2 e^{(\beta+1)x_2}(1-\cos s)\right.\\
&\quad \left. -(2\beta+1)e^{\beta x_2}u_1K\sin s + e^{(\beta-1)x_2}u_1^2K^2\cos s\right\}dx_2\Big|\\
&=: G(u_1).
\end{aligned}$$

Substituting $z = Ke^{-x_2} - \epsilon = \frac{s}{u_1}$ we rewrite $G(u_1)$ as

$$\begin{aligned}
G(u_1) &= \frac{1}{2\pi u_1}\Big|\int_0^\infty \left(\frac{K}{\epsilon+z}\right)^{iu_2}\left\{(\beta+1)^2\left(\frac{K}{\epsilon+z}\right)^{\beta+1}(1-\cos(u_1z))\right.\\
&\quad -(2\beta+1)u_1K\left(\frac{K}{\epsilon+z}\right)^{\beta}\sin(u_1z)\\
&\quad \left. + u_1^2K^2\left(\frac{K}{\epsilon+s}\right)^{\beta-1}\cos(u_1z)\right\}\frac{ds}{\epsilon+z}\Big|\\
&\le C_4 + \frac{K^{\beta+1}u_1}{2\pi}\Big|\int_0^\infty\left(\frac{1}{\epsilon+z}\right)^{\beta+iu_2}\cos u_1z dz\Big|\\
&= C_4 + \frac{K^{\beta+1}}{2\pi}\Big|\int_0^\infty(\beta+iu_2)\left(\frac{1}{\epsilon+z}\right)^{\beta+1+iu_2}\sin u_1z dz\Big|\\
&\le C_4 + \frac{K^{\beta+1}}{2\pi\epsilon^{\beta+1}} =: C_4 + C_5 < \infty.
\end{aligned}$$

Now (1.40) holds with $C := \sum_{i=1}^5 C_i$. □

Corollary 1.4.4. *The Fourier transform $\hat{f}_1$ belongs to $L^1(\mathbb{R}^2)$, i.e. (H3) is satisfied.*

Proof. By Lemma 1.4.3 we have

$$\begin{aligned}
\int_{\mathbb{R}^2}|\hat{f}_1(u_1,u_2)|du_1du_2 &\le C\int_{\mathbb{R}^2}\frac{1}{1+u_2^2(1+|u_1|)+u_1^2}du_2du_1\\
&= 2\pi C\int_0^\infty\frac{1}{\sqrt{(1+u_1^2)(1+u_1)}}du_1 < \infty.
\end{aligned}$$

□

Hence all assumptions necessary to apply Theorem 1.2.3 and to calculate P_t^2 with a damping parameter $\alpha = 0$ are satisfied, and we can compute prices of put options including heavy tail distributed catastrophe losses. Pricing of catastrophe call and spread options can then be obtained by using call-put parity arguments as in Examples 1.3.2–1.3.3.

Chapter 2

Pricing of catastrophe options under assumption of immediate loss reestimation

2.1 Modeling the loss index

Motivated by the index structure (2) elaborated in the introductory section of Part I, we now model the stochastic process $L = (L_t)_{0\leq t\leq T_2}$ representing the loss index as

$$L_t = \sum_{j=1}^{N_{t\wedge T_1}} Y_j A^j_{t-\tau_j}, \qquad t \in [0, T_2] \,, \tag{2.1}$$

where

(L1) $N_s, s \in [0, T_2]$, is a Poisson process with intensity $\lambda > 0$ and jump times τ_j , that models the number of catastrophes occurring during the loss period.

(L2) Y_j, $j = 1, 2, \ldots$, are positive i.i.d. random variables with distribution function F^Y, that represent the first loss estimation at the time of occurrence of the j-th catastrophe.

(L3) $A^j_s, s \in [0, T_2],\ j = 1, 2, \ldots$, are positive i.i.d. martingales such that

$$A^j_0 = 1, \quad \forall j = 1, 2, \ldots.$$

(L4) $A^j, Y_j,\ j = 1, 2, \ldots$, and N are independent.

In the sequel we will often drop the index j and simply write Y and A in some formulas, when only the probability distribution of the objects matters.

The martingales A^j_t represent the *unbiased reestimation factors*. Here we assume that reestimation begins *immediately* after the occurrence of the j-th catastrophe with initial loss estimate Y_j, individually for each catastrophe.

We here suppose that market participants observe the evolution of the individual catastrophe losses. Note that observing the market quotes of the catastrophe index L alone is in general not sufficient for the knowledge of the single reestimation factors A. However, it might not be unrealistic to assume that market participants are able to obtain additional information about the evolution of individual catastrophes. Therefore, we assume the market information filtration $(\mathcal{F}_t)_{0 \le t \le T_2}$ to be the right continuous completion of the filtration generated by the catastrophe occurrences N, the corresponding initial loss estimates Y_j, and the corresponding reestimation factors A^j.

2.1.1 Pricing of insurance derivatives

We consider again the problem of pricing a European option with payoff depending on the value L_{T_2} of the loss index at maturity T_2. In the catastrophe insurance market the underlying index L is not traded. Hence the market is highly incomplete and the choice of the pricing measure is not obvious.

As in Section 1.2.1 we adopt here the common approach that the risk neutral pricing measure is structure preserving for the model, except that the pricing measure might introduce a drift into the reestimation martingales $A^j,\ j = 1, 2, \ldots$. At this place we don't discuss further the choice of the pricing measure. Therefore, to simplify the notation and without loss of generality, we perform pricing with the model specification given under $\mathbb{P}$, where we substitute the hypothesis (L3) with

(L3') $A_s^j, s \in [0, T_2]$, $j = 1, 2, \ldots$, are positive i.i.d. semimartingales such that

$$A_0^j = 1, \quad \forall j = 1, 2, \ldots.$$

Recall that a catastrophe insurance option is a European derivative H written on the loss index with maturity T_2 and payoff

$$h(L_{T_2}) > 0$$

for a payoff function $h : \mathbb{R} \mapsto \mathbb{R}_+$. Analogously to Section 1.2.2 we consider the price processes of the catastrophe option H in discounted terms, i.e the price process of H is given by

$$\pi_t = E\left[h(L_{T_2})|\mathcal{F}_t\right], \quad t \in [0, T_2]. \tag{2.2}$$

In the following we will calculate the expected payoff in (2.2) by using Fourier transform techniques. To this end, we impose the following conjectures:

(C1) The payoff function h is continuous.

(C2) There exists $k \in \mathbb{R}$ such that

$$h - k \in L^2(\mathbb{R}) = \left\{ g : \mathbb{R} \to \mathbb{C} \text{ measurable} \,\middle|\, \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty \right\}.$$

Remark 2.1.1. In Chapter 1 we were able to consider more general options that did not necessarily fulfill (C2) by considering dampened payoffs. However, as we have seen in Subsection 1.4, the cost of this approach is that treating heavy tailed distributions of claim sizes Y becomes more complicated. The approach in this section allows for general claim size modeling, including distributions with heavy tails. Further, as we will see in Subsection 2.2, Conjecture (C2) is satisfied by call and put spread catastrophe insurance options, the typical options traded in the market.

Now let

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz}(h(z) - k)dz, \quad \forall u \in \mathbb{R},$$

be the Fourier transform of $h - k$ and assume that

(C3) $\hat{h} \in L^1(\mathbb{R})$.

Note that Conjecture (C2) does not necessarily imply (C3). Since (C2) and (C3) are in force, the following inversion formula holds (see cf. [33], Section 8.2)

$$h(x) - k = \int_{-\infty}^{\infty} e^{iux} \hat{h}(u)du\,. \tag{2.3}$$

Note that Remark 1.2.1 is also in force here because of (C1).
By (2.3) and (C3) we obtain

$$\begin{aligned} \pi_t &= E[h(L_{T_2})|\mathcal{F}_t] = E[h(L_{T_2}) - k|\mathcal{F}_t] + k \\ &= E\left[\int_{-\infty}^{\infty} e^{iuL_{T_2}} \hat{h}(u)du|\mathcal{F}_t\right] + k \qquad (2.4) \\ &= \int_{-\infty}^{\infty} E\left[e^{iuL_{T_2}}|\mathcal{F}_t\right] \hat{h}(u)du + k, \qquad (2.5) \end{aligned}$$

where in the last equation we could apply Fubini's theorem because of (C3).

Hence, in order to calculate the price process $(\pi_t)_{t\in[0,T_2]}$ in (2.5), the essential task is to compute the conditional characteristic function of L_{T_2}

$$c_t(u) := E\left[e^{iuL_{T_2}}|\mathcal{F}_t\right] = E\left[\exp\left\{iu\sum_{j=1}^{N_{T_1}} Y_j A^j_{T_2-\tau_j}\right\}\middle|\mathcal{F}_t\right], \quad u \in \mathbb{R}, \tag{2.6}$$

for $t \in [0, T_2]$. We define the conditional characteristic function of the reestimation martingale A^j as

$$\psi_t^{A^j}(s,u) := E\left[e^{iuA^j_s}\middle|\mathcal{F}_t^{A^j}\right], \quad 0 \le t \le s \le T_2\,, \tag{2.7}$$

where $\mathcal{F}_t^{A^j} := \sigma(A^j_s\,,\, 0 \le s \le t)$ is the filtration generated by the j-th reestimation factor. The main result of this part is

Theorem 2.1.2. *The conditional characteristic function* (2.6) *of the loss index* L_{T_2} *is given*

1. *for* $t < T_1$ *by*

$$c_t(u) = \exp\left\{-\lambda(T_1 - t)\left(1 - E\left[\psi_0^A(T_2 - U, uY)\right]\right)\right\} \times \prod_{j=1}^{N_t} \psi_{t-s_j}^{A^j}(T_2 - s_j, uy_j)\Big|_{s_j=\tau_j\,, y_j=Y_j}, \qquad u \in \mathbb{R};$$

2. *for* $t \in [T_1, T_2]$ *by*

$$c_t(u) = \prod_{j=1}^{N_{T_1}} \psi_{t-s_j}^{A^j}(T_2 - s_j, uy_j)\Big|_{s_j=\tau_j\,, y_j=Y_j}, \quad u \in \mathbb{R}\,.$$

Here U *is a uniformly distributed on* $[t, T_1]$ *random variable, and* Y *is a random variable with the distribution function* F^Y *and independent of* U.

Note that in Theorem 2.1.2, the times of catastrophe occurrence τ_j and the initial loss estimates Y_j up to time t are known data.

2.1.2 Proof of Theorem 2.1.2

We distinguish the computations over the two periods.
1) For $t \in [0, T_1]$ we get by Assumption (L4) and by the independent increments of N_t that

$$\begin{aligned} c_t(u) &= E\left[\exp\left\{iu\left(\sum_{j=1}^{N_t} Y_j A^j_{T_2-\tau_j} + \sum_{j=N_t+1}^{N_{T_1}} Y_j A^j_{T_2-\tau_j}\right)\right\}\Bigg|\, \mathfrak{F}_t\right] \\ &= \underbrace{E\left[\exp\left\{iu\sum_{j=1}^{N_t} Y_j A^j_{T_2-\tau_j}\right\}\Bigg|\, \mathfrak{F}_t\right]}_{:=a_t(u)} \underbrace{E\left[\exp\left\{iu\sum_{j=N_t+1}^{N_{T_1}} Y_j A^j_{T_2-\tau_j}\right\}\Bigg|\, \mathfrak{F}_t\right]}_{:=b_t(u)}. \end{aligned} \tag{2.8}$$

We compute separately the terms a_t and b_t in (2.8). By Assumption (L4) for $a_t(u)\,, u \in \mathbb{R}$, we have

$$\begin{aligned}
a_t(u) &= E\left[\exp\left\{iu\sum_{j=1}^{N_t} Y_j A^j_{T_2-\tau_j}\right\}\middle|\,\mathcal{F}_t\right] \\
&= E\left[\exp\left\{iu\sum_{j=1}^{n} y_j A^j_{T_2-s_j}\right\}\middle|\,\mathcal{F}_t\right]_{n=N_t\,,\,s_j=\tau_j\,,\,y_j=Y_j} \\
&= \prod_{j=1}^{N_t} E\left[\exp\left\{iuy_j A^j_{T_2-s_j}\right\}\middle|\,\mathcal{F}_t\right]_{s_j=\tau_j\,,\,y_j=Y_j} \\
&= \prod_{j=1}^{N_t} E\left[\exp\left\{iuy_j A^j_{T_2-s_j}\right\}\middle|\,\mathcal{F}^{A^j}_{t-s_j}\right]_{s_j=\tau_j\,,\,y_j=Y_j} \\
&= \prod_{j=1}^{N_t} \psi^{A^j}_{t-s_j}(T_2-s_j, uy_j)\Big|_{s_j=\tau_j\,,\,y_j=Y_j}\,.
\end{aligned}$$

Note that for this first term the Y_j's, τ_j's, and N_t are known data, because the corresponding catastrophes have happened before t.

For the second term $b_t(u)\,, u \in \mathbb{R}$, we get again by (L4) and the independent increments of the Poisson process N

$$b_t(u) = E\left[\exp\{iu\sum_{j=N_t+1}^{N_{T_1}} Y_j A^j_{T_2-\tau_j}\}\middle|\,\mathcal{F}_t\right] = E\left[\exp\{iu\sum_{j=N_t+1}^{N_{T_1}} Y_j A^j_{T_2-\tau_j}\}\right]$$

$$\begin{aligned}
&= E\left[E\Big[e^{iu\sum_{j=1}^{n} y_j A^j_{T_2-s_j}}\Big|N_{T_1}-N_t, Y_1,\ldots,Y_{N_{T_1}-N_t}\tau_1,\ldots,\tau_{N_{T_1}-N_t}\Big]_{\substack{n=N_{T_1}-N_t,\\ y_j=Y_j,\\ s_j=\tau_j}}\right] \\
&= E\left[E\Big[\prod_{j=1}^{n}\psi^{A^j}_0(T_2-s_j,uy_j)\Big|N_{T_1}-N_t, Y_1,\ldots,Y_{N_{T_1}-N_t}\tau_1,\ldots,\tau_{N_{T_1}-N_t}\Big]_{\substack{n=N_{T_1}-N_t,\\ y_j=Y_j,\\ s_j=\tau_j}}\right] \\
&= E\left[\prod_{j=N_t+1}^{N_{T_1}}\psi^{A}_0(T_2-\tau_j,uY_j)\right]\,. \qquad (2.9)
\end{aligned}$$

By Theorem 5.2.1 of [47] we obtain the following result:

Lemma 2.1.3. *Let N_t be a Poisson process with jump times τ_j, $j = 1, 2, \ldots$. Then for all $0 \le t \le T$,*

$$(\tau_{N_t+1}, \ldots, \tau_{N_T} | \, N_T - N_t = n)$$

has the same distribution as the order statistics $(U_{(1)}, \ldots, U_{(n)})$, where U_j, $j = 1, \ldots, n$ are i.i.d. uniformly distributed on the interval $[t, T]$.

Using Lemma 2.1.3 and again Assumption (L4), we can replace the τ_j's in (2.9) with the order statistics $U_{(j)}$ of i.i.d. uniformly distributed random variables on the interval $[t, T_1]$ and get

$$b_t(u) = E\left[\prod_{j=N_t+1}^{N_{T_1}} \psi_0^A(T_2 - U_{(j)}, uY_j) \right], \quad u \in \mathbb{R}\,. \tag{2.10}$$

Next, we need the following simple auxiliary lemma

Lemma 2.1.4. *Consider the order statistics $U_{(1)}, \ldots, U_{(n)}$ of n i.i.d. random variables $U_1, \ldots, U_n$ and a bounded measurable function $f(x_1, \ldots, x_n)$ symmetric in its arguments. Then*

$$E\left[f(U_{(1)}, \ldots, U_{(n)})\right] = E\left[f(U_1, \ldots, U_n)\right].$$

Proof. We denote by Σ_n the set of all permutations. of $\{1, \ldots, n\}$

$$\begin{aligned} E\left[f(U_{(1)}, \ldots, U_{(n)})\right] &= E\left[\sum_{\sigma \in \Sigma_n} f(U_{\sigma(1)}, \ldots, U_{\sigma(n)}) I_{\{U_{\sigma(1)} < \cdots < U_{\sigma(n)}\}} \right] \\ &= E\Big[f(U_1, \ldots, U_n) \underbrace{\sum_{\sigma \in \Sigma_n} I_{\{U_{\sigma(1)} < \cdots < U_{\sigma(n)}\}}}_{1} \Big] \\ &= E[f(U_1, \ldots, U_n)]. \end{aligned}$$

□

By the i.i.d. assumption of the Y_j's and A^j's, we see that the function

$$f_u^n(s_1, \ldots, s_n) := E\left[\prod_{j=1}^{n} \psi_0^A(T_2 - s_j, uY_j) \right], \quad u \in \mathbb{R}\,,$$

is symmetric in $s_1 \dots, s_n$. It is then not difficult to see, using Lemma 2.1.4, that

$$\begin{aligned}
b_t(u) &= E\left[E\Big[\prod_{j=1}^{n} \psi_0^A(T_2 - s_j, uY_j) \Big| N_{T_1} - N_t,\, U_{(1)}, \dots, U_{(N_{T_1}-N_t)} \Big]_{\substack{n=N_{T_1}-N_t,\\ s_j=U_{(j)}}} \right] \\
&= E\left[f_u^n(s_1,\dots,s_n)\Big|_{\substack{n=N_{T_1}-N_t,\\ s_j=U_{(j)}}} \right] = E\left[f_u^n(U_{(1)},\dots,U_{(n)})\Big|_{n=N_{T_1}-N_t} \right] \\
&= E\left[f_u^n(U_1,\dots,U_n)\Big|_{n=N_{T_1}-N_t} \right] = E\left[\prod_{j=N_t+1}^{N_{T_1}} \psi_0^A(T_2 - U_j, uY_j) \right] \\
&= E\left[\exp\left\{ iu \sum_{j=N_t+1}^{N_{T_1}} Y_j A^j_{T_2-U_j} \right\} \right], \qquad (2.11)
\end{aligned}$$

where we have substituted the order statistics $U_{(j)}$ with the i.i.d. uniform variables U_j.

Note that (2.11) coincides with the characteristic function of a compound Poisson process of the form

$$\sum_{j=N_t+1}^{N_{T_1}} Z^j,$$

where $Z^j = Y_j A^j_{T_2-U_j}$, $j = 1, 2, \dots,$ are i.i.d. The form of the characteristic function is in this case well-known. Thus we can rewrite (2.11) as

$$\begin{aligned}
E\left[\exp\{iu \sum_{j=N_t+1}^{N_{T_1}} Y_j A^j_{T_2-U_j}\} \right] &= \exp\left\{ -\lambda(T_1 - t)\left(1 - E\left[e^{iuZ^1}\right]\right)\right\} \\
&= \exp\left\{ -\lambda(T_1 - t)\left(1 - E\left[\psi_0^A(T_2 - U, uY)\right]\right)\right\} .
\end{aligned}$$

This completes the proof for the case $t \le T_1$.

2) For the case when $t > T_1$, we get

$$c_t(u) = \prod_{j=1}^{N_{T_1}} \psi_{t-s_j}^{A^j}(T_2 - s_j, uy_j)\Big|_{s_j=\tau_j,\, y_j=Y_j}; \quad u \in \mathbb{R},$$

as for the term a_t in the case $0 \le t \le T_1$.

This completes the proof of Theorem 2.1.2.

Remark 2.1.5. In [49] a special case of our model is presented where the reestimation factor A is a geometric Brownian motion. In this case, the conditional characteristic function of the reestimation factor can be computed by numeric integration via

$$\begin{aligned}
\psi_t^A(s,u) &= E\left[e^{iu\exp(B_s - \frac{1}{2}s)} \middle| \mathcal{F}_t \right] = E\left[e^{iu\exp(B_t - \frac{1}{2}t)\exp\{B_s - B_t - \frac{1}{2}(s-t)\}} \middle| \mathcal{F}_t \right] \\
&= E\left[e^{iuw_t\exp(B_{s-t} - \frac{1}{2}(s-t))} \right] \Big|_{w_t = e^{B_t - \frac{1}{2}t}} \\
&= \int e^{iuw_t e^y} e^{-\frac{(y - \frac{1}{2}(s-t))^2}{2(s-t)}} dy \Big|_{w_t = e^{B_t - \frac{1}{2}t}}.
\end{aligned}$$

Here we note that no closed-form expression is known for the lognormal characteristic function. Moreover, the numerical computation of lognormal characteristic functions is a fairly challenging problem because the defining integral formulas are not well suited to the common numerical integration techniques. However, several approaches have been proposed to calculate the characteristic function of a lognormal random variable. For instance, in [38] two main methods were introduced. The first one is to solve a functional differential equation, applying the fact that the Fourier transform of the lognormal characteristic functions is known, and therefore the solution is unique. Another approach of [38] is to evaluate the characteristic function as a convergent series of Hermite functions. See [38] for more details. We refer also to [2] and [24] for further issues on the numerical computation of the characteristic function of a lognormal random variable.

In Section 2.3 we turn our attention to a class of reestimation processes where the conditional characteristic function is numerically tractable and in some cases analytically obtainable: affine processes. For further information on affine processes and their applications to mathematical finance, we refer to [16], [15] and [21].

2.2 Pricing of call and put spreads

We conclude Chapter 2 by applying the developed pricing method to call and put spread options, which are the typical catastrophe options traded in the market.

A call spread option with strike prices $0 < K_1 < K_2$ is a European derivative with the payoff function at maturity given by

$$h(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq K_1; \\ x - K_1, & \text{if } K_1 < x \leq K_2; \\ K_2 - K_1, & \text{if } x > K_2. \end{cases}$$

The integrability condition $h - k \in L^2(\mathbb{R}_+)$ is satisfied for $k := K_2 - K_1$. In particular, $h - k \in L^1(\mathbb{R}_+)$.

To satisfy (C1) and (C3) we continuously extend h from $\mathbb{R}_+$ to $\mathbb{R}$ by

$$\bar{h}(x) := \begin{cases} h(-x), & \text{if } x < 0; \\ h(x), & \text{if } x \geq 0. \end{cases}$$

Note that the price processes of the two corresponding options with payoffs $h(L_{T_2})$ and $\bar{h}(L_{T_2})$ remain the same, because $L_{T_2} \geq 0$.

Let

$$\hat{\bar{h}}(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz}(\bar{h}(z) - k)dz, \quad \forall u \in \mathbb{R},$$

be the Fourier transform of $\bar{h} - k$. Then

$$\begin{aligned} \hat{\bar{h}}(u) &= \frac{1}{2\pi} \left[\int_{-K_2}^{-K_1} e^{-iux}(-x - K_2)dx \right. \\ &\quad \left. + \int_{-K_1}^{K_1} e^{-iux}(K_1 - K_2)dx + \int_{K_1}^{K_2} e^{-iux}(x - K_2)dx \right] \\ &= \frac{1}{2\pi u^2} \left[e^{-iuK_2} + e^{iuK_2} - e^{-iuK_1} - e^{iuK_1} \right] \\ &= \frac{1}{\pi u^2} (\Re e^{iuK_2} - \Re e^{iuK_1}) = \frac{1}{\pi u^2} (\cos uK_2 - \cos uK_1) \in L^1(\mathbb{R}), \end{aligned}$$

and by applying the inversion formula (2.3) to $\bar{h}(x)$ for $x \geq 0$, we obtain that (2.3) holds also for h, since $h(x) = \bar{h}(x)$ for $x \geq 0$.

In particular since $L_{T_2} \geq 0$ a.s., for the price of the call spread we can write

$$\begin{aligned}
\pi_t^{CS} &= E[h(L_{T_2}) - k|\mathcal{F}_t] + k = E[\bar{h}(L_{T_2}) - k|\mathcal{F}_t] + k \\
&= E\left[\int_{-\infty}^{\infty} e^{iuL_{T_2}}\hat{\bar{h}}(u)du|\mathcal{F}_t\right] + k \\
&= \int_{-\infty}^{\infty} E\left[e^{iuL_{T_2}}|\mathcal{F}_t\right]\hat{\bar{h}}(u)du + k \\
&= \int_{-\infty}^{\infty} c_t(u)\hat{\bar{h}}(u)du + k \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{c_t(u)}{u^2}(\cos uK_2 - \cos uK_1)du + K_2 - K_1, \qquad (2.12)
\end{aligned}$$

where $c_t(u)$ is defined in (2.6). Note that the integral in (2.12) is real-valued, since $\Im c_t(-u) = -\Im c_t(u)$ by definition of c_t.

Analogously, for the put spread catastrophe option with payoff at the maturity given by

$$g(x) = \begin{cases} K_2 - K_1, & \text{if } 0 \leq x \leq K_1; \\ K_2 - x, & \text{if } K_1 < x \leq K_2; \\ 0, & \text{if } x > K_2, \end{cases}$$

we obtain

$$\pi_t^{PS} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{c_t(u)}{u^2}(\cos uK_1 - \cos uK_2)du. \qquad (2.13)$$

Note that the call-put parity is satisfied:

$$\pi_t^{PS} = K_2 - K_1 - \pi_t^{CS}.$$

2.3 Reestimation with positive affine processes

In this section we suppose that the reestimation factors are given by positive affine processes. Affine processes constitute a rich class of processes suitable to model a wide range of phenomena. At the same time the advantage is that the conditional characteristic function can be obtained explicitly up to the solution of two Riccati equations.

Definition 2.3.1. *A Markov process* $A = (A_t, \mathbb{P}_x)$ *on* $[0, \infty]$ *is called an* affine process *if there exist* $\mathbb{C}$*-valued functions* $\phi(t,u)$ *and* $\psi(t,u)$*, defined on* $\mathbb{R}_+ \times \mathbb{R}$*, such that for* $t \geq 0$

$$E\left[e^{iuA_{T_2}} \middle| \mathcal{F}_t\right] = e^{\phi(T_2-t,u)+\psi(T_2-t,u)A_t}. \tag{2.14}$$

We assume that

(A1) A is conservative, i.e. for every $t > 0$ and $x \geq 0$

$$\mathbb{P}_x[A_t < \infty] = 1.$$

(A2) A is stochastically continuous for every $\mathbb{P}_x$.

By Proposition 1.1 in [30] Assumption (A2) is equivalent to the assumption that $\phi(t,u)$ and $\psi(t,u)$ are continuous in t for each u.

In the framework of our model, the computation of the conditional characteristic function reduces to the computation of ϕ and ψ. In some cases these are explicitly known, otherwise they can be obtained numerically. In the particular case when the reestimation factors remain positive affine martingales under the pricing measure we are able to prove the following characterization, which provides a useful simplification of the conditional characteristic function.

Theorem 2.3.2. *Let* A *be an affine process, satisfying Assumptions (A1) and (A2). Then* A *is a* positive local martingale *if and only if* A *admits the following semimartingale characteristics* (B, C, ν)*:*

$$\begin{aligned} B_t &= \beta \int_0^t A_s ds, \\ C_t &= \alpha \int_0^t A_s ds, \quad \text{and} \\ \nu(dt, dy) &= A_t \mu(dy) dt, \end{aligned}$$

where

$$\beta = \mu[1, \infty) - \int_1^\infty y \mu(dy),$$

$\alpha \geq 0$*, and* μ *is a Lévy measure on* $(0, \infty)$.

Proof. Since A satisfies Assumptions (A1) and (A2), by Theorem 1.1 in [30] and Theorem 2.12 in [15] A is a positive affine semimartingale if and only if A_t admits the following characteristics (B, C, ν):

$$B_t = \int_0^t (\tilde{b} + \beta A_s) ds,$$
$$C_t = \alpha \int_0^t A_s ds, \quad \text{and}$$
$$\nu(dt, dy) = (m(dy) + A_t \mu(dy))dt,$$

for every $\mathbb{P}_x$, where

$$\tilde{b} = b + \int_{(0,\infty)} (1 \wedge y) m(dy),$$

$\alpha, b \geq 0, \beta \in \mathbb{R}$, m and μ are Lévy measures on $(0, \infty)$, such that

$$\int_{(0,\infty)} (y \wedge 1) m(dy) < \infty.$$

By (A2) and Theorem 7.16 in [11] the following operator L

$$\begin{aligned} Lf(x) = \frac{1}{2}\alpha x f''(x) + (b + \beta x) f'(x) + \int_{(0,\infty)} (f(x+y) - f(x)) m(dy) \\ + x \int_{(0,\infty)} (f(x+y) - f(x) - f'(x)(1 \wedge y)) \mu(dy) \end{aligned} \tag{2.15}$$

on $C^2(\mathbb{R}_+)$ is a version of the restriction of the extended infinitesimal generator[1] of A to $C^2(\mathbb{R}_+)$. Then A is a local martingale, if and only if

$$Lf(x) \equiv 0 \quad \text{for } f(x) = x.$$

Substituting $f(x) = x$ in (2.15), we get

$$\begin{aligned} Lx &= b + \beta x + \int_{(0,\infty)} y m(dy) + x \int_{(0,\infty)} (y - (1 \wedge y)) \mu(dy) \\ &= \Big(\beta + \int_1^\infty (y-1) \mu(dy)\Big) x + b + \int_{(0,\infty)} y m(dy). \end{aligned}$$

[1] An operator L with domain $\mathcal{D}_L$ is said to be an extended infinitesimal generator for A if $\mathcal{D}_L$ consists of those Borel functions f for which there exists a Borel function Lf such that the process

$$L_t^f = f(A_t) - f(A_0) - \int_0^t Lf(X_s) ds$$

is a local martingale.

Hence, A is a local martingale if and only if

$$(\beta + \int_1^\infty (y-1)\mu(dy))x + b + \int_{(0,\infty)} ym(dy) = 0. \tag{2.16}$$

for any $x \in \mathbb{R}_+$. Since $b \geq 0$ and m is a non-negative measure, condition (2.16) means that

$$b = 0, \quad m \equiv 0, \quad \text{and} \quad \beta = \mu[1,\infty) - \int_1^\infty y\mu(dy). \tag{2.17}$$

□

Let A be an affine process, satisfying Assumptions (A1) and (A2). By Theorem 4.3 in [21] the conditional characteristic function of A satisfies (2.14), where $\phi(t,u)$ and $\psi(t,u)$ solve the equations

$$\partial_t \psi(t,u) = R(\psi(t,u)), \quad \psi(0,u) = iu, \quad \text{and} \tag{2.18}$$

$$\phi(t,u) = \int_0^t F(\psi(s,u))ds, \tag{2.19}$$

where, for $z \in \{\mathbb{C} \mid \Re z \leq 0\}$,

$$R(z) = \frac{1}{2}\alpha z^2 + \beta z + \int_{(0,\infty)} (e^{zy} - 1 - z(y \wedge 1))\mu(dy), \tag{2.20}$$

$$F(z) = bz + \int_{(0,\infty)} (e^{zy} - 1)m(dy), \tag{2.21}$$

and α, β, b, m, μ are the parameters of the infinitesimal generator (2.15) of A. If A is a local martingale, then by (2.17) we can simplify (2.21) and (2.20) as follows:

$$R(z) = \frac{1}{2}\alpha z^2 + \int_{(0,\infty)} (e^{zy} - zy - 1)\mu(dy), \quad \text{and} \tag{2.22}$$

$$F(z) \equiv 0. \tag{2.23}$$

From (2.23) and (2.19) we immediately obtain for positive affine local martingales that

$$\phi(t,u) \equiv 0.$$

In order to determine ψ, one has in general to solve (2.19) numerically. For some special cases, however, it is possible to compute ψ analytically. We give two examples.

Example 2.3.3. If A has no jump part then A is called *Feller diffusion* (see e.g. [15]). In that case the positive affine martingale dynamics is given by

$$dA_t = \sqrt{\alpha A_t} dW_t \,,$$

where W_t is a standard Brownian motion. Consequently, we have $\mu = 0$ in (2.22) and we can rewrite (2.18) as

$$\psi_t' = \frac{1}{2}\alpha\psi_t^2. \tag{2.24}$$

Solving the differential equation (2.24) we get

$$\psi(t,u) \equiv 0 \quad \text{or} \quad \psi(t,u) = -\frac{1}{\frac{1}{2}\alpha t + C(u)}, \qquad u \in \mathbb{R},$$

where $C(u)$ can be found from the boundary condition $\psi(0,u) = iu$. Substituting $C(u)$ into ψ, we get

$$\psi(t,u) \equiv 0 \quad \text{or} \quad \psi(t,u) = -\frac{1}{\frac{1}{2}\alpha t + \frac{i}{u}}, \qquad u \in \mathbb{R}.$$

Note that if we have no jump part, then A has positive probability to be absorbed at 0. However, it may still be of interest to consider also the case of positive probability of absorption at zero, if we wish to include the possibility of fraud or falsified reporting of claims into the model. In this case, reestimation might discover the fraud and the previous fake evaluation will be set to zero.

Example 2.3.4. In order to give an example of a positive affine martingale with jumps where we can solve for ψ explicitly, we specify the jump density $\mu(dy)$ in the semimartingale characteristics in Theorem 2.3.2 as

$$\mu(dy) = \frac{3}{4\sqrt{\pi}} \frac{dy}{y^{5/2}} \,.$$

Then some calculations give $R(z)$ in (2.20)

$$R(z) = \frac{1}{2}\alpha z^2 + \frac{3}{4\sqrt{\pi}} \int_{(0,\infty)} (e^{zy} - zy - 1) \frac{dy}{y^{5/2}} \tag{2.25}$$

$$= \frac{1}{2}\alpha z^2 + (-z)^{3/2} \tag{2.26}$$

for $z \in \{\mathbb{C} \mid \Re z \leq 0\}$. Consider $\eta(t,u) := -\psi(t,u)$. By (2.18) we have

$$-\eta_t' = \frac{1}{2}\alpha\eta^2 + \eta^{3/2}. \tag{2.27}$$

The solutions to (2.27) are $\eta(t,u) \equiv 0$ and

$$\eta(t,u) = \frac{4}{\alpha^2}(1 + W(-C(u)e^{-\frac{t}{\alpha}}))^{-2}, \tag{2.28}$$

where $W(\cdot)$ is the Lambert W function[2]. The boundary condition $\eta(0,u) = -\psi(0,u) = iu$ yields

$$C(u) = -\left(-1 + \frac{2}{\alpha}\sqrt{\frac{i}{u}}\right)\exp\left(-1 + \frac{2}{\alpha}\sqrt{\frac{i}{u}}\right)$$

Substituting $C(u)$ into (2.28), we get for $\psi(t,u) = -\eta(t,u)$:

$$\psi(t,u) \equiv 0 \quad \text{or} \quad \psi(t,u) = -\frac{4}{\alpha^2}\left(1 + W((-1 + \frac{2}{\alpha}\sqrt{\frac{i}{u}})e^{-\frac{t}{\alpha} - 1 + \frac{2}{\alpha}\sqrt{\frac{i}{u}}})\right)^{-2}.$$

[2]The *Lambert W function* $W(z)$ is defined to be the function satisfying $W(z)e^{W(z)} = z$, $z \in \mathbb{C}$. See [13] for more details on the Lambert function.

Part II

Pricing of electricity options

Outline of Part II

This part of the thesis is organized as follows. In Chapter 3 we follow the method of [26] for pricing electricity contracts, which converts an electricity futures and spot market into a money market applying an appropriate change of numeraire transformation. We point out that in [26] all price processes involved were assumed to be continuous and the classical Heath-Jarrow-Morton (HJM) approach was proposed to model a bond market. We generalize the approach of [26] replacing, in the dynamics of the asset prices, the Brownian motion by a general Lévy process taking into account the occurrence of spikes. We show in Chapter 4 that this method combined with the Fourier transform techniques introduced in Chapter 1 provides explicit pricing formulas for European electricity options. Moreover, in this framework the spot price dynamics becomes Markovian, and therefore, complicated path-dependent derivatives such as electricity swing options can be valuated.

To begin with, in Section 3.1 we explain a connection between electricity and fixed-income markets. Then, in Section 3.2 we introduce an electricity market model derived by a Lévy term structure. In particular, we consider the corresponding measure transformation in Section 3.3. Thereafter, in Section 3.4 we examine the Markov property of the spot price process in our framework. Moreover, in Section 3.4 we show that our framework contains as a special case the commonly accepted model for an electricity market, where the spot price process is an exponential of an Ornstein-Uhlenbeck process. Finally, we apply the results of Sections 3.2 and 3.4 to valuation of electricity derivatives in Chapter 4.

Chapter 3

Electricity markets derived by Lévy term structure models

3.1 Connection between electricity market and money market

Let $F(t,\tau)$, $0 \le t \le \tau$, be the futures price of electricity at time t and T be a finite time horizon, $\tau \le T$. Denote the set of chronological time pairs by

$$\mathcal{D} := \{(t,\tau) : 0 \le t \le \tau \le T\}.$$

We model the futures market starting from the following axioms:
C1: For every $\tau \in [0,T]$ the futures price evolution $(F(t,\tau))_{(t,\tau)\in\mathcal{D}}$ is a positive-valued adapted stochastic process realized on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$.
C2: There exists a martingale measure $\mathbb{Q}^F$ equivalent to $\mathbb{P}$ such that for all $\tau \in [0,T]$ the futures price process $(F(t,\tau))_{(t,\tau)\in\mathcal{D}}$ is a $\mathbb{Q}^F$-martingale.
C3: At $t = 0$ futures prices start at deterministic positive values $(F(0,\tau))_{\tau\in[0,T]}$.
C4: Terminal prices form a *spot price process* $S_t := F(t,t)$, $t \in [0,T]$.

Following the approach of [26] we now convert the electricity market into a *money market* consisting of bonds $(P(t,\tau))_{0\le t\le\tau}$ equipped with an additional risky asset $(N_t)_{t\in[0,T]}$ by using the following transformation:

$$P(t,\tau) := \frac{F(t,\tau)}{S_t}, \tag{3.1}$$

$$N_t := \frac{1}{S_t}. \tag{3.2}$$

The money market defined by the currency change (3.1)–(3.2) satisfies the following axioms:

M1: $(N_t)_{t\in[0,T]}$ and $(P(t,\tau))_{(t,\tau)\in\mathcal{D}}$ are positive, adapted stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$.

M2: There exist a positive-valued, adapted numéraire process $(C_t)_{t\in[0,T]}$ and a martingale measure $\mathbb{Q}^M$ equivalent to $\mathbb{P}$, such that for all $\tau \in [0,T]$ the discounted price processes $\hat{P}(t,\tau) := \frac{P(t,\tau)}{C_t}$, $(t,\tau) \in \mathcal{D}$, and $\hat{N}_t := \frac{N_t}{C_t}$, $0 \le t \le T$, are $\mathbb{Q}^M$-martingales.

M3: Prices start at deterministic values N_0 and $(P(0,\tau))_{\tau\in[0,T]}$.

M4: Bond prices finish at one, i.e. $P(t,t) = 1$, for every $t \in [0,T]$.

We now need a slight generalization of Theorem 1 in [26].

Theorem 3.1.1. *i) Suppose that the commodity market* $(F(t,\tau))_{(t,\tau)\in\mathcal{D}}$ *fulfills C1–C4 with an initial futures curve* $(F(0,\tau))_{\tau\in[0,T]}$ *and a martingale measure* $\mathbb{Q}^F$*. Then the transformation* (3.1) – (3.2) *provides a money market satisfying M1–M4 with initial values*

$$P(0,\tau) := \frac{F(0,\tau)}{S_0}, \quad \forall\, \tau \in [0,T], \quad \textit{and} \quad N_0 = \frac{1}{S_0},$$

where the discounting process and the martingale measure are given by

$$C_t = P(t,T), \quad t \in [0,T], \quad \textit{and} \quad d\mathbb{Q}^M = \frac{S_T}{F(0,T)} d\mathbb{Q}^F. \tag{3.3}$$

ii) Suppose that the money market $(P(t,\tau))_{(t,\tau)\in\mathcal{D}}$*,* $(N_t)_{t\in[0,T]}$ *fulfills M1–M4 with initial values* $(P(0,\tau))_{\tau\in[0,T]}$ *and* N_0*, a discounting process* $(C_t)_{t\in[0,T]}$*, and a martingale measure* $\mathbb{Q}^M$*. Then the transformation*

$$F(t,\tau) := \frac{P(t,\tau)}{N_t}, \quad (t,\tau) \in \mathcal{D}, \tag{3.4}$$

gives an electricity market with the deterministic initial futures curve

$$F(0,\tau) := \frac{P(0,\tau)}{N_0},$$

for all $\tau \in [0,T]$*, and the martingale measure*

$$d\mathbb{Q}^F := \frac{N_T}{C_T}\frac{C_0}{N_0} d\mathbb{Q}^M. \tag{3.5}$$

Note that in Theorem 1 of [26] all price processes were assumed continuous. In our proof we will only use the integrability properties of the processes involved.

Proof.

i) It is easy to see that the properties M1, M3, and M4 are consequences of C1, C2, and C4 due to (3.1) and (3.2). To prove M2 we define the discounting process C_t and the money market measure $\mathbb{Q}^M$ as in (3.3). Then the Radon-Nikodym density of $\mathbb{Q}^M$ w.r.t. $\mathbb{Q}^F$ conditioned on $\mathcal{F}_t$ is given by

$$\frac{d\mathbb{Q}^M}{d\mathbb{Q}^F}\Big|_{\mathcal{F}_t} := E^{\mathbb{Q}^F}\left[\frac{d\mathbb{Q}^M}{d\mathbb{Q}^F}\Big|\mathcal{F}_t\right] = \frac{F(t,T)}{F(0,T)}.$$

For the discounted bond price process we get

$$\hat{P}(t,\tau) := \frac{P(t,\tau)}{C_t} = \frac{F(t,\tau)}{F(t,t)}\frac{F(t,t)}{F(t,T)} = \frac{F(t,\tau)}{F(t,T)}. \tag{3.6}$$

Conjecture C2 yields the integrability of $\hat{P}(t,\tau)$ under $\mathbb{Q}^M$, since

$$\begin{aligned} E^{\mathbb{Q}^M}[\hat{P}(t,\tau)] &= E^{\mathbb{Q}^F}\left[\hat{P}(t,\tau)\frac{d\mathbb{Q}^M}{d\mathbb{Q}^F}\Big|_{\mathcal{F}_t}\right] = E^{\mathbb{Q}^F}\left[\frac{F(t,\tau)}{F(0,T)}\right] \\ &= \frac{F(0,\tau)}{F(0,T)} < \infty. \end{aligned}$$

Furthermore, by Bayes rule for conditional expectations we get due to Conjecture C2 and equality (3.6) that

$$\begin{aligned} E^{\mathbb{Q}^M}[\hat{P}(t,\tau)|\mathcal{F}_s] &= \frac{E^{\mathbb{Q}^F}\left[\hat{P}(t,\tau)\frac{d\mathbb{Q}^M}{d\mathbb{Q}^F}\Big|_{\mathcal{F}_t}\Big|\mathcal{F}_s\right]}{\frac{d\mathbb{Q}^M}{d\mathbb{Q}^F}\Big|_{\mathcal{F}_s}} = \frac{E^{\mathbb{Q}^F}\left[F(t,\tau)\Big|\mathcal{F}_s\right]}{F(s,T)} \\ &= \frac{F(s,\tau)}{F(s,T)} = \hat{P}(s,\tau). \end{aligned}$$

Hence $(\hat{P}(t,\tau))_{(t,\tau)\in\mathcal{D}}$ is a $\mathbb{Q}^M$-martingale. For the process $\hat{N}_t$ we analogously get

$$\hat{N}_t := \frac{N_t}{C_t} = \frac{N_t}{P(t,T)} = \frac{1}{S_t}\frac{S_t}{F(t,T)} = \frac{1}{F(t,T)}, \tag{3.7}$$

and hence

$$E^{\mathbb{Q}^M}[\hat{N}_t] = E^{\mathbb{Q}^F}\left[\hat{N}_t\frac{F(t,T)}{F(0,T)}\right] = \frac{1}{F(0,T)} < \infty.$$

Using Bayes rule and equality (3.7) we obtain

$$E^{\mathbb{Q}^M}[\hat{N}_t|\mathcal{F}_s] = \frac{E^{\mathbb{Q}^F}\left[\hat{N}_t\frac{F(t,T)}{F(0,T)}\Big|\mathcal{F}_s\right]}{\frac{F(s,T)}{F(0,T)}} = \frac{1}{F(s,T)} = \hat{N}_s.$$

Hence $(\hat{N}_t)_{0\leq t\leq T}$ is a $\mathbb{Q}^M$-martingale.

ii) Define the futures price process $F(t,\tau)$ as in (3.4). Then $F(t,\tau)$ is positive and adapted by M1. Consider the equivalent probability measure $\mathbb{Q}^F$ given by (3.5). $F(t,\tau)$ is integrable w.r.t. $\mathbb{Q}^F$, since by Assumption M2,

$$\begin{aligned} E^{\mathbb{Q}^F}[F(t,\tau)] &= E^{\mathbb{Q}^M}\left[\frac{P(t,\tau)}{N_t}\frac{d\mathbb{Q}^F}{d\mathbb{Q}^M}\Big|_{\mathcal{F}_t}\right] = \frac{C_0}{N_0}E^{\mathbb{Q}^M}\left[\frac{P(t,\tau)}{N_t}\frac{N_t}{C_t}\right] \\ &= \frac{C_0}{N_0}E^{\mathbb{Q}^M}\left[\frac{P(t,\tau)}{C_t}\right] < \infty. \end{aligned}$$

Furthermore, M2 yields

$$\begin{aligned} E^{\mathbb{Q}^F}[F(t,\tau)|\mathcal{F}_s] &= \frac{E^{\mathbb{Q}^M}[F(t,\tau)\frac{N_t}{C_t}|\mathcal{F}_s]}{\frac{N_s}{C_s}} = \frac{E^{\mathbb{Q}^M}[\frac{P(t,\tau)}{C_t}|\mathcal{F}_s]}{\frac{N_s}{C_s}} \\ &= \frac{P(s,\tau)}{N_s} = F(s,\tau), \quad \forall\ 0\leq s\leq t\leq\tau. \end{aligned}$$

Hence, $(F(t,\tau))_{(t,\tau)\in\mathcal{D}}$ is a $\mathbb{Q}^F$-martingale.

□

In the following sections we apply this approach and study electricity markets derived by term structure models driven by general Lévy processes.

3.2 Money market construction

We follow the HJM approach and specify the term structure by modeling the (instantaneous) forward rate $f(t,\tau)$, $(t,\tau) \in \mathcal{D}$. Let $P(t,\tau)$, $(t,\tau) \in \mathcal{D}$, be the market price at the moment t of a bond paying 1 at the maturity time τ, $\tau \leq T$. Given the forward rate curve $f(t,\tau)$ the bond prices are defined by

$$P(t,\tau) = \exp\{-\int_t^\tau f(t,s)ds\}, \tag{3.8}$$

while the instantaneous short rate r at time t is given by

$$r(t) := f(t,t). \tag{3.9}$$

A general introduction to fixed-income markets is given in [6].

Let $L = (L^1, \ldots, L^n)$ be an n-dimensional Lévy process with independent components, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q}^M)$ endowed with the completed canonical filtration $(\mathcal{F}_t)_{t\in[0,T]}$ associated with L. We denote by (b_i, c_i, ν_i) the *characteristic triplet* of each component L^i, $i = 1, \ldots, n$.

We assume that

A1: we are given an $\mathbb{R}$-valued and $\mathbb{R}^n$-valued stochastic processes $\alpha(t,\tau)$ and $\eta(t,\tau) = (\eta^1(t,\tau), \ldots, \eta^n(t,\tau))$, $(t,\tau) \in \mathcal{D}$, respectively, such that $\alpha(t,\tau)$ and $\eta(t,\tau)$ are continuous and adapted.

A2: $\int_0^T \int_0^T E|\alpha(s,u)|dsdu < \infty$, $\int_0^T \int_0^T E\|\eta(s,u)\|^2 dsdu < \infty$.

A3: $P(\tau,\tau) = 1, \quad \forall \tau \in [0,T]$.

A4: The initial forward curve is given by a deterministic and continuously differentiable function $\tau \mapsto f(0,\tau)$ on the interval $[0,T]$.

For the forward rate we consider a generalized HJM model, i.e. we assume that the forward rate process follows the dynamics

$$f(t,\tau) = f(0,\tau) + \int_0^t \alpha(s,\tau)ds + \sum_{i=1}^n \int_0^t \eta^i(s,\tau)dL_s^i, \quad t \leq \tau. \tag{3.10}$$

In terms of short rates we can rewrite (3.10) and (3.9) as

$$r(t) = r(0) + \int_0^t \alpha(s,t)ds + \sum_{i=1}^n \int_0^t \eta^i(s,t)dL_s^i, \tag{3.11}$$

where $r(0) = f(0,t)$, $t \leq T$.

Lévy term structures of the type (3.10)–(3.11) are frequently considered in the literature (see e.g. [19], [17], [22] or [28]).

We now consider the bank account process as a discounting factor, i.e.

$$C_t = \exp\{\int_0^t r(s)ds\}. \tag{3.12}$$

In order to provide a condition which ensures that $\mathbb{Q}^M$ is a local martingale measure for

$$\hat{P}(t,\tau) := \frac{P(t,\tau)}{C_t}, \quad t \in [0,\tau], \tag{3.13}$$

we assume that there exist $a_i < 0 < d_i$ such that the Lévy measures ν_i of L^i satisfy

$$\int_{\{|x|>1\}} e^{ux}\nu_i(dx) < \infty, \quad u \in [a_i, d_i], \quad i = 1,\ldots,n, \tag{3.14}$$

(see [19] or [22]).

Lemma 3.2.1. *Under Assumption* (3.14), $L = (L_t)_{0\leq t\leq T}$ *is a special semimartingale admitting the canonical representation:*

$$L_t = bt + \sqrt{c}B_t + \int_0^t \int_{\mathbb{R}} x(J_L(dx \times ds) - \nu(dx)ds),$$

where $b = (b_1,\ldots,b_n) \in \mathbb{R}^n$, c *is a positive definite* $n \times n$ *matrix,* B *is a standard* n*-dimensional Brownian motion,* J_L *is the random measure of jumps, and* ν *is its compensator.*

Note that, since $L^1,\ldots,L^n$ are independent, c is a diagonal matrix with elements $c_1,\ldots,c_n > 0$ on the main diagonal.

Proof of Lemma 3.2.1. In view of II.2.29 in [27] it is sufficient to show that $(|x|^2 \wedge |x|) * \nu \in \mathcal{A}_{\text{loc}}$, i.e. that $(|x|^2 \wedge |x|) * \nu$ is an adapted process with locally integrable variation. Since $(|x|^2 \wedge |x|) * \nu$ is increasing and deterministic, we only need to show the finiteness of

$$
\begin{aligned}
(|x|^2 \wedge |x|) * \nu &= \int_{\mathbb{R}} (|x|^2 \wedge |x|)\nu(dx) \\
&= \int_{\{|x|<1\}} (|x|^2 \wedge |x|)\nu(dx) + \int_{\{|x|>1\}} (|x|^2 \wedge |x|)\nu(dx) \\
&= \int_{\{|x|<1\}} |x|^2\nu(dx) + \int_{\{|x|>1\}} |x|\nu(dx).
\end{aligned}
$$

The first term is finite, since ν is a Lévy measure, and the second summand is finite by Assumption (3.14). □

Furthermore, condition (3.14) ensures the existence of the cumulant generating function

$$\Theta^i(u) := \log E[\exp(uL_1^i)] \tag{3.15}$$

at least on the set $\{u \in \mathbb{C} |\, \Re u \in [a_i, d_i]\}$, where $\Re u$ denotes the real part of $u \in \mathbb{C}$, $i = 1, \dots, n$. By Lemma 26.4 in [48], Θ^i is continuously differentiable and has the representation:

$$\Theta^i(u) = b_i u + \frac{c_i}{2}u^2 + \int_{\mathbb{R}} (e^{ux} - 1 - ux)\nu_i(dx), \quad i = 1, \dots, n. \tag{3.16}$$

As a consequence, the Lévy processes L^i, $i = 1, \dots, n$, have finite moments of arbitrary order.

Putting (3.8), (3.10), and (3.11) together we derive the following representation for the bond price:

$$
\begin{aligned}
P(t,\tau) &= \exp\Big\{ - \int_t^\tau \Big[f(0,u) + \int_0^t \alpha(s,u)ds + \sum_{i=1}^n \int_0^t \eta^i(s,u)dL_s^i \Big] du \Big\} \\
&= P(0,\tau)\exp\Big\{ \int_0^t f(0,u)du - \int_t^\tau \Big[\int_0^t \alpha(s,u)ds + \sum_{i=1}^n \int_0^t \eta^i(s,u)dL_s^i \Big] du \Big\} \\
&= P(0,\tau)\exp\Big\{ \int_0^t r(u)du - \int_0^t \Big[\int_0^u \alpha(s,u)ds + \sum_{i=1}^n \int_0^u \eta^i(s,u)dL_s^i \Big] du \\
&\qquad - \int_t^\tau \Big[\int_0^t \alpha(s,u)ds + \sum_{i=1}^n \int_0^t \eta^i(s,u)dL_s^i \Big] du \Big\}.
\end{aligned} \tag{3.17}
$$

It is convenient to assume that

$$\alpha(t,\tau) = \eta(t,\tau) = 0 \quad \text{for} \quad t > \tau, \tag{3.18}$$

so that the forward rate (3.10) is defined for all $t, \tau \in [0, T]$. Then by (3.18) and Assumption A2, we can rewrite (3.17) in a more compact form

$$\begin{aligned}
P(t,\tau) &= P(0,\tau)\exp\Big\{\int_0^t r(u)du - \int_0^t \Big[\int_0^t \alpha(s,u)ds + \sum_{i=1}^n \int_0^t \eta^i(s,u)dL_s^i\Big]du \\
&\quad - \int_t^\tau \Big[\int_0^t \alpha(s,u)ds + \sum_{i=1}^n \int_0^t \eta^i(s,u)dL_s^i\Big]du\Big\} \\
&= P(0,\tau)\exp\Big\{\int_0^t r(u)du - \int_0^\tau \Big[\int_0^t \alpha(s,u)ds + \sum_{i=1}^n \int_0^t \eta^i(s,u)dL_s^i\Big]du\Big\} \\
&= P(0,\tau)\exp\Big\{\int_0^t r(u)du - \int_0^t \int_0^\tau \alpha(s,u)duds \\
&\quad - \sum_{i=1}^n \int_0^t \int_0^\tau \eta^i(s,u)dudL_s^i\Big\},
\end{aligned} \tag{3.19}$$

where in the last equality we could apply Fubini's theorem, because Assumption A2 holds. Provided

$$-\int_0^\tau \eta^i(s,u)du \in (a_i, d_i) \quad \text{for} \quad i = 1, \dots, n,$$

for any $\tau \leq T$, the HJM condition on the drift

$$\alpha(t,x) = \sum_{i=1}^n \frac{\partial}{\partial x}\Theta^i\Big(-\int_0^x \eta^i(t,u)du\Big) \tag{3.20}$$

implies that $\mathbb{Q}^M$ is a local martingale measure. The drift condition (3.20) is derived in [17] and [19]. For an analogous drift condition in the infinite dimensional Lévy setting see [28] and [22].

Denoting by

$$\sigma^i(t,\tau) := -\int_0^\tau \eta^i(t,u)du, \quad i = 1, \dots, n, \tag{3.21}$$

we can rewrite the HJM drift condition (3.20) as

$$\begin{aligned}
\int_0^\tau \alpha(s,u)du &= \sum_{i=1}^n \int_0^\tau \frac{\partial}{\partial u}\Theta^i(\sigma^i(s,u))du \\
&= \sum_{i=1}^n \Theta^i(\sigma^i(s,\tau)).
\end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.19), we get the same representation for $P(t,\tau)$ as in [19]:

$$P(t,\tau) = P(0,\tau)\exp\Big\{\int_0^t r(u)du - \sum_{i=1}^n \int_0^t \Theta^i(\sigma^i(s,\tau))ds + \sum_{i=1}^n \int_0^t \sigma^i(s,\tau)dL_s^i\Big\}. \tag{3.23}$$

To complete the modeling of the arbitrage-free money market satisfying Assumptions M1–M4, we assume that the risky asset N_t is given by

$$N_t = \exp\Big\{\int_0^t r(u)du - \sum_{i=1}^n \int_0^t \Theta^i(v^i(s))ds + \sum_{i=1}^n \int_0^t v^i(s)dL_s^i\Big\}, \tag{3.24}$$

where $v = (v^1,\dots,v^n)$ is a continuous function, such that

$$\hat{N}_t = \frac{N_t}{C_t}$$

is a well-defined local martingale under $\mathbb{Q}^M$.

Now we consider the futures price process

$$F(t,\tau) = \frac{P(t,\tau)}{N_t}, \quad (t,\tau) \in \mathcal{D}, \tag{3.25}$$

where $P(t,\tau)$ and N_t are now given by (3.23) and (3.24).

According to Theorem 3.1.1 the transformation (3.25) gives an arbitrage-free electricity futures market with the deterministic initial futures curve

$$F(0,\tau) := \frac{P(0,\tau)}{N_0} = P(0,\tau).$$

By the same theorem,

$$\begin{aligned} d\mathbb{Q}^F &= \frac{N_T}{C_T}\frac{C_0}{N_0} d\mathbb{Q}^M \\ &= \exp\left\{\sum_{i=1}^n \int_0^T v^i(s)dL_s^i - \sum_{i=1}^n \int_0^T \Theta^i(v^i(s))ds\right\} d\mathbb{Q}^M \end{aligned} \tag{3.26}$$

is a martingale measure for $F(t,\tau)$, $(t,\tau) \in \mathcal{D}$. Indeed, by (3.23) and (3.24) we get that

$$F(t,\tau) = \frac{P(t,\tau)}{N_t} = \frac{\hat{P}(t,\tau)}{\frac{d\mathbb{Q}^F}{d\mathbb{Q}^M}\Big|_{\mathcal{F}_t}}, \tag{3.27}$$

and hence, $F(t,\tau)$ is a $\mathbb{Q}^F$-martingale.

Furthermore, by (3.27)

$$\begin{aligned} F(t,\tau) &= F(0,\tau)\exp\{\sum_{i=1}^n \int_0^t (\sigma^i(s,\tau)-v^i(s))dL_s^i \\ &\quad -\sum_{i=1}^n \int_0^t (\Theta^i(\sigma^i(s,\tau))-\Theta^i(v^i(s)))ds\}. \end{aligned} \tag{3.28}$$

Setting $\tau = t$ in (3.28) we obtain the electricity spot price process

$$S(t) = F(t,t) = F(0,t)\exp\{\sum_{i=1}^n \int_0^t \delta^i(s,t)dL_s^i - \sum_{i=1}^n \int_0^t \psi^i(s,t)ds\} \tag{3.29}$$

$$=: F(0,t)E_t, \tag{3.30}$$

where

$$E_t := \exp\Big\{\sum_{i=1}^n \int_0^t \delta^i(s,t)dL_s^i - \sum_{i=1}^n \int_0^t \psi^i(s,t)ds\Big\}, \tag{3.31}$$

$$\delta^i(s,t) := \sigma^i(s,t) - v^i(s), \quad \text{and} \tag{3.32}$$

$$\psi^i(s,t) := \Theta^i(\sigma^i(s,t)) - \Theta^i(v^i(s)). \tag{3.33}$$

In order to study the electricity market (3.28) – (3.29) under the measure $\mathbb{Q}^F$ defined by (3.26) we need some technical results given in Section 3.3.

3.3 Measure transformation

Let us consider now the density process

$$Z_t := \frac{d\mathbb{Q}^F}{d\mathbb{Q}^M}\Big|_{\mathcal{F}_t}, \quad 0 \le t \le T.$$

Since L is a process with independent increments, by (3.26) we get

$$Z_t = \exp\{\sum_{i=1}^n \int_0^t v^i(s)dL_s^i - \sum_{i=1}^n \int_0^t \Theta^i(v^i(s))ds\}. \tag{3.34}$$

By (3.16), (3.34), and Lemma 3.2.1 we obtain for the density process (3.34):

Lemma 3.3.1.

$$Z_t = \mathcal{E}_t\left(\sum_{i=1}^{n}\sqrt{c_i}\int_0^{\cdot} v^i(s)dB_s^i + \int_0^{\cdot}\int_{\mathbb{R}^n}(e^{\langle v(s),x\rangle}-1)(J_L(dx\times ds)-\nu(dx)ds)\right),$$

where $\mathcal{E}_t(\cdot)$ *is the Doleans exponential.*

Proof. Since the components $L^1,\ldots,L^n$ of the Lévy process L are independent, in order to prove Lemma 3.3.1 it is enough to show that

$$Z_t^i := \exp\left\{\int_0^t v^i(s)dL_s^i - \int_0^t \Theta^i(v^i(s))ds\right\} \tag{3.35}$$

$$= \mathcal{E}_t\left(\sqrt{c_i}\int_0^{\cdot} v^i(s)dB_s^i + \int_0^{\cdot}\int_{\mathbb{R}}(e^{v^i(s)x}-1)(J_{L^i}(dx\times ds)-\nu_i(dx)ds)\right), \tag{3.36}$$

where J_{L^i} is the jump measure of L^i, $i=1,\ldots,n$.

Furthermore, by representation for Θ^i (3.16) and by Lemma 3.2.1 we get

$$\begin{aligned}
Z_t^i &= \exp\left\{\int_0^t v^i(s)dL_s^i - \int_0^t \Theta^i(v^i(s))ds\right\}\\
&= \exp\left\{\int_0^t v^i(s)dL_s^i - b_i\int_0^t v^i(s)ds - \frac{c_i}{2}\int_0^t (v^i(s))^2ds\right.\\
&\quad \left. - \int_0^t\int_{\mathbb{R}}(e^{v^i(s)x}-1-v^i(s)x)\nu_i(dx)ds\right\}\\
&= \exp\left\{\sqrt{c_i}\int_0^t v^i(s)dB_s + \int_0^t\int_{\mathbb{R}} v^i(s)x(J_{L^i}-\nu_i)(dx\times ds)\right.\\
&\quad \left. - \frac{c_i}{2}\int_0^t (v^i(s))^2ds - \int_0^t\int_{\mathbb{R}}(e^{v^i(s)x}-1-v^i(s)x)\nu_i(dx)ds\right\}\\
&= \exp\left\{\sqrt{c_i}\int_0^t v^i(s)dB_s^i - \frac{c_i}{2}\int_0^t (v^i(s))^2ds\right\}\\
&\quad \times \exp\left\{\int_0^t\int_{\mathbb{R}} v^i(s)xJ_{L^i}(dx\times ds) - \int_0^t\int_{\mathbb{R}}(e^{v^i(s)x}-1)\nu_i(dx)ds\right\}\\
&= \mathcal{E}_t\Big(\sqrt{c_i}\int_0^t v^i(s)dB_s^i\Big)\mathcal{E}_t\Big(\int_0^t\int_{\mathbb{R}} v^i(s)xJ_{L^i}(dx\times ds)\Big),
\end{aligned}$$

where for the last equality we applied Propositions 3.6–3.7 in [12]. □

As an application of the preceding lemma we obtain the following proposition, that is essential in order to examine the Markov property of the spot price process under $\mathbb{Q}^F$.

Proposition 3.3.2. *L is a (non-homogeneous) Lévy process with respect to $\mathbb{Q}^F$ with the characteristic function given by*

$$E^{\mathbb{Q}^F}[e^{i\langle u,L_t\rangle}] = \exp\Big\{\sum_{j=1}^{n} \Phi_j^{\mathbb{Q}^F}(t,u_j)\Big\}, \quad u=(u_1,\dots,u_n)\in\mathbb{R}^n, \tag{3.37}$$

where

$$\begin{aligned}\Phi_j^{\mathbb{Q}^F}(t,u_j) = iu_j\int_0^t b_j^{\mathbb{Q}^F}(s)ds - \frac{u_j^2}{2}\int_0^t c_j^{\mathbb{Q}^F}(s)ds \\ + \int_0^t\int_{\mathbb{R}}(e^{iu_jx}-1-iu_jxI_{|x|\le 1})\nu_j^{\mathbb{Q}^F}(ds,dx),\end{aligned} \tag{3.38}$$

and

$$b_j^{\mathbb{Q}^F}(t) := b_j + c_j v^j(t) + \int_{\mathbb{R}}(e^{v^j(t)x}-1)I_{|x|\le 1}(x)\nu_j(dx), \tag{3.39}$$

$$c_j^{\mathbb{Q}^F}(t) := c_j, \tag{3.40}$$

$$\nu_j^{\mathbb{Q}^F}(dt,dx) := e^{v^j(t)x}\nu_j(dx)dt. \tag{3.41}$$

Remark 3.3.3. *Note that if $v(t)$ is a constant function, then by Proposition 3.3.2 L is a time-homogeneous Lévy process under $\mathbb{Q}^F$.*

Proof of Proposition 3.3.2. Consider the j-th component of L, $j \in \{1,\dots,n\}$. We first show that the characteristic triplet of L^j with respect to (w.r.t.) $\mathbb{Q}^F$ associated with the truncation function $h(x) = I_{|x|\le 1}(x)$ is given by (3.39)–(3.41).

In order to find the semimartingale characteristics of L^j w.r.t. $\mathbb{Q}^F$, we consider $\beta_t := c_j v^j(t)$ and $Y(t,x) := e^{v^j(t)x}$ and show that Y and β meet all the conditions of Girsanov's Theorem for semimartingales (cf. Theorem III.3.24 in [27]).

Consider the process Z^j defined in (3.35). Denote by Z^{jc} the continuous martingale part of the process Z^j and by L^{jc} the continuous martingale part of L^j relative to $\mathbb{Q}^M$. By representation (3.36), $Z_t^{jc} = \sqrt{c_j}\int_0^t Z_s^j v^j(s)dB_s^j$, and by Lemma 3.2.1 $L^{jc}(t) = \sqrt{c_j}B_t^j$. Since

$$\left\langle L^{jc}, Z^{jc}\right\rangle_t = c_j \int_0^t Z_s^j v^j(s)ds = \int_0^t Z_s^j \beta_s ds,$$

where $\langle\cdot,\cdot\rangle$ is the bracket relative to $\mathbb{Q}^M$, and β satisfies condition (III.3.28) in [27].

Let $\tilde{\mathcal{P}} := \mathcal{P}\times\mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}$ denotes the predictable σ-field on $\Omega\times[0,T]$. For any nonnegative and $\tilde{\mathcal{P}}$-measurable U we have

$$\begin{aligned} &E\Big[\int_0^T \int_{\mathbb{R}} Y(s,x)U(s,x)J_{L^j}(dx\times ds)\Big] \\ &= E\Big[\sum_{0\le s\le T} e^{v^j(s)\Delta L_s^j} U(s,\Delta L_s^j) I_{\{\Delta L_s^j\neq 0\}}\Big] \\ &= E\Big[\int_0^T \int_{\mathbb{R}} \frac{Z_s^j}{Z_{s-}^j} I_{\{Z_{s-}^j>0\}} U(s,x)J_{L^j}(dx\times ds)\Big], \end{aligned}$$

since $\frac{Z_s^j}{Z_{s-}^j} I_{\{Z_{s-}^j>0\}} = e^{v^j(s)\Delta L_s^j}$. Hence Y satisfies the conditions of Girsanov's Theorem (Theorem III.3.24 in [27]), which justifies (3.39)–(3.41).

By Theorem II.4.15 in [27] L^j is a process with independent increments under $\mathbb{Q}^F$. Moreover, by the same theorem, L^j is a (non-homogeneous) Lévy process with respect to $\mathbb{Q}^F$, since its characteristic function is given by (3.37) – (3.38). □

3.4 Markov property of the spot price

In this section we examine the Markov property of the spot price process S given by (3.29). To begin with, applying Proposition 3.3.2, we compute the dynamics of S under $\mathbb{Q}^F$ as follows.

Lemma 3.4.1. *The dynamics of S under $\mathbb{Q}^F$ is given by*

$$\begin{aligned} dS(t) = S(t)[-r(t) + \frac{1}{2}\sum_{i=1}^n c_i(v^i(t))^2 + \sum_{i=1}^n \Theta^i(v^i(t))]dt - S(t-)\sum_{i=1}^n v^i(t)dL_t^i \\ + \int_{R^n} S(t-)(e^{\langle v(t-),x\rangle} - 1 + \langle v(t-),x\rangle)J_L^{\mathbb{Q}^F}(dx\times dt), \qquad (3.42) \end{aligned}$$

where $J_L^{\mathbb{Q}^F}$ *is the jump measure of* L *under* $\mathbb{Q}^F$.

Proof. By Itô formula and Assumption (3.18) we obtain the dynamics of the spot prices (3.30) as

$$\begin{aligned}
dS(t) &= E_t \frac{\partial}{\partial t} F(0,t)dt + F(0,t)dE_t \\
&= S(t)\left(\frac{\partial}{\partial t}\ln F(0,t) + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial t}\delta^i(s,t)dL_s^i - \sum_{i=1}^n \int_0^t \frac{\partial}{\partial t}\psi^i(s,t)ds\right)dt \\
&\quad + F(t-,t)\left(-\sum_{i=1}^n \psi^i(t,t)dt + \sum_{i=1}^n \delta^i(t,t)dL_t^i + \frac{1}{2}\sum_{i=1}^n c_i\Big(\delta^i(t,t)\Big)^2 dt\right) \\
&\quad + S(t) - F(t-,t) - F(t-,t)\sum_{i=1}^n \delta^i(t,t)\triangle L_t^i.
\end{aligned}$$

Since $\delta^i(t,t) = -v^i(t)$ and $\psi^i(t,t) = -\Theta^i(v^i(t))$ by (3.32), (3.33), (3.21), and (3.18), we get

$$\begin{aligned}
dS(t) &= -S(t)r(t)dt + S(t)\frac{1}{2}\sum_{i=1}^n c_i(v^i(t))^2 dt + \sum_{i=1}^n \Theta^i(v^i(t))dt \\
&\quad - F(t-,t)\sum_{i=1}^n v^i(t)dL_t^i + S(t) - F(t-,t) + F(t-,t)\sum_{i=1}^n v^i(t)\triangle L_t^i.
\end{aligned}$$

Since $F(0,t)$, $\delta^i(t,t)$, and $\psi^i(t,t)$, $i = 1,\dots,n$, are continuous in t by Assumptions A2 – A4, we have $F(t-,t) = F(t-,t-) = S(t-)$, and thus we can rewrite the dynamics of S as

$$\begin{aligned}
dS(t) &= S(t)\big[-r(t) + \frac{1}{2}\sum_{i=1}^n c_i(v^i(t))^2 + \sum_{i=1}^n \Theta^i(v^i(t))\big]dt \\
&\quad - S(t-)\sum_{i=1}^n v^i(t)dL_t^i + \Delta S(t) + S(t-)\sum_{i=1}^n v^i(t)\triangle L_t^i. \qquad (3.43)
\end{aligned}$$

By (3.29) and (3.18) we have

$$\triangle S(t) = S(t-)(e^{\sum_{i=1}^n v^i(t-)\triangle L_t^i} - 1). \qquad (3.44)$$

Inserting (3.44) into (3.43), we can obtain (3.42). □

Hence, since v is deterministic, we get the following result:

Proposition 3.4.2. *Suppose the short rate process r is a Markov process. Then the vector process (S, r) is a Markov process.*

Proof. Since r is a Markov process and v is deterministic, (S, r) is a Markov process by (3.42). □

Remark 3.4.3. *Note that if the volatility η is deterministic, the short rate process r is a Markov process by* (3.11).

We consider now some examples. In particular, we show that our model for the electricity market contains the case, where the spot price process is an exponential of an Ornstein-Uhlenbeck process.

Example 3.4.4. Suppose the spot price process S is the exponential (e^X) of an Ornstein-Uhlenbeck process X, i.e. X is a solution of the following stochastic differential equation

$$dX(t) = \theta(\mu - X(t))dt + \varsigma dW_t^{\mathbb{Q}^F}, \quad X(0) = 1, \tag{3.45}$$

where $\mu \in \mathbb{R}$, $\theta, \varsigma > 0$, and $W_t^{\mathbb{Q}^F}$ is a one-dimensional standard Brownian motion under $\mathbb{Q}^F$. Now we find the corresponding short rate process r under the assumptions that the volatility $v(t)$ appearing in (3.42) is constant, i.e. $v(t) \equiv v < 0$, and

$$dL_t = vdt + dW_t^{\mathbb{Q}^F}, \quad t \in [0, T]. \tag{3.46}$$

By Itô formula and equation (3.45),

$$\begin{aligned} dS(t) &= S(t)(dX(t) + \frac{1}{2}\varsigma^2 dt) \\ &= S(t)\Big(\big[\theta(\mu - X(t)) + \frac{1}{2}\varsigma^2\big]dt + \varsigma dW_t^{\mathbb{Q}^F}\Big). \end{aligned} \tag{3.47}$$

On the other hand, by (3.42) the dynamics of the electricity spot price is given by

$$dS(t) = S(t)[-r(t) + \frac{1}{2}v^2 + \Theta(v)]dt - S(t)vdL_t, \quad t \in [0, T].$$

Note that, according to Proposition 3.3.2, (3.46) means that L is a one-dimensional standard Brownian motion under $\mathbb{Q}^M$. Hence, by (3.16) we have $\Theta(v) = \frac{1}{2}v^2$ in this case. Applying (3.46), we can simplify the dynamics of S as

$$\begin{aligned} dS(t) &= S(t)[-r(t) + v^2]dt - S(t)v dL_t \\ &= S(t)(-r(t) + v^2)dt - S(t)v(vdt + dW_t^{\mathbb{Q}^F}) \\ &= S(t)(-r(t)dt - vdW_t^{\mathbb{Q}^F}), \quad t \in [0, T]. \end{aligned} \tag{3.48}$$

Putting (3.47) and (3.48) together we obtain

$$v = -\varsigma, \quad \text{and} \tag{3.49}$$

$$r(t) = -\theta(\mu - X(t)) - \frac{\varsigma^2}{2}. \tag{3.50}$$

In particular, (3.50) and (3.45) yield

$$r(0) = -\theta(\mu - 1) - \frac{\varsigma^2}{2}. \tag{3.51}$$

Since the solution of (3.45) is

$$X(t) = e^{-\theta t} + \mu(1 - e^{-\theta t}) + \varsigma \int_0^t e^{\theta(s-t)} dW_s^{\mathbb{Q}^F}, \quad t \le T, \tag{3.52}$$

substituting (3.52) into (3.50) we obtain,

$$\begin{aligned} r(t) &= -\theta\mu + \theta(e^{-\theta t} + \mu(1 - e^{-\theta t}) + \varsigma \int_0^t e^{\theta(s-t)} dW_s^{\mathbb{Q}^F}) - \frac{\varsigma^2}{2} \\ &= \theta e^{-\theta t}(1 - \mu) - \frac{\varsigma^2}{2} + \varsigma\theta \int_0^t e^{\theta(s-t)} dW_s^{\mathbb{Q}^F}. \end{aligned}$$

On the other hand, by (3.11) we have the following dynamics for the short rate process

$$r(t) = r(0) + \int_0^t (\alpha(s,t) + v\eta(s,t))ds + \int_0^t \eta(s,t) dW_s^{\mathbb{Q}^F}, \quad t \le T. \tag{3.53}$$

Hence, by (3.53) and (3.51)

$$\eta(s,t) = \varsigma\theta e^{\theta(s-t)} \tag{3.54}$$

and

$$-\theta(\mu-1)+\int_0^t \alpha(s,t)ds-\varsigma^2\theta\int_0^t e^{\theta(s-t)}ds=\theta e^{-\theta t}(1-\mu),$$

i.e.

$$\begin{aligned}\int_0^t \alpha(s,t)ds &= \theta(\mu-1)(1-e^{-\theta t})+\varsigma^2(1-e^{-\theta t})\\ &= (1-e^{-\theta t})(\varsigma^2+\theta\mu-\theta). \qquad (3.55)\end{aligned}$$

Furthermore, by (3.28) and (3.29) the futures price process F satisfies

$$\begin{aligned}F(s,t) &= S(s)\exp\{-\int_s^t \delta(u,t)dL_u+\int_s^t \psi(u,t)du\}\\ &= \exp\{X(s)+\int_s^t(\psi(u,t)+\varsigma\delta(u,t))du-\int_s^t \delta(u,t)dW_u^{\mathbb{Q}^F}\}, \quad (3.56)\end{aligned}$$

where by (3.32) and (3.33) we have

$$\begin{aligned}\psi(u,t) &= \frac{1}{2}(\sigma^2(u,t)-\varsigma^2),\\ \delta(u,t) &= \sigma(u,t)+\varsigma,\end{aligned}$$

and by (3.21)

$$\sigma(u,t)=-\int_0^t \eta(u,s)ds=-\varsigma e^{\theta u}(1-e^{-\theta t}).$$

In particular, the corresponding bond price process is given by $P(s,t)=\frac{F(s,t)}{S(t)}$, $s\le t$. Hence, by (3.56)

$$P(s,t)=\exp\{X(s)-X(t)+\int_s^t(\psi(u,t)+\varsigma\delta(u,t))du-\int_s^t \delta(u,t)dW_u^{\mathbb{Q}^F}\}.$$

Now we introduce an example, where the spot price S is an exponential of an Ornstein-Uhlenbeck process driven by a pure jump Lévy process.

Example 3.4.5. Let L be a pure jump integral under $\mathbb{Q}^F$, i.e.

$$L_t=\int_0^t\int_{\mathbb{R}} xJ_L^{\mathbb{Q}^F}(dx\times ds)$$

for some Poisson random measure $J_L^{\mathbb{Q}^F}$ on $\mathbb{R} \times (0, \infty)$, and let X_t be an Ornstein-Uhlenbeck process satisfying

$$dX_t = -X_t dt + dL_t, \quad X_0 = x_0 \in \mathbb{R}, \quad t \le T.$$

Then

$$X_t e^{-(T-t)} = \underbrace{x_0 e^{-T}}_{=:k} + \int_0^t e^{-(T-s)} dL_s, \tag{3.57}$$

where $\nu^{\mathbb{Q}^F}$ is the Lévy measure of L under $\mathbb{Q}^F$.

Further, we assume that the spot price S is the exponential of X, i.e. $S(t) = e^{X_t}$, $t \le T$. We show in this example that we can find a forward rate structure such that the futures price process $F(t,T) := S(t)P(t,T)$, $t \le T$, is a $\mathbb{Q}^F$-martingale.

Assume that

$$\int_{\mathbb{R}} e^x (1 + |x|) \nu^{\mathbb{Q}^F}(dx) < \infty, \tag{3.58}$$

and define

$$P(t,T) := \exp\Big\{ (e^{-(T-t)} - 1) X_t + \int_t^T \int_{\mathbb{R}} \left(\exp\{e^{-(T-s)} x\} - 1 \right) \nu^{\mathbb{Q}^F}(dx) ds \Big\}. \tag{3.59}$$

Since $P(T,T) = 1$, we can consider $P(t,T)$, for $t \le T$, as a bond price process. Furthermore,

$$\begin{aligned} F(t,T) &= S(t)P(t,T) \\ &= F(0,T) \exp\Big\{ \int_0^t e^{-(T-s)} dL_s - \int_0^t \int_{\mathbb{R}} \left(e^{e^{-(T-s)}x} - 1 \right) \nu^{\mathbb{Q}^F}(dx) ds \Big\} \\ &= F(0,T) \exp\Big\{ \int_0^t \int_{\mathbb{R}} e^{-(T-s)} x dJ_L^{\mathbb{Q}^F}(dx \times ds) \\ &\quad - \int_0^t \int_{\mathbb{R}} \left(e^{e^{-(T-s)}x} - 1 \right) \nu^{\mathbb{Q}^F}(dx) ds \Big\}, \quad t \le T, \end{aligned} \tag{3.60}$$

where

$$F(0,T) = \exp\Big\{k + \int_0^T \int_{\mathbb{R}} \big(\exp\{e^{-(T-s)}x\} - 1\big)\, \nu^{\mathbb{Q}^F}(dx)ds\Big\},$$

and $k \in \mathbb{R}$ is defined in (3.57). By the exponential formula for Poisson random measures (see e.g. [12], Proposition 3.6) the process $F(t,T)$, given in (3.60), is a $\mathbb{Q}^F$-martingale. We now derive the forward rate that gives us the bond $P(t,T)$ as in (3.59):

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T} \ln P(t,T) \\ &= e^{-(T-t)}X_t + \int_t^T \int_{\mathbb{R}} \exp\{e^{-(T-s)}x\} e^{-(T-s)} x \nu^{\mathbb{Q}^F}(dx)ds\} \\ &\quad + \int_{\mathbb{R}} (e^x - 1)\nu^{\mathbb{Q}^F}(dx). \end{aligned} \tag{3.61}$$

In particular, the corresponding short rate process is then given by

$$r(t) = f(t,t) = X_t + \int_{\mathbb{R}} (e^x - 1)\nu^{\mathbb{Q}^F}(dx).$$

Note that condition (3.58) guarantees that $f(t,T)$ in (3.61) and $P(t,T)$ in (3.59) are well-defined.

In the next section we consider the Markov property of the spot price S under $\mathbb{Q}^F$ in the special case, where δ and ψ appearing in (3.29) are deterministic.

3.4.1 Case of the deterministic coefficients

For the sake of simplicity we will only consider the one-dimensional case, i.e. we assume $n = 1$. However, all results of this subsection still hold in the case of multidimensional non-homogeneous Lévy process with independent components.

We examine the Markov property of the spot price process S given by

$$S(t) = F(0,t) \exp\{\int_0^t \delta(s,t)dL_s - \int_0^t \psi(s,t)ds\}, \quad t \in [0,T], \tag{3.62}$$

under the futures martingale measure $\mathbb{Q}^F$ when δ and ψ are deterministic continuous functions. Because $F(0,t)$ is also deterministic by assumptions, S is a Markov process iff the process

$$Z_t = \int_0^t \delta(s,t)dL_s, \quad t \in [0,T], \tag{3.63}$$

is Markovian. Recall that L is a non-homogeneous Lévy process under $\mathbb{Q}^F$ by Proposition 3.3.2.

Proposition 3.4.6. *We assume that there are constants $\epsilon, \eta > 0$ and functions $c(t), \gamma(t) : [0,T] \to \mathbb{R}^+$, such that for all $t \in [0,T]$*

1. $\int_0^t c(s)ds < \infty$,
2. $\gamma(t) \geq \epsilon$,
3. $\Re\Phi^{\mathbb{Q}^F}(t,u) \leq c(t) - \gamma(t)|u|^\eta$*, for every* $u \in \mathbb{R}$*, where* $\Phi^{\mathbb{Q}^F}(t,u)$ *is the characteristic exponent of* L_t *under* $\mathbb{Q}^F$ *defined by* (3.37).

Then the spot price process S is Markovian iff for all fixed w and u with $0 < w < u \leq T$ there exists a real constant $\xi = \xi_u^w$ (which may depend on w and u) such that

$$\delta(t,u) = \xi_u^w \delta(t,w), \quad \forall t \in [0,T],$$

where δ is the volatility structure of S in (3.62).

The proof of Proposition 3.4.6 uses the idea of the proofs of Lemmas 4.1 and 4.2 in [19]. We start with the following lemma, generalizing Lemma 4.1 in [19] to the case of inhomogeneous Lévy processes.

Lemma 3.4.7. *Suppose $t \in [0,T]$ and that $f, g : [0,t] \to \mathbb{R}$ are continuous linearly independent functions. Then, under the hypothesis of Proposition 3.4.6, the joint distribution of the random variables $X := \int_0^t f(s)dL_s$ and $Y := \int_0^t g(s)dL_s$ is absolutely continuous w.r.t. the Lebesgue measure λ^2 on $\mathbb{R}^2$.*

Proof of Lemma 3.4.7. A probability distribution on $\mathbb{R}^d$ is absolutely continuous w.r.t. λ^d iff its characteristic function is integrable over $\mathbb{R}^d$. Thus it is enough to prove the λ^2 integrability of the joint characteristic function $\phi(x,y)$ of X and Y.

According to Proposition 1.9 in [33]

$$\phi(x,y) := E^{\mathbb{Q}^F}[e^{ixX+iyY}] = \exp\{\int_0^t \Phi_s^{\mathbb{Q}^F}(xf(s)+yg(s))ds\}.$$

Hence, by assumption,

$$\begin{aligned}|\phi(x,y)| &= \exp\{\int_0^t \Re\Phi_s^{\mathbb{Q}^F}(xf(s)+yg(s))ds\}\\ &\le \exp\{\int_0^t (c(s)-\gamma(s)|xf(s)+yg(s)|^\eta)ds\}\\ &\le C_t\exp\{-\gamma\int_0^t |xf(s)+yg(s)|^\eta ds\},\end{aligned}$$

where $C_t := e^{\int_0^t c(s)ds} < \infty$. Consider the normed vector $(x_0,y_0) := \frac{(x,y)}{\|(x,y)\|}$. Since $xf(s)+yg(s)$ is the Euclidean scalar product in $\mathbb{R}^2$ of the vectors (x,y) and $(f(s),g(s))$, we obtain

$$\int_0^t |xf(s)+yg(s)|^\eta ds = \|(x,y)\|^\eta \int_0^t |x_0f(s)+y_0g(s)|^\eta ds. \tag{3.64}$$

The integral on the right hand side of (3.64) is a continuous function of the vector (x_0,y_0). Therefore, it has a minimum m on the unit circle in $\mathbb{R}^2$. It is obvious that $m \ge 0$. Suppose $m = 0$. This would imply that the integrand vanishes for all s; but this is impossible, because f and g are linearly independent by assumption. Hence we must have $m > 0$. From $m > 0$ follows

$$\int_{\mathbb{R}^2} |\phi(x,y)|d\lambda^2(x,y) \le C_t\exp\{-\gamma m\|(x,y)\|^\eta d\lambda^2(x,y)\} < \infty.$$

□

Proof of Proposition 3.4.6. The proof of Proposition 3.4.6 actually repeats the proof of Lemma 4.2 in [19]. We include it to make the text self-contained.

Proof of the necessity: Assume first that S is a Markov process. Then, according to the preliminary consideration above, the process Z defined by (3.63) is Markovian. This implies

$$E[Z_u|\mathcal{F}_w] = E[Z_u|Z_w], \quad 0 < w < u \leq T.$$

By (3.63) the last equation becomes

$$E\Big[\int_0^w \delta(s,u)dL_s\Big|\mathcal{F}_w\Big] + E\Big[\int_w^u \delta(s,u)dL_s\Big|\mathcal{F}_w\Big]$$
$$= E\Big[\int_0^w \delta(s,u)dL_s\Big|Z_w\Big] + E\Big[\int_w^u \delta(s,u)dL_s\Big|Z_w\Big].$$

Since $\delta(\cdot,\cdot)$ is deterministic and L is a process with independent increments, $\int_w^u \delta(s,u)dL_s$ is independent of the σ-field $\mathcal{F}_w$ and, in particular, of Z_w. This implies that the second summands on both sides are equal. Additionally, $\int_0^w \delta(s,u)dL_s$ is measurable with respect to $\mathcal{F}_w$. Thus,

$$\int_0^w \delta(s,u)dL_s = E\Big[\int_0^w \delta(s,u)dL_s\Big|Z_w\Big] = E\Big[\int_0^w \delta(s,u)dL_s\Big|\int_0^w \delta(s,w)dL_s\Big].$$

But this means that the integral $\int_0^w \delta(s,u)dL_s$ can be expressed as some measurable function G applied to the integral $\int_0^w \delta(s,w)dL_s$. Hence, the joint distribution of these two random variables is concentrated on the Lebesgue null set

$$\{(x, G(x))|\, x \in \mathbb{R}\} \subset \mathbb{R}^2,$$

and thus cannot be continuous with respect to λ^2. Hence, by Lemma 3.4.7 $\delta(\cdot,w)$ and $\delta(\cdot,u)$ restricted to $[0,w]$ are linearly dependent.

Proof of the sufficiency: It is enough to show that the process Z defined by (3.63) is Markovian. Suppose that w and u satisfy $0 < w < u \leq T$. Then we have

$$Z_u = \int_0^w \delta(s,u)dL_s + \int_w^u \delta(s,u)dL_s.$$

By assumption, the first term on the right-hand side is equal to

$$\int_0^w \xi\delta(s,w)dL_s = \xi Z_w$$

for some real constant ξ. Hence it is measurable w.r.t. $\mathcal{F}_w$. The second term is independent of $\mathcal{F}_w$. These two facts yield

$$\mathbb{P}[Z_u \in A|\mathcal{F}_w] = \mathbb{P}[Z_u \in A|Z_w], \quad \text{for every} \quad A \in \mathcal{B}(\mathbb{R}).$$

□

From Proposition 3.4.6 follows that

Corollary 3.4.8. *Under the hypotheses of Proposition 3.4.6 the spot price process S is Markovian iff its volatility structure δ admits the representation*

$$\delta(t,\tau) = \zeta(t)\rho(\tau), \quad (t,\tau) \in \mathcal{D}, \tag{3.65}$$

where $\zeta, \rho : [0,T] \to \mathbb{R}$ are continuously differentiable functions.

See Theorem 4.3 in [19] for the proof of Corollary 3.4.8.

Now we consider two examples of the volatility function δ that satisfies (3.65).

Example 3.4.9 (Vasicek volatility structure)**.** Recall that

$$\delta(t,\tau) = \sigma(t,\tau) - v(t),$$

where σ is the volatility of the corresponding bond and v is a deterministic function. Let

$$\sigma(t,\tau) = \frac{\hat{\sigma}}{a}(1 - e^{-a(\tau - t)}) \quad \textit{(Vasicek volatility)},$$

where $\hat{\sigma} > 0$ and $a \neq 0$. Then by Corollary 3.4.8 the spot price process S is Markovian iff there exist continuously differentiable functions $\zeta, \rho : [0,T] \to \mathbb{R}$, such that

$$v(t) = \frac{\hat{\sigma}}{a}(1 - e^{-a(\tau - t)}) - \zeta(t)\rho(\tau).$$

Since v is constant in τ, by deriving we obtain

$$\zeta(t)\rho'(\tau) = \hat{\sigma}e^{at}e^{-a\tau},$$

and consequently

$$\zeta(t) = \lambda \hat{\sigma} e^{at},$$
$$\rho'(\tau) = \frac{1}{\lambda} e^{-a\tau}$$

for $(t, \tau) \in \mathcal{D}$ and some $\lambda \neq 0$. Then $\rho(\tau) = -\frac{1}{a\lambda} e^{-a\tau} + c$ for some $c \in \mathbb{R}$, $\lambda \neq 0$. Hence, in this example the spot price process S is Markovian iff $v(t)$ is of the form

$$v(t) = \frac{\hat{\sigma}}{a} - \hat{\sigma} c e^{at}$$

for some $c \in \mathbb{R}$.

Example 3.4.10 (Ho-Lee volatility structure)**.** In case the bond volatility structure σ satisfies

$$\sigma(t, \tau) = \hat{\sigma}(\tau - t) \quad \text{with} \quad \hat{\sigma} > 0 \quad \textit{(Ho-Lee volatility)},$$

Corollary 3.4.8 yields that the spot price S is a Markov process iff $v(t)$ is of the form $v(t) = \hat{\sigma}(c - t)$ for some $c \in \mathbb{R}$.

Now we show that Corollary 3.4.8 enables us to characterize the class of stationary volatility structures δ that lead to Markovian spot price process S.

Proposition 3.4.11. *Suppose the volatility structure δ is stationary, that means, there exists a twice continuously differentiable function $\tilde{\delta} : [0, T] \to \mathbb{R}^+$ such that $\delta(t, \tau) = \tilde{\delta}(\tau - t)$ for all $(t, \tau) \in \mathcal{D}$. Then, under the hypotheses of Proposition 3.4.6, S is a Markov process iff δ is of the form*

$$\delta(t, \tau) = \hat{\delta} e^{a(\tau - t)} \tag{3.66}$$

with $a \in \mathbb{R}$ and $\hat{\delta} > 0$.

Proof. If δ is of the form (3.66), then S is a Markov process by Corollary 3.4.8.

Assume now that S is Markovian. As $\delta(t,\tau)$ is stationary by assumption, the partial derivatives satisfy

$$\frac{\partial}{\partial\tau}\delta(t,\tau) = \tilde{\delta}'(\tau - t) = -\frac{\partial}{\partial t}\delta(t,\tau).$$

Corollary 3.4.8 yields then

$$\zeta'(t)\rho(\tau) = -\zeta(t)\rho'(\tau),$$

i.e.

$$(\log\rho)'(\tau) = -(\log\zeta')(t)$$

for all $(t,\tau) \in \mathcal{D}$. Since t and τ are independent variables, neither of the last equality sides can actually depend on t or τ. Hence both sides are constant. Denoting their common value by a, we obtain

$$\rho(\tau) = e^{a\tau + K_1} \quad \text{and} \quad \zeta(t) = e^{-at+K_2}$$

with two real constants K_1 and K_2, and hence

$$\delta(t,\tau) = e^{K_1+K_2}e^{a(\tau-t)}.$$

Defining $\hat{\delta} := e^{K_1+K_2}$, we get (3.66). □

The volatility structure (3.66) picks up the maturity effect for $a < 0$: the volatility increases when a future contract comes to delivery, since temperature forecasts, outages and other specifics about the delivery period become more and more precise. However, the model (3.66) does not include seasonality: futures during winter months show higher prices than comparable contracts during the summer. See [5], [36], and [32] for a description of electricity futures and options markets. In order to include the seasonality we can use, for example, the volatility model suggested in [20]:

$$\delta(t,\tau) = a(t)e^{-b(\tau-t)}, \quad b \geq 0.$$

The seasonal part $a(t)$ can be modeled, for example, as a truncated Fourier series

$$a(t) = a + \sum_{j=1}^{J}(d_j\sin(2\pi jt) - f_j\cos(2\pi jt)),$$

where $a \geq 0$, $d_j, f_j \in \mathbb{R}$, and t is measured in years. See [20] and [5] for more details on the modeling of volatility.

Chapter 4

Valuation of electricity derivatives

4.1 Pricing of European options

For the valuation of the European options on the spot price we use Fourier transform method applied to the dampened payoff introduced in Section 1.2.2. We consider the pricing of the options only on the example of an electricity floor contract. Electricity calls, puts and caps can be priced similarly. See also [26] for the pricing of European options on the electricity spot price under the assumption of continuous futures and spot price processes.

A floor is a European type contract that protects against low commodity prices within $[\tau_1, \tau_2]$. It ensures a cash flow at intensity $((K - S(t))^+)_{t \in [\tau_1, \tau_2]}$ with strike price $K > 0$ at any time $t \in [\tau_1, \tau_2]$ of the contract.

In the remainder of this subsection we suppose that the riskless interest rate r is constant. The fair price at time t of the floor option with strike price $K > 0$ is equal to

$$\text{Floor}(t, K) = E^{\mathbb{Q}^F}\Big[\int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)}(K - S(\tau))^+ d\tau \Big| \mathcal{F}_t\Big].$$

By Fubini's Theorem we get

$$\text{Floor}(t, K) = \int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} E^{\mathbb{Q}^F}\Big[(K - S(\tau))^+ \Big| \mathcal{F}_t\Big] d\tau. \tag{4.1}$$

To simplify the notation we only consider the one-dimensional case under assumption of the deterministic coefficients, i.e. we assume the spot price process $S(t)$ to be given by (3.62), where δ and ψ are deterministic.

Recall that by (3.56) the spot price process S satisfies

$$S(\tau) = F(t,\tau)\exp\{\int_t^\tau \delta(s,\tau)dL_s - \int_t^\tau \psi(s,\tau)ds\} =: F(t,\tau)U_t^\tau, \tag{4.2}$$

where $F(t,\tau)$, for $0 \le t \le \tau$, is a $\mathbb{Q}^F$-martingale, and L is a non-homogeneous Lévy process. Since $F(t,\tau)$ is $\mathcal{F}_t$-measurable and U_t^τ is independent of $\mathcal{F}_t$, by substituting (4.2) into (4.1) we obtain

$$\begin{aligned}\text{Floor}(t,K) &= \int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)}E^{\mathbb{Q}^F}\left[(K - F(t,\tau)U_t^\tau)^+ d\tau \Big| \mathcal{F}_t\right]d\tau \\ &= \int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)}F(t,\tau)e^{-\int_t^\tau \psi(s,\tau)ds}E^{\mathbb{Q}^F}\left[(K(f) - e^{\int_t^\tau \delta(s,\tau)dL_s})^+\right]\Big|_{f:=F(t,\tau)}d\tau,\end{aligned} \tag{4.3}$$

where $K(f) := \frac{K}{f}\exp\{\int_t^\tau \psi(s,\tau)ds\}$, $f > 0$. In order to compute the expectation in (4.3), consider the integrable dampened pay-off function

$$g(x) := e^x(K(f) - e^x)^+ \in L^1(\mathbb{R}).$$

Denote by $\hat{g}$ its Fourier transform:

$$\hat{g}(u) := \int_{\mathbb{R}} e^{iux}g(x)dx = K(f)^{2+iu}\frac{1}{(1+iu)(2+iu)} \in L^1(\mathbb{R}). \tag{4.4}$$

Using the Inversion Theorem for Fourier transform (cf. [37], Section 8.2) we get

$$\begin{aligned}E^{\mathbb{Q}^F}\left[\left(K(f) - e^{\int_t^\tau \delta(s,\tau)dL_s}\right)^+\right] &= E^{\mathbb{Q}^F}\left[e^{-\int_t^\tau \delta(s,\tau)dL_s}g(\int_t^\tau \delta(s,\tau)dL_s)\right] \\ &= E^{\mathbb{Q}^F}\left[e^{-\int_t^\tau \delta(s,\tau)dL_s}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iu\int_t^\tau \delta(s,\tau)dL_s}\hat{g}(u)du\right] \\ &= \frac{1}{2\pi}E^{\mathbb{Q}^F}\left[\int_{\mathbb{R}} e^{-(1+iu)\int_t^\tau \delta(s,\tau)dL_s}\hat{g}(u)du\right] \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} E^{\mathbb{Q}^F}\left[e^{-(1+iu)\int_t^\tau \delta(s,\tau)dL_s}\right]\hat{g}(u)du,\end{aligned} \tag{4.5}$$

where (4.4) allows to apply Fubini's Theorem in the last equality. By Proposition 3.3.2 and Proposition 1.9 in [33]

$$E^{\mathbb{Q}^F}[e^{-\int_t^\tau (1+iu)\delta(s,\tau)dL_s}] = \exp\{\int_t^\tau \Theta_s^{\mathbb{Q}^F}(-(1+iu)\delta(s,\tau))ds\}, \qquad (4.6)$$

where $\Theta_s^{\mathbb{Q}^F}$ is given by

$$\Theta_s^{\mathbb{Q}^F}(z) = zb_s^{\mathbb{Q}^F} + \frac{z^2}{2}c_s^{\mathbb{Q}^F} + \int_{\mathbb{R}} (e^{zx} - 1 - zxI_{|x|\leq 1})e^{v(s)x}\nu(dx), \quad s \leq T.$$

Substituting (4.5), (4.4), and (4.6) into (4.3), we obtain

$$\begin{aligned} \text{Floor}(t, K) = & \int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)}F(t,\tau)e^{-\int_t^\tau \psi(s,\tau)ds} \\ & \times \int_{\mathbb{R}} \exp\{\int_t^\tau \Theta_s^{\mathbb{Q}^F}(-(1+ix)\delta(s,\tau))ds\} \\ & \times \left(\frac{K}{F(t,\tau)}e^{\int_t^\tau \psi(s,\tau)ds}\right)^{2+ix} \frac{1}{(1+ix)(2+ix)}dxd\tau \\ = & K^2 e^{rt} \int_{t\vee\tau_1}^{\tau_2} e^{-r\tau} \int_{\mathbb{R}} \exp\{\int_t^\tau \Theta_s^{\mathbb{Q}^F}(-(1+ix)\delta(s,\tau))ds\} \\ & \times \left(\frac{e^{\int_t^\tau \psi(s,\tau)ds}}{F(t,\tau)}\right)^{1+ix} \frac{K^{ix}}{(1+ix)(2+ix)}dxd\tau. \end{aligned}$$

4.2 Pricing of swing options

In this section we illustrate how the spot price model (3.42) can be used to valuate path dependent derivatives on an example of electricity swing options. For the sake of simplicity we consider a special case, where the process L is a one-dimensional standard Brownian motion under $\mathbb{Q}^M$, as in Example 3.4.4. Analogously to (3.48) we get the following dynamics of the electricity spot price under the measure $\mathbb{Q}^F$:

$$dS(t) = -S(t)r(t)dt - S(t)v(t)dW_t^{\mathbb{Q}^F}, \quad t \in [0,T]. \qquad (4.7)$$

Recall that by (3.11) and (3.46) the short rate process r satisfies

$$\begin{aligned} dr(t) &= \alpha(t,t)dt + \eta(t,t)dL_t \\ &= (v(t)\eta(t,t) + \alpha(t,t))dt + \eta(t,t)dW_t^{\mathbb{Q}^F}, \quad t \in [0,T], \qquad (4.8) \end{aligned}$$

under $\mathbb{Q}^F$. Now we assume that the volatility η is deterministic, and hence r is a Markov process.

Moreover, we assume that there exists a unique solution $(S(t), r(t))$ of (4.7) – (4.8) satisfying the initial condition $(S(u), r(u)) = (s, r) \in \mathbb{R}^2$, and such that

$$E^{\mathbb{Q}^F}[S^2(t)] < \infty \quad \text{for all } t \in [0, T].$$

Recall that, since r is Markovian, by Proposition 3.4.2 $(S(t), r(t))$ is a Markov process.

Let us consider a swing option on the spot price process (4.7). A swing option is an agreement to purchase energy at a certain fixed price over a specified time interval. In short, the payoff of a swing option settled at time T is defined as

$$\int_0^T \nu(t)(S(t) - K)dt, \tag{4.9}$$

where $\nu(t)$ is the production intensity and $K > 0$ is the strike price of the contract. The holder of the contract has the right (within specified limits), to control the intensity of electricity production at any moment. The goal of the option holder is to maximize the value of the contract by selecting the optimal intensity process ν among the processes that are limited by contract specific lower and upper bounds:

$$\nu_{low} \leq \nu(t) \leq \nu^{up} \quad \text{a.e. } t,$$

under the constraint that the optimal intensity process ν is such that the total volume produced

$$C^\nu(t) = c + \int_u^t \nu(x)dx, \quad u \leq t \leq T, \tag{4.10}$$

does not exceed the maximum amount $\bar{C}$ that can be produced during the contract life time. Hence the option holder tries to maximize the expected profit, i.e. to find

$$V(u,s,r,c) := \sup_{\nu\in N} E^{\mathbb{Q}^F}\left[\int_u^{T\wedge\tau_{\bar{C}}} \nu(t)(S(t)-K)dt\right] \tag{4.11}$$

$$= E^{\mathbb{Q}^F}\left[\int_u^{T\wedge\tau_{\bar{C}}} \nu^*(t)(S(t)-K)dt\right], \tag{4.12}$$

where

$$N := \{\nu \text{ progressively measurable: } \nu(t)\in[\nu_{low},\nu^{up}] \text{ for a.e. } t\in[0,T]\}$$

is the control set, and

$$\tau_{\bar{C}} := \inf\{t>0 |\ C^\nu(t) = \bar{C}\}$$

is the first time when all of production rights are used up. Note that the value function V satisfies the boundary conditions

$$V(T,s,r,c) = 0 \quad \text{and} \quad V(u,s,r,\bar{C}) = 0. \tag{4.13}$$

If we assume that V in (4.11) is sufficiently smooth, then by Itô formula, (4.7), and (4.8) we get

$$V(T\wedge\tau_{\bar{C}},S,r,C^\nu) - V(u,s,r,c) + \int_u^{T\wedge\tau_{\bar{C}}} \nu(t)(S(t)-K)dt$$
$$= \int_u^{T\wedge\tau_{\bar{C}}} \Big[\partial_t V + \nu(t)\partial_c V - S(t)r(t)\partial_s V + (v(t)\eta(t,t)+\alpha(t,t))\partial_r V$$
$$+ S^2(t)v^2(t)\partial^2_{ss}V - 2S(t)v(t)\eta(t,t)\partial^2_{sr}V + \eta^2(t,t)\partial^2_{rr}V\Big]dt$$
$$- \int_u^{T\wedge\tau_{\bar{C}}} (S(t)v(t)\partial_s V - \eta(t,t)\partial_r V)dW_t^{\mathbb{Q}^F} + \int_u^{T\wedge\tau_{\bar{C}}} \nu(t)(S(t)-K)dt. \tag{4.14}$$

Denote

$$A^\nu V(t,s,r,c) := \partial_t V(t,s,r,c) - sr\partial_s V(t,s,r,c) + (v(t)\eta(t,t)$$
$$+ \alpha(t,t))\partial_r V(t,s,r,c) + s^2v^2(t)\partial^2_{ss}V(t,s,r,c) - 2sv(t)\eta(t,t)\partial^2_{sr}V(t,s,r,c)$$
$$+ \eta^2(t,t)\partial^2_{rr}V(t,s,r,c) + \nu(t)(\partial_c V(t,s,r,c) + s - K),$$

for $\nu \in N$. Since

$$V(T \wedge \tau_{\bar{C}}, S(T \wedge \tau_{\bar{C}}), r(T \wedge \tau_{\bar{C}}), C^{\nu}(T \wedge \tau_{\bar{C}})) = 0,$$

we can rewrite (4.14) as

$$\begin{aligned} E^{\mathbb{Q}^F}\Bigg[\int\limits_u^{T\wedge\tau_{\bar{C}}} \nu(t)(S(t)-K)dt\Bigg] - V(u,s,r,c) \\ = E^{\mathbb{Q}^F} \int_u^{T\wedge\tau_{\bar{C}}} A^{\nu} V(t, S(t), r(t), C^{\nu}(t))dt \\ - E^{\mathbb{Q}^F}\Bigg[\int\limits_u^{T\wedge\tau_{\bar{C}}} (S(t)v(t)\partial_s V - \eta(t,t)\partial_r V)dW_t^{\mathbb{Q}^F}\Bigg]. \end{aligned} \tag{4.15}$$

Note that the left hand side of equality (4.15) is non-positive for every $\nu \in N$. Furthermore, it vanishes if the pair (ν, V) is a solution of the stochastic control problem (4.11).

Let $\mathcal{S} = [0,T) \times \mathbb{R}_+^2 \times [0,\bar{C})$. Assume that there exists a solution $(\hat{\nu}, \hat{V})$ of the Hamilton-Jacobi-Bellman equation

$$A^{\hat{\nu}}\hat{V}(x) = 0, \quad \text{for each } x \in \mathcal{S}, \tag{4.16}$$

where $\hat{\nu} \in N$ and $\hat{V} \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ satisfies the terminal and boundary conditions (4.13). Moreover, suppose that

$$E^{\mathbb{Q}^F}\Big[\int_u^{T\wedge\tau_{\bar{C}}} |A^{\hat{\nu}}\hat{V}(t,S(t),r(t),C^{\hat{\nu}}(t))|dt\Big] < \infty. \tag{4.17}$$

Then by Dynkin formula (see Theorem 1.24 in [44]) and by (4.15) we have

$$\begin{aligned} E^{\mathbb{Q}^F}\Bigg[\int\limits_u^{T\wedge\tau_{\bar{C}}} \hat{\nu}(t)(S(t)-K)dt\Bigg] - \hat{V}(u,s,r,c) \\ = E^{\mathbb{Q}^F} \int_u^{T\wedge\tau_{\bar{C}}} A^{\hat{\nu}}\hat{V}(t,S(t),r(t),C^{\hat{\nu}}(t))dt = 0, \end{aligned}$$

and hence the pair $(\hat{\nu}, \hat{V})$ is a solution of (4.11). Note that we could apply Dynkin formula because of the Markov property of the process (S, r). We obtained the verification theorem for the optimal control problem (4.11) in the following form:

Proposition 4.2.1. *Assume that there exist* $\hat{V} \in C^2(\mathbb{S}) \cap C(\bar{\mathbb{S}})$ *and* $\hat{\nu} \in N$*, such that* $(\hat{\nu}, \hat{V})$ *is a solution of the Hamilton-Jacobi-Bellman equation* (4.16) *satisfying* (4.17)*. Moreover, suppose that* $\hat{V}$ *fulfills the terminal and boundary conditions* (4.13)*. Then* $\hat{V}$ *is the value function of the swing option defined in* (4.11)*.*

Note that Proposition 4.2.1 also follows from the classical verification theorem, but the direct derivation is less technical and more illustrative. We refer to [44] for more details on stochastic optimal control problems.

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