Bigroupoid 2-torsors

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig-Maximilians-Universität München
eingereicht von

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2008

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Tag der mündlichen Prüfung: 27.06.2008

Zusammenfassung

In der vorliegenden Doktorarbeit werden zwei fundamentale Konzepte der *höher dimensionalen Algebra*, die *Kategorifizierung* und *Internalisierung*, verfolgt. Von der geometrischen Perspektive waren die bis jetzt allgemeinsten Torsoren mittels der *Wirkungen von Kategorien und Gruppoiden* in der Dimension $n = 1$ definiert. In der Dimension $n = 2$ haben Mauri and Tierney, und neulich Baez und Bartels von einem anderen Gesichtwinkel weniger allgemeine 2-Torsoren mit einer 2-Strukturgruppe definiert. In der Sprache der simplizialen Algebra haben Duskin und Glenn Wirkungen und Torsoren, die zur jeder Barrschen exakten Kategorie $\mathcal{E}$ intern sind, in einer beliebigen Dimension n definiert. Diese Wirkungen sind simpliziale Abbildungen, die in Dimensionen $m \geq n$ *exakte Faserungen* über speziellen simplizialen Objekten, so genannten *n-dimensionalen Kanschen Hypergroupoiden*, sind. Die Korrespondenz zwischen der geometrischen und algebraischen Theorie in der Dimension $n = 1$ ist durch die Grothendiecksche Konstruktion vom Nerv gegeben, da das Grothendiecksche Nerv von einem Gruppoid genau ein 1-dimesionales Kansches Gruppoid ist. Ein Hauptresultat ist, dass die Wirkungen von Gruppoiden und die Torsoren der Gruppoide zu simplizialen Wirkungen und simplizialen Torsoren über den entsprechenden 1-dimesionalen Kanschen Gruppoid werden, nach dem die Grothendiecksche Nerv-Konstruktion angewandt wird.
Das Hauptergebnis der vorliegenden Doktorarbeit ist eine Verallgemeinerung dieser Korrespondenz auf die Dimension $n = 2$. Dieses Resultat wurde durch die Einfhrung von zwei neuen geometrischen und algebraischen Konzepten, *Wirkungen von Bikategorien* und *2-Torsoren von Bigruppoiden*, die eine Kategorifizierung und Internalisierung der Wirkungen von Kategorien und Torsoren von Gruppoiden darstellen, erreicht. Wir liefern die Klassifizierung von 2-Torsoren von Bigruppoiden mittels der *zweiten nichtabelschen Kohomologie* mit Koeffizienten im Struktur-Bigruppoid. Die zweite nichtabelsche Kohomologie wird mittels eines dritten neuen Konzepts, das in der Doktorarbeit eingeführt wird und das eine *kleine 2-Faserung* die einem internen Bigruppoid in der Kategorie $\mathcal{E}$ entspricht, definiert. Die Korrespondenz in der Dimension $n = 2$ ist durch die Nerv-Konstruktion für Bikategorien und Bigruppoiden von Duskin gegeben, da diese genau ein 2-dimesionales Kansches Gruppoid ergibt. Das letzte Hauptresultat der Doktorarbeit sagt, dass die Bigruppoidwirkungen und Bigruppoid-2-Torsoren zu simplizialen Wirkungen und simplizialen 2-Torsoren über den entsprechenden 2-dimesionalen Kanschen Gruppoid werden, nach dem die Duskinsche Nerv-Konstruktion angewandt wird.

The summary

In this thesis we follow two fundamental concepts from the *higher dimensional algebra*, the *categorification* and the *internalization*. From the geometric point of view, so far the most general torsors were defined in the dimension $n = 1$, by *actions of categories and groupoids*. In the dimension $n = 2$, Mauri and Tierney, and more recently Baez and Bartels from the different point of view, defined less general 2-torsors with the structure 2-group. Using the language of simplicial algebra, Duskin and Glenn defined actions and torsors internal to any Barr exact category $\mathcal{E}$, in an arbitrary dimension n. This actions are simplicial maps which are *exact fibrations* in dimensions $m \geq n$, over special simplicial objects called *n-dimensional Kan hypergroupoids*. The correspondence between the geometric and the algebraic theory in the dimension $n = 1$ is given by the Grothendieck nerve construction, since the Grothendieck nerve of a groupoid is precisely a 1-dimensional Kan hypergroupoid. One of the main results is that groupoid actions and groupoid torsors become simplicial actions and simplicial torsors over the corresponding 1-dimensional Kan hypergroupoids, after the application of the Grothendieck nerve functor.

The main result of the thesis is a generalization of this correspondence to the dimension $n = 2$. This result is achieved by introducing two new algebraic and geometric concepts, *actions of bicategories* and *bigroupoid 2-torsors*, as a categorification and an internalization of actions of categories and groupoid torsors. We provide the classification of bigroupoid 2-torsors by *the second nonabelian cohomology* with coefficients in the structure bigroupoid. The second nonabelian cohomology is defined by means of the third new concept in the thesis, a *small 2-fibration* corresponding to an internal bigroupoid in the category $\mathcal{E}$. The correspondence between the geometric and the algebraic theory in the dimension $n = 2$ is given by the Duskin nerve construction for bicategories and bigroupoids since the Duskin nerve of a bigroupoid is precisely a 2-dimensional Kan hypergroupoid. Finally, the main results of the thesis is that bigroupoid actions and bigroupoid 2-torsors become simplicial actions and simplicial 2-torsors over the corresponding 2-dimensional Kan hypergroupoids, after the application of the Duskin nerve functor.

Acknowledgements

The saga which led to the completion of this thesis began early in 2000, when Prof. Julius Wess, a great man and physicist, who unfortunately died last year, invited me to join his group in the Department of Theoretical Physics at Ludwig-Maximilians-University in Munich. In the following three years, I spent most of my time by reading fundamental texts in mathematics and physics, trying to achieve some initial results in physics, but my biggest discovery during that time was that deep inside, I was really a mathematician. Together with this realization, which was the most important in my carrier, also came a burden of guilt, amplified by the death of Prof. Wess, from whom I should have learn more.

During my time in Munich, I started to collaborate with my supervisor Prof. Branislav Jurčo, a physicist who recognized my wish to work in the foundations of mathematics. This goal, hard as it is, become even harder after I needed to return from Munich to the Rudjer Boškovic Institute in Zagreb, when my postgraduate fellowship expired in 2004. My ideas about categories, topos theory and homotopy theory were naturally met with skepticism by professors at the Theoretical Physics Department, due to the fact that I was forced to start a research as a student, in the surrounding without any tradition in such fields. Fortunately, Prof. Josip Trampetić, who comes from research area of noncommutative gauge field theories and related phenomenology, invited me to his project at Rudjer Boškovic Institute [1], and gave me complete freedom and support to work on my ideas. One of the greatest sources of my inspiration at the Institute was, and still is, my dear friend and colleague Zoran Škoda, who deepen my knowledge in many other areas of mathematics, which were not of my primer interest. We initiated together a bilateral project on nonabelian cohomology and applications, with Prof. Schweigert as a principal investigator from the German side [2], and I was lucky to meet my friend and collaborator Urs Schreiber, with who I share many common interests and to who I thank for many great discussions.

I would like to express my gratitude specially to my supervisor Prof. Branislav Jurčo, who showed enormous patience which allowed me to develop ideas present in this thesis.

I would like to thank to my mother Slavica, who will probably be the happiest by the fact that this record delay in the publication of a thesis has come to an end. I also thank to my father Duško, who developed the sense for reading and knowledge in my earliest age. I thank to my brother Boris, and my best friends Mustafa Mehmedović and Saša Marić, for their encouragement in the hard times.

And the last but not least, this thesis would never be finished without constant support by my dear wife Zrinka, who crowned our love by the birth of our two precious children, Lovre and Hana.

[1] This thesis was in part supported by the Croatian Ministry of Science, Education and Sport, Project No. 098-0982930-2990.

[2] The author acknowledge support from the project "Applications of nonabelian cohomology to Geometry, Algebra and Physics", Fonds DAAD06-346

[3] The author acknowledge support from EU under the MRTN-CT-2006-035505 network programme.

Contents

Part I
Introduction

1 The introduction

One of the central themes of Grothendieck's epic text [43], is the deep relation between topos theory and homotopy theory, where he emphasized the importance of the sheaf theoretical objects corresponding to higher categorical structures. His main motivation for introduction of such categorical structures, as (weak) n-categories and (weak) n-groupoids, was to provide algebraic models for homotopy n-types. Since a homotopy n-type X is a topological space with trivial homotopy groups $\pi_k(X)$ for $k > n$, it can be conveniently described by a simplicial set $\Pi_n(X)$, called *fundamental n-dimensional hypergroupoid* of X.

Simplicial sets were introduced by Eilenberg and Zilber in 1950 [33], and soon after that simplicial homotopy theory was developed by Kan [58], [59], [60], followed by the more general homotopy theories associated to closed model categories, developed by Quillen in 1960's [77]. A simplicial set $X_\bullet$ is the presheaf

$$X_\bullet \colon \Delta^{op} \to Set$$

on the skeletal simplicial category Δ in which objects are given by finite nonempty ordinals $[n] = \{0 < 1 < ... < n\}$, and morphisms are monotone maps between these. There are certain incidence relations between canonical maps $\partial_i \colon [n-1] \to [n]$ for $0 \leq i \leq n-1$, called coface maps, which are injective maps that omit i in the image, and the maps $\sigma_i \colon [n] \to [n-1]$ for $0 \leq i \leq n-1$, called codegeneracy maps, which are surjective maps which repeat i in the image. This relations allows the description of the simplicial set $X_\bullet$ by the diagram

$$X_0 \underset{d_0}{\overset{d_1}{\leftleftarrows}} X_1 \underset{d_0}{\overset{d_2}{\leftleftarrows}} X_2 \underset{d_0}{\overset{d_3}{\leftleftarrows}} X_3 ...$$

in which elements of the set X_n are called n-simplices, and they satisfy *simplicial identities*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && (i < j) \\ s_i s_j &= s_{j+1} s_i && (i \leq j) \\ d_i s_j &= s_{j-1} d_i && (i < j) \\ d_i s_j &= id && (i = j, i = j+1) \\ d_i s_j &= s_{j+1} d_i && (i > j+1) \end{aligned}$$

where maps $d_i := X(\partial_i)$ and $s_i := X(\sigma_i)$ are images of coface and codegeneracy maps. Simplicial sets are objects of the category $\mathcal{S}Set$ whose morphisms are *simplicial maps*, and they are given by natural transformations between presheaves which define simplicial sets. We have the Yoneda embedding

$$y \colon \Delta \to \mathcal{S}Set \tag{1.1}$$

which takes any ordinal $[n]$ to the representable simplicial set $\Delta[n]$, whose m-simplices are given by the set $\Delta[n]_m = Hom_{\mathcal{S}Set}([m],[n])$ of *singular m-simplices*.

The decisive step which brought together category theory and simplicial theory was done by Grothendieck [41] in 1960's, when he realized that to any small category $\mathcal{C}$ one can associate a simplicial set $N\mathcal{C}$, which he called the *nerve* of the category $\mathcal{C}$, in analogy to the construction of the nerve of the covering of a topological space. The set $N\mathcal{C}_n$ of n-simplices consists of all composable strings of n morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \ldots x_{n-2} \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

in the category $\mathcal{C}$. Simplicial face operators $d_i^n \colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ are given by projections for extremals $i = 0, n$, and by composing out i^{th} morphism in a string for inner ones $0 < i < n$. The degeneracy operators are given by inserting identity morphism of i^{th} indexed object. Then the associativity and the identity law in the category $\mathcal{C}$ are encoded in the simplicial identities between face and degeneracy operators on $N\mathcal{C}$. This construction gives a fully faithful functor

$$N \colon \mathrm{Cat} \to \mathcal{S}Set \tag{1.2}$$

from the category Cat of small categories to the category $\mathcal{S}Set$ of simplicial sets, so that the fundamental definitions of the category theory are all inherent in simplicial sets.

The fact that the nerve functor is fully faithful is the reflection of the fact that the skeletal category Δ of finite ordinals (non-empty totally ordered sets) and monotonic maps between them, is a dense subcategory of the category Cat of small categories, or an adequate subcategory in the sense of Isbell [46],[47]. More precisely, if we regard the category $\mathcal{C}$ as an object the category Cat of small categories, then we can interpret the nerve $N\mathcal{C}$ of the category $\mathcal{C}$ as the special case of the so called *geometric nerve* construction. Given a functor from the skeletal category

$$i \colon \Delta \to \mathcal{E} \tag{1.3}$$

to any category $\mathcal{E}$, a geometric nerve NC of an object C of the category $\mathcal{E}$ is the simplicial set whose set of n-simplices is defined by $N\mathcal{C}_n := Hom_{\mathcal{E}}(i[n], C)$. By this construction we obtain the geometric nerve functor

$$N \colon \mathcal{E} \to \mathcal{S}Set \tag{1.4}$$

and if this functor is fully faithful then we say that the functor (1.3) is dense [62]. Consequently, the skeletal category Δ is dense subcategory of the category Cat of small categories, since

$$i \colon \Delta \to \mathrm{Cat} \tag{1.5}$$

the natural embedding of ordinals as non-empty totally ordered sets is dense. For any functor

$$D \colon \mathcal{J} \to \mathcal{E} \tag{1.6}$$

from a small category $\mathcal{J}$ to a cocomplete category $\mathcal{E}$, the *singular functor* of the functor D

$$\mathcal{E}(D, 1) \colon \mathcal{E} \to [\mathcal{J}^{op}, Set] \tag{1.7}$$

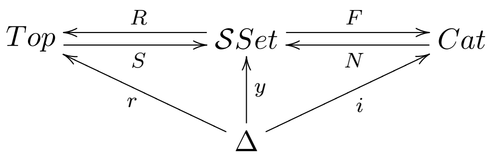
sending an object E of $\mathcal{E}$ to the presheaf $\mathcal{E}(D(-),E)\colon \mathcal{J}^{op} \to Set$, has a left adjoint $L\colon Set^{\mathcal{J}^{op}} \to \mathcal{E}$ defined for each presheaf $P\colon \mathcal{J}^{op} \to Set$ by the colimit

$$L(P) = \varinjlim(\textstyle\int_{\mathcal{J}} P \xrightarrow{\pi_P} \mathcal{J} \xrightarrow{D} \mathcal{E}) \tag{1.8}$$

where $\pi_P\colon \int_{\mathcal{J}} P \to \mathcal{J}$ is a discrete fibration obtained from the Grothendieck construction [42] applied to the presheaf P. Using this construction, the nerve functor for categories may be seen as the *singular functor* of the functor (1.5) and it has a left adjoint

$$F\colon \mathcal{S}Set \to Cat \tag{1.9}$$

so called *fundamental category functor*, which is a part of the diagram of functors



and for any simplicial set $X_\bullet$, the fundamental category $FX_\bullet$ is the quotient of the free category generated by the 1-skeleton of $X_\bullet$, with respect to congruence relation given by 2-simplices. The other pair of adjoint functors in the above diagram is given by the singular functor $S\colon Top \to \mathcal{S}Set$ of the functor $r\colon \Delta \to Top$, which takes any ordinal $[n]$ to the so called topological standard n-simplex $\Delta^n = \{(t_0,\ldots,t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^n t_i = 1, t_i \geq 0\}$. Again by the geometric nerve construction, n-simplices of the simplicial set $S(X)$ are given by the set $Hom_{Top}(r[n],X)$ of *singular n-simplices* of the topological space X. Its left adjoint is defined for any simplicial set $X_\bullet$ by the colimit $\varinjlim(\int_\Delta X_\bullet \xrightarrow{\pi} \Delta \xrightarrow{r} Top)$ where $\int_\Delta X_\bullet$ is a *simplex category* of $X_\bullet$, constructed by the Grothendieck construction, and we call it the *geometric realization functor*

$$R\colon \mathcal{S}Set \to Top. \tag{1.10}$$

The geometric realization $RX_\bullet$ of the simplicial set $X_\bullet$ is first described by Milnor in [74], as the topological space obtained from the coproduct $\coprod_{n\geq 0} X_n \times \Delta^n$, where X_n is supplied with the discrete topology, factored by the equivalence relation generated by identifications $(X\alpha(x),t) \sim (x, r\alpha(t))$, for any morphism $\alpha\colon [n] \to [m]$ in Δ, and any $x \in X_m$ and $t \in \Delta^n$. This construction is later generalized by Segal [80], to the geometric realization functor

$$S\colon \mathcal{S}Top \to Top \tag{1.11}$$

from the category $\mathcal{S}Top$ whose objects are *simplicial spaces*, which are defined by presheaves

$$X_\bullet\colon \Delta^{op} \to Top$$

with values in the category Top of topological spaces, where each X_n is a topological space.

There are several different ways to characterize those simplicial sets which arise as nerves of categories, and the most of this (equivalent) ways rely on the Quillen closed model structure on the category $\mathcal{S}Set$ of simplicial sets. Simplicial sets which are fibrant objects for the closed model structure on $\mathcal{S}Set$ are called Kan complexes, and they are characterized by certain *horn filling* conditions describing their exactness properties. This conditions for a simplicial set $X_\bullet$ explicitly use a *simplicial kernel* $K_n(X_\bullet)$ in dimension n

$$K_n(X_\bullet) = \{(x_0, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_{n-1}, x_n) | d_i(x_j) = d_{j-1}(x_i), i < j\} \subseteq X_{n-1}^{n+1}$$

which is interpreted as the set of all possible sequences of (n-1)-simplices which could possibly be the boundary of any n-simplex. There exists a natural *boundary map*

$$\partial_n \colon X_n \to K_n(X_\bullet) \tag{1.12}$$

which takes any n-simplex $x \in X_n$ to the sequence $\partial_n(x) = (d_0(x), d_1(x), \ldots, d_{n-1}(x), d_n(x))$ of its (n-1)-faces. The set $\bigwedge_n^k(X_\bullet)$ of k-horns in dimension n

$$\bigwedge_n^k(X_\bullet) = \{(x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}, x_n) | d_i(x_j) = d_{j-1}(x_i), i < j, i, j \neq k\} \subseteq X_{n-1}^n$$

is the set of all possible sequences of (n-1)-simplices which could possibly be the boundary of any n-simplex, except that we k^{th} face is missing. The k-horn map in dimension n

$$p_n^k(x) \colon X_n \to \bigwedge_n^k(X_\bullet) \tag{1.13}$$

is defined by the composition of the boundary map (1.12), with the natural projection $q_n^k(x) \colon K_n(X_\bullet) \to \bigwedge_n^k(X_\bullet)$, which just omits the k^{th} (n-1)-simplex from the sequence. Then we say that for $X_\bullet$ the k^{th} *Kan condition in dimension* n is satisfied (exactly) if the k-horn map (1.13) is surjection (bijection). If Kan conditions are satisfied for all $0 < k < n$ and for all n, then we say that $X_\bullet$ is a *weak Kan complex*, and if Kan conditions are satisfied for extremal horns as well $0 \leq k \leq n$ and for all n, then we say that $X_\bullet$ is a *Kan complex*.

One of the above mentioned characterizations of nerves of categories, first observed by Street, is that the simplicial set $X_\bullet$ is the nerve of a category if and only if it is a weak Kan complex in which the weak Kan conditions are satisfied exactly. Weak Kan complexes were introduced by Boardman and Vogt [19] in their work on homotopy invariant algebraic structures. These objects are fundamental in the recent work of Joyal [53], which is so far the most advanced form of the interplay between the category theory and the simplicial theory. He even used the name *quasicategory*, instead of the weak Kan complex, in order to emphasize that *"most concepts and results of category theory can be extended to quasicategories"*. Joyal introduced different closed model structure on $\mathcal{S}Set$, called the *model structure for quasicategories*, in which quasicategories are fibrant objects. In the language of quasicategories, Lurie recently formulated his work on higher topoi in [69] in which he also extended a considerable amount category theory to quasicategories.

Similar characterization of nerves of groupoids leads to the fundamental simplicial objects introduced by Duskin in [30]. An *n-dimensional Kan hypergroupoid*, is a Kan complex $X_\bullet$ in which Kan conditions (1.13) are satisfied exactly for all $m > n$ and $0 \leq k \leq m$. Glenn used the name *n-dimensional hypergroupoid* in [36] for any simplicial set in which Kan conditions are satisfied exactly above dimension n, while Beke called them in [16] *exact n-types*, in order to emphasize their homotopical meaning. These simplicial sets morally play the role of nerves of weak n-groupoids, which is known to be valid for small n. Consequently, a simplicial set $X_\bullet$ is the nerve of a groupoid if and only if it is a 1-dimensional Kan hypergroupoid, and similar characterization holds for nerves of bigroupoids.

Bigroupoids and bicategories, introduced by Bénabou [15] in 1967, are weakest possible generalization of ordinary groupoids and categories, respectively, to the immediate next level. In a bicategory (bigroupoid), Hom-sets become categories (groupoids) and the composition becomes functorial instead of functional. This changes properties of associativity and identities which only hold up to *coherent natural isomorphisms*. The *coherence laws* which this natural isomorphisms satisfy, are the deep consequence of the process called *categorification*, invented by Crane [26], [27], in which we find category theoretic analogs of set theoretic concepts by replacing sets with categories, equations between elements of the sets by isomorphisms between objects of the category, functions by functors and equations between functions by natural isomorphisms between functors.

The categorification become an essential tool in many areas of modern mathematics. By generalizing algebraic concepts from the classical set theory to the context of higher category theory, Baez developed a program of *higher dimensional algebra* in an attempt to unify quantum field theory with traditional algebraic topology. The algebraic concepts which arose from this program include braided monoidal 2-categories [4], 2-Hilbert spaces [5], 2-Tangles [7], 2-groups [8], and Lie 2-algebras [9]. Bartels developed a concept of a principal 2-bundle with the structure 2-group [13] and Baez and Schreiber used this concept in order to develop a *higher gauge theory* [10], [11] which describes the parallel transport of strings using 2-connections on principal 2-bundles, as the categorification of the usual gauge theory which describes the parallel transport of point particles using connections on principal bundles. Vector 2-spaces arose as a categorification of vector spaces in the work of Kapranov and Voevodsky [61], and they were used by Baas, Dundas and Rognes [2], who defined vector 2-bundles in a search for a geometrically defined elliptic cohomology. Later, Baas, Bökstedt and Kro used topological bicategories and vector 2-bundles [3] in order to develop 2-categorical K-theory as the categorification of the usual K-theory.

Another essential tool which we use in the thesis is an *internalization*. This is a process of generalizing concepts from the category *Set* of sets, which are described in terms of sets, functions and commutative diagrams, to concepts in another category $\mathcal{E}$ by describing them in terms of objects, morphisms, and commutative diagrams in $\mathcal{E}$. The internalization of the particular algebraic or geometric structure in the category $\mathcal{E}$ rely on exactness properties of $\mathcal{E}$ needed to describe corresponding commutative diagrams. Therefore, the choice of the category $\mathcal{E}$ will depend on the algebraic or geometric structure one wants to describe.

The most natural choice for an internalization and a categorification of algebraic and geometric structures is a *topos*, which is according to Grothendieck, the ultimate generalization of the concept of space. Topos theory has its origins in two separate lines of mathematical development, *sheaf theory* and the categorical foundations of the set theory.

The sheaf theory was born in the work of Leray in 1945, and it become an essential tool for a cohomology theory of non-simply connected spaces by providing an axiomatization of "local coefficient system", mostly within the context of algebraic topology. The usual notion of a sheaf on a topological space X used the topology of open subsets of the space X. But it was soon discovered by Grothendieck, that in the context of algebraic geometry, the topological notion of sheaf was not entirely adequate. Motivated by the Galois theory and Serre fibrations, he replaced the usual topology of topological spaces, by the more general *Grothendieck topology* [1] of categories, and he invented a generalized notion of a sheaf over a *site*, which is a category supplied with a Grothendieck topology. He defined the *Grothendieck topos* as the collection of all sheaves over a fixed site, and these objects were central in the development of étale cohomology, built up during the "Seminaire de Géometrie Algébrique du Bois Marie" held during 1963-1964 by Grothendieck with the assistance of Artin, Giraud, Verdier and others at Institut des Hautes Études Scientifiques.

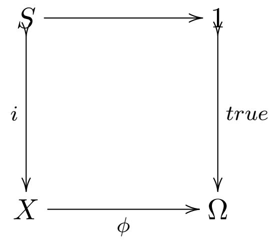
The second line of development of the topos theory can be traced to the Freyd-Mitchell embedding theorem for abelian categories, which showed that there exist a set of elementary axioms implying all the finitary exactness properties of module categories. But the true development started with Lawvere's pioneering paper [65], setting out a list of elementary axioms which were sufficient to characterize the category *Set* of sets. Then he began to investigate an idea that the two element set $\{true, false\}$ in the category *Set* of sets can be seen as an "object of truth-values" in *Set*. In his subsequent paper [66], Lawvere observed that a presence of an "object of truth-values" Ω in an arbitrary category $\mathcal{E}$, reduces the *comprehension axiom* (which essentially says that given a property, there is a set consisting exactly of the elements having that property) to an elementary statement about adjoint functors. Gray described analogous result [39] in the category *Cat* of small categories.

One of the most important results of the Seminaire de Géometrie Algébrique (SGA) was the famous Giraud's theorem, which characterized Grothendieck toposes purely by exactness properties and size conditions of categories. This exactness properties says that any Grothendieck topos is an *exact category*, that is a finitely complete category with pullback stable coequalizers and effective equivalence relations. Exact categories were defined by Barr [12] who used them as the basis of a non-additive embedding theorem, which represents the first coming-together of the two lines of development of the topos theory. Barr observed that Giraud's theorem may be seen as tittle more then a special case of his embedding theorem. One consequence of Barr's embedding theorem is that for any small exact category $\mathcal{E}$, there is a family of left exact epimorphism preserving functors

$$F_i \colon \mathcal{E} \to Set \tag{1.14}$$

which are collectively faithful and collectively limit and epimorphism reflecting.

By an additional stage of abstraction, Lawvere and Tierney began to investigate the axiom of existence of truth value object Ω in any category, after Lawvere observed that every Grothendieck topos has such an object, which was later called a *subobject classifier*. In a finitely complete category $\mathcal{E}$, a subobject classifier is a monomorphism $true\colon 1 \to \Omega$, such that any monomorphism $i\colon S \to X$ is a pullback



by the unique morphism $\phi\colon X \to \Omega$, called a classifying morphism of a subobject S. Lawvere and Tierney proposed a concept of a magnificent simplicity, an *elementary topos*, which is a finitely complete category $\mathcal{E}$ together with a subobject classifier Ω in $\mathcal{E}$, in which any object X of $\mathcal{E}$ is exponential. An object X in a finitely complete category $\mathcal{E}$ is exponential if the functor

$$X\times\colon \mathcal{E} \to \mathcal{E}$$

which takes any object Y in $\mathcal{E}$ to the product $X \times Y$, has a right adjoint.

We could have chosen any topos as a carrier for an internalization and a categorification of algebraic and geometric structures which we describe in the thesis. Most of mathematical structures are described in terms of axioms, operations and relations. A first order formula $\phi(x_1, \ldots, x_n)$ is called *geometric formula*, if it is built up from atomic formulas by using conjunction, disjunction and existential quantification. For any kind of a mathematical structure, which can be described by geometric formulas, there exists a *classifying topos*, which we will illustrate later on an example of a topos of presheves on a small category. All hypotheses and desired conclusions in the thesis can be phrased in the language of sets, membership, ordered tuples and projections, and unions and intersections, in the syntax of *geometric logic*. By results of Joyal, Deligne and others, the theorems whose hypotheses and conclusions can be phrased in finitary geometric logic, and even in a countable geometric logic by results of Makkai and Reyes [72], stay valid in an arbitrary topos.

However, we decided to choose exact categories, and sometimes even more general finitely complete categories, as an ambient for the description of our algebraic and geometric structures. For any diagram in an exact category $\mathcal{E}$ involving finite limits and coequalizers we can apply arbitrary limit and epimorphism preserving functor $F\colon \mathcal{E} \to Set$ which yields a diagram in *Set* with the same limits and epimorphisms as the original diagram. Another consequence of Barr's embedding theorem (1.14), which we will use in the thesis, is that any conclusion one may come to about the diagram in a category *Set* of sets must hold also for an original diagram in an exact category $\mathcal{E}$. Therefore, all proofs in the thesis will be done in the category *Set* of sets, without loosing generality for exact categories.

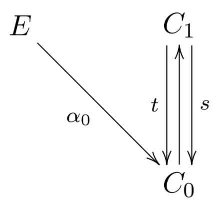
Let us now describe the content and the main results of the thesis which is divided in two main parts. After the introduction, Part II is a recollection of the well known one-dimensional theory of (internal) categories and their relation with (internal) simplicial objects. On the other side, Part III describes a two-dimensional theory of (internal) bicategories and their relation with (internal) simplicial objects and pseudo simplicial categories, obtained by the categorification and the internalization of the corresponding one-dimensional theory.

In Chapter 2 we recall some basic simplicial methods which we will extensively use in the thesis. Most of this material is standard and can be found in a classical book [73] by May, or in a modern treatment in [37]. However, we also recall some more exotic endofunctors on a category $\mathcal{S}Set$ of simplicial sets, such as the *n-Coskeleton* $Cosk^n$ and the *shift functor* or *décalage* Dec which can be find in [29]. Actions and n-torsors over n-dimensional Kan hypergroupoids are defined by Glenn in [36] using simplicial maps which we call *exact fibrations*. A simplicial map $\lambda_\bullet \colon \mathcal{E}_\bullet \to \mathcal{B}_\bullet$ is an exact fibration in dimension n, if for all $0 \leq k \leq n$, the diagrams

$$\begin{array}{ccc} E_n & \xrightarrow{\lambda_n} & B_n \\ \Big\downarrow{\scriptstyle p_{\bar{k}}} & & \Big\downarrow{\scriptstyle p_{\bar{k}}} \\ \bigwedge_n^k(\mathcal{E}_\bullet) & \longrightarrow & \bigwedge_n^k(\mathcal{B}_\bullet) \end{array}$$

are pullbacks. It is called an exact fibration if it is an exact fibration in all dimensions. At the end of this chapter, we describe two crucial concepts from [36] which we will use later in the thesis. An action of the n-dimensional hypergroupoid $\mathcal{B}_\bullet$ is given in Definition 2.13 as a simplicial map $\lambda_\bullet \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ which is an exact fibration for all $m \geq n$, and an n-dimensional hypergroupoid n-torsor over X in $\mathcal{E}$ is given in Definition 2.14 as a simplicial map $\lambda_\bullet \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ such that $P_\bullet$ is augmented over X, aspherical and $n-1$-coskeletal.

Chapter 3 is a review of internal categories and groupoids and Chapter 4 describe their relation with internal simplicial objects via the *nerve* functor. In Chapter 5, Definition 5.1 recalls an action of an internal category $\mathcal{C}$ on an object E in a finitely complete category $\mathcal{E}$



(1.15)

given by an *action morphism*

$$\alpha_1 \colon E \times_{C_0} C_1 \to E \tag{1.16}$$

satisfying the usual axioms for quasiassociativity, identity and equivariance of the action.

In Theorem 5.1 we show how one associates to an action (1.15) an *action category* $\mathcal{E} \triangleleft \mathcal{C}$, and we give a proof in Proposition 5.1 that a naturally induced internal functor

$$P\colon \mathcal{E} \triangleleft \mathcal{C} \to \mathcal{C} \tag{1.17}$$

is a *discrete fibration.* At the end of this chapter, we describe two important results from [36], whose categorified versions will be the main results of the thesis. When action (1.15) is restricted to an action of the groupoid $\mathcal{G}$, we provide a simplicial characterization of an action (1.15) in Theorem 5.3 where we prove that the nerve of the canonical projection functor (1.17) is a simplicial action of the 1-dimensional Kan hypergroupoid $G_\bullet$ which is the nerve of $\mathcal{G}$. Also, in Theorem 5.4 we state the result from [36], that the the action of the groupoid $\mathcal{G}$ is principal, if the corresponding simplicial map is a simplicial 1-torsor, in the sense of Glenn.

Chapter 6 recalls how a *small fibration* corresponding to an internal category is constructed in Theorem 6.1. Although this result is well known, its proof is hard to find in the literature but it will provide a good basis for categorification methods developed later in Chapter 11 in the construction of a small 2-fibration.

The two dimensional theory in Part III starts with a Chapter 7 where definitions of a bicategory, their homomorphisms, pseudonatural transformations and modifications are given as they were defined by Bénabou in his classical paper [15]. Then Chapter 8 describes the Duskin nerve for bicategories as a geometric nerve defined by the singular functor of the fully faithful embedding

$$i\colon \Delta \to Bicat \tag{1.18}$$

of the skeletal simplicial category Δ into the category $Bicat$ of bicategories and strictly unital homomorphism of bicategories, constructed by Bénabou in [15]. This embedding regards any ordinal $[n]$ as the locally discrete 2-category, in the sense that Hom-categories are discrete, so there exist only trivial 2-cells. We show that the Duskin nerve functor

$$N_2\colon Bicat \to \mathcal{S}Set \tag{1.19}$$

is fully faithful in Theorem 8.1 based on the result that the geometric nerve provides a fully faithful functor on the category $2 - Cat_{lax}$ of 2-categories and normal lax 2-functors given in [17]. The sets of n-simplices of the nerve $N_2\mathcal{B}$ of a bicategory $\mathcal{B}$ are defined by $Hom_{Bicat}(i[n], \mathcal{B})$, which were explicitly described by Duskin [32] in a geometric form.

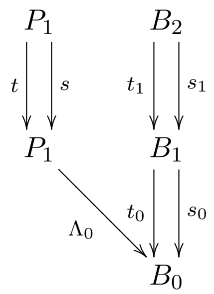
In Chapter 9 we recall how internal bicategories were defined by Bénabou in [15] and Chapter 10 shows how we can associate to any (internal) bicategory $\mathcal{B}$ a pseudosimplicial category, which may be sees as a *supercoherent nerve* following Jardine [49]. Then in Chapter 10 we introduce *the first new concept* in the thesis, a *small 2-fibration* corresponding to an internal bicategory, and in Theorem 11.1 we state the more general result which says that a small 2-fibration is an example of a *fibration of bicategories*, whose definition is proposed by Hermida in [44].

The first explicit definition of the second nonabelian cohomology with coefficients in a bicategory $\mathcal{B}$ is given in Chapter 12, following a general approach described by Street in [82]. We give an explicit description of the bicategory of 2-descent data in Theorem 12.1, which Street calls a *cohomology bicategory of X with values in a bicategory $\mathcal{B}$*. Then in Definition 12.1 we define *the second nonabelian cohomology* $\mathcal{H}^2(\mathcal{U},\mathcal{B})$ with coefficients in $\mathcal{B}$

$$\mathcal{H}^2(\mathcal{U},\mathcal{B}) = Desc_2(\mathcal{E}(\mathcal{U},\mathcal{B})) \tag{1.20}$$

as the bicategory of 2-descent data which corresponds to the cosimplicial bicategory $\mathcal{E}(\mathcal{U},\mathcal{B})$ naturally defined by a covering $\mathcal{U}$ of a topological space X, and a small 2-fibration corresponding to $\mathcal{B}$.

In Chapter 13, we introduce *the second new concept* of the thesis, *action of a bicategory* in Definition 13.1 as a categorification of an action of a category. For an internal bicategory $\mathcal{B}$ given by a bigraph in a finitely complete category $\mathcal{E}$, and an internal category $\mathcal{P}$



(1.21)

together with the *momentum functor* $\Lambda\colon \mathcal{P} \to \mathcal{B}_0$ to a discrete category $\mathcal{B}_0$ of objects of the bicategory $\mathcal{B}$, an *action functor*

$$A\colon \mathcal{P} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{P} \tag{1.22}$$

is a categorification of an action (1.16) of the category. We introduce coherence laws for this action, which express the fact that categories with an action of the bicategory $\mathcal{B}$ are *pseudoalgebras* over a *pseudomonad* [45], [63], [67] naturally defined by $\mathcal{B}$. We give a description of an Eilenberg-Moore 2-category of actions of the bicategory $\mathcal{B}$, without details of the construction for corresponding pseudoalgebras over a pseudomonad. For each action (1.21) of a bicategory $\mathcal{B}$ on a category $\mathcal{P}$, we define *the third new concept* in the thesis, an *action bicategory* $\mathcal{P} \triangleleft \mathcal{B}$ whose construction is given in Theorem 13.2. Then we see in Proposition 13.1 that an action bicategory $\mathcal{P} \triangleleft \mathcal{B}$ comes with a canonical projection

$$\Lambda\colon \mathcal{P} \triangleleft \mathcal{B} \to \mathcal{B} \tag{1.23}$$

to the bicategory $\mathcal{B}$, which is a strict homomorphism of bicategories.

In Chapter 14 we define *the fourth new concept* in the thesis, and our main geometric object - a *bigroupoid 2-torsor.* In Definition 14.2 we define a bigroupoid 2-torsor as a bundle of groupoids $\pi\colon \mathcal{P} \to X$ over an object X in the category $\mathcal{E}$, for which the induced functor

$$(Pr_1, A)\colon \mathcal{P} \times_{\mathcal{B}_0} B_1 \to \mathcal{P} \times_X \mathcal{P} \tag{1.24}$$

for an action (1.21) is a strong equivalence of groupoids. This means that this functor has a weak inverse, whose nontrivial component is given by the *division functor*, which we define as a categorification of the usual divison map corresponding to a principal action of a groupoid, which Moerdijk called in [75] a cocycle valued in a groupoid. These objects extend the well known theory of Grothendieck in the dimension $n = 1$ since bigroupoid 2-torsors are equivalent to *bigroupoid principal 2-bundles* defined by the condition that these are precisely groupoids which are locally equivalent to the trivial bigroupoid principal 2-bundle given by the target functor of the structure bigroupoid. The division functor can be thought of as a generator of the cohomology class of $\mathcal{P}$ in the second nonabelian cohomology $\mathcal{H}^2(X, \mathcal{B})$, and in Theorem 14.1 we prove a classification of bigroupoid 2-torsors by means of cohomology classes in $\mathcal{H}^2(\mathcal{U}, \mathcal{B})$. Then in Theorem 14.2 we outline an inverse construction of gluing of trivial $\mathcal{B}$-torsors by 2-cocycles, which would ultimately provide a full classification of 2-torsors by classes in $\mathcal{H}^2(X, \mathcal{B})$ *the second Čech nonabelian cohomology*

$$\mathcal{H}^2(X, \mathcal{B}) = \underrightarrow{lim}\mathcal{H}^2(\mathcal{U}, \mathcal{B}) \tag{1.25}$$

where such colimit of cohomology bicategories $\mathcal{H}^2(\mathcal{U}, \mathcal{B})$ is taken over the cofiltered category $[Cov^2]$ of Čech 2-covers, described by Beke in [15].

The first main result of the thesis is Theorem 15.1 in Chapter 15 which proves that for an action (1.21) of an internal bigroupoid $\mathcal{B}$ on groupoid $\mathcal{P}$, the simplicial map $\Lambda_\bullet = N_2(\Lambda)\colon \mathcal{Q}_\bullet \to \mathcal{B}_\bullet$ which arise as an application of a Duskin nerve for bicategories (1.22) on a canonical homomorphism of bicategories (1.23) is a (simplicial) action of the bigroupoid $\mathcal{B}$ on the groupoid $\mathcal{P}$, i.e. it is an exact fibration for all $n \geq 2$.

The second main result of the thesis is Theorem 15.3 which proves that for any $\mathcal{B}$-2-torsor $\mathcal{P}$ over X, the simplicial map $\Lambda_\bullet = N_2(\Lambda)\colon \mathcal{Q}_\bullet \to \mathcal{B}_\bullet$ is a Glenn's 2-torsor, which is an internal simplicial map $\Lambda_\bullet\colon P_\bullet \to \mathcal{B}_\bullet$ in $\mathcal{S}(\mathcal{E})$, which is an exact fibration for all $n \geq 2$, and where $P_\bullet$ is augmented over X, aspherical and 1-coskeletal ($P_\bullet \simeq Cosk^1(P_\bullet)$).

We would like to emphasize why we think that these results and the theory of 2-torsors developed in the thesis might be important.

Most of the classical cohomology theories have had associated with them some sort of an *intrinsic interpretation theory* only in *low dimensions.* However, any such generally satisfactory theory in *high dimensions*, which would provide such interpretation by intrinsic cohomological classification, remained elusive for a long time. It is the intention of this thesis to remedy this, by proposing a unified treatment of *nonabelian cohomology theory*, using the theory of 2-torsors and their simplicial interpretation as the basis of the theory of nonabelian *higher torsors.*

Let $U\colon \mathcal{E} \to \mathcal{B}$ be a functor together with a left adjoint $F\colon \mathcal{B} \to \mathcal{E}$ and an adjunction

$$F\colon \mathcal{B} \rightleftarrows \mathcal{E}\colon U \tag{1.26}$$

given by the unit $\eta\colon Id_{\mathcal{B}} \to UF$ and the counit $\epsilon\colon FU \to Id_{\mathcal{E}}$ natural transformation (satisfying *coherence conditions* described by two *triangle identities*). Then these data may be used to produce an augmented simplicial object in $\mathcal{E}$

$$X \xleftarrow{\epsilon_X} G(X) \overset{d_0}{\underset{d_1}{\rightrightarrows}} G^2(X) \overset{d_0}{\underset{d_2}{\rightrightarrows}} G^3(X) \overset{d_0}{\underset{d_3}{\rightrightarrows}} G^4(X) \tag{1.27}$$

which defines a *standard G-resolution of the object X in* $\mathcal{E}$ as an object $G^{\bullet}_{+}(X) \to X$ in the category $\mathcal{S}_{aug}(\mathcal{E})$ of internal augmented simplicial objects $\mathcal{E}$. From the diagram (1.27) we see that the n-simplices $G^{\bullet}(X)_n$ of the augmented simplicial object $G^{\bullet}(X)$ are defined by $G^{\bullet}(X)_n = G \circ G \ldots \circ G(X) = G^{n+1}(X)$ for any $n \geq -1$ with $G^0(X) = X$, and $G = FU$. Then the n^{th} comonadic cohomology $H^n_G(X, \pi)$ of X corresponding to an adjunction (1.26), with coefficients in an abelian group π in $\mathcal{E}$ is defined as the n^{th} cohomology

$$H^n_G(X, \pi) = H^n(\Sigma Hom_E(G^{\bullet}(X), \pi)) \tag{1.28}$$

of the cochain complex $\Sigma Hom_E(G^{\bullet}(X), \pi)$ associated to the cosimplicial abelian group $Hom_E(G^{\bullet}(X), \pi)$. This cosimplicial abelian group may be seen as the restriction of a small 2-fibration $F_{\Sigma^2(\pi)}\colon \mathcal{F}\Sigma^2(\pi) \to \mathcal{E}$ (11.10) associated to the strict 2-groupoid $\Sigma^2(\pi)$ whose nerve is given by an internal simplicial Eilenberg-MacLane object $K(\pi, n)$ in $\mathcal{S}(\mathcal{E})$.

Now, by using main objects and results of the nonabelian cohomology theory, as it is developed in the thesis in dimension $n = 2$, we extend the "triple" or comonadic cohomology theory outlined in [28] and further developed in [29] by Duskin, for an equivalent interpretation of (1.28). An abelian 2-cocycle in $H^n(\Sigma Hom_E(G^{\bullet}(X), \pi))$

$$\alpha\colon G^3(X) \to \pi \tag{1.29}$$

is identified with an object in the bicategory $\mathcal{H}^2(G^{\bullet}(X), \Sigma^2(\pi))$ of 2-descent data (12.3), represented by the same 2-cocycle. This correspondence allows us to introduce the n^{th} nonabelian comonadic cohomology $H^n_G(X, \mathcal{B})$ of an object X with coefficients in a weak n-groupoid $\mathcal{B}$ in $\mathcal{E}$ by

$$H^n_G(X, \mathcal{B}) = Desc^n(\mathcal{F}\mathcal{B}_{G^{\bullet}(X)}) \tag{1.30}$$

a weak n-groupoid of n-descent data of a cosimplicial weak n-groupoid $Hom_{\mathcal{E}}(G^{\bullet}(X), \mathcal{F}\mathcal{B})$. This is just the restriction to the standard G-resolution $G^{\bullet}(X)$ of the object X in $\mathcal{E}$, of a small n-fibration

$$F_{\mathcal{B}}\colon \mathcal{F}\mathcal{B} \to \mathcal{E} \tag{1.31}$$

associated to the weak n-groupoid $\mathcal{B}$ in $\mathcal{E}$, which naturally generalize the small 2-fibration (11.10), for an arbitrary n.

The (generalization of the) correspondence between 2-torsors and corresponding 2-cocycles from Theorem 14.1 may be seen as the part of the *characteristic n-cocycle weak n-functor*

$$Z^n_G(X,\mathcal{B})\colon TORS^n_{Uqc}(X,\mathcal{B}) \to H^n_G(X,\mathcal{B}) \tag{1.32}$$

from the weak n-groupoid $TORS_{Uqc}(X,\mathcal{B})$ of n-torsors under the weak-n-groupoid $\mathcal{B}$ and their quasicoherent weak n-functors, to the *cohomology weak n-groupoid* $H^n_G(X,\mathcal{B})$ [82] admits a left weak-n-adjoint right inverse

$$S^n_G(X,\mathcal{B})\colon H^n_G(X,\mathcal{B}) \to TORS^n_{Uqc}(X,\mathcal{B}) \tag{1.33}$$

which we will call the *standard n-torsor weak-n-functor*. The cohomology weak n-category $H^n_G(X,\mathcal{A})$ should be interpreted by the weak n-category

$$H^n_G(X,\mathcal{B}) \sim TORS_{Uqc}(X,\mathcal{B})[\mathcal{W}^{-1}] \tag{1.34}$$

of fractions, which in dimension $n=1$ correspond to the Gabriel's localization $\mathcal{C}[\mathcal{W}^{-1}]$ of the category $\mathcal{C}$ (see [34]), and in dimension $n=2$ to the bicategory $\mathcal{B}[\mathcal{W}^{-1}]$ of fractions of a bicategory $\mathcal{B}$ introduced by Pronk in [76], with respect to the class $\mathcal{W}$ of quasicoherent weak n-functors. It would follow then that the cohomology weak n-groupoid $H^n_G(X,\mathcal{B})$

$$H^n_G(X,\mathcal{B}) \simeq \begin{cases} Hom_{\mathcal{E}}(X,\mathcal{B}) & n=0 \\ TORS^n_U[X,\mathcal{B}] & n \geq 1 \end{cases} \tag{1.35}$$

where $Hom_{\mathcal{E}}(X,\mathcal{B})$ is (the fiber over X of) the *small n-fibration* corresponding to a weak n-category $\mathcal{A}$, and $TORS^n_U[X,\mathcal{B}]$ is the set $\pi_0(TORS^n_U(X,\mathcal{B}))$ of n-equivalence classes of the n-stack $TORS^n_U(X,\mathcal{B})$ of n-torsors under $\mathcal{B}$.

A weak k-groupoid $\mathcal{H}^k_G(X,\mathcal{A})$ for $0 \leq k \leq n$ is defined by the fiber over X of an *associated* $(n-k)$*-tuply weakly monoidal k-stack* $Ass^k_{n-k}(\mathcal{A})$ for a weak n-category $\mathcal{A}$

$$H^k_G(X,\mathcal{A}) \simeq \begin{cases} L^0(\mathcal{A})_X \sim Hom_{\mathcal{E}}(X,\mathcal{A}) & k=0 \\ L^k(\mathcal{A})_X \sim Ass^k_{n-k}(\mathcal{A})_X & 0<k<n \\ L^n(\mathcal{A})_X \sim TORS^n_U(X,\mathcal{A}) & k=n \end{cases} \tag{1.36}$$

where $L\colon Fib_n \to St^n$ is a left n-adjoint to an inclusion $J\colon St^n \to Fib_n$ of n-stacks St^n into fibered weak n-categories Fib_n.

For strict 2-groupoids, it is known that these are equivalent to crossed modules of groupids. Therefore for any strict 2-groupoid $\mathcal{G}$ there exists an equivalence

$$H^1_G(X,\mathcal{G}) \sim TORS^1(\tilde{\mathcal{G}})_X \tag{1.37}$$

where $TORS^1(\tilde{\mathcal{G}})_X$ is the fiber over X of a gpd-stack $TORS^1(\tilde{\mathcal{G}})$ of 1-torsors under the corresponding crossed module $\tilde{\mathcal{G}}$ of groupids. In the case of the strict 2-group $\mathcal{G}$, objects in the corresponding associated gr-stack are explicitly described by Jurčo in [57], where he called them crossed module $\mathcal{G}$-bundles. Also, Jurčo described [57] objects in corresponding associated 2-stack of $\mathcal{G}$ under the name *crossed module bundle gerbes.*

To conclude the introduction, let us give a few words on some of the things which are not contained in the thesis but they are naturally connected with its main results.

Although we mainly used methods of higher category theory or higher dimensional algebra, there is a different approach to the theory of torsors, motivated by the homotopy theory, which we didn't use in the thesis. At the heart of this approach is the fact that the classifying space functor

$$B\colon \mathrm{Cat} \to Top \tag{1.38}$$

defined as the composition of the nerve functor (1.2) followed by the geometric realization functor (1.10) is a fundamental construction of algebraic topology and algebraic K-theory. Quillen defined in [78] higher algebraic K-theory by taking higher homotopy groups of the classifying spaces of suitably defined categories. His construction raised interest in the relation between categories and homotopy types of their classifying spaces since it became apparent that classifying space functor (1.38) transports categorical coherence to homotopical coherence. Quillen's work was followed by Thomason's result in [83] who shown that after an application of the other fundamental homotopy construction, *the homotopy colimit* to the diagram of categories, the result has the homotopy type of the Grothendieck construction applied to the diagram. Bullejos and Cegarra used these results as the basis for their proof that geometric realizations of geometric nerves are classifying spaces for (strict) 2-categories in [22] and (weak) monoidal categories in [23]. Such classifying spaces are defined by the functor

$$B\colon Bicat \to Top \tag{1.39}$$

which is the composition of the Duskin nerve functor (1.19) for bicategories followed by the geometric realization functor (1.10).

Consequently, the classifying space construction became the main source of homotopy classification theorems for objects with a specified geometrical or topological structure. The generalization of the Schreier theory of extensions of groups, done by Breen in [21], Ulbrich in [84] or Blanco, Bullejos and Faro in [18] was used by Cegarra and Garzon in [25] to obtain the cohomological classification of categorical torsors. Along these lines, it would be natural to obtain the cohomological classification of topological bigroupoid 2-torsors by extending the classical result which says that for any topological group G and any CW-complex X there exists a natural bijection

$$H^1(X, G) \sim [X, BG] \tag{1.40}$$

between the set $[X, BG]$ of homotopy classes of maps from X to the classifying space BG and the set $H^1(X, G)$ of isomorphism classes of G-torsors over X. The analog of (1.40) would be a natural bijection

$$H^2(X, \mathcal{B}) \sim [X, B\mathcal{B}] \tag{1.41}$$

between a set $H^2(X, \mathcal{B})$ of equivalence classes of $\mathcal{B}$-torsors and a set $[X, B\mathcal{B}]$ of homotopy classes of maps from X to the classifying space $B\mathcal{B}$ of a topological bigroupoid $\mathcal{B}$.

The systematic study of homotopy theory in an arbitrary Grothendieck topos $\mathcal{E}$ was initiated by Joyal and Tierney in [55] and [56] were they developed a theory of classifying spaces for sheaves of simplicial groupoids, or more precisely, sheaves of groupoids enriched in simplicial sets. This theory was based on their discovery in [54] of a Quillen closed model structure on the category of internal categories $Cat(\mathcal{E})$ and internal groupoids $Gpd(\mathcal{E})$ in a Grothendieck topos $\mathcal{E}$.

Jardine shown in [52] that the Joyal-Tierney theory has an analogue for presheaves K of groupoids enriched in simplicial sets. Earlier, he proved in [48] that for any sheaf of groups G on a Grothendieck site $\mathcal{C}$, the set $H^1(\mathcal{C}, G)$ of isomorphism classes of G-torsors is in bijective correspondence with the set of morphisms $ho_{\mathcal{S}(\mathcal{E})}(*, BG)$ in the homotopy category of the category $\mathcal{S}(\mathcal{E})$ of simplicial sheaves, where $\mathcal{E}$ is a Grothendieck topos $Sh(\mathcal{C})$. Following this result, he proved that the set of morphisms $ho_{\mathcal{S}(\mathcal{C}^{\Delta^{op}})}(*, BK)$ in the local homotopy category of simplicial presheaves is in bijective correspondence with the set of path components of a category of K-torsors, where K-torsors are K-diagrams which have trivial homotopy colimits.

In this context, the most general torsors for a presheaf of categories enriched in simplicial sets is given by Jardine in [51], see also [50]. He also gave homotopy classification results for A-torsors, in a wide variety of settings which includes motivic homotopy theory. To relate these results to our classification of 2-torsors, one should first note that any strict 2-category $\mathcal{C}$ can be seen as a category $\tilde{\mathcal{C}}$ enriched in simplicial sets, by taking the nerve of the category $\mathcal{C}_1$ of morphisms as the simplicial set of morphisms of $\tilde{\mathcal{C}}$. In this special case, one could possibly relate Jardine's results with the classification of strict 2-category 2-torsors. However, the main results in [51] is a bijection

$$\pi_0(Tors_A) \sim [*, dBA]$$

which classify it the set $\pi_0(Tors_A)$ of isomorphism classes $\pi_0(Tors_A)$ of torsors for a for a presheaf A of categories enriched in simplicial sets. On the other side, our classification takes into account higher dimensional information by means of it the bicategory $\mathcal{H}^2(X, \mathcal{B})$ which would correspond to Jardine's set of of isomorphism classes of torsors after an application of π_0 functor.

Part II
One-dimensional theory

2 Simplicial objects

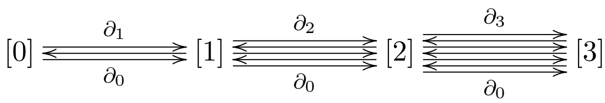
In this section we will review some standard notions from the theory of simplicial sets. Most of the statements and proofs may be found in standard textbooks, such as [37] or [73].

Definition 2.1. *Skeletal simplicial category Δ consists of the following data:*

- *objects are finite nonempty ordinals $[n] = \{0 < 1 < ... < n\}$,*
- *morphisms are monotone maps $f\colon [n] \to [m]$, which for all $i, j \in [n]$ such that $i \leq j$, satisfy $f(i) \leq f(j)$.*

We also call Δ the topologist's simplicial category, and this is a full subcategory of the algebraist's simplicial category $\bar{\Delta}$, which has an additional object $[-1] = \emptyset$, given by a zero ordinal, that is an empty set.

Skeletal simplicial category Δ may be also given by means of generators given by the diagram



and relations given by the maps $\partial_i\colon [n-1] \to [n]$ for $0 \leq i \leq n-1$, called coface maps, which are injective maps that omit i in the image, and the maps $\sigma_i\colon [n] \to [n-1]$ for $0 \leq i \leq n-1$, called codegeneracy maps, which are surjective maps which repeat i in the image. These maps satisfy following cosimplicial identities:

$$\begin{aligned}
\partial_j \partial_i &= \partial_i \partial_{j-1} && (i < j) \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1} && (i \leq j) \\
\sigma_j \partial_i &= \partial_i \sigma_{j-1} && (i < j) \\
\sigma_j \partial_i &= id && (i = j, i = j+1) \\
\sigma_j \partial_i &= \partial_i \sigma_{j+1} && (i > j+1)
\end{aligned}$$

We will use the following factorization of monotone maps by means of cofaces and codegeneracies.

Lemma 2.1. *Any monotone map $f\colon [m] \to [n]$ has a unique factorization given by*

$$f = \partial^{n}_{i_1} \partial^{n-1}_{i_2} ... \partial^{n-s+1}_{i_s} \sigma^{m-t}_{j_t} ... \sigma^{m-2}_{j_2} \sigma^{m-1}_{j_1}$$

where $0 \leq i_s < i_{s-1} < ... < i_1 \leq n$, $0 \leq j_t < j_{t-1} < ... < j_1 \leq m$ and $n = m - t + s$.

Proof. The proof follows directly from the injective-surjective factorization in Set and simplicial identities. □

Definition 2.2. *Simplicial object $X_\bullet$ in a category $\mathcal{C}$ is a functor $X\colon \Delta^{op} \to \mathcal{C}$. This is an object of the category $\mathcal{S}(\mathcal{C})$ whose morphisms are natural transformations, which we call internal simplicial morphisms. In the case when the category $\mathcal{C} =$ Set is the category of sets (in a fixed Grothendieck universe), then we call $X_\bullet$ a simplicial set, and we denote the corresponding category of simplicial sets by $\mathcal{S}Set$.*

Thus we can view a simplicial object $X_\bullet$ in $\mathcal{C}$ as a diagram

$$X_0 \;\underset{d_0}{\overset{d_1}{\leftleftarrows\!\rightrightarrows}}\; X_1 \;\underset{d_0}{\overset{d_2}{\leftleftarrows\!\rightrightarrows}}\; X_2 \;\underset{d_0}{\overset{d_3}{\leftleftarrows\!\rightrightarrows}}\; X_3 ...$$

in $\mathcal{C}$, where we denoted just extremal face operators, and left the signature for inner face operators, and degeneracies.

Then the following simplicial identities hold:

$$\begin{array}{ll} d_i d_j = d_{j-1} d_i & (i < j) \\ s_i s_j = s_{j+1} s_i & (i \leq j) \\ d_i s_j = s_{j-1} d_i & (i < j) \\ d_i s_j = id & (i = j, i = j+1) \\ d_i s_j = s_{j+1} d_i & (i > j+1) \end{array}$$

where $d_i := X(\partial_i)$ and $s_i := X(\sigma_i)$.

Definition 2.3. *An augmented simplicial object $X_\bullet \to X_{-1}$ in a category $\mathcal{C}$ is a functor $X\colon \bar{\Delta}^{op} \to \mathcal{C}$. This is an object of the category $\mathcal{S}_a(\mathcal{C})$ whose morphisms are natural transformations, which we call simplicial maps of augmented simplicial objects.*

In order to define basic endofunctors on the category $\mathcal{S}(\mathcal{C})$, which we will use in the thesis, we first need to describe the process of a truncation of internal simplicial objects. For any natural number n, we have the full subcategory Δ_n of the simplicial category Δ, whose objects are the first $n+1$ ordinals. Then we have the following definition.

Definition 2.4. *Let $X_\bullet$ be a simplicial object in $\mathcal{C}$. An n-truncated simplicial object $tr_n(X_\bullet)$ in a category $\mathcal{C}$ is a functor $Xi_n\colon \Delta_n^{op} \to \mathcal{C}$ given by the precomosition with an embedding $i_n\colon \Delta_n \to \Delta$. This is an object of the category $\mathcal{S}^n(\mathcal{C})$, and we have an n-truncation functor*

$$tr^n\colon \mathcal{S}(\mathcal{C}) \to \mathcal{S}^n(\mathcal{C})$$

from the category $\mathcal{S}s(\mathcal{C})$ of simplicial objects in $\mathcal{C}$, to the category $\mathcal{S}s^n(\mathcal{C})$ of n-truncated simplicial objects in $\mathcal{C}$.

If $\mathcal{C}$ is a finitely complete category, an n-truncation functor $tr^n\colon \mathcal{S}(\mathcal{C}) \to \mathcal{S}^n(\mathcal{C})$ has a right adjoint $cosk^n\colon \mathcal{S}^n(\mathcal{C}) \to \mathcal{S}(\mathcal{C})$, and if $\mathcal{C}$ is a finitely cocomplete category, it has a left adjoint $sk^n\colon \mathcal{S}^n(\mathcal{C}) \to \mathcal{S}(\mathcal{C})$.

The corresponding comonad $Sk^n = sk^n tr^n \colon \mathcal{S}Set \to \mathcal{S}Set$ for $\mathcal{C} = \text{Set}$ is easy to describe. For any simplicial set $X_\bullet$, its skeleton $Sk^n(X_\bullet)$ is a simplicial subset of $X_\bullet$, which is identical to $X_\bullet$ in all dimensions $k \leq n$, and has only degenerate simplices in all higher dimensions.

The monad $Cosk^n = cosk^n tr^n \colon \mathcal{S}(\mathcal{C}) \to \mathcal{S}(\mathcal{C})$ is described by the simplicial kernel.

Definition 2.5. *The n^{th} simplicial kernel of the simplicial object $X_\bullet$ is an object $K_n(X_\bullet)$ in $\mathcal{C}$, together with morphisms $pr_j \colon K_n(X_\bullet) \to X_{n-1}$ for $j = 0, \dots, n$, which is universal with respect to relations $d_i pr_j = pr_{j-1} d_i$, for all $0 \leq i < j \leq n$.*

Now, let we describe in more detail the monad $Cosk^n = cosk^n tr^n \colon \mathcal{S}Set \to \mathcal{S}Set$ in the case $\mathcal{C} = \text{Set}$, that is when we deal with simplicial sets.

The simplicial kernel of the simplicial set $X_\bullet$ in dimension n is a set $K_n(X_\bullet)$ defined by

$$K_n(X_\bullet) = \{(x_0, x_1, \dots, x_i, \dots, x_j, \dots, x_{n-1}, x_n) | d_i(x_j) = d_{j-1}(x_i), i < j\} \subseteq X_{n-1}^{n+1}$$

so that we can interpret it as the set of all possible sequences of (n-1)-simplices which could possibly be the boundary of any n-simplex. If $x \in X_n$ is an n-simplex in a simplicial set $X_\bullet$, its boundary $\partial_n(x)$ is a sequence of its (n-1)-faces

$$\partial_n(x) = (d_0(x), d_1(x), \dots, d_{n-1}(x), d_n(x)).$$

Then, for the simplicial set $X_\bullet$, the simplicial set $Cosk^n(X_\bullet)$ is identical to $X_\bullet$ in all dimensions $k \leq n$, and the set of (n+1)-simplices of $Cosk^n(X_\bullet)$ is defined by

$$Cosk^n(X_\bullet)_{n+1} = K_{n+1}(X_\bullet)$$

while the face operators are given by the projections $d_i = pr_i \colon K_{n+1}(X_\bullet) \to X_n$ for all $0 \leq i \leq n+1$. All of the higher dimensional set of simplices of $Cosk^n(X_\bullet)$ are obtained just by inductively iterating the simplicial kernels

$$Cosk^n(X_\bullet)_{n+2} = K_{n+2}(tr^{n+1} Cosk^n(X_\bullet))$$

and so on.

From the universal property of the n^{th} simplicial kernel $K_n(X_\bullet)$, we have a canonical morphism $\delta_n = (d_0, d_1, \dots, d_{n-1}, d_n) \colon X_n \to K_n(X_\bullet)$, called the *boundary* of the object of n-simplices, or briefly the n^{th} boundary morphism.

The first nontrivial component of the unit $\eta \colon Id_{\mathcal{S}s} \to Cosk^n$ of the adjunction is given by $(n+1)^{th}$ boundary morphism

$$\delta_{n+1} = (d_0, d_1, \dots, d_n, d_{n+1}) \colon X_{n+1} \to Cosk^n(X_\bullet)_{n+1} = K_{n+1}(X_\bullet)$$

and we have following definitions.

Definition 2.6. *We say that the simplicial object $X_\bullet$ in $\mathcal{C}$ is coskeletal in dimension n, or n-coskeletal, if the unit $\eta\colon Id_{SSet} \to Cosk^n$ of the adjunction is a natural isomorphism. Similarly, we say that the simplicial object $X_\bullet$ in $\mathcal{C}$ is skeletal in dimension n, or n-skeletal, if the counit $\epsilon\colon Sk^n \to Id_{SSet}$ of the adjunction is a natural isomorphism.*

Definition 2.7. *We say that the simplicial object $X_\bullet$ in $\mathcal{C}$ is aspherical in dimension n if the n^{th} boundary morphism $\delta_n\colon X_n \to K_n(X_\bullet)$ is an epimorphism. If $X_\bullet$ is aspherical in all dimensions, then we say that it is aspherical.*

In order to define Kan complexes later, we use another universal construction which formally describe 'hollow' simplices, or simplices in which the k^{th} face is missing.

Definition 2.8. *The k-horn in dimension n of the simplicial object $X_\bullet$ is an object $\bigwedge_n^k(X_\bullet)$ in $\mathcal{C}$, together with morphisms $p_i\colon \bigwedge_n^k(X_\bullet) \to X_{n-1}$ for $i = 0, \dots, n$ and $i \neq k$, which is universal with respect to relations $d_i p_j = p_{j-1} d_i$, for all $0 \leq i < j \leq n$ and $i, j \neq k$.*

The set $\bigwedge_n^k(X_\bullet)$ of k-horns in dimension n

$$\bigwedge_n^k(X_\bullet) = \{(x_0, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n-1}, x_n) | d_i(x_j) = d_{j-1}(x_i), i < j, i, j \neq k\} \subseteq X_{n-1}^n$$

is the set of all possible sequences of (n-1)-simplices which could possibly be the boundary of any n-simplex, except that we k^{th} face is missing. Then for the simplicial set $X_\bullet$, the k-horn map in dimension n

$$p_n^k(x)\colon X_n \to \bigwedge_n^k(X_\bullet)$$

is defined by the composition of the boundary map $\partial_n\colon X_n \to K_n(X_\bullet)$, with the projection $q_n^k(x)\colon K_n(X_\bullet) \to \bigwedge_n^k(X_\bullet)$, and it just omits the k^{th} (n-1)-simplex from the sequence.

If $x \in X_n$ is an n-simplex, its k-horn $p_n^k(x)$ is defined by the image of the projection of its boundary to the sequence of faces in which the k^{th} face is omitted

$$p_n^k(x) = (d_0(x), d_1(x), \dots, d_{k-1}(x), d_{k+1}(x), \dots, d_{n-1}(x), d_n(x))$$

Let $(x_0, x_1, \dots, x_{k-1}, -, x_{k+1}, \dots, x_{n-1}, x_n) \in \bigwedge_n^k(X_\bullet)$ be a k-horn in dimension n. If there exists an n-simplex $x \in X_n$ such that

$$p_n^k(x) = (x_0, x_1, \dots, x_{k-1}, -, x_{k+1}, \dots, x_{n-1}, x_n)$$

then we say that n-simplex x is a filler of the horn.

Definition 2.9. *Let $X_\bullet$ be an simplicial object in the category $\mathcal{C}$. We say that the k^{th} Kan condition in dimension n is satisfied for $X_\bullet$ if the k-horn morphism*

$$p_n^k(x)\colon X_n \to \bigwedge_n^k(X_\bullet)$$

is an epimorphism. The condition is satisfied exactly if the above morphism is an isomorphism. If Kan conditions are satisfied for all $0 < k < n$ *and for all* n*, then we say that* $X_\bullet$ *is a weak Kan complex. Finally, if Kan conditions are satisfied for extremal horns as well* $0 \leq k \leq n$ *and for all* n*, then we say that* $X_\bullet$ *is a Kan complex.*

This condition can be stated entirely in the topos theoretic context by using the sieves

$$\overset{k}{\bigwedge}[n] \hookrightarrow \overset{\bullet}{\Delta}[n] \hookrightarrow \Delta[n]$$

in $\mathcal{S}Set$, where $\Delta[n]$ is the *standard n-simplex*, which is just the simplicial set represented by the ordinal $[n]$. The simplicial set $\overset{\bullet}{\Delta}[n]$ is the *boundary of the standard n-simplex* which is identical to standard n-simplex in all dimensions bellow n, and has only degenerate simplices in higher dimensions. It is defined by the (n-1)-skeleton $\overset{\bullet}{\Delta}[n] = Sk^{n-1}(\Delta[n])$ of the standard n-simplex. The simplicial set $\bigwedge^k[n]$ is the *k-horn of the standard n-simplex*, which is identical to $\overset{\bullet}{\Delta}[n]$ except that it is not generated by the simplex $\delta_k \colon [n-1] \to [n]$.

Using the Yoneda lemma

$$Hom_{\mathcal{S}Set}(\Delta[n], X_\bullet) \simeq X_n$$

the n^{th} Kan condition says that for any simplicial map $\bar{x} \colon \bigwedge^k[n] \to X_\bullet$, there exist a simplicial map $x \colon \Delta[n] \to X_\bullet$ such that the diagram

$$\begin{array}{ccc} \bigwedge^k[n] & \xrightarrow{\bar{x}} & X_\bullet \\ \big\downarrow & \nearrow_{x} & \\ \Delta[n] & & \end{array}$$

commutes.

Remark 2.1. *The* n^{th} *Kan condition is equivalent to the injectivity of the simplicial set* $X_\bullet$ *with respect to monomorphisms* $\bigwedge^k[n] \hookrightarrow \Delta[n]$ *for all* $0 \leq k \leq n$*. In this terms, Kan complex* $X_\bullet$ *is a simplicial set which is injective with respect to all monomorphisms* $\bigwedge^k[n] \hookrightarrow \Delta[n]$ *for all* $0 \leq k \leq n$*, and all* $n \geq 0$*.*

Proposition 2.1. *Every aspherical simplicial object* $X_\bullet$ *is a Kan simplicial object.*

Proof. We will use the Barr embedding theorem and prove it in Set. Consider the diagram

$$\begin{array}{ccccc} & & X_n & & \\ & \swarrow^{\delta_n} & & \searrow^{p_n^k} & \\ K_n(X_\bullet) & & \xrightarrow[q_n^k]{} & & \bigwedge_n^k(X_\bullet) \end{array}$$

and a k-horn $(x_0, x_1, \dots, x_{k-1}, -, x_{k+1}, \dots, x_n, x_{n+1}) \in \bigwedge^k_{n+1}(X_\bullet)$. If there exists a filler $x \in X_{n+1}$ for which $p^k_{n+1}(x) = (x_0, x_1, \dots, x_{k-1}, -, x_{k+1}, \dots, x_n, x_{n+1})$ then its k-face $d_k(x) = x_k$ has a boundary uniquely determined by the simplices x_i for $i \neq k$ since

$$d_i(x_k) = \begin{cases} d_{k-1}(x_i) & 0 \leq i < k \leq n+1 \\ d_k(x_{i+1}) & 0 \leq k \leq i \leq n+1 \end{cases}$$

and therefore $(d_0(x_k), d_1(x_k), \dots, d_{n-1}(x_k), d_n(x_k)) \in K_n(X_\bullet)$. Since we supposed that $\delta_n \colon X_n \to K_n(X_\bullet)$ is an epimorphism, then such a simplex $x_k \in X_n$ really exists, and we conclude that the morphism $q^k_{n+1} \colon K_{n+1}(X_\bullet) \to \bigwedge^k_{n+1}(X_\bullet)$ is also an epimorphism. But this is true for all n, and it follows that $p^k_{n+1} \colon X_{n+1} \to \bigwedge^k_{n+1}(X_\bullet)$ is an epimorphism as a composition of epimorphisms, and therefore $X_\bullet$ is a Kan simplicial set. □

Remark 2.2. *For any simplicial set $X_\bullet$ the simplicial kernel $K_1(X_\bullet)$ in dimension 1 is equal to the product $K_1(X_\bullet) = X_0 \times X_0$. For the augmented simplicial set $X_\bullet \to X_{-1}$, when we have $K_1(X_\bullet) = X_0 \times_{X_{-1}} X_0$. The set of k-horns is given by $\bigwedge^k_1(X_\bullet) = X_0$ for $k = 0, 1$, and in each case maps $p^k_1 \colon X_1 \to \bigwedge^k_1(X_\bullet)$ and $q^k_1 \colon K_1(X_\bullet) \to \bigwedge^k_1(X_\bullet)$ are always epimorphisms.*

Definition 2.10. *A simplicial object $X_\bullet$ in $\mathcal{C}$ is said to be split if there exist a family of morphisms $s_{n+1} \colon X_n \to X_{n+1}$ for all $n \geq 0$, called the contraction for $X_\bullet$, which satisfy all the simplicial identities involving degeneracies. When a simplicial object is augmented $p \colon X_0 \to X_{-1}$ then the contraction includes also a morphism $s_0 \colon X_{-1} \to X_0$ such that $ps_0 = id_{X_{-1}}$.*

Remark 2.3. *Any augmented split simplicial set $X_\bullet \to X_{-1}$ may be seen as the simplicial set $X_\bullet$ together with the homotopy equivalence $d_\bullet \colon X_\bullet \to K(X_{-1}, 0)$ to the constant simplicial set $K(X_{-1}, 0)$ which has X_{-1} at each dimension and the identity maps for faces and degeneracies. This means that there exists a simplicial map $s_\bullet \colon K(X_{-1}, 0) \to X_\bullet$ such that the compositions $s_\bullet d_\bullet \simeq id_{X_\bullet}$ and $d_\bullet s_\bullet \simeq id_{K(X_{-1},0)}$ are homotopic to respective identity simplicial maps.*

Proposition 2.2. *Every augmented aspherical simplicial set $X_\bullet \to X_{-1}$ is split.*

Proof. The proof follows by induction. Let's take any section $s_0 \colon X_{-1} \to X_0$ and we assume that we have the n^{th} contraction $s_n \colon X_{n-1} \to X_n$. Let $q_i(x) \colon X_n \to K_{n+1}(X_\bullet)$ be the i^{th} degeneracy for the n^{th} simplicial kernel of $X_\bullet$, and we define $q_{n+1}(x) \colon X_n \to K_{n+1}(X_\bullet)$ by

$$q_{n+1}(x) = (s_n d_0(x), s_n d_1(x), \dots, s_n d_{n-1}(x), s_n d_n(x)).$$

Now let's choose the splitting $s \colon K_{n+1}(X_\bullet) \to X_{n+1}$ of the $(n+1)^{th}$ boundary map $\delta_{n+1}(x) \colon X_{n+1} \to K_{n+1}(X_\bullet)$, which is a surjection by assumption, such that $s_i = sq_i$ for all $0 \leq i \leq n$. Then the contraction $s_{n+1} \colon X_n \to X_{n+1}$ defined by $s_{n+1} = sq_{n+1}$ satisfy all the identities involving degeneracies since $q_{n+1} = \delta_{n+1} s q_{n+1} = \delta_{n+1} s_{n+1}$. □

An n-truncation functor has the extension to the *augmented n-truncation functor*

$$tr_a^n \colon \mathcal{S}_a(\mathcal{C}) \to \mathcal{S}_a^n(\mathcal{C})$$

from the category $\mathcal{S}_a(\mathcal{C})$ of augmented simplicial objects in $\mathcal{C}$ to the category $\mathcal{S}_a^n(\mathcal{C})$ of n-truncated augmented simplicial objects in $\mathcal{C}$. Since $\mathcal{C}$ is finitely complete, it has a right adjoint $cosk_a^n \colon \mathcal{S}_a^n(\mathcal{C}) \to \mathcal{S}_a(\mathcal{C})$, called the *augmented n-coskeleton functor*. If we regard any augmented simplicial object $X_\bullet \to X_{-1}$ in $\mathcal{C}$ as the ordinary simplicial object in the slice category $(\mathcal{C}, X_{-1})$, then the augmented n-coskeleton functor becomes ordinary n-coskeleton functor in the slice category $(\mathcal{C}, X_{-1})$.

Example 2.1. *The category $\mathcal{C}$ may be identified with the category $\mathcal{S}_a^{-1}(\mathcal{C})$ of -1-truncated augmented simplicial objects in $\mathcal{C}$, and the augmented -1-truncation functor $tr_a^{-1} \colon \mathcal{S}_a(\mathcal{C}) \to \mathcal{S}_a^{-1}(\mathcal{C})$ assigns to any augmented simplicial object $X_\bullet \to X_{-1}$ the object X_{-1} of $\mathcal{C}$. Its right adjoint is augmented -1-coskeleton functor $cosk_a^{-1} \colon \mathcal{S}_a^{-1}(\mathcal{C}) \to \mathcal{S}_a(\mathcal{C})$ which assigns to any object X in $\mathcal{C}$ the constant augmented simplicial object*

$$X \xleftarrow{\ id\ } X \overset{id}{\underset{id}{\leftleftarrows\!\!\!\rightarrow}} X_1 \overset{id}{\underset{id}{\Longleftrightarrow}} X \overset{id}{\underset{id}{\Longleftrightarrow}} X \ldots$$

denoted by $K(X,0) \to X$.

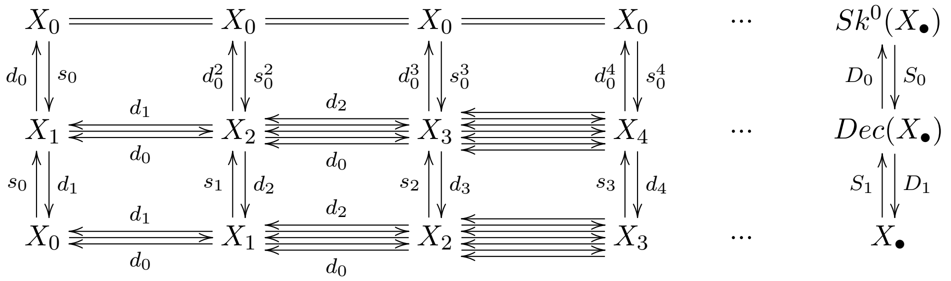
Example 2.2. *The category of morphisms $\mathcal{C}^I$ of $\mathcal{C}$ may be identified with the category $\mathcal{S}_a^0(\mathcal{C})$ of 0-truncated augmented simplicial objects in $\mathcal{C}$, and the augmented 0-truncation functor $tr_a^0 \colon \mathcal{S}_a(\mathcal{C}) \to \mathcal{S}_a^0(\mathcal{C})$ assigns to any augmented simplicial object $X_\bullet \to X_{-1}$ the morphism $d \colon X_0 \to X_{-1}$ of $\mathcal{C}$. Its right adjoint is augmented 0-coskeleton functor $cosk_a^0 \colon \mathcal{S}_a^0(\mathcal{C}) \to \mathcal{S}_a(\mathcal{C})$ which assigns to any morphism $d \colon X_0 \to X_{-1}$ in $\mathcal{C}$ the simplicial kernel of the morphism*

$$X_{-1} \xleftarrow{\ d\ } X_0 \overset{pr_1}{\underset{pr_2}{\leftleftarrows\!\!\!\rightarrow}} X_0 \times_{X_{-1}} X_0 \overset{pr_{12}}{\underset{pr_{23}}{\Longleftrightarrow}} X_0 \times_{X_{-1}} X_0 \times_{X_{-1}} X_0$$

denoted by $cosk_a^0(X_0 \to X_{-1})$.

The corresponding monad and the comonad on the category $\mathcal{S}_a(\mathcal{C})$ of augmented simplicial objects in $\mathcal{C}$ are denoted by $Cosk_a \colon \mathcal{S}_a(\mathcal{C}) \to \mathcal{S}_a(\mathcal{C})$ and $Sk_a \colon \mathcal{S}_a(\mathcal{C}) \to \mathcal{S}_a(\mathcal{C})$ respectively, in accordance with the case of nonaugmented simplicial objects in $\mathcal{C}$.

Another important construction on simplicial objects is given by the so called *shift functor*. For any simplicial object $X_\bullet$ in $\mathcal{C}$), we restrict the corresponding functor $X \colon \Delta^{op} \to \mathcal{C}$ to the subcategory of Δ^{op} with the same objects, and with the same generators except for the injections $\partial_n \colon [n-1] \to [n]$. If we renumber the objects in Δ^{op}, so that the ordinal $[n-1]$ becomes $[n]$, we obtain a simplicial object in $\mathcal{C}$, denoted by $Dec(X_\bullet)$, which is augmented to the object X_0 (or to the constant simplicial object $Sk^0(X_\bullet)$ in $\mathcal{C}$) and is contractible with respect to the simplicial map obtained from the family $(s_n)_{n\geq 0}$ of extremal degeneracies, as is shown in the diagram



(2.1)

where the simplicial map $S_0\colon Sk^0(X_\bullet) \to Dec(X_\bullet)$ on the right side of the diagram is defined by $(S_0)_n = (s_0)^n = s_0 s_0 \dots s_0$, and the simplicial map $D_0\colon Dec(X_\bullet) \to Sk^0(X_\bullet)$ is defined by $(D_0)_n = (d_0)^n = d_0 d_0 \dots d_0$, for each level n. The other two simplicial maps $S_1\colon X_\bullet \to Dec(X_\bullet)$ and $D_1\colon Dec(X_\bullet) \to X_\bullet$ are defined by $(S_1)_n = s_n$ and $(D_1)_n = d_n$ respectively.

The above construction extends to a functor

$$Dec\colon \mathcal{S}(\mathcal{C}) \to \mathcal{S}_{as}(\mathcal{C})$$

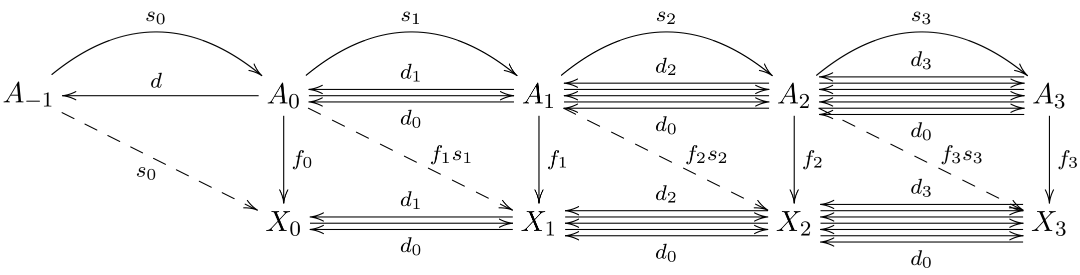
from the category of simplicial objects in $\mathcal{C}$, to the category $\mathcal{S}s_{as}(\mathcal{C})$ of augmented split simplicial objects in $\mathcal{C}$. This functor has a left adjoint, given by the forgetful functor

$$U\colon \mathcal{S}_{ac}(\mathcal{C}) \to \mathcal{S}(\mathcal{C})$$

which forgets the augmentation and a splitting. Thus, for any split augmented simplicial object $A_\bullet \to A_{-1}$ in $\mathcal{S}_{ac}(\mathcal{C})$, and any simplicial object $X_\bullet$ in $\mathcal{S}(\mathcal{C})$, we have a natural bijection

$$\theta_{A_\bullet,X_\bullet}\colon Hom_{\mathcal{S}(\mathcal{C})}(U(A_\bullet), X_\bullet) \overset{\simeq}{\to} Hom_{\mathcal{S}_{as}(\mathcal{C})}(A_\bullet, Dec(X_\bullet))$$

which takes any simplicial map $f_\bullet\colon U(A_\bullet) \to X_\bullet$ to its composite with the splitting



as in the above diagram.

In order to compare later our 2-torsors with Glenn's simplicial 2-torsors we will recall some basic definitions from [36].

Definition 2.11. *A simplicial map $\Lambda_\bullet \colon \mathcal{E}_\bullet \to \mathcal{B}_\bullet$ is said to be an exact fibration in dimension n, if for all $0 \le k \le n$, the diagrams*

$$\begin{array}{ccc} E_n & \xrightarrow{\lambda_n} & B_n \\ \downarrow{\scriptstyle p_{\bar{k}}} & & \downarrow{\scriptstyle p_{\bar{k}}} \\ \bigwedge_n^k(\mathcal{E}_\bullet) & \longrightarrow & \bigwedge_n^k(\mathcal{B}_\bullet) \end{array}$$

are pullbacks. It is called an exact fibration if it is an exact fibration in all dimensions n.

Using the language of simplicial algebra, Glenn defined actions and n-torsors over n-dimensional hypergroupoids. This objects morally play the role of the n-nerve of weak n-groupoids, and we give their formal definition.

Definition 2.12. *An n-dimensional Kan hypergroupoid is a Kan simplicial object $G_\bullet$ in $\mathcal{E}$ such that the canonical map $G_m \to \bigwedge_m^k(G_\bullet)$ is an isomorphism for all $m > n$ and $0 \le k \le m$.*

Remark 2.4. *The term n-dimensional hypergroupoid was introduced by Duskin [30], for any simplicial object satisfying the above condition without being Kan simplicial object. One of his motivational examples was the standard simplicial model for an Eilenberg-MacLane space $K(A, n)$, for any abelian group object A in $\mathcal{E}$. In [15], Beke used the term an exact n-type to emphasize the meaning of these objects as algebraic models for homotopy n-types.*

Definition 2.13. *An action of the n-dimensional hypergroupoid is an internal simplicial map $\Lambda_\bullet \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ in $\mathcal{E}$ which is an exact fibration for all $m \ge n$.*

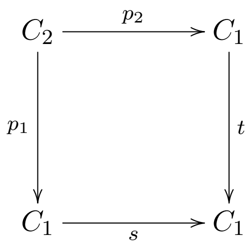
Definition 2.14. *An action $\Lambda_\bullet \colon P_\bullet \to \mathcal{B}_\bullet$ is the n-dimensional hypergroupoid n-torsor over X in $\mathcal{E}$ if $P_\bullet$ is augmented over X, aspherical and n-1-coskeletal ($P_\bullet \simeq Cosk^{n-1}(P_\bullet)$).*

3 Internal categories and internal groupoids

In this section we recall some basic notions from *internal category theory*, which are standard and can be find, for example, in a classical book by MacLane [70], or in a more modern treatment in [71]. We will start by defining categories $Cat(\mathcal{E})$ and $Gpd(\mathcal{E})$ of internal categories and internal groupoids, respectively, in the category $\mathcal{E}$ with finite limits. Although we will not use any model-theoretic arguments, we describe the class of weak equivalence which is a part of a closed model structure in categories $Cat(\mathcal{E})$ and $Gpd(\mathcal{E})$, discovered by Joyal and Tierney in [54].

Definition 3.1. *An internal category $\mathcal{C}$ in $\mathcal{E}$ consists of the following data:*

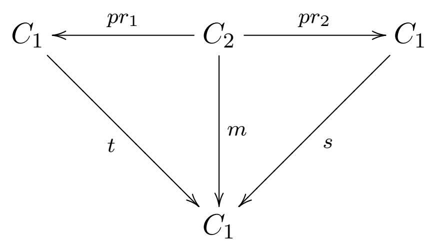
- *two objects C_1 and C_0 called respectively the object of arrows and the object of objects,*
- *two morphisms $s, t\colon C_1 \to C_0$ called respectively the source morphism and the target morphism,*
- *a morphism $u\colon C_0 \to C_1$ called the unit morphism,*
- *a morphism $m\colon C_2 \to C_1$ from the object C_2 defined by the pullback*



(which is in the discrete case when the category $\mathcal{E}$ is a category Set of sets, isomorphic to the set $C_1 \times_{C_0} C_1 := \{(g, f) \in C_1 \times C_1 \colon s(g) = t(f)\}$, and we denote $m(g, f) = gf$)

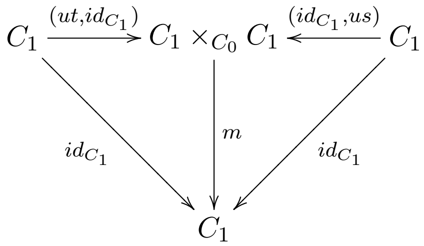
such that the following diagrams commute:

- *left and right invariance law of the source and the target respectively:*



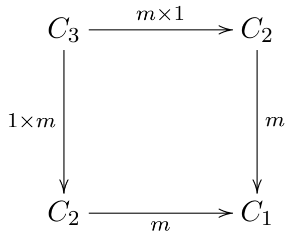
(in the case $\mathcal{E} = Set$ for any $(g, f) \in C_1 \times_{C_0} C_1$, this means $t(gf) = t(g), s(gf) = s(f)$)

- *left and right unit laws*

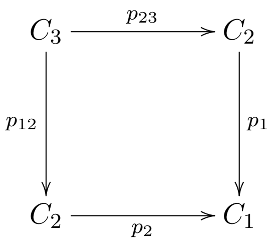


(in the case $\mathcal{E} = Set$, for any $f \in C_1$, we have an identity $u(t(f))f = f = fu(s(f))$),

- *the associativity law*



where an object C_3 in $\mathcal{E}$ is defined by the pullback



(in the case $\mathcal{E} = Set$ for any composable triple $(h, g, f) \in C_1 \times_{C_0} C_1 \times_{C_0} C_1$, i.e. any triple which satisfy $t(g_3) = s(g_2), t(g_2) = s(g_1)$, we have the identity $(hg)f = h(gf)$).

Definition 3.2. *An internal category $\mathcal{G}$ in $\mathcal{E}$ is an internal groupoid if there exists*

- *a morphism $i \colon G_1 \to G_1$ called an inversion,*

such that the following axiom is satisfied:

- *left and right inverse laws*

$$\begin{array}{ccc} G_1 & \xrightarrow{(i, id_{G_1})} & G_1 \times_{G_0} G_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle m} \\ G_0 & \xrightarrow[u]{} & G_1 \end{array} \qquad \begin{array}{ccc} G_1 & \xrightarrow{(id_{G_1}, i)} & G_1 \times_{G_0} G_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle m} \\ G_0 & \xrightarrow[u]{} & G_1 \end{array}$$

(in the case $\mathcal{E} = Set$, for any $g \in C_1$, for which we denote $g^{-1} = i(g)$, the above two diagrams give two identities $g^{-1}g = u(s(g)), gg^{-1} = u(t(g))$, respectively).

Definition 3.3. *Given two internal groupoids $\mathcal{G}$ and $\mathcal{H}$ in $\mathcal{E}$, a homomorphism from $F\colon \mathcal{G} \to \mathcal{H}$, consists of the following morphisms:*

- *a morphism $F_0\colon G_0 \to H_0$,*
- *a morphism $F_1\colon G_1 \to H_1$,*

such that the following axioms are satisfied:

- *compatibility laws between the groupoid structures*

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ G_0 & \xrightarrow[F_0]{} & H_0 \end{array} \qquad \begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ G_0 & \xrightarrow[F_0]{} & H_0 \end{array} \qquad \begin{array}{ccc} G_0 & \xrightarrow{F_0} & H_0 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ G_1 & \xrightarrow[F_1]{} & H_1 \end{array}$$

- *functoriality law*

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{F_1 \times F_1} & H_1 \times_{H_0} H_1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ G_1 & \xrightarrow[F_1]{} & H_1 \end{array}$$

(in the case $\mathcal{E} = Set$ for any $(g, f) \in G_1 \times_{G_0} G_1$, we have $F_1(gf) = F_1(g)F_1(f)$).

Definition 3.4. *An internal functor $F\colon \mathcal{G} \to \mathcal{H}$ in $\mathcal{E}$ is fully faithful if the diagram*

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ {\scriptstyle (t,s)}\downarrow & & \downarrow{\scriptstyle (t,s)} \\ G_0 \times G_0 & \xrightarrow[F_0 \times F_0]{} & H_0 \times H_0 \end{array}$$

is a pullback, and it is essentially surjective if in the diagram where the square is a pullback

$$\begin{array}{ccccc} G_0 \times_{H_0} H_1 & \xrightarrow{pr_2} & H_1 & \xrightarrow{s} & H_0 \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle t} & & \\ G_0 & \xrightarrow[F_0]{} & H_0 & & \end{array}$$

the top composite $spr_2\colon G_0 \times_{H_0} H_1 \to H_0$ is an epimorphism in $\mathcal{E}$.

If the functor $F\colon \mathcal{G} \to \mathcal{H}$ is both, fully faithful and essentially surjective, we call it an essential equivalence or weak equivalence.

Definition 3.5. *We say that two internal groupoids $\mathcal{G}$ and $\mathcal{H}$ in $\mathcal{E}$ are Morita equivalent if there exists a third groupoid $\mathcal{K}$ and two weak equivalences as in the diagram*

$$\mathcal{G} \xleftarrow{F} \mathcal{K} \xrightarrow{F'} \mathcal{H}$$

Definition 3.6. *Let $F_1, F_2\colon \mathcal{G} \to \mathcal{H}$ be two homomorphisms of groupoids in $\mathcal{E}$. A natural transformation $\alpha\colon F_1 \Rightarrow F_2$ is given by:*

- *a morphism $\alpha\colon G_0 \to H_1$*

such that the following axiom is satisfied:

- *naturality law*

$$\begin{array}{ccc} G_1 & \xrightarrow{(F_2,\alpha s)} & H_1 \times_{H_0} H_1 \\ {\scriptstyle (\alpha t, F_1)}\downarrow & & \downarrow{\scriptstyle m} \\ H_1 \times_{H_0} H_1 & \xrightarrow[m]{} & H_1 \end{array}$$

(in the case $\mathcal{E} = Set$, for any $f\colon x \to y$ in G_1, we have $\alpha(y)F_1(f) = F_2(f)\alpha(x)$).

4 Nerves of categories

Any ordinal $[n] = \{0 < 1 < ... < n\}$ may be seen as the category in which there exists a (unique) morphism between i and j if and only if $i \leq j$ and any monotonic map may be seen as the functor between such categories. Thus we have a full embedding

$$i\colon \Delta \to \mathrm{Cat},$$

and the nerve of the category $\mathcal{C}$ is a simplicial set $N(\mathcal{C})_\bullet$ defined via this embedding by

$$N(\mathcal{C})_n := Hom_{\mathrm{Cat}}(i[n], \mathcal{C}).$$

Thus the 0-simplices of $N(\mathcal{C})$ are the objects of $\mathcal{C}$ and the n-simplices are given by composable sequences of morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \dots x_{n-2} \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

in $\mathcal{C}$. We will use the usual "face opposite vertex" convention for simplices and for any morphism $f_1\colon x_0 \to x_1$ in $\mathcal{C}$, the source $s(f_1)$ is given by $d_1(f_1) = x_0$ and the target $t(f_1)$ by $d_0(f_1) = x_1$. Then the face operators are defined by composing out i^{th} object

$$d_i(f_n, f_{n-1}, \dots, f_2, f_1) = \begin{cases} (f_n, f_{n-1}, \dots, f_3, f_2) & i = 0 \\ (f_n, \dots, f_{i+1}f_i, \dots, f_1) & 0 < i < n \\ (f_{n-1}, f_{n-2}, \dots, f_2, f_1) & i = n \end{cases} \tag{4.1}$$

and the degeneracy operators are defined by

$$s_i(f_n, f_{n-1}, \dots, f_2, f_1) = \begin{cases} (f_n, f_{n-1}, \dots, f_1, id_{x_0}) & s = 0 \\ (f_n, \dots, f_{i+1}, id_{x_i}, f_i, \dots, f_1) & 0 < s < n \\ (id_{x_n}, f_n, \dots, f_2, f_1) & s = n \end{cases} \tag{4.2}$$

expanding the i^{th} object by its identity morphism. Then it is easy to see that the simplicial identities are either consequences of the construction or are equivalent to the associativity and identity axioms for a category. For example, the associativity law is given by the simplicial identity

$$d_1 d_1 = d_1 d_2$$

since for any three composable morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3$$

in $\mathcal{C}$, we have an identity

$$d_1 d_1(f_3, f_2, f_1) = f_3(f_2 f_1) = (f_3 f_2) f_1 = d_1 d_2(f_3, f_2, f_1)$$

and the left and right identity laws are given by simplicial identities

$$d_1 s_0 = id = d_1 s_1$$

since for any morphism $f_1 \colon x_0 \to x_1$ we have an identity

$$d_1 s_0(f_1) = d_1(f_1, id_{x_0}) = f_1 id_{x_0} = f_1 = id_{x_1} f_1 = d_1(id_{x_1}, f_1) = d_1 s_1(f_1)$$

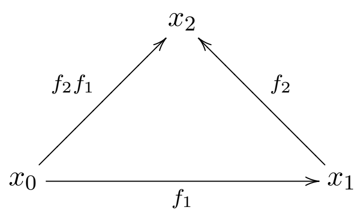
Since the Yoneda lemma applies also in this case, for any contravariant representable functor denoted by $\Delta[n] \colon \mathcal{C}^{op} \to Set$, we have

$$Hom_{\mathcal{S}Set}(\Delta[n], N(\mathcal{C})_\bullet) \simeq N(\mathcal{C})_n$$

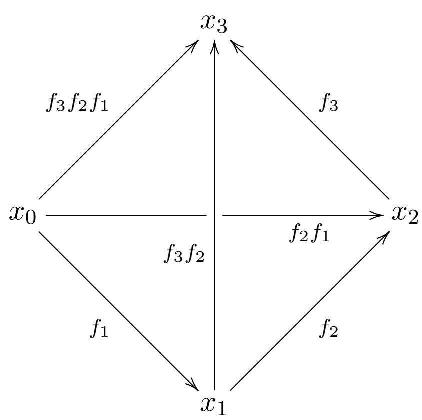
and this allows us to interpret simplices of $N(\mathcal{C})_\bullet$ in a more geometric way. The 0-simplices are just described by vertices and 1-simplices are directed line segments

$$x_0 \xrightarrow{f_1} x_1$$

A typical 2-simplex $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2$ may be geometrically described by the triangle



and a typical 3-simplex $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3$ may be geometrically described by the tetrahedron



and in this way it is easy to read off faces of such geometric simplices using "face opposite vertex" convention. For example, the last triangle is clearly d_3 face of the above tetrahedron since it lies opposite to the x_3 vertex.

The nerve construction is clearly functorial with respect to functors of categories and we have a well known result.

Theorem 4.1. *The nerve functor* $\mathcal{N}\colon \mathrm{Cat} \to \mathcal{S}s$ *is fully faithful.*

Proof. The proof lies on the fact that the skeletal category of ordinal numbers Δ is an adequate subcategory of the category of categories Cat (in the sense of Isbell) and it is an elementary exercise. □

We have simplicial characterizations of nerves of categories and groupoids, which we will only state, and the proof can be find in [32].

Theorem 4.2. *Let* $X_\bullet$ *be a simplicial set. Then the following is equivalent:*

- $X_\bullet$ *is the nerve of the category*
- $X_\bullet$ *is the weak Kan complex in which the weak Kan conditions are satisfied exactly*

Theorem 4.3. *Let* $X_\bullet$ *be a simplicial set. Then the following is equivalent:*

- $X_\bullet$ *is the nerve of the groupoid*
- $X_\bullet$ *is a Kan complex in which the Kan conditions are satisfied exactly, that is* $X_\bullet$ *is a 1-dimensional Kan hypergroupoid in the terminology of Glenn in [36] (or an exact 1-type in the terminology of Beke in [15])*

5 Actions of categories and groupoids

When $\mathcal{E}$ is a category Set of sets, we are accustomed to consider not only functors between small categories, but also functors from a small category to a large one, like presheaves, which are functors to Set itself. To internalize this concept, in this chapter we will describe actions of categories and groupoids internal to some finitely complete category $\mathcal{E}$. These actions are also called *internal presheaves* and the first elementary characterization of categories of actions $\mathcal{E}^{\mathcal{C}}$ for some internal category $\mathcal{C}$ in $\mathcal{E}$ was given by Bunge in [24]. When $\mathcal{E}$ is an (elementary) topos, then $\mathcal{E}^{\mathcal{C}}$ is also a topos, called an *internal presheaf topos*.

Definition 5.1. *Let $\mathcal{E}$ be a finitely complete category and $\mathcal{C}$ an internal category in $\mathcal{E}$. A right action of the category $\mathcal{C}$ on an object E in $\mathcal{E}$ consists of the following data:*

- *a morphism $\alpha_0 \colon E \to C_0$ called a momentum of the action*
- *a morphism $a \colon E \times_{C_0} C_1 \to E$, called an action, whose domain is defined by the pullback*

$$\begin{array}{ccc} E \times_{C_0} C_1 & \xrightarrow{pr_2} & C_1 \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle t} \\ E & \xrightarrow[\alpha_0]{} & C_0 \end{array} \qquad (5.1)$$

(in the case $\mathcal{E} = Set$ isomorphic to the set $E \times_{C_0} C_1 := \{(e, g) \in \mathcal{C} \times E | t(g) = \alpha_0(e)\}$)

This data are such that the following diagrams commute:

- *a momentum invariance*

$$\begin{array}{ccc} E \times_{C_0} C_1 & \xrightarrow{pr_2} & C_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle s} \\ E & \xrightarrow[\alpha_0]{} & C_0 \end{array} \qquad (5.2)$$

(which in the case $\mathcal{E} = Set$ gives an identity $\alpha_0(eg) = s(g)$, $\forall (e, g) \in E \times_{C_0} C_1$ where we denoted $eg := a(e, g)$)

- *a (quasi)associativity law*

$$
\begin{array}{ccc}
E \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{a \times id_{C_1}} & E \times_{C_0} C_1 \\
{\scriptstyle id_E \times m}\downarrow & & \downarrow{\scriptstyle a} \\
E \times_{C_0} C_1 & \xrightarrow[a]{} & E
\end{array}
\tag{5.3}
$$

(which in the case $\mathcal{E} = Set$ for any $(e, h, g) \in E \times_{G_0} G_1 \times_{G_0} G_1$ gives $(eh)g = e(hg)$)

- *a unit law*

$$
\begin{array}{ccc}
E & = & E \\
{\scriptstyle (id_E, \alpha_0)}\downarrow & & \uparrow{\scriptstyle a} \\
E \times_{C_0} C_0 & \xrightarrow[id_E \times u]{} & E \times_{C_0} C_1
\end{array}
\tag{5.4}
$$

(which in the discrete case $\mathcal{E} = Set$, for any $e \in E$ gives an identity $ei_{\alpha_0(e)} = e$.)

Theorem 5.1. *For an action of an internal category $\mathcal{C}$ on an object E in $\mathcal{E}$, there exists an action category $E \triangleleft \mathcal{C}$ whose underlying graph consists of the following data:*

- *objects of $E \triangleleft \mathcal{C}$ are given by an object E of the category $\mathcal{E}$,*
- *morphisms of $E \triangleleft \mathcal{C}$ are given by an object $E \times_{C_0} C_1$ in the pullback (5.1)*
- *source is an action $a \colon E \times_{C_0} C_1 \to E$ and target is a projection $pr_1 \colon E \times_{C_0} C_1 \to E$*

Proof. In terms of elements, any (e, g) in $E \times_{C_0} C_1$ is seen as an arrow

$$eg \xrightarrow{(e,g)} e$$

and the target and source $d_0, d_1 \colon E_1 \to E_0$ are defined by the following two identities

$$
\begin{aligned}
d_0(e, g) &= e \\
d_1(e, g) &= eg.
\end{aligned}
\tag{5.5}
$$

For any composable pair of morphisms in $E \times_{C_0} C_1$

$$egh \xrightarrow{(eg,h)} eg \xrightarrow{(e,g)} e$$

their composition is induced by a composition in the category $\mathcal{C}$ and is defined by

$$(e, g)(eg, h) := (e, gh)$$

The associativity and identity axioms for $E \triangleleft \mathcal{C}$ follows directly from those of $\mathcal{C}$. □

That internal presheaves or actions of internal categories are the right internalization of preshaves follows from the well known equivalence

$$[\mathcal{E}^{op}, \mathrm{Set}] \sim DFib_{\mathcal{E}} \tag{5.6}$$

between the category $[\mathcal{E}^{op}, \mathrm{Set}]$ of presheaves on the category $\mathcal{E}$ and the category $DFib_{\mathcal{E}}$ of *discrete fibrations* over $\mathcal{E}$ (see [68] for example). The discrete fibration is a special case of *fibered categories* (see [38]) introduced by Grothendieck in [42], and it is defined by the functor

$$P \colon \mathcal{F} \to \mathcal{E}$$

which has the property that for any morphism $f \colon F \to E$ in $\mathcal{E}$ and any object X in $\mathcal{F}$, such that $F(X) = E$, there exists a unique morphism $\tilde{f} \colon Y \to X$ in $\mathcal{F}$, such that $F(\tilde{f}) = f$. In order to give an internal characterization of equivalence (5.6) we use the following definition.

Definition 5.2. *An internal functor $F \colon \mathcal{A} \to \mathcal{C}$ in $\mathcal{E}$ is a discrete fibration in $\mathcal{E}$, if the diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{F_1} & C_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ A_0 & \xrightarrow[F_0]{} & C_0 \end{array} \tag{5.7}$$

involving targets is a pullback.

Proposition 5.1. *Let an internal category $\mathcal{C}$ acts on an object E in $\mathcal{E}$. Then there exists a canonical internal functor*

$$P \colon E \triangleleft \mathcal{C} \to \mathcal{C} \tag{5.8}$$

in $\mathcal{E}$ which is a discrete fibration.

Proof. The components of an internal functor $P\colon E \triangleleft \mathcal{C} \to \mathcal{C}$ are given by the diagram

$$\begin{array}{ccc} E \times_{C_0} C_1 & \xrightarrow{pr_2} & C_1 \\ {\scriptstyle pr_1}\downarrow\downarrow{\scriptstyle a} & & {\scriptstyle d_0}\downarrow\downarrow{\scriptstyle d_1} \\ E & \xrightarrow[\alpha_0]{} & C_0 \end{array}$$

and the fact that is a discrete fibration is equivalent to (5.1). □

The following theorem is an internal characterization of the equivalence (5.6) between presheaves and discrete fibrations over $\mathcal{E}$.

Theorem 5.2. *Let $\mathcal{C}$ be an internal category in $\mathcal{E}$. An internal functor $F\colon \mathcal{A} \to \mathcal{C}$ is isomorphic to the functor (5.8) for some action of the category $\mathcal{C}$ if and only if it is a discrete fibration. Therefore, there exists an equivalence*

$$\mathcal{E}^{\mathcal{C}} \sim DFib(\mathcal{E}) \tag{5.9}$$

between the category $\mathcal{E}^{\mathcal{C}}$ of internal presheaves in $\mathcal{E}$ and the category $DFib(\mathcal{E})$ of discrete fibrations in $\mathcal{E}$.

Proof. If $F\colon \mathcal{A} \to \mathcal{C}$ is a discrete fibration, then A_1 is isomorphic to $A_0 \times_{C_0} C_1$ in a pullback

$$\begin{array}{ccc} A_0 \times_{C_0} C_1 & \xrightarrow{pr_2} & C_1 \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle d_0} \\ A_0 & \xrightarrow[F_0]{} & C_0 \end{array}$$

by the unique isomorphism $(d_0, F_1)\colon A_1 \to A_0 \times_{C_0} C_1$ which, on the level of elements, sends any morphism $f\colon x \to y$ in $\mathcal{A}$ to the pair $(F_1(f), y)$ in $A_0 \times_{C_0} C_1$. Then we define an action of the morphism $f\colon x \to y$ on an element y by $yf := x$. It easy follows that such action is well defined. Conversely, any action of the category $\mathcal{C}$ on an object E gives a discrete fibration by Proposition 5.1. □

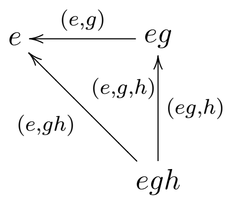
Now we will restrict our attention to actions of internal groupoids. For any action of an internal groupoid $\mathcal{G}$ on an object E in $\mathcal{E}$ the nerve of an action groupoid $E \triangleleft \mathcal{G}$ is an internal simplicial object $E_\bullet$ in $\mathcal{E}$ whose terms are given by objects

$$\begin{gathered} E_0 := E \\ E_1 := E \times_{G_0} G_1 \\ E_2 := E \times_{G_0} G_1 \times_{G_0} G_1 \\ \dots \\ E_k := E \times_{G_0} \underbrace{G_1 \times_{G_0} \dots \times_{G_0} G_1}_{k \text{ times}} \end{gathered}$$

In terms of elements, the set E_0 of vertices is given by elements of E and the set $E \times_{C_0} C_1$ of 1-simplices is given by pairs (e, g) for which degeneracy operators are defined by

$$\begin{gathered} d_0^1(e, g) = e \\ d_1^1(e, g) = eg. \end{gathered} \tag{5.10}$$

A composable pair $(e, g), (eg, h)$ defines a 2-simplex $(e, g, h) \in E_2$ which we see as a triangle



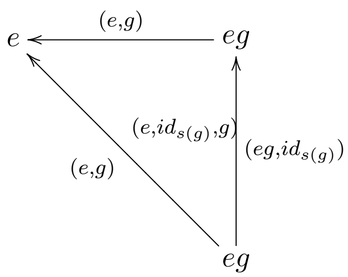
and face operators $d_i^2 \colon E_2 \to E_1$ are given by

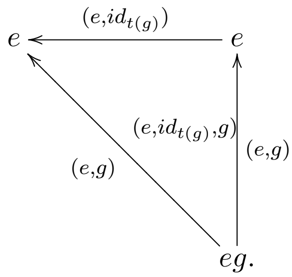
$$\begin{gathered} d_0^2(e, g, h) = (e, g) \\ d_1^2(e, g, h) = (e, gh) \\ d_2^2(e, g, h) = (eg, h). \end{gathered} \tag{5.11}$$

Also we define for any 1-simplex $(e, g) \in E_1$ degeneracy operators $s_i^1 \colon E_1 \to E_2$, $i = 0, 1$ by

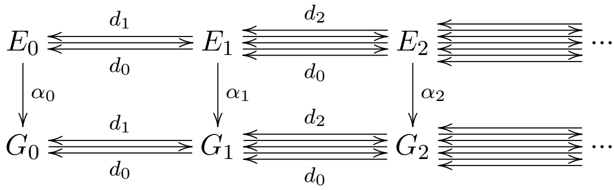
$$\begin{gathered} s_0^1(e, g) = (e, g, id_{s(g)}) \\ s_1^1(e, g) = (e, id_{t(g)}, g) \end{gathered} \tag{5.12}$$

which we respectively see as two triangles





Let $\alpha_\bullet = NP\colon E_\bullet \to G_\bullet$ be the simplicial map defined as the nerve of the canonical functor (5.8) from the nerve $E_\bullet$ of an action groupoid $\mathcal{E}$, to the nerve $G_\bullet$ of the groupoid $\mathcal{G}$



where $\alpha_0\colon E_0 \to G_0$ is a momentum of the action, and maps $\alpha_n\colon E_n \to G_n$ are defined by

$$\alpha_n(e, g_n, \ldots, g_1) = (g_n, \ldots, g_1) \tag{5.13}$$

for $n \geq 1$. It is an elementary consequence that this construction really defines a simplicial map, by the fact that nerve is a functor. Nevertheless we will give a direct proof of this fact to shed some light to simplicial techniques which we will use later.

Proposition 5.2. *The map $\alpha_\bullet\colon E_\bullet \to G_\bullet$ is a simplicial map from the nerve of $E_\bullet$ the action groupoid $\mathcal{E}$, to the nerve $G_\bullet$ of of the groupoid $\mathcal{G}$.*

Proof. Let $(e, g) \in E_1$, which means that $t(g) = \alpha_0(e)$. The first two simplicial identities $\alpha_0 d_i = d_i \alpha_1$ for $i = 0, 1$ are verified by evaluation on a general element $(e, g) \in E_1$

$$\alpha_0 d_0(e, g) \overset{(5.10)}{=} \alpha_0(e) \overset{(5.1)}{=} d_0(g) \overset{(5.13)}{=} d_0 \alpha_1(e, g)$$
$$\alpha_0 d_1(e, g) \overset{(5.10)}{=} \alpha_0(eg) \overset{(5.2)}{=} d_1(g) \overset{(5.13)}{=} d_1 \alpha_1(e, g)$$

where the first identity is equivalent to the definition of (the domain of) the action and the second follows from the momentum invariance. The last identity in dimension $n = 1$ is $\alpha_1 s_0 = s_0 \alpha_0$ which is verified for any element $e \in E_0$ by

$$\alpha_1 s_0(e) = \alpha_1(e, id_{\alpha_0(e)}) = id_{\alpha_0(e)} = s_0 \alpha_0(e).$$

In the dimension $n = 2$ identities $\alpha_1 d_i = d_i \alpha_2$ for $i = 0, 1, 2$ are verified by evaluation on the general element $(e, g, h) \in E_2$

$$\alpha_1 d_0(e, g, h) \overset{(5.11)}{=} \alpha_1(e, g) \overset{(5.13)}{=} g \overset{(4.1)}{=} d_0(g, h) \overset{(5.13)}{=} d_0 \alpha_2(e, g, h)$$
$$\alpha_1 d_1(e, g, h) \overset{(5.11)}{=} \alpha_1(e, gh) \overset{(5.13)}{=} gh \overset{(4.1)}{=} d_1(g, h) \overset{(5.13)}{=} d_1 \alpha_2(e, g, h)$$
$$\alpha_1 d_2(e, g, h) \overset{(5.11)}{=} \alpha_1(eg, h) \overset{(5.13)}{=} h \overset{(4.1)}{=} d_2(g, h) \overset{(5.13)}{=} d_2 \alpha_2(e, g, h)$$

and two relations in dimension $n = 2$ involving degeneracy operators, $\alpha_2 s_i = s_i \alpha_1$ for $i = 0, 1, 2$ are verified by evaluation on the general element $(e, g) \in E_1$

$$\alpha_2 s_0(e, g) \overset{(5.12)}{=} \alpha_2(e, g, id_{s(g)}) \overset{(5.13)}{=} (g, id_{s(g)}) \overset{(4.2)}{=} s_0(g) \overset{(5.13)}{=} s_0 \alpha_1(e, g)$$
$$\alpha_2 s_1(e, g) \overset{(5.12)}{=} \alpha_2(e, id_{t(g)}, g) \overset{(5.13)}{=} (id_{t(g)}, g) \overset{(4.2)}{=} s_1(g) \overset{(5.13)}{=} s_1 \alpha_1(e, g)$$

Clearly, the similar pattern repeats in all higher dimensions which concludes the proof. □

Now, we will provide a simplicial characterization of groupoid actions.

Theorem 5.3. *Let an internal groupoid $\mathcal{G}$ acts on an object E in $\mathcal{C}$. Then the simplicial map $\alpha_\bullet = NP\colon E_\bullet \to G_\bullet$ is an exact fibration for all $n \geq 1$.*

Proof. Suppose that the groupoid $\mathcal{G}$ acts on a set E. We first check conditions in dimension $n = 1$, namely that the two squares

$$\begin{array}{ccc} E_1 & \xrightarrow{\alpha_1} & G_1 \\ \downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle q_0} \\ \bigwedge_1^0(E_\bullet) & \longrightarrow & \bigwedge_1^0(G_\bullet) \end{array} \qquad \begin{array}{ccc} E_1 & \xrightarrow{\alpha_1} & G_1 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_1} \\ \bigwedge_1^1(E_\bullet) & \longrightarrow & \bigwedge_1^1(G_\bullet) \end{array}$$

are pullbacks. Since the set of 0-horns and 1-horns of the simplicial set $E_\bullet$ in dimension 1 is just $\bigwedge_1^0(E_\bullet) = E = \bigwedge_1^1(E_\bullet)$ and the set of 0-horns and 1-horns of the simplicial set $G_\bullet$ is just $\bigwedge_1^0(E_\bullet) = G_0 = \bigwedge_1^0(E_\bullet)$, two squares are just two pullbacks

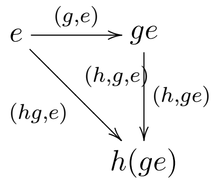
$$\begin{array}{ccc} E \times_{G_0} G_1 & \xrightarrow{pr_2} & G_1 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle s} \\ E & \xrightarrow[\alpha_0]{} & G_0 \end{array} \qquad \begin{array}{ccc} E \times_{G_0} G_1 & \xrightarrow{pr_2} & G_1 \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle t} \\ E & \xrightarrow[\alpha_0]{} & G_0 \end{array}$$

given by the momentum invariance (5.2) and the definition (5.1) of a domain of an action, respectively.

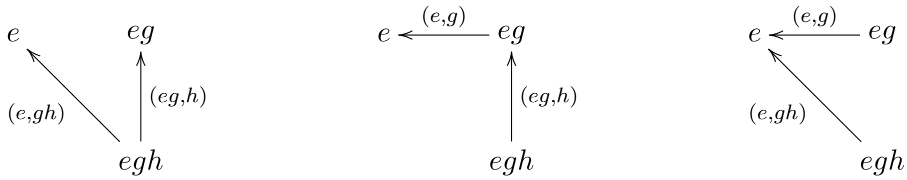
In the dimension $n = 2$, the object of 0-horns is $\bigwedge_2^0(E_\bullet) = (E \times_{G_0} G_1) \times_E (E \times_{G_0} G_1)$, where the pullback is obtained by the map $pr_1\colon E \times_{G_0} G_1 \to E$, and similarly the object of 2-horns is $\bigwedge_2^2(E_\bullet) = (E \times_{G_0} G_1) \times_E (E \times_{G_0} G_1)$, where now we use the map $a\colon E \times_{G_0} G_1 \to E$ to define the pullback. The object of 1-horns is $\bigwedge_2^1(E_\bullet) = E \times_{G_0} G_1 \times_{G_0} G_1$, and diagrams

$$\begin{array}{ccc} E_2 & \xrightarrow{\alpha_2} & G_2 \\ \downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle q_0} \\ \bigwedge_2^0(E_\bullet) & \longrightarrow & \bigwedge_2^0(G_\bullet) \end{array} \qquad \begin{array}{ccc} E_2 & \xrightarrow{\alpha_2} & G_2 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_1} \\ \bigwedge_2^1(E_\bullet) & \longrightarrow & \bigwedge_2^1(G_\bullet) \end{array} \qquad \begin{array}{ccc} E_2 & \xrightarrow{\alpha_2} & G_2 \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_2} \\ \bigwedge_2^2(E_\bullet) & \longrightarrow & \bigwedge_2^2(G_\bullet) \end{array}$$

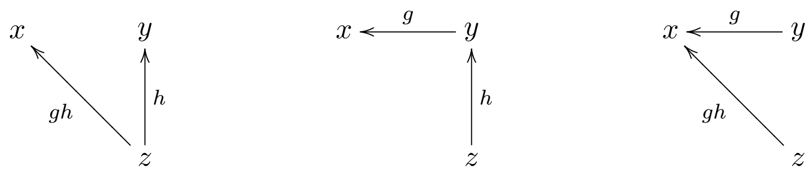
are pullbacks. For a general 2-simplex $(e, g, h) \in E_2$



three maps $q_i \colon E_2 \to \bigwedge_2^i(E_\bullet)$, for $i = 0, 1, 2$, are given by the three projections, pictured as



respectively, and three bottom maps between corresponding horns $\bigwedge_2^i(E_\bullet)$ and $\bigwedge_2^i(G_\bullet)$, induced by the simplicial map $\alpha_\bullet \colon E_\bullet \to G_\bullet$, transform above horns into three diagrams



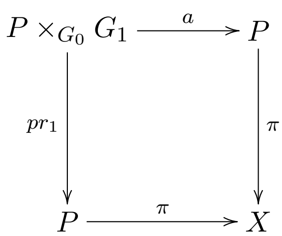
respectively. These three horns have the unique filler $(h, hg, g) \in G_2$ by the invertibility of arrows in the groupoid $\mathcal{G}$, it follows that maps $\nu_i \colon E_2 \to \bigwedge_2^i(E_\bullet) \times_{\bigwedge_2^i(G_\bullet)} G_2$, for $i = 0, 1, 2$ defined by

$$\begin{aligned}
\nu_0(e, g, h) &:= ((-, (e, gh), (e, g)), (g, gh, h)) \\
\nu_1(e, g, h) &:= ((eg, h), -, (e, g)), (g, gh, h)) \\
\nu_2(e, g, h) &:= ((eg, h), (e, gh), -), (g, gh, h))
\end{aligned}$$

are all isomorphisms, which is just equivalent to the quasiassociativity $(eg)h = e(gh)$. □

Definition 5.3. *Let $\mathcal{G}$ be an internal groupoid in the category $\mathcal{C}$. A right $\mathcal{G}$-bundle P over an object X is defined by the following data:*

- *left $\mathcal{G}$-object P along the momentum morphism $\alpha_0 \colon P \to G_0$,*
- *a $\mathcal{G}$-invariant epimorphism $\pi \colon P \to X$*



We say that the $\mathcal{G}$-bundle $\pi\colon P \to X$ is principal, or that it is a right $\mathcal{G}$-torsor, if the naturally induced morphism

$$(pr_1, a)\colon P \times_{G_0} G_1 \longrightarrow P \times_X P$$

is an isomorphism.

Theorem 5.4. *Let be $\mathcal{G}$ an internal groupoid in $\mathcal{C}$ which acts on an object E. Then the nerve of the corresponding action groupoid is a simplicial map $\alpha_\bullet\colon E_\bullet \to G_\bullet$ which is an exact fibration for all $n \geq 1$.*

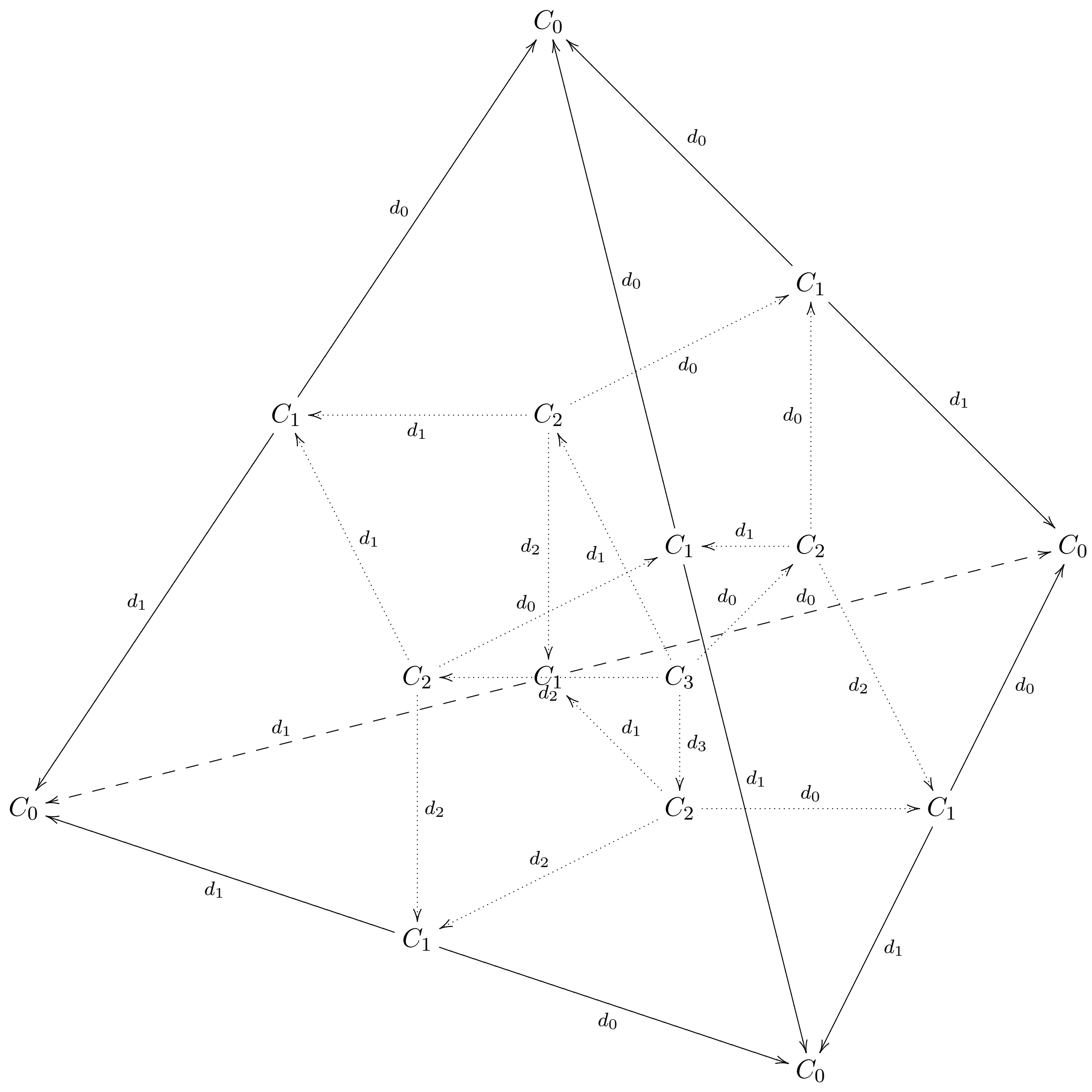
Proof. The proof is straightforward and it follows a similar pattern of the previous theorem. □

6 Small fibrations

Let $\mathcal{C}$ be an object of $\mathrm{Cat}(\mathcal{E})$. Thus $\mathcal{C}$ is given by the 3-truncation of the internal simplicial object in $\mathcal{E}$

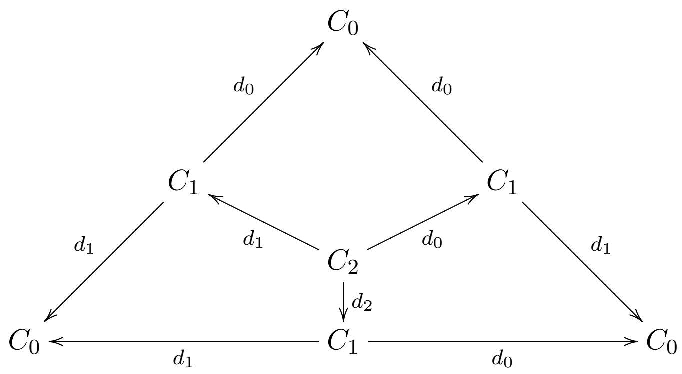
$$C_0 \underset{d_0}{\overset{d_1}{\leftleftarrows}} C_1 \underset{d_0}{\overset{d_2}{\leftleftarrows}} C_2 \underset{d_0}{\overset{d_3}{\leftleftarrows}} C_3 \tag{6.1}$$

and this data is equivalent to the barycentric division of the 3-simplex

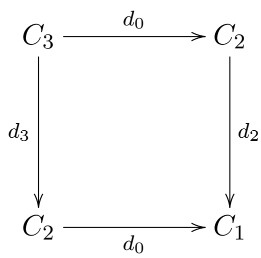


where we keep in mind that internally objects are given by C_0, morphisms by C_1, composable pairs of morphisms by C_2 and composable triples of morphisms by C_3.

The faces of the above 3-simplex are given by the barycentric subdivision of the 2-simplex



in which the lower right square is a pullback (which represents C_2 as an object of composable pairs of morphisms). Also we have the pullback

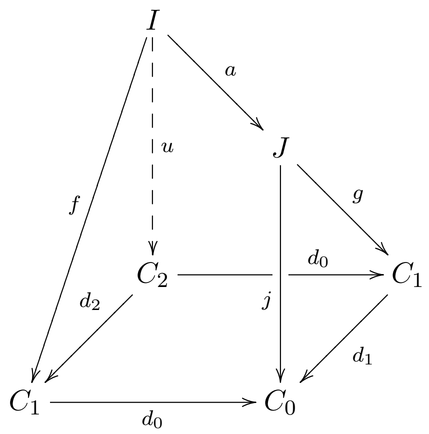


in the interior of the 3-simplex, which represents (internally) composable triples of morphisms.

We will construct the fibered category $F_{\mathcal{C}}\colon \mathcal{FC} \to \mathcal{E}$ as follows. The objects of $\mathcal{FC}$ are pairs (I, i), where I is an object in $\mathcal{E}$, and $i\colon I \to C_0$ is a morphism in $\mathcal{E}$. For any two such objects (I, i) and (J, j), a morphism in $\mathcal{FC}$ is given by a pair $(a, f)\colon (I, i) \to (J, j)$, which consists of the morphism $a\colon I \to J$, and the morphism $f\colon I \to C_1$ in $\mathcal{E}$, such that $d_1 f = i$ and $d_0 f = ja$. For any two composable morphisms in $\mathcal{FC}$

$$(I, i) \xrightarrow{(a,f)} (J, j) \xrightarrow{(b,g)} (K, k)$$

the composition is defined by $(b, g)(a, f) := (ba, g \circ f)$ where the morphism $g \circ f\colon I \to C_1$ is defined by $g \circ f := d_1 u$, and $u\colon I \to C_2$ is the unique morphism given by the universal property of the pullback



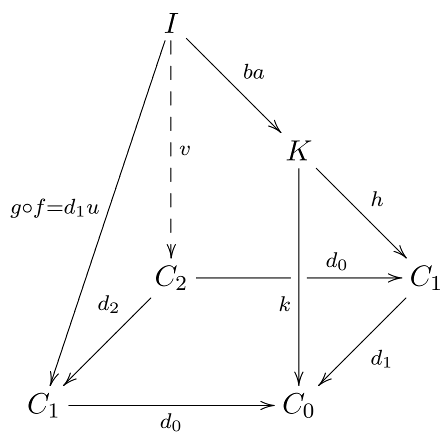
obtained from the factorization $d_0 f = ja = d_1 ga$. Thus we have the following result.

Theorem 6.1. *The above construction defines a fibred category $F_{\mathcal{C}}\colon \mathcal{FC} \to \mathcal{E}$ which we call the small fibration induced by $\mathcal{C}$.*

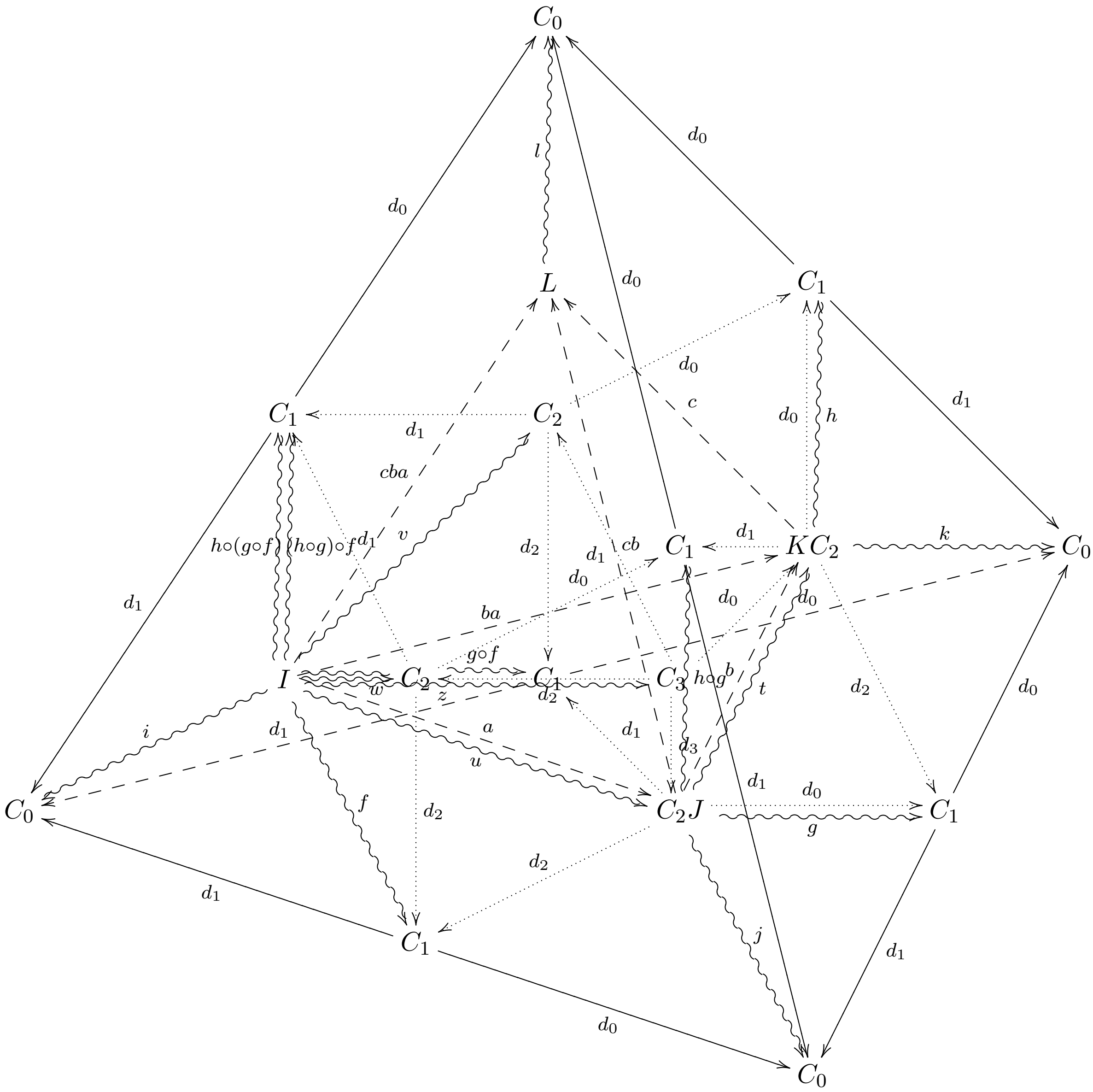
Proof. We need to show that the above composition is associative and unital. Let's consider composable triple of morphisms

$$(I, i) \xrightarrow{(a,f)} (J, j) \xrightarrow{(b,g)} (K, k) \xrightarrow{(c,h)} (L, l)$$

in $\mathcal{FC}$. We need to prove that $[(c, h)(b, g)](a, f) = (c, h)[(b, g)(a, f)]$. The right hand side is given by $(c, h)(ba, g \circ f) := (cba, h \circ (g \circ f))$ where the second component is defined by $h \circ (g \circ f) := d_1 v$ and the morphism $v\colon I \to C_2$ is the unique one given by the universal property of the pullback



obtained from the factorization $d_0 d_1 u = kba = d_1 hba$. This is described by the diagram

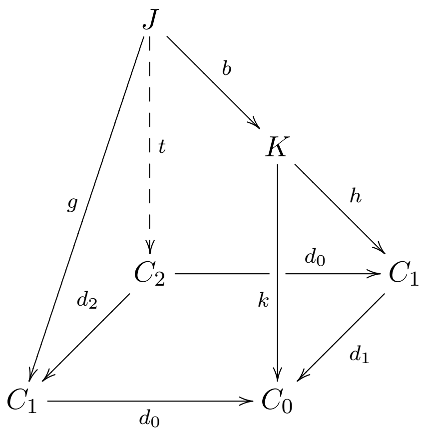


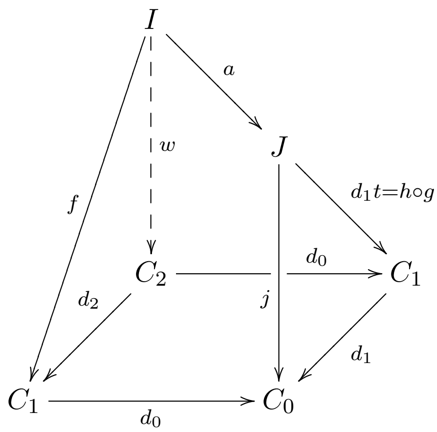
where objects and morphisms of the category $\mathcal{FC}$ are drawn as curved arrows, and the 3-simplex that is an element of $N\mathcal{E}_3$ in the nerve of $\mathcal{E}$ corresponding to the composable triple

$$I \xrightarrow{a} J \xrightarrow{b} K \xrightarrow{c} L$$

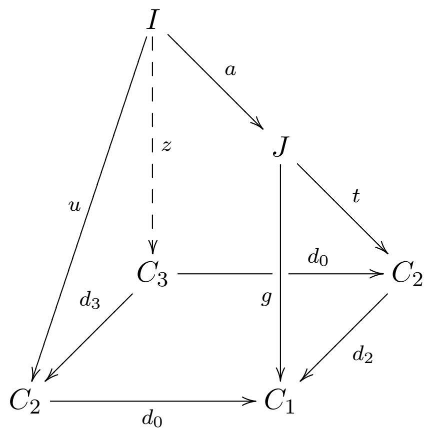
of morphisms in $\mathcal{E}$, sits inside the barycentic division of the 3-simplex.

On the other side, first composing $(c,h)(b,g) := (cb, h \circ g)$ we have $[(c,h)(b,g)](a,f)$ which is equal to $(cb,(h\circ g))(a,f) = (cba,(h\circ g)\circ f)$, where $h\circ g := d_1 t$ and $(h\circ g)\circ f := d_1 w$ are morphisms obtained from two pullbacks





whose diagonals are given by $d_0 g = kb = d_1 hb$ and $d_0 f = ja = d_1 d_1 ta$, respectively. Now, we use the universal property of the pullback



to obtain a unique morphism $z\colon I \to C_3$ from the factorization $d_0 u = ga = d_2 ta$. But then we have

$$d_1 v = d_1 d_1 z = d_1 d_2 z = d_1 w$$

which finally gives the associativity of composition.

The functor $F\colon \mathcal{F} \to \mathcal{E}$ is defined on objects by $F(I,i) = I$ and on morphisms by $F(a,f) = a$, and it is straightforward to prove that it is a fibration. $\square$

Part III
Two-dimensional theory

7 Bicategories

Bicategories were defined by Benabou [15], and from the modern perspective, we could call them weak 2-categories. Instead of stating their original definition we will use Batanin's approach to weak n-categories given in [14]. In this approach a bicategory $\mathcal{B}$, given by the reflexive 2-graph

$$\mathcal{B} \equiv (B_2 \underset{d_0^1}{\overset{d_1^1}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} B_1 \underset{d_0^0}{\overset{d_1^0}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} B_0)$$

is a 1-skeletal monoidal globular category, given by the diagram of categories and functors

$$\mathcal{B}_1 \underset{D_0}{\overset{D_1}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} \mathcal{B}_0$$

where the category $\mathcal{B}_1$ is the category of morphisms of the bicategory $\mathcal{B}$ and the category $\mathcal{B}_0$ is the image $\mathcal{D}(B_0)$ of the discrete functor $\mathcal{D}\colon \mathrm{Set} \to \mathrm{Cat}$ which just turns an object of $\mathcal{E}$ into a discrete internal category in $\mathcal{E}$. Source functor D_1 is defined by $D_1 := d_1^0\colon B_1 \to B_0$ and $D_1 := d_1^0 d_1^1 = d_1^0 d_0^1\colon B_2 \to B_0$, and a target functor D_0 is defined by $D_0 := d_1^0\colon B_1 \to B_0$ and $D_1 := d_0^0 d_1^1 = d_0^0 d_1^1\colon B_2 \to B_0$, where we used the same notation for objects and morphisms parts of the functor. Also, the unit functor $I\colon B_0 \to B_1$ is defined by $I := s_0\colon B_0 \to B_1$ on the level of objects, and $I := s_1\colon B_1 \to B_2$ on the level of morphisms, where $s_0\colon B_0 \to B_1$ and $s_1\colon B_1 \to B_2$ are section morphisms in the above 2-graph from left to right, which we didn't label to avoid too much indices.

In the lower definition of a bicategory we will denote the vertex $\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ of the following pullback of functors

$$\begin{array}{ccc}
\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{Pr_2} & \mathcal{B}_1 \\
{\scriptstyle Pr_1}\downarrow & & \downarrow{\scriptstyle D_0} \\
\mathcal{B}_1 & \xrightarrow[D_1]{} & \mathcal{B}_0
\end{array}$$

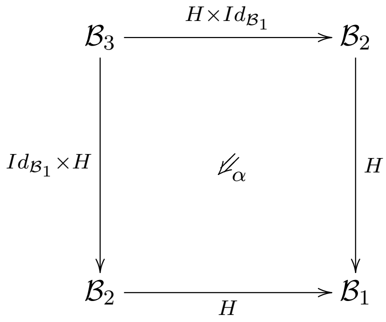
by $\mathcal{B}_2 := \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ and likewise $\mathcal{B}_3 := \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$, and so on. Thus we will adopt the following convention: for any functor $P\colon \mathcal{E} \to B_0$, the first of the symbols

$$\mathcal{E} \times_{\mathcal{B}_0} \mathcal{B}_1 \ \textit{and} \ \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{E}$$

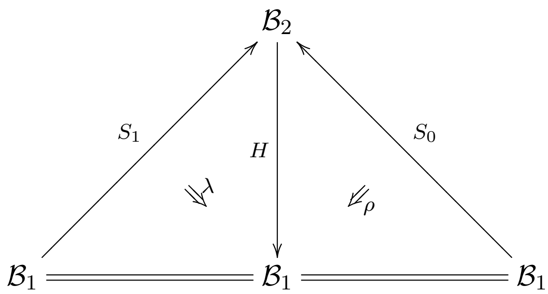
will denote the pullback of P and D_0, and the second one that of D_1 and P.

Definition 7.1. *A bicategory $\mathcal{B}$ consists of the following data:*

- *two categories, a discrete category $\mathcal{B}_0$ of objects, and a category $\mathcal{B}_1$ of morphisms of the weak 2-category $\mathcal{B}$,*
- *functors $D_0, D_1 \colon \mathcal{B}_1 \to \mathcal{B}_0$, called target and source functors, respectively, a functor $I \colon \mathcal{B}_0 \to \mathcal{B}_1$, called unit functor, and a functor $H \colon \mathcal{B}_2 \to \mathcal{B}_1$, called the horizontal composition functor,*
- *natural isomorphism*



- *natural isomorphisms*



  where the functor $S_0 \colon \mathcal{B}_1 \to \mathcal{B}_2$ is defined by the composition

$$\mathcal{B}_1 \xrightarrow{(D_0, Id_{\mathcal{B}_1})} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_0 \xrightarrow{I \times Id_{\mathcal{B}_1}} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1,$$

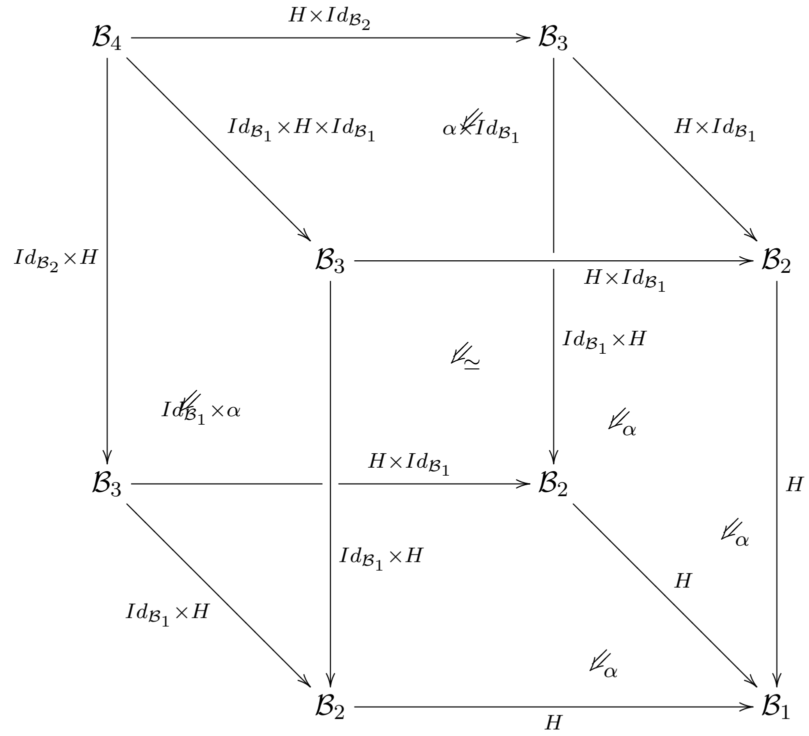
  and the functor $S_1 \colon \mathcal{B}_1 \to \mathcal{B}_2$ is defined by the composition

$$\mathcal{B}_1 \xrightarrow{(Id_{\mathcal{B}_1}, D_1)} \mathcal{B}_0 \times_{\mathcal{B}_0} \mathcal{B}_1 \xrightarrow{Id_{\mathcal{B}_1} \times I} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1,$$

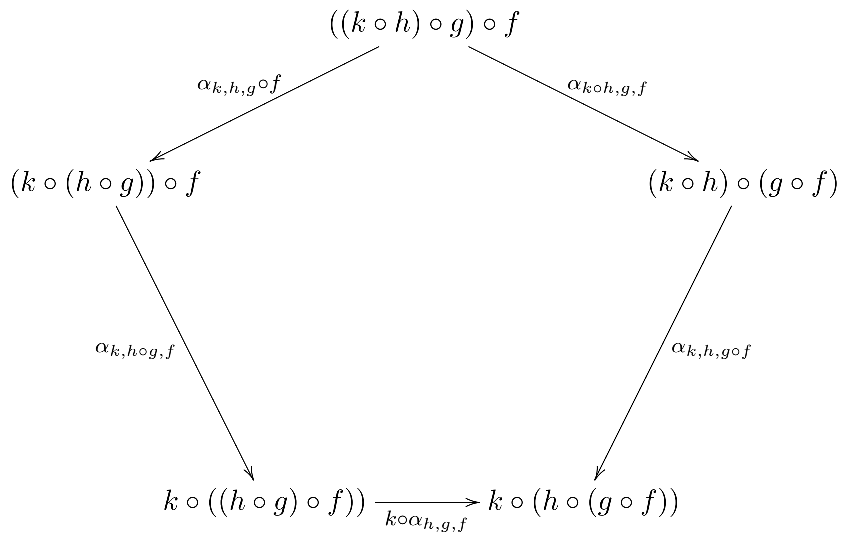
  or more explicitly for any 1-morphism $f \colon x \to y$ in $\mathcal{B}$ (i.e. object in $\mathcal{B}_1$) we have $S_0(f) = (f, i_x)$ and $S_1(f) = (i_y, f)$,

such that following axioms are satisfied:

- *associativity 3-cocycle*

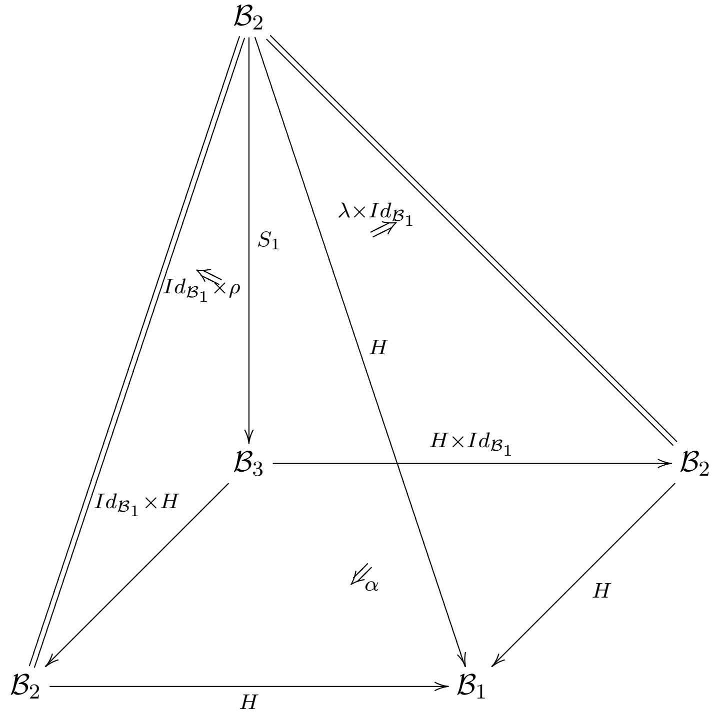


which for any object (k,h,g,f) in $\mathcal{B}_4$ becomes the commutative pentagon

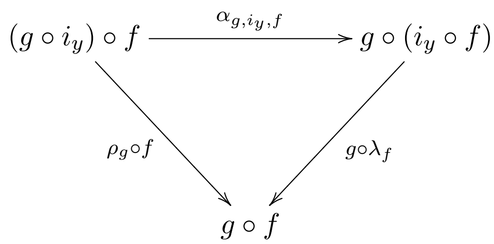


of components of natural transformations

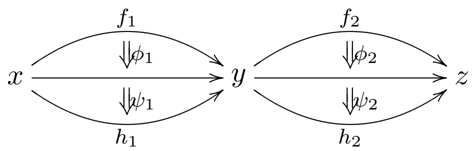
- *the commutative pyramid*



which for any object (g, f) in $\mathcal{B}_2$ becomes the triangle diagram



Remark 7.1. *Note that in the above definition of the horizontal composition functor $H \colon \mathcal{B}_2 \to \mathcal{B}_1$, for any diagram of 2-arrows (i.e. a morphism in a category $\mathcal{B}_2 \times_{\mathcal{B}_1} \mathcal{B}_2$)*



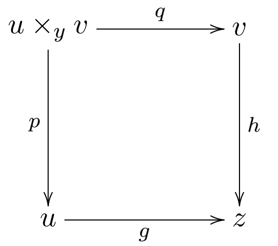
by functoriality we immediately have a Godement interchange law

$$(\psi_2 \circ \psi_1)(\phi_2 \circ \phi_1) = (\psi_2 \psi_1) \circ (\phi_2 \phi_1).$$

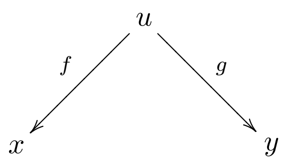
Example 7.1. *(Strict 2-categories) A weak 2-category in which associativity and left and right identity natural isomorphisms are identities is called (strict) 2-category.*

Example 7.2. *(Monoidal categories) Monoidal category is a bicategory $\mathcal{B}$ in which $\mathcal{B}_0 = 1$ is terminal discrete category (or one point set). Strict monoidal category is a one object strict 2-category.*

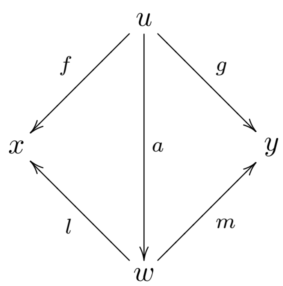
Example 7.3. *(Bicategory of spans) Let $\mathcal{C}$ be a cartesian category (that is a category with pullbacks). First we make a choice of the pullback*



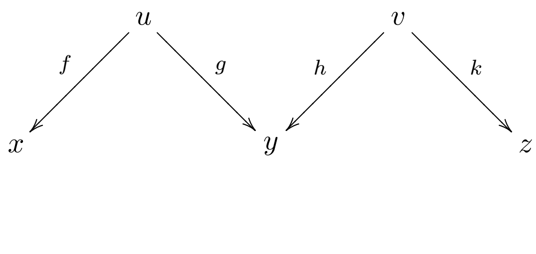
for any such diagram $x \xrightarrow{f} z \xleftarrow{g} y$ in a category $\mathcal{C}$. We construct the weak 2-category $Span(\mathcal{C})$ of spans in the category $\mathcal{C}$. The objects of $Span(\mathcal{C})$ are the same as objects of $\mathcal{C}$. For any two objects x, y in $Span(\mathcal{C})$, a 1-morphism $u\colon x \nrightarrow y$ is a span



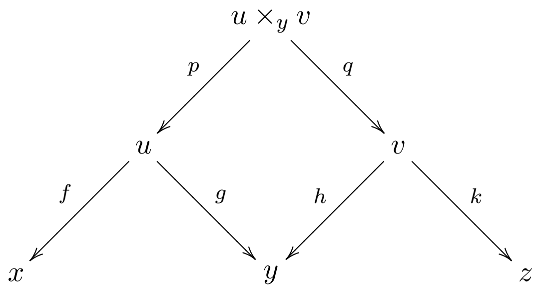
and a 2-morphism $a\colon z \nRightarrow w$ is given by the commutative diagram



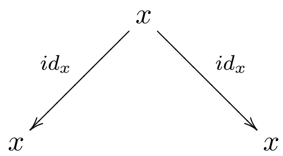
from which we easily see that vertical composition of 2-morphisms is given by the composition in $\mathcal{C}$. Horizontal composition of composable 1-morphisms



is given by the pullback



and from here we have obvious horizontal identity $i_x \colon x \nrightarrow x$



Example 7.4. *(Bimodules) Let Bim denote the bicategory whose objects are rings with identity. For any two rings A and B, Bim(A, B) will be a category of A − B bimodules and their homomorphisms. Horizontal composition is given by the tensor product, and associativity and identity constraints are the usual ones for the tensor product.*

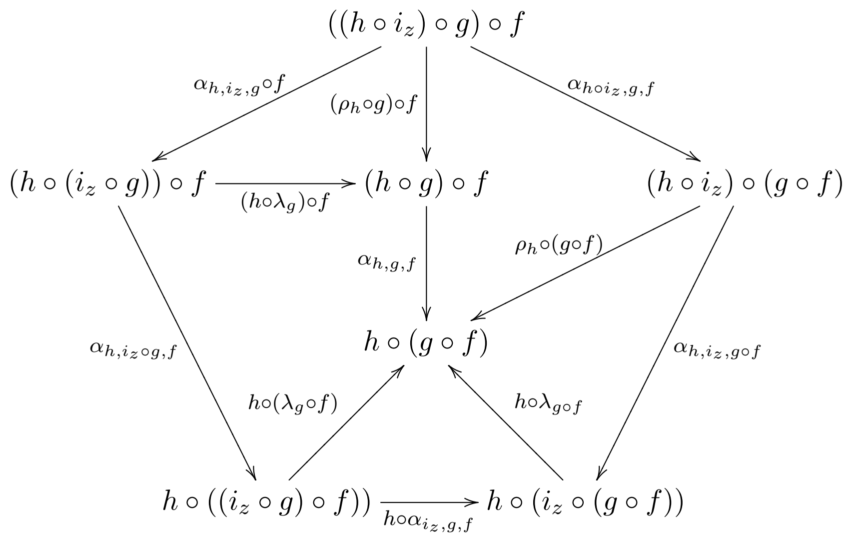
The following result is a typical example how new commutative diagrams arise from the associativity coherence and left and right identity coherence.

Proposition 7.1. *Let $\mathcal{B}$ be a bicategory. Then the diagrams*

$$\begin{array}{ccc} (i_z \circ g) \circ f & \xrightarrow{\alpha_{i_z,g,f}} & i_z \circ (g \circ f) \\ & {\scriptstyle \lambda_g \circ f} \searrow \quad \swarrow {\scriptstyle i_z \circ \lambda_{g \circ f}} & \\ & g \circ f & \end{array} \qquad \begin{array}{ccc} (g \circ f) \circ i_x & \xrightarrow{\alpha_{g,f,i_x}} & g \circ (f \circ i_x) \\ & {\scriptstyle \rho_{g \circ f}} \searrow \quad \swarrow {\scriptstyle g \circ \rho_f} & \\ & g \circ f & \end{array}$$

commute for any pair of 1-morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ *in* $\mathcal{B}$.

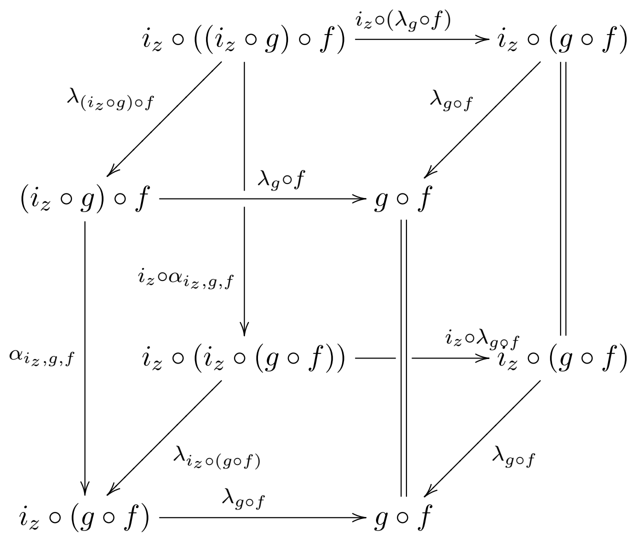
Proof. For any triple of 1-morphisms $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t$, we consider the diagram



in which two triangles (beside the bottom one) commute because of the triangle coherence for identities, and two deformed squares commute by the naturality of associativity coherence. Since all the terms are 2-isomorphisms, then the bottom triangle also commutes. By taking $h = i_z$, we obtain the identity

$$i_z \circ (\lambda_{g\circ f}\alpha_{i_z,g,f}) = i_z \circ (\lambda_g \circ f)$$

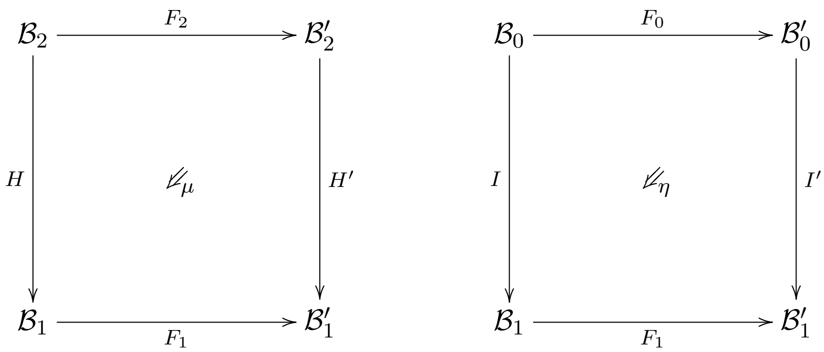
from which it follows that the back face of the cube



commutes. The top, bottom and right faces commute from the naturality of the left identity coherence, and the right face commutes trivially. Since all edges are 2-isomorphisms we conclude that the front face also commutes, which proves that the first triangle in lemma commutes. Similarly, we prove the commutativity of the other triangle. ☐

Definition 7.2. *A homomorphism $F\colon \mathcal{B} \to \mathcal{B}'$ between bicategories consists of the following data:*

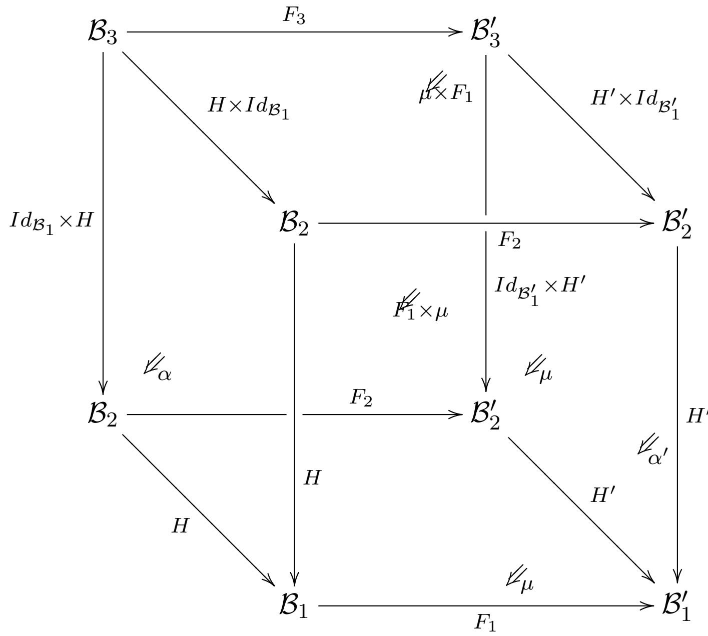
- *a (discrete) functor $F_0\colon \mathcal{B}_0 \to \mathcal{B}'_0$, and a functor $F_1\colon \mathcal{B}_1 \to \mathcal{B}'_1$,*
- *natural transformations*



given by components $\mu_{g,f}\colon F(g) \circ F(f) \to F(g \circ f)$ and $\eta_x\colon i'_{F(x)} \to F(i_x)$, respectively (in which we omitted the subscripts on functor signs in order to avoid too much indices),

such that following axioms are satisfied:

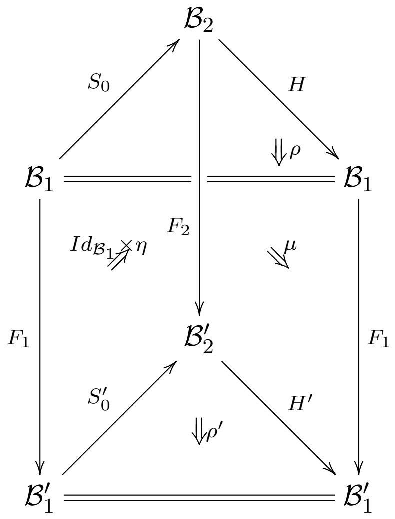
- *commutative cube*



which when evaluated at the object (h,g,f) *in* $\mathcal{B}_3$ *becomes a commutative diagram*

$$\begin{array}{ccccc}
(F(h)\circ F(g))\circ F(f) & \xrightarrow{\mu_{h,g}\circ F(f)} & F(h\circ g)\circ F(f) & \xrightarrow{\mu_{h\circ g,f}} & F((h\circ g)\circ f) \\
\downarrow {\scriptstyle a'_{F(h),F(g),F(f)}} & & & & \downarrow {\scriptstyle F(a_{h,g,f})} \\
F(h)\circ(F(g)\circ F(f)) & \xrightarrow[F(h)\circ\mu_{g,f}]{} & F(h)\circ F(g\circ f) & \xrightarrow[\mu_{h,g\circ f}]{} & F(h\circ(g\circ f))
\end{array}$$

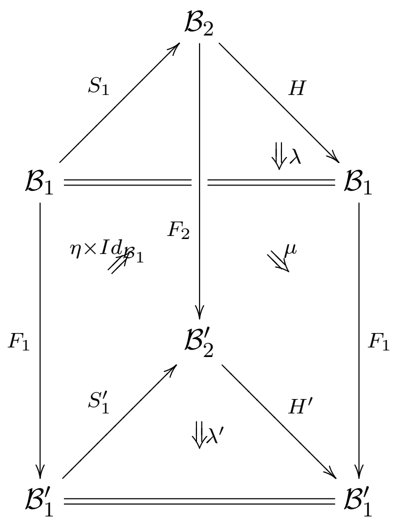
- *a commutative diagram*



which when evaluated at the object f in $\mathcal{B}_1$ becomes a commutative diagram

$$\begin{array}{ccccc} F(f)\circ i'_{F(x)} & \xrightarrow{F(f)\circ\eta_x} & F(f)\circ F(i_x) & \xrightarrow{\mu_{f,i_x}} & F(f\circ i_x) \\ {\scriptstyle \rho'_{F(f)}}\downarrow & & & & \downarrow{\scriptstyle F(\rho_f)} \\ F(f) & & = & & F(f) \end{array}$$

- *a commutative diagram*



which when evaluated at the object f in $\mathcal{B}_1$ becomes a commutative diagram

$$\begin{array}{ccccc}
i'_{F(y)} \circ F(f) & \xrightarrow{\eta_y \circ F(f)} & F(i_y) \circ F(f) & \xrightarrow{\mu_{i_y,f}} & F(i_y \circ f) \\
\downarrow{\scriptstyle \lambda'_{F(f)}} & & & & \downarrow{\scriptstyle F(\lambda_f)} \\
F(f) & & = & & F(f)
\end{array}$$

Remark 7.2. *If both $\mathcal{B}$ and $\mathcal{B}'$ are strict 2-categories then the coherence for composition becomes*

$$\begin{array}{ccc}
F(h) \circ F(g) \circ F(f) & \xrightarrow{\mu_{h,g} \circ F(f)} & F(h \circ g) \circ F(f) \\
\downarrow{\scriptstyle F(h) \circ \mu_{g,f}} & & \downarrow{\scriptstyle \mu_{h \circ g, f}} \\
F(h) \circ F(g \circ f) & \xrightarrow[\mu_{h, g \circ f}]{} & F(h \circ g \circ f)
\end{array}$$

and the coherence for identities become two commutative triangles

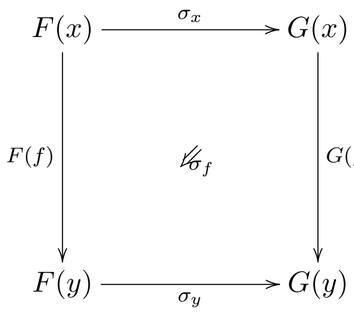
$$\begin{array}{ccc}
 & F(f) \circ F(i_x) & \\
{\scriptstyle F(f) \circ \eta_x} \nearrow & & \searrow {\scriptstyle \mu_{f, i_x}} \\
F(f) & = & F(f)
\end{array}
\qquad
\begin{array}{ccc}
 & F(i_y) \circ F(f) & \\
{\scriptstyle \eta_y \circ F(f)} \nearrow & & \searrow {\scriptstyle \mu_{i_y, f}} \\
F(f) & = & F(f)
\end{array}$$

Definition 7.3. *A (left) lax natural transformation $\sigma \colon F \Longrightarrow G$ is defined by the following data:*

- *a natural transformation $\sigma_0 \colon F_0 \to G_0$ between (discrete) functors (which just amounts to the family of morphisms $\sigma_x \colon F(x) \to G(x)$),*
- *natural transformation*

$$\begin{array}{ccc}
\mathcal{B}_1 & \xrightarrow{G_1} & \mathcal{B}'_1 \\
\downarrow{\scriptstyle F_1} & \swarrow\!\!\!\swarrow {\scriptstyle \sigma_1} & \downarrow{\scriptstyle \sigma_0^*} \\
\mathcal{B}'_1 & \xrightarrow[\sigma_{0*}]{} & \mathcal{B}'_1
\end{array}$$

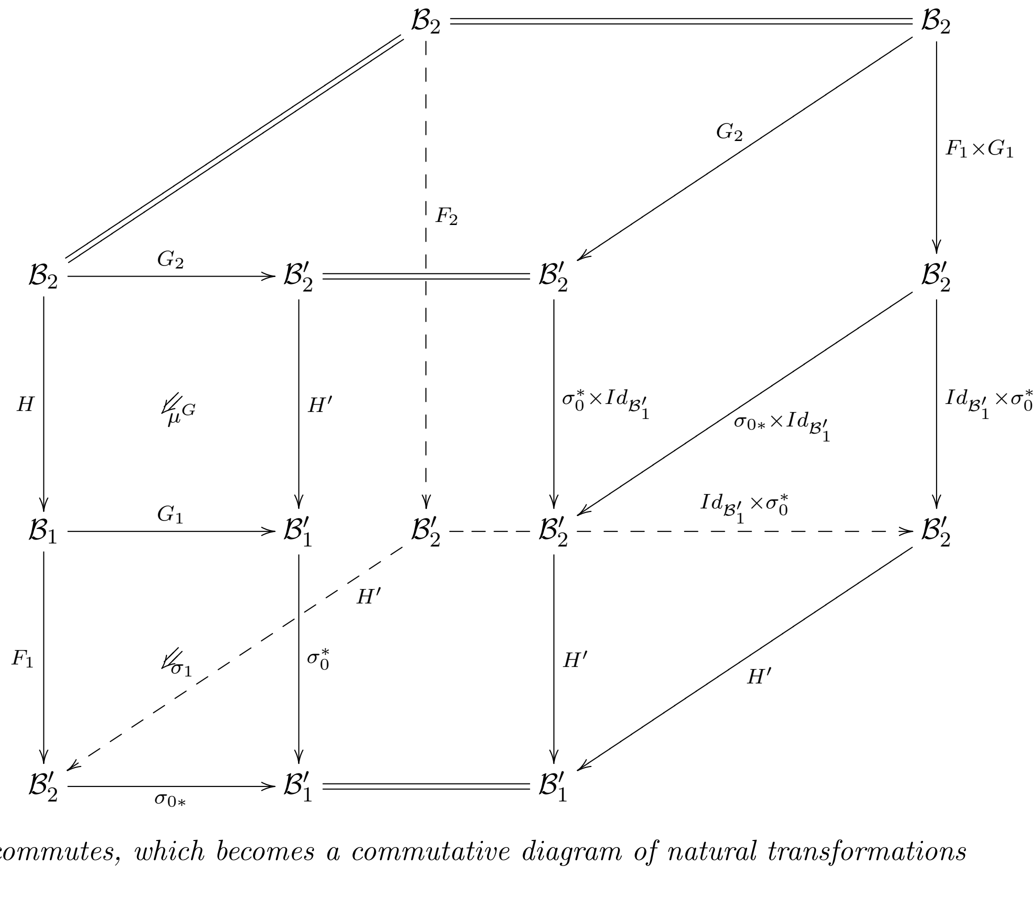
whose component at the object $f\colon x \to y$ in $\mathcal{B}_1$ is given by the square



which is a 2-morphism $\sigma_f\colon G(f)\circ\sigma_x \Longrightarrow \sigma_y \circ F(f)$ *,*

such that the following axioms are satisfied:

- *the following cube of functors and natural transformations*



commutes, which becomes a commutative diagram of natural transformations

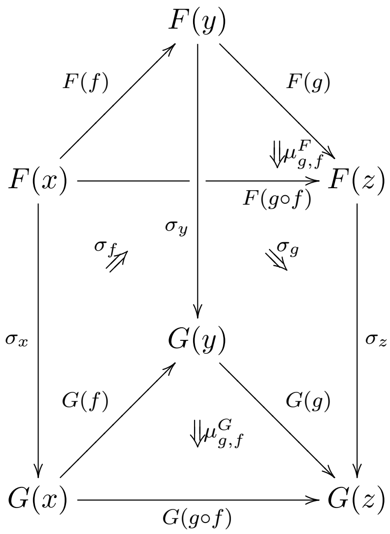
$$
\begin{array}{ccccc}
(G(g)\circ G(f))\circ\sigma_x & \xrightarrow{\alpha'_{G(g),G(f),\sigma_x}} & G(g)\circ(G(f)\circ\sigma_x) & \xrightarrow{G(g)\circ\sigma_f} & G(g)\circ(\sigma_y\circ F(f)) \\
{\scriptstyle \mu^G_{g,f}\circ\sigma_x}\downarrow & & & & \downarrow{\scriptstyle \alpha'^{-1}_{G(g),\sigma_y,F(f)}} \\
G(g\circ f)\circ\sigma_x & & & & (G(g)\circ\sigma_y)\circ F(f) \\
{\scriptstyle \sigma_{g\circ f}}\downarrow & & & & \downarrow{\scriptstyle \sigma_g\circ F(f)} \\
\sigma_z\circ F(g\circ f) & \xleftarrow[\sigma_z\circ\mu^F_{g,f}]{} & \sigma_z\circ(F(g)\circ F(f)) & \xleftarrow[\alpha'_{\sigma_z,F(g),F(f)}]{} & (\sigma_z\circ F(g))\circ F(f)
\end{array}
$$

when it is evaluated at the object (g, f) *in* $\mathcal{B}_2$*,*

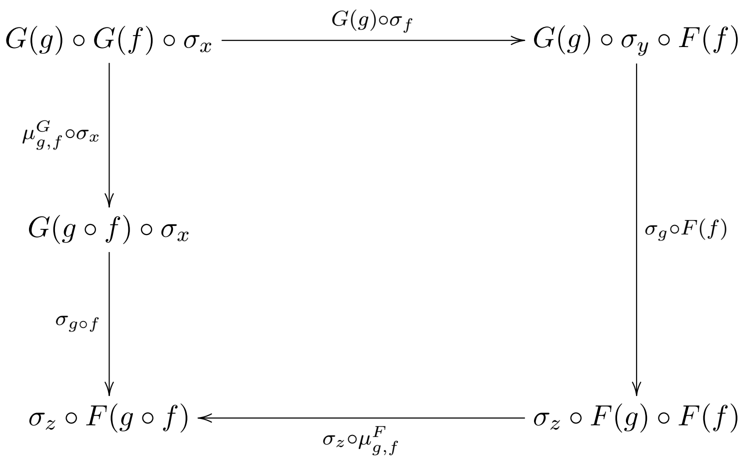
- *a commutative diagram*

$$
\begin{array}{ccccc}
i'_{G(x)}\circ\sigma_x & \xrightarrow{\lambda'_{\sigma_x}} & \sigma_x & \xrightarrow{\rho'^{-1}_{\sigma_x}} & \sigma_x\circ i'_{F(x)} \\
{\scriptstyle \eta^G_x\circ\sigma_x}\downarrow & & & & \downarrow{\scriptstyle \sigma_x\circ\eta^F_x} \\
G(i_x)\circ\sigma_x & & \xrightarrow[\sigma_{i_x}]{} & & \sigma_x\circ F(i_x)
\end{array}
$$

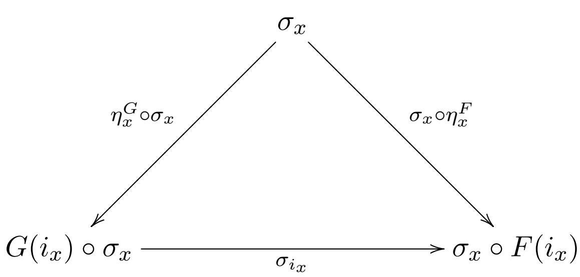
Remark 7.3. *If both $\mathcal{B}$ and $\mathcal{B}'$ are strict 2-categories then the above coherence becomes*



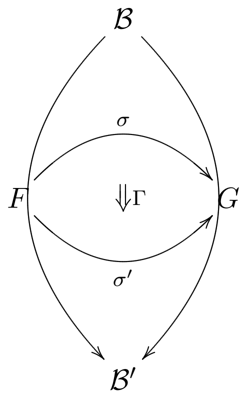
which is equivalent to the commutative diagram



The second coherence becomes the commutative diagram



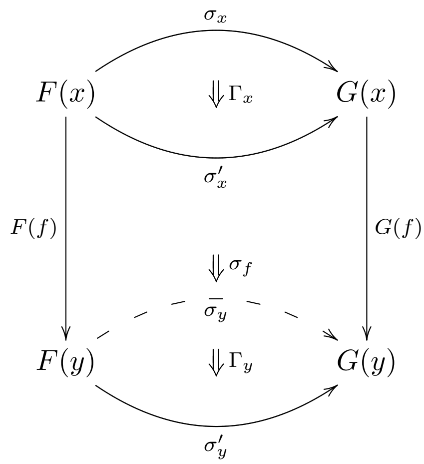
Definition 7.4. *A modification* $\Gamma\colon \sigma \to \sigma'$



consists of the following data:

- *a 2-morphism* $\Gamma_x\colon \sigma_x \to \sigma'_x$ *for each object* x *in* $\mathcal{B}$

such that the following diagram



which becomes a diagram

$$\begin{array}{ccc} G(f)\circ\sigma_x & \xrightarrow{G(f)\circ\Gamma_x} & G(f)\circ\sigma'_x \\ \Big\downarrow{\scriptstyle \sigma_f} & & \Big\downarrow{\scriptstyle \sigma'_f} \\ \sigma_y\circ F(f) & \xrightarrow[\Gamma_y\circ F(f)]{} & \sigma'_y\circ F(f) \end{array}$$

commutes.

8 Nerves of bicategories

In this section, we describe the nerve construction for bicategories, first given by Duskin in [32]. This construction is a natural outcome of various attempts to describe nerves of higher dimensional categories and groupoids, whose origin is a conjecture on a characterization of the nerve of *strict* n-category, in an unpublished work of Roberts. This conjecture was published by Street in [81], and it was finally proved by Verity [85], who characterized nerves of strict n-categories by means of special simplicial sets, which he called *complicial sets.*

We will derive the construction of the Duskin nerve for bicategories from the standard description of the geometric nerve (1.4). First we have a fully faithful functor

$$i\colon \Delta \to Bicat \tag{8.1}$$

where *Bicat* is a category of bicategories and their homomorphisms, as it is given in [15], so we consider each ordinal as a locally discrete 2-category. Thus the nerve of the bicategory $\mathcal{B}$ is a simplicial set $N_2\mathcal{B}_\bullet$ which is defined via the embedding (8.1) by

$$N_2\mathcal{B}_n := Hom_{Bicat}(i[n], \mathcal{B}). \tag{8.2}$$

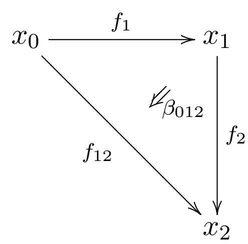
The 0-simplices of $N_2(\mathcal{B})$ are the objects of $\mathcal{B}$ and 1-simplices are directed line segments

$$x_0 \xrightarrow{f_{01}} x_1$$

which may be seen as homomorphisms $f\colon [1] \to \mathcal{B}$ from the locally discrete bicategory [1] to $\mathcal{B}$. Face maps are defined by $d_0(f_{01}) = x_1$ and $d_1(f_{01}) = x_0$. If x_0 is a 0-cell of $\mathcal{B}$ then we define the corresponding degenerate 1-simplex $s_0(x_0)$ by

$$x_0 \xrightarrow{id_{x_0}} x_0.$$

A typical 2-simplex is given by the triangle filled with a 2-morphism $\beta_{012}\colon f_{12} \circ f_{01} \Rightarrow f_{02}$



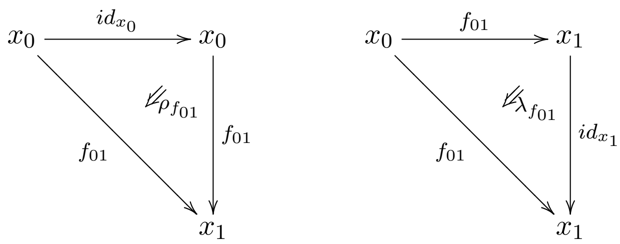
where $f_{ij}\colon [1] \to \mathcal{B}$ is a homomorphism for which $f_{ij}(0) = x_i$ and $f_{ij}(1) = x_j$. The face operators are defined as usual by

$$d_i(f_{12}, f_{02}, f_{01}, \beta_{012}) = \begin{cases} f_{12} & i = 0 \\ f_{02} & i = 1 \\ f_{01} & i = 2 \end{cases}$$

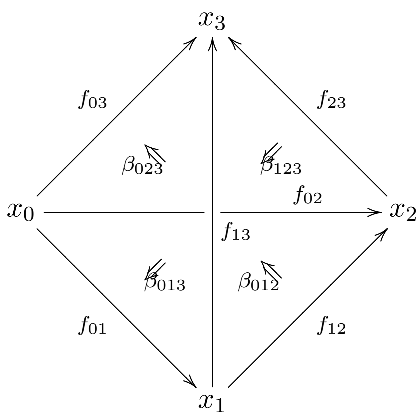
while for a 1-cell $x_0 \xrightarrow{f_{01}} x_1$ the degeneracy operators are defined by

$$s_0(f_{01}) = \rho_{f_{01}}$$
$$s_1(f_{01}) = \lambda_{f_{01}}$$

which are the two 2-simplices



respectively, where the 1-morphisms $\rho_{f_{01}} : f_{01} \circ id_{x_0} \to f_{01}$ and $\lambda_{f_{01}} : id_{x_1} \circ f_{01} \to f_{01}$ are the components of the right and left identity natural isomorphisms in $\mathcal{B}$. The general 3-simplex is of the form



such that we have an identity

$$\beta_{023}(\beta_{012} \circ f_{23})\alpha_{0123} = \beta_{013}(\beta_{123} \circ f_{01})$$

where $\alpha_{0123} : (f_{23} \circ f_{12}) \circ f_{01} \Rightarrow f_{23} \circ (f_{12} \circ f_{01})$, and this condition follows directly from the coherence for the composition. Since this construction is given by the geometric nerve (8.2) it follows immediately that the Duskin nerve is functorial with respect to homomorphisms of bicategories, which leads us to the following result.

Theorem 8.1. *The Duskin nerve functor $N_2 : Bicat \to \mathcal{S}Set$ is fully faithful.*

Proof. An analogous proof that the geometric nerve provides a fully faithful functor on the category $2 - Cat_{lax}$ of 2-categories and normal lax 2-functors is given in [17]. Then the statement of the theorem follows immediately for a category *Bicat* of bicategories and normal homomorphisms. □

9 Internal bicategories

When he introduced bicategories, Bénabou also internalized the notion, so that he gave the definition of an internal category by a long list of diagrams. All the diagrams in this chapter are borrowed from his paper [15] which was necessary in order to define later a small 2-fibration corresponding to an internal bicategory. Throughout this section, $\mathcal{E}$ will denote a finitely complete category.

Definition 9.1. *A bigraph $\mathcal{B}$ in $\mathcal{E}$ is the diagram of objects and morphisms in $\mathcal{C}$*

$$B_2 \overset{s_1}{\underset{t_1}{\rightrightarrows}} B_1 \overset{s_0}{\underset{t_0}{\rightrightarrows}} B_0 \tag{9.1}$$

such that two identities $s_0 s_1 = s_0 t_1$ and $t_0 s_1 = t_0 t_1$ hold. If we have additionally two morphisms

$$B_2 \xleftarrow{i_1} B_1 \xleftarrow{i_0} B_0$$

such that $s_0 i_0 = id_{B_0} = t_0 i_0$ and $s_1 i_1 = id_{B_1} = t_1 i_1$, we call a diagram $\mathcal{B}$ a reflexive bigraph.

From the definition it follows that by composing structure morphisms we have only two morphisms from B_2 to B_0, for which we will sometimes abuse the notation and write $s_0, t_0 \colon B_2 \to B_0$. Thus we will have pullbacks

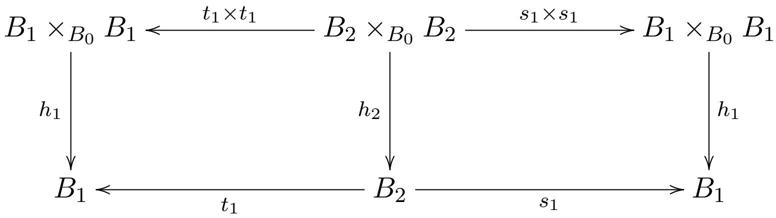
$$\begin{array}{ccc} B_1 \times_{B_0} B_1 & \xrightarrow{pr_1} & B_1 \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle s_0} \\ B_1 & \xrightarrow[t_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} B_2 \times_{B_0} B_2 & \xrightarrow{pr_1} & B_2 \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle s_0} \\ B_2 & \xrightarrow[t_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} B_2 \times_{B_1} B_2 & \xrightarrow{pr_1} & B_2 \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle s_1} \\ B_2 & \xrightarrow[t_1]{} & B_1 \end{array}$$

Definition 9.2. *A composition on a bigraph $\mathcal{B}$ in $\mathcal{E}$ consists of morphisms*

$$\begin{aligned} h_1 &\colon B_1 \times_{B_0} B_1 \to B_1 \\ h_2 &\colon B_2 \times_{B_0} B_2 \to B_2 \\ v &\colon B_2 \times_{B_1} B_2 \to B_2 \end{aligned} \tag{9.2}$$

such that the following diagrams commute:

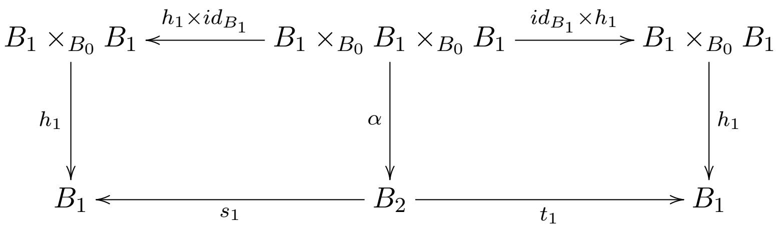
$$\begin{array}{ccccc} B_2 & \xleftarrow{pr_1} & B_2 \times_{B_1} B_2 & \xrightarrow{pr_2} & B_2 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle s_1} \\ B_1 & \xleftarrow[t_1]{} & B_2 & \xrightarrow[s_1]{} & B_1 \end{array}$$



Definition 9.3. *Let $\mathcal{B}$ be a bigraph in $\mathcal{E}$ with a composition. An associator is a morphism*

$$\alpha \colon B_1 \times_{B_0} B_1 \times_{B_0} B_1 \to B_2 \tag{9.3}$$

such that the diagram

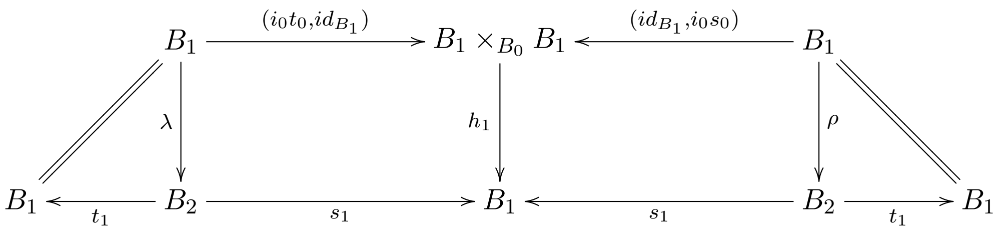


commutes.

Definition 9.4. *Let $\mathcal{B}$ be a reflexive bigraph (9.1) with a composition (9.2). Left and right identities are morphisms*

$$\begin{aligned} \lambda &\colon B_1 \to B_2 \\ \rho &\colon B_1 \to B_2 \end{aligned} \tag{9.4}$$

such that the diagram



commutes.

We use the above data in order to define internal categories.

Definition 9.5. *An internal bicategory $\mathcal{B}$ in $\mathcal{E}$ is the reflexive bigraph (9.1) with a composition (9.2), associator (9.3) and left and right identities (9.4) satisfying following coherence conditions:*

i) associativity law for vertical composition

$$\begin{array}{ccc} B_2 \times_{B_1} B_2 \times_{B_1} B_2 & \xrightarrow{v \times id_{B_2}} & B_2 \times_{B_1} B_2 \\ {\scriptstyle id_{B_2} \times v}\downarrow & & \downarrow{\scriptstyle v} \\ B_2 \times_{B_1} B_2 & \xrightarrow[v]{} & B_2 \end{array} \tag{9.5}$$

ii) left and right identity laws for the vertical composition

$$\begin{array}{ccccc} B_2 & \xrightarrow{(i_1 t_1, id_{B_2})} & B_2 \times_{B_1} B_2 & \xrightarrow{(id_{B_2}, i_1 s_1)} & B_2 \\ & \searrow{=} & \downarrow{\scriptstyle v} & {=}\swarrow & \\ & & B_2 & & \end{array} \tag{9.6}$$

iii) (Godement) interchange law

$$\begin{array}{ccccc} (B_2 \times_{B_0} B_2) \times_{B_1 \times_{B_0} B_1} (B_2 \times_{B_0} B_2) & & \xrightarrow{\tau} & & (B_2 \times_{B_1} B_2) \times_{B_0} (B_2 \times_{B_1} B_2) \\ {\scriptstyle h_2 \times_{B_1} h_2}\downarrow & & & & \downarrow{\scriptstyle v \times_{B_0} v} \\ B_2 \times_{B_1} B_2 & \xrightarrow[v]{} & B_2 & \xleftarrow[h_2]{} & B_2 \times_{B_0} B_2 \end{array} \tag{9.7}$$

where τ is the canonical morphism given by $((s_1, t_1), (s_0, t_0)) \mapsto ((s_1, s_0), (t_1, t_0))$, and the morphism $h_2 \times_{B_1} h_2 \colon (B_2 \times_{B_0} B_2) \times_{B_1 \times_{B_0} B_1} (B_2 \times_{B_0} B_2) \to B_2 \times_{B_1} B_2$ is the unique one making the diagram

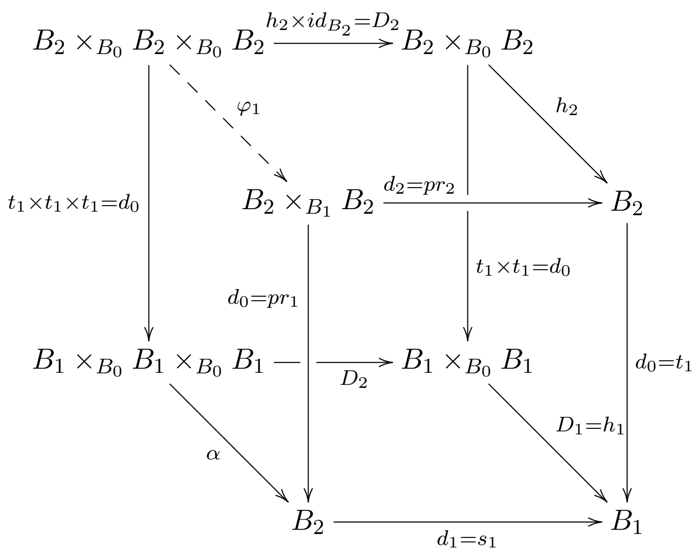
$$\begin{array}{ccc} (B_2 \times_{B_0} B_2) \times_{B_1 \times_{B_0} B_1} (B_2 \times_{B_0} B_2) & \hookrightarrow & (B_2 \times_{B_0} B_2) \times (B_2 \times_{B_0} B_2) \\ {\scriptstyle h_2 \times_{B_1} h_2}\downarrow & & \downarrow{\scriptstyle h_2 \times h_2} \\ B_2 \times_{B_1} B_2 & \hookrightarrow & B_2 \times B_2 \end{array}$$

commutative, where the horizontal arrows are the canonical monomorphisms of pullbacks into products.

iv) compatibility of horizontal composition with vertical identities

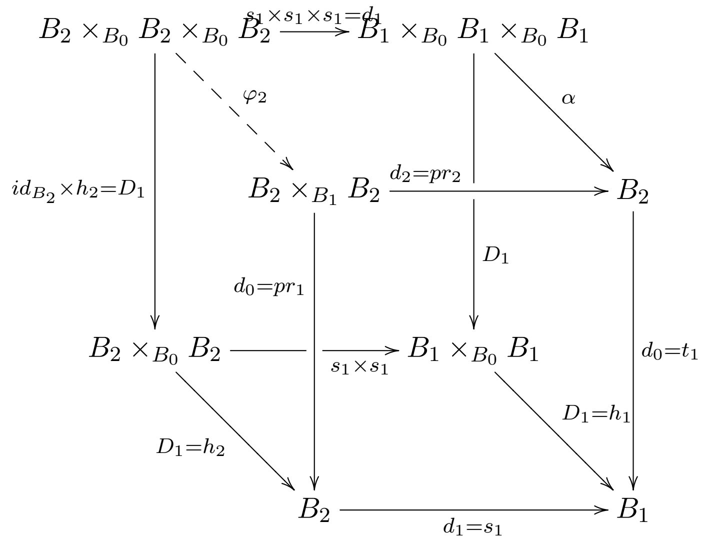
$$\begin{array}{ccc} B_1 \times_{B_0} B_1 & \xrightarrow{h_1} & B_2 \times_{B_1} B_2 \\ {\scriptstyle i_1 \times i_1}\downarrow & & \downarrow{\scriptstyle i_1} \\ B_2 \times_{B_0} B_2 & \xrightarrow[h_2]{} & B_2 \end{array} \tag{9.8}$$

For the next axiom, we consider a unique morphism $\varphi_1 \colon B_2 \times_{B_0} B_2 \times_{B_0} B_2 \to B_2 \times_{B_1} B_2$ *obtained from the universal property of the pullback in the front face of the diagram*



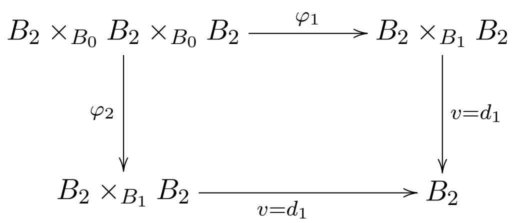
such that the diagram commutes, and a morphism $\varphi_2 \colon B_2 \times_{B_0} B_2 \times_{B_0} B_2 \to B_2 \times_{B_1} B_2$

obtained from the same universal property as in the diagram



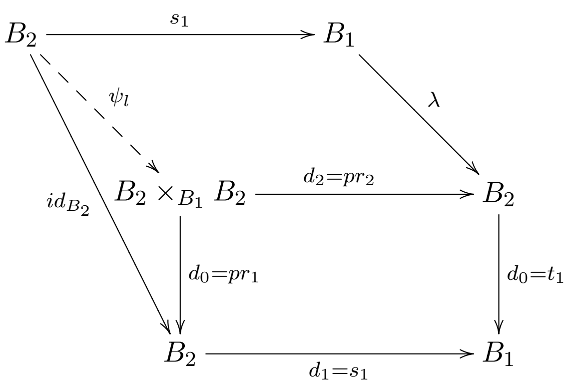
such that the diagram commutes. Then we can express the next axiom:

v) Naturality law for the associativity

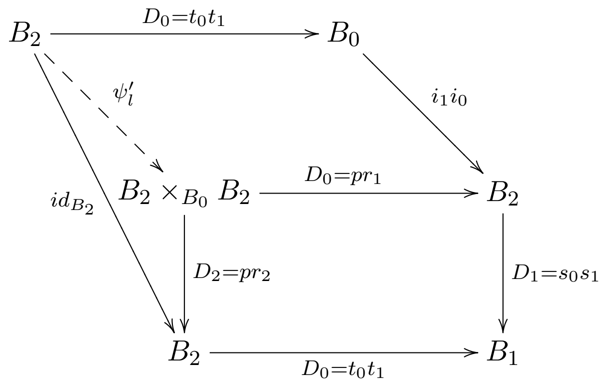


(9.9)

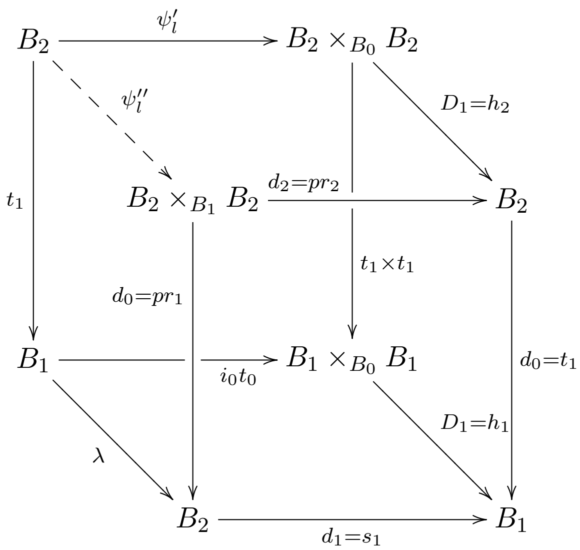
The next axiom, uses the morphism $\psi_l \colon B_2 \to B_2 \times_{B_1} B_2$ *obtained from the universal property of the pullback in the diagram*



and the morphism $\psi_l' \colon B_2 \to B_2 \times_{B_1} B_2$ *obtained from the universal property of the pullback*

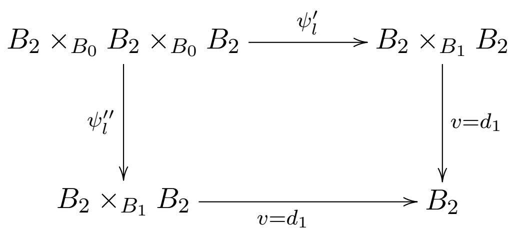


This two morphisms generate a unique morphism $\psi_l'' \colon B_2 \to B_2 \times_{B_1} B_2$ *from the pullback*



From this data we have a new axiom:

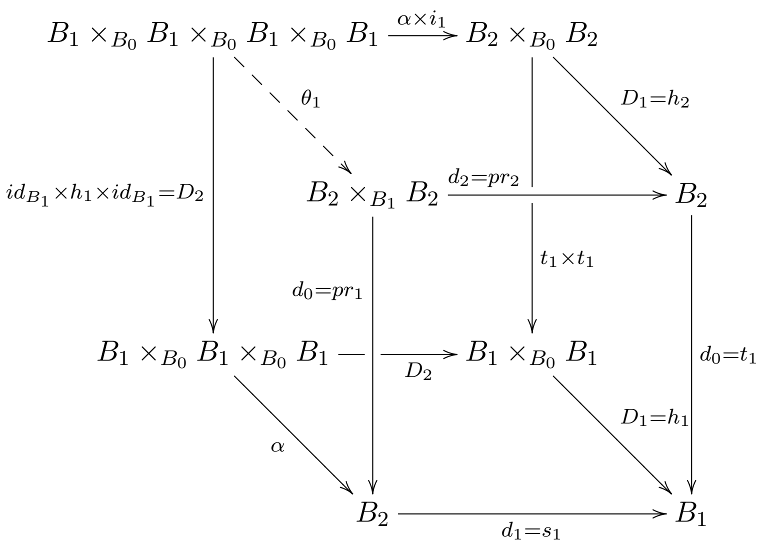
$vi)_l$ *naturality of the left identity*



(9.10)

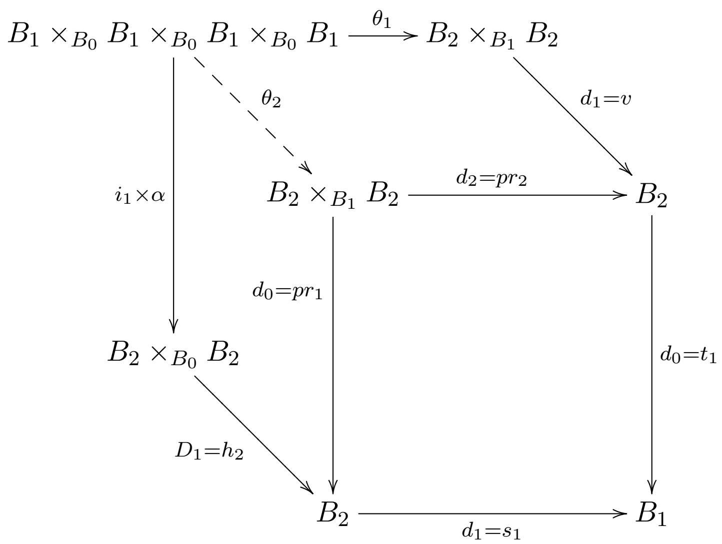
There exists also a similar axiom vi)$_r$ which says that the right identity is natural.

The next axiom, uses the morphism $\theta_1\colon B_1\times_{B_0}B_1\times_{B_0}B_1\times_{B_0}B_1\to B_2\times_{B_1}B_2$ *obtained from the universal property of the pullback in the diagram*



(9.11)

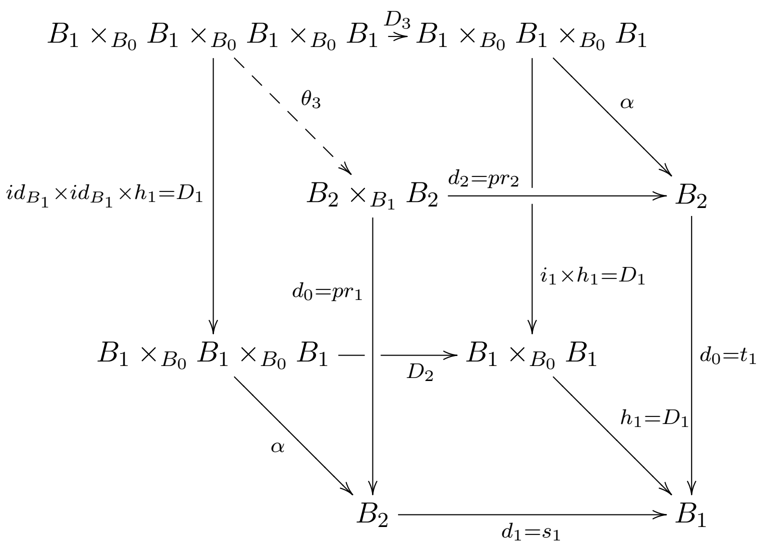
such that the diagram commutes, and a morphism $\theta_2\colon B_1\times_{B_0}B_1\times_{B_0}B_1\times_{B_0}B_1\to B_2\times_{B_1}B_2$ *obtained from the same universal property as in the diagram*



(9.12)

such that the diagram commutes.

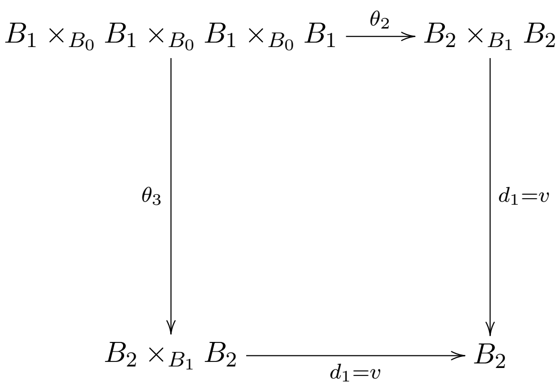
We also use a morphism $\theta_3\colon B_1\times_{B_0}B_1\times_{B_0}B_1\times_{B_0}B_1\to B_2\times_{B_1}B_2$ *obtained from the same universal property as in the diagram*



(9.13)

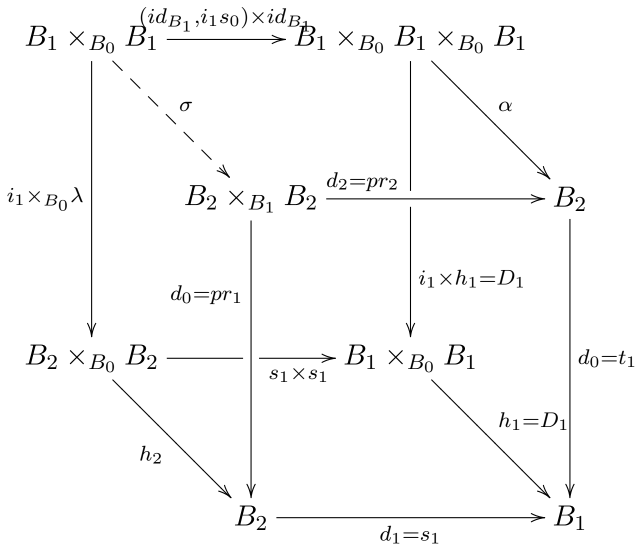
such that the diagram commutes. Then we can express the next axiom:

vii) the associativity coherence law



(9.14)

From the commutativity of the exterior of the diagram



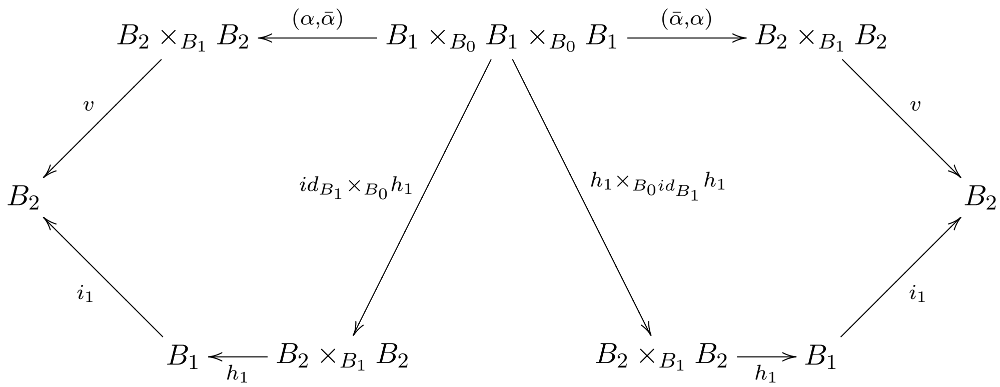
we have another axiom:

viii) the coherence for left and right identity

$$\begin{array}{ccc} B_1 \times_{B_0} B_1 & \xrightarrow{\sigma} & B_2 \times_{B_1} B_2 \\ \Big\downarrow{\scriptstyle \rho \times_{B_0} i_1} & & \Big\downarrow{\scriptstyle d_1 = v} \\ B_2 \times_{B_1} B_2 & \xrightarrow[d_1 = v]{} & B_2 \end{array} \tag{9.15}$$

There exists a unique $\bar{\alpha}\colon B_1 \times_{B_0} B_1 \times_{B_0} B_1 \to B_2$ *such that* $d_0\bar{\alpha} = d_1\alpha$ *and* $d_1\bar{\alpha} = d_0\alpha$*.*

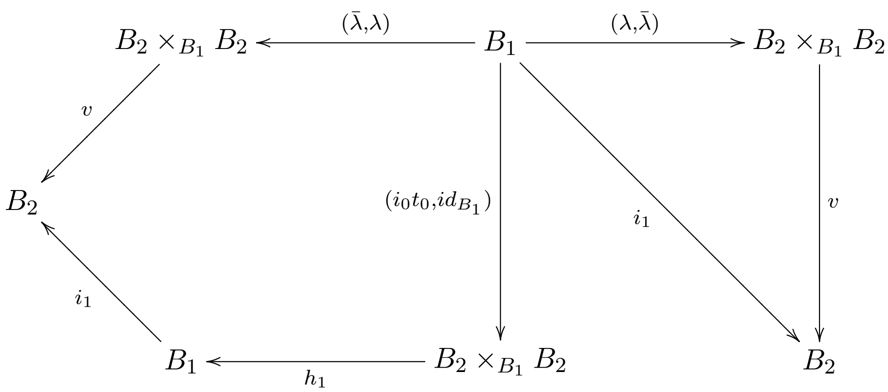
ix) invertibility of associativity



(9.16)

There exists a unique morphism $\bar{\lambda}\colon B_1 \to B_2$ *such that* $d_0\bar{\lambda} = d_1\lambda$ *and* $d_1\bar{\lambda} = d_0\lambda$.

$x)_l$ invertibility of the left identity



(9.17)

Also there exists a similar axiom $x)_r$ for the invertibility of the right identity.

10 Pseudosimplicial categories

In this chapter, we use a *supercoherence* developed by Jardine in [50] which associates to an internal bicategory $\mathcal{B}$ given by the diagram of categories and functors

$$\mathcal{B}_1 \underset{D_0}{\overset{D_1}{\rightleftarrows\!\!\rightarrow}} \mathcal{B}_0 \tag{10.1}$$

a pseudosimplicial category called the *pseudosimplicial nerve* or *supercoherent nerve* of $\mathcal{B}$

$$\mathcal{B}_0 \underset{D_0}{\overset{D_1}{\Longleftarrow\!\!\Rightarrow}} \mathcal{B}_1 \underset{D_0}{\overset{D_2}{\Lleftarrow\!\!\Rrightarrow}} \mathcal{B}_2 \underset{D_0}{\overset{D_3}{\Lleftarrow\!\!\Rrightarrow}} \mathcal{B}_3 \ldots \tag{10.2}$$

Here, the category $\mathcal{B}_1$ is the category of morphisms of the bicategory $\mathcal{B}$ and the category $\mathcal{B}_0$ is the image $\mathcal{D}(B_0)$ of the discrete functor $\mathcal{D}\colon \mathcal{E} \to \mathrm{Cat}(\mathcal{E})$ which just turns an object of $\mathcal{E}$ into a discrete internal category in $\mathcal{E}$. Source functor D_1 is defined by $D_1 := d_1^0\colon B_1 \to B_0$ and $D_1 := d_1^0 d_1^1 = d_1^0 d_0^1\colon B_2 \to B_0$, and a target functor D_0 is defined by $D_0 := d_1^0\colon B_1 \to B_0$ and $D_1 := d_0^0 d_1^1 = d_0^0 d_1^1\colon B_2 \to B_0$, where we used the same notation for both components of the functor, and we will constantly use this convention elsewhere. Also, the unit functor $S_0\colon \mathcal{B}_0 \to \mathcal{B}_1$ is defined by $S_0 := i_0\colon B_0 \to B_1$ and $S_0 := i_0 i_1\colon B_1 \to B_2$ on the level of objects and morphisms respectively, where $i_0\colon B_0 \to B_1$ and $i_1\colon B_1 \to B_2$ are unit sections. The vertex of the following pullback of functors

$$\begin{array}{ccc} \mathcal{B}_2 & \xrightarrow{D_0} & \mathcal{B}_1 \\ {\scriptstyle D_2}\downarrow & & \downarrow{\scriptstyle D_1} \\ \mathcal{B}_1 & \xrightarrow[D_0]{} & \mathcal{B}_0 \end{array} \tag{10.3}$$

is (isomorphic to) the category $\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ whose objects and morphisms are horizontally composable pairs of 1-morphisms and 2-morphisms respectively, with vertical composition. We always use the following convention: for any functor $F\colon \mathcal{C} \to B_0$, the first of the symbols

$$\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \qquad\qquad \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{C}$$

will denote the pullback of F and D_0, and the second one that of D_1 and F, so that two projections in the above diagram are defined by $D_0 = Pr_1$ and $D_2 = Pr_2$. The third functor $D_1\colon \mathcal{B}_2 \to \mathcal{B}_1$ from $\mathcal{B}_2$ is given by the horizontal composition $H\colon \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{B}_1$.

These three functors $D_0, D_1, D_2 \colon \mathcal{B}_2 \to B_1$ satisfy the following three simplicial identities

$$\begin{aligned} D_0 D_1 &= D_0 D_0 \\ D_1 D_2 &= D_1 D_1 \\ D_0 D_2 &= D_1 D_0 \end{aligned} \tag{10.4}$$

with target and source functors $D_0, D_1 \colon \mathcal{B}_1 \to B_0$, where the first and the second identity is the compatibility of the horizontal composition with the target and source functors respectively, and the third identity is given by the pullback (10.3).

Two degeneracy functors $S_0, S_1 \colon \mathcal{B}_1 \to B_2$ are defined respectively by two compositions

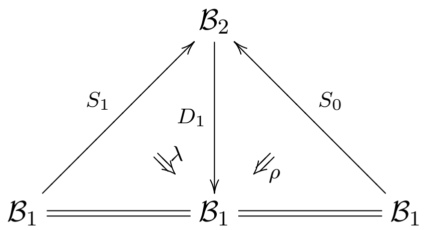
$$\mathcal{B}_1 \xrightarrow{(D_0, Id_{\mathcal{B}_1})} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_0 \xrightarrow{S_0 \times Id_{\mathcal{B}_1}} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$$

$$\mathcal{B}_1 \xrightarrow{(Id_{\mathcal{B}_1}, D_1)} \mathcal{B}_0 \times_{\mathcal{B}_0} \mathcal{B}_1 \xrightarrow{Id_{\mathcal{B}_1} \times S_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$$

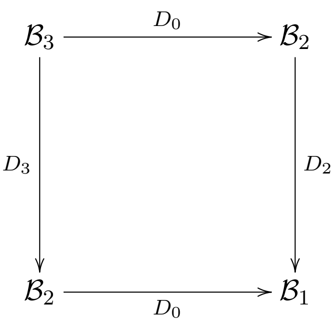
so that for any 1-morphism $f \colon x \to y$ in $\mathcal{B}$ we have $S_0(f) = (f, i_x)$ and $S_1(f) = (i_y, f)$. The left identity $\lambda \colon H(IT \times Id_{\mathcal{B}_1}) \Rightarrow Id_{\mathcal{B}_1}$ and the right identity $\rho \colon H(Id_{\mathcal{B}_1} \times IS) \Rightarrow Id_{\mathcal{B}_1}$ give two pseudosimplicial identities

$$\begin{aligned} \lambda &\colon D_1 S_1 \Rightarrow Id_{\mathcal{B}_1} \\ \rho &\colon D_1 S_0 \Rightarrow Id_{\mathcal{B}_1} \end{aligned} \tag{10.5}$$

which are described by the diagram



The category $\mathcal{B}_3$ of horizontally composable triples of morphisms is defined by the pullback



(10.6)

whose vertex is (isomorphic to) the category $\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$, and the two functors $D_0, D_3 \colon \mathcal{B}_3 \to B_1$ are defined by projections $Pr_{12}, Pr_{23} \colon \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \to B_1$ respectively. The associativity coherence is a natural isomorphism $\alpha \colon H(H \times Id_{\mathcal{B}_1}) \Rightarrow H(Id_{\mathcal{B}_1} \times H)$ given by the diagram

$$\begin{array}{ccc} \mathcal{B}_3 & \xrightarrow{D_2} & \mathcal{B}_2 \\ {\scriptstyle D_1}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle D_1} \\ \mathcal{B}_2 & \xrightarrow[D_1]{} & \mathcal{B}_1 \end{array} \tag{10.7}$$

in which two functors $D_1, D_2 \colon \mathcal{B}_3 \to B_2$ are defined by $H(Id_{\mathcal{B}_1} \times H), H(H \times Id_{\mathcal{B}_1}) \colon \mathcal{B}_3 \to B_2$ respectively. The associativity natural isomorphism give a pseudosimplicial isomorphism

$$\alpha \colon D_1 D_2 \Rightarrow D_1 D_1 \tag{10.8}$$

which is the only nontrivial relation among face pseudosimplicial identities from $\mathcal{B}_3$ to B_1

$$D_i D_j = D_{j-1} D_i \qquad (i < j, i \neq 1) \tag{10.9}$$

The sequence of categories $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$ is a part of a diagram of categories and functors

$$\mathcal{B}_0 \underset{D_0}{\overset{D_1}{\leftleftarrows\!\!\rightrightarrows}} \mathcal{B}_1 \underset{D_0}{\overset{D_2}{\equiv}} \mathcal{B}_2 \underset{D_0}{\overset{D_3}{\equiv}} \mathcal{B}_3 \ldots \tag{10.10}$$

in $\mathrm{Cat}(\mathcal{E})$, where we denoted just extremal face functors $D_0, D_n \colon \mathcal{E}_n \to \mathcal{E}_{n-1}$, while we omitted degeneracy functors $S_i \colon \mathcal{E}_n \to \mathcal{E}_{n+1}$ for $0 \leq i \leq n$. These functors do not satisfy simplicial identities on the nose, but they constitute the so called *pseudosimplicial category.*

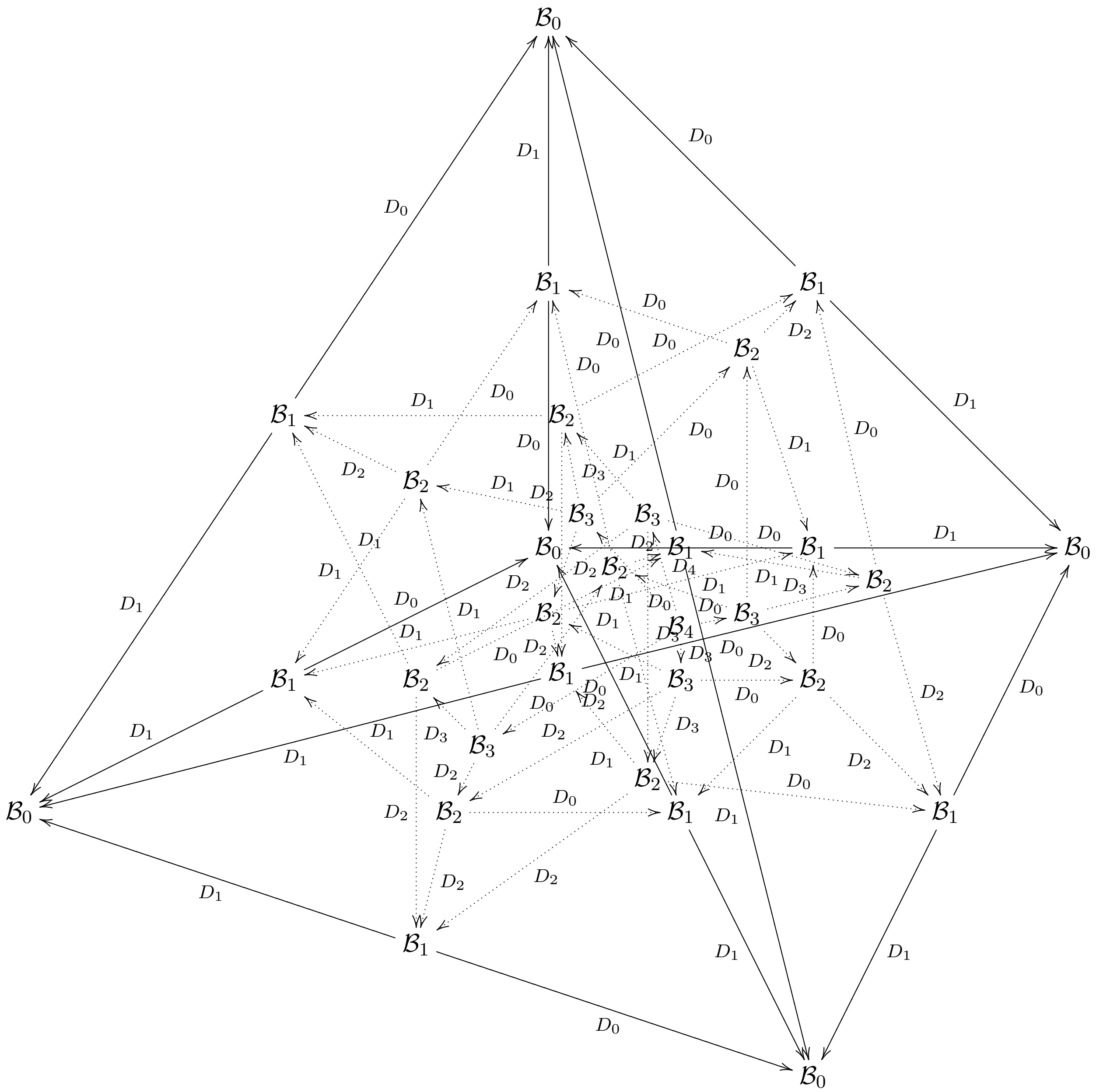
Definition 10.1. *A pseudosimplicial category* $\mathcal{B}_\bullet$ *is a pseudofunctor* $\mathcal{B} \colon \Delta^{op} \to \mathrm{Cat}$ *from the skeletal simplicial category* Δ *to the 2-category* Cat *of small 2-categories.*

The sequence (10.10) is the pseudosimplicial category $\mathcal{B}_\bullet$ called the *pseudosimplicial nerve* of the bicategory $\mathcal{B}$ and is such that it satisfies *pseudosimplicial identities*

$$\begin{array}{ll} \alpha \colon D_i D_j \Rightarrow D_{j-1} D_i & (i < j) \\ \alpha \colon S_i S_j \Rightarrow S_{j+1} S_i & (i \leq j) \\ \alpha \colon D_i S_j \Rightarrow S_{j-1} D_i & (i \leq j) \\ \alpha \colon D_i S_j \Rightarrow Id & (i = j, i = j+1) \\ \alpha \colon D_i S_j \Rightarrow S_{j-1} D_i & (i > j+1) \end{array} \tag{10.11}$$

The only nontrivial simplicial natural isomorphisms in the pseudosimplicial nerve $\mathcal{B}_\bullet$ of the bicategory $\mathcal{B}$ are provided with associativity and left and right identity isomorphisms, and they satisfy coherence conditions appropriate for those in the definition of the bicategory.

The sequence of categories (10.10) may be seen as a barycentric subdivision of the 4-simplex



in a similar way by which we have seen a sequence of objects (6.1) as a data for an internal category $\mathcal{C}$. This time, certain faces of the above 4-simplex will again be pullbacks corresponding to categories (10.3) and (10.6) of horizontally composable morphisms, but some other faces which do not commute correspond to an associativity coherence (10.7).

11 Small 2-fibrations

From an internal bicategory $\mathcal{B}$, we will construct the fibered bicategory $F_{\mathcal{B}}\colon \mathcal{FB} \to \mathcal{E}$ as follows. The objects of $\mathcal{FB}$ are pairs (I, i), where I is an object in $\mathcal{E}$, and $i\colon I \to B_0$ is a morphism in $\mathcal{E}$. For any two such objects (I, i) and (J, j), a 1-morphism in $\mathcal{FB}$ is given by a pair $(a, f)\colon (I, i) \to (J, j)$, which consists of two morphisms $a\colon I \to J$ and $f\colon I \to B_1$ in $\mathcal{E}$, such that $D_1 f = i$ and $D_0 f = ja$. A 2-morphism $\phi\colon (a, f) \Rightarrow (a', f')\colon (I, i) \to (J, j)$ in $\mathcal{FB}$ is a morphism $\phi\colon I \to B_2$ in $\mathcal{E}$ such that $d_1\phi = f$ and $d_0\phi = f'$. It is then necessary that we have $ja = ja'$ since we have an identity

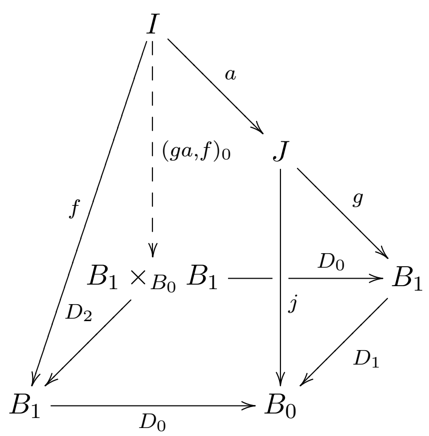
$$ja = D_0 f = D_0 d_1 \phi = D_0 d_0 \phi = D_0 f' = ja'$$

Remark 11.1. *The above definition of 1-morphisms is not entirely appropriate because a general 1-morphism $(a, f)\colon (I, i) \to (J, j)$ is fully determined by a triple (a, f, j), since we cannot extract its 0-target (specially a morphism $j\colon J \to B_0$) by the structure of $\mathcal{E}$ and $\mathcal{B}$, like we could for the 0-source, by defining $s_0(a, f) = (s(f), s_0 f)$. Similar remark holds for 2-morphisms also. However, we will use an abbreviated form for morphisms in $\mathcal{FB}$ in order to avoid to many labels.*

For any two composable 1-morphisms in $\mathcal{FB}$

$$(I, i) \xrightarrow{(a,f)} (J, j) \xrightarrow{(b,g)} (K, k)$$

the composition is defined by $(b, g) \circ (a, f) := (ba, g \circ f)$ where the morphism $g \circ f\colon I \to B_1$ is defined by $g \circ f := D_1(ga, f)_0$, and $(ga, f)_0\colon I \to B_1 \times_{B_0} B_1$ is the unique morphism given by the universal property of the pullback



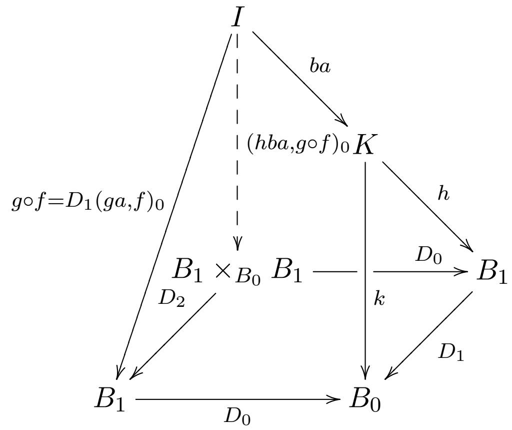
obtained from the factorization $D_0 f = ja = D_1 ga$, for which we have following identities

$$\begin{aligned} D_0(ga, f)_0 &= ga \\ D_1(ga, f)_0 &= g \circ f \\ D_2(ga, f)_0 &= f. \end{aligned} \tag{11.1}$$

Let's consider composable triple of 1-morphisms

$$(I,i) \xrightarrow{(a,f)} (J,j) \xrightarrow{(b,g)} (K,k) \xrightarrow{(c,h)} (L,l)$$

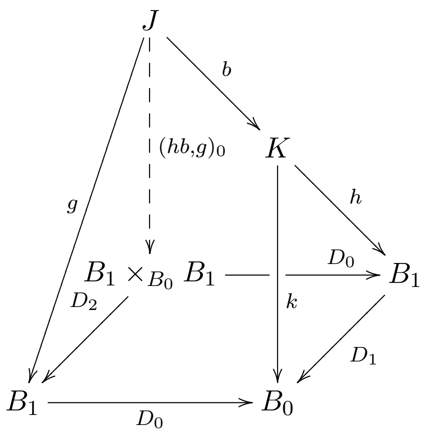
The 1-morphism $(c,h)[(b,g)(a,f)]$ is given by $(c,h)(ba, g\circ f) := (cba, h\circ(g\circ f))$ where $h\circ(g\circ f) := D_1(hba, g\circ f)_0$ and the morphism $(hba, g\circ f)_0 \colon I \to B_1 \times_{B_0} B_1$ is the unique one given by the universal property of the pullback

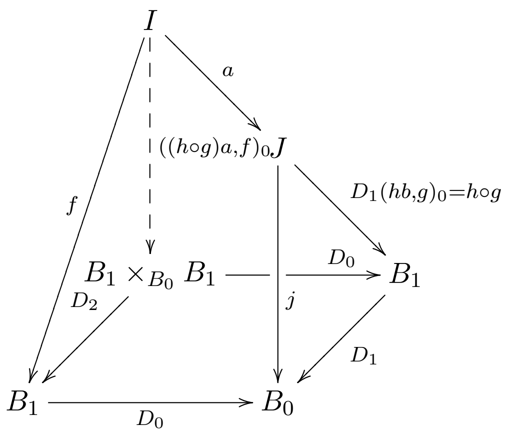


obtained from the factorization $D_0D_1(ga,f)_0 \overset{(10.4)}{=} D_0D_0(ga,f)_0 \overset{(11.1)}{=} D_0ga = kba = D_1hba$, such that we have following identities

$$\begin{aligned} &D_0(hba, g\circ f)_0 = hba \\ &D_1(hba, g\circ f)_0 = D_1(hba, D_1(ga,f)_0)_0 = h\circ(g\circ f) \\ &D_2(hba, g\circ f)_0 = D_1(ga,f)_0 = g\circ f. \end{aligned} \tag{11.2}$$

On the other side, On the other side, from $(c,h)(b,g) := (cb, h\circ g)$ we have an identity $[(c,h)(b,g)](a,f) = (cb, h\circ g)(a,f) = (cba, (h\circ g)\circ f)$, where $h\circ g := D_1(hb,g)_0$ and $(h\circ g)\circ f := D_1((h\circ g)a, f)_0$ are two 1-morphisms in $\mathcal{FB}$ obtained from two pullbacks





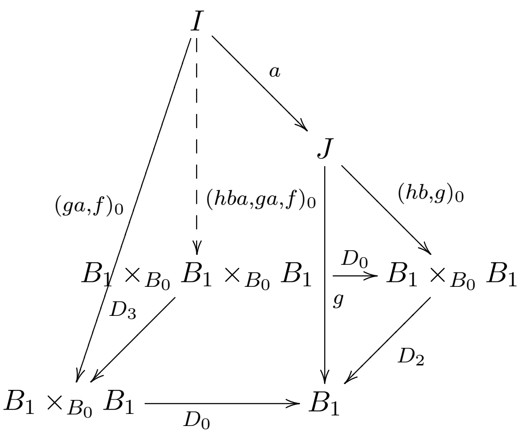
whose diagonals are $D_0 g = kb = D_1 hb$ and $D_0 f = ja = D_1 ga = D_1 D_2(hb, g)_0 a \overset{(10.4)}{=} D_1 D_1(hb, g)_0 a$, respectively. From the first diagram we have the following identities

$$\begin{aligned} D_0(hb, g)_0 &= hb \\ D_1(hb, g)_0 &= h \circ g \\ D_2(hb, g)_0 &= g \end{aligned} \tag{11.3}$$

and from the second diagram we have the following identities

$$\begin{aligned} D_0((h \circ g)a, f)_0 &= D_1(hb, g)_0 = h \circ g \\ D_1((h \circ g)a, f)_0 &= D_1(D_1(hb, g)_0 a, f) = (h \circ g) \circ f \\ D_2((h \circ g)a, f)_0 &= f. \end{aligned} \tag{11.4}$$

The morphism $(hba, ga, f)_0 \colon I \to B_1 \times_{B_0} B_1 \times_{B_0} B_1$ is obtained from the following pullback



and the factorization $D_1(ga, f)_0 = ga = D_2(hb, g)_0 a$, and it is a unique one for which identities

$$\begin{aligned} D_0(hba, ga, f)_0 &= (hb, g)_0 a \\ D_3(hba, ga, f)_0 &= (ga, f)_0 \end{aligned} \tag{11.5}$$

are satisfied. Then we use this morphism to define the corresponding component of an *associativity coherence 2-morphism* $\alpha_{h,g,f} \colon [(c, h) \circ (b, g)] \circ (a, f) \Rightarrow (c, h) \circ [(b, g) \circ (a, f)]$

$$\alpha_{h,g,f} := \alpha(hba, ga, f)_0 \tag{11.6}$$

where $\alpha \colon B_1 \times_{B_0} B_1 \times_{B_0} B_1 \to B_2$ is defined in (9.3). The morphism $D_1(hba, ga, f)_0$ satisfies

$$\begin{aligned} D_0 D_1(hba, ga, f)_0 &\overset{(10.9)}{=} D_0 D_0(hba, ga, f)_0 \overset{(11.5)}{=} D_0(hb, g)_0 a \overset{(11.3)}{=} hba \\ D_2 D_1(hba, ga, f)_0 &\overset{(10.9)}{=} D_1 D_3(hba, ga, f)_0 \overset{(11.5)}{=} D_1(ga, f)_0 \overset{(11.1)}{=} g \circ f \end{aligned}$$

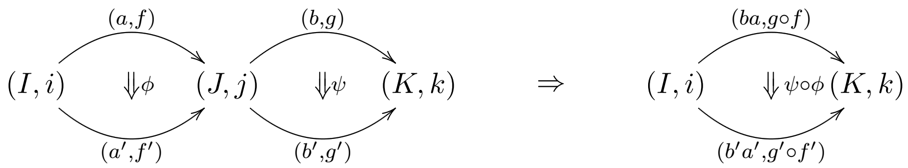
and since from (11.2) we know that the morphism $(hba, g\circ f)_0$ is the unique one satisfying these identities, we conclude $D_1(hba, ga, f)_0 = (hba, g\circ f)_0$. The morphism $D_2(hba, ga, f)_0$ satisfies following identities

$$\begin{aligned} D_0D_2(hba, ga, f)_0 &\overset{(10.9)}{=} D_1D_0(hba, ga, f)_0 \overset{(11.5)}{=} D_1(hb, g)_0 a \overset{(11.3)}{=} h\circ g \\ D_2D_2(hba, ga, f)_0 &\overset{(10.9)}{=} D_2D_3(hba, ga, f)_0 \overset{(11.5)}{=} D_2(ga, f)_0 \overset{(11.1)}{=} f \end{aligned}$$

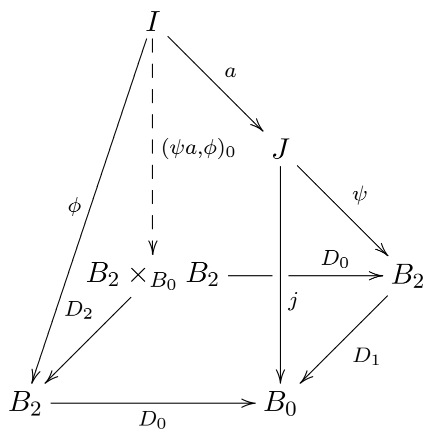
and since from (11.4) we know that the morphism $((h\circ g)a, f)_0$ is the unique one satisfying these identities, we conclude $D_2(hba, ga, f)_0 = ((h\circ g)a, f)_0$. Therefore we have identities

$$\begin{aligned} D_1(hba, ga, f)_0 &= (hba, g\circ f)_0 \\ D_2(hba, ga, f)_0 &= ((h\circ g)a, f)_0 \end{aligned} \tag{11.7}$$

The horizontal composition of 2-morphisms in $\mathcal{FB}$



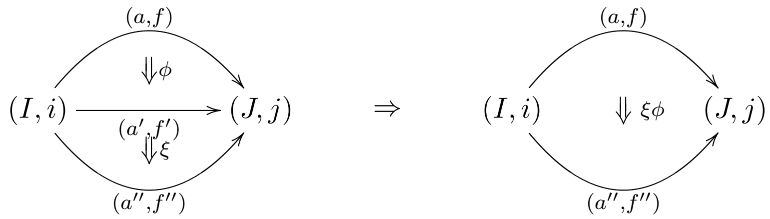
is defined by $\psi\circ\phi := d_1(\psi a, \phi)_0$, where $(\psi a, \phi)_0 \colon I \to B_2 \times_{B_0} B_2$ is the unique morphism obtained from the factorization $D_0\phi = ja = D_1\psi a$ in the pullback



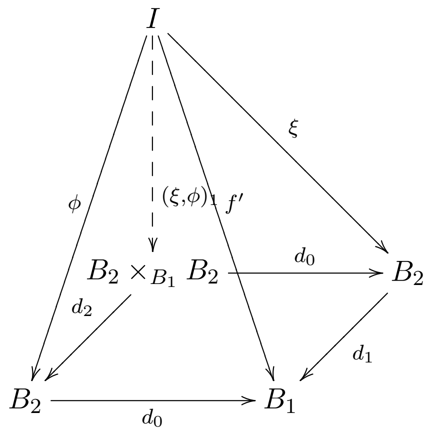
satisfying the following identities

$$\begin{aligned} D_0(\psi a, \phi)_0 &= \psi a \\ D_1(\psi a, \phi)_0 &= \psi\circ\phi \\ D_2(\psi a, \phi)_0 &= \phi. \end{aligned} \tag{11.8}$$

The vertical composition of 2-morphisms in $\mathcal{FB}$



is given by the morphism $\xi\phi\colon I \to B_2$ defined by $\xi\phi := d_1(\xi,\phi)_1$ where the morphism $(\xi,\phi)_1\colon I \to B_2 \times_{B_1} B_2$ is the unique one obtained from the factorization $d_0\phi = f' = d_1\xi$



in the above pullback satisfying the following identities

$$\begin{aligned} d_0(\xi,\phi)_1 &= \xi \\ d_1(\xi,\phi)_1 &= \xi\phi \\ d_2(\xi,\phi)_1 &= \phi. \end{aligned} \tag{11.9}$$

Remark 11.2. *The statement of the following theorem, will use a notion of a fibration of bicategories or fibred bicategory. Hermida defined a fibered 2-category in [44] as a strict 2-functor $F\colon \mathcal{E} \to \mathcal{B}$ between strict 2-categories which has enough cartesian 1-cells and 2-cells, defined by universal properties which generalize those for cartesian morphisms in usual fibrations of categories. Also he gave a slightly different characterization of fibered 2-categories in [45] where he proposed the definition of the fibred bicategory by means of the bireflection of 2-categories and their homomorphisms into 2-categories and 2-functors. Therefore, a homomorphism $F\colon \mathcal{E} \to \mathcal{B}$ between bicategories must be a 2-fibration if its associated strict 2-functor $\tilde{F}\colon \tilde{\mathcal{E}} \to \tilde{\mathcal{B}}$ between strict 2-categories is such. We will use this notion without going into details, which will be given elsewhere.*

Theorem 11.1. *The above construction defines a fibred bicategory*

$$F_{\mathcal{B}}\colon \mathcal{FB} \to \mathcal{E} \tag{11.10}$$

which we call the small 2-fibration induced by $\mathcal{B}$.

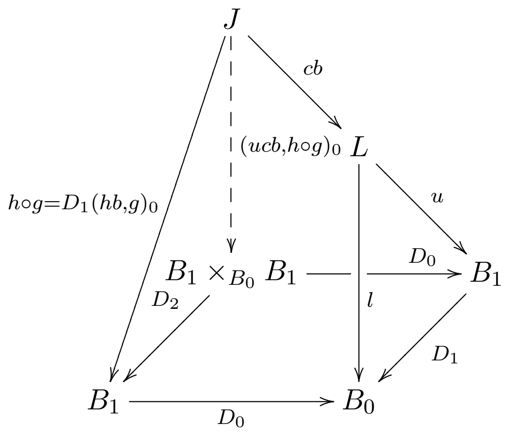
Proof. Let's consider composable string of 1-morphisms

$$(I,i) \xrightarrow{(a,f)} (J,j) \xrightarrow{(b,g)} (K,k) \xrightarrow{(c,h)} (L,l) \xrightarrow{(d,u)} (M,m).$$

First we will show that the horizontal composition is coherently associative, which means that we have the following identity

$$(u \circ \alpha_{h,g,f})\alpha_{u,h\circ g,f}(\alpha_{u,h,g} \circ f) = \alpha_{u,h,g\circ f}\alpha_{u\circ h,g,f}.$$

The 1-morphism $(d,u)[(c,h)(b,g)]$ is given by $(d,u)(cb, h\circ g) := (dcb, u\circ(h\circ g))$ where $u\circ(h\circ g) := D_1(ucb, h\circ g)_0$ and the morphism $(ucb, h\circ g)_0\colon I \to B_1 \times_{B_0} B_1$ is the unique one given by the universal property of the pullback



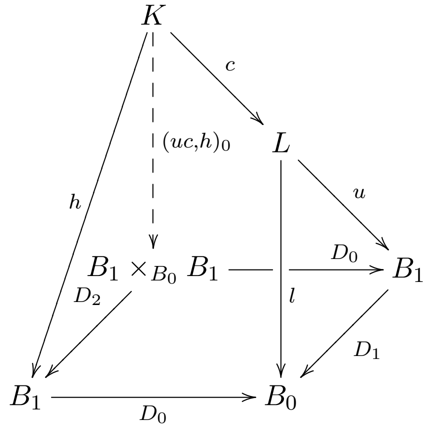
obtained from the factorization $D_0D_1(hb,g)_0 \overset{(10.4)}{=} D_0D_0(hb,g)_0 = D_0hb = lcb = D_1ucb$, such that the identities

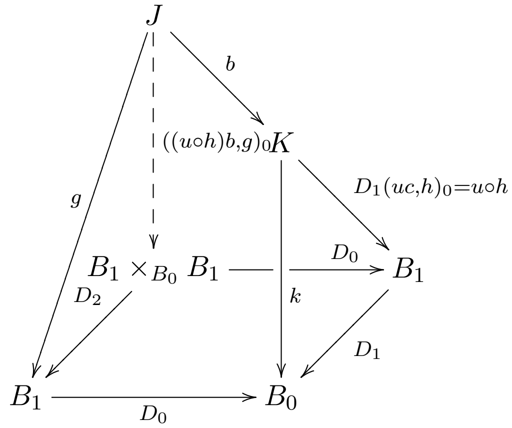
$$\begin{aligned} &D_0(ucb, h\circ g)_0 = ucb \\ &D_1(ucb, h\circ g)_0 = D_1(ucb, D_1(hb,g)_0)_0 = u\circ(h\circ g) \\ &D_2(ucb, h\circ g)_0 = D_1(hb,g)_0 = h\circ g \end{aligned} \tag{11.11}$$

are satisfied. Also, from an identity $(d,u)\circ(c,h) := (dc, u\circ h)$ it follows

$$[(d,u)\circ(c,h)]\circ(b,g) = (dc, u\circ h)(b,g) = (dcb, (u\circ h)\circ g)$$

where $u\circ h := D_1(uc,h)_0$ and $(u\circ h)\circ g = D_1((u\circ h)b, g)_0 = D_1(D_1(uc,h)_0 b, g)_0$ are morphisms obtained from two pullbacks





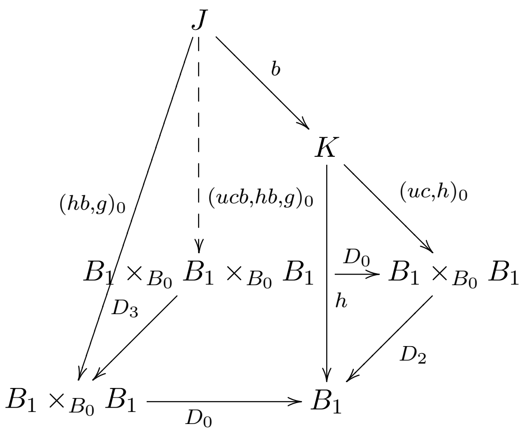
where $D_0 h = lc = D_1 uc$ and $D_0 g = kb = D_1 hb = D_1 D_2(uc,h)_0 b \overset{(10.4)}{=} D_1 D_1(uc,h)_0 b$ are factorizations represented by two diagonals, respectively. Therefore, we have identities

$$\begin{aligned} D_0(uc,h)_0 &= uc \\ D_1(uc,h)_0 &= u \circ h \\ D_2(uc,h)_0 &= h \end{aligned} \tag{11.12}$$

from the first diagram and from the second diagram we have following identities

$$\begin{aligned} D_0((u \circ h)b, g)_0 &= D_1(uc,h)_0 = u \circ h \\ D_1((u \circ h)b, g)_0 &= D_1(D_1(uc,h)_0 b, g) = (u \circ h) \circ g \\ D_2((u \circ h)b, g)_0 &= g. \end{aligned} \tag{11.13}$$

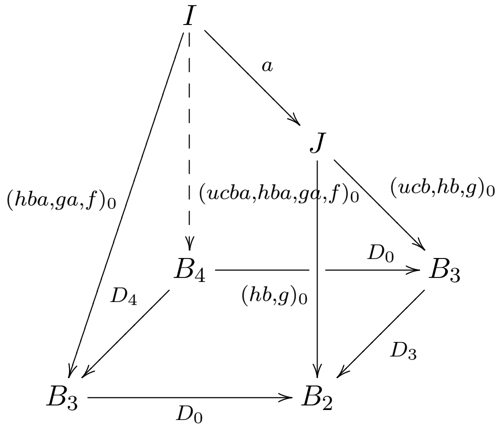
By an analogy with (11.5) we have a morphism $(ucb, hb, g)_0 \colon J \to B_1 \times_{B_0} B_1 \times_{B_0} B_1$ obtained



from a factorization $D_0(hb,g)_0 \overset{(11.3)}{=} hb \overset{(11.12)}{=} D_2(uc,h)_0 b$, which is the unique one such that

$$\begin{aligned} D_0(ucb,hb,g)_0 &= (uc,h)_0 b \\ D_1(ucb,hb,g)_0 &= (ucb,h\circ g)_0 \\ D_2(ucb,hb,g)_0 &= ((u\circ h)b,g)_0 \\ D_3(ucb,hb,g)_0 &= (hb,g)_0. \end{aligned} \tag{11.14}$$

We use a factorization $D_0(hba,ga,f)_0 \overset{(11.5)}{=} (hb,g)_0 a \overset{(11.14)}{=} D_3(ucb,hb,g)_0 a$ in the pullback



to obtain a unique morphism $(ucba,hba,ga,f)_0\colon I\to B_4$ which satisfies following identities

$$\begin{aligned} D_0(ucba,hba,ga,f)_0 &= (ucb,hb,g)_0 a \\ D_4(ucba,hba,ga,f)_0 &= (hba,ga,f)_0. \end{aligned} \tag{11.15}$$

The morphism $D_1(ucba,hba,ga,f)_0\colon I\to B_3$ satisfies following identities

$$D_0D_1(ucba,hba,ga,f)_0 \overset{(10.11)}{=} D_0D_0(ucba,hba,ga,f)_0 \overset{(11.15)}{=} D_0(ucb,hb,g)_0 a \overset{(11.14)}{=} (uc,h)_0 b$$
$$D_3D_1(ucba,hba,ga,f)_0 \overset{(10.11)}{=} D_1D_4(ucba,hba,ga,f)_0 \overset{(11.15)}{=} D_1(hba,ga,f)_0 \overset{(11.7)}{=} (hba,g\circ f)_0$$

and by an analogy with (11.5), the morphism $(ucba,hb,g\circ f)_0\colon I\to B_3$ is the unique one satisfying these identities, we conclude

$$D_1(ucba,hba,ga,f)_0 = (ucba,hba,g\circ f)_0. \tag{11.16}$$

By a similar argument, from the uniqueness of the morphism $(ucba,(h\circ g)a,f)_0\colon I\to B_3$ we get an identity

$$D_2(ucba,hba,ga,f)_0 = (ucba,(h\circ g)a,f)_0 \tag{11.17}$$

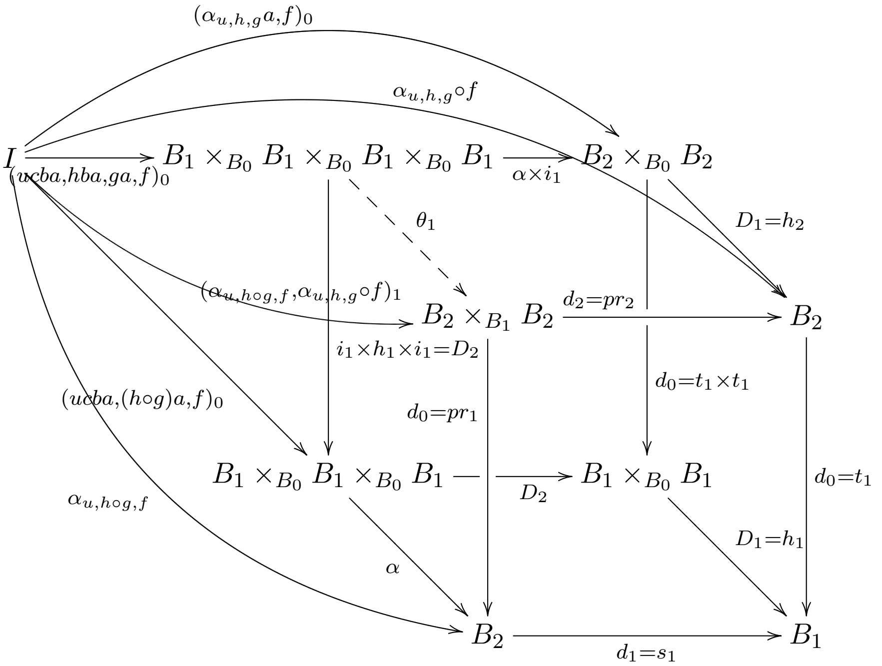
and from the uniqueness of the morphism $((u\circ h)ba,ga,f)_0\colon I\to B_3$ we get an identity

$$D_3(ucba,hba,ga,f)_0 = ((u\circ h)ba,ga,f)_0. \tag{11.18}$$

From the definition (9.3) we have the following identities:

$$\begin{aligned}
\alpha_{u,h,g}a &= \alpha(ucb, hb, g)_0 a \overset{(11.15)}{=} \alpha D_0(ucba, hba, ga, f)_0\\
\alpha_{u,h,g\circ f} &= \alpha(ucba, hba, g\circ f)_0 \overset{(11.16)}{=} \alpha D_1(ucba, hba, ga, f)_0\\
\alpha_{u,h\circ g,f} &= \alpha(ucba, (h\circ g)a, f)_0 \overset{(11.17)}{=} \alpha D_2(ucba, hba, ga, f)_0\\
\alpha_{u\circ h,g,f} &= \alpha((u\circ h)ba, ga, f)_0 \overset{(11.18)}{=} \alpha D_3(ucba, hba, ga, f)_0\\
\alpha_{h,g,f} &= \alpha(hba, ga, f)_0 \overset{(11.15)}{=} \alpha D_4(ucba, hba, ga, f)_0
\end{aligned}$$

Also, from the definition (9.11) a morphism $\theta_1\colon B_1\times_{B_0} B_1\times_{B_0} B_1\times_{B_0} B_1 \to B_2\times_{B_1} B_2$



$$(11.19)$$

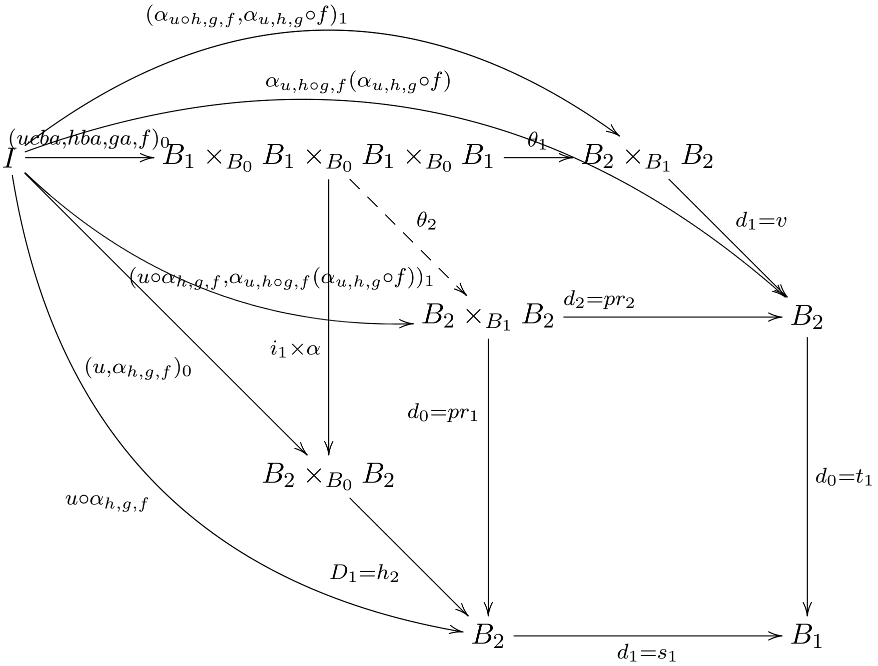
is the unique one such that two identities $d_0\theta_1 = \alpha D_2$ and $d_2\theta_1 = D_1(\alpha\times i_1)$ are satisfied. The first identity implies

$$d_0\theta_1(ucba, hba, ga, f)_0 = \alpha D_2(ucba, hba, ga, f)_0 \overset{(11.17)}{=} \alpha(ucba, (h\circ g)a, f)_0 = \alpha_{u,h\circ g,f}$$

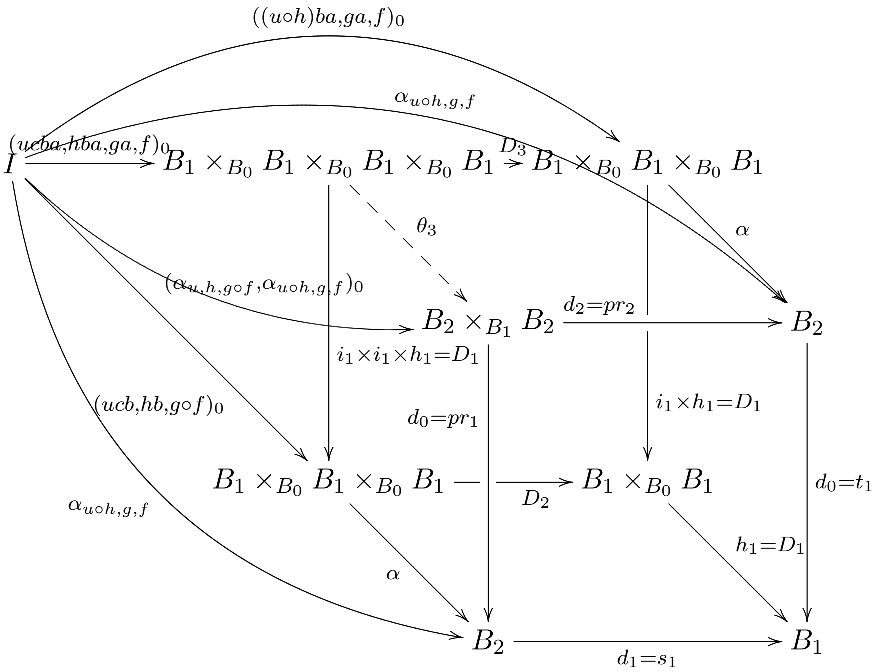
and we see that the morphism $\alpha_{u,h\circ g,f}$ factors through $\theta_1(ucba, hba, ga, f)_0$. But from the universal property of the pullback at the front face of the diagram (11.19) we know that a morphism $(\alpha_{u,h\circ g,f}, \alpha_{u,h,g}\circ f)_1\colon I\to B_2\times_{B_1} B_2$ is the unique one with this property, and we conclude that

$$\theta_1(ucba, hba, ga, f)_0 = (\alpha_{u,h\circ g,f}, \alpha_{u,h,g}\circ f)_1. \tag{11.20}$$

Definitions (9.12) and (9.13) of morphisms $\theta_2, \theta_3 \colon B_1 \times_{B_0} B_1 \times_{B_0} B_1 \times_{B_0} B_1 \to B_2 \times_{B_1} B_2$



(11.21)



(11.22)

in diagrams (11.21) and (11.22), provide by a similar argument the following two identities

$$\theta_2(ucba, hba, ga, f)_0 = (u \circ \alpha_{h,g,f}, \alpha_{u,h\circ g,f}(\alpha_{u,h,g} \circ f))_1 \tag{11.23}$$

$$\theta_3(ucba, hba, ga, f)_0 = (\alpha_{u,h,g\circ f}, \alpha_{u\circ h,g,f})_1 \tag{11.24}$$

so that (11.20), (11.23) and (11.24) together with the associativity coherence law (9.14) imply

$$(u \circ \alpha_{h,g,f})[\alpha_{u,h\circ g,f}(\alpha_{u,h,g} \circ f)] = d_1(u \circ \alpha_{h,g,f}, \alpha_{u,h\circ g,f}(\alpha_{u,h,g} \circ f))_1 \overset{(11.23)}{=}$$
$$\overset{(11.23)}{=} d_1\theta_2(ucba, hba, ga, f)_0 \overset{(9.14)}{=} d_1\theta_3(ucba, hba, ga, f)_0 \overset{(11.24)}{=} d_1(\alpha_{u,h,g\circ f}, \alpha_{u\circ h,g,f})_1 =$$
$$= \alpha_{u,h,g\circ f}\alpha_{u\circ h,g,f}$$

and we conclude that the horizontal composition is coherently associative. The coherence for the left and right identity follows similar pattern and it is implied by an axiom (9.15).
□

12 The second nonabelian cohomology

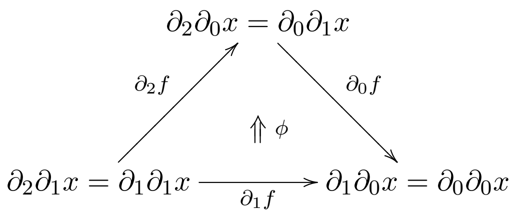
In this chapter we will give an explicit definition of the second nonabelian cohomology, following the general approach described by Street in [82]. The crucial step is the constructive proof of the existence of the *bicategory of 2-descent data*, associated to any 3-truncated cosimplicial bicategory

$$\mathcal{B}_0 \underset{\partial_0}{\overset{\partial_1}{\rightrightarrows}} \mathcal{B}_1 \underset{\partial_0}{\overset{\partial_2}{\rightrightarrows}} \mathcal{B}_2 \underset{\partial_0}{\overset{\partial_3}{\rightrightarrows}} \mathcal{B}_3 \tag{12.1}$$

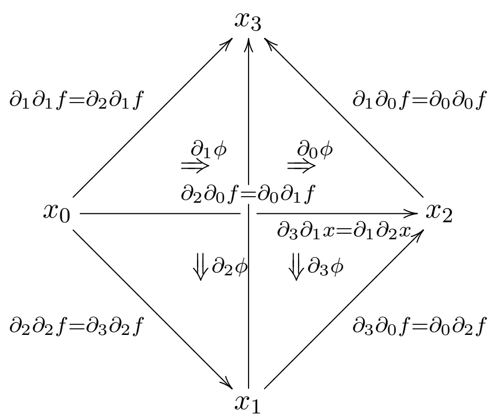
in the category $Bicat_s$ of bicategories and strict morphisms of bicategories. The category $Bicat_s$ is an Eilenberg-Moore category of algebras over a monad $K_2\colon 2{-}Graph \to 2{-}Graph$ on the category $2{-}Graph$ of 2-graphs. This monad was first explicitly described by Batanin in [14], who called it an *initial contractible monad with a system of compositions*, inspired by ideas from the homotopy theory. Since K_2 preserves filtered colimits, its algebras are models of a finite-limit theory, so that we can take models of bicategories in any finitely complete category $\mathcal{E}$.

Theorem 12.1. *For any 3-truncated cosimplicial bicategory (12.1) $Desc_2(\mathcal{B})$ consisting of the following data:*

- *any object is a triple (x, f, ϕ) where x is an object in $\mathcal{B}_0$, $f\colon \partial_1 x \to \partial_0 x$ is a 1-morphism in $\mathcal{B}_1$, and $\phi\colon \partial_1 f \Rightarrow \partial_0 f \circ \partial_0 f$ is a 2-morphism in $\mathcal{B}_2$*



such that for $x_0 = \partial_3\partial_2\partial_1 x$, $x_1 = \partial_3\partial_2\partial_0 x$, $x_2 = \partial_3\partial_1\partial_0 x$, $x_3 = \partial_2\partial_1\partial_0 x$ the 3-simplex



commutes. The pasting composite of the above 3-simplex is a commutative diagram

$$\begin{array}{ccc}
\partial_1\partial_1 f = \partial_2\partial_1 f \overset{\partial_2\phi}{\Longrightarrow} \partial_2\partial_0 f \circ \partial_2\partial_2 f = \partial_0\partial_1 f \circ \partial_2\partial_2 f & \xRightarrow{\partial_0\phi\circ\partial_2\partial_2 f} & (\partial_0\partial_0 f \circ \partial_0\partial_2 f) \circ \partial_2\partial_2 f \\
\Big\Downarrow{\scriptstyle \partial_1\phi} & & \Big\Downarrow{\scriptstyle \alpha_{\partial_0\partial_0 f, \partial_0\partial_2 f, \partial_2\partial_2 f}} \\
\partial_1\partial_0 f \circ \partial_1\partial_2 f = \partial_0\partial_0 f \circ \partial_3\partial_1 f & \xRightarrow[\partial_0\partial_0 f \circ \partial_3\phi]{} & \partial_0\partial_0 f \circ (\partial_3\partial_0 f \circ \partial_3\partial_2 f) = \partial_0\partial_0 f \circ (\partial_0\partial_2 f \circ \partial_2\partial_2 f)
\end{array}$$

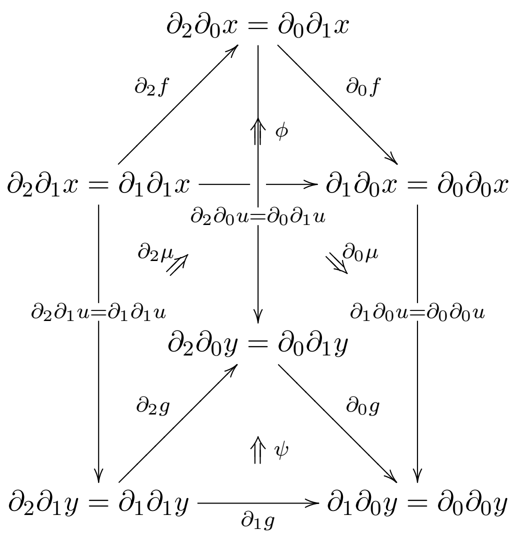
that represents a nonabelian 2-cocycle condition

$$\alpha_{\partial_0\partial_0 f, \partial_0\partial_2 f, \partial_2\partial_2 f}(\partial_0\phi \circ \partial_2\partial_2 f)\partial_2\phi = (\partial_0\partial_0 f \circ \partial_3\phi)\partial_1\phi. \tag{12.2}$$

- *any 1-morphism* $(u, \mu)\colon (x, f, \phi) \to (y, g, \psi)$ *is a pair consisting of a 1-morphism* $u\colon x \to y$ *in* $\mathcal{B}_0$*, together with the 2-morphism in* $\mathcal{B}_1$

$$\begin{array}{ccc}
\partial_1 x & \xrightarrow{\partial_1 u} & \partial_1 y \\
{\scriptstyle f}\Big\downarrow & \swarrow_{\mu} & \Big\downarrow{\scriptstyle g} \\
\partial_0 x & \xrightarrow[\partial_0 u]{} & \partial_0 y
\end{array}$$

such that the prism in the bicategory $\mathcal{B}_2$

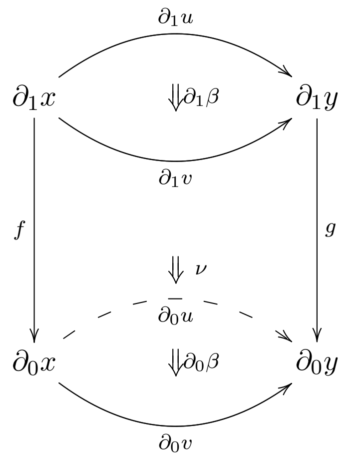


commutes. This means that we have a commutative diagram

$$
\begin{array}{ccccc}
\partial_1 g \circ \partial_2\partial_1 u = \partial_1 g \circ \partial_1\partial_1 u & \xRightarrow{\partial_1\mu} & \partial_1\partial_0 u \circ \partial_1 f = \partial_0\partial_0 u \circ \partial_1 f & \xRightarrow{\partial_0\partial_0 u \circ \phi} & \partial_0\partial_0 u \circ (\partial_0 f \circ \partial_2 f) \\
\Downarrow {\scriptstyle \psi \circ \partial_2\partial_1 u} & & & & \Uparrow {\scriptstyle \alpha_{\partial_0\partial_0 u, \partial_0 f, \partial_2 f}} \\
(\partial_0 g \circ \partial_2 g) \circ \partial_2\partial_1 u & & & & (\partial_0\partial_0 u \circ \partial_0 f) \circ \partial_2 f \\
\Downarrow {\scriptstyle \alpha_{\partial_0 g, \partial_2 g, \partial_2\partial_1 u}} & & & & \Uparrow {\scriptstyle \partial_0\mu \circ \partial_2 f} \\
\partial_0 g \circ (\partial_2 g \circ \partial_2\partial_1 u) & \xRightarrow[\partial_0 g \circ \partial_2\mu]{} & \partial_0 g \circ (\partial_2\partial_0 u \circ \partial_2 f) = \partial_0 g \circ (\partial_0\partial_1 u \circ \partial_2 f) & \xRightarrow[\alpha^{-1}_{\partial_0 g, \partial_0\partial_1 u, \partial_2 f}]{} & (\partial_0 g \circ \partial_0\partial_1 u) \circ \partial_2 f
\end{array}
$$

in the category of morphisms of the bicategory $\mathcal{B}_2$*.*

- *a 2-morphisms* $\beta\colon (u,\mu) \Rightarrow (v,\nu)$ *is a 2-morphism* $\beta\colon u \Rightarrow v$ *in* $\mathcal{B}_0$*, such that the diagram*



commutes. This diagram becomes in a 1-dimensional form a commutative diagram

$$
\begin{array}{ccc}
g \circ \partial_1 u & \xRightarrow{g \circ \partial_1\beta} & g \circ \partial_1 v \\
\Downarrow {\scriptstyle \mu} & & \Downarrow {\scriptstyle \nu} \\
\partial_0 u \circ f & \xRightarrow[\partial_0\beta \circ f]{} & \partial_0 v \circ f
\end{array}
$$

Proof. For any two composable 1-morphisms in $Desc_2(\mathcal{B})$

$$(x,f,\phi) \xrightarrow{(u,\mu)} (y,g,\psi) \xrightarrow{(v,\nu)} (w,h,\xi)$$

we define the composition by $(v,\nu)\circ(u,\mu) = (v\circ u, \nu\Box\mu)$ where $\nu\Box\mu$ is a 2-morphism obtained by the pasting of the diagram

$$\begin{array}{ccccc}
\partial_1 x & \xrightarrow{\partial_1 u} & \partial_1 y & \xrightarrow{\partial_1 v} & \partial_1 w \\
{\scriptstyle f}\downarrow & \swarrow_{\mu} & {\scriptstyle g}\downarrow & \swarrow_{\nu} & \downarrow{\scriptstyle h} \\
\partial_0 x & \xrightarrow[\partial_0 u]{} & \partial_0 y & \xrightarrow[\partial_0 v]{} & \partial_0 w
\end{array}$$

in the bicategory $\mathcal{B}_1$. This means that the 2-morphism $\nu\Box\mu\colon h\circ\partial_1(v\circ u)\Rightarrow \partial_0(v\circ u)\circ f$ is defined by the diagram

$$\begin{array}{ccccc}
h\circ\partial_1(v\circ u) = h\circ(\partial_1 v\circ\partial_1 u) & \xRightarrow{\alpha^{-1}_{h,\partial_1 v,\partial_1 u}} & (h\circ\partial_1 v)\circ\partial_1 u & \xRightarrow{\nu\circ\partial_1 u} & (\partial_0 v\circ g)\circ\partial_1 u \\
{\scriptstyle \nu\Box\mu}\Downarrow & & & & \Downarrow{\scriptstyle \alpha_{\partial_0 v,g,\partial_1 u}} \\
\partial_0(v\circ u)\circ f = (\partial_0 v\circ\partial_0 u)\circ f & \xLeftarrow[\alpha^{-1}_{\partial_0 v,\partial_0 u,f}]{} & \partial_0 v\circ(\partial_0 u\circ f) & \xLeftarrow[\partial_0 v\circ\mu]{} & \partial_0 v\circ(g\circ\partial_1 u)
\end{array}$$

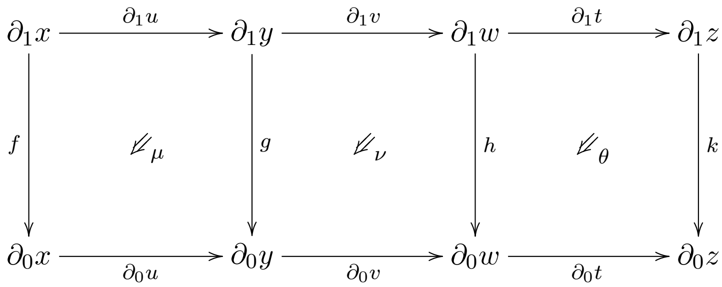
so that we have an identity

$$\nu\Box\mu := \alpha^{-1}_{\partial_0 v,\partial_0 u,f}(\partial_0 v\circ\mu)\alpha_{\partial_0 v,g,\partial_1 u}(\nu\circ\partial_1 u)\alpha^{-1}_{h,\partial_1 v,\partial_1 u}$$

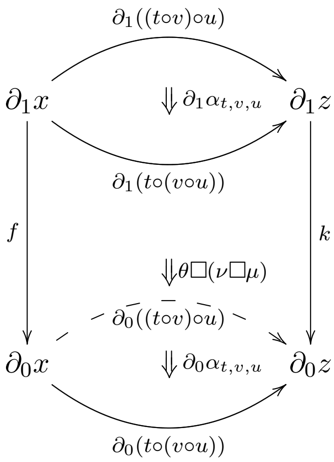
The horizontal and vertical compositions of 2-morphisms in $Desc_2(\mathcal{B})$ are inherited from the bicategory $\mathcal{B}_0$. So the associativity and left and right identity coherence are also inherited from the bicategory $\mathcal{B}_0$, and we will prove that for any three composable 1-morphisms in $Desc_2(\mathcal{B})$

$$(x,f,\phi) \xrightarrow{(u,\mu)} (y,g,\psi) \xrightarrow{(v,\nu)} (w,h,\xi) \xrightarrow{(t,\theta)} (z,k,\zeta)$$

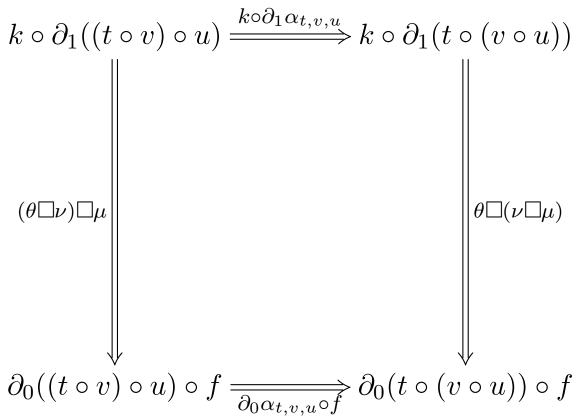
represented by the diagram



the component $\alpha_{t,v,u}\colon [(t,\theta)\circ(v,\nu)]\circ(u,\mu) \Rightarrow (t,\theta)\circ[(v,\nu)\circ(u,\mu)]$ of the associativity isomorphism satisfy



which means that the following diagram in the category of morphisms of the bicategory $\mathcal{B}_1$



commutes, so that we have an identity

$$(\theta\Box(\nu\Box\mu))(k\circ\partial_1\alpha_{t,v,u}) = (\partial_0\alpha_{t,v,u}\circ f)((\theta\Box\nu)\Box\mu)$$

The proof of the commutativity of the above diagram follows from the sequence of identities

$$
\begin{aligned}
&(\theta\Box(\nu\Box\mu))(k\circ\partial_1\alpha_{t,v,u}) \overset{(def.)}{=} \\
&\overset{(def.)}{=} \alpha^{-1}_{\partial_0 t,\partial_0(v\circ u),f}(\partial_0 t\circ(\nu\Box\mu))\alpha_{\partial_0 t,h,\partial_1(v\circ u)}(\theta\circ\partial_1(v\circ u))[\alpha^{-1}_{k,\partial_1 t,\partial_1(v\circ u)}(k\circ\alpha_{\partial_1 t,\partial_1 v,\partial_1 u})] \overset{(a.c.)}{=} \\
&\overset{(a.c.)}{=} \alpha^{-1}_{\partial_0 t,\partial_0(v\circ u),f}(\partial_0 t\circ(\nu\Box\mu))\alpha_{\partial_0 t,h,\partial_1(v\circ u)}[(\theta\circ\partial_1(v\circ u))\alpha_{k\circ\partial_1 t,\partial_1 v,\partial_1 u}](\alpha^{-1}_{k,\partial_1 t,\partial_1 v}\circ\partial_1 u)\alpha^{-1}_{k,\partial_1(t\circ v),\partial_1 u} \overset{(a.n.)}{=} \\
&\overset{(a.n.)}{=} \dots[(\partial_0 t\circ\alpha^{-1}_{h,\partial_1 v,\partial_1 u})\alpha_{\partial_0 t,h,\partial_1(v\circ u)}\alpha_{\partial_0 t\circ h,\partial_1 v,\partial_1 u}]((\theta\circ\partial_1 v)\circ\partial_1 u)(\alpha^{-1}_{k,\partial_1 t,\partial_1 v}\circ\partial_1 u)\alpha^{-1}_{k,\partial_1(t\circ v),\partial_1 u} \overset{(a.c.)}{=} \\
&\overset{(a.c.)}{=} \dots[(\partial_0 t\circ(\nu\circ\partial_1 u))\alpha_{\partial_0 t,h\circ\partial_1 v,\partial_1 u}](\alpha_{\partial_0 t,h,\partial_1 v}\circ\partial_1 u)((\theta\circ\partial_1 v)\circ\partial_1 u)(\alpha^{-1}_{k,\partial_1 t,\partial_1 v}\circ\partial_1 u)\alpha^{-1}_{k,\partial_1(t\circ v),\partial_1 u} \overset{(a.n.)}{=} \\
&\overset{(a.n.)}{=} \dots[(\partial_0 t\circ\alpha_{\partial_0 v,g,\partial_1 u})\alpha_{\partial_0 t,\partial_0 v\circ g,\partial_1 u}]((\partial_0 t\circ\nu)\circ\partial_1 u)(\alpha_{\partial_0 t,h,\partial_1 v}\circ\partial_1 u)((\theta\circ\partial_1 v)\circ\partial_1 u)(\alpha^{-1}_{k,\partial_1 t,\partial_1 v}\circ\partial_1 u)\dots \overset{(a.c.)}{=} \\
&\overset{(a.c.)}{=} \dots\alpha_{\partial_0(t\circ v),g,\partial_1 u}[(\alpha^{-1}_{\partial_0 t,\partial_0 v,g}\circ\partial_1 u)((\partial_0 t\circ\nu)\circ\partial_1 u)(\alpha_{\partial_0 t,h,\partial_1 v}\circ\partial_1 u)((\theta\circ\partial_1 v)\circ\partial_1 u)(\alpha^{-1}_{k,\partial_1 t,\partial_1 v}\circ\partial_1 u)]\dots \overset{(def.)}{=} \\
&\overset{(def.)}{=} \alpha^{-1}_{\partial_0 t,\partial_0(v\circ u),f}(\partial_0 t\circ\alpha^{-1}_{\partial_0 v,\partial_0 u,f})[(\partial_0 t\circ(\partial_0 v\circ\mu))\alpha_{\partial_0 t,\partial_0 v,g\circ\partial_1 u}]\alpha_{\partial_0(t\circ v),g,\partial_1 u}((\theta\Box\nu)\circ\partial_1 u)\alpha^{-1}_{k,\partial_1(t\circ v),\partial_1 u} \overset{(a.n.)}{=} \\
&\overset{(a.n.)}{=} [\alpha^{-1}_{\partial_0 t,\partial_0(v\circ u),f}(\partial_0 t\circ\alpha^{-1}_{\partial_0 v,\partial_0 u,f})\alpha_{\partial_0 t,\partial_0 v,\partial_0 u\circ f}](\partial_0(t\circ v)\circ\mu)\alpha_{\partial_0(t\circ v),g,\partial_1 u}((\theta\Box\nu)\circ\partial_1 u)\alpha^{-1}_{k,\partial_1(t\circ v),\partial_1 u} \overset{(a.c.)}{=} \\
&\overset{(a.c.)}{=} (\alpha_{\partial_0 t,\partial_0 v,\partial_0 u}\circ f)[\alpha^{-1}_{\partial_0(t\circ v),\partial_0 u,f}(\partial_0(t\circ v)\circ\mu)\alpha_{\partial_0(t\circ v),g,\partial_1 u}((\theta\Box\nu)\circ\partial_1 u)\alpha^{-1}_{k,\partial_1(t\circ v),\partial_1 u})] \overset{(def)}{=} \\
&\overset{(def)}{=} (\partial_0\alpha_{t,v,u}\circ f)((\theta\Box\nu)\Box\mu)
\end{aligned}
$$

where each expression in the square brackets transforms by the associativity coherence (a.c.) in the bicategory $\mathcal{B}_0$ or by the fact that the associativity is a natural isomorphism (a.n.). The coherence for such associativity follows straight from the coherence for associativity in the bicategory $\mathcal{B}_0$. □

Therefore, to any 3-truncated cosimplicial bicategory (12.1) we associate a bicategory

$$Desc_2(\mathcal{B}) \tag{12.3}$$

called the *bicategory of 2-descent data* associated to $\mathcal{B}$. Any internal simplicial object $X\colon \Delta^{op}\to\mathcal{E}$ and any internal bicategory $\mathcal{B}$ in $\mathcal{E}$, may be use to produce a cosimplicial bicategory

$$\mathcal{E}(X,\mathcal{B})\colon \Delta\to Bicat_s \tag{12.4}$$

by the composition

$$\Delta \xrightarrow{X^{op}} \mathcal{E}^{op} \xrightarrow{Hom_{\mathcal{E}}(-,\mathcal{B})} Bicat_s$$

where $Hom_{\mathcal{E}}(-,\mathcal{B})\colon \mathcal{E}^{op}\to Bicat_s$ is a presheaf of bicategories, and we denote by $Hom_{\mathcal{E}}(Y,\mathcal{B})$ the fiber of the small 2-fibration (11.10)

$$\begin{array}{c}\mathcal{FB}\\ \big\downarrow{\scriptstyle F_{\mathcal{B}}}\\ \mathcal{E}\end{array}$$

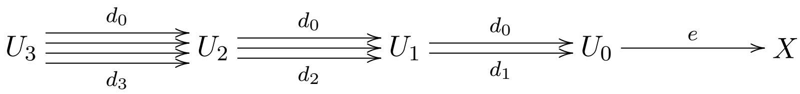
over an object Y in $\mathcal{E}$. These construction allows us to define the second nonabelian cohomology of simplicial objects in $\mathcal{E}$, with coefficients in an internal bicategory $\mathcal{B}$.

Definition 12.1. *Let $\mathcal{B}$ be an internal bicategory in a finitely complete category in $\mathcal{E}$, and let $X\colon \Delta^{op} \to \mathcal{E}$ be a simplicial object in $\mathcal{E}$. The cohomology bicategory $\mathcal{H}^2(X, \mathcal{B})$ of the simplicial object X with coefficient in a bicategory $\mathcal{B}$ is defined by*

$$\mathcal{H}^2(X, \mathcal{B}) = Desc_2(\mathcal{E}(X, \mathcal{B})) \tag{12.5}$$

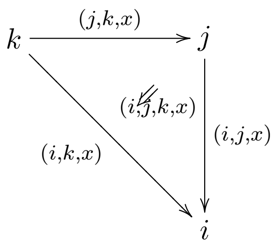
the bicategory of 2-descent data of the cosimplicial bicategory $\mathcal{E}(X, \mathcal{B})$.

Example 12.1. *The second Čech nonabelian cohomology $\mathcal{H}^2(\mathcal{U}, \mathcal{B})$ is defined with respect to the covering $\mathcal{U} = \{U_i\}_{i\in I}$ of the topological space X. The epimorphism $e = (e_i)_{i\in I}\colon \coprod_{i\in I} U_i \to X$, induced by the family of embeddings $e_i\colon U_i \to X$, gives a 3-truncation of the simplicial resolution $U_\bullet$*



where $U_0 = \coprod_{i\in I} U_i$, $U_1 = \coprod_{i,j\in I} U_{ij}$, $U_2 = \coprod_{i,j,k\in I} U_{ijk}$ and $U_3 = \coprod_{i,j,k,l\in I} U_{ijkl}$ (where U_{ij} denotes the double intersection $U_{ij} = U_i \cap U_j$ and so on).

This is just the 3-truncation of the nerve of the Čech groupoid associated to the covering $e\colon U \to X$, whose objects are given by the elements (i, x) of U, and for which there exists a unique morphism $(i, j, x)\colon (j, x) \to (i, x)$ for any element $x \in U_{ij}$. Thus, target and source morphisms defines face operators $d_0^1, d_1^1\colon U_1 \to U_0$ which are given by the first and the second projection, respectively. The 2-simplex (i, j, k, x) in U_2 may be seen as the diagram



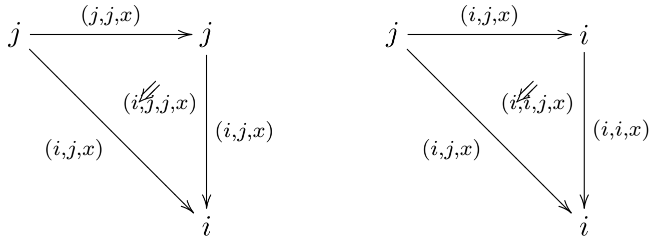
from which we see that the face operators $d_0^2, d_1^2, d_2^2\colon U_2 \to U_1$ are defined by

$$\begin{aligned} d_0^2(i,j,k,x) &= (i,j,x)\\ d_1^2(i,j,k,x) &= (i,k,x)\\ d_2^2(i,j,k,x) &= (j,k,x) \end{aligned}$$

and they are just three possible inclusions of triple intersections into double intersections. The degeneracy operators $s_0^2, s_1^2\colon U_1 \to U_2$ are given by

$$\begin{aligned} s_0^2(i,j,x) &= (i,j,j,x)\\ s_1^2(i,j,x) &= (i,i,j,x) \end{aligned}$$

and these two degenerate 2-simplices may be seen as the two diagrams

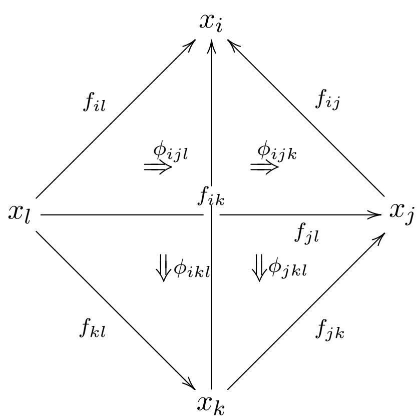


respectively. The 3-truncation of the simplicial resolution of the covering $\mathcal{U} = \{U_i\}_{i \in I}$ *defines a cosimplicial bicategory*

$$\mathcal{B}_0 \underset{\partial_0}{\overset{\partial_1}{\rightrightarrows}} \mathcal{B}_1 \underset{\partial_0}{\overset{\partial_2}{\rightrightarrows}} \mathcal{B}_2 \underset{\partial_0}{\overset{\partial_3}{\rightrightarrows}} \mathcal{B}_3$$

where each bicategory $\mathcal{B}_n$ *has objects given by the discrete category* $(\mathcal{B}_i)_0$ *defined by the set* $Hom_{\mathcal{E}}(U_n, B_0)$*, and whose category of 1-morphisms and 2-morphisms is given by the fiber of the small fibration* $\mathcal{F}_{\mathcal{B}}U_n$ *over the object* U_n *in* $\mathcal{E}$*. On the level of objects, coface operators are defined by the precomposition* $\partial_i^n(f) = fd_i^n$ *for any object* $f\colon U_{n-1} \to B_0$ *of the bicategory* $\mathcal{B}_{n-1}$*, so that these are the strict homomorphisms of bicategories.*

Thus the 2-cocycle in the second Čech nonabelian cohomology is given by the triple $(\mathbf{x}, \mathbf{f}, \phi)$*, where* $\mathbf{x} = (x_i)_{i \in I}$ *is the family of morphisms* $x_i\colon U_i \to B_0$ *together with the family* $\mathbf{f} = (f_{ij})_{i,j \in I}$ *of morphisms* $f_{ij}\colon U_{ij} \to B_1$ *such that* $s_0 f_{ij} = x_j$ *and* $t_0 f_{ij} = x_i$*. The family* $\phi = (\phi_{ijk})_{i,j,k \in I}$ *is given by morphisms* $\phi_{ijk}\colon U_{ijk} \to B_2$ *which satisfy* $s_1\phi_{ijk} = f_{ik}$ *and* $t_1\phi_{ijk} = f_{ij} \circ f_{jk}$ *and we can view it as the 2-simplex*



commutes, which means that we have an identity

$$(f_{ij} \circ \phi_{jkl})\phi_{ijl} = \alpha_{ijkl}(\phi_{ijk} \circ f_{kl})\phi_{ikl} \tag{12.6}$$

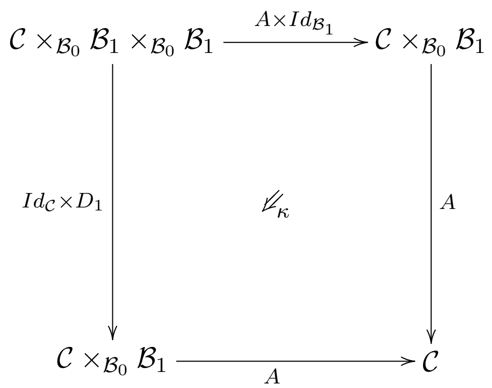
for the Čech nonabelian 2-cocycle $(x_i, f_{ij}, \phi_{ijk})$ *with values in the bicategory* $\mathcal{B}$*.*

13 Actions of bicategories

In this section, we will introduce actions of bicategories. It will be clear from the definition that such actions are categorification of actions of categories.

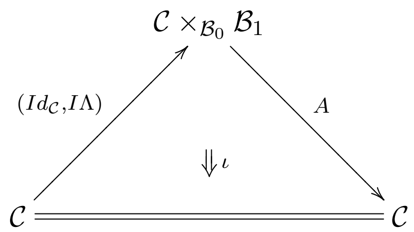
Definition 13.1. *A right action of a bicategory $\mathcal{B}$ is quintuple $(\mathcal{C}, \Lambda, A, \kappa, \iota)$ given by:*

- *a category $\mathcal{C}$ and a functor $\Lambda\colon \mathcal{C} \to \mathcal{B}_0$ to the discrete category of objects $\mathcal{B}_0$ of the bicategory $\mathcal{B}$, called the momentum functor,*
- *a functor $A\colon \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{C}$, called the action functor, and we usually write $A(p, f) := p \triangleleft f$, for any object (p, f) in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$, and $A(a, \phi) := a \triangleleft \phi$ for any morphism $(a, \phi)\colon (p, f) \to (q, g)$ in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$,*
- *a natural isomorphism*



whose components are denoted by $\kappa_{p,f,g}\colon (p \triangleleft f) \triangleleft g \to p \triangleleft (f \circ g)$ for any object (p, f, g) in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$

- *a natural isomorphism*



whose components are denoted by $\iota_p\colon p \triangleleft i_{\Lambda(p)} \to p$ for each object p in $\mathcal{C}$

such that following axioms are satisfied:

- *equivariance of the action*

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{A} & \mathcal{C} \\ {\scriptstyle Pr_2}\downarrow & & \downarrow{\scriptstyle \Lambda} \\ \mathcal{B}_1 & \xrightarrow[D_1]{} & \mathcal{B}_0 \end{array}$$

which means that for any object (p, f) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$*, we have* $\Lambda(p \triangleleft f) = D_1(f)$*, and for any morphism* $(a, \phi)\colon (p, f) \to (q, g)$ *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$*, we have* $\Lambda(a \triangleleft \phi) = D_1(\phi)$*,*

- *for any object* (p, f, g, h) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ *the following diagram*

$$\begin{array}{ccc} & ((p \triangleleft f) \triangleleft g) \triangleleft h & \\ {\scriptstyle \kappa_{p,f,g} \triangleleft h}\swarrow & & \searrow{\scriptstyle \kappa_{p \triangleleft f,g,h}} \\ (p \triangleleft (f \circ g)) \triangleleft h & & (p \triangleleft f) \triangleleft (g \circ h) \\ {\scriptstyle \kappa_{p,f \circ g,h}}\downarrow & & \downarrow{\scriptstyle \kappa_{p,f,g \circ h}} \\ p \triangleleft ((f \circ g) \circ h) & \xrightarrow[p \triangleleft \alpha_{f,g,h}]{} & p \triangleleft (f \circ (g \circ h)) \end{array}$$

commutes,

- *for any object* (p, f) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$ *following diagrams*

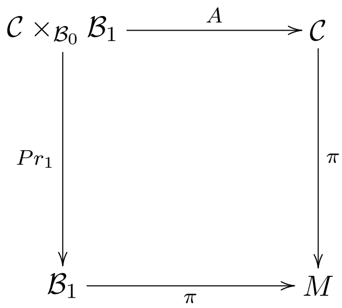
$$\begin{array}{ccc} (p \triangleleft i_{\Lambda_0(p)}) \triangleleft f & \xrightarrow{\kappa_{p,i_{\Lambda_0(p)},f}} & p \triangleleft (i_{\Lambda_0(p)} \circ f) \\ {\scriptstyle \iota_p \triangleleft f}\searrow & & \swarrow{\scriptstyle p \triangleleft \lambda_f} \\ & p \triangleleft f & \end{array} \qquad \begin{array}{ccc} (p \triangleleft f) \triangleleft i_{s_0(f)} & \xrightarrow{\kappa_{p,f,i_{s_0(f)}}} & p \triangleleft (f \circ i_{s_0(f)}) \\ {\scriptstyle \iota_{p \triangleleft f} \triangleleft i_{s_0(f)}}\searrow & & \swarrow{\scriptstyle p \triangleleft \rho_f} \\ & p \triangleleft f & \end{array}$$

commute.

Remark 13.1. *Note the fact that* $A\colon \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{C}$ *is a functor, immediately implies an interchange law*

$$(b \triangleleft \psi)(a \triangleleft \phi) = (ba) \triangleleft (\psi\phi)$$

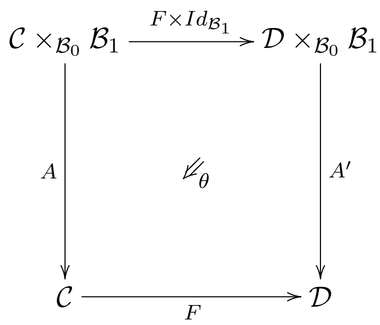
Definition 13.2. *Let* $\pi\colon \mathcal{C} \to M$ *be a bundle of categories over an object* M *in* $\mathcal{E}$. *A (fiberwise) right action of a bicategory* $\mathcal{B}$ *on a bundle of categories* $\pi\colon \mathcal{C} \to M$ *is given by the action of the bicategory* $\mathcal{B}$ *on a category* $\mathcal{C}$ *for which the diagram*



commute. We call a bundle $\pi\colon \mathcal{C} \to M$, *a* $\mathcal{B}$*-2-bundle over* M.

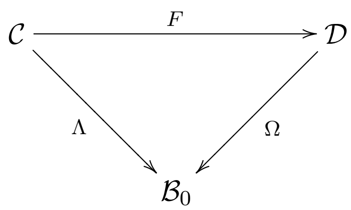
Definition 13.3. *Let* $(\mathcal{C}, \Lambda, A, \kappa, \iota)$ *and* $(\mathcal{D}, A', \Omega, \kappa', \iota')$ *be two* $\mathcal{B}$*-categories. A* $\mathcal{B}$*-equivariant functor is a pair* $(F, \theta)\colon (\mathcal{C}, \Lambda, A, \kappa, \iota) \to (\mathcal{D}, A', \Omega, \kappa', \iota')$ *consisting of*

- *a functor* $F\colon \mathcal{C} \to \mathcal{D}$
- *a natural transformations* $\theta\colon A' \circ (F \times Id_{\mathcal{B}_1}) \Rightarrow F \circ A$

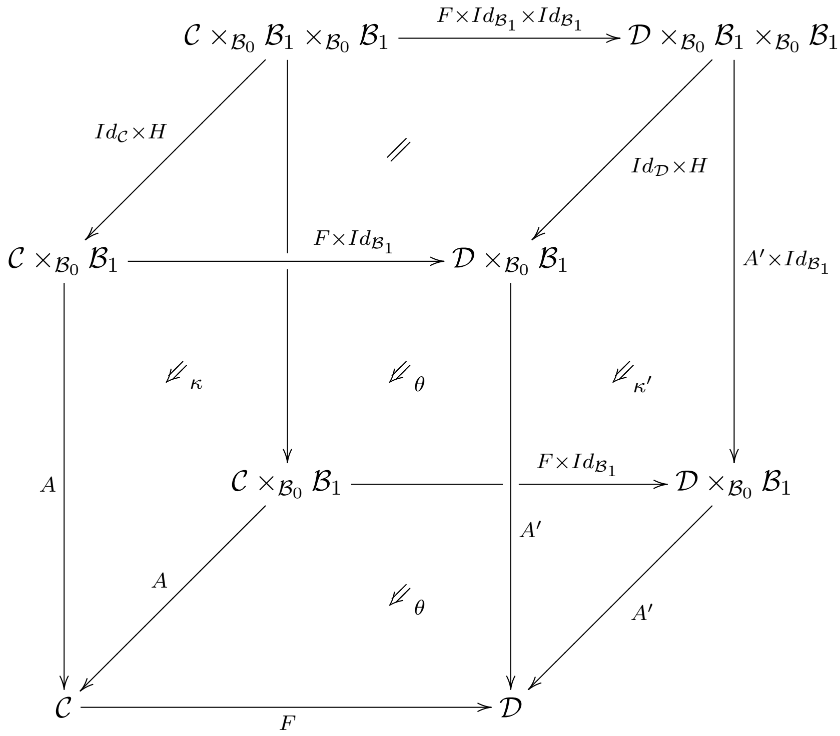


such that following conditions are satisfied:

- $\Omega \circ F = \Lambda$



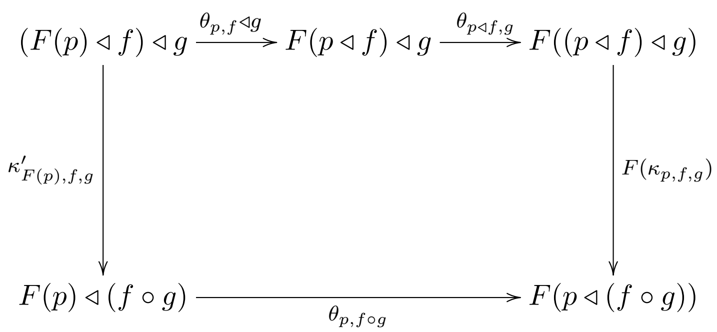
- *the diagram*



commutes, which means that we have an identity of natural transformations

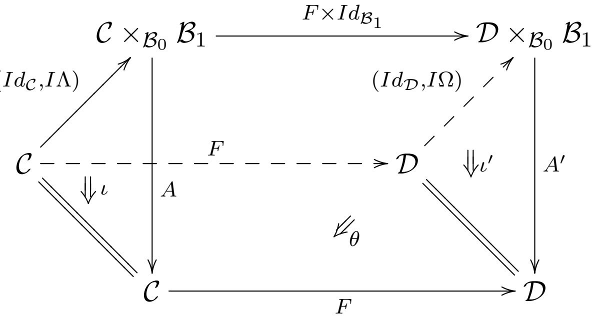
$$(F \circ \kappa)[\theta \circ (A \times Id_{\mathcal{B}_1})][A' \circ (\theta \times Id_{\mathcal{B}_1})] = [\theta \circ (Id_{\mathcal{C}} \times H)][\kappa' \circ (F \times Id_{\mathcal{B}_1} \times Id_{\mathcal{B}_1})]$$

when evaluated at object (p, f, g) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$, *becomes a commutative diagram*



in the category $\mathcal{D}$.

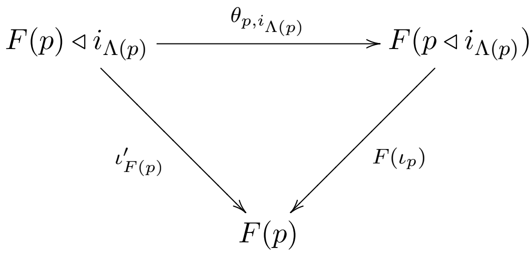
- *the diagram*



commutes, which means that we have identity of natural transformations

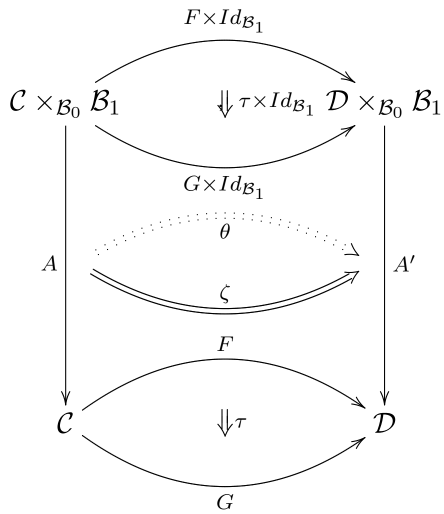
$$(\iota' \circ Id_F)Id_F = (F \circ \iota)[\theta \circ (Id_{\mathcal{C}}, I\Lambda)]Id_{(F,I\Lambda)}$$

when evaluated at object p in $\mathcal{C}$, becomes a commutative diagram



in the category $\mathcal{D}$.

Definition 13.4. *A $\mathcal{B}$-equivariant natural transformation $\tau\colon (F, \theta) \Rightarrow (G, \zeta)$ between $\mathcal{B}$-covariant functors $(F, \theta), (G, \zeta)\colon (\mathcal{C}, \Lambda, \Phi, \alpha, \iota) \to (\mathcal{D}, \Psi, \Omega, \beta, \kappa)$ is a natural transformation $\tau\colon F \Rightarrow G$ such that diagram*



(13.1)

commutes, which means that we have a following identity

$$\zeta[A' \circ (\tau \times Id_{\mathcal{B}_1})] = (\tau \circ A)\theta$$

that becomes a commutative diagram

$$\begin{array}{ccc} F(p) \triangleleft f & \xrightarrow{\theta_{p,f}} & F(p \triangleleft f) \\ \Big\downarrow{\scriptstyle \tau_p \triangleleft f} & & \Big\downarrow{\scriptstyle \tau_{p \triangleleft f}} \\ G(p) \triangleleft f & \xrightarrow[\zeta_{p,f}]{} & G(p \triangleleft f) \end{array}$$

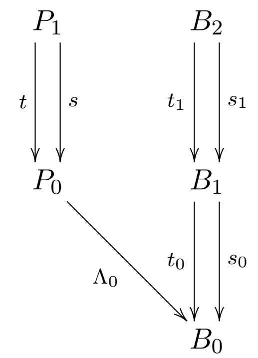
in the category $\mathcal{D}$, when evaluated at object p in $\mathcal{C}$.

The above construction gives rise to the 2-category in an obvious way, so we have a following theorem.

Theorem 13.1. *The class of $\mathcal{B}$-categories, $\mathcal{B}$ equivariant functors and their natural transformations form a 2-category.*

Proof. The vertical and horizontal composition in a 2-category is induced from the composition in Cat. □

Let $\mathcal{B}$ be a bicategory and $\mathcal{P}$ a category together with a momentum functor $\Lambda\colon \mathcal{P} \to \mathcal{B}_0$



(13.2)

and let $\mathcal{B}$ acts on $\mathcal{P}$ via an action functor

$$A\colon \mathcal{P} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{P} \tag{13.3}$$

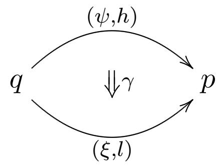
which satisfies coherence axioms from Definition 13.1. Such actions allows us to introduce a fundamental objects which we will use later.

Theorem 13.2. *For any action (13.2) of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$, there exists an action bicategory $\mathcal{P} \triangleleft \mathcal{B}$ consisting of the following data:*

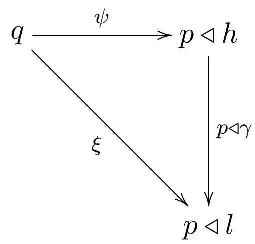
- *Objects of $\mathcal{P} \triangleleft \mathcal{B}$ are given by objects P_0 of the category $\mathcal{P}$*
- *a 1-morphism is a pair $(\psi, h)\colon q \to p$ which we draw as an arrow*

$$q \xrightarrow{(\psi,h)} p$$

  where $h\colon \Lambda_0(q) \to \Lambda_0(p)$ is a 1-morphism in the bicategory $\mathcal{B}$, and $\psi\colon q \to p \triangleleft h$ is a morphism in the category $\mathcal{P}$, thus it is an element of P_1.
- *a 2-morphism $\gamma\colon (\psi, h) \Rightarrow (\xi, l)$*



  is a 2-morphism $\gamma\colon h \Rightarrow l$ in B_2, such that the diagram of morphisms in $\mathcal{P}$



  commutes.

Proof. We define the composition for any two composable 1-morphisms

$$r \xrightarrow{(\phi,g)} q \xrightarrow{(\psi,h)} p$$

by $(\psi, h) \circ (\phi, g) = (\psi \circ \phi, h \circ g)\colon r \to p$, where $\psi \circ \phi\colon r \to p \triangleleft (h \circ g)$ is a morphism in $\mathcal{P}$, defined by the composition

$$r \xrightarrow{\phi} q \triangleleft g \xrightarrow{\psi \triangleleft g} (p \triangleleft h) \triangleleft g \xrightarrow{\kappa_{p,h,g}} p \triangleleft (h \circ g)$$

and we will show that this composition is a coherently associative. For any three composable 1-morphisms

$$s \xrightarrow{(\varphi,f)} r \xrightarrow{(\phi,g)} q \xrightarrow{(\psi,h)} p$$

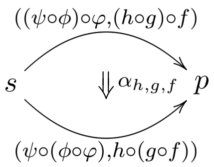
first we have a morphism $((\psi \circ \phi) \circ \varphi, (h \circ g) \circ f)$, where the first term is a composite of

$$s \xrightarrow{\varphi} r \lhd f \xrightarrow{(\psi\circ\phi)\lhd f} (p \lhd (h \circ g)) \lhd f \xrightarrow{\kappa_{p,h\circ g,f}} p \lhd ((h \circ g) \circ f)$$

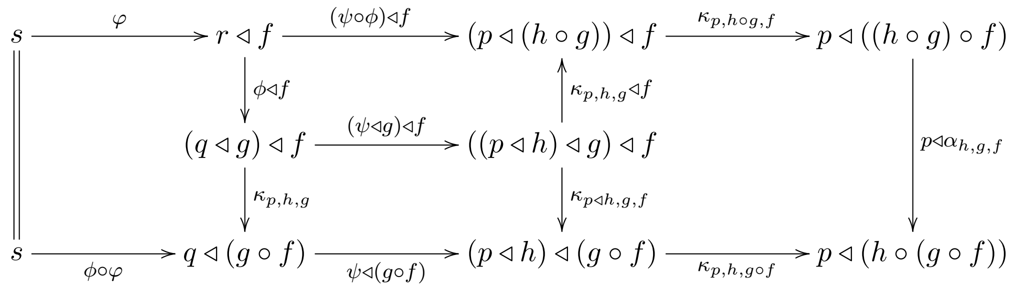
Also we have the composition $(\psi \circ (\phi \circ \varphi), h \circ (g \circ f))$, and the first term is given by a composite

$$s \xrightarrow{\phi\circ\varphi} q \lhd (g \circ f) \xrightarrow{\psi\lhd(g\circ f)} (p \lhd h) \lhd (g \circ f) \xrightarrow{\kappa_{p,h,g\circ f}} p \lhd (h \circ (g \circ f))$$

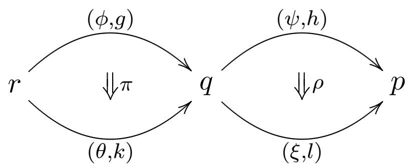
and the component of the associativity $\alpha_{h,g,f} \colon (h \circ g) \circ f \to h \circ (g \circ f)$, defines a 2-morphism



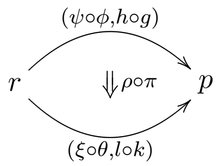
which we see from the commutativity of the diagram



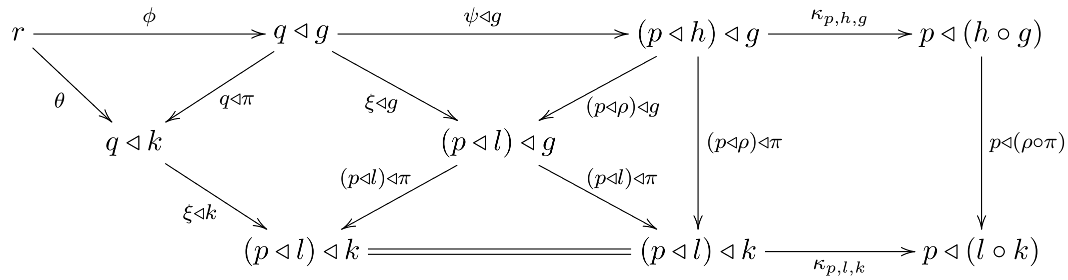
that follows from the definition of the horizontal composition, the naturality and the coherence for quasiassociativity of the action. The horizontal composition of 2-morphisms



is given by the horizontal composition in B_2



since we have a commutative diagram



which follows from the interchange law and the naturality of the coherence for the quasi-associativity of the action. The vertical composition of 2-morphisms in $\mathcal{P} \triangleleft \mathcal{B}$ is similarly induced from the one in $\mathcal{B}$. The coherence of the horizontal composition in $\mathcal{P} \triangleleft \mathcal{B}$ is immediately given by the coherence of the horizontal composition in $\mathcal{B}$. □

Proposition 13.1. *There exists a canonical projection*

$$\Lambda\colon \mathcal{P} \triangleleft \mathcal{B} \to \mathcal{B} \tag{13.4}$$

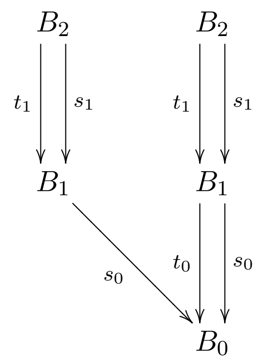
which is a strict homomorphism of bicategories.

Proof. A homomorphism $\Lambda\colon \mathcal{P} \triangleleft \mathcal{B} \to \mathcal{B}$ is defined by (the component of) the momentum functor $\Lambda_0(p) = \lambda_0(p)$, for any object p in $\mathcal{P} \triangleleft \mathcal{B}$. For any 1-morphism (ψ, h) it is defined by $\Lambda_1(\psi, h) = h$, and for any 2-morphism $\gamma\colon (\psi, h) \Rightarrow (\xi, l)$ in $\mathcal{P} \triangleleft \mathcal{B}$, it is given simply by $\Lambda_2(\gamma) = \gamma$. Then we have a following identity

$$\Lambda((\psi, h) \circ (\phi, g)) = \Lambda(\psi \circ \phi, h \circ g) = h \circ g = \Lambda(\psi, h) \circ \Lambda(\phi, g)$$

which means that this homomorphism is strict (it preserves a composition strictly). □

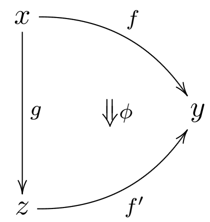
Example 13.1. *The right action of a bicategory $\mathcal{B}$ on itself is given by a diagram*



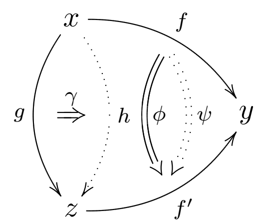
where a momentum functor is given by the source $S\colon \mathcal{B}_1 \to \mathcal{B}_0$ and an action functor is given by a horizontal composition $H\colon \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{B}_1$. Any object of an action bicategory $\mathcal{B}_1 \triangleleft \mathcal{B}$ is an element of B_1, which which is a 1-morphism

$$x \xrightarrow{f} y.$$

A 1-morphism from an object f to an object f' is a pair $(\phi, g)\colon f \to f'$ as in the diagram



where $\phi\colon f \Rightarrow f' \circ g$ is a 2-morphism in $\mathcal{B}$. A 2-morphism $\gamma\colon (\phi, g) \Rightarrow (\psi, h)$ is a diagram



where $\gamma\colon g \Rightarrow h$ is a 2-morphism in $\mathcal{B}$ such that identity $\psi = (f' \circ \gamma)\phi$ holds. We will denote an action bicategory $\mathcal{B}_1 \triangleleft \mathcal{B}$ by $T\mathcal{B}$, and we call it a tangent bicategory because the 2-bundle

$$T\colon T\mathcal{B} \to B_0 \tag{13.5}$$

(which associates to all above diagrams an object y) is a generalization of a tangent 2-bundle introduced by Roberts and Schreiber in [79] in the case of strict 2-categories. This example of an action bicategory plays a crucial role in understanding of universal 2-bundles. We will later in Example 15.1 relate the construction of a tangent 2-bundle with a décalage construction (2.1) introduced in Chapter 2.

14 Bigroupoid 2-torsors

Definition 14.1. *A right action of a bigroupoid $\mathcal{B}$ on a groupoid $\mathcal{P}$ is given by the action of the underlying bicategory $\mathcal{B}$ on a category $\mathcal{P}$ given as previously by $(\mathcal{P}, \mathcal{B}, \Lambda, A, \alpha, \iota)$.*

Definition 14.2. *Let $\mathcal{B}$ be an internal bigroupoid in $\mathcal{E}$, and $\pi \colon \mathcal{P} \to X$ a right $\mathcal{B}$-2-bundle of groupoids over X in $\mathcal{E}$. We say that $(\mathcal{P}, \pi, \Lambda, A, X)$ is a right $\mathcal{B}$-principal-2-bundle (or a right $\mathcal{B}$-torsor) over X if the following conditions are satisfied:*

- *the projection morphism $\pi_0 \colon P_0 \to X$ is an epimorphism,*
- *the action morphism $\lambda_0 \colon P_0 \to B_0$ is an epimorphism,*
- *the induced internal functor*

$$(Pr_1, A) \colon \mathcal{P} \times_{B_0} \mathcal{B}_1 \to \mathcal{P} \times_X \mathcal{P} \tag{14.1}$$

  is a (strong) equivalence of internal groupoids over $\mathcal{P}$ (where both groupoids are seen as objects over $\mathcal{P}$ by the first projection functor).

Example 14.1. *(The trivial 2-torsor) The trivial 2-torsor is given by the triple $(\mathcal{B}_1, T, S, \mathcal{H}, B_0)$ where the momentum is given by the source functor $S \colon \mathcal{B}_1 \to \mathcal{B}_0$, and the action is given by the horizontal composition $H \colon \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{B}_1$.*

Example 14.2. *For any $\mathcal{B}$-2-torsor $(\mathcal{P}, \pi, \Lambda, A, X)$ over X, and any morphism $f \colon M \to B_0$, we have a pullback $\mathcal{B}$-2-torsor over M, defined by the quadruple $(f^*(\mathcal{P}), Pr_1, \Lambda \circ Pr_2, f^*(A), X)$.*

Since we assumed that the functor (14.1) is an equivalence, we choose its weak inverse

$$(Pr_1, D) \colon \mathcal{P} \times_X \mathcal{P} \to \mathcal{P} \times_{B_0} \mathcal{B}_1$$

together with natural isomorphisms

$$(Pr_1, \mu) \colon Id_{\mathcal{P} \times_{B_0} \mathcal{B}_1} \Rightarrow (Pr_1, D) \circ (Pr_1, A), \qquad (Pr_1, \nu) \colon (Pr_1, A) \circ (Pr_1, D) \Rightarrow Id_{\mathcal{P} \times_X \mathcal{P}}.$$

The second component of the above weak inverse is (what we call) the *division functor*

$$D \colon \mathcal{P} \times_X \mathcal{P} \to \mathcal{B}_1 \tag{14.2}$$

and its value on any object $(p, q) \in \mathcal{P} \times_X \mathcal{P}$ is a 1-morphism $D(p, q)$ of $\mathcal{B}$ which we denote

$$p^*q \colon \lambda_0(q) \to \lambda_0(p)$$

and for any morphism $(\gamma, \delta) \colon (p, q) \to (r, s)$ in $\mathcal{P} \times_X \mathcal{P}$ we have a 2-morphism $D(\gamma, \delta)$ of $\mathcal{B}$

$$\gamma^*\delta \colon p^*q \Rightarrow r^*s.$$

The component of the natural isomorphism $\nu\colon A\circ(\mathrm{Pr}_1, D) \Rightarrow \mathrm{Pr}_2$ between the two functors from $\mathcal{P} \times_X \mathcal{P}$ to $\mathcal{P}$, indexed by an object (p, q) in $\mathcal{P} \times_X \mathcal{P}$ is given by an isomorphism

$$\nu_{p,q}\colon p \triangleleft p^* q \to q.$$

The component of the natural isomorphism $\mu\colon \mathrm{Pr}_1 \Rightarrow D\circ(Pr_1, A)$ between the two functors from $\mathcal{P} \times_{B_0} \mathcal{B}_1$ to $\mathcal{P}$ indexed by an object (p, f) in $\mathcal{P} \times_{B_0} \mathcal{B}_1$ is given by an isomorphism

$$\mu_{p,f}\colon p \to p^*(p \triangleleft f).$$

When the category $\mathcal{E}$ is the category Top of topological spaces, we use local sections $\sigma_i\colon U_i \to P_0$ of the map $\pi_0\colon P_0 \to X$ over some covering $\mathcal{U} = (U_i)_{i\in I}$ of the base space X. We use the division functor to define morphisms $g_{ij} = \sigma_i^* \sigma_j\colon U_{ij} \to B_1$, and the morphisms

$$f_{ij}\colon \sigma_j \to \sigma_i \triangleleft g_{ij}$$

are defined by the inverse of the component $\nu_{\sigma_i,\sigma_j}\colon \sigma_i \triangleleft \sigma_i^* \sigma_j \to \sigma_j$. The following diagram

$$\begin{array}{ccc}
\sigma_k & \xrightarrow{\quad f_{jk} \quad} & \sigma_j \triangleleft g_{jk} \\
 & & \downarrow{\scriptstyle f_{ij}\triangleleft g_{jk}} \\
{\scriptstyle f_{ik}}\downarrow & & (\sigma_i \triangleleft g_{ij}) \triangleleft g_{jk} \\
 & & \downarrow{\scriptstyle \kappa_{ijk}} \\
\sigma_i \triangleleft g_{ik} & \xrightarrow[\sigma_i\triangleleft\beta_{ijk}]{} & \sigma_i \triangleleft (g_{ij} \circ g_{jk})
\end{array}$$

defines a morphism in $\psi \in Hom_{\mathcal{P}\times_X\mathcal{P}}(\sigma_i \triangleleft g_{ik}, \sigma_i \triangleleft (g_{ij} \circ g_{jk}))$ by the composition

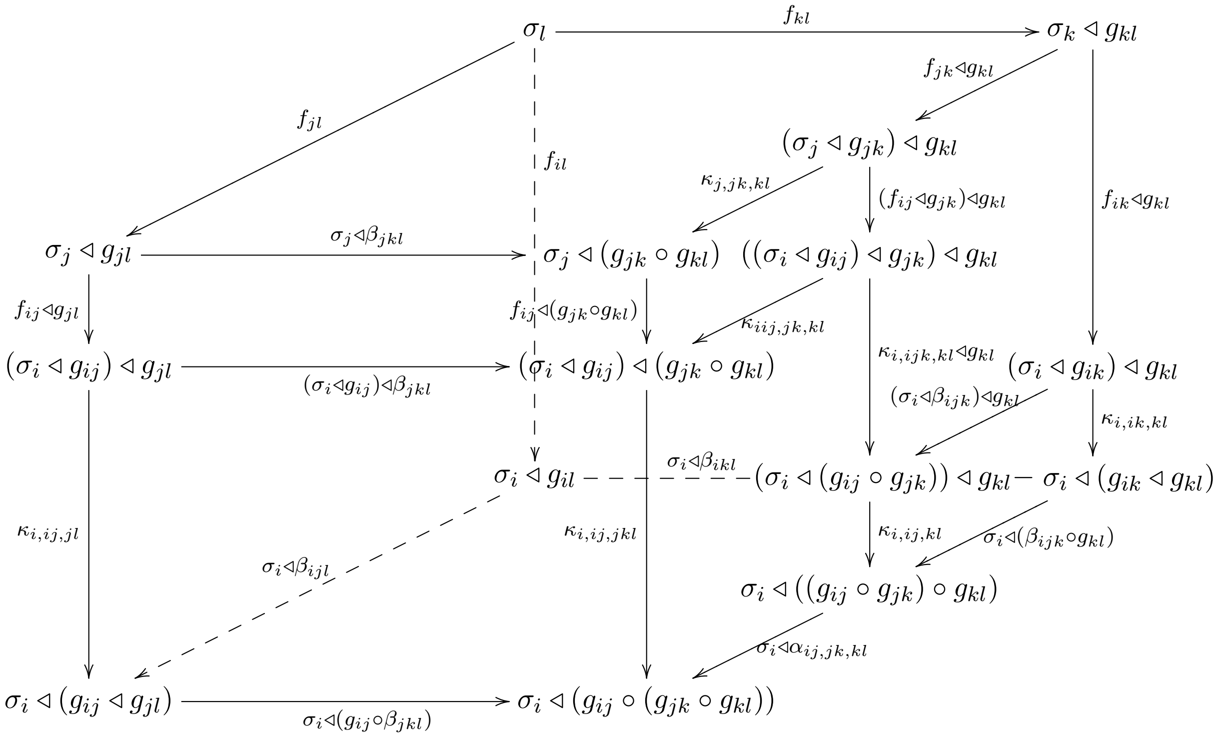
$$\sigma_i \triangleleft g_{ik} \xrightarrow{f_{ik}^{-1}} \sigma_k \xrightarrow{f_{jk}} \sigma_j \triangleleft g_{jk} \xrightarrow{f_{ij}\triangleleft g_{jk}} (\sigma_i \triangleleft g_{ij}) \triangleleft g_{jk} \xrightarrow{\kappa_{ijk}} \sigma_i \triangleleft (g_{ij} \circ g_{jk})$$

and since the set $Hom_{\mathcal{P}\times_X\mathcal{P}}(\sigma_i \triangleleft g_{ik}, \sigma_i \triangleleft (g_{ij} \circ g_{jk}))$ is an image of the induced functor (Pr_1, Φ) which defines a bijective correspondence with the set $Hom_{\mathcal{P}\times_{\mathcal{B}_0}\mathcal{B}_1}((\sigma_i, g_{ik}), (\sigma_i, g_{ij}\circ g_{jk}))$ the inverse image of ψ defines sections $\beta_{ijk}\colon g_{ik} \to g_{ij}\circ g_{jk}$ in B_2, such that the diagram becomes the identity

$$(\sigma_i \triangleleft \beta_{ijk}) f_{ik} = \kappa_{ijk}(f_{ij} \triangleleft g_{jk}) f_{jk}.$$

Theorem 14.1. *Any $\mathcal{B}$-2-torsor $\pi\colon \mathcal{P} \to X$ gives rise to the Čech 2-cocycle in $\mathcal{H}^2(\mathcal{U},\mathcal{B})$ for some covering $\mathcal{U} = (U_i)_{i\in I}$ of the base space X.*

Proof. Let's take local sections $\sigma_i\colon U_i \to P_0$ of the map $\pi_0\colon P_0 \to X$ over some covering $\mathcal{U} = (U_i)_{i\in I}$ of the base space X. We define local sections $\tau_i\colon U_i \to B_0$ as objects of the small 2-fibration $\mathcal{FB}$ over the fiber U_i by $\tau_i = \lambda_0\sigma_i$. Then consider the following cube



in which the top, left and back faces are the defining diagrams for nonabelian cocycles. The top right part of the right face consists of one such diagram acted by g^{η}_{kl}, and the top left and bottom right part of the right face are two instances of naturality of the action, while the bottom left part of the right face as well as bottom part of the front face is the coherence for an action. The top part of the front face is the commutativity of an action obtained by factoring in two (equal) ways the morphism $f_{ij} \lhd \beta_{jkl}\colon \sigma_j \lhd g_{jl} \to (\sigma_i \lhd g_{ij}) \lhd (g_{jk} \circ g_{kl})$. Since these five faces of the cube in which all arrows are invertible commute, it follows that the sixth (bottom) face

$$\begin{array}{ccc}
\sigma_i \lhd g_{il} & \xrightarrow{\sigma_i \lhd \beta_{ikl}} & \sigma_i \lhd (g_{ik} \circ g_{kl}) \\
 & & \downarrow {\scriptstyle \sigma_i \lhd (\beta_{ijk} \circ g_{kl})} \\
{\scriptstyle \sigma_i \lhd \beta_{ijl}} \downarrow & & \sigma_i \lhd ((g_{ij} \circ g_{jk}) \circ g_{kl}) \\
 & & \downarrow {\scriptstyle \sigma_i \lhd \alpha_{ij,jk,kl}} \\
\sigma_i \lhd (g_{ij} \circ g_{jl}) & \xrightarrow[\sigma_i \lhd (g_{ij} \circ \beta_{ijk})]{} & \sigma_i \lhd (g_{ij} \circ (g_{jk} \circ g_{kl}))
\end{array}$$

also commutes. Since the functor $(Pr_1, A)\colon \mathcal{P} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{P} \times_X \mathcal{P}$ is fully faithful, the inverse image of the diagonal 2-morphism from $\sigma_i \lhd g_{il}$ to $\sigma_i \lhd (g_{ij} \circ (g_{jk} \circ g_{kl}))$ in the above diagram, consists of the single 2-morphism between g_{il} and $(g_{ij} \circ (g_{jk} \circ g_{kl}))$ which gives the identity

$$(g_{ij} \circ \beta_{jkl})\beta_{ijl} = \alpha_{ijkl}(\beta_{ijk} \circ g_{kl})\beta_{ikl}$$

for the nonabelian 2-cocycle (g_{ij}, β_{ijk}) with values in the bigroupoid $\mathcal{B}$. □

Now we describe the gluing construction which is inverse to the construction from the previous theorem.

Theorem 14.2. *For any 2-cocycle $(\tau_i, g_{ij}, \beta_{ijk})$ in $\mathcal{H}^2(\mathcal{U}, \mathcal{B})$, there exists a $\mathcal{B}$-2-torsor $\pi\colon \mathcal{P} \to X$ over X together with an equivalence*

$$\phi\colon \tau^*(\mathcal{B}_1) \longrightarrow \mathcal{P}|_U$$

over $\mathcal{U}$.

Proof. We take the 2-cocycle $(\tau_i, g_{ij}, \beta_{ijk})$ in $\mathcal{H}^2(\mathcal{U}, \mathcal{B})$, with respect to some covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, and a 2-truncation of the simplicial resolution $U_\bullet$

$$U \times_X U \times_X U \overset{d_0}{\underset{d_2}{\Rrightarrow}} U \times_X U \overset{d_0}{\underset{d_1}{\rightrightarrows}} U \xrightarrow{\;e\;} X$$

of the epimorphism $e = (e_i)_{i \in I}\colon U = \coprod_{i \in I} U_i \to X$, induced by a family of embeddings $e_i\colon U_i \to X$. This is just the nerve of the Čech groupoid associated to the covering $e\colon U \to X$, whose objects are given by the elements (i, x) of U, and unique morphisms $(i, j, x)\colon (j, x) \to (i, x)$ between any two elements in the same fiber. Thus, target and source morphisms $d_0, d_1 \colon U \times_X U \to U$ are given by the first and the second projection, respectively.

The construction of the 2-torsor $\mathcal{P}$ is given by the pseudocolimit of the pseudosimplicial category over the simplicial resolution $U_\bullet$ of the covering $\tau \colon U \to X$

$$\begin{array}{ccccccc}
\mathcal{R}_2 & \overset{D_0}{\underset{D_2}{\Rrightarrow}} & \mathcal{R}_1 & \overset{D_0}{\underset{D_1}{\rightrightarrows}} & \mathcal{R}_0 & \xrightarrow{\eta} & \mathcal{P} \\
\downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \pi} \\
U_2 & \overset{d_0}{\underset{d_2}{\Rrightarrow}} & U_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & U_0 & \xrightarrow{e} & X
\end{array}$$

where $U_0 = U$, $U_1 = U \times_X U$, $U_2 = U \times_X U \times_X U$, and each groupoid $\mathcal{R}_n$ is a pullback

$$\begin{array}{ccc}
\mathcal{R}_n & \xrightarrow{\pi_n} & \mathcal{B}_1 \\
\downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle D_0} \\
U_n & \xrightarrow[\tau_n]{} & \mathcal{B}_0
\end{array} \tag{14.3}$$

of the *trivial right* $\mathcal{B}$*-torsor* $D_0 \colon \mathcal{B}_1 \to B_0$ by the unique morphism $\tau_n = \tau d^n \colon U_n \to B_0$ and the morphism $d^n \colon U_n \to U_0$ is defined by $d^n = d_n d_{n-1} \ldots d_1$ for $n \geq 1$, and $d^0 = id_U$.

Explicitly, on the level of objects, the category $\mathcal{R}_0$ is given by the pullback $\tau^*(\mathcal{B}_1)$ of the trivial $\mathcal{B}$-2-torsor $T \colon \mathcal{B}_1 \to \mathcal{B}_0$. Object of the category $\mathcal{R}_0$ are triples (i, x, f) where $\tau_i(x) = t_0(f)$, and any morphism is given by a triple $(i, x, \phi) \colon (i, x, f) \to (i, x, f')$ where $\phi \colon f \Rightarrow f'$ is a 2-morphism in B_2, such that $\tau_i(x) = T(\phi)$. The composition in $\mathcal{R}_0$ is inherited from the vertical composition of 2-morphisms in $\mathcal{B}$, and the functor $\rho_0 \colon \mathcal{R}_0 \to U$ is given by the projection on the first two factors.

The category $\rho_1 \colon \mathcal{R}_1 \to U \times_X U$ over $U \times_X U$ is defined by the pullback $\mathcal{R} = (\tau d_1)^*(\mathcal{B}_1)$. Objects of the category $\mathcal{R}_1$ are quadruples (i, j, x, g) where $\sigma_j(x) = t_0(g)$, and any morphism is given by a quadruple $(i, j, x, \psi) \colon (i, j, x, g) \to (i, j, x, g')$ where $\psi \colon g \Rightarrow g'$ is again a 2-morphism in B_2, such that $\sigma_j(x) = T(\psi)$.

The category $\rho_2 \colon \mathcal{R}_2 \to U \times_X U \times_X U$ over $U \times_X U \times_X U$ is defined by the pullback $\mathcal{R} = (\tau d_2 d_1)^*(\mathcal{B}_1)$, so its objects and morphisms are given by quintuples as above.

The first two face functors $D_0, D_1 \colon \mathcal{R}_1 \to \mathcal{R}_0$ are defined for any 1-simplex (i, j, x, g) in $\mathcal{R}_1$ by

$$D_i(i, j, x, g) = \begin{cases} (i, x, g_{ij}(x) g) & i = 0 \\ (j, x, g) & i = 1 \end{cases} \tag{14.4}$$

on the level of objects and similarly on the level of morphisms.

The next three face functors $D_0, D_1, D_2\colon \mathcal{R}_2 \to \mathcal{R}_1$ are defined by

$$D_i(i,j,k,x,h) = \begin{cases} (i,j,x,g_{jk}(x)h) & i=0 \\ (i,k,x,h) & i=1 \\ (j,k,x,h) & i=2 \end{cases} \tag{14.5}$$

on the level of objects and similarly on the level of morphisms.

The following simplicial identities of functors hold on the nose

$$D_1D_1(i,j,k,x,h) = D_1(i,k,x,h) = (k,x,h) = D_1(j,k,x,h) = D_1D_2(i,j,k,x,h)$$
$$D_0D_2(i,j,k,x,h) = D_0(j,k,x,h) = (j,x,g_{jk}(x)h) = D_0(i,j,x,g_{jk}(x)h) = D_1D_0(i,j,k,x,h)$$

The nontrivial simplicial identity is given by a natural isomorphism $\beta\colon D_0D_0 \Rightarrow D_0D_1$, whose component indexed by an object (i,j,k,x,h) of $\mathcal{R}_2$ is given by a morphism (i,x,β_{ijk}^{-1}) from the object $D_0D_0(i,j,k,x,h) = D_0(i,j,x,g_{jk}(x)h) = (i,x,g_{ij}(x)g_{jk}(x)h)$ to the object $D_0D_1(i,j,k,x,h) = D_0(i,k,x,h) = (i,x,g_{ik}(x)h)$.

We construct the category $\mathcal{P}$ as a pseudocolimit of the pseudosimplicial category $\mathcal{R}_\bullet$. It is given by a version of the Grothendieck construction, and it goes as follows.

The objects of $\mathcal{P}$ are given by the union of objects of $\mathcal{R}_n$. We describe morphisms in $\mathcal{P}$ by means of a particular example. A morphism $(m,\phi)\colon (i,x,f) \to (i,j,k,x,g)$ from an object (i,x,f) in $\mathcal{R}_0$ to an object (i,j,k,x,g) in $\mathcal{R}_2$ is given by a pair of morphisms, where $m\colon [0] \to [2]$ is a monotonic map in Δ, whose canonical factorization in Δ is given by $m = \delta_1\delta_0$ (so that we have $U(m)(i,j,k,x) = (i,x)$ in U_1). Then the second component of the above pair is given by a morphism $\phi\colon (i,x,f) \to \mathcal{R}(m)(i,j,k,x,g) = (i,x,g_{ik}(x)g)$ in $\mathcal{R}_0$. For another morphism $(n,\psi)\colon (i,j,k,x,g) \to (i,j,k,l,x,h)$, where $n = \delta_1\colon [2] \to [3]$ and $\psi\colon (i,j,k,x,g) \to \mathcal{R}(n)(i,j,k,l,x,h) = (i,j,k,x,g_{kl}(x)h)$, the composition is defined by a pair $(nm, \psi\circ\phi)\colon (i,x,f) \to (i,j,k,l,x,h)$, where the morphism $\psi\circ\phi\colon (i,x,f) \to (i,k,l,x,h)$ is defined by the composition

$$(i,x,f) \xrightarrow{\phi} \mathcal{R}(m)(i,j,k,x,g) \xrightarrow{\mathcal{R}(m)(\psi)} \mathcal{R}(m)\mathcal{R}(n)(i,j,k,l,x,h) \xrightarrow{\sim} \mathcal{R}(nm)(i,j,k,l,x,h)$$

where the last isomorphism is obtained from the component of the natural isomorphism $\beta\colon D_0D_0 \Rightarrow D_0D_1$.

The projection $\pi\colon \mathcal{P} \to X$ is explicitly described by $\pi_0(i,j,x,h) = x$ on the level of objects. Also we have a momentum functor $\lambda\colon \mathcal{P} \to B_0$, defined by $\pi_0(i,j,x,h) = s_0(h)$, and the action functor is naturally defined by the horizontal composition

$$(i,j,x,h) \triangleleft g = (i,j,x,h\circ g). \tag{14.6}$$

It follows that the functor $\pi\colon \mathcal{P} \to M$ is a $\mathcal{B}$-2-torsor over X, with respect to an action (14.6). $\square$

15 Simplicial interpretation of bigroupoid 2-torsors

Let we describe the simplicial set $\mathcal{P}_\bullet$ arising by an application of the Duskin nerve functor

$$N_2 \colon Bicat \to \mathcal{S}Set$$

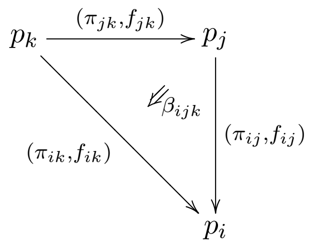
to an action bicategory $\mathcal{P} \triangleleft \mathcal{B}$. The set of 0-simplices is P_0 and any 1-simplex is an arrow

$$p_j \xrightarrow{(\pi_{ij}, f_{ij})} p_i$$

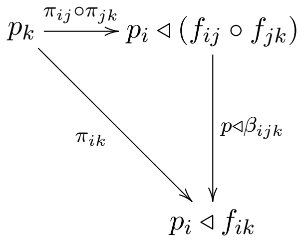
and face operators are defined by $d_0^1(\pi_{ij}, f_{ij}) = p_i$ and $d_1^1(\pi_{ij}, f_{ij}) = p_j$, while the degeneracy is defined by $s_0^1(p_i) = (\iota_{p_i}, i_{p_i})$ and it is given by the arrow

$$p_i \xrightarrow{(\iota_{p_i}, i_{p_i})} p_i$$

where the morphism $\iota_{p_i} \colon p_i \to p_i \triangleleft i_{\Lambda_0(p_i)}$ is an identity coherence of the action. A 2-simplex in $\mathcal{P}_\bullet$ is of the form



where the diagram



of morphisms in $\mathcal{P}$ commutes, and the morphism $\pi_{ij} \circ \pi_{jk} \colon p_k \to p_i \triangleleft (f_{ij} \circ f_{jk})$ is the composite of

$$p_k \xrightarrow{\pi_{jk}} p_j \triangleleft f_{jk} \xrightarrow{\pi_{ij} \triangleleft f_{jk}} (p_i \triangleleft f_{ij}) \triangleleft f_{jk} \xrightarrow{\kappa_{i,j,k}} p_i \triangleleft (f_{ij} \circ f_{jk})$$

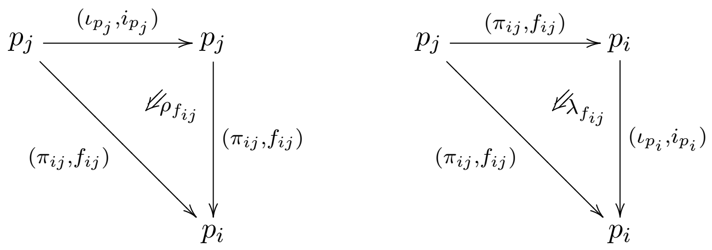
of morphisms in $\mathcal{P}$. Face operators are defined by

$$d_0^2(\beta_{ijk}) = (\pi_{jk}, f_{jk})$$
$$d_1^2(\beta_{ijk}) = (\pi_{ik}, f_{ik})$$
$$d_2^2(\beta_{ijk}) = (\pi_{ij}, f_{ij})$$

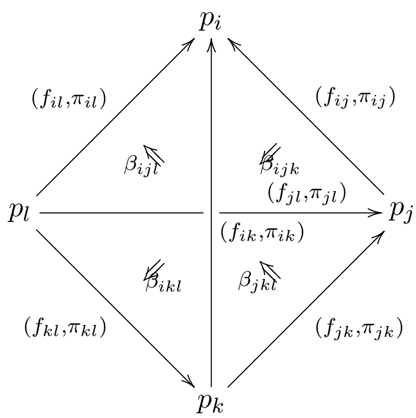
and the degeneracy operators are given by

$$s_0^2(\pi_{ij}, f_{ij}) = \rho_{f_{ij}}$$
$$s_1^2(\pi_{ij}, f_{ij}) = \lambda_{f_{ij}}$$

which are the two 2-simplices



respectively, where the 1-morphisms $\rho_{f_{ij}}\colon f_{ij} \circ i_{p_j} \to f_{ij}$ and $\lambda_{f_{ij}}\colon i_{p_i} \circ f_{ij} \to f_{ij}$ are the components of the right and left identity natural isomorphisms in $\mathcal{B}$.
A general 3-simplex is of the form



where we have an identity

$$\beta_{ikl}(\beta_{ijk} \circ f_{kl}) = \alpha_{ijkl}\beta_{ijl}(\beta_{jkl} \circ f_{ij})$$

which is just a nonabelian 2-cocycle condition.

Example 15.1. *Let $B_\bullet$ be Duskin nerve for a bicategory $\mathcal{B}$. The tangent bicategory $T\mathcal{B}$ from Example 13.1. is action bicategory for the right action of $\mathcal{B}$ on itself and a décalage construction (2.1) from Chapter 2 becomes the diagram of simplicial sets*

$$
\begin{array}{ccccccccccc}
B_0 & = & B_0 & = & B_0 & = & B_0 & \cdots & Sk^0(B_\bullet) \\
{\scriptstyle d_0}\uparrow\downarrow{\scriptstyle s_0} & & {\scriptstyle d_0^2}\uparrow\downarrow{\scriptstyle s_0^2} & & {\scriptstyle d_0^3}\uparrow\downarrow{\scriptstyle s_0^3} & & {\scriptstyle d_0^4}\uparrow\downarrow{\scriptstyle s_0^4} & & {\scriptstyle D_0}\uparrow\downarrow{\scriptstyle S_0} \\
B_1 & \underset{d_0}{\overset{d_1}{\leftrightarrows}} & B_2 & \underset{d_0}{\overset{d_2}{\leftrightarrows}} & B_3 & \leftrightarrows & B_4 & \cdots & Dec(B_\bullet) \\
{\scriptstyle s_0}\uparrow\downarrow{\scriptstyle d_1} & & {\scriptstyle s_1}\uparrow\downarrow{\scriptstyle d_2} & & {\scriptstyle s_2}\uparrow\downarrow{\scriptstyle d_3} & & {\scriptstyle s_3}\uparrow\downarrow{\scriptstyle d_4} & & {\scriptstyle S_1}\uparrow\downarrow{\scriptstyle D_1} \\
B_0 & \underset{d_0}{\overset{d_1}{\leftrightarrows}} & B_1 & \underset{d_0}{\overset{d_2}{\leftrightarrows}} & B_2 & \leftrightarrows & B_3 & \cdots & B_\bullet
\end{array}
$$

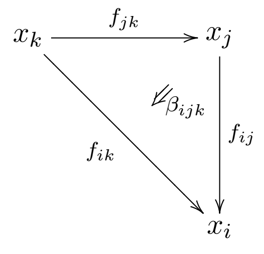
in which $D_1\colon Dec(B_\bullet) \to B_\bullet$ is a simplicial map which is the Duskin nerve of the canonical projection $\Lambda\colon T\mathcal{B} \to \mathcal{B}$ and $D_0\colon Dec(B_\bullet) \to B_\bullet$ is a simplicial map which is the Duskin nerve of the tangent 2-bundle $T\colon T\mathcal{B} \to B_0$

Theorem 15.1. *Let the bigroupoid $\mathcal{B}$ acts on a groupoid $\mathcal{P}$. Then the Duskin nerve of the canonical projection (13.4) is a simplicial map $\Lambda_\bullet = \mathcal{N}_2(\Lambda)\colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ which is a simplicial action of the Duskin nerve $B_\bullet$ on the bigroupoid $\mathcal{B}$, i.e. it is an exact fibration for all $n \geq 2$.*

Proof. We need to show that for any $n \geq 2$ and for any k such that $0 \leq k \leq n$, the diagram

$$
\begin{array}{ccc}
P_n & \xrightarrow{\lambda_n} & B_n \\
{\scriptstyle p_{\bar{k}}}\downarrow & & \downarrow{\scriptstyle p_{\bar{k}}} \\
\bigwedge_n^k(\mathcal{P}_\bullet) & \xrightarrow[\lambda_n^k]{} & \bigwedge_n^k(\mathcal{B}_\bullet)
\end{array}
$$

is a pullback. A k-horn $((f_{ij}, \pi_{ij}), ..., (f_{j,k-1}, \pi_{j,k-1}), (f_{k,k+1}, \pi_{k,k+1}), ..., (f_{n-1,n}, \pi_{n-1,n}))$ in $\bigwedge_n^k(\mathcal{P}_\bullet)$ is given by the n-tuple of 1-morphisms in $\mathcal{A}_{\mathcal{B}}\mathcal{P}$, and its image by $\lambda_2^k\colon \bigwedge_2^k(\mathcal{P}_\bullet) \to \bigwedge_2^k(\mathcal{P}_\bullet)$ is a k-horn in $\bigwedge_n^k(\mathcal{B}_\bullet)$, given by the n-tuple $(f_{ij}, ..., f_{j,k-1}, f_{k,k+1}, ..., f_{n-1,n})$ of 1-morphisms in $\mathcal{B}$. For example, in the case $n = 2$, any filler of a 1-horn $(f_{ij}, -, f_{jk})$ in $\bigwedge_2^1(\mathcal{B}_\bullet)$, is the 2-simplex



in B_2. A 2-simplex in $\mathcal{P}_\bullet$ is a lifting of the previous 2-simplex if it is of the form

$$\begin{array}{ccc} p_k & \xrightarrow{(\pi_{jk},f_{jk})} & p_j \\ & \searrow^{(\pi_{ik},f_{ik})} \quad \Swarrow \beta_{ijk} & \downarrow{\scriptstyle (\pi_{ij},f_{ij})} \\ & & p_i \end{array}$$

where the diagram

$$\begin{array}{ccc} p_k & \xrightarrow{\pi_{ij}\circ\pi_{jk}} & p_i \lhd (f_{ij} \circ f_{jk}) \\ & \searrow^{\pi_{ik}} & \downarrow{\scriptstyle p\lhd\beta_{ijk}} \\ & & p_i \lhd f_{ik} \end{array}$$

of morphisms in $\mathcal{P}$ commutes, and the morphism $\pi_{ij} \circ \pi_{jk} \colon p_k \to p_i \lhd (f_{ij} \circ f_{jk})$ is the composite of

$$p_k \xrightarrow{\pi_{jk}} p_j \lhd f_{jk} \xrightarrow{\pi_{ij}\lhd f_{jk}} (p_i \lhd f_{ij}) \lhd f_{jk} \xrightarrow{\kappa_{i,j,k}} p_i \lhd (f_{ij} \circ f_{jk})$$

so we see that a pair $((f_{ij}, \pi_{ij}), -, (f_{jk}, \pi_{jk}), \beta_{ijk})$ in $\bigwedge_2^1(\mathcal{P}_\bullet) \times_{\bigwedge_2^1(\mathcal{B}_\bullet)} B_2$ uniquely determines above 2-simplex in $\mathcal{P}_2$. Since $\mathcal{P}$ is a groupoid, any pair consisting of a k-horn in $\bigwedge_2^k(\mathcal{B}_\bullet)$, for $k = 0, 2$, and a 2-simplex in $\mathcal{B}_2$ which covers the k-horn, uniquely determines a 2-simplex in $\mathcal{P}_2$, and thus provides a canonical isomorphism $P_2 \simeq \bigwedge_2^k(\mathcal{P}_\bullet) \times_{\bigwedge_2^k(\mathcal{B}_\bullet)} B_2$. Since both simplicial objects are 2-coskeletal, the assertion follows for all $n \geq 2$. □

Observe that even in the case when we just have an action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$, the above condition for an exact fibration is still satisfied for inner horns $0 < k < n$. Thus it is sensible to introduce weakened concept of an exact fibration.

Definition 15.1. *A simplicial map $\Lambda_\bullet \colon \mathcal{E}_\bullet \to \mathcal{B}_\bullet$ is a weak exact fibration in dimension n if diagrams*

$$\begin{array}{ccc} E_n & \xrightarrow{\lambda_n} & B_n \\ \downarrow{\scriptstyle p_{\bar{k}}} & & \downarrow{\scriptstyle p_{\bar{k}}} \\ \bigwedge_n^k(\mathcal{E}_\bullet) & \longrightarrow & \bigwedge_n^k(\mathcal{B}_\bullet) \end{array}$$

are pullbacks for all $0 < k < n$. We call it a weak exact fibration if it is a weak exact fibration in all dimensions.

With respect to this definition we generalize the simplicial actions of n-dimensional hypergroupoids to the case of weak n-dimensional Kan complexes. First we give their formal definition.

Definition 15.2. *A weak n-dimensional Kan hypergroupoid $G_\bullet$ in $\mathcal{E}$ is a weak Kan complex such that the canonical map $G_m \to \bigwedge_m^k(G_\bullet)$ is an isomorphism for all $m > n$ and $0 < k < m$.*

Now we generalize actions with respect to this simplicial objects.

Definition 15.3. *An action of the n-dimensional Kan complex is an internal simplicial map $\Lambda_\bullet \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ in $\mathcal{E}$ which is a weak exact fibration for all $m \geq n$.*

This concept provides a following simplicial characterization of an action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$.

Theorem 15.2. *Let the bicategory $\mathcal{B}$ acts on a category $\mathcal{P}$. Then the simplicial map $\Lambda_\bullet = \mathcal{N}_2(\Lambda) \colon P_\bullet \to B_\bullet$ is a simplicial action of the Duskin nerve $B_\bullet$ of the bicategory $\mathcal{B}$, i.e. it is a weak exact fibration for all $n \geq 2$.*

In the case of the bigroupoid $\mathcal{B}$, the Duskin nerve functor is a 2-dimensional hypergroupoid $\mathcal{B}_\bullet = \mathcal{N}_2(\mathcal{B})$ and let $\mathcal{P}_\bullet = \mathcal{N}_2(\mathcal{A}_\mathcal{B}\mathcal{P})$ be the Duskin nerve of an action bigroupoid associated to the action of the bigroupoid $\mathcal{B}$ on the groupoid $\mathcal{P}$. Glenn introduced in [36] a simplicial definition of an n-dimensional hypergroupoid n-torsor in $\mathcal{E}$.

Definition 15.4. *An action $\Lambda_\bullet \colon P_\bullet \to \mathcal{B}_\bullet$ is the n-dimensional hypergroupoid n-torsor over X in $\mathcal{E}$ if $P_\bullet$ is augmented over X, aspherical and n-1-coskeletal ($P_\bullet \simeq Cosk^{n-1}(P_\bullet)$).*

In the case of the bigroupoid $\mathcal{B}$, the above definition reduces to the following definition.

Definition 15.5. *A bigroupoid $\mathcal{B}_\bullet$ 2-torsor over an object X in $\mathcal{E}$ is an internal simplicial map $\Lambda_\bullet \colon P_\bullet \to \mathcal{B}_\bullet$ in $\mathcal{S}(\mathcal{E})$, which is an exact fibration for all $n \geq 2$, and where $P_\bullet$ is augmented over X, aspherical and 1-coskeletal ($P_\bullet \simeq Cosk^1(P_\bullet)$).*

Thus in the case when an action of $\mathcal{B}$ on $\mathcal{P}$ is principal, we have the following result.

Theorem 15.3. *Let $\mathcal{P}$ be a $\mathcal{B}$-2-torsor over X. Then simplicial map $\Lambda_\bullet = \mathcal{N}_2(\Lambda) \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ is a Duskin-Glenn 2-torsor.*

Proof. The simplicial complex $\mathcal{P}_\bullet$ is augmented over X because the action of $\mathcal{B}$ is fiberwise, since for any 1-simplex $(f_{ij}, \pi_{ij}) \colon p_j \to p_i$ in P_0, where $\pi_{ij} \colon p_j \to p_i \triangleleft f_{ij}$ we have

$$\pi_0 d_0(f_{ij}, \pi_{ij}) = \pi_0(p_i) = \pi_0(p_i \triangleleft f_{ij}) = \pi_1(\pi_{ij}) = \pi_0(p_j) = \pi_0 d_1(f_{ij}, \pi_{ij}).$$

The simplicial complex $\mathcal{P}_\bullet$ is obviously aspherical and we prove now that it is also 1-coskeletal. A general 2-simplex in $Cosk^1(P_\bullet)_2$ is a triple $((f_{ij},\pi_{ij}),(f_{ik},\pi_{ik}),(f_{jk},\pi_{jk}))$ which we see as the triangle

$$\begin{array}{ccc} p_k & \xrightarrow{(\pi_{jk},f_{jk})} & p_j \\ & {\scriptstyle (\pi_{ik},f_{ik})}\searrow & \big\downarrow {\scriptstyle (\pi_{ij},f_{ij})} \\ & & p_i \end{array}$$

from which we have morphisms $\pi_{ij}\circ\pi_{jk}\colon p_k \to p_i \triangleleft (f_{ij}\circ f_{jk})$ and $\pi_{ik}\colon p_k \to p_i \triangleleft f_{ik}$ in $\mathcal{P}$. Now we use the fact that the induced functor

$$(Pr_1,\mathcal{A})\colon \mathcal{P}\times_{B_0}\mathcal{B}_1 \longrightarrow \mathcal{P}\times_X\mathcal{P}$$

is a (strong) equivalence of internal groupoids over $\mathcal{P}$, and therefore fully faithful. Specially, for the two objects $(p_i, f_{ij}\circ f_{jk})$ and (p_i,f_{ik}) of $\mathcal{P}\times_{B_0}\mathcal{B}_1$, this equivalence induces a bijection

$$Hom_{\mathcal{P}\times_{B_0}\mathcal{B}_1}((p_i,f_{ij}\circ f_{jk}),(p_i,f_{ik})) \simeq Hom_{\mathcal{P}\times_X\mathcal{P}}((p_i,p_i\triangleleft(f_{ij}\circ f_{jk})),(p_i,p_i\triangleleft f_{ik}))$$

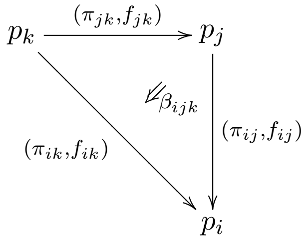
and therefore for a morphism $(id_{p_i},\pi_{ik}\circ(\pi_{ij}\circ\pi_{jk})^{-1})\colon (p_i,p_i\triangleleft(f_{ij}\circ f_{jk}))\to(p_i,p_i\triangleleft f_{ik}))$

$$\begin{array}{ccc} p_k & \xleftarrow{(\pi_{ij}\circ\pi_{jk})^{-1}} & p_i\triangleleft(f_{ij}\circ f_{jk}) \\ & {\scriptstyle \pi_{ik}}\searrow & \\ & & p_i\triangleleft f_{ik} \end{array}$$

there exists a unique 2-morphism $\beta_{ijk}\colon f_{ij}\circ f_{jk}\to f_{ik}$ in $\mathcal{B}$, such that the diagram

$$\begin{array}{ccc} p_k & \xrightarrow{\pi_{ij}\circ\pi_{jk}} & p_i\triangleleft(f_{ij}\circ f_{jk}) \\ & {\scriptstyle \pi_{ik}}\searrow & \big\downarrow {\scriptstyle p\triangleleft\beta_{ijk}} \\ & & p_i\triangleleft f_{ik} \end{array}$$

commutes, and this uniquely determines a 2-simplex



in $\mathcal{P}_2$, which proves that we have a bijection $\mathcal{P}_2 \simeq Cosk^1(P_\bullet)_2$. From here it follows immediately that $\mathcal{P}_\bullet \simeq Cosk^1(P_\bullet)$. $\square$

Part IV
The bibliography and biographical data

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Education

Graduate studies

- Faculty of Sciences and Mathematics, Split 1993-1996,
- Faculty of Sciences and Mathematics, Zagreb 1997-1999

Diploma thesis: Rational points on elliptic curves (prof. Dujella, adviser, Faculty of Sciences and Mathematics, Zagreb, 1999)

Postgraduate studies in physics

- Ludwig-Maximilians University, Munich 2000-2003,

Doctorate studies in mathematics

- Rudjer Boskovic Institute, Zagreb, 2004-2007

Ph.D. thesis: Bigroupoid 2-torsors (prof. Jurčo, adviser, Ludwig-Maximilians University, Munich 2007)

Conferences

- Noncommutativity - Geometry and Probability, July 10-14, 2000, University of Nottingham, United Kingdom
- XIII International Congress on Mathematical Physics, July 17-22, 2000, Imperial College, London, United Kingdom
- Mathematical Challenges of the 21st Century, August 07-12, 2000, UCLA, Los Angeles, USA
- Noncommutative Yang-Mills theories, Bayrischzell Workshop 2001, March 30 - April 02, 2001, Bayrischzell, Germany
- Noncommutative Geometry and Quantum Groups, September 17-29, 2001, Banach Institute, Warsaw
- Bayrischzell Workshop 2002, April 26-29, 2002, Bayrischzell,
- Noncommutativity: From mathematics to phenomenology, Bayrischzell Workshop 2003, May 02-05, 2002, Bayrischzell
- Groupoids and stacks in physics and geometry, June 28 - July 02, 2004, Centre International des Recontres Mathématiques, Marseilles

- Mathematische Arbeitstagung 2005, June 10-16, Max Planck Institute for Mathematics, Bonn, Germany
- Gerbes, Groupoids and Quantum field theory, May 9-13, 2006, Erwin Schroedinger Institute, Vienna, Austria
- Workshop on Higher categories and their applications, January 9-13, 2007, Fields Institute, Toronto
- Arbeitsgemeinschaft: Conformal Field Theory, 1 - 7 April 7, 2007, Mathematisches Forschungsinstitut Oberwolfach
- IV Seminar on Categories and Applications, 6 - 9 June, 2007, Centre de Recerca Matemàtica, Barcelona

Talks

- Nonabelian bundle gerbes, talk at Ludwig-Maximilians University, April, 2006, Munich
- Bigroupoid 2-torsors, talk at the program Gerbes, Groupoids and Quantum field theory, May 9-13, 2006, Erwin Schroedinger Institute, Vienna
- 2-groupoid 2-torsors, talk at the University of Hamburg, October 31, 2006, Hamburg
- The second nonabelian cohomology, talk at Ludwig-Maximilians University, December 15, 2006, Munich
- Bigroupoid principal 2-bundles, talk at Workshop on Higher categories and their applications, January 9-13, 2007, Fields Institute, Toronto
- Bigroupoid principal 2-bundles and gerbes, talk at IV Seminar on Categories and Applications, 6 - 9 June, 2007, Centre de Recerca Matemàtica, Barcelona

Research

- nonabelian cohomology
- nonabelian algebraic topology
- higher category theory
- homotopy theory