An Effective D-Brane Action in Quantized Anti de Sitter Space

and

The Local Renormalization Group of $\mathcal{N} = 1$ Supersymmetric Gauge Theories

Ludwig–Maximilians–Universität München
Fakultät für Physik

Dissertation

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Stephan Höhne
geboren am 8. April 1978 in Zittau

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erste Gutachterin: PD Dr. Johanna Erdmenger

zweiter Gutachter: Prof. Dr. Dieter Lüst

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Preface

This thesis is an account of my work as a doctoral student at the Max–Planck–Institut für Physik between October 2005 and May 2008. I performed research in two lines of investigation.

In a joint project with Martin Ammon, Johanna Erdmenger, Dieter Lüst and René Meyer we were studying the noncommutative $U(1)$ instanton in type IIB string theory on $AdS_5 \times S^5$ background space. The resulting effective action can be interpreted in terms of the AdS/CFT correspondence. The results are reported in chapters 2 and 4. Related work is published in

M. Ammon, J. Erdmenger, S. Höhne, D. Lüst and R. Meyer,
"Fayet–Iliopoulos terms in AdS/CFT with flavour,"
JHEP **07** (2008) 068, arXiv:0805.1917 [hep-th].

In a joint project with Johanna Erdmenger and Johannes Große the renormalization group flow of $\mathcal{N} = 1$ supersymmetric gauge theories was studied in view of finding a central function along the flow. Chapter 6 contains the results of my research.

Zusammenfassung

Im ersten Teil dieser Dissertation wird ein geometrisches Modell in der Typ IIB Stringtheorie untersucht. Die Einbettung einer einzelnen $D7$-Bran Probe in einen $AdS_5 \times S^5$ Hintergrundraum mit konstantem antisymmetrischem B-Feld wird betrachtet. Die effektive Wirkung dieser Konfiguration wird berechnet und auf einem statischen nichtkommutativen Instanton Eichfeld ausgewertet. Das resultierende Potential legt einen Vakuumerwartungswert für das B-Feld fest. Dieses phänomenologische Modell wird im Rahmen der AdS/CFT Korrespondenz interpretiert.

Im zweiten Teil werden renormierbare supersymmetrische Eichtheorien im Funktionalintegralformalismus der Quantenfeldtheorie untersucht. Sie sind an einen vierdimensionalen klassischen gekrümmten Hintergrundraum gekoppelt. Dimensionslose lokale Theta-Kopplungen und ein $U(1)$ R-Vektorfeld werden als äußere Felder im Vakuumenergiefunktional eingeführt. Sie sind Quellen für lokale zusammengesetzte Operatoren und R-Ströme. Die lokalen Ward-Identitäten für die Weyl-Symmetrie und die R-Symmetrie in $\mathcal{N} = 1$ supersymmetrischen Eichtheorien werden hingeschrieben. Die Wess–Zumino Konsistenzbedingungen werden ausgewertet. Das Ziel ist, eine Zentralfunktion für den Koeffizienten einer Gravitationsanomalie zu finden.

Abstract

In the first part of this thesis, a geometric model in type IIB string theory is studied. The embedding of a single $D7$-brane probe in an $AdS_5 \times S^5$ background space with constant antisymmetric B-field is considered. The effective action of this configuration is calculated. The action is evaluated on a static noncommutative instanton gauge field. The resulting potential determines a vacuum expectation value for the B-field. This phenomenological model is interpreted in terms of the AdS/CFT correspondence.

In the second part, renormalizable supersymmetric gauge theories are investigated in the functional integral formalism of quantum field theory. They are coupled to a four dimensional classical curved background space. Dimensionless local theta couplings and an $U(1)$ R-vector field are introduced as auxiliary fields in the vacuum energy functional. They are sources for local composite operators and R-currents. The local Ward identities for Weyl symmetry and R-symmetry in $\mathcal{N} = 1$ supersymmetric gauge theories are written down. The Wess–Zumino consistency conditions are evaluated. The aim is to find a central function for the coefficient of a gravitational anomaly.

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Chapter 1

Introduction

This thesis consists of two parts, both of them are about supersymmetric gauge theories.

Part I: String Theory Phenomenology and Quantized Spacetime

Superstring theory provides modern physics with phenomenological models of relevance to several areas.

In elementary particle physics, orientifold compactification of the extra dimensions is expected to give a geometric interpretation of the free parameters of the minimal supersymmetric standard model. Forthcoming particle accelerators, in particular the LHC at CERN can in turn give restrictions on the string theory models [1]. In cosmology, Calabi–Yau compactifications with fluxes can describe four dimensional spacetimes with small positive cosmological constant and inflationary phase [2].

In hadronic physics, the AdS/CFT correspondence provides an additional tool besides lattice gauge theory for computations in the nonperturbative regime of quantum chromodynamics. Examples are the mass spectra of mesons and the thermodynamic properties of the quark gluon plasma observed at the RHIC collider [3].

In the first part of this thesis we study the consequences of quantized spacetime for phenomenological AdS/CFT models.

The AdS/CFT correspondence is a duality between ten dimensional type IIB string theory on $AdS_5 \times S^5$ curved background space and four dimensional $\mathcal{N} = 4$ Yang–Mills theory with gauge group $SU(N_C)$ in the large N_C limit [4, 5]. It relates a gravity theory at weak coupling to a gauge theory at strong coupling.

Computations in strongly coupled QCD simplify considerably in the 't Hooft limit, where the number of colors N_C is taken to be large. The 't Hooft coupling $\lambda = g_{YM}^2 N_C$ provides then a constant parameter for perturbative expansions [6].

The first important evidence for the duality is that the symmetries match on both sides. The Lie group $SO(4,2)$ is the isometry group of five dimensional anti de Sitter space. It is also the conformal symmetry group of four dimensional Minkowski space. The $SO(6)$ rotations of the five sphere on the gravity side are translated to the $SU(4)$ rotations of the fermionic superspace coordinates on the gauge theory side.

The second important evidence for the correspondence comes from the comparison of the perturbative diagrammatic expansions on both sides. In the 't Hooft limit, the vacuum polarization diagrams composed of gluon loops can be written down in a double line notation. The propagators of gluons in the adjoint representation $N_C^2 - 1$ of the gauge group are replaced by oriented double lines denoting quark–antiquark pairs in the product representation $N_C \otimes \bar{N}_C$ (fundamental times its conjugate).

The double line diagrams are topologically equivalent to orientable Riemann surfaces representing closed string interactions with coupling g_S. This is visualized in table 1.1.

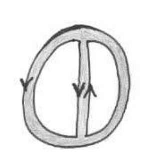
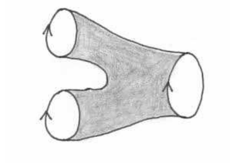
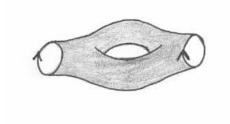
	 λN_C^2	 $\chi = -1$
 $\chi = +2$	 λ^2	 $\chi = -2$
simplex $\chi = v - e + f$	double line diagram $\lambda = g_{YM}^2 N_C$	orientable Riemann surface $\chi = 2 - 2g - b - c$
	constant expansion parameter $\sim \lambda^{e-v}$	closed string interaction $\sim g_S^{-\chi}$

Table 1.1 – Double line diagrams in the large N_C limit are topologically equivalent to Riemann surfaces in string theory. We regard the compact domain in between the quark lines as "made of chewing gum," no compactification is needed. The left column shows simplices apparently similar to the double line diagrams. The edges e can be thought of as gluon propagators. They meet at interaction vertices v and enclose faces f. Their Euler characteristic χ can be found in the mathematics literature [7]. The change of the Euler number indicates a different topology of these gluon diagrams.

The double line diagrams in the middle column are composed of oriented quark and antiquark propagators. In the right column, the oriented quark lines are interpreted as strings, and the shaded areas represent the string worldsheet. These Riemann surfaces are characterized by their holes g, boundaries b and cross caps c.

The planar diagrams are weighted by N_C^2 and dominate the perturbative expansion. The nonplanar diagrams are suppressed by a factor N_C^{-2}. The large N_C expansion and the closed string genus expansion are of the same weight if the number of colors is inversely proportional to the string coupling g_S.

The concepts visualized in these pictures are of crucial importance to the AdS/CFT

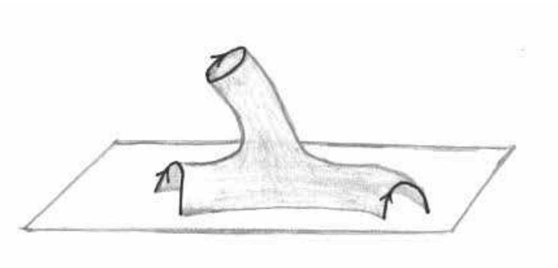


Figure 1.1 – A propagating oriented open string emits a closed string graviton. The plane at the bottom represents a D-brane.

correspondence. A phenomenologically interesting effect appears when we cut the world-sheet of the closed string, so that a boundary is introduced. The result is an open string ending on a D-brane, see for example figure 1.1. Open strings with one end on the D-brane are interpreted as quarks in the fundamental representation of the gauge group.

The embedding of D-branes in the curved background space on the gravity side allows to incorporate matter fields with flavor degrees of freedom on the gauge theory side [8]. This construction is not yet completely rigorous since the probe branes extend over noncompact anti de Sitter space and so their charge is not conserved. A consistent treatment would need to compactify $AdS_5 \times S^5$ and include orientifold planes. As proposed in [8], an alternative solution to this problem is to consider an energy regime where the open string interactions are decoupled from the closed string gravity interactions.

The AdS/CFT correspondence is intended as a tool to describe nonperturbative strong coupling phenomena of hadronic physics quantitatively. The relevant gauge theory QCD is not supersymmetric and its renormalization group flow exhibits asymptotic freedom at high energies and confinement at low energies. In order to come closer to QCD, the conformal symmetry and the supersymmetry of the $\mathcal{N} = 4$ Yang–Mills theory must be broken.

The AdS/CFT correspondence provides a geometric description of the symmetries on the gauge theory side in terms of the ten dimensional space $AdS_5 \times S^5$. Therefore the symmetry breaking has to be realized geometrically on the gravity side. The S^5 is embedded in the six directions transversal to Minkowski space. The spherical symmetry is translated to the rotation symmetry of the fermionic superspace coordinates on the gauge theory side. The mechanism of supersymmetry breaking is achieved by placing geometric objects in these six dimensions. Their spatial positions and orientations are interpreted as vacuum expectation values of fluctuation fields.

A manifestation of this mechanism is the embedding of $D7$-brane probes into the $AdS_5 \times S^5$ geometry. For each brane, the two transverse fluctuation scalars and the six polarizations of the gauge field along the brane comprise the eight bosonic degrees of freedom of the ten dimensional $\mathcal{N} = 1$ vector supermultiplet in the massless type IIB open string excitation spectrum.

The starting point for our work is the paper [9]. We summarize here those results that are of relevance to our work. Two parallel $D7$-brane probes were embedded in the $AdS_5 \times S^5$ geometry in static gauge. The two branes were placed at the same position in the transverse coordinates z^8 and z^9. The $U(2) \cong SU(2) \times U(1)$ gauge theory on the worldvolume of this D-brane stack was studied. This setup has interesting effect that the gauge symmetry is spontaneously broken. All four transverse scalars have a vacuum expectation value. They play the role of Higgs bosons of translation symmetry.

Another important feature of this setup is of great relevance for phenomenology. The coinciding $D7$-brane probes were placed at a distance $z^9 = (2\pi\alpha')m$ from the background generating stack of $D3$-branes located at the origin. The two stacks of branes can form bound states with an interaction potential. The fluctuations of the tense $3-7$ strings between the branes give massive fields. These fields are translated with the AdS/CFT dictionary in two fundamental and two anti-fundamental quark hypermultiplets with mass m.

The center of mass of the $D7$ stack fixes the $U(1)$ factor in the unitary gauge group. The nonabelian $SU(2)_f$ gauge theory on the D-branes is then translated to a global isospin degree of freedom on the field theory side.

Mass terms for the fundamental hypermultiplets play an important phenomenological role, because they break the conformal symmetry on the field theory side by introducing a mass scale m. The vacuum expectation value for the massive fields is determined by evaluating the $SU(2)_f$ gauge theory on a static instanton configuration. The expectation value of a condensate of scalar quark superpartners is determined by the size parameter of the instanton.

The interplay between spontaneous symmetry breaking and an instanton solution motivated us to study an alternative setup that exhibits similar features.

The first part of this thesis is organized as follows. In chapter 2 we use the solitonic 3-brane type IIB supergravity solution found by Horowitz and Strominger [10]. Our additional ingredient is a constant antisymmetric Kalb–Ramond B-field. We study two vacuum expectation values with rank four, one is selfdual (2.10a), the other is anti-selfdual (2.10b).

We embed in this background a single $D7$-brane probe in static gauge, at the same position as the $D3$-brane stack in the transverse directions. We evaluate the effective D-brane action (2.35) for this setup and discuss its properties.

In chapter 3 we review the for us relevant aspects of the Seiberg–Witten map [11] between ordinary and noncommutative gauge theories. The closed string B-field is translated to the open string Θ coupling. The coordinates along the B-field are quantized via the boundary propagator of two open string vertices. We start in section 3.1 by writing down the bosonic and fermionic part of the classical type IIB sigma model. The aim is to obtain curvature corrections (3.11) for the open string vertex correlator. The B-field interpolates between Neumann and Dirichlet boundary conditions and therefore gives rise to a $D3$-brane instanton dissolved within the worldvolume of the $D7$-brane probe. This is discussed in section 3.2.

The relevant properties of the noncommutative $U(1)$ instanton we use are summarized in section 3.3. In short, the moduli space of the noncommutative $U(1)$ instanton has the same dimension as the moduli space of the ordinary $SU(2)$ instanton. With the Seiberg–Witten map, the $U(1)$ instanton on noncommutative $\mathbb{R}^4_{NC}$ found by Nekrasov and Schwarz [12] is translated to the $D3$-brane instanton due to the B-field.

In the case of broken supersymmetry, the interaction potential of the bound state between $D7$ probe and $D3$ instanton contains a Fayet–Iliopoulos term $V \sim \zeta^2$, where ζ is the size parameter of the noncommutative gauge theory instanton on $\mathbb{R}^4_{NC}$. The paper [13] investigates how the FI term can be translated via the AdS/CFT correspondence.

The main results of the first part of this work can be found in chapter 4. We argue in section 4.1 that the coordinates y^m along the B-field obtain a vacuum expectation value due to their noncommutative nature. So we conjecture that they may play the same role as the transverse scalar fluctuations and may give rise to an alternative mechanism of spontaneous symmetry breaking.

In sections 4.2 and 4.3 we evaluate the supersymmetric effective D-brane action from chapter 2 on a static instanton configuration, then we integrate out the noncommutative coordinates along the B-field. The resulting potential distinguishes a vacuum expectation value of the B-field, see figures 4.3 and 4.4. We discuss the parameter range of validity of this result. In order to interpret our results in terms of the AdS/CFT correspondence, we propose to investigate fluctuations (4.34) around the static instanton.

At the end of chapter 4 and in the conclusions chapter 7 we discuss possible deformations of the background metric. The AdS/CFT correspondence translates them to renormalization group flows that break supersymmetry on the field theory side.

At a more fundamental level, in this work we study the implications of the idea that spacetime is not a continuum at short distances [14, 15].

I would like to motivate this. The assumption that there exists a fundamental scale at very high energies relative to the system under consideration is often very useful. In quantum field theory, ultraviolet divergences are regularized with a high momentum cutoff. Due to the particle–wave duality, this high energy scale is translated to a short distance scale. In statistical physics, there is the atomic distance cutoff for phonons. Lattice gauge theory computations are based on the assumption that spacetime itself is discrete.

In practice, the coordinates of the discrete spacetime are quantized using the canonical rules. They are promoted to Hilbert space operators that obey the Heisenberg algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathtt{i}\zeta\Theta^{\mu\nu}\,. \tag{1.1}$$

This formula states that positions can not be measured simultaneously in different directions, just like the Heisenberg uncertainty relation states that position and momentum are not measurable simultaneously. It also introduces a fundamental quantum ζ of area. If existing, this scale would be a constant of nature.

In mathematical terms, the commutator (1.1) is a Lie algebra with central charge ζ. The structure constants $\Theta^{\mu\nu}$ are equal to one in the directions that do not commute.

The mathematical formulation of quantized spacetimes is in terms of noncommutative geometry [16].

The work presented in the first part of this thesis arose out of a joint project with Martin Ammon, Johanna Erdmenger, Dieter Lüst and René Meyer [13].

Part II: Central Functions in Quantum Field Theory

The predictions of quantum field theory are most accurately confirmed by collider experiments in elementary particle physics.

The Wilson duality allows for the application of quantum field theory methods to thermodynamics [17]. In this way scaling laws of correlation functions near second order phase transitions can be predicted. Due to universality the results can be applied to many thermodynamic systems, for example ferromagnets near the Curie temperature.

The variation of the partition functional with respect to the external fields gives correlation functions of the associated currents. A ferromagnet in the presence of a magnetic field is an example. An approach to do quantum field theory in curved spacetime is to regard the gravitational field as external. The metric is the source of the energy-momentum tensor current. In the second part of this thesis we study renormalizable supersymmetric quantum field theories in the presence of a gravitational background field.

We are especially interested in $\mathcal{N} = 1$ supersymmetric gauge theories. They are of importance in elementary particle physics because the minimal supersymmetric extension of the standard model is based on them. Here we investigate the local renormalization group of this theory.

The aim of our work is to find a function of the dimensionless local couplings that decreases monotonically along the renormalization group flow parametrized by the mass scale M,

$$M\frac{\partial}{\partial M}\ (\text{central function})\ = -\ (\text{positive quantity})\ . \tag{1.2}$$

We use the Wilson approach to the renormalization group [17]. In colloquial terms, we probe the system at the energy M, or equivalently, we look at it from the distance M^{-1}. The renormalization process consists of two steps. First a thin shell of high momentum Fourier modes is integrated out, and then the configuration space is rescaled. We look at the system from larger distances, or probing it with less energy. The flow generated by these steps leads to a coarse graining of degrees of freedom. We search for a monotonic function that describes this information loss.

We are especially interested in the conformal window of $\mathcal{N} = 1$ supersymmetric gauge theories with gauge group $SU(N_C)$ and N_f matter fields. In the parameter regime

$$\frac{N_C}{2} < \frac{N_f}{3} < N_C \tag{1.3}$$

they possess interacting infrared fixed points besides the ultraviolet fixed point. It is expected that the behavior near the critical points can be characterized by a central charge, a number that counts the massless degrees of freedoms of the system [18]. At the fixed points, a central function coincides with the central charges. Along the flow it interpolates continuously between them.

We study the behavior of the theory under conformal transformations. The global renormalization group rescalings $M \to (1 + \sigma)M$ are promoted to local Weyl transformations of the background metric,

$$g_{\mu\nu} \to \exp\left(-2\sigma\left(x\right)\right) g_{\mu\nu}\,. \tag{1.4}$$

We use functional integral methods to make statements about the symmetry properties of the quantum theory coupled to classical curved background. The response of the vacuum energy functional to infinitesimal conformal transformations is encoded in the Ward identity,

$$\Delta W = \int (\text{ anomalies } + \Delta \text{ counterterms }) \, . \tag{1.5}$$

The derivative operator Δ contains the variations with respect to the external fields. The anomalies and counterterms of the quantum theory appear on the right hand side. The conformal symmetry is broken explicitly by the renormalization scale and by the gravitational anomalies from the curved background. The gravitational anomalies are made of curvature tensor combinations. The numerical coefficients of the anomalies are conjectured to be the central charges of the quantum theory [18].

Anomalies are organized into the structure of a cohomology, "an anomaly is not a derivative and it does not have a derivative." The conformal anomalies belong to the Weyl cohomology, which is constructed in [19, 20] in the same manner as the de Rham cohomology for the exterior derivative of differential forms. The conformal anomalies are the 1-cocycles without coboundary, and the conformal variation operator is nilpotent $\Delta^2 = 0$.

An anomaly $\mathcal{A}$ is closed, but not exact. So it is defined as the nontrivial solution of

$$\Delta \int \mathcal{A} = 0 \, . \tag{1.6}$$

The integral sign indicates that the geometry is "integrated out," and only the topological properties of the anomaly remain. An example is the integral of the Euler density anomaly. It is equal to the Euler number that characterizes the topology of the background space. The trivial solutions of (1.6) are the counterterms.

In order to find a central function, we have to treat the dynamical renormalization scale anomalies on the same footing as the external gravitational anomalies. To achieve this, the dimensionless coupling parameters of the renormalizable quantum field theory are promoted to auxiliary fields.

These local couplings $\lambda^i(x, M)$ are functions of the coordinates of the background space and the renormalization scale. They act as sources for local composite dimension four operators.

There arise local anomalies in the Ward identity, which are built up from covariant derivatives of the dimensionless local couplings. The task is to write down a basis of independent tensor monomials, consistent with dimensional analysis. The local anomalies have the same tensor structure as the finite counterterms, so that scheme independence is ensured.

Our approach is inspired by Osborn [21]. The contribution of our work is to restrict the formalism to supersymmetric $\mathcal{N} = 1$ supersymmetric gauge theories. In the eight dimensional classical superspace background, the bosonic coordinates are accompanied by four fermionic coordinates. In order to investigate the abelian rotation symmetry of these fermionic coordinates, we promote the instanton angle of the Yang–Mills theory to a set of local couplings $\hat{\theta}^i(x)$. We introduce the external R-vector field $V_\mu(x)$ as the source for the local R-current.

We impose local symmetries, write down their anomalies and investigate the resulting constraints on the theory. The most important restriction on the anomalies of the theory is the Wess–Zumino consistency condition [22]. For Weyl cohomology, it states that two conformal transformations have to commute,

$$[\Delta', \Delta]\, W = 0\,. \tag{1.7}$$

We evaluate this important condition for the local Ward identities of R-symmetry and conformal symmetry. Further restrictions come from the demand that the theory be globally supersymmetric.

The resulting consistency relations between the coefficients of the local anomalies can lead to an algebraic relation of the form

$$M\frac{\partial}{\partial M}\left(\beta^{grav}+a_i\beta^i\right)=-c_{ij}\beta^i\beta^j\,. \tag{1.8}$$

The central function on the left hand side would consist of a gravitational anomaly coefficient β^{grav} and combinations of local coefficients a_i contracted with beta functions, so that they vanish at the fixed points. The right hand side would be a positive definite local coefficient matrix c_{ij}.

The second part of this thesis is organized as follows. In chapter 5 we discuss the method of local couplings. In section 5.1 we explain the strategy leading to central functions at hand of the conformal anomalies of Osborn [21]. In section 5.2 we present our ansatz for the local theta couplings and R-symmetry sources of $\mathcal{N}=1$ supersymmetric gauge theories, and we discuss the supersymmetry constraints.

In chapter 6 we write down the new local renormalization group of $\mathcal{N}=1$ supersymmetric gauge theories. In section 6.1 we present the Ward identity for local R-symmetry and its Wess–Zumino consistency condition. In section 6.2 we write down the conformal local renormalization group equation and evaluate the Wess–Zumino consistency condition. We present a special solution with interesting properties. We discuss in section 6.3 how the supersymmetry constraints may be implemented in our formalism. In section 6.4 we use our special solution to outline Osborn's recipe [21] leading to central functions.

The results are discussed in the conclusions chapter 7. Appendix A contains the variations of all the local anomalies. They are important for future reference.

Part I

An Effective D-Brane Action in Quantized Anti de Sitter Space

Chapter 2

The Effective D-Brane Action

In this chapter we calculate an effective D-brane action in type IIB string theory to second order in the Regge slope. We use the static embedding of a single $D7$-brane in the background generated by the solitonic 3-brane supergravity solution found by Horowitz and Strominger [10].

In section 2.1 we write down our setup, in section 2.2 we present the calculation of the Dirac–Born–Infeld as well as Chern–Simons action for the $D7$-brane to second order in α'. In section 2.3 we discuss the properties of the solution.

We begin with a brief introduction following the review by Blumenhagen et. al. [1] and the book of Polchinski [23]. In the microscopic interpretation, a Dp-brane is an extended p-dimensional object representing the boundary of the worldsheet of fundamental strings. To leading order in the string coupling g_S, the dynamics of massless open string modes contribute to the effective action

$$S_{eff} = S_{DBI} + S_{CS} \,. \tag{2.1}$$

The effective action contains the coupling of the open string modes to the various background form fields in the Neveu–Schwarz and Ramond–Ramond sectors of the closed string.

In type IIB superstring theory, the massless excitation spectrum of the closed bosonic string in the Neveu–Schwarz sector is organized into the symmetric graviton g_{MN}, the antisymmetric Kalb–Ramond field B_{MN} and the scalar dilaton Φ. These fields can acquire vacuum expectation values and are then regarded as background fields.

The Dirac–Born–Infeld action consists of the open string coupling to the background fields in the Neveu–Schwarz sector,

$$S_{DBI} = -\mu_p \int \mathrm{d}^{p+1}\xi \ \exp\left(-\Phi\right) \sqrt{-\det\left(P\left[g\right]_{ab} + 2\pi\alpha' \mathcal{F}_{ab}\right)} \, . \tag{2.2}$$

The DBI action depends on the gauge invariant combination

$$2\pi\alpha' \mathcal{F}_{ab} = P\left[B\right]_{ab} + 2\pi\alpha' F_{ab} \, . \tag{2.3}$$

The fundamental string couples electrically to the antisymmetric Kalb–Ramond field, which gives rise to a gauge theory with field strength F_{ab} on the worldvolume of the brane. The DBI action for a single brane can be seen as the generalization of the Born–Infeld action of electrodynamics. Since the graviton is part of the spectrum, the DBI action can also be interpreted as encoding the influence of the gravitational forces on the massive Dp-brane.

In the Ramond–Ramond sector, Dp-branes are considered as generalizations of electromagnetic sources, objects with charge

$$\mu_p = \frac{1}{\left(2\pi\right)^p \sqrt{\alpha'}^{p+1}} \tag{2.4}$$

with respect to the Ramond–Ramond potentials,

$$\mu_p \int_{Vol(Dp)} C_{p+1} \, . \tag{2.5}$$

In compact transverse spaces, the conservation of Ramond–Ramond charge is imposed in analogy to the Gauss law. In addition, configurations of multiple D-branes have to fulfill the consistency condition that their Ramond–Ramond charges cancel. Only then they are regarded as stable.

The Chern–Simons action describes the coupling of the open string modes to the Ramond–Ramond potentials and to the Neveu–Schwarz 2-forms. The multiplication in the exterior algebra of anticommuting forms is given by the wedge product. In order to perform the integration, the rank of the form fields has to be equal to the dimension of the worldvolume. The Chern–Simons action of a single Dp-brane is given by

$$S_{CS} = -\mu_p \int_{Vol(Dp)} \sum_r P\left[C_r \wedge \exp B_2\right] \wedge \exp\left(2\pi\alpha' F_2\right) \, . \tag{2.6}$$

From a macroscopic perspective, Dp-branes are solitonic solutions of the low energy supergravity equations of motion. From this geometric viewpoint they determine the classical background space of superstring theory, a manifold of dimension ten.

Into this background we can put a D-brane probe with finite energy density as "test object", so that it has no backreaction on the geometry of the target space. Then the D-brane probe can be seen as a hypersurface embedded in the ambient space manifold. The effective action can then be seen as a functional measuring the area. The couplings between the D-brane and the various form fields are interpreted as surface tension forces. The effective action is proportional to the tension μ_p/g_S of the D-brane. There exists an energetically favored shape, the surface of minimal tension [24].

2.1 Our Setup

We use the classical type IIB supergravity solution found by Horowitz and Strominger [10]. This extended black hole solution consists of N_C parallel $D3$-branes at coincident

$AdS_5 \times S^5$									
0	1	2	3	4	5	6	7	8	9
X^M									
x^μ				y_m				z^i	
$D7$-brane, ξ^a									
				B-field					

Table 2.1 – Our setup of background fields and probe brane.

positions in the transverse space. The branes are both electric and magnetic sources for the C_4 potential. The 5-form field strength is selfdual and the string coupling is given by the constant vacuum expectation value of the dilaton field,

$$g_S = \langle \exp \Phi \rangle \,. \tag{2.7}$$

The form of the metric solution is constrained by Poincaré invariance along the branes and rotational symmetry in the transverse directions. There is still the freedom to rescale the metric by a warp factor,

$$g_{MN}\, \mathrm{d}X^M \mathrm{d}X^N = \frac{\eta_{\mu\nu}}{\sqrt{H}}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + \sqrt{H} \left(\delta_{mn} \mathrm{d}y^m \mathrm{d}y^n + \delta_{ij} \mathrm{d}z^i \mathrm{d}z^j \right) \,. \tag{2.8}$$

The warp factor H in the $D3$-brane metric solution is a harmonic function of the transverse coordinates, characterized by the curvature parameter R,

$$H = 1 + \frac{R^4}{\left(y^2 + z^2\right)^2} \,. \tag{2.9}$$

We study two vacuum expectation values B^+ and B^- for the Kalb–Ramond background field. Both B-fields are extended in the y^m directions, see table 2.1. The components of the 2-forms in coordinate basis are

$$B^+_{MN}\, \mathrm{d}X^M \wedge \mathrm{d}X^N = b \left(\mathrm{d}y^4 \wedge \mathrm{d}y^5 + \mathrm{d}y^6 \wedge \mathrm{d}y^7 \right) \,, \tag{2.10a}$$

$$B^-_{MN}\, \mathrm{d}X^M \wedge \mathrm{d}X^N = b \left(\mathrm{d}y^4 \wedge \mathrm{d}y^6 + \mathrm{d}y^5 \wedge \mathrm{d}y^7 \right) \,. \tag{2.10b}$$

The dimensionless parameter b is a constant real number. The Pfaffians of the component matrices are $Pf(B^+) = b^2 > 0$ for the selfdual ansatz and $Pf(B^-) = -b^2 < 0$ for the antiselfdual ansatz. Both sets of background fields are solutions to the supergravity equations of motion.

We embed a single $D7$-brane probe into this background geometry. We consider the static gauge embedding

$$X^\mu (\xi^a) = x^\mu \,, \tag{2.11a}$$

$$X^m (\xi^a) = y^m \,, \tag{2.11b}$$

$$\begin{aligned} X^i (\xi^a) &= z^i \\ &= 0 \,. \end{aligned} \tag{2.11c}$$

The $D7$-brane is situated at the origin in the transversal directions, at the same position as the $D3$-branes generating the background.

2.2 Calculation of Dirac–Born–Infeld and Chern–Simons Action

Here we perform the calculation of the effective action at hand of the particularly interesting case of selfdual B-field. We use the solitonic $D3$-brane background given in (2.8) and (2.10). The Dirac–Born–Infeld action for the embedded $D7$-brane is given by

$$S_{DBI} = -\frac{\mu_7}{g_S} \int \mathrm{d}^8\xi \; e^{-\Phi} \sqrt{-\det \left(P\left[g + B\right]_{ab} + 2\pi\alpha' F_{ab} \right)} \,. \tag{2.12}$$

The field strength tensor F_{ab} for the abelian gauge theory living on the $D7$-brane is chosen to be nontrivial only in the y^m directions,

$$F_{ab} = \left(\begin{array}{c|cccc} 0_{4\times4} & & 0_{4\times4} & & \\ \hline & 0 & F_{45} & F_{46} & F_{47} \\ 0_{4\times4} & -F_{45} & 0 & F_{56} & F_{57} \\ & -F_{46} & -F_{56} & 0 & F_{67} \\ & -F_{47} & -F_{57} & -F_{67} & 0 \end{array} \right) . \tag{2.13}$$

For the calculation of the effective action, we leave the components of the field strength generic. In chapter 4 we will specify the gauge field.

The form fields g and B live in the 2-cotangent space of the ambient manifold. In order to evaluate the DBI action, we have to take their pullback P to the hypersurface 2-cotangent space by differentiating the embedding functions with respect to the D-brane coordinates. In our case the pullback is the restriction map and we can directly start with the components in the basis $\mathrm{d}y^m \otimes \mathrm{d}y^n$ on the $D7$-brane 2-cotangent space. We calculated the DBI action for both B-fields, here we write down the selfdual case. We abbreviate

$$E_{ab} := P\left[g + B^+\right]_{ab} . \tag{2.14}$$

With the background 2-forms g and B as in equations (2.8) and (2.10), the component matrix E is given explicitly by

$$E = \left(\begin{array}{cccc|cccc} -\sqrt{H}^{-1} & 0 & 0 & 0 & & & & \\ 0 & \sqrt{H}^{-1} & 0 & 0 & & 0_{4\times4} & & \\ 0 & 0 & \sqrt{H}^{-1} & 0 & & & & \\ 0 & 0 & 0 & \sqrt{H}^{-1} & & & & \\ \hline & & & & \sqrt{H} & b & 0 & 0 \\ & 0_{4\times4} & & & -b & \sqrt{H} & 0 & 0 \\ & & & & 0 & 0 & \sqrt{H} & b \\ & & & & 0 & 0 & -b & \sqrt{H} \end{array}\right) . \tag{2.15}$$

We expand the DBI action (2.12) to second order in α'. We make use of the canonical algebraic properties of determinants,

$$\sqrt{-\det\left(E + 2\pi\alpha' F\right)} = \sqrt{-\det E}\sqrt{\det\left(1 + 2\pi\alpha' E^{-1}F\right)} . \tag{2.16}$$

With $\sqrt{-\det E} = 1 + b^2/H$ we are left with the calculation of the second square root. Defining the matrix

$$A := 2\pi\alpha' E^{-1} F \tag{2.17}$$

we use the expansion formula

$$\sqrt{\det\left(1+A\right)} = 1 + \tfrac{1}{2}\mathrm{Tr}A + \tfrac{1}{8}\left(\mathrm{Tr}A\right)^2 - \tfrac{1}{4}\mathrm{Tr}\left(A^2\right) + (\text{higher powers of } A). \tag{2.18}$$

In the limit $\alpha' \to 0$ it is justified to neglect the terms of higher powers in A. The component matrix A in the four y_m directions is given explicitly by

$$A = \frac{2\pi\alpha' b}{H+b^2}\begin{pmatrix} F_{45} & F_{45}h & F_{46}h - F_{56} & F_{47}h - F_{57} \\ -F_{45}h & F_{45} & F_{46} + F_{56}h & F_{47} + F_{57}h \\ -F_{46}h + F_{47} & -F_{56}h + F_{57} & F_{67} & F_{67}h \\ -F_{46} - F_{47}h & -F_{56} - F_{57}h & -F_{67}h & F_{67} \end{pmatrix} \tag{2.19}$$

$$h = \frac{\sqrt{H}}{b}\,. \tag{2.20}$$

All other entries involving x^μ directions are zero. The relevant matrix trace for (2.18) is

$$\mathrm{Tr}\left(A^2\right) = \left(\frac{2\pi\alpha' b}{H+b^2}\right)^2 \Big(2\left(F_{45}^2 + F_{67}^2 - 2F_{47}F_{56} + 2F_{46}F_{57}\right) - 2h^2\left(F_{45}^2 + F_{46}^2 + F_{47}^2 + F_{56}^2 + F_{57}^2 + F_{67}^2\right)\Big)\,. \tag{2.21}$$

Together with the other terms of (2.18) we obtain the DBI action expanded to second order in α',

$$S_{DBI}^{(2)} = -\frac{\mu_7}{g_S}\int \mathrm{d}^4x \int \mathrm{d}^4y \, \left(\mathcal{L}_{DBI}^{(0)} + \left(2\pi\alpha'\right)\mathcal{L}_{DBI}^{(1)} + \tfrac{1}{2}\left(2\pi\alpha'\right)^2\mathcal{L}_{DBI}^{(2)}\right), \tag{2.22}$$

$$\mathcal{L}_{DBI}^{(0)} = 1 + \frac{b^2}{H}\,, \tag{2.23}$$

$$\mathcal{L}_{DBI}^{(1)} = \frac{b}{H}\left(F_{45} + F_{67}\right), \tag{2.24}$$

$$\mathcal{L}_{DBI}^{(2)} = \frac{1}{H+b^2}\left[\tfrac{1}{2}F_{mn}F_{mn} + \tfrac{1}{4}\left(\frac{b^2}{H}\right)\epsilon_{mnkl}F_{mn}F_{kl}\right]. \tag{2.25}$$

We continue with the Chern–Simons action. In the massless type II excitation spectrum, the 64 spacetime bosons in the Neveu–Schwarz sector are accompanied by 64

Ramond fields, in order to match the 128 fermions of the mixed sector. In the type IIB theory, we have the antisymmetric tensor fields C_0, C_2 and C_4, consisting of 1, 28, 35 bosonic components, respectively.

In the supergravity solution of Horowitz and Strominger [10] the $D3$-branes are both electric and magnetic sources for C_4. We use the Freund-Rubin ansatz [25],

$$C_4 = (g_S H)^{-1} \, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \, . \tag{2.26}$$

The charge of the $D3$-branes with respect to C_4 is conserved,

$$\oint_{S^5} {}^*F_5 = N_C \, . \tag{2.27}$$

The branes are point particles in the perpendicular directions, and the integral is over the compact five sphere. We refer to the unit of $D3$-brane charge as positive, and the total charge N_C is equal to the number of $D3$-branes.

We consider the $D7$ as a test charge in the static potential C_4 generated by the 3-branes. The $D7$ can either have positive charge, $q := +1$, or negative charge, $q := -1$. We take over the usual interpretation of electrodynamics that like charges repel and opposite charges attract each other.

In noncompact spaces the volume integral is divergent, so there is no straightforward way to verify whether Ramond–Ramond charge conservation is obeyed. We assume here that we still have the freedom to choose the relative charge.[1]

[1] Naturally, the $D7$-brane sources the Ramond–Ramond potential C_8. Alternatively, we take as an instance the ansatz $F_9 = q\mathrm{d}(B \wedge B \wedge C_4)$ for the 9-form field strength. The letter d denotes the exterior derivative. The charge of the $D7$-brane is then calculated by integrating the Ramond–Ramond flux through the circle at transverse infinity,

$$\oint_{S^1} {}^*F_9 = q 16\pi \frac{b^2}{g_S H} \, . \tag{2.28}$$

The Chern–Simons action of our configuration is given by

$$S_{CS} = -q\mu_7 \int_{Vol(D7)} \sum_r P\left[C_r \wedge \exp B_2\right] \wedge \exp\left(2\pi\alpha' F_2\right) . \tag{2.29}$$

It consists of three parts for the fields we used,

$$\begin{aligned} S_{CS} &= -\tfrac{1}{2} q\mu_7 \int_{D7} P\left[C_4 \wedge B_2 \wedge B_2\right] && (2.30)\\ &\quad -2\pi\alpha' q\mu_7 \int_{D7} P\left[C_4 \wedge B_2\right] \wedge F_2 && (2.31)\\ &\quad -\tfrac{1}{2}\left(2\pi\alpha'\right)^2 q\mu_7 \int_{D7} P\left[C_4\right] \wedge F_2 \wedge F_2 && (2.32)\\ S_{CS} &= -q\mu_7 \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\frac{b^2}{H} + 2\pi\alpha' \frac{b}{H}\left(F_{45} + F_{67}\right) + \right. && (2.33)\\ &\quad \left. + \tfrac{1}{2}\left(2\pi\alpha'\right)^2 \frac{1}{4H} \epsilon_{mnkl} F_{mn} F_{kl}\right) . && (2.34) \end{aligned}$$

By comparison with the terms in the DBI action, we see that in the case of negative $D7$-brane, with $q = -1$, we enjoy the cancellation of contributions from the two actions. The resulting effective action is

$$\boxed{S_{\overline{D7}} = -\frac{\mu_7}{g_S} \int \mathrm{d}^4x \int \mathrm{d}^4y \left(1 + \tfrac{1}{2} \frac{\left(2\pi\alpha'\right)^2}{H + b^2} F^-_{mn} F^-_{mn}\right) .} \tag{2.35}$$

The interesting property of this action is that the parameters factorize out of the $U(1)$ field strength. The curvature of the background space is parametrized by the warp factor H, the B-field by the number b. In the Maxwell action, the antiselfdual part of the $U(1)$ field strength appears

$$F^-_{mn} = \tfrac{1}{2}\left(F_{mn} - {}^*F_{mn}\right) . \tag{2.36}$$

Our configuration of D-branes and the B-field is a particular instance of the relation between noncommutative gauge theory and string theory found by Seiberg and Witten

[11]. In the next two chapters we discuss our result in the context of noncommutative geometry, especially we evaluate the action on a static noncommutative instanton configuration.

2.3 Properties of our Effective Action

We considered two constant vacuum expectation values B^+ and B^- for the Kalb–Ramond field, differing in the sign of the Pfaffian. We can choose the sign $q = \pm 1$ of the $D7$-brane charge relative to the stack of positively charged $D3$-branes. For all four combinations we calculated the effective action to second order in α',

$$S_{eff} = S_{DBI} + S_{CS} \,. \tag{2.37}$$

The Neveu–Schwarz and Ramond–Ramond form fields in the effective action are potentials for gravitational and electrostatic forces acting between the stack of N_C solitonic $D3$-branes the $D7$-brane probe. In the two cases where the Pfaffian of the B-field and the charge of the $D7$-brane have different sign, the contributions from the Dirac–Born–Infeld and the Chern–Simons actions cancel each other out and give the force free configurations

$$Pf(B^+) > 0, \qquad S_{\overline{D7}} = -\frac{\mu_7}{g_S}\int \mathrm{d}^4x \int \mathrm{d}^4y \, \left(1 + \tfrac{1}{2}\frac{(2\pi\alpha')^2}{H+b^2} F^-_{mn}F^-_{mn}\right), \tag{2.38}$$

$$Pf(B^-) < 0, \qquad S_{D7} = -\frac{\mu_7}{g_S}\int \mathrm{d}^4x \int \mathrm{d}^4y \, \left(1 + \tfrac{1}{2}\frac{(2\pi\alpha')^2}{H+b^2} F^+_{mn}F^+_{mn}\right). \tag{2.39}$$

The interpretation is that the balance of attractive gravitational forces and repulsive electrostatic forces leads to a force free configuration of D-branes, see table 2.2.

B-Field	$Pf(B^+) > 0$	$Pf(B^-) < 0$
effective action	$\sim 1 + F^-F^-$	$\sim 1 + F^+F^+$
charge q of $D7$ probe	negative ($\overline{D7}$)	positive ($D7$)
instanton solution [12]	$\hat{F}^+_A = 0$	SIS, $A_m = 0$

Table 2.2 – The properties of the two force free effective actions. SIS means small instanton singularity.

For the other two combinations of signs, no cancellation takes place. The interpretation is that target space supersymmetry is broken geometrically. The effective action has the form

$$S_{eff} = -\frac{\mu_7}{g_S}\int \mathrm{d}^4x \int \mathrm{d}^4y \, \left(1 + 2\frac{b^2}{H} + (2\pi\alpha')\,\mathcal{L}^{(1)} + \tfrac{1}{2}\,(2\pi\alpha')^2\,\mathcal{L}^{(2)}\right) . \tag{2.40}$$

The Lagrangians for $q = +1$, $Pf(B^+) > 0$ are

$$\mathcal{L}^{(1)} = 2\frac{b}{H}\,(F_{45} + F_{67}) \, , \tag{2.41a}$$

$$\mathcal{L}^{(2)} = \frac{1}{H + b^2}\left[\tfrac{1}{2}F_{mn}F_{mn} + \tfrac{1}{4}\left(\frac{2b^2}{H}\right)\epsilon_{mnkl}F_{mn}F_{kl}\right] . \tag{2.41b}$$

The Lagrangians for $q = -1$, $Pf(B^-) < 0$ are

$$\mathcal{L}^{(1)} = 2\frac{b}{H}\,(F_{46} + F_{57}) \, , \tag{2.42a}$$

$$\mathcal{L}^{(2)} = \frac{1}{H + b^2}\left[\tfrac{1}{2}F_{mn}F_{mn} - \left(\frac{H + 2b^2}{H}\right)\tfrac{1}{4}\epsilon_{mnkl}F_{mn}F_{kl}\right] . \tag{2.42b}$$

In order to explain the repulsive electromagnetic forces, we must take the presence of the B-field into account. According to Seiberg and Witten [11], a $D3$-brane is dissolved within the worldvolume $\overline{D7}$-brane (we discuss the effects of the B-field in detail in the next chapter).

Our effective actions listed in table 2.2 correspond to force free supersymmetric half BPS configurations of the $D3$-brane background and dissolved $D3$-brane, they break half of the target space supersymmetry. In chapter 4 we evaluate the supersymmetric effective action (2.38) for selfdual B-field on a static instanton configuration. We use the antiselfdual noncommutative $U(1)$ instanton found by Nekrasov and Schwarz [12]. We adopt their viewpoint that the antiselfdual instanton preserves supersymmetry.

Our effective action can be applied to phenomenological models via the AdS/CFT correspondence. Our setup of D-branes and B-field is on the gravity side of the duality. The symmetries are realized geometrically, the isometries of the transversal five sphere are translated to the R-symmetry on the gauge theory side. Due to the choice of a constant B-field and the embedding of a D-brane, the configuration is invariant under a subgroup of the isometries of the five sphere.

Here we write down the quantum numbers of our configuration under the residual symmetries. The relevant tensor for this discussion is the gauge invariant combination $\mathcal{F}$ defined in (2.3). It is an antisymmetric tensor in the fifteen dimensional adjoint representation of the Lie algebra $\mathfrak{su}(4)$ of the isometry group $SO(6)$.

The subalgebra of residual symmetries is determined by the geometry of our configuration. We use the properties of our static embedding. It is invariant under rotations in the z^i directions, and the geometry of the $D7$-brane is $AdS_5 \times S^3$. So the residual isometry group of our D-brane configuration is the product manifold $SO(4) \times SO(2)_{89}$.

In order to determine the corresponding Lie algebra, we avail us of the isomorphisms between orthogonal and unitary Lie algebras.[2] The representations of $\mathfrak{so}(4)$ are classified in terms of the tensor product of two angular momentum algebras via the isomorphism

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \otimes \mathfrak{su}(2)\,, \tag{2.43}$$

and we also use $\mathfrak{so}(2) \cong \mathfrak{u}(1)$. So we need the quantum numbers in terms of the subalgebra

$$\mathfrak{su}(4) \supset \mathfrak{su}(2)_L \otimes \mathfrak{su}(2)_R \otimes \mathfrak{u}(1)_{89}\,. \tag{2.44}$$

The branching rules for the decomposition of the fifteen dimensional representation into this subalgebra can be found in [27],

$$15 \cong (1,1)_0 \oplus (3,1)_0 \oplus (1,3)_0 \oplus (2,2)_1 \oplus (2,2)_{-1}\,. \tag{2.45}$$

This is a direct sum of vector spaces. The numbers in brackets denote the dimension of the representation spaces of the angular momentum algebra. For instance, 2 corresponds to the spin 1/2 representation. The subscripts denote the $\mathfrak{u}(1)_{89}$ charges.

In the gauge/gravity duality, these quantum numbers have to match with the appropriate field theory operators. This is discussed in [13]. At the end of chapter 4 we discuss further applications of our effective action to phenomenological models in the context of the AdS/CFT correspondence.

We continue with a discussion of the properties of the effective action. In natural units, the action is dimensionless and the integrand is a density. Our effective action is different in this respect, since the dimension of the measure cancels with the $[\text{mass}]^8$ of $\mu_7 \sim 1/\alpha'^4$, so the integrand is dimensionless.

[2]For the representation theory I used the book of Fuchs and Schweigert [26].

We reproduce the situation with vanishing B-field.[3] We partially integrate the topological B-field in the classical action (3.1b). This results in a Maxwell interaction on the worldsheet boundary between the open string and the gauge field on the D-brane. We make the ansatz $A_m = (1/2)B_{km}X^k$ (3.13), which is a rotation vector field with singularity at the origin. We put the corresponding field strength is $B_{mn} = F_{mn}$ into our effective action and the result is

$$S_{\overline{D7}} = -\frac{\mu_7}{g_S}\int \mathrm{d}^4x \int \mathrm{d}^4y \, \left(1 + \frac{2b^2}{H + b^2}\right) . \tag{2.46}$$

For $b = 0$ we obtain the expected volume element of the $\overline{D7}$-brane as a hypersurface. The interpretation is that $b = 0$ gives the embedding of the D-brane with minimal tension. The B-field contributes to the energy density of the D-brane.

[3]Here we anticipate the discussion in sections 3.1 and 3.2.1.

Chapter 3

String Theory and Noncommutative Geometry

In this chapter I describe how the dynamics of massless strings give rise to noncommutative gauge theory on the worldvolume of D-branes. The main source I used is the paper by Seiberg and Witten [11]. In addition I used the review article by Douglas and Nekrasov on noncommutative field theory [15].

In section 3.1 I discuss the classical bosonic sigma model of type IIB theory including the topological B-field term. Following Freedman [28], I write down the fermionic action in $AdS_5 \times S^5$ target space. The aim is to obtain the worldsheet boundary propagator of two open string vertex operators in $AdS_5 \times S^5$ target space.

I discuss important aspects of the Seiberg-Witten map [11] in section 3.2, in particular the effects of the B-field on the D-brane, the point splitting regularization of the sigma model, and the Seiberg–Witten limes.

In our setup, the noncommutative $U(1)$ gauge theory on the worldvolume of the $D7$-brane has an instanton solution found by Nekrasov and Schwarz [12]. In section 3.3 we highlight some interesting properties of this instanton, in particular we discuss the Wick symbol of the instanton action along the lines of [29].

3.1 The Sigma Model of Type IIB String Theory

In this section I write down the bosonic and fermionic part of the classical type IIB worldsheet sigma model action in the $D3$-brane background introduced in the previous chapter. The aim is to calculate the boundary correlator of two open string vertex operators. I start with a short discussion of the classical type IIB action for strings in the presence of background fields, following the review [1] and the book of Polchinski [30].

In the classical model of string theory, the dynamical fields map the worldsheet of the string to the spacetime. In geometric terms they embed the two dimensional hypersurface Σ into the Riemannian manifold $\mathcal{M}$ of dimension ten. The geometry of the curved target space is characterized by the background fields. The metric is interpreted as a coherent state of gravitons.

The most general local action invariant under Poincaré transformations in target space as well as diffeomorphisms and Weyl rescalings of the worldsheet is given by [31]

$$S_b = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}\sigma\mathrm{d}\tau\sqrt{-\gamma}\;\gamma^{ab}\partial_a X^M \partial_b X^N g_{MN}(X) \tag{3.1a}$$

$$+\tfrac{1}{2}\int_\Sigma \mathrm{d}\sigma\mathrm{d}\tau\sqrt{-\gamma}\;\epsilon^{ab}\partial_a X^M \partial_b X^N B_{MN}(X) \tag{3.1b}$$

$$+\frac{1}{4\pi}\int_\Sigma \mathrm{d}\sigma\mathrm{d}\tau\sqrt{-\gamma}\;R_\Sigma(\gamma)\,\Phi(X)\;. \tag{3.1c}$$

The bosonic fields $X^M(\sigma,\tau)$ are the ten coordinates of the manifold $\mathcal{M}$. They are harmonic maps, therefore the action S_b is also referred to as the nonlinear worldsheet sigma model of the bosonic string.

The Polyakov action (3.1a) is proportional to the area of the worldsheet. The auxiliary worldsheet metric γ_{ab} has Lorentz signature and is contracted with the pullback of the target space metric g_{MN}.

The second term (3.1b) in the bosonic action represents the string interaction mediated by the antisymmetric Kalb–Ramond B-field [32]. It is nonzero if the target space $\mathcal{M}$ has a nontrivial topology or a boundary $\partial\Sigma$. This part of the classical action is important for our work, I discuss it in section 3.2.1.

The dilaton term (3.1c) in S_b is renormalizable but not Weyl invariant. It represents the conformal anomaly in two dimensions and couples to the Euler density. The Ricci scalar of the worldsheet R_Σ is zero, since we assume a flat worldsheet.

In natural units, the action S_b is dimensionless from the worldsheet point of view. The background fields are dimensionless couplings, so that the sigma model is renormalizable, and the embedding functions are ten massless Klein–Gordon fields. In order to make S_b dimensionless in spacetime, we associate to the B-field the spacetime dimension[1]

$$[B_{MN}] = \text{ mass }^2 . \tag{3.2}$$

The embedding functions are vectors with dimension length. So the action S_b is dimensionless in target space as well. I discuss the units associated to the quantities we use further in section 4.3.

In order to define a consistent string theory, the renormalization process has to preserve target space diffeomorphisms and worldsheet conformal symmetry. The action S_b consists of all renormalizable terms. The bosonic model can be regularized in different ways. A model with curved background space is called nonlinear, since the solutions

[1] In the previous chapter we had rescaled the B-field with α' so that it is dimensionless like the metric.

X^M do not form a linear space, but are harmonic instead. Friedan studied the renormalization with dimensional regularization [33, 34]. In our case $dB = 0$ the renormalization group equation of the metric is the Ricci flow and the leading order beta functions are

$$\beta^g_{MN} = \tfrac{1}{2} R_{MN}\,, \tag{3.3a}$$

$$\beta^B_{MN} = 0\,, \tag{3.3b}$$

$$\beta^\Phi = 0\,. \tag{3.3c}$$

We demand worldsheet supersymmetry. We follow the conventions of [1, 23, 30]. We map the worldsheet Σ conformally to the upper half complex plane $\mathbb{H}$. This means we introduce complex coordinates $z = \tau + \mathtt{i}\sigma$ and $\bar{z} = \tau - \mathtt{i}\sigma$. They are accompanied by fermionic worldsheet coordinates θ, $\bar\theta$ to form two dimensional $\mathcal{N} = 1$ superspace.

The bosonic embedding fields X^M are supplemented by fermionic embedding functions ψ^M that are Weyl spinors on the world sheet and vectors in target space. Together with the auxiliary fields F^M they are components of the supersymmetry multiplet

$$X^M + \mathtt{i}\theta\psi^M_L + \mathtt{i}\bar\theta\psi^M_R + \theta\bar\theta F^M\,. \tag{3.4}$$

The supersymmetric completion of the classical bosonic action was explored in [28, 35]. Here we follow the discussion of the book of Polchinski [23]. On a superconformally extended flat worldsheet the fermionic component action is

$$S_f = \tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\; g_{MN}\left(\psi^M_+\bar{D}\psi^N_+ + \psi^M_- D\psi^N_-\right) + \tfrac{1}{8\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\; R_{MNAB}\psi^M_+\psi^N_+\psi^A_-\psi^B_-\,. \tag{3.5}$$

The partial derivative is replaced by the covariant derivative,

$$\bar{D}\psi^M_+ = \bar\partial\psi^M_+ + \left(\Gamma^M_{AB} + \tfrac{1}{2}dB^M_{AB}\right)\bar\partial X^A\psi^B_+\,, \tag{3.6a}$$

$$D\psi^M_- = \partial\psi^M_- + \left(\Gamma^M_{AB} - \tfrac{1}{2}dB^M_{AB}\right)\partial X^A\psi^B_-\ . \tag{3.6b}$$

The above action together with the Polyakov action is the component expansion of the supersymmetric $\mathcal{N} = 1$ sigma model consisting of the Kähler potential of one scalar superfield. The action is invariant under the supersymmetry transformations, parametrized by the spinor ϵ [28],

$$\delta X^M = \bar{\epsilon}\psi^M \,, \tag{3.7a}$$

$$\delta\psi^M = -\mathtt{i}\partial X^M \epsilon - \Gamma^M_{AB}\left(\delta X^A\right)\psi^B \,. \tag{3.7b}$$

Four our setup of $D3$-brane background metric (2.8) and static embedding (2.11) we have for the covariant derivatives of the fermionic embedding functions the results

$$\begin{aligned} \bar{D}\psi^0_+ &= \bar{\partial}\psi^0_+ + \Gamma^0_{AB}\bar{\partial}X^A\psi^B_+ \\ &= \bar{\partial}\psi^0_+ - \frac{1}{R^2}\left(X_m\bar{\partial}X^0\psi^m_+ - X_m\bar{\partial}X^m\psi^0_+\right) \,, \end{aligned} \tag{3.8a}$$

$$\begin{aligned} \bar{D}\psi^\mu_+ &= \bar{\partial}\psi^\mu_+ + \Gamma^\mu_{AB}\bar{\partial}X^A\psi^B_+ \\ &= \bar{\partial}\psi^\mu_+ + \frac{1}{R^2}\left(X_n\bar{\partial}X^\mu\psi^n_+ + X_n\bar{\partial}X^n\psi^\mu_+\right) \,, \end{aligned} \tag{3.8b}$$

$$\begin{aligned} \bar{D}\psi^m_+ &= \bar{\partial}\psi^m_+ + \Gamma^m_{AB}\bar{\partial}X^A\psi^B_+ \\ &= \bar{\partial}\psi^m_+ + +\frac{1}{R^2}\left(X_m\bar{\partial}X^0\psi^0_+ - 2X_m\delta_{\mu\nu}\bar{\partial}X^\mu\psi^\nu_+\right) \\ &\quad + 2\frac{R^2}{y^4}\left(-X_n\bar{\partial}X^n\psi^m_+ - X_n\bar{\partial}X^m\psi^n_+ + X^m\bar{\partial}X_n\psi^n_+\right) \,. \end{aligned} \tag{3.8c}$$

These terms are to be supplemented by their counterparts $D\psi^0_-$. The components of the fermionic action contracted with the Riemann tensor are

$$\begin{aligned} R_{MNAB}\psi^M_+\psi^N_+\psi^A_-\psi^B_- = &-\frac{2}{R^2}\psi^0_+\psi^m_+\psi_{-m}\psi^0_- + \frac{4}{y^4}X_mX_n\psi^0_+\psi^m_+\psi^n_-\psi^0_- \\ &+ \left(\frac{1}{y^2} - \frac{1}{R^2}\right)\psi^m_+\psi^\mu_+\psi_{-m}\psi_{-\mu} + \frac{3}{R^4}X_mX_n\psi^m_+\psi^\mu_+\psi^n_-\psi_{-\mu} \,. \end{aligned} \tag{3.9}$$

So the complete fermionic classical action on our $AdS_5 \times S^5$ background reads

$$
\begin{aligned}
S_f \;=\; & \tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\; g_{MN}\left(\psi_+^M\bar{\partial}\psi_+^N+\psi_-^M\partial\psi_-^N\right)\\
&+\tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{y^2}{R^4}\left(\psi_+^0X_m\bar{\partial}X^0\psi_+^m+\psi_+^0X_m\bar{\partial}X^m\psi_+^0\right)\\
&+\tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{y^2}{R^4}\left(\psi_-^0X_m\partial X^0\psi_-^m+\psi_-^0X_m\partial X^m\psi_-^0\right)\\
&+\tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{y^2}{R^4}\left(\psi_+^\mu X_n\bar{\partial}X_\mu\psi_+^n+\psi_+^\mu X_n\bar{\partial}X^n\psi_{+\mu}\right)\\
&+\tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{y^2}{R^4}\left(\psi_-^\mu X_n\partial X_\mu\psi_-^n+\psi_-^\mu X_n\partial X^n\psi_{-\mu}\right)\\
&+\tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;y^2\left(\psi_+^mX_m\bar{\partial}X^0\psi_+^0-2\psi_+^mX_m\delta_{\mu\nu}\bar{\partial}X^\mu\psi_+^\nu\right)\\
&+\tfrac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{R^4}{y^6}\psi_+^m\left(-X_n\bar{\partial}X^n\psi_{+m}-X_n\bar{\partial}X_m\psi_+^n+X_m\bar{\partial}X_n\psi_+^n\right)\\
&+\tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;y^2\left(\psi_-^mX_m\partial X^0\psi_-^0-2\psi_-^mX_m\delta_{\mu\nu}\partial X^\mu\psi_-^\nu\right)\\
&+\tfrac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{R^4}{y^6}\psi_-^m\left(-X_n\partial X^n\psi_{-m}-X_n\partial X_m\psi_-^n+X_m\partial X_n\psi_-^n\right)\\
&-\tfrac{1}{8\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{2}{R^2}\psi_+^0\psi_+^m\psi_{-m}\psi_-^0\\
&+\tfrac{1}{8\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{4}{y^4}X_mX_n\psi_+^0\psi_+^m\psi_-^n\psi_-^0\\
&+\tfrac{1}{8\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\left(\frac{1}{y^2}-\frac{1}{R^2}\right)\psi_+^m\psi_+^\mu\psi_{-m}\psi_{-\mu}\\
&+\tfrac{1}{8\pi\alpha'}\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\;\frac{3}{R^4}X_mX_n\psi_+^m\psi_+^\mu\psi_-^n\psi_{-\mu}\,.
\end{aligned}
\tag{3.10}
$$

I expect this action to be an important starting point for the calculation of open string vertex correlators with conformal field theory methods. In the limit $\alpha' \to 0$, an ansatz for the boundary correlation function of two open string vertex operators in $AdS_5 \times S^5$ background space is

$$
\langle X^m(\kappa)\,,X^n(\upsilon)\rangle=\frac{\mathtt{i}}{2}\Theta^{mn}\,\mathrm{sgn}\,(\kappa-\upsilon)+f(R^2,\kappa,\upsilon)t^{mn}\,. \tag{3.11}
$$

In flat space, this correlator is given in equation (2.6) in [11]. We search for curvature corrections parametrized by the radius R. The function f stands for the curvature corrections, the generic tensor t^{mn} catches the index structure, and κ and υ are boundary coordinates. This correlation function should be important for the formulation of the Seiberg–Witten map in curved background space.

3.2 The Seiberg–Witten Map

In this section we discuss several aspects of the Seiberg–Witten map [11]. Section 3.2.1 starts with the basics. We explain how the the B-field interpolates between boundary conditions and gives rise to noncommutative D-brane coordinates. We continue with point splitting regularization in section 3.2.2, and finally we summarize the Seiberg–Witten limes in section 3.2.3.

3.2.1 D-Brane, B-Field and Noncommutative Coordinates

In the presence of a Dp-brane as a worldsheet boundary, we have the interaction

$$S_{int} = \int_{\partial\Sigma} \mathrm{d}\tau \; A_m \dot{X}^m \,. \tag{3.12}$$

Starting from the bosonic sigma model action S_b, we can obtain this bosonic interaction term by integrating the topological term (3.1b) partially. Small Latin indices $m, n, \ldots$ denote coordinates along the brane, the worldsheet coordinates are σ and τ.

Here we discuss in general how a rank r Kalb–Ramond background gives rise to $D(p-r)$-branes dissolved within Dp-branes. This discussion follows the book of Zwiebach [36].

The boundary interaction (3.12) represents the Maxwell coupling of the open string endpoints to an electromagnetic field on a Dp-brane. The physical interpretation is that the magnetic background gauge field B_{MN} induces a Maxwell current in the strings. In order to retain charge conservation we have to add this second term in (3.1a). So the current can "flow into the brane."

How does the induced electromagnetic field F_{mn} on the $D7$-brane look like? For a constant B-field we can partially integrate the bosonic action and replace the B-field by a Maxwell interaction term with the gauge field

$$A_m = \tfrac{1}{2} B_{km} X^k \,. \tag{3.13}$$

This results in a constant field strength tensor $F_{mn} = B_{mn}$. Following Seiberg and Witten [11] we require that F_{mn} is zero at infinity.

We investigate how the boundary conditions are changed by the presence of the Kalb–Ramond field. We start with evaluating the variation of the action (3.1a).

Because the Polyakov action is invariant under diffeomorphism as well as Weyl transformations of the worldsheet, we can always cast the world sheet metric γ into a symmetric form with Lorentz signature ($\gamma_{\tau\tau} = -1$ and $\gamma_{\sigma\sigma} = +1$). In this case the sigma model Lagrangian takes the form

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \left(-\dot{X}^M \dot{X}^N + X'^M X'^N \right) g_{MN} + \dot{X}^M X'^N B_{MN} \,. \tag{3.14}$$

We vary the classical action with this Lagrangian and the constant gauge field (3.13),

$$\delta S_{IIB} = \int_\Sigma \mathrm{d}\tau \, \mathrm{d}\sigma \; \delta\mathcal{L}\left(\dot{X}, X'\right) + \tfrac{1}{2} \int_{\partial\Sigma} \mathrm{d}\tau \; F_{km} \; \delta\left(\dot{X}^m X^k\right) \,, \tag{3.15}$$

We define the canonical impulses by

$$\mathcal{P}^\sigma_M = \frac{\delta\mathcal{L}}{\delta X'^M} \,, \quad \mathcal{P}^\tau_M = \frac{\delta\mathcal{L}}{\delta \dot{X}^M} \,, \tag{3.16}$$

so the variation is

$$\delta S_{IIB} = \int_\Sigma \mathrm{d}\tau\,\mathrm{d}\sigma\,\left(\mathcal{P}^\tau_M \delta\dot{X}^M + \mathcal{P}^\sigma_M \delta X'^M\right) \tag{3.17}$$

$$+\tfrac{1}{2}\int_{\partial\Sigma} \mathrm{d}\tau\; F_{km}\,\left(\delta\dot{X}^m X^k + \dot{X}^m \delta X^k\right). \tag{3.18}$$

The derivatives with respect to the world sheet coordinates σ, τ are partially integrated. The wave equation

$$\dot{\mathcal{P}}^\tau_M + \mathcal{P}'^\sigma_M = 0 \tag{3.19}$$

is used to simplify the expression,

$$\delta S_{IIB} = \int_\Sigma \mathrm{d}\tau\,\mathrm{d}\sigma\,\left(\partial_\tau\left(\mathcal{P}^\tau_M \delta X^M\right) + \partial_\sigma\left(\mathcal{P}^\sigma_M \delta X^M\right)\right) \tag{3.20}$$

$$+\tfrac{1}{2}\int_{\partial\Sigma} \mathrm{d}\tau\; F_{km}\,\left(\partial_\tau\left(\delta X^m X^k\right) - \delta X^m \dot{X}^k + \dot{X}^m \delta X^k\right). \tag{3.21}$$

The total derivatives ∂_τ are surface integrals. We use Stoke's theorem to argue that their value is zero. The variations δX vanish at the temporal boundaries. The last two terms in the second line are equal because the Faraday tensor is antisymmetric. So we are left with

$$\delta S_{IIB} = \int_\Sigma \mathrm{d}\tau\,\mathrm{d}\sigma\,\partial_\sigma\left(\mathcal{P}^\sigma_M \delta X^M\right) \tag{3.22}$$

$$+\int \mathrm{d}\tau\; F_{km}\,\left[\dot{X}^m \delta X^k\right]_{\sigma=0}^{\sigma=\pi}. \tag{3.23}$$

Stokes theorem is used again in the first line. For the $(10-p)$ coordinates normal to the brane, indicated by a, the familiar boundary condition is unchanged,

$$\int \mathrm{d}\tau\;\left[\mathcal{P}^\sigma_a \delta X^a\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,. \tag{3.24}$$

So the Dirichlet boundary conditions in the normal directions remain,

$$\left[\delta X^a\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,, \tag{3.25}$$

but the p Neumann boundary conditions along the Dp-brane are altered to

$$\int \mathrm{d}\tau \, \left[\left(\mathcal{P}^{\sigma}_{m} + F_{mn}\dot{X}^{n}\right)\delta X^{m}\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,, \tag{3.26}$$

$$\left[\mathcal{P}^{\sigma}_{m} + F_{mn}\dot{X}^{n}\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,. \tag{3.27}$$

We choose light cone gauge, impose the Virasoro constraints and have for the momentum

$$\mathcal{P}^{\sigma}_{m} = -\frac{X'_{m}}{2\pi\alpha'}\,. \tag{3.28}$$

So the p boundary conditions along the brane are now mixed:

$$\left[X'_{m} - 2\pi\alpha' F_{mn}\dot{X}^{n}\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,. \tag{3.29}$$

We choose as an instance the constant electromagnetic field F_{mn} on the brane to be purely magnetic. We have $F_{0i} = 0$ for $i = 1, \ldots, p$ and we choose for example $F_{p-1,p} = B$ as the only nontrivial component, so the Faraday tensor has column rank $r = 2$. In this case, the boundary condition for X^0 is still Neumann, but we get two additional Dirichlet boundary conditions,

$$[X'_{0}]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,, \tag{3.30}$$

$$\left[X'_{p-1} - 2\pi\alpha' B\dot{X}_{p}\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,, \tag{3.31}$$

$$\left[X'_{p} + 2\pi\alpha' B\dot{X}_{p-1}\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,. \tag{3.32}$$

So for a large magnetic field we have a $D(p-2)$-brane dissolved within the Dp-brane. In the general case, where the magnetic field has always even rank $r \in 2\mathbb{N}$, we have $p-r$ additional Dirichlet conditions. For our setup introduced in the previous chapter, we have $p = 7$ and $r = 4$ for the purely magnetic Faraday tensor in the y-directions. So we have an electric $D3$ dissolved within the magnetic $D7$. Another interesting case discussed in Seiberg and Witten [11] is $p = 3$, $r = 4$. The $D(-1)$ instanton has Dirichlet boundary condition in all space direction as well as the time direction, with the Lagrangian

$$\mathcal{L}_{D(-1)} = g_s^{-1}\,. \tag{3.33}$$

How do the mixed boundary conditions (3.29) give rise to noncommutative coordinates? We have to calculate the propagator of open string vertex operators in the presence of the mixed boundary condition, according to [37]. The propagator is evaluated at the boundary of the disc shaped worldsheet, located at $z = \bar{z}$ in complex coordinates on the upper half plane $\mathbb{H}$.

Using the background field method, the embedding functions are split into a solution of the equations of motion and a fluctuation,

$$X^M(\tau,\sigma) = \bar{X}^M(\tau,\sigma) + \xi^M(\tau,\sigma)\ . \tag{3.34}$$

The ten wave equations are

$$\Box \bar{X}^M = 0\,, \quad \Box = \partial_\tau^2 + \partial_\sigma^2 = \partial\bar{\partial}\,, \tag{3.35}$$

and the ten boundary conditions for the background field are

$$\partial_\sigma \bar{X}_M + \mathtt{i} B_{MN} \partial_\tau \bar{X}^N \big|_{\partial\Sigma} \stackrel{!}{=} 0\,. \tag{3.36}$$

The solutions of the respective inhomogeneous wave equations are the ten Green's functions

$$G^M = -\alpha' \left(A^M \ln|z-w| + B^M \ln|z-\bar{w}| \right) + C\bar{X}^M\,. \tag{3.37}$$

The propagator of two open string vertex operators on the boundary is (with $\kappa, \upsilon \in \mathbb{R}$) given by

$$\langle X^m(\kappa)\,, X^n(\upsilon)\rangle = -\alpha' G^{mn} \ln(\kappa-\upsilon)^2 + \frac{\mathtt{i}}{2}\Theta^{mn}\,\mathrm{sgn}(\kappa-\upsilon)\ . \tag{3.38}$$

The signum function is defined by

$$\mathrm{sgn}(\kappa-\upsilon) = \begin{cases} +1 & \kappa > \upsilon\,, \\ -1 & \kappa < \upsilon\,. \end{cases} \tag{3.39}$$

In the important zero slope limit only the second term of (3.38) is left over and the propagator is not singular. In particular the correlation function of two fields is constant at the fixed point, which is a familiar fact in statistical physics. In the Seiberg–Witten limes $\alpha' \to 0$ we arrive at

$$\langle X^m(\kappa), X^n(\upsilon)\rangle = \frac{\mathtt{i}}{2}\Theta^{mn}\,\mathrm{sgn}(\kappa-\upsilon)\ . \tag{3.40}$$

This propagator can be viewed as a time ordering instruction,

$$: X^m(\kappa)\cdot X^n(\upsilon) := \langle X^m(\kappa), X^n(\upsilon)\rangle\,. \tag{3.41}$$

Then Seiberg and Witten [11] interpret time ordering as operator ordering.

The next step is to introduce the regulator ϵ, a small positive real number interpreted as a time. With the time ordering instruction (3.41), the commutator of two X-operators becomes

$$[X^m(\epsilon), X^n(0)] = \langle X^m(0), X^n(-\epsilon)\rangle - \langle X^n(0), X^m(\epsilon)\rangle\,. \tag{3.42}$$

In the first propagator on the right hand side the time arguments are shifted by $-\epsilon$ and the chronology is $-\epsilon < 0 < \epsilon$. Therefore both propagators contribute with positive sign, and we have for the commutator in the limit $\epsilon \to 0$,

$$\lim_{\epsilon\to 0}[X^m(\epsilon), X^n(0)] = \mathtt{i}\Theta^{mn}\,. \tag{3.43}$$

This is how the “noncommutative coordinates” arise. The bracket above has the properties of a central extension of a Lie algebra, since Θ is a constant. Then the Baker–Campbell–Hausdorff formula becomes especially easy. The equation

$$e^C = e^A\cdot e^B\,,\quad [A,B] = c\,, \tag{3.44}$$

has the solution

$$C = A + B + \tfrac{1}{2}c\,. \tag{3.45}$$

Now we apply this BCH formula to two plane wave operators evaluated at $X^m(\epsilon)$ and $X^n(0)$. We leave the time arguments away,

$$\exp\left(\mathtt{i}p_m X^m\right) \cdot \exp\left(\mathtt{i}q_n X^n\right) \tag{3.46}$$

$$= \exp\left(-\tfrac{1}{2}p_m q_n\left[X^m, X^n\right] + \mathtt{i}p_m X^m + \mathtt{i}q_n X^n\right) \tag{3.47}$$

$$= \exp\left(-\tfrac{\mathtt{i}}{2}p_m q_n \theta^{mn}\right)\exp\left(\mathtt{i}p_m X^m + \mathtt{i}q_n X^n\right)\ . \tag{3.48}$$

If we interpret the momentum as a derivative, the right hand side already bears similarity to the Moyal product [16]. Later on we want to define what happens when two of these plane wave operators come close together in time. Therefore we will evaluate them at small positive ϵ and zero, respectively,

$$\lim_{\epsilon\to 0} : \exp\left(\mathtt{i}p_m X^m\left(\epsilon\right)\right) \cdot \exp\left(\mathtt{i}q_n X^n\left(0\right)\right) := \exp\left(\mathtt{i}p_m X^m\right) \star \exp\left(\mathtt{i}q_n X^n\right)\ . \tag{3.49}$$

3.2.2 Point Splitting Regularization

The quantum field theory defined by the bosonic part (3.1a) needs to be regularized in order to yield finite correlation functions of open string vertex operators. Here we follow the discussion Seiberg and Witten [11] for flat space.

The constant B-field gives rise to the Maxwell coupling of the string to the brane. We study the gauge transformation of the path integral of the boundary interaction

$$\mathcal{Z} = \int \mathcal{D}X \exp S_{int}\,, \quad \delta\mathcal{Z} = \mathcal{Z}\ \delta S_{int}\,, \tag{3.50}$$

where S_{int} is given by (3.12). We consider the abelian gauge transformation $\delta A_m = \partial_m \lambda$. The parameter λ is a functional of the open string vertex operators X^m. Then both F and $P[B]$ are gauge invariant and we are left with the variation

$$\delta S_{int} = \int_{\upsilon}^{\kappa} \mathrm{d}\tau\ \delta A_m \partial_\tau X^m = \int_{\upsilon}^{\kappa} \mathrm{d}\tau\ \partial_\tau \lambda = \lambda\left[X\left(\kappa\right)\right] - \lambda\left[X\left(\upsilon\right)\right]\ . \tag{3.51}$$

According to (3.50) the path integral changes under the gauge transformation as

$$\delta\mathcal{Z} = \left(\int \mathcal{D}X \exp S_{int}\right) \cdot (\lambda_\kappa - \lambda_\upsilon) \ . \tag{3.52}$$

The subscript on the operator λ indicates the time of the evaluation. Expanding the exponential of the Maxwell coupling term S_{int} (3.12) gives

$$\delta\mathcal{Z} = \int \mathcal{D}X \left(1 + \int \mathrm{d}\tau\ A\dot{X} + \text{ higher } \mathcal{O}\left(A\right)\right) \cdot (\lambda_\kappa - \lambda_\upsilon) \ . \tag{3.53}$$

The most interesting contribution to the variation $\delta\mathcal{Z}$ is the operator product

$$\int \mathrm{d}\tau\ A\dot{X} \cdot (\lambda_\kappa - \lambda_\upsilon) = \int \mathrm{d}\tau\ \left(A\dot{X}\left(\tau\right) \cdot \lambda_\kappa - A\dot{X}\left(\tau\right) \cdot \lambda_\upsilon\right) \ . \tag{3.54}$$

These two operator products become singular at those points in time when the integration variable τ becomes equal to either κ or υ. The operator λ is evaluated at these times. So this product needs to be regularized. Seiberg and Witten [11] are doing this by inventing the point splitting regularization.

The product of two generic operators f, g is supplemented by a time ordering. Instead of $f \cdot g$ one evaluates

$$: f(t_f) \cdot g(t_g) : = \begin{cases} f \cdot g & t_f < t_g \,, \\ g \cdot f & t_g < t_f \,. \end{cases} \tag{3.55}$$

This means that the operators are written down in the order of decreasing time. In our case the time ordering instruction is actually hidden in the propagator (3.40). For (3.54) with $\upsilon < \kappa$ we have three possible sequences,

$$\upsilon < \tau < \kappa \qquad A\dot{X} \cdot \lambda - \lambda \cdot A\dot{X} \,, \tag{3.56}$$

$$\upsilon < \kappa < \tau \qquad \lambda \cdot A\dot{X} - \lambda \cdot A\dot{X} = 0 \,, \tag{3.57}$$

$$\tau < \upsilon < \kappa \qquad A\dot{X} \cdot \lambda - A\dot{X} \cdot \lambda = 0 \,. \tag{3.58}$$

Now Seiberg and Witten [11] define the point splitting regularization by demanding that two operators shall never be at the same point in time. They introduce the regulator ϵ by cutting out the two regions

$$|\tau - \kappa| < \epsilon\,, \quad |\tau - \upsilon| < \epsilon \tag{3.59}$$

from the integration region $\int \mathrm{d}\tau$. This cut operation shall not alter the results of the calculation, so we take the limit of small ϵ. As motivated before in equations (3.43) and (3.49), in this limit the operator product is replaced by the Moyal product,

$$\lim_{\epsilon \to 0} : f(\epsilon) \cdot g(0) : = f \star g\,. \tag{3.60}$$

We regularize the operator product (3.54). We put the later operator to the first place and the earlier operator to the last place, then replacing the ordinary product by the star product. In this way we obtain the variation of the path integral (3.52),

$$\delta \mathcal{Z} = \int \mathcal{D}X \left(\text{ cte } + \int \mathrm{d}\tau \ \left(\lambda \star A\dot{X} - A\dot{X} \star \lambda \right) + \text{ higher } \mathcal{O}(A) \right)\,. \tag{3.61}$$

The $\dot{X}^m$ factorizes out in the zero slope limit $\alpha' \to 0$, as can be seen from the propagator (3.38). The path integral is supposed to be invariant under gauge transformations,

$$\delta \mathcal{Z} \overset{!}{=} 0\,. \tag{3.62}$$

The gauge transformation is altered to

$$\delta A_m = \partial_m \lambda + A_m \star \lambda - \lambda \star A_m\,. \tag{3.63}$$

This equation defines noncommutative gauge invariance. In the literature, the gauge fields as well as the parameter λ are denoted with a hat. The variation of S_{int} becomes

$$\delta S_{int} = \lambda_\kappa - \lambda_\upsilon + \int_\upsilon^\kappa \mathrm{d}\tau \ \left(A\dot{X} \star \lambda - \lambda \star A\dot{X} \right)\,. \tag{3.64}$$

CFT on $\mathbb{H}$	closed strings	open strings
regularization	Pauli–Villars	point splitting
gauge invariance	standard abelian	noncommutative (3.63)
couplings	g_{MN}, B_{MN}, g_S, A_m	G_{MN}, Θ_{MN}, G_S, $\hat{A}_m$

Table 3.1 – The two regularizations of the conformal field theory on the upper half complex plane $\mathbb{H}$.

This new δS_{int} is put in $\delta\mathcal{Z}$ in equation (3.53). The additional terms are multiplied with the 1 of the expansion of $\exp S_{int}$, and cancel out with the regulated operator products in (3.61). So, to first order in the gauge fields, we obtain a gauge invariant path integral. In summary, we evaluated the gauge transformation of the path integral with point splitting regularization. The gauge invariance of the path integral requires us to modify the gauge transformations according to (3.63).

Alternatively we could have used Pauli–Villars regularization. Two regularization schemes are related by a redefinition of the coupling parameters. In our case noncommutative gauge fields arise out of the choice between two regularizations of the same quantum field theory (3.1a). This statement is summarized in table 3.1.

The remaining task is to find the redefinition of the couplings. The characteristic feature of sigma models like (3.1a) is that the couplings are geometric objects in target space. So the redefinition is a matter of geometry.

Now we discuss the fermionic boundary term

$$S_{int} = \tfrac{1}{4}\int_{\partial\Sigma} \mathrm{d}\tau\ F_{mn}\Psi^m\Psi^n\,. \tag{3.65}$$

It changes the $p-1$ boundary conditions (3.29) to

$$\left[(\psi_L - \psi_R)_m + 2\pi\alpha' F_{mn}\Psi^n\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,. \tag{3.66}$$

Without B-field the boundary condition is $\psi_R = \psi_L$ and the rotational symmetries are broken down to a diagonal $SO(4)$. We need the supersymmetry transformation of the diagonal spinor Ψ. It is given by the combination of the transformations of ψ_L and ψ_R above,

$$\Psi \to \dot{X} \,. \tag{3.67}$$

The variation of the bosonic boundary interaction (3.12) under a global $\mathcal{N} = 1$ supersymmetry transformation is a total derivative,

$$\begin{aligned} A\dot{X} + \tfrac{1}{4}F\Psi^2 \quad &\overset{\Psi^2 \to 2\Psi\dot{X}}{\longrightarrow} \quad A\dot{\Psi} + \tfrac{1}{2}F\dot{X}\,\Psi \\ &= \partial_\tau \left(A\Psi\right) \,. \end{aligned} \tag{3.68}$$

So the boundary variation picks up surface terms, depending on the regularization.

The next task is to establish the global supersymmetry invariance of the path integral of our sigma model. Having finished that, we can go on to the influence of D-branes on the supersymmetry.

As in the bosonic case, the constant B-field gives rise to a background gauge field. We study the supersymmetry transformation of the path integral of the boundary interaction (3.12),

$$\mathcal{Z} = \int \mathcal{D}X \exp S_{int} \,, \tag{3.69}$$

$$\delta\mathcal{Z} = \mathcal{Z}\ \delta S_{int} \,, \tag{3.70}$$

$$\delta S_{int} = \int_\upsilon^\kappa \mathrm{d}\tau\ \partial_\tau \left(A\Psi\right) = \left(A\Psi\right)_\kappa - \left(A\Psi\right)_\upsilon \,, \tag{3.71}$$

$$\delta\mathcal{Z} = \left(\int \mathcal{D}X \exp S_{\mathrm{int}}\right) \cdot \left(\left(A\Psi\right)_\kappa - \left(A\Psi\right)_\upsilon\right) \,. \tag{3.72}$$

Expanding the exponential of the Maxwell coupling term (3.12) gives

$$\delta\mathcal{Z} = \int \mathcal{D}X \left(1 + \int \mathrm{d}\tau\ \left(A\dot{X} + F\Psi^2\right) + \text{ higher } \mathcal{O}\left(A\right)\right) \cdot \left(\left(A\Psi\right)_\kappa - \left(A\Psi\right)_\upsilon\right) \,. \tag{3.73}$$

The most interesting contribution to the variation $\delta\mathcal{Z}$ is the operator product

$$\int \mathrm{d}\tau \ \left(A\dot{X}\left(\tau\right)\cdot\left(A\Psi\right)_{\kappa} - A\dot{X}\left(\tau\right)\cdot\left(A\Psi\right)_{\upsilon}\right) . \tag{3.74}$$

These two operator products become singular at the points in time when the integration variable τ becomes equal to one of the times κ, υ where the operator $A\Psi$ is evaluated. So this product needs to be regularized with point splitting regularization.

We do the same time ordering as in the discussion above. When sending the regulator ϵ to zero according to (3.60), we have to replace the ordinary product with the star product. So we have obtained the variation of the path integral,

$$\delta\mathcal{Z} = \int \mathcal{D}X \left(\text{ cte } + \int \mathrm{d}\tau \ \left(A\Psi \star A\dot{X} - A\dot{X} \star A\Psi\right) + \text{ higher } \mathcal{O}\left(A\right)\right) . \tag{3.75}$$

We want the path integral to be invariant under a supersymmetry transformation. This time we have to replace not the gauge transformation as above, but the boundary interaction,

$$S_{int} = \int_{\partial\Sigma} \mathrm{d}\tau \ \left(A\dot{X} + \tfrac{1}{4}F_{mn}\Psi^m\Psi^n + A_m\Psi^m \star A_n\Psi^n \right) . \tag{3.76}$$

The variation of the first two terms gives the surface terms, and the supersymmetry variation of the last term is antisymmetric. Then, the variation of S_{int} is

$$\delta S_{int} = \left(A\Psi\right)_{\kappa} - \left(A\Psi\right)_{\upsilon} + \int \mathrm{d}\tau \ \left(A\dot{X} \star A\Psi - A\Psi \star A\dot{X}\right) . \tag{3.77}$$

This new δS_{int} must be put in $\delta\mathcal{Z}$. The additional terms must be multiplied with the 1 of the expansion of $\exp S_{int}$, and cancel out with the regulated operator products in (3.75). Finally, the path integral is invariant not only under gauge transformations, but under supersymmetry transformations as well.

3.2.3 The Seiberg–Witten Limes

The following two formulas are of crucial importance to the relation between string theory and noncommutative gauge theory.

a) In the limes, the boundary propagator of two open string vertex operators gives rise to noncommutative coordinates,

$$\langle X^m(\kappa), X^n(\upsilon)\rangle = \frac{\mathtt{i}}{2}\Theta^{mn}\,\mathrm{sgn}(\kappa-\upsilon)\ . \tag{3.78}$$

b) In the limes, the relation between the noncommutativity matrix Θ and the Kalb–Ramond field is the inversion,

$$\Theta = B^{-1}\,. \tag{3.79}$$

For these formulas to hold, we need the relations

a) $\alpha' \to 0$, while Θ_{mn} is at a finite value,

b) $g << 2\pi\alpha' B$.

To accomplish this, Seiberg and Witten [11] perform the following steps. For the boundary propagator of open string vertex operators to hold, they

- keep the components of the open string metric G_{mn}, the open string anticommuting tensor Θ_{mn} and the closed string Kalb–Ramond field B_{MN} at a finite fixed value,
- and take the limit $\alpha' \to 0$.

For the inverse relation between Θ and B to hold it is necessary that the components of the metric "go faster to zero than $2\pi\alpha' B_{MN}$" when taking $\alpha' \to 0$. They accomplish this by choosing an ansatz for the metric

$$g \sim \left(\frac{\alpha'}{\alpha'_{cte}}\right)^2 g_{cte}\,. \tag{3.80}$$

In particular they set

$$g \sim \varepsilon\ g_{cte}\,, \tag{3.81a}$$

$$\alpha' \sim \sqrt{\varepsilon}\ \alpha'_{cte}\,, \tag{3.81b}$$

and take $\varepsilon \to 0$ afterwards.

3.3 The Noncommutative $U(1)$ Instanton

In this section I review briefly the $U(1)$ instanton on noncommutative $\mathbb{R}^4_{NC}$ found by Nekrasov and Schwarz [12]. I used the literature [29, 15].

Noncommutative geometry is based on the assumption that there exists a fundamental quantum ζ, a minimal area, in analogy with Planck's quantum $\hbar$ for the action. The coordinates of noncommuative space are quantized according to the canonical rules. The noncommutative space $\mathbb{R}^4_{NC}$ is the algebra generated by the operators $\hat{y}_m$ with commutator

$$[\hat{y}_m, \hat{y}_n] = \mathtt{i}\Theta_{mn} \,. \tag{3.82}$$

Nekrasov and Schwarz take the matrix Θ to be constant, so that (3.82) becomes a Heisenberg algebra. They define the "central charge" of this algebra to be a real number ζ with $\zeta > 0$.

According to the rules of quantum mechanics, we have two recipes to perform practical calculations. We can either view the generators $\hat{y}_m$ as an oscillator algebra acting on a Fock space, or we translate them into phase space functions with the Wick symbol $\Omega\left(\hat{y}_m\right)$.

3.3.1 Oscillator Algebra, Coherent States, Normal Ordering

Nekrasov and Schwarz rewrite the Heisenberg algebra generators $\hat{y}_m$ into creation operators $\hat{z}_0$, $\hat{z}_1$ and annihilation operators $\hat{\bar{z}}_0$, $\hat{\bar{z}}_1$,

$$\hat{z}_0 = \hat{y}_4 + \mathtt{i}\hat{y}_5 \,, \tag{3.83}$$

$$\hat{\bar{z}}_0 = \hat{y}_4 - \mathtt{i}\hat{y}_5 \,, \tag{3.84}$$

$$\hat{z}_1 = \hat{y}_6 + \mathtt{i}\hat{y}_7 \,, \tag{3.85}$$

$$\hat{\bar{z}}_1 = \hat{y}_6 - \mathtt{i}\hat{y}_7 \,. \tag{3.86}$$

The oscillator algebra is then

$$\left[\hat{z}_0, \hat{\bar{z}}_0\right] = \left[\hat{z}_1, \hat{\bar{z}}_1\right] = -\frac{\zeta}{2} \,. \tag{3.87}$$

The operators act on the Fock space (3.3) in [12]. In our work we rescale the step operators,

$$a^\dagger = \sqrt{\tfrac{2}{\zeta}}\,\hat{z}_0 \quad , \quad a = \sqrt{\tfrac{2}{\zeta}}\,\hat{\bar{z}}_0 \,, \tag{3.88}$$

$$b^\dagger := \sqrt{\tfrac{2}{\zeta}}\,\hat{z}_1 \quad , \quad b := \sqrt{\tfrac{2}{\zeta}}\,\hat{\bar{z}}_1 \,, \tag{3.89}$$

in order to obtain the canonical oscillator algebra

$$\left[a, a^\dagger\right] = \left[b, b^\dagger\right] = 1 \,. \tag{3.90}$$

and the usual eigenvalue equations

$$a^\dagger \left|n, m\right\rangle = \sqrt{n+1}\left|n+1, m\right\rangle \,, \tag{3.91}$$

$$a \left|n, m\right\rangle = \sqrt{n}\left|n-1, m\right\rangle \,, \tag{3.92}$$

$$b^\dagger \left|n, m\right\rangle = \sqrt{m+1}\left|n, m+1\right\rangle \,, \tag{3.93}$$

$$b \left|n, m\right\rangle = \sqrt{m}\left|n, m-1\right\rangle \,. \tag{3.94}$$

If we interpret the states in this Fock space as "bosonic particles" created by $b^\dagger$ and "fermions" created by $a^\dagger$, we can write down the Hamiltonian of supersymmetric quantum mechanics, and $\hat{d}$ is the number operator in this Fock space,

$$\hat{d} = \hat{z}_0 \hat{\bar{z}}_0 + \hat{z}_1 \hat{\bar{z}}_1 + \zeta/2\,. \tag{3.95}$$

In our context it is more appropriate to call $\hat{d}$ the distance operator. In terms of the Heisenberg algebra generators we have

$$\hat{d} = \sum_{m=4}^{7} (\hat{y}_m)^2\,. \tag{3.96}$$

Expressed in the rescaled creation and annihilation operators, the distance operator is given by

$$\hat{d} = \tfrac{\zeta}{2} \left(a^\dagger a + b^\dagger b + 1 \right)\,. \tag{3.97}$$

The Wick symbol of an operator is given by the expectation value in coherent states. Since the coherent states are the most classical states of a quantum system, it is a reasonable assumption to identify the eigenvalues with the classical coordinates, in our case the D-brane coordinates y_m along the B-field. Coherent states are eigenstates of the annihilation operators,

$$a \left|\bar{\alpha}\bar{\beta}\right\rangle = \bar{\alpha} \left|\bar{\alpha}\bar{\beta}\right\rangle\,, \tag{3.98}$$

$$b \left|\bar{\alpha}\bar{\beta}\right\rangle = \bar{\beta} \left|\bar{\alpha}\bar{\beta}\right\rangle\,. \tag{3.99}$$

We denote the eigenvalues as

$$\bar{\alpha} = \sqrt{\tfrac{2}{\zeta}} \left(y_4 - \mathtt{i} y_5 \right)\,, \tag{3.100}$$

$$\bar{\beta} = \sqrt{\tfrac{2}{\zeta}} \left(y_6 - \mathtt{i} y_7 \right)\,. \tag{3.101}$$

This amounts to the following eigenvalue equations for the respective number operators,

$$a^\dagger a \, |\alpha,\beta\rangle \;=\; \tfrac{2}{\zeta}\left(y_4^2+y_5^2\right)|\alpha,\beta\rangle \;, \tag{3.102}$$

$$b^\dagger b \, |\alpha,\beta\rangle \;=\; \tfrac{2}{\zeta}\left(y_6^2+y_7^2\right)|\alpha,\beta\rangle \;, \tag{3.103}$$

$$\left(a^\dagger a + b^\dagger b\right)|\alpha,\beta\rangle \;=\; \tfrac{2}{\zeta}\, y^2 \, |\alpha,\beta\rangle \;. \tag{3.104}$$

In order to perform explicit calculations with the instanton action, it is necessary to introduce the inverse of the number operator. We define it via the eigenvalue equation in the Fock states,

$$\frac{1}{a^\dagger a}\,|n\rangle = \frac{1}{n}\,|n\rangle \;. \tag{3.105}$$

We discuss whether this ansatz is well defined. Formula (B.27) for the inverse of a sum of operators gives

$$\frac{1}{a^\dagger a} = \sum_{k=0}^{\infty} (-1)^k \left(a^\dagger a - 1\right)^k \;, \tag{3.106}$$

where we have used $A = 1$ and $B = a^\dagger a - 1$. The operator on the left hand side is bounded, so the operator on the right hand side should have the same property. To check this, we take the expectation value in Fock states.

Since we have not normal ordered yet, we can first expand $\left(1 - a^\dagger a\right)^p$ with the binomial formula, then make use of the eigenvalues n^p of $\left(a^\dagger a\right)^p$, and finally use the binomial formula backwards. In this way, the expectation value of the right hand side becomes

$$\langle n|\left(1 - a^\dagger a\right)^p |n\rangle = (1-n)^p \;. \tag{3.107}$$

With the substitution $q = 1 - n$ and use of the geometric sum formula (B.29), we obtain the sum

$$\langle n|\,\frac{1}{a^\dagger a}\,|n\rangle = \sum_{k=0}^{\infty} (1-n)^k = \frac{1}{n} \,. \tag{3.108}$$

Unfortunately, this geometric sum converges only for $0 < n < 2$. So the only "allowed" occupation number $n \in \mathbb{N}$ is 1. The next step would be to include the projection operator in the definition,

$$N^{-1} = \frac{1}{a^\dagger a} : \exp\left(-a^\dagger a\right) : \, . \tag{3.109}$$

We leave this for future work.

The aim of the following discussion is to find the operator symbol for the inverse number operator (3.105). Since the eigenvalue equations of the coherent states are well defined for annihilators only, we have to put composite operators in order before taking the expectation value. We define the normal order of the left hand side by the normal order of the right hand side,

$$: \frac{1}{a^\dagger a} := \sum_{k=0}^{\infty} (-1)^k : \left(a^\dagger a - 1\right)^k : \, . \tag{3.110}$$

We aim at taking the expectation value of the normal ordered inverse number operator in coherent states,

$$\langle \alpha | : \frac{1}{a^\dagger a} : | \alpha \rangle = \sum_{k=0}^{\infty} \langle \alpha | : \left(1 - a^\dagger a\right)^k : | \alpha \rangle \, . \tag{3.111}$$

The expectation value of the normal ordered number operator in coherent states is

$$\langle \alpha | : \left(a^\dagger a\right)^p : | \alpha \rangle = |\alpha|^{2p} \, . \tag{3.112}$$

Therefore we have a similar formula as in the case with the Fock states,

$$\langle \alpha | : \frac{1}{a^\dagger a} : | \alpha \rangle = \sum_{k=0}^{\infty} \left(1 - |\alpha|^2\right)^k \, . \tag{3.113}$$

This sum is not convergent. So we conclude that the inverse number operator (3.105) is not well defined.

We continue with a discussion of the instanton found by Nekrasov and Schwarz [12]. There exists no instanton of ordinary $U(1)$ gauge theory in four dimensions. Nevertheless, there is an antiselfdual instanton in selfdual noncommutative $\mathbb{R}^4_{NC}$. Under assumption that all operators are diagonal and no normal ordering is necessary, we can express the instanton in terms of our coordinates along the B-field,

$$A_m = \mathtt{i}\zeta \frac{\omega_{mn} y^n}{(y^2 + \zeta/2)\,(y^2 - \zeta/2)}\,. \tag{3.114}$$

The components of the matrix

$$\omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.115}$$

are given by the signature of the components of B^+.

This gauge field configuration is a rotation vector field. It is singular along a circle with radius ζ. The singularity can be shifted to spatial infinity by performing a gauge transformation. This gauge field is therefore a finite action solution and a proper instanton.

3.3.2 Wick Symbol of the Instanton Action

The Wick symbol of an Hilbert space operator is given by the expectation value of the normal ordered operator in the coherent states [29],

$$\Omega\left(\hat{O}\right) = \left\langle \alpha, \beta \right| : \hat{O} : \left| \bar{\alpha}, \bar{\beta} \right\rangle\,. \tag{3.116}$$

We circumvent the issue of normal ordering by assuming that the operators under consideration are diagonal.

In chapter 4 we use the instanton action given in formula (4.8) of [12]. Following [29] we review the calculation of the Wick symbol of the operator

$$\begin{aligned}\hat{F}_A\hat{F}_A &= \frac{\zeta^2}{\left(\hat{d}-\zeta/2\right)\hat{d}^2\left(\hat{d}+\zeta/2\right)}\hat{\Pi} \\ &= \frac{32}{\zeta^2\left(a^\dagger a+b^\dagger b\right)\left(a^\dagger a+b^\dagger b+1\right)^2\left(a^\dagger a+b^\dagger b+2\right)}\hat{\Pi}\,.\end{aligned} \tag{3.117}$$

The projection operator is given by

$$\hat{\Pi} := \mathbb{1} - |00\rangle\langle 00|\;. \tag{3.118}$$

We compute the Wick symbol under the assumption that the action (3.117) is diagonal in the Fock states. From the spectral decomposition

$$\hat{F}_A\hat{F}_A = \sum_{(n_1,m_1)=(0,0)}^{\infty}\sum_{(n_2,m_2)=(0,0)}^{\infty}|n_1m_1\rangle\langle n_1m_1|\,\hat{F}_A\hat{F}_A\,|n_2m_2\rangle\langle n_2m_2| \tag{3.119}$$

we can see immediately that $\hat{\Pi}$ projects the zero out,

$$\hat{F}_A\hat{F}_A = \frac{32}{\zeta^2}\sum_{(n,m)\neq(0,0)}^{\infty}\frac{|nm\rangle\langle nm|}{(n+m)\,(n+m+1)^2\,(n+m+2)}\,. \tag{3.120}$$

In the coherent state basis we have the expectation values

$$\left\langle\alpha\beta\,|nm\rangle\langle nm|\,\bar{\alpha}\bar{\beta}\right\rangle = |\langle nm|\alpha\beta\rangle|^2 = \exp\left(-|\alpha|^2-|\beta|^2\right)\frac{|\alpha|^{2n}}{n!}\frac{|\beta|^{2m}}{m!}\,. \tag{3.121}$$

So the full expression for the calculation to perform reads

$$\begin{aligned}\Omega\left(\hat{F}_A\hat{F}_A\right) &= \frac{32}{\zeta^2}\sum_{(n,m)\neq(0,0)}^{\infty}\frac{|c_{nm}|^2}{(n+m)\,(n+m+1)^2\,(n+m+2)} \\ &= \frac{32}{\zeta^2}\exp\left(-\tfrac{2}{\zeta}y^2\right)S(y^2)\,,\end{aligned} \tag{3.122}$$

$$S(y^2) = \sum_{(n,m)\neq(0,0)}^{\infty} \frac{|\alpha|^{\,2n}}{n!} \frac{|\beta|^{\,2m}}{m!} \frac{1}{(n+m)\,(n+m+1)^{\,2}\,(n+m+2)} \,. \tag{3.123}$$

We substitute $N = n + m$,

$$S(y^2) = \sum_{(N-m,m)\neq(0,0)}^{\infty} \frac{|\alpha|^{2(N-m)}}{(N-m)\,!} \frac{|\beta|^{2m}}{m!} \frac{1}{N\,(N+1)^2\,(N+2)} \,. \tag{3.124}$$

We use the resummation rule

$$\sum_{(N-m,m)\neq(0,0)}^{\infty} \to \sum_{N-m>0}^{\infty} \sum_{m=0}^{\infty} \to \sum_{N=1}^{\infty} \sum_{m=0}^{N} , \tag{3.125}$$

which gives

$$S(y^2) = \sum_{N=1}^{\infty} \frac{1}{N\,(N+1)^2\,(N+2)} \sum_{m=0}^{N} \frac{|\alpha|^{2(N-m)}}{(N-m)\,!} \frac{|\beta|^{2m}}{m!} \,. \tag{3.126}$$

The last sum is the binomial formula, so we have

$$S(y^2) = \sum_{N=1}^{\infty} \frac{1}{N\,(N+1)^2\,(N+2)} \frac{\left(|\alpha|^2 + |\beta|^2\right)^N}{N!} \,. \tag{3.127}$$

The result for the Wick symbol is

$$\Omega\left(\hat{F}_A \hat{F}_A\right) = \frac{32}{\zeta^2} \exp\left(-\tfrac{2}{\zeta} y^2\right) \sum_{N=1}^{\infty} \frac{\left(\frac{2}{\zeta} y^2\right)^N}{N!} \frac{1}{N\,(N+1)^{\,2}\,(N+2)} \,, \tag{3.128}$$

in agreement with the literature [29].

3.3.3 D-Branes as Instantons

In this section I comment on the equivalence between

a) the moduli space of string theory vacua in the gauge theory of the $D3-D7$ strings as described in Seiberg and Witten [11], see formula (5.18),

b) and the moduli space of instanton parameters (collective coordinates), characterized by the ADHM equations with FI term ζ, see Nekrasov and Schwarz [12].

Here I repeat briefly the relevant statements of these two papers.

- *Nekrasov and Schwarz [12].*

  They describe an antiselfdual instanton for $U(1)$ gauge theory on selfdual noncommutative $\mathbb{R}^4_{NC}$. The moduli space is parametrized by four complex functions B_0, B_1, and I, J. These collective coordinates are subject to the ADHM equations

$$\text{D-term:} \quad \left[B_0, B_0^\dagger\right] + \left[B_1, B_1^\dagger\right] + I\,I^\dagger - J^\dagger J \;= \zeta\,, \tag{3.129}$$

$$\text{F-term:} \quad [B_0, B_1] + I\,J \;= 0\,. \tag{3.130}$$

  The structure of these equations is equivalent to the supersymmetry breaking equations.

- *Seiberg and Witten [11].*

  An open string theory instanton is a configuration of Dirichlet branes and background B-field. The constant rank r Kalb–Ramond field is extended along the worldvolume of a Dp-brane. It interpolates between Neumann and Dirichlet boundary conditions,

$$\left[X'_m - 2\pi\alpha' B_{mn}\dot{X}^n\right]_{\sigma=0}^{\sigma=\pi} \stackrel{!}{=} 0\,. \tag{3.131}$$

  The Dirichlet boundary conditions for large B-field give rise to a $D(p-r)$-brane dissolved within the Dp-brane. Such a configuration of D-branes is called a string theory instanton [38]. The number k of dissolved branes corresponds to the winding number of the instantons, and the number of parent D-branes determines the rank $N-1$ of the gauge group $SU(N)$. The Higgs branch describes the $D3$ as an instanton of the $SU(N)$ gauge theory on the $D7$, the $D3$ is stuck on the worldvolume of the $D7$. We make use of the single instanton solution $N = k = 1$, so we have one $D3$-brane dissolved within the $D7$-brane. In our case there is no Higgs branch.

The sigma model action of type IIB string theory can be regularized consistently in different ways. The B-field gives rise to a propagator of open string vertex operators,

$$\langle X^m(\kappa), X^n(\upsilon)\rangle = -\alpha' G^{mn} \ln(\kappa-\upsilon)^2 + \frac{\mathtt{i}}{2}\Theta^{mn}\,\mathrm{sgn}(\kappa-\upsilon)\,. \tag{3.132}$$

So the coordinates along the B-field are noncommutative.

For selfdual B-field with rank four, the quantization of the string stretching between Dp and $D(p-4)$ is consistent in the $\alpha' \to 0$ limit. It gives rise to a potential of the form

$$V(\zeta) \sim \left(\left[X, X^\dagger\right] + \left[Y, Y^\dagger\right] + q\,q^\dagger - p^\dagger p - \zeta\right)^2 + \left([X,Y] + qp\right)^2 . \tag{3.133}$$

The two chiral superfields p and q are components of an $\mathcal{N} = 2$ hypermultiplet $H = (p, q)$.

- The main message is that the moduli space of the Nekrasov Schwarz instanton parametrized by the collective coordinates,

$$\left(\left[B_0, B_0^\dagger\right] + \left[B_1, B_1^\dagger\right] + I\,I^\dagger - J^\dagger J - \zeta\right)^2 + \left([B_0, B_1] + IJ\right)^2 , \tag{3.134}$$

is the same as the space of zeros of the potential V (3.133). So we can translate

$$I = q\,, \quad J = p\,, \quad B_0 = X\,, \quad B_1 = Y\,. \tag{3.135}$$

- We identify the instanton "size" parameters with the fundamental quark hypermultiplets Q, $\tilde{Q}$ and the "position" parameters with the scalar superfields Φ_1, Φ_2

in the adjoint representation of the gauge group. All these superfields are chiral, according to Seiberg and Witten [11].

Furuuchi [29] considers the solution $I = \sqrt{\zeta}$, $J = 0$, to the ADHM equations. The scalar fields B_0, B_1, and I, J can be interpreted as the position and the size of the instanton. This instanton gives rise to a supersymmetric vacuum,

$$V(\zeta) = 0\,. \tag{3.136}$$

On the other hand, the zero solution $X = Y = p = q = 0$ is a nonsupersymmetric vacuum with a potential proportional to the noncommutativity parameter,

$$V \sim \zeta^2\,. \tag{3.137}$$

This is not a solution to the ADHM equation and the corresponding pointlike instanton does not exist [11].

Chapter 4

The Potential for the Noncommutative Instanton

In this chapter the main results of the first part of this thesis can be found. We obtain a potential for the B-field by evaluating the effective $D7$-brane action on a static instanton configuration. We carry out the integration over the noncommutative coordinates along the B-field.

In section 4.1 we motivate this integration by studying how the B-field interpolates between Neumann and Dirichlet boundary condition. Inspired by [11] we argue that the noncommutative coordinates obtain a vacuum expectation value in the regime of Dirichlet boundary condition.

In section 4.2 we evaluate the effective action on a static gauge field configuration, the noncommutative $U(1)$ instanton found by Nekrasov and Schwarz [12]. In the Dirichlet regime we integrate out the noncommutative y_m directions transversal to the dissolved $D3$-brane.

In section 4.3 we present the resulting potential for the B-field. It distinguishes a vacuum expectation value, as shown in figure 4.3. We discuss this result and propose possible projects for future research.

4.1 Limits and Boundary Conditions

The B-field interpolates between Neumann and Dirichlet boundary conditions of the $\overline{D7}$-brane. The variation of the sigma model action $S_b + S_{int}$ with worldsheet boundary gives the conditions

$$g_{mn}X'^n + 2\pi\alpha'\mathcal{F}^0{}_{mn}\ \dot{X}^n|_{\sigma=0} = 0\,, \tag{4.1a}$$

$$g_{mn}X'^n + 2\pi\alpha'\mathcal{F}^\pi{}_{mn}\ \dot{X}^n|_{\sigma=\pi} = 0\,. \tag{4.1b}$$

We rescale the B-field to have dimension [mass]^2, so the gauge invariant combination is $\mathcal{F}_{mn} = B_{mn} + F_{mn}$. The conditions can be applied independently at each boundary $\partial\Sigma_0$ and $\partial\Sigma_\pi$ of the worldsheet [1]. We choose the same gauge at both ends of the open string, $A^0_M = A^\pi_M$. The boundary interaction terms cancel out each other, so we have $\mathcal{F}_{mn} = B_{mn}$. The relevant conditions along the directions of the B-field are

$$g_{mn}X'^n + 2\pi\alpha' B_{mn}\ \dot{X}^n|_{\partial\Sigma_0,\partial\Sigma_\pi} = 0\,. \tag{4.2}$$

For large B-field, the boundary conditions become Dirichlet. Each endpoint is attached to a four dimensional subspace, the open string is tied to a $D3$-brane inside the $\overline{D7}$-brane [11]. Now we apply our background fields g_{MN} (2.8) and B^+_{MN} (2.10a) to the boundary conditions,

$$\sqrt{H}\delta_{mn}X'^n + 2\pi\alpha' b\ \omega_{mn}\ \dot{X}^n|_{\partial\Sigma_0,\partial\Sigma_\pi} = 0\,. \tag{4.3}$$

The matrix ω is defined in (3.115). It catches the signature of the B^+-field.

In view of future applications as a phenomenological model, we investigate now the compatibility of the boundary conditions with the decoupling limes. This is necessary for the interpretation of our setup as a probe brane with worldvolume flux. The AdS/CFT correspondence relates the string theory parameters to the gauge theory parameters via the important relation

$$\frac{R^4}{(2\pi\alpha')^{\,2}} = \lambda\,. \tag{4.4}$$

R is the radius of $AdS_5 \times S^5$ and α' is the Regge slope. We have rescaled the 't Hooft coupling by a factor of π.

In the decoupling limit, the 't Hooft coupling λ has a large but finite value and α' is taken to be small but nonzero, in order to keep R fixed. We have then

$$R^4 >> y^4\,. \tag{4.5}$$

We approximate the warp factor in the solitonic 3-brane solution by $\sqrt{H} \approx R^2/y^2$, so for our static gauge embedding we obtain

$$\frac{\sqrt{\lambda}}{y^2}\delta_{mn}{y'}^n + b\;\omega_{mn}\;\dot{y}^n\,|_{\partial\Sigma_0,\partial\Sigma_\pi} = 0\,. \tag{4.6}$$

We assume that α' is not exactly zero and factorize out $2\pi\alpha'$. We conclude that the decoupling limit is compatible with Dirichlet boundary conditions, but not with Neumann boundary conditions, since we have

$$\text{Dirichlet b. c. in decoupling limit} \quad \frac{\sqrt{\lambda}}{y^2} << b\,, \tag{4.7}$$
$$\text{Neumann b. c. for} \quad \frac{\sqrt{\lambda}}{y^2} >> b\,.$$

The Dirichlet boundary conditions $\dot{y}^m|_{\partial\Sigma_0,\partial\Sigma_\pi} = 0$ determine the time derivative of the open string vertex, but not its position within the D-brane. Figure 4.1 shows a two dimensional projection of the $\overline{D7}$-brane with the open string vertices. Due to the

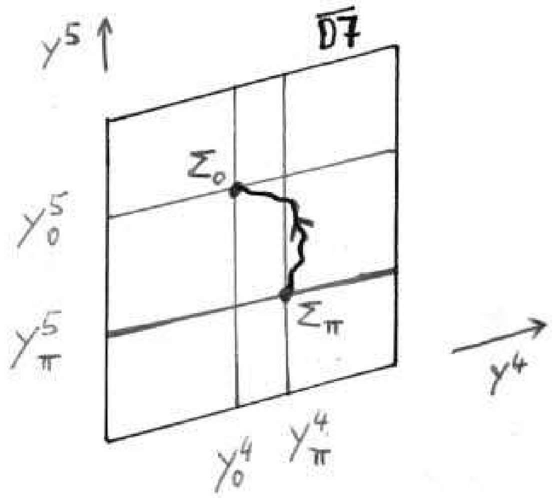


Figure 4.1 – Open string vertices and boundary conditions on a two dimensional projection of the $\overline{D}7$-brane.

noncommutative nature of the directions along the B-field, the two endpoints cannot coincide. Instead, the noncommutativity parameter determines the minimal radius

$$(y_0 - y_\pi)^2 \geq \zeta \, . \tag{4.8}$$

The $D3$-brane is not fully localized in the transverse coordinates, but dissolved within the worldvolume of the probe brane.

If all the endpoints merged, a joining interaction of the open string vertices would be possible. The resulting closed string would escape in the transverse directions as a D-instanton, giving rise to the small instanton singularity [11].

The condition (4.8) can be solved by giving vacuum expectation values to the transverse fluctuations, for example

$$\langle y_0^m - y_\pi^m \rangle = \sqrt{\zeta} \text{ and all other vevs zero.} \tag{4.9}$$

It is interesting to notice that this line of reasoning bears similarity to the Higgs mechanism for translation symmetry.

For two parallel coincident $D7$-branes, the four vacuum expectation values of the transverse scalars break the nonabelian gauge symmetry spontaneously. These degrees of freedom are missing in our setup, but the four vacuum expectation values of the y^m fluctuations can replace them.

4.2 Integrating out Noncommutative Coordinates

Static instanton configurations of the $SU(2)_f$ flavor gauge theory were studied in [9, 39]. Here we investigate the case where a worldvolume flux on the probe brane is sourced by a constant Kalb–Ramond B-field.

We consider the parameter regime of Dirichlet boundary condition. Our aim is to evaluate the effective action for the $D3$-brane dissolved within the worldvolume of the probe brane. We concentrate on the selfdual B-field,

$$S_{\overline{D7}} = -\frac{\mu_7}{g_S} \int \mathrm{d}^4x \int \mathrm{d}^4y \left(1 + \tfrac{1}{2} \left(\frac{\lambda}{y^4} + b^2 \right)^{-1} F^-_{mn} F^-_{mn} \right) . \tag{4.10}$$

The parameters characterizing the curvature of $AdS_5 \times S^5$ and the B-field factorize out of the Maxwell action. The same effect was observed in [9] in the absence of worldvolume flux.

We evaluate the effective action on a static gauge field configuration. We use the noncommutative $U(1)$ instanton solution in flat space $\mathbb{R}^4_{NC}$ found by Nekrasov and Schwarz [12]. We separate the instanton contribution from the volume element,

$$S_{\overline{D7}} = \left(-\frac{\mu_7}{g_S} \int \mathrm{d}^4x \int \mathrm{d}^4y \right) - \frac{1}{g_S} \int \mathrm{d}^4x \, V\,(b, \lambda) \; . \tag{4.11}$$

The dissolved $D3$-brane is extended in the x^μ directions. Along the lines of [39] we integrate out the noncommutative y_m coordinates perpendicular to it. We interpret the resulting function $V(b,\lambda)$ as a potential for the B-field vacuum expectation value. The potential function depends on the 't Hooft coupling λ parameterizing the curvature of $AdS_5 \times S^5$ background space.

The Seiberg–Witten map makes no statement on the relation between the integration measure on the D-brane worldvolume and the measure on $\mathbb{R}^4_{NC}$. We assume that the correspondence is the identity. In our effective action (4.10) we replace the measure, the coordinates along the B-field and the field strength according to

$$\int_{\mathrm{D-brane}} \mathrm{d}^4 y \to \int_{\mathbb{R}^4_{NC}} \mathrm{d}^4 \hat{y}\,, \tag{4.12a}$$

$$y_m \to \hat{y}_m\,, \tag{4.12b}$$

$$F^-_{mn} \to \hat{F}_A\,. \tag{4.12c}$$

We replace the antiselfdual part F^-_{mn} of our generic field strength by the antiselfdual curvature $\hat{F}_A$ for the instanton solution of Nekrasov and Schwarz [12]. Assuming the validity of these exchanges in the effective action, we solve the integral

$$V\left(b,\lambda\right) = -\mu_7 \int_{\mathbb{R}^4_{NC}} \mathrm{d}^4\hat{y}\, \left(\frac{\lambda}{\hat{y}^4} + b^2\right)^{-1} \left(\tfrac{1}{2}\hat{F}_A \hat{F}_A\right)\,. \tag{4.13}$$

For the integration over noncommutative coordinates, there are two methods available. We can either take the trace over Hilbert space operators or integrate over the corresponding phase space functions. These methods are described in [40] and [29].

Integration with Operator Trace

The integration over $\mathbb{R}^4_{NC}$ can be performed by taking the trace of operators in the Fock space $\mathcal{H}$ with the rule

$$\int_{\mathbb{R}^4_{NC}} \mathrm{d}^4\hat{y} \; = (2\pi)^{\,2} Pf\,(\Theta)\, \mathrm{Tr}_{\mathcal{H}}\,. \tag{4.14}$$

This method is applied in [12] to integrate the instanton action (3.117), the result is

$$\int_{\mathbb{R}^4_{NC}} \mathrm{d}^4\hat{y}\left(\tfrac{1}{2}\hat{F}_A\hat{F}_A\right) = (2\pi)^{\,2}\,. \tag{4.15}$$

We integrate (4.13) with the rule (4.14) and with $Pf(\Theta) = \zeta^2/4$. In the regime of Dirichlet boundary condition we can work with the approximation[1]

$$V\,(b,\lambda) \approx -\mu_7 \frac{1}{b^2}\int_{\mathbb{R}^4_{NC}} \mathrm{d}^4\hat{y}\,\left(1 - \frac{\lambda}{b^2\hat{y}^4}\right)\left(\tfrac{1}{2}\hat{F}_A\hat{F}_A\right)\,. \tag{4.16}$$

The first summand is the instanton action (4.15), and for the second term

$$\int_{\mathbb{R}^4_{NC}} \mathrm{d}^4\hat{y}\,\frac{1}{\hat{d}^2}\left(\tfrac{1}{2}\hat{F}_A\hat{F}_A\right) \tag{4.17}$$

we use the operator trace formula (4.14). When taking the trace we assume the operators $\hat{d}^{-2}$ and $\hat{F}_A$ to be diagonal. We use the spectral decomposition of these operators, and we apply the summation techniques explained in section 3.3.2. With formula (3.97) for the distance operator expressed in number operators $a^\dagger a$ we have

$$\mathrm{Tr}_{\mathcal{H}}\frac{1}{\hat{d}^2}\left(\tfrac{1}{2}\hat{F}_A\hat{F}_A\right) = \frac{16}{\zeta^2}\frac{4}{\zeta^2}\sum_{(n,m)\neq(0,0)}^{\infty}\frac{1}{(n+m)\,(n+m+1)^4\,(n+m+2)}\,. \tag{4.18}$$

[1]We keep only the linear term of the binomial series

$$(1+x)^{\,r} = \sum_{n=0}^{\infty}\binom{r}{n}x^n$$

for $|x| < 1$, $r < 0$, $x := \lambda/\left(y^4b^2\right)$.

We sum over $N = n + m$ and multiply each summand with $N + 1$,

$$\mathrm{Tr}_{\mathcal{H}} \frac{1}{\hat{d}^2} \left(\tfrac{1}{2} \hat{F}_A \hat{F}_A \right) = \frac{4^3}{\zeta^4} \sum_{N=1}^{\infty} \frac{1}{N\,(N+1)^3\,(N+2)} \,. \tag{4.19}$$

We used the software `Mathematica` to calculate the sum. The result is

$$\mathrm{Tr}_{\mathcal{H}} \frac{1}{\hat{d}^2} \left(\tfrac{1}{2} \hat{F}_A \hat{F}_A \right) = \frac{64}{\zeta^4} \left(\frac{5}{4} - \mathrm{RiemannZeta}\,(3) \right) \,. \tag{4.20}$$

The number RiemannZeta (3) is also known as Apéry's constant [41],

$$\mathrm{RiemannZeta}\,(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx \frac{6}{5} \,. \tag{4.21}$$

So the integral (4.17) is given by

$$\int_{\mathbb{R}^4_{NC}} \mathrm{d}^4 \hat{y} \, \frac{1}{\hat{d}^2} \left(\tfrac{1}{2} \hat{F}_A \hat{F}_A \right) \approx (2\pi)^2 \frac{16}{20 \zeta^2} \,. \tag{4.22}$$

We use $\zeta = 4/b$ and put these outcomes in the potential (4.16). With the method of operator trace we have obtained

$$V_{ot}\,(b, \lambda) \approx -\mu_7 \frac{(2\pi)^2}{b^2} \left(1 - \frac{\lambda}{20} \right) \,. \tag{4.23}$$

This result is valid in the parameter regime of Dirichlet boundary conditions and in the Seiberg–Witten limit. It is not a reasonable potential since it shows a "run away" behavior. This should be due to the fact that we approximated the binomial series by its first two terms. In the following we compare this result with the potential obtained with Wick symbol.

Integration with Wick Symbol

Fock space operators are represented by their phase space Wick symbols according to the discussion in section 3.3. We apply the standard integration rule for operators $\hat{f}$ and $\hat{g}$ [15],

$$\begin{aligned} \int_{\mathbb{R}^4_{NC}} \mathrm{d}^4\hat{y}\ \hat{f}\,\hat{g} &= \int_{\mathbb{R}^4} \mathrm{d}^4y\, \Omega(\hat{f}) \star \Omega(\hat{g}) \\ &= \int_{\mathbb{R}^4} \mathrm{d}^4y\, \Omega(\hat{f}) \cdot \Omega(\hat{g})\,. \end{aligned} \tag{4.24}$$

The multiplication in the algebra of operator symbols is given by the Groenewold–Moyal star product. In our work, we only use this product only under the integral, where it is equivalent to the ordinary multiplication of functions.

We evaluate our potential (4.13) by integrating over Wick symbols,

$$V\left(b,\lambda\right) = -\mu_7 \int_{\mathbb{R}^4} \mathrm{d}^4y \left(\frac{\lambda}{y^4} + b^2\right)^{-1} \Omega\left(\tfrac{1}{2}\hat{F}_A\hat{F}_A\right)\,. \tag{4.25}$$

Despite the necessity of normal ordering, we treat the prefactor as a number. We are aware that this assumption is contentious, arguments supporting this point of view are given in section 3.3.1. We use the Wick symbol (3.128) of the instanton action and change to spherical coordinates,

$$V\left(b,\lambda\right) = -\mu_7 2\pi^2 \int_0^\infty \mathrm{d}\rho\ \frac{\rho^3}{\frac{\lambda}{\rho^4}+b^2} \left(\frac{16}{\zeta^2}\right) \exp\left(-\tfrac{2}{\zeta}\rho^2\right) \sum_{N=1}^{\infty} \frac{\left(\frac{2}{\zeta}\rho^2\right)^N}{N!} \frac{1}{N\left(N+1\right)^2\left(N+2\right)}\,. \tag{4.26}$$

In the above formula, both the noncommutativity scale ζ and the B-field parameter b appear under the integral. They are related via the Seiberg–Witten map summarized

$\alpha' \to 0$	$2\pi\alpha' = 1$
$\Theta = B^{-1}$	$\Theta = E_A^{-1}$
$\zeta = \frac{4}{b}$	$\zeta = \frac{4b}{1+b^2}$

Table 4.1 – The Seiberg–Witten map between open and closed string parameters. E_A is the antisymmetric part of the matrix $g + B$.

in table 4.1. The relation between ζ and b depends on the zero slope limit. We start by taking the limit and discuss the situation for finite α' in the next section. In order to perform the integration, we substitute

$$\xi = \frac{2\rho^2}{\zeta} . \tag{4.27}$$

This amounts to an implicit relation between the radial coordinate ρ of AdS space and the noncommutativity parameter. In the zero slope limit, the potential is

$$V_{zero}(b, \lambda) = -\mu_7 \frac{4\pi^2}{b^2} \int_0^\infty \mathrm{d}\xi \, \frac{\xi^3}{\frac{\lambda}{4} + \xi^2} \exp(-\xi) \sum_{N=1}^{\infty} \frac{\xi^N}{N!\, N\,(N+1)^2\,(N+2)} . \tag{4.28}$$

The integral over ξ is solved numerically with `Mathematica`[2]. The resulting numerical function $I_{zero}(\lambda)$ is finite for all positive λ and does not depend on b. So the potential is of the form

$$V_{zero}(b, \lambda) = -\mu_7 \frac{(2\pi)^2}{b^2} I_{zero}(\lambda) . \tag{4.29}$$

This potential shows a "run away" behavior. No expectation value of the B-field is distinguished. The energy of the system is lowered with increasing B-field. We interpret this behavior as due to the fact that the $3-3$ strings at large B-field have less degrees of freedom than the $3-7$ strings at small B-field.

[2] As of May 2008, the notebook is available online at `http://www.mppmu.mpg.de/~hoehne/`.

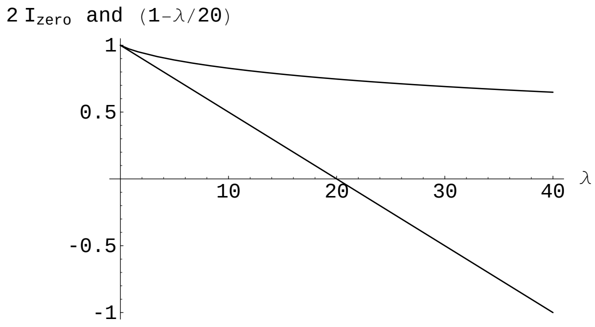


Figure 4.2 – Comparison of the two methods in the zero slope limit. The numerical function $2I_{zero}(\lambda)$ is plotted against $(1-\lambda/20)$ coming from the operator trace result (4.23).

We use the results for V_{zero} and V_{ot} (4.23) to compare the methods of Wick symbol and operator trace to each other. They should give exactly the same results. The function $(1 - \lambda/20)$ in V_{ot} switches sign due to the approximation we made. It is necessary to calculate higher terms in the binomial formula. We regard the method of Wick symbol as more advantageous since the potential does not change sign. The function I_{zero} does not influence the shape of the potential significantly. Both functions are shown in plot 4.2.

4.3 The Potential

In this section I discuss the situation with Regge slope fixed at finite value $2\pi\alpha' = 1$. The potential possesses a minimum, the plots are shown in figures 4.3 and 4.4.

For finite Regge slope, the relation $\zeta(b)$ is not invertible and possesses an extremum. We expect now a different behavior of the potential. We check this expectation by numerical integration of the Wick symbol potential,

$$\boxed{V_{finite}\,(b,\lambda) = -\frac{1}{2\pi b^2}\, I_{finite}\,(b,\lambda)\ .} \tag{4.30}$$

This function is plotted in figure 4.3, a minimum is present at $b \approx 1$. We consider the integral $I_{finite}(b, \lambda)$ as a function of the B-field parameter and solve it numerically with `Mathematica`,

$$I_{finite}\,(b, \lambda) = \int_{20}^{\infty} \mathrm{d}\xi \; \frac{\xi^3}{\left(\frac{1}{b^2}+1\right)^2 \left(\frac{\lambda}{4}+\xi^2\right)} \exp\left(-\xi\right) \sum_{N=1}^{\infty} \frac{\xi^N}{N!\, N\,(N+1)^2\,(N+2)} \,. \quad (4.31)$$

The most interesting property of this potential is that it distinguishes a particular value for the closed string B-field parameter. The minimum arises only for finite value of α'. In performing the integration we assumed two implicit dependencies. The AdS radius and the noncommutativity scale are related via our substitution (4.27), and the Seiberg–Witten map relates the closed string b parameter to the open string ζ parameter.

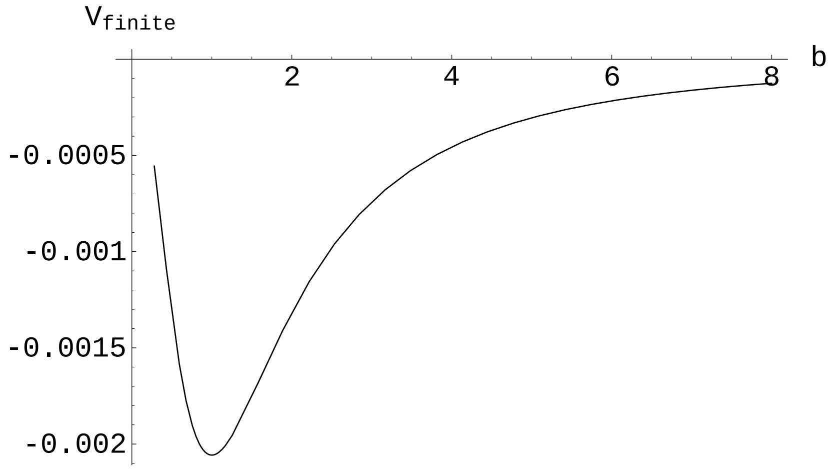


Figure 4.3 – The potential V_{finite} from equation (4.30) is shown at the value $\lambda = 100$ and in the range $0.28 < b < 8$.

Now we discuss whether the parameter range we investigated is physically significant. For the integration of I_{finite} we used the integral borders $20 < \xi < \infty$. The Dirichlet boundary conditions are fulfilled in the vicinity of the minimum $b \approx 1$. We use both

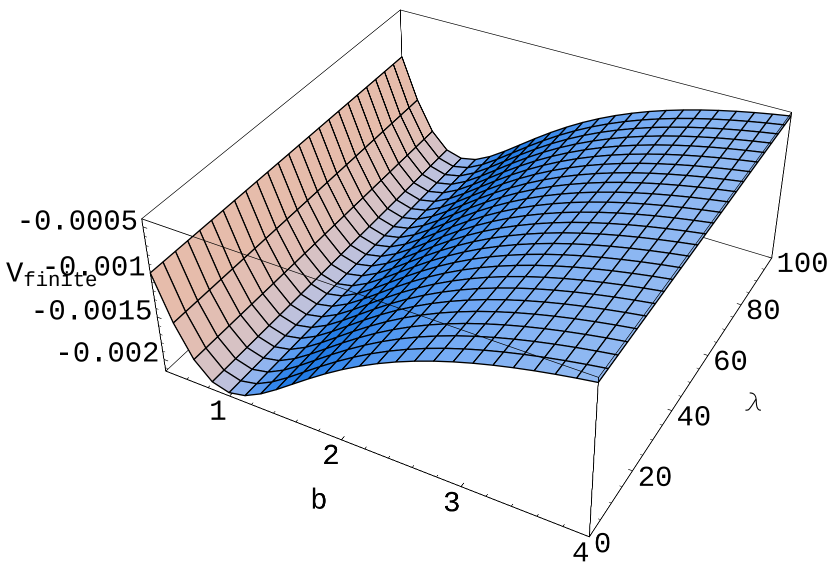


Figure 4.4 – The potential V_{finite} at finite α' from equation (4.30) is shown in the ranges $0.4 < b < 4$ and $0 < \lambda < 100$.

implicit dependencies to get $\xi \approx \rho$, yielding $1/2 < 1$ at $\lambda = 100$. The result of the computation does not depend significantly on the cutoff, and it was also possible to integrate over the full range $0 < \xi < \infty$.

For plotting figure 4.3 we used the numerical value $\lambda = 100$. From the numerical calculation shown in figure 4.4 we infer that the value of the parameter λ and therefore the curvature of $AdS_5 \times S^5$ does not influence the shape of the potential significantly. At a fixed value of b, the potential increases slightly with λ. Near the minimum of the potential we have the values

$$V_{finite}(1,0) \approx -2.101 \cdot 10^{-3}\,,$$
$$V_{finite}(1,100) \approx -2.057 \cdot 10^{-3}\,, \tag{4.32}$$
$$V_{finite}(1,800) \approx -1.821 \cdot 10^{-3}\,.$$

The numerical calculation becomes imprecise at values $\lambda > 800$.

In the process of calculating the potential, we have set all dimensionful constants to one. The aim of the following considerations is to restore the SI units of the potential as a consistency check. In SI units, the potential is an energy, and in natural units it has dimension $[\text{mass}]^4$. The B-field may be interpreted as a generalization of the magnetic field in electrodynamics. The components of the magnetic field B are measured in the SI unit tesla. The noncommutativity scale is an area.

quantity	α'	ζ	B	A_m	field strength
natural units	length^2	length^2	mass^2	mass	mass^2
SI derived units	$\frac{1}{J^2}$	m^2	$\frac{Vs}{m^2} = \frac{kg}{As^2}$	$\frac{Vs}{m}$	$\frac{Vs}{m^2}$

Table 4.2 – The natural and SI units of the quantities we use.

The interplay of spacetime translations and internal gauge transformations is characteristic to noncommutative geometry [15],

$$\partial_m \hat{f} = [A_m, \hat{f}] \,. \tag{4.33}$$

This property is used in the derivation of the noncommutative instanton, where the gauge potential is set equal to quantities of length dimensions.

In our model, the relation between B-field parameter and noncommutativity scale due to the Seiberg–Witten map implies a connection between electromagnetic and spacetime units. The electric charge carried by electrons is measured in coulomb. Since in practice charge transport always amounts to mass transport, it is in my opinion natural to associate the ampere with the dimension mass. On the other hand, the voltage is the difference in an electric potential over a certain distance, and therefore has inverse length. This gives the right units for the instanton gauge field.

I would like to put our result in the context of string theory by comparing our potential to approaches where potentials stabilize the moduli fields of the six compact extra dimensions. In the interesting work of Haack et. al. [42], metastable four dimensional de Sitter spacetimes are obtained by uplifting stabilized supersymmetric AdS vacua of type IIB string theory compactified on a Calabi–Yau manifold. To achieve this, two stacks of $D7$-branes are considered. A worldvolume flux leads to a D-term potential in one stack, and gaugino condensation stabilizes some of the Kähler moduli on the other. The D-term potential can uplift the vacuum from AdS to dS and therefore give rise to a small positive cosmological constant.

In our model, the transverse space is spanned by the noncompact fifth direction ρ of AdS_5 and the compact five sphere. The shape of our potential arises due to the noncommutative nature of the transverse spacetime directions.

Our results may be of relevance for phenomenological models in the AdS/CFT correspondence. For this it would be interesting consider fluctuations around the static instanton gauge field along the lines of [43]. The simplest ansatz should be

$$A_m = \exp\left(\mathtt{i}\tfrac{\pi}{2}\right)\zeta\frac{\omega_{mn}y^n}{(y^2+\zeta/2)\,(y^2-\zeta/2)} + f(y)\xi_m(k_\mu)\exp(\mathtt{i}kx)\,. \tag{4.34}$$

The fluctuations are plane waves with wave vector k_μ in the directions along the dissolved $D3$-brane. The spectral functions $f(y)\xi_m(k)$ are vectors in the transversal directions.

The next steps would be to calculate the equations of motion for the fluctuations, and to consider non-static embeddings.

A mechanism for supersymmetry breaking on the field theory side is to deform the $AdS_5 \times S^5$ metric on the gravity side. Since the constant B-fields investigated here do not cause such an backreaction effect, it should be promising to look at the most general ansatz for a B-field with dynamical components in the z^8, z^9 directions,

$$B_{MN}\,\mathrm{d}X^M \wedge \mathrm{d}X^N = b\,\mathrm{d}X^4 \wedge \mathrm{d}X^5 + b\,\mathrm{d}X^6 \wedge \mathrm{d}X^7 + b_z\,(X)\,\mathrm{d}X^8 \wedge \mathrm{d}X^9\,, \tag{4.35}$$

$$dB = db_z\,(X^\mu, X^m) \wedge \mathrm{d}X^8 \wedge \mathrm{d}X^9\,. \tag{4.36}$$

Our embedding is the restriction to the submanifold $AdS_5 \times S^3$, in particular $X^\mu = x^\mu$, $X^m = y^m$, $X^8 = X^9 = 0$. None of the instanton calculations presented here is altered by this ansatz.

It would be interesting to evaluate the backreaction of this B-field on the Polchinski–Strassler background [44]. Flavor branes are added to this background in [45]. An alternative possibility is that the quantization of the coordinates gives rise to a metric deformation. We discuss this further in the conclusions chapter 7.

Part II

The Local Renormalization Group of $\mathcal{N} = 1$ Supersymmetric Gauge Theories

Chapter 5

The Local Renormalization Group

In this chapter we describe the strategy to obtain information on central functions of renormalizable quantum field theories coupled to a classical four dimensional curved background space.

Starting from the work of Osborn [21], we explain the structure of the conformal Ward identity and outline how it can lead to central functions in section 5.1.

We are particularly interested in $\mathcal{N} = 1$ supersymmetric gauge theories with matter fields. Our aim is to find a monotonically decreasing central function for the coefficient of the Euler anomaly. This leads us to introduce new local theta couplings and external vector sources for $U(1)$ R-symmetry in section 5.2. We also explain why we expect supersymmetry to give further restrictions on the anomaly coefficients.

5.1 The Strategy of Local Couplings

The aim of this work is to investigate the properties of central functions in four dimensional Euclidean quantum field theories. These functions of the couplings count the massless degrees of freedom at a particular renormalization scale and interpolate between the central charges at the fixed points.

Our starting point is the work of Osborn [21]. His ansatz is to promote the renormalized couplings of composite dimension four scalar operators to auxiliary fields $\lambda^i(x, M)$ that depend on the coordinates of the background space and the renormalization scale. Under functional differentiation of the vacuum energy functional $W[g_{\mu\nu}, \lambda^i]$, the local couplings act as external sources for finite local quantum operators $[\mathcal{O}^i(x)]$, just like the metric sources the energy-momentum tensor,

$$\langle [T_{\mu\nu}(x)] \rangle = \frac{2}{\sqrt{g(x)}} \frac{\delta W}{\delta g^{\mu\nu}(x)}, \tag{5.1a}$$

$$\langle [\mathcal{O}_i(x)] \rangle = \frac{1}{\sqrt{g(x)}} \frac{\delta W}{\delta \lambda^i(x)}. \tag{5.1b}$$

The introduction of local couplings allows for studying local conformal anomalies away from the fixed points of the renormalization group flow. As explained in detail in the rest of this chapter, the response of the vacuum energy functional on a Weyl transformation Δ_σ with local parameter $\sigma(x)$ is encoded in the conformal Ward identity

$$\begin{aligned}\Delta_\sigma W = &\int \sqrt{g}\, \mathrm{d}^4x\, \sigma \Big(\beta^{\mathscr{A}} F + \beta^{\mathscr{B}} G + \tfrac{1}{9}\beta^{\mathscr{C}} R^2 + \tfrac{1}{2}\chi^a_{ij} \Box\lambda^i \Box\lambda^j + \tfrac{1}{2}\chi^b_{ijk}\Lambda^{ij}\Box\lambda^k + \tfrac{1}{4}\chi^c_{ijkl}\Lambda^{ij}\Lambda^{kl} \\ &+ \tfrac{1}{3}\chi^e_i \partial_\mu\lambda^i\partial^\mu R + \tfrac{1}{6}\chi^f_{ij}\Lambda^{ij} R + \tfrac{1}{2}\chi^g_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j G^{\mu\nu}\Big) \\ &+ \int \sqrt{g}\, \mathrm{d}^4x\, \partial_\mu\sigma \left(w_i \partial_\nu \lambda^i G^{\mu\nu} + \tfrac{1}{3}\partial^\mu (d\, R) + \partial^\mu \left(U_i \Box\lambda^i\right) + \tfrac{1}{2}\partial^\mu\left(V_{ij}\Lambda^{ij}\right) \right. \\ &\left. + \tfrac{1}{3} Y_i \partial^\mu\lambda^i R + S_{ij}\partial^\mu\lambda^i\Box\lambda^j + \tfrac{1}{2} T_{ijk}\Lambda^{ij}\partial^\mu\lambda^k \right).\end{aligned} \tag{5.2}$$

On the right hand side, the gravitational anomalies consisting of the square of the Weyl tensor F, the Euler density G and the Ricci scalar R, are accompanied by a complete basis of independent local anomaly monomials build from spacetime curvature tensors and coupling derivatives. The anomaly coefficients can be calculated perturbatively. We investigate all of them in detail in chapter 6, here we are interested in the coefficient $\beta^{\mathscr{B}}$ of the Euler anomaly.

The Wess–Zumino consistency condition [22] states that two conformal transformations have to commute,

$$\left[\Delta'_{\sigma}, \Delta_{\sigma}\right] W = 0 \,. \tag{5.3}$$

Evaluating it gives the important relations between anomaly coefficients needed for the central function. Osborn obtains in this way a function $C^{\mathscr{B}}(\lambda^i, M)$ satisfying the slope equation

$$M \frac{\partial}{\partial M} C^{\mathscr{B}} = -\tfrac{1}{8} \chi^g_{ij} \beta^i \beta^j \tag{5.4a}$$

and the central charge equation

$$C^{\mathscr{B}} = \beta^{\mathscr{B}} + \tfrac{1}{8} w_i \beta^i \,. \tag{5.4b}$$

The anomaly coefficients are defined in (5.2). At the fixed points, this function coincides with the Euler coefficient $\beta^{\mathscr{B}}$, and the relation

$$\chi^g_{ij} = -2\chi^a_{ij} \tag{5.5}$$

between local anomaly coefficients follows from the Wess–Zumino consistency condition. It is shown in [46] that the symmetric form χ^a_{ij} is negative definite in any renormalization scheme by comparing it to the scalar 2-point function $\langle [\mathcal{O}_i][\mathcal{O}_j] \rangle$. Our aim is to establish $C^{\mathscr{B}}$ as a central function. Our ansatz of restricting to supersymmetric gauge theory can be found in section 5.2.

Here we discuss the local renormalization group equation (5.2). In short, there are gravitational anomalies since the quantum theory is coupled to curved space, internal anomalies are caused by renormalization and local anomalies arise due to the local couplings.

Classically, the traceless symmetric energy-momentum tensor is the conserved current of conformal symmetry. But in the quantum theory the corresponding Ward identity receives contributions from gravitational anomalies,

$$g^{\mu\nu}\langle[T_{\mu\nu}]\rangle = \beta^{\mathscr{A}} F + \beta^{\mathscr{B}} G + \beta^{\mathscr{C}} R^2 \,. \tag{5.6}$$

As explored by mathematicians [20], the gravitational anomalies in four dimensions are the Euler density G and the square of the Weyl tensor F. We use here the conventions of Osborn [21],

$$F = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2 \,, \tag{5.7a}$$

$$G = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \,. \tag{5.7b}$$

The $\beta^{\mathscr{C}} R^2$ anomaly arises in perturbation theory, cf. [47, 48, 49]. Riegert [50] shows that it cannot be derived from a functional obeying the axiom of locality. He argues that perturbation theory R^2 ultraviolet divergences have to cancel out in any quantum field theory coupling matter to curved background. This is a constraint on the theory, and the coefficient equation $\beta^{\mathscr{C}} = 0$ puts restrictions on the anomaly coefficients. In this work, we regard the coefficient $\beta^{\mathscr{C}}$ as being zero at the fixed points, but nonvanishing elsewhere.

In the quantum theory, renormalization introduces a scale M and therefore breaks conformal symmetry (and even dilatation invariance in flat space). This is reflected in the internal anomalies of the dynamical fields. For example the dimension four operator composed of the field strength,

$$\beta^F(g)\,\mathrm{Tr}\langle[F_{\mu\nu}F^{\mu\nu}]\rangle \;\text{ is one term in }\; \beta^i(\lambda)\,\langle[\mathcal{O}_i]\rangle\,. \tag{5.8}$$

The anomalies we have discussed so far are summarized in the operator equation

$$g^{\mu\nu}\langle[T_{\mu\nu}]\rangle - \beta^i\langle[\mathcal{O}_i]\rangle = \beta^{\mathscr{A}}F + \beta^{\mathscr{B}}G + \beta^{\mathscr{C}}R^2\,. \tag{5.9}$$

Each renormalization step consists of an integration and a global rescaling, either in momentum or position space. Under such a rescaling $\delta_\sigma M = \sigma M$ with constant parameter σ, the couplings transform as

$$\delta_\sigma\lambda^k = \sigma\beta^k\,. \tag{5.10}$$

The beta functions are defined as the renormalization group derivative of the couplings

$$\beta^i\left(\lambda^i\right) = M\frac{\partial}{\partial M}\lambda^i(x,M)\;. \tag{5.11}$$

The crucial point of the local coupling formalism is to take into account local rescalings with coordinate dependent parameter $\sigma(x)$. The dilatation is promoted to a conformal transformation, a local rescaling that preserves angles.

In mathematical terms, a conformal transformation is a diffeomorphism of the background space that respects the equivalence relation [51]

$$\bar{g}\sim g \quad\Leftrightarrow\quad \bar{g}_{\mu\nu}(x) = \exp\left(-2\sigma(x)\right)g_{\mu\nu}(x)\;. \tag{5.12}$$

This transformation is called Weyl rescaling of the metric with local scale factor $\sigma(x)$. The infinitesimal transformation is determined by the action of the Lie derivative $\mathcal{L}_k$ along the symmetry vector field k, as expressed in the conformal Killing equation

$$\left(\mathcal{L}_k g\right)_{\mu\nu}(x) = -2\sigma(x)\,g_{\mu\nu}(x)\;. \tag{5.13}$$

We use this infinitesimal Weyl rescaling of our external metric in the subsequent variation formulas. We assume that the classical theory is invariant under Weyl transformations of the metric, which puts strong restrictions on the theory [21].

The homogeneous Callan–Symanzik equation for the renormalized connected correlation function states that global shifts of the couplings, expressed in the beta functions, compensate for the change of the renormalization scale, in a way that the bare correlation function remains unchanged. We assume that a local version of the homogeneous Callan–Symanzik equation can be applied to the vacuum energy functional,

$$\left(M \frac{\partial}{\partial M} + \int \mathrm{d}^4 x \, \beta^i(x) \frac{\delta}{\delta \lambda^i(x)} \right) W = 0 \,, \tag{5.14}$$

and from the renormalization group invariance of the vacuum energy functional we have the relation

$$\left(M \frac{\partial}{\partial M} - \int \mathrm{d}^4 x \, 2 g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} \right) W = 0 \,. \tag{5.15}$$

In summary, the response of the vacuum energy functional to a local Weyl rescaling is expressed in the local renormalization group equation,

$$\begin{aligned} \Delta_\sigma W &= \int \sqrt{g} \, \mathrm{d}^4 x \, \sigma(x) \left(g^{\mu\nu} \langle [T_{\mu\nu}] \rangle - \beta^i \langle [\mathcal{O}_i] \rangle \right) \\ &= \int \sqrt{g} \, \mathrm{d}^4 x \, \sigma(x) \, (\text{gravitational and local anomalies}) \,. \end{aligned} \tag{5.16a}$$

with the operator

$$\Delta_\sigma = \int \mathrm{d}^4 x \, \sigma(x) \left(2 g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^i(x) \frac{\delta}{\delta \lambda^i(x)} \right) . \tag{5.16b}$$

This local renormalization group equation can be seen as a local version of the Callan–Symanzik equation and neatly summarizes the gravitational quantum anomalies due to the curved background and the internal anomalies coming from renormalization, with the help of local anomalies.

Together with the new local renormalization group equation for R-symmetry introduced in section 5.2, this Ward identity is the starting point of our evaluation of the Wess–Zumino consistency condition. The mutual consistency conditions and the relationship of the gravitational and R-anomalies as superpartners is expected to give new information about the local renormalization group of $\mathcal{N} = 1$ supersymmetric gauge theories.

5.2 Local R-Symmetry in $\mathcal{N} = 1$ Supersymmetric Gauge Theories

Our aim is to establish $C^{\mathscr{B}}$ as a central function. Our expectation is that new information comes from specializing to $\mathcal{N} = 1$ supersymmetric gauge theories. The Euler anomaly coefficient then satisfies $\beta^{\mathscr{B}}_{UV} > \beta^{\mathscr{B}}_{IR}$ and therefore provides a reasonable central charge [52].

In this work we make steps towards showing the monotonicity of $C^{\mathscr{B}}$. We want to find the relation (5.5), $\chi^g_{ij} = -2\chi^a_{ij}$, by evaluating the Wess–Zumino consistency condition for conformal symmetry as well as R-symmetry. We expect further restrictions on the anomaly coefficients from supersymmetry.

In four dimensional renormalizable supersymmetric gauge theory, the Yang–Mills coupling and the instanton angle are combined to the complex gauge coupling

$$\tau = \frac{\theta}{2\pi} + \mathtt{i}\frac{4\pi}{g^2}\,. \tag{5.17}$$

The complex gauge coupling is renormalized and therefore a function of the chiral superfields Φ^I. Supersymmetry restricts it to be holomorphic.

In [53], Witten adds a total derivative term to the Yang–Mills action that violates CP but preserves renormalizability. The instanton angle θ is the coupling parameter of this topological term. The transformation $\theta \to \theta + 2\pi$ is enhanced to a $SL(2,\mathbb{Z})$ duality by Montonen and Olive [54].

We promote the complex gauge coupling to a set of external bosonic fields,

$$\tau^i(x, M) = \hat{\theta}^i(x) + \mathtt{i}\lambda^i(x, M) , \tag{5.18}$$

consisting of the dimensionless local lambda couplings and the newly introduced local theta couplings. The dimensionless theta couplings are then the external sources of composite local scalar operators $[\hat{\mathcal{O}}_i(x)]$.

$\mathcal{N} = 1$ supersymmetric gauge theories are invariant under an $U(1)$ R-symmetry rotating the fermionic coordinates of superspace. We introduce new local external $U(1)$ vector fields $V_\mu(x)$ as sources for the R-current $R^\mu(x)$.

This allows for treating the internal, the gravitational as well as the R-symmetry anomalies on the same footing. The external sources and the local operators they couple to are summarized in table 5.1.

We discuss the properties of our ansatz. The starting point is the vacuum energy functional of the external fields,

$$\exp W\left[g^{\mu\nu}, \lambda^i, V_\mu, \hat{\theta}^i\right] = \int \mathcal{D}\Phi \, \exp\left(-S\left[\Phi^I, g^{\mu\nu}, \lambda^i, V_\mu, \hat{\theta}^i\right] + \int \mathrm{d}^4x \, J_I(x) \, \Phi^I(x)\right) . \tag{5.19}$$

metric	$g_{\mu\nu}(x)$	$[T^{\mu\nu}(x)]$
local lambda coupling	$\lambda^i(x, M)$	$[\mathcal{O}_i(x)]$
R-vector	$V_\mu(x)$	$[R^\mu(x)]$
local theta coupling	$\hat{\theta}^i(x)$	$[\hat{\mathcal{O}}_i(x)]$

Table 5.1 – Our external sources and the local operators they couple to. The renormalization scale is denoted with M.

The dynamical fields Φ^I couple to the matter sources J^I. The local R-current and the composite dimension four scalar operators are generated by functional differentiation of W with respect to the external sources,

$$\langle [R^\mu(x)] \rangle = -\frac{1}{\sqrt{g(x)}} \frac{\delta W}{\delta V_\mu(x)} , \tag{5.20a}$$

$$\langle [\hat{\mathcal{O}}_i(x)] \rangle = \frac{1}{\sqrt{g(x)}} \frac{\delta W}{\delta \hat{\theta}^i(x)} . \tag{5.20b}$$

The connected correlation functions of these finite operator insertions obey the axioms of regularity, rotational invariance and reflection positivity [55, 56].

The rigid superspace is spanned by four bosonic and four fermionic coordinates, $z^M = \{x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\}$, and a R-symmetry transformation rotates the fermionic coordinates with global parameter ρ,

$$\theta^\alpha \to \exp(-\mathtt{i}\rho)\, \theta^\alpha , \tag{5.21a}$$

$$\bar{\theta}^{\dot{\alpha}} \to \exp(+\mathtt{i}\rho)\, \bar{\theta}^{\dot{\alpha}} . \tag{5.21b}$$

We would like to express the conformal anomalies and the R-anomalies in a unified formalism, therefore we introduce external sources with the appropriate tensor structures. We think of R-symmetry as a local abelian $U(1)$ gauge symmetry. The external source $V_\mu(x)$ is the associated gauge field. We take the Weyl weight of the vector field to be zero. The infinitesimal local transformations are

$$\delta_\rho V_\mu\left(x\right) = \partial_\mu \rho\left(x\right)\,, \tag{5.22a}$$

$$\delta_\sigma V_\mu\left(x\right) = 0\,. \tag{5.22b}$$

In the spirit of Osborn, we promote the instanton angle to local theta couplings. In analogy with the conformal lambda couplings, we assume that the change of the local theta couplings under R-symmetry is given by

$$\delta_\rho \hat{\theta}^k\left(x\right) = \rho\left(x\right) \hat{\beta}^k(\hat{\theta}^i)\,. \tag{5.23}$$

They change continuously under a local R-symmetry rotation of the fermionic superspace coordinates. In the evaluation of the Wess–Zumino consistency condition we make use of generic beta functions $\hat{\beta}^i(\hat{\theta}^i)$ associated to the theta couplings.

A global transformation of the form (5.23) is used in supersymmetric QCD [57], where the global R-symmetry rotation of the fermionic matter fields changes the path integral measure and gives rise to a shift in the instanton angle.

In accord with Osborn [58], we associate to our couplings a “parity” property, we impose that two odd theta couplings have to square to an even lambda coupling. This puts restrictions on the form of the local anomalies in the Ward identities.

We assume that the theta couplings are not renormalized and have Weyl weight zero, $\delta_\sigma \hat{\theta}^k = 0$. We assume the lambda couplings to be invariant under R-symmetry, $\delta_\rho \lambda^k = 0$. The requirement that the complex gauge couplings $\tau^i(\Phi^I)$ have to be holomorphic functions of the chiral matter fields gives the restriction that the bosonic component functions λ^i and $\hat{\theta}^i$ have to fulfill the Cauchy–Riemann equations.

In addition to the local Weyl anomalies studied by Osborn [21], we investigate the anomalies of local abelian $U(1)$ R-symmetry. It has an internal anomaly of similar structure to the Adler–Bell–Jackiw anomaly [57],

$$\beta^{ABJ}(g)\,\mathrm{Tr}\langle[F_{\mu\nu}\ {}^{*}F^{\mu\nu}]\rangle \ \text{ is one term in } \ \hat{\beta}^{i}\langle[\hat{\mathcal{O}}_{i}]\rangle\,. \tag{5.24}$$

The anomalous operator equation for the divergence of the axial current is exact to all orders in perturbation theory [59], so the beta function is a constant number. We define an external field strength $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, which we assume to give rise to an additional external anomaly of the same tensor structure and parity as the internal Adler–Bell–Jackiw anomaly. The anomalies are summarized in the operator equation

$$\partial_\mu\langle[R^{\mu}(x)]\rangle - \hat{\beta}^{i}(x)\,\langle[\hat{\mathcal{O}}_{i}(x)]\rangle = -\tfrac{1}{2}\hat{\beta}^{\mathscr{V}}V_{\mu\nu}\,{}^{*}V^{\mu\nu}\,. \tag{5.25}$$

If we make a global shift of the instanton angle of the form (5.23) and assume that the Adler–Bell–Jackiw anomaly is the only contribution to the internal anomaly, we have the anomaly free R-current [57]. This gives restrictions for our generic beta functions.

We postulate that there exists a local renormalization group equation for R-symmetry,

$$\begin{aligned}\Delta_\rho W &= \int\!\sqrt{g}\,\mathrm{d}^4x\,\rho(x)\Big(\partial_\mu\langle[R^{\mu}(x)]\rangle - \hat{\beta}^{i}\langle[\hat{\mathcal{O}}_{i}(x)]\rangle\Big) \qquad (5.26\text{a})\\ &= \int\!\sqrt{g}\,\mathrm{d}^4x\,\rho(x)\,(\text{local R-anomalies})\ ,\end{aligned}$$

and with (5.20) and partial integration we define the operator

$$\Delta_\rho = \int\!\mathrm{d}^4x\,\left(\partial_\mu\rho(x)\,\frac{\delta}{\delta V_\mu(x)} - \rho(x)\,\hat{\beta}^{i}(x)\,\frac{\delta}{\delta\hat{\theta}^{i}(x)}\right)\,. \tag{5.26b}$$

We accompany the external anomaly by a basis of independent tensor monomials build from field strength, curvature tensors and derivatives of local couplings. We write down the explicit form of the local R-anomalies and the Wess–Zumino consistency condition in section 6.1.

$\mathcal{N} = 1$ supersymmetric gauge theories including chiral matter fields [60, 61] possess two fixed points in the "conformal window"

$$\frac{N_C}{2} < \frac{N_f}{3} < N_C \,. \tag{5.27}$$

So they are the prime example where the existence of central functions can be established. In particular it is shown in [52] that the coefficient of the Euler anomaly satisfies the inequality

$$\beta^{\mathscr{B}}_{UV} > \beta^{\mathscr{B}}_{IR} \,, \tag{5.28}$$

so it is to be expected that the function $C^{\mathscr{B}}$ (5.4) can be established as a central function. This motivates us to search for additional restrictions coming from supersymmetry, besides the relations coming from the Wess–Zumino consistency condition.

In superspace formalism, our external sources, their currents and anomalies are components of superfields. We expect new information from taking into account their multiplet structures.

The presence of sources for the energy-momentum tensor and R-current take us into the realm of external field supergravity. The difference to the approach to supergravity as the gauge theory of global supersymmetry is that the metric does not propagate, but is considered as a coherent state of gravitons. The symmetries under consideration lead us to couple the gauge theory to a classical $\mathcal{N} = 1$ conformal supergravity background,

$$S = S_{SUGRA} + S_{YM} \,. \tag{5.29}$$

S_{YM} is the action of $\mathcal{N} = 1$ $SU(N_C)$ Yang–Mills theory with N_f chiral and anti-chiral matter superfields. The conformal supergravity action can be interpreted as the volume integral of the curved superspace.

In order to make S_{SUGRA} invariant under superconformal transformations, auxiliary compensating fields are introduced. These are the vector superfield H_μ and the chiral and antichiral compensating superfields ϕ and $\bar{\phi}$. A discussion of the prepotentials can be found in [62, 63].

We assume the existence of a supergravity partition function Γ. The derivative of the generating functional with respect to the prepotentials gives the supercurrent and the supertrace,

$$J_\mu = \frac{\delta\Gamma}{\delta H^\mu}\,, \quad J = \phi\frac{\delta\Gamma}{\delta\phi}\,, \quad \bar{J} = \bar{\phi}\frac{\delta\Gamma}{\delta\bar{\phi}}\,. \tag{5.30}$$

Superconformal currents and their Ward identities are discussed by Erdmenger and Rupp [63]. Now I explain how we expect supersymmetry to give restrictions on the coefficients of our anomalies. The starting point is the conservation equation of superconformal symmetry,

$$\bar{\mathscr{D}}_{\dot{\alpha}}\, J^{\alpha\dot{\alpha}} \; - \mathscr{D}^\alpha J = 0\,. \tag{5.31}$$

This equation determines whether superconformal symmetry is broken or not, just like conformal symmetry is broken by contributions to the energy-momentum tensor trace. The superconformal local anomalies, their local renormalization group equation as well as the consistency conditions are given in the dissertation of Johannes Große [43]. Curved superspace geometry is covered in [62], where also a derivation of the supercovariant derivatives can be found.

Here we express the vector prepotential and the supercurrent in component language, which allows for direct comparison with the explicit calculation of the Wess–Zumino consistency conditions. We follow the approach presented in [63], which is based on a metric expansion around rigid superspace. Our vacuum energy functional is then given by the linearized functional

$$W = \tfrac{1}{8}\int \mathrm{d}^8 z\; H^a J_a + \int \mathrm{d}^6 z\; J\phi + \int \mathrm{d}^6 \bar{z}\; \bar{J}\bar{\phi}\,. \tag{5.32}$$

We are interested in the vector prepotential, the supercurrent and the supertrace, because they contain all the sources, currents and anomalies in their component expansions. The vector prepotential multiplet consists of the graviphoton, the gravitinos, and the graviton [63],

$$H^{\mu}\,(z) = \theta^2\, V^{\mu}\bar{\theta}^2 \; - \mathtt{i}\theta^{\alpha}\; \psi^{\mu}{}_{\alpha}\,\bar{\theta}^2 - \mathtt{i}\theta^2\,\bar{\psi}^{\mu}{}_{\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}} + e_a{}^{\mu}\theta^{\alpha}\,\sigma^a{}_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}\;, \tag{5.33}$$

where the sigma matrices convert between Weyl spinor indices α, $\dot{\alpha}$ and tangent space vector indices a, b, c, d. We have rotated the metric with the tetrads $\delta_{ab} = e_a{}^{\mu} e_b{}^{\nu} g_{\mu\nu}$.

We write down the local anomalies of the currents sourced by the graviphoton and the graviton in the next chapter. We must also take into account that the gravitinos are the source for the supersymmetry currents,

$$\langle[Q^{a\alpha}\;(x)]\rangle = e^{-1}\frac{\delta W}{\delta\psi_{a\alpha}\;(x)}\,, \tag{5.34a}$$

$$\langle[\bar{Q}^{a\dot{\alpha}}\;(x)]\rangle = e^{-1}\frac{\delta W}{\delta\bar{\psi}_{a\dot{\alpha}}\;(x)}\,. \tag{5.34b}$$

I discuss in section 6.3 how the gravitinos and their supersymmetry currents may be implemented in our formalism.

The supercurrent is an axial vector superfield. Its components are related to our external sources as

$$J_b(z) = R_b(x) + \theta^\alpha \chi_{b\alpha}(x) - \bar{\chi}_{b\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + v_{ab}(x)\theta^\alpha \sigma^a{}_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}} \,, \tag{5.35a}$$

$$T_{ab} = \tfrac{1}{8}\left(v_{ab} + v_{ba} - 2\delta_{ab}\delta_{cd}v^{cd}\right) \,, \tag{5.35b}$$

$$Q^{a\alpha} = \mathtt{i}\left(\chi^{a\alpha} - \sigma^a{}_{\alpha\dot{\alpha}}\,\sigma^{b\beta\dot{\alpha}}\,\chi_{b\beta}\right) . \tag{5.35c}$$

The supertrace J is a chiral superfield obtained by functional derivative with respect to the chiral compensator. We expect that the trace anomalies and the R-anomalies are components of the supertrace. The energy-momentum tensor and R-current couple to components of both prepotentials H^a and ϕ [63], therefore they appear as components of supercurrent and supertrace.

The conservation equation (5.31) determines the superconformal anomalies. In rigid superspace, the corresponding supersymmetry conservation equation is[1]

$$\bar{D}_{\dot{\alpha}}\, J^{\alpha\dot{\alpha}} - D^\alpha J = 0 \,. \tag{5.36}$$

Since the existence of two fixed points in $\mathcal{N}=1$ supersymmetric gauge theories relies on supersymmetry, we impose that this conservation equation is free of anomalies. This requirement is less restrictive than imposing local superconformal symmetry. Evaluated in components, we expect it to give constraints on the anomaly coefficients.

In the next chapter we evaluate in components the Wess–Zumino consistency conditions for Weyl symmetry and R-symmetry. Thereby we make progress towards implementing the following constraints.

[1] In rigid superspace, the holonomic coordinate basis $\partial_M(z)$ of tangent space is replaced by the anholonomic basis of covariant derivatives, $D_A = E_A{}^M \partial_M$. A review of rigid superspace geometry can be found in the lecture notes of Gieres [64].

a) We impose in components the mutual consistency of conformal symmetry and R-symmetry. This is reflected in the Wess–Zumino consistency conditions [22]

$$\left[\Delta'_{\rho}\,,\Delta_{\rho}\right] W = 0\,, \tag{5.37a}$$

$$\left[\Delta'_{\sigma}\,,\Delta_{\sigma}\right] W = 0\,. \tag{5.37b}$$

b) We treat the Weyl transformations of the background metric and lambda couplings as being independent from the rotations of the fermionic superspace coordinates. This is imposed via the consistency condition

$$\left[\Delta_{\sigma}\,,\Delta_{\rho}\right] W = 0\,. \tag{5.38}$$

c) We impose that rigid supersymmetry is conserved. In the linearized formalism we expect to be able to calculate explicitly how the coefficients are constrained. The conservation equation (5.36) has to be fulfilled, in particular the supersymmetry currents should have a vanishing derivative.

Chapter 6

Consistency Conditions

This chapter reports the results of the evaluation of the Wess–Zumino consistency condition for local Weyl anomalies and local R-anomalies of $\mathcal{N} = 1$ supersymmetric gauge theories.

The main results of my work can be found in sections 6.1 and 6.2. I write down the local Ward identity for R-Symmetry and the conformal local renormalization group equation. I evaluate their Wess–Zumino consistency condition and discuss them.

In section 6.3 I propose how the supersymmetry constraints stated at the end of the last chapter could be implemented explicitly. I present a local Ward identity and consistency condition for a generic tensor structure that may be appropriate to the supersymmetry current. These recent results are under active investigation and therefore subject to change.

Section 6.4 is an outline of Osborn's recipe [21] leading to a central function. I apply it to my special solution of the conformal consistency conditions.

6.1 R-Symmetry Consistency Conditions

In this section we write down the local Ward identity and the consistency conditions for R-symmetry. As described in the last chapter, we promote the complex gauge coupling of $\mathcal{N} = 1$ supersymmetric gauge theories,

$$\tau = \frac{\theta}{2\pi} + \mathtt{i}\frac{4\pi}{g^2} \,, \tag{6.1}$$

to a set of bosonic external sources,

$$\tau^i\left(x, M\right) = \hat{\theta}^i\left(x\right) + \mathtt{i}\lambda^i\left(x, M\right) \,. \tag{6.2}$$

The renormalized lambda coupling is a local version of the Yang–Mills coupling. The introduction of the local instanton angle $\hat{\theta}^i(x)$ is inspired by Osborn [58].

We write down a basis of eight independent tensor monomials for the local Ward identity of R-symmetry,

$$\Delta_\rho W = \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\mathcal{E} + \partial_\mu\rho\mathcal{J}^\mu\right) \,, \tag{6.3a}$$

$$\begin{aligned} \mathcal{E} = {} & \hat{\beta}^{\mathscr{R}} R^\mu{}_{\nu\alpha\beta}\;{}^*R_\mu{}^{\nu\alpha\beta} + \tfrac{1}{2}\hat{\beta}^{\mathscr{V}} V_{\mu\nu}\,{}^*V^{\mu\nu} \\ & + \left(a_i\hat{a}_j\partial_\mu\lambda^i\partial_\nu\hat{\theta}^j + \hat{b}_i\partial_\mu\hat{\theta}^i V_\nu\right){}^*V^{\mu\nu} + \hat{c}_i\partial_\mu\hat{\theta}^i\nabla_\nu\,{}^*V^{\mu\nu} \,, \end{aligned} \tag{6.3b}$$

$$\mathcal{J}^\mu = \left(\hat{d}_i\partial_\nu\hat{\theta}^i + \hat{f}V_\nu\right){}^*V^{\mu\nu} + \hat{e}\nabla_\nu\,{}^*V^{\mu\nu} \,. \tag{6.3c}$$

The local anomaly terms are tensor monomials built up from the external sources $g^{\mu\nu}$, λ^i, $\hat{\theta}^i$, V_μ, the spacetime Riemann tensor and the field strength of the R-vector. We use the covariant derivative ∇_μ associated to the metric $g^{\mu\nu}$. Except for $a_i(\lambda)$, the anomaly coefficients are labeled by a hat to indicate the dependence on the theta coupling.

The form of the local anomalies is restricted by locality, dimensional analysis and gauge invariance. The R-symmetry anomalies have the same spacetime parity as the Adler–Bell–Jackiw anomaly, therefore either the Hodge dual of the field strength tensor ${}^{*}V^{\mu\nu}$ defined in (B.31) or the dual Riemann tensor has to be present in each anomaly. In accord with Osborn [58], we apply the empirical "parity rule" that an odd number of theta couplings have to appear in an anomaly term. Due to this rule, the coefficient $a_i(\lambda)$ appears in the mixed local anomaly.

In [52], the coefficients of the Riemann tensor monomial and the external gauge anomaly were related to the Euler and Weyl coefficients of the gravitational anomalies,

$$\hat{\beta}^{\mathscr{R}} = \frac{2}{3}\left(\beta^{\mathscr{A}} + \beta^{\mathscr{B}}\right) , \tag{6.4a}$$

$$\hat{\beta}^{\mathscr{V}} = -\left(\frac{40}{3}\beta^{\mathscr{B}} - 8\beta^{\mathscr{A}}\right) . \tag{6.4b}$$

Of these four coefficients, only the Euler central charge is shown to satisfy $\beta^{\mathscr{B}}_{UV} > \beta^{\mathscr{B}}_{IR}$. The methods described in this chapter are a tool to construct slope equations of the form

$$\frac{\delta\hat{\beta}^{\mathscr{V}}}{\delta\hat{\theta}^{k}} \sim \text{ (anomaly coefficients times beta functions) } . \tag{6.5}$$

The Wess–Zumino consistency condition [22] states that two consecutive symmetry transformations have to commute,

$$\left[\Delta'_{\rho}, \Delta_{\rho}\right] W = 0 \,. \tag{6.6}$$

The commutator bracket refers to the exchange of the variation parameter. We evaluate the Wess–Zumino consistency condition by applying the local variation operator introduced in section 5.2,

$$\Delta'_{\rho} = \int \mathrm{d}^4 y \left(\partial_\mu \rho'(y) \frac{\delta}{\delta V_\mu(y)} - \rho'(y)\, \hat{\beta}^i(y) \frac{\delta}{\delta \hat{\theta}^i(y)} \right) , \tag{6.7}$$

to the Ward identity (6.3a). The variation at different positions gives a delta function $\delta^{(4)}(x-y)$ that is used to carry out the integration over y. This operator is equivalent to the infinitesimal transformations

$$\delta_\rho V_\mu = \partial_\mu \rho\,, \quad \delta_\rho g_{\mu\nu} = 0\,, \quad \delta_\rho \hat{\theta}^i = \rho \hat{\beta}^i\,, \quad \delta_\rho \lambda^i = 0\,. \tag{6.8}$$

The transformation law for the for the theta coupling reflects the fact that the rotation of fermionic coordinates is related to a shift in the instanton angle.

Since the second variation is to be commuted, we can make use of the symmetry properties of the resulting tensor monomials. Most importantly we use the vanishing of those combinations of rho derivatives that are symmetric with respect to their spacetime indices, so in the following variations

$\cong$ means "equal up to terms with vanishing rho commutator."

We evaluate the action of the variation operator on the local anomaly terms. The field strength and its Hodge dual are gauge invariant,

$$\delta_\rho V_{\mu\nu} = \delta_\rho\, {}^*V^{\mu\nu} = 0\,, \tag{6.9}$$

therefore we have

$$\Delta'_{\rho} \int \sqrt{g}\, \mathrm{d}^4 x \left(-\rho \tfrac{1}{2} \hat{\beta}^{\mathscr{V}} V_{\mu\nu}\, {}^*V^{\mu\nu} \right) = \int \sqrt{g}\, \mathrm{d}^4 x\, \rho\rho' \tfrac{1}{2} \frac{\delta \hat{\beta}^{\mathscr{V}}}{\delta \hat{\theta}^k} \hat{\beta}^k V_{\mu\nu}\, {}^*V^{\mu\nu} \cong 0\,, \tag{6.10}$$

and we can not obtain a slope equation like (6.5) from the R-symmetry consistency condition. If the future evaluation of the mixed consistency conditions gives a negative result as well, the coefficient $\hat{\beta}^{\mathscr{V}}$ can not be established as a central function with our methods. The same is true for the coefficient of the Riemann tensor monomial, since the variation is

$$\Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho \hat{\beta}^{\mathscr{R}} R^\mu{}_{\nu\alpha\beta} \; {}^*R_\mu{}^{\nu\alpha\beta}\right) \cong 0 \,. \tag{6.11}$$

Nevertheless it is important to check the consistency of the local Ward identity. The remaining variations in $\mathcal{E}$ are

$$\Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho a_i \hat{a}_j \partial_\mu \lambda^i \partial_\nu \hat{\theta}^j \, {}^*V^{\mu\nu}\right) \cong \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho \partial_\mu \rho'\right) a_i \hat{a}_j \partial_\nu \lambda^i \hat{\beta}^j \, {}^*V^{\mu\nu} \,, \tag{6.12}$$

$$\Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho \hat{b}_i \partial_\mu \hat{\theta}^i V_\nu \, {}^*V^{\mu\nu}\right) \cong \int \sqrt{g}\, \mathrm{d}^4x \, \left(\rho \partial_\mu \rho'\right) \hat{b}_i \left(\partial_\nu \hat{\theta}^i + \hat{\beta}^i V_\nu\right) {}^*V^{\mu\nu} \,, \tag{6.13}$$

$$\Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho \hat{c}_i \partial_\mu \hat{\theta}^i \nabla_\nu \, {}^*V^{\mu\nu}\right) \cong \int \sqrt{g}\, \mathrm{d}^4x \, \left(\rho \partial_\mu \rho'\right) \hat{c}_i \hat{\beta}^i \nabla_\nu \, {}^*V^{\mu\nu} \,, \tag{6.14}$$

and for the current $\mathcal{J}^\mu$ we have the variations

$$\begin{aligned} \Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4x \, \partial_\mu \rho \hat{d}_i \partial_\nu \hat{\theta}^i \, {}^*V^{\mu\nu} \cong & \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho' \partial_\mu \rho\right) \mathcal{L}_{\hat{\beta}} \, \hat{d}_i \partial_\nu \hat{\theta}^i \, {}^*V^{\mu\nu} \\ & + \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\partial_\nu \rho' \partial_\mu \rho\right) \hat{d}_i \hat{\beta}^i \, {}^*V^{\mu\nu} \,, \end{aligned} \tag{6.15a}$$

$$\Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4x \, \partial_\mu \rho \hat{e} \nabla_\nu \, {}^*V^{\mu\nu} \cong \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho' \partial_\mu \rho\right) \mathcal{L}_{\hat{\beta}} \, \hat{e} \nabla_\nu \, {}^*V^{\mu\nu} \,, \tag{6.15b}$$

$$\begin{aligned} \Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4x \, \partial_\mu \rho \hat{f} V_\nu \, {}^*V^{\mu\nu} \cong & \int \sqrt{g}\, \mathrm{d}^4x \, \left(\partial_\mu \rho \partial_\nu \rho'\right) \hat{f} \, {}^*V^{\mu\nu} \\ & + \int \sqrt{g}\, \mathrm{d}^4x \, \left(-\rho' \partial_\mu \rho\right) \mathcal{L}_{\hat{\beta}} \, \hat{f} \, V_\nu \, {}^*V^{\mu\nu} \,. \end{aligned} \tag{6.15c}$$

The variation $\delta_\rho \hat{\chi}_{ij}(\hat{\theta})$ of a generic anomaly coefficient gives the Lie derivative along the vector field defined by the beta functions,

$$\mathcal{L}_{\hat{\beta}}\, \hat{\chi}_{ij} = \hat{\beta}^k \frac{\delta \hat{\chi}_{ij}}{\delta \hat{\theta}^k} + \hat{\chi}_{kj} \frac{\delta \hat{\beta}^k}{\delta \hat{\theta}^i} + \hat{\chi}_{ik} \frac{\delta \hat{\beta}^k}{\delta \hat{\theta}^j} \,. \tag{6.16}$$

We assume that the volume integral over a total derivative is zero, so the variations are not unique but can be integrated partially. Without doing so, we group the variations according to the rho commutator they are proportional to,

$$\begin{aligned}\left[\Delta'_\rho\,, \Delta_\rho\right] W = &\int\!\sqrt{g}\, \mathrm{d}^4x\ [\rho, \partial_\mu \rho'] \Big(-a_i \hat{a}_j \partial_\nu \lambda^i \hat{\beta}^j\, {}^*V^{\mu\nu} + \Big(\hat{b}_i + \mathcal{L}_{\hat{\beta}}\, \hat{d}_i\Big)\, \partial_\nu \hat{\theta}^i\, {}^*V^{\mu\nu} \\ &\quad + \Big(\hat{b}_i \hat{\beta}^i + \mathcal{L}_{\hat{\beta}}\, \hat{f}\Big)\, V_\nu\, {}^*V^{\mu\nu} + \Big(\hat{c}_i \hat{\beta}^i + \mathcal{L}_{\hat{\beta}}\, \hat{e}\Big)\, \nabla_\nu\, {}^*V^{\mu\nu} \Big) \\ &+ \int\!\sqrt{g}\, \mathrm{d}^4x\ [\partial_\nu \rho, \partial_\mu \rho'] \Big(\hat{d}_i \hat{\beta}^i - \hat{f} \Big)\ {}^*V^{\mu\nu} \\ = &\ 0\,. \end{aligned} \tag{6.17}$$

For a fixed partial integration, the Wess–Zumino consistency condition amounts to solving a linear algebraic equation in the vector space spanned by the rho commutators. Since this equation has to hold for every value of the parameters, the commutators are linearly independent, so each sum has to vanish separately. The variables of the resulting two consistency equations are tensor monomials. We factorize out the anomaly coefficients, and the Wess–Zumino consistency condition is solved by the coefficient relations

$$a_i \partial_\nu \lambda^i \hat{a}_j \hat{\beta}^j = 0\,, \tag{6.18a}$$

$$\hat{b}_i = -\mathcal{L}_{\hat{\beta}}\, \hat{d}_i\,, \tag{6.18b}$$

$$\hat{b}_i \hat{\beta}^i = -\mathcal{L}_{\hat{\beta}}\, \hat{f}\,, \tag{6.18c}$$

$$\hat{c}_i \hat{\beta}^i = -\mathcal{L}_{\hat{\beta}}\, \hat{e}\,, \tag{6.18d}$$

$$\hat{d}_i \hat{\beta}^i = \hat{f}\,. \tag{6.18e}$$

The vanishing of the first monomial demands that we set either $\hat{a}_j$ or a_i to zero. The relations (6.18) then determine an algebraically consistent set of local anomalies for $U(1)$ R-symmetry. By putting them back we obtain the consistent local Ward identity

$$\Delta_\rho W = \int\sqrt{g}\,\mathrm{d}^4x\,\Big(-\rho\Big(\hat{\beta}^{\mathscr{R}} R^\mu{}_{\nu\alpha\beta}\;{}^*R_\mu{}^{\nu\alpha\beta} + \tfrac{1}{2}\hat{\beta}^{\mathscr{V}} V_{\mu\nu}\,{}^*V^{\mu\nu} - \mathcal{L}_{\hat{\beta}}\,\hat{d}_i\partial_\mu\hat{\theta}^i V_\nu\,{}^*V^{\mu\nu} \\ + \hat{c}_i\partial_\mu\hat{\theta}^i\nabla_\nu\,{}^*V^{\mu\nu}\Big) + \partial_\mu\rho\Big(\hat{d}_i\partial_\nu\hat{\theta}^i + \hat{d}_i\hat{\beta}^i V_\nu + \hat{e}\nabla_\nu\Big)\,{}^*V^{\mu\nu}\Big)\,. \tag{6.19}$$

The result for the consistent Ward identity depends on the precise form of the beta functions. If they depended continuously on the local theta couplings, they could be interpreted as vector fields in the space of couplings. It is hard to interpret the beta functions in this way, since the instanton angle is not renormalized. We expect that the mixed consistency conditions give us further information.

6.2 Weyl Consistency Conditions

In this section I write down explicitly the renormalization group equation and the consistency condition for the local Weyl anomalies. Then I write down one particular solution of the consistency condition and discuss its consequences.

Osborn [21] has written down a canonical basis of 16 independent spacetime tensor monomials for the conformal Ward identity (5.2). These local anomaly terms are built up from the external sources $g^{\mu\nu}$, λ^i, and spacetime curvature tensors, in accord with dimensional analysis. So they consist of up to four spacetime derivatives of the dimensionless local couplings.

These anomalies are our starting point. Our additional sources V_μ and $\hat{\theta}^i$ make it possible to construct new independent monomials. We build up new anomaly terms successively by replacing the spacetime derivative of lambda couplings by the vector field V_μ of mass one or its field strength $V_{\mu\nu}$.

In this way we obtain a local Ward identity consisting of 68 independent tensor monomials. We group them in twelve sums denoted with calligraphic letters, so we write down the conformal local renormalization group equation as

$$\begin{aligned}\Delta_\sigma W = &\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\mathcal{A}+\hat{\mathcal{A}}+\mathcal{B}+\hat{\mathcal{B}}+\mathcal{C}+\hat{\mathcal{C}}+\mathcal{D}+\hat{\mathcal{D}}\right)\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\left(\mathcal{Z}^\mu+\hat{\mathcal{Z}}^\mu+\mathcal{Y}^\mu+\hat{\mathcal{Y}}^\mu\right).\end{aligned} \tag{6.20}$$

The gravitational anomalies from the operator equation (5.6) are summarized in

$$\mathcal{A} = \beta^{\mathscr{A}}F + \beta^{\mathscr{B}}G + \tfrac{1}{9}\beta^{\mathscr{C}}R^2 + \tfrac{1}{12}\beta^{\mathscr{D}}\Box R\,. \tag{6.21}$$

The notation for the coefficient functionals of the gravitational anomalies follows [21], where the formalism of local couplings is applied to string theory sigma models. All the gravitational coefficients are functionals of the lambda couplings. I included the exact gravitational anomaly $\Box R$, where the box denotes the second derivative operator, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. The coefficient functional $\beta^{\mathscr{D}}(\lambda)$ is used in section 6.4 to demonstrate the argument leading to the necessary equations for a central function.

The local anomalies in $\mathcal{B}$ and the current $\mathcal{Z}^\mu$ are taken from Osborn [21],

$$\begin{aligned}\mathcal{B} = &\tfrac{1}{2}\chi^a_{ij}\Box\lambda^i\Box\lambda^j + \tfrac{1}{2}\chi^b_{ijk}\Lambda^{ij}\Box\lambda^k + \tfrac{1}{4}\chi^c_{ijkl}\Lambda^{ij}\Lambda^{kl}\\ &+\tfrac{1}{3}\chi^e_i\partial_\mu\lambda^i\partial^\mu R + \tfrac{1}{6}\chi^f_{ij}\Lambda^{ij}R + \tfrac{1}{2}\chi^g_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j G^{\mu\nu}\,,\end{aligned} \tag{6.22a}$$

$$\begin{aligned}\mathcal{Z}^\mu = &w_i\partial_\nu\lambda^i G^{\mu\nu} + \tfrac{1}{3}\partial^\mu\left(d\,R\right) + \partial^\mu\left(U_i\Box\lambda^i\right) + \tfrac{1}{2}\partial^\mu\left(V_{ij}\Lambda^{ij}\right)\\ &+\tfrac{1}{3}Y_i\partial^\mu\lambda^i R + S_{ij}\partial^\mu\lambda^i\Box\lambda^j + \tfrac{1}{2}T_{ijk}\Lambda^{ij}\partial^\mu\lambda^k\,.\end{aligned} \tag{6.22b}$$

We use the abbreviation $\Lambda^{ij} = g^{\mu\nu}\partial_\mu\lambda^i\partial_\nu\lambda^j$. Within $\mathcal{Z}^\mu$, the symbol V_{ij} denotes the anomaly coefficient, not the field strength. Most of the anomaly coefficients are tensors in the space of couplings. The exceptions are χ^b_{ijk}, χ^c_{ijkl} V_{ij}, T_{ijk}, κ^d_{ijk}, which do not fulfill the tensor transformation law. The spacetime symmetry of a monomial determines the symmetry of the associated coefficient matrix in the space of couplings.

Now we write down our new monomials including the external sources V^μ and $\hat{\theta}^i$. We begin with the terms of the same tensor structure as in $\mathcal{B}$ and $\mathcal{Z}^\mu$. We replace every pair of lambda couplings by two theta couplings,

$$\begin{aligned}\hat{\mathcal{B}} &= \tfrac{1}{2}\hat{\chi}^a_{ij}\Box\hat{\theta}^i\Box\hat{\theta}^j + \tfrac{1}{2}\hat{\chi}^b_{ij}\omega^b_k\hat{\Lambda}^{ij}\Box\lambda^k + \tfrac{1}{2}\omega^d_i\hat{\chi}^d_{jk}\partial_\mu\lambda^i\partial^\mu\hat{\theta}^j\Box\hat{\theta}^k \\ &\quad + \tfrac{1}{4}\chi^h_{ij}\hat{\chi}^h_{kl}\Lambda^{ij}\hat{\Lambda}^{kl} + \tfrac{1}{4}\hat{\chi}^c_{ijkl}\hat{\Lambda}^{ij}\hat{\Lambda}^{kl} + \tfrac{1}{6}\hat{\chi}^f_{ij}\hat{\Lambda}^{ij}R + \tfrac{1}{2}\hat{\chi}^g_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j G^{\mu\nu}\,,\end{aligned} \tag{6.23a}$$

$$\hat{\mathcal{Z}}^\mu = \tfrac{1}{2}\partial^\mu\left(\hat{V}_{ij}\hat{\Lambda}^{ij}\right) + \hat{S}_{ij}\partial^\mu\hat{\theta}^i\Box\hat{\theta}^j + \tfrac{1}{2}\hat{T}_{ij}T_k\hat{\Lambda}^{ij}\partial^\mu\lambda^k + \tfrac{1}{2}t_i\hat{t}_{jk}\partial_\nu\lambda^i\partial^\nu\hat{\theta}^j\partial^\mu\hat{\theta}^k\,. \tag{6.23b}$$

We assume that the coefficients of anomalies consisting of both types of couplings can depend on one type of coupling only. We split up the regarding tensors accordingly. The dependence on the type of coupling is indicated by a hat, for instance $\hat{\chi}^b_{ij}(\hat{\theta})$ and $\omega^b_k(\lambda)$. We use the abbreviation $\hat{\Lambda}^{ij} = g^{\mu\nu}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j$.

Now we write down all the monomials including the R-vector. The sums $\mathcal{C}$ and $\hat{\mathcal{C}}$ contain the tensor monomials of vanishing Weyl weight, in $\mathcal{D}$ and $\hat{\mathcal{D}}$ we write down the terms including either curvature tensors or the second derivative operator. Using the abbreviations $(v\cdot w) = g^{\mu\nu}v_\mu w_\nu$ and $v^2 = v\cdot v$ we have

$$\begin{aligned}\mathcal{C} &= \tfrac{1}{4}\beta^{\mathscr{V}}V^{\mu\nu}V_{\mu\nu} + \tfrac{1}{4}v^a V^4 + \tfrac{1}{2}\kappa^a_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j V^{\mu\nu} + \tfrac{1}{3}\kappa^b_i\partial_\mu\lambda^i\nabla_\nu V^{\mu\nu} \\ &\quad + \tfrac{1}{3}\kappa^c_i\partial_\mu\lambda^i V_\nu V^{\mu\nu} + \tfrac{1}{2}\kappa^d_{ijk}\Lambda^{ij}\left(\partial\lambda^k\cdot V\right) + \tfrac{1}{2}\kappa^e_{ij}\Lambda^{ij}V^2 \\ &\quad + \tfrac{1}{2}\kappa^f_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j V^\mu V^\nu + \kappa^g_i\left(\partial\lambda^i\cdot V\right)V^2 + \tfrac{1}{2}\kappa^h_i\left(\partial\lambda^i\cdot\partial\right)V^2\,,\end{aligned} \tag{6.24a}$$

$$\hat{\mathcal{C}} = \tfrac{1}{2}\hat{\kappa}^a_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j V^{\mu\nu} + \tfrac{1}{2}\hat{\zeta}^d_{ij}\zeta^d_k\hat{\Lambda}^{ij}\left(\partial\lambda^k\cdot V\right) + \tfrac{1}{2}\xi^d_i\hat{\xi}^d_{jk}\left(\partial\lambda^i\cdot\partial\hat{\theta}^j\right)\left(\partial\hat{\theta}^k\cdot V\right) + \tfrac{1}{2}\hat{\kappa}^e_{ij}\hat{\Lambda}^{ij}V^2 + \tfrac{1}{2}\hat{\kappa}^f_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j V^\mu V^\nu\,, \tag{6.24b}$$

$$\mathcal{D} = \tfrac{1}{2}\kappa^m_{ij}\left(\partial\lambda^i\cdot V\right)\Box\lambda^j + \tfrac{1}{3}\kappa^n_i\left(\partial\lambda^i\cdot V\right)R + \tfrac{1}{2}v^b\left(\nabla\cdot V\right)V^2 + v^c\left(\nabla\cdot V\right)R + \tfrac{1}{2}\kappa^o_i\Box\lambda^i V^2 + \tfrac{1}{2}v^d\Box V^2 + \tfrac{1}{6}v^e V^2 R + \tfrac{1}{2}\kappa^p_i\Box\lambda^i\left(\nabla\cdot V\right) + \tfrac{1}{2}\kappa^q_i\partial_\mu\lambda^i V_\nu G^{\mu\nu} + \tfrac{1}{2}v^f V_\mu V_\nu G^{\mu\nu}\,, \tag{6.24c}$$

$$\hat{\mathcal{D}} = \hat{\kappa}^m_{ij}\left(\partial\hat{\theta}^i\cdot V\right)\Box\hat{\theta}^j\,, \tag{6.24d}$$

$$\mathcal{Y}^\mu = \kappa^r_i\partial_\nu\lambda^i V^{\mu\nu} + v^g\nabla_\nu V^{\mu\nu} + v^h V_\nu V^{\mu\nu} + v^m V^\mu V^2 + \tfrac{1}{2}\kappa^s_i\partial^\mu\lambda^i V^2 + \kappa^t_i\left(\partial\lambda^i\cdot V\right)V^\mu + \tfrac{1}{2}\kappa^u_i\Box\lambda^i V^\mu + \tfrac{1}{2}\kappa^v_i\partial^\mu\lambda^i\left(\nabla\cdot V\right) + \kappa^w_{ij}\Lambda^{ij}V^\mu + \kappa^y_{ij}\partial^\mu\lambda^i\left(\partial\lambda^j\cdot V\right) + v^n R V^\mu + \tfrac{1}{2}v^o V_\nu G^{\mu\nu}\,, \tag{6.24e}$$

$$\hat{\mathcal{Y}}^\mu = \hat{\kappa}^w_{ij}\hat{\Lambda}^{ij}V^\mu + \hat{\kappa}^y_{ij}\partial^\mu\hat{\theta}^i\left(\partial\hat{\theta}^j\cdot V\right)\,. \tag{6.24f}$$

The first term in $\mathcal{C}$ is the external gauge anomaly of the same structure as the internal anomaly (5.8), determined by gauge invariance, locality, and parity [65]. For $\mathcal{N}=1$ supersymmetric gauge theories the relation $8\beta^{\mathscr{A}} = 3\beta^{\mathscr{V}}$ was found in [52] by nonperturbative calculations.

For this conformal local renormalization group equation we evaluate the Wess–Zumino consistency condition

$$\left[\Delta'_\sigma\,, \Delta_\sigma\right]W = 0 \tag{6.25}$$

by applying the local variation operator introduced in section 5.1,

$$\Delta'_\sigma = \int \mathrm{d}^4 y\,\sigma'(y)\left(2g^{\mu\nu}(y)\frac{\delta}{\delta g^{\mu\nu}(y)} - \beta^i(y)\frac{\delta}{\delta\lambda^i(y)}\right)\,. \tag{6.26}$$

The variation at different positions gives again a delta function $\delta^{(4)}(x-y)$ that is used to carry out the integration over y. This operator is equivalent to the infinitesimal transformations

$$\delta_\sigma \lambda^i = \sigma \beta^i \,, \quad \delta_\sigma \hat{\theta}^i = 0 \,, \quad \delta_\sigma g^{\mu\nu} = +2\sigma g^{\mu\nu} \,, \quad \delta_\sigma V_\mu = 0 \,. \tag{6.27}$$

The variations of all the conformal anomaly terms are written down in appendix A.1. The resulting integral is arranged into a set of commutators of parameter derivatives, denoted $\mathcal{F}_I$, multiplied by sums of tensor monomials, denoted $\mathcal{S}_I$,

$$\left[\Delta'_\sigma \,, \Delta_\sigma\right] W = \int \sqrt{g}\, \mathrm{d}^4 x \, \left(\mathcal{F}_I \cdot \mathcal{S}_I\right) \,. \tag{6.28}$$

These sets depend on the choice of partial integration. For the partial integrations written down in appendix A.1, it is possible to group the variations proportional to four independent commutators,

$$\mathcal{F}_I = \left\{[\sigma', \partial_\mu \sigma] \,, \quad [\sigma', \Box\sigma] \,, \quad [\partial_\mu \sigma', \partial_\nu \sigma] \,, \quad [\partial_\mu \sigma', \Box\sigma]\right\} . \tag{6.29}$$

Now we write down the integrand for the variations listed in section A.1. The commutator $\mathcal{F}_1 = [\sigma', \partial_\mu \sigma]$ multiplies the sum

$$\mathcal{S}_1 =$$

$$\left(-8\frac{\delta \beta^{\mathscr{B}}}{\delta \lambda^i} - \mathcal{L}_\beta\, w_i + \chi^g_{ij} \beta^j\right) \partial_\nu \lambda^i G^{\mu\nu} \tag{6.30a}$$

$$+\tfrac{1}{3}\left(\tfrac{1}{2}\beta^{\mathscr{D}} + \chi^e_i \beta^i\right) \partial^\mu R \tag{6.30b}$$

$$+\left(2\chi^a_{ij} + \chi^g_{ij} - \mathcal{L}_\beta\, S_{ij} + \chi^b_{kij}\beta^k\right) \partial^\mu \lambda^i \Box \lambda^j \tag{6.30c}$$

$$+\tfrac{1}{3}\left(\chi^f_{ij}\beta^j - 2\chi^e_i - \mathcal{L}_\beta\, Y_i\right) \partial^\mu \lambda^i R \tag{6.30d}$$

$$+\left(\chi^c_{ijkl}\beta^k + \chi^b_{ijl} - \tfrac{1}{2}\mathcal{L}_\beta\, T_{ijl}\right) \Lambda^{ij} \partial^\mu \lambda^l \tag{6.30e}$$

$$+\nabla_\nu \left(\chi^g_{ij} \partial^\mu \lambda^i + \kappa^q_j V^\mu\right) \partial^\nu \lambda^j \tag{6.30f}$$

$$+\nabla_\nu \left(v^f V^\mu V^\nu\right) - 8\beta^{\mathscr{B}} \nabla_\nu G^{\mu\nu} \tag{6.30g}$$

(terms including R-vectors)

$$+\left(\kappa^{e}_{ij}\beta^{j}+\kappa^{o}_{i}-\tfrac{1}{2}\mathcal{L}_{\beta}\,\kappa^{s}_{i}\right)\partial^{\mu}\lambda^{i}V^{2} \tag{6.30h}$$

$$-\left(\mathcal{L}_{\beta}\,\kappa^{r}_{i}+\kappa^{a}_{ji}\beta^{j}\right)\partial_{\nu}\lambda^{i}V^{\mu\nu} \tag{6.30i}$$

$$+\left(\tfrac{1}{3}\kappa^{b}_{i}\beta^{i}-\mathcal{L}_{\beta}\,v^{g}\right)\nabla_{\nu}V^{\mu\nu} \tag{6.30j}$$

$$+\left(\kappa^{c}_{i}\beta^{i}-\mathcal{L}_{\beta}\,v^{h}\right)V_{\nu}V^{\mu\nu} \tag{6.30k}$$

$$+\left(\kappa^{d}_{kji}\beta^{k}+\kappa^{m}_{ij}-\mathcal{L}_{\beta}\,\kappa^{y}_{ji}\right)\partial^{\mu}\lambda^{j}\left(\partial\lambda^{i}\cdot V\right) \tag{6.30l}$$

$$+\left(\tfrac{1}{2}\kappa^{d}_{ijk}\beta^{k}-\mathcal{L}_{\beta}\,\kappa^{w}_{ij}\right)\Lambda^{ij}V^{\mu} \tag{6.30m}$$

$$+\left(\kappa^{f}_{ij}\beta^{i}-\mathcal{L}_{\beta}\,\kappa^{t}_{i}\right)\left(\partial\lambda^{j}\cdot V\right)V^{\mu} \tag{6.30n}$$

$$+\left(\kappa^{p}_{j}+\kappa^{q}_{j}+\tfrac{1}{2}\kappa^{m}_{ij}\beta^{i}-\tfrac{1}{2}\mathcal{L}_{\beta}\,\kappa^{u}_{i}\right)\Box\lambda^{j}V^{\mu} \tag{6.30o}$$

$$+\left(\kappa^{g}_{i}\beta^{i}+v^{b}-\mathcal{L}_{\beta}\,v^{m}\right)V^{2}V^{\mu} \tag{6.30p}$$

$$+\left(\tfrac{1}{3}\kappa^{n}_{i}\beta^{i}+2v^{c}-\mathcal{L}_{\beta}\,v^{n}\right)V^{\mu}R \tag{6.30q}$$

$$+\left(v^{d}+\tfrac{1}{2}\kappa^{h}_{i}\beta^{i}\right)\partial^{\mu}V^{2} \tag{6.30r}$$

$$+\left(\kappa^{p}_{i}-\tfrac{1}{2}\mathcal{L}_{\beta}\,\kappa^{v}_{i}\right)\partial^{\mu}\lambda^{i}\left(\nabla\cdot V\right) \tag{6.30s}$$

$$+\left(\kappa^{q}_{i}\beta^{i}-\mathcal{L}_{\beta}\,v^{o}\right)V_{\nu}G^{\mu\nu} \tag{6.30t}$$

(terms with theta couplings)

$$+\left(2\hat{\chi}^{a}_{ij}+\hat{\chi}^{g}_{ij}+\tfrac{1}{2}\beta^{k}\omega^{d}_{k}\hat{\chi}^{d}_{ij}\right)\partial^{\mu}\hat{\theta}^{i}\Box\hat{\theta}^{j} \tag{6.30u}$$

$$+\left(\omega^{d}_{i}\hat{\chi}^{d}_{lj}-\tfrac{1}{2}\hat{t}_{lj}\mathcal{L}_{\beta}\,t_{i}\right)\partial_{\nu}\lambda^{i}\partial^{\nu}\hat{\theta}^{l}\partial^{\mu}\hat{\theta}^{j} \tag{6.30v}$$

$$+\left(\hat{\chi}^{b}_{ij}\omega^{b}_{k}-\tfrac{1}{2}\hat{T}_{ij}\mathcal{L}_{\beta}\,T_{k}\right)\hat{\Lambda}^{ij}\partial^{\mu}\lambda^{k} \tag{6.30w}$$

$$+\left(\tfrac{1}{2}\xi^{d}_{k}\beta^{k}\hat{\xi}^{d}_{ij}+2\hat{\kappa}^{m}_{ji}\right)\partial^{\mu}\hat{\theta}^{i}\left(\partial\hat{\theta}^{j}\cdot V\right) \tag{6.30x}$$

$$+\left(\tfrac{1}{2}\hat{\zeta}^{d}_{ij}\zeta^{d}_{k}\beta^{k}\partial_{\nu}\hat{\theta}^{i}V^{\mu}+\nabla_{\nu}\left(\hat{\chi}^{g}_{ij}\partial^{\mu}\hat{\theta}^{i}\right)\right)\partial^{\nu}\hat{\theta}^{j}\,. \tag{6.30y}$$

The commutator $\mathcal{F}_2 = [\sigma', \Box\sigma]$ multiplies the sum

$$\mathcal{S}_2 =$$

$$\tfrac{1}{3}\left(-\tfrac{1}{2}\beta^{\mathscr{D}} - 4\beta^{\mathscr{C}} + \mathcal{L}_\beta\, d\right) R \tag{6.31a}$$

$$+\left(\chi^g_{ij} - \chi^f_{ij} + \tfrac{1}{2}\chi^b_{ijk}\beta^k + \tfrac{1}{2}\mathcal{L}_\beta\, V_{ij}\right)\Lambda^{ij} \tag{6.31b}$$

$$-\tfrac{1}{2}\Box\beta^{\mathscr{D}} + 2\nabla_\mu\left(\chi^e_i\partial^\mu\lambda^i\right) \tag{6.31c}$$

$$+\left(\chi^a_{ij}\beta^i + \mathcal{L}_\beta\, U_j\right)\Box\lambda^j \tag{6.31d}$$

$$+\left(\tfrac{1}{2}\kappa^m_{ij}\beta^j - 2\kappa^n_i\right)\left(\partial\lambda^i\cdot V\right) \tag{6.31e}$$

$$+\left(-6v^c + \tfrac{1}{2}\kappa^p_i\beta^i\right)(\nabla\cdot V) \tag{6.31f}$$

$$+\left(\tfrac{1}{2}\kappa^o_i\beta^i - v^d - v^e\right)V^2 \tag{6.31g}$$

$$+\left(v^f V_\mu + \kappa^q_i\partial_\mu\lambda^i\right)V^\mu \tag{6.31h}$$

$$+\left(\hat{\chi}^g_{ij} - \hat{\chi}^f_{ij} + \tfrac{1}{2}\omega^b_k\beta^k\hat{\chi}^b_{ij}\right)\hat{\Lambda}^{ij}\,. \tag{6.31i}$$

The commutator $\mathcal{F}_3 = [\partial_\mu\sigma', \partial_\nu\sigma]$ multiplies the sum

$$\mathcal{S}_3 =$$

$$\kappa^r_i\beta^i V^{\mu\nu} + 2S_{[ij]}\;\partial^\mu\lambda^i\partial^\nu\lambda^j \tag{6.32a}$$

$$+\nabla^\mu\left(v^o V^\nu + 2w_i\partial^\nu\lambda^i\right) \tag{6.32b}$$

$$+\left(\kappa^v_i - \kappa^u_i + \kappa^q_i + \kappa^y_{ij}\beta^j - 2\kappa^w_{ij}\beta^j\right)\partial^\mu\lambda^i V^\nu \tag{6.32c}$$

$$+\left(2\hat{S}_{[ij]}\;+\tfrac{1}{2}t_k\beta^k\hat{t}_{ji}\right)\partial^\mu\hat{\theta}^i\partial^\nu\hat{\theta}^j\,. \tag{6.32d}$$

The commutator $\mathcal{F}_4 = [\partial_\mu\sigma', \Box\sigma]$ multiplies the sum

$$\mathcal{S}_4 =$$

$$2\left(\chi^e_i - Y_i + U_i + \frac{\delta U_l}{\delta\lambda^i}\beta^l + U_l\frac{\delta\beta^l}{\delta\lambda^i}\right)\partial^\mu\lambda^i \tag{6.33a}$$

$$+\left(S_{ij} + V_{(ij)}\;\right)\partial^\mu\lambda^i\beta^j \tag{6.33b}$$

$$+\left(\tfrac{1}{2}\kappa^u_i\beta^i - 6v^n\right)V^\mu\,. \tag{6.33c}$$

The Wess–Zumino consistency condition is fulfilled if the integrand vanishes for any value of the functions $\sigma(x)$ and $\sigma'(x)$. So it can only be solved by demanding that each sum $\mathcal{S}_I$ is zero separately. Within each sum $\mathcal{S}_I$, we have factorized out the independent tensor monomials, and arranged the resulting relations between anomaly coefficients accordingly. We take the relation (6.30c) as an example. The statement of the Wess–Zumino consistency condition is that

$$2\chi^a_{ij} + \chi^g_{ij} - \mathcal{L}_\beta\, S_{ij} + \chi^b_{kij}\beta^k = 0 \tag{6.34}$$

has to be true for all renormalizable four dimensional quantum field theories in the presence of the external sources. The Lie derivative along the beta vector field is

$$\mathcal{L}_\beta\, S_{ij} = \beta^k \frac{\delta S_{ij}}{\delta\lambda^k} + S_{kj}\frac{\delta\beta^k}{\delta\lambda^i} + S_{ik}\frac{\delta\beta^k}{\delta\lambda^j}\,. \tag{6.35}$$

Another important relation is the functional derivative of the Euler coefficient,

$$\frac{\delta\beta^{\mathscr{B}}}{\delta\lambda^i} = \tfrac{1}{8}\left(\chi^g_{ij}\beta^j - \mathcal{L}_\beta\, w_i\right)\,. \tag{6.36}$$

Osborn uses it in the central charge equation (5.4b) to obtain the slope equation (5.4a). We outline in section 6.4 the technique used to obtain central functions.

For the relation $\chi^g_{ij} = -2\chi^a_{ij}$ we need additional restrictions on the coefficients in (6.34). We expect them to come out of the supersymmetry conservation equation $\bar{D}_{\dot\alpha}\, J^{\alpha\dot\alpha} = D^\alpha\, J$, as explained in the next section.

Here I attempt to make the physical meaning of the Wess–Zumino consistency condition more explicit by specializing to a particular quantum field theory. I assume that the anomaly coefficients can be calculated explicitly with perturbative methods, and that this calculation results in the relation $\chi^g_{ij} = -2\chi^a_{ij}$. I write down the associated algebraic solution to the four linear equations $\mathcal{S}_I = 0$ and investigate its algebraic consistency.

The sum (6.30) multiplied by $[\sigma', \partial_\mu\sigma]$ is set to zero by the following relations.

$$\chi^g_{ij}\beta^j = \mathcal{L}_\beta\, w_i\,, \tag{6.37a}$$

$$\tfrac{1}{2}\beta^{\mathscr{D}} = -\chi^e_i\beta^i\,, \tag{6.37b}$$

$$\chi^g_{ij} = -2\chi^a_{ij} \quad \text{and} \quad \mathcal{L}_\beta\, S_{ij} = \chi^b_{kij}\beta^k\,, \tag{6.37c}$$

$$2\chi^e_i = \chi^f_{ij}\beta^j - \mathcal{L}_\beta\, Y_i\,, \tag{6.37d}$$

$$\chi^b_{ijl} = \tfrac{1}{2}\mathcal{L}_\beta\, T_{ijl} - \chi^c_{ijkl}\beta^k\,, \tag{6.37e}$$

$$\chi^g_{ij}\partial^\mu\lambda^i = -\kappa^q_j V^\mu\,, \tag{6.37f}$$

$$8\beta^{\mathscr{B}}G^{\mu\nu} = v^f V^\nu V^\mu\,, \tag{6.37g}$$

(terms including R-vectors)

$$2\kappa^e_{ij}\beta^j = \mathcal{L}_\beta\, \kappa^s_i - 2\kappa^o_i\,, \tag{6.37h}$$

$$-\kappa^a_{ji}\beta^j = \mathcal{L}_\beta\, \kappa^r_i\,, \tag{6.37i}$$

$$\tfrac{1}{3}\kappa^b_i\beta^i = \mathcal{L}_\beta\, v^g\,, \tag{6.37j}$$

$$\kappa^c_i\beta^i = \mathcal{L}_\beta\, v^h\,, \tag{6.37k}$$

$$\kappa^d_{kji}\beta^k = \mathcal{L}_\beta\, \kappa^y_{ji} - \kappa^m_{ij}\,, \tag{6.37l}$$

$$\tfrac{1}{2}\kappa^d_{ijk}\beta^k = \mathcal{L}_\beta\, \kappa^w_{ij}\,, \tag{6.37m}$$

$$\kappa^f_{ij}\beta^i = \mathcal{L}_\beta\, \kappa^t_i\,, \tag{6.37n}$$

$$\kappa^m_{ij}\beta^i = -2\kappa^p_j \quad \text{and} \quad \mathcal{L}_\beta\, \kappa^u_i = 2\kappa^q_j\,, \tag{6.37o}$$

$$\kappa^g_i\beta^i = \mathcal{L}_\beta\, v^m - v^b\,, \tag{6.37p}$$

$$\tfrac{1}{3}\kappa^n_i\beta^i = \mathcal{L}_\beta\, v^n - 2v^c\,, \tag{6.37q}$$

$$\tfrac{1}{2}\kappa^h_i\beta^i = -v^d\,, \tag{6.37r}$$

$$2\kappa^p_i = \mathcal{L}_\beta\, \kappa^v_i\,, \tag{6.37s}$$

$$\kappa^q_i\beta^i = \mathcal{L}_\beta\, v^o\,, \tag{6.37t}$$

(terms with theta couplings)

$$\hat{\chi}^g_{ij} = -2\hat{\chi}^a_{ij} - \tfrac{1}{2}\beta^k\omega^d_k\hat{\chi}^d_{ij}\,, \tag{6.37u}$$

$$\omega^d_i\hat{\chi}^d_{lj} = \tfrac{1}{2}\hat{t}_{lj}\mathcal{L}_\beta\, t_i\,, \tag{6.37v}$$

$$\hat{\chi}^b_{ij}\omega^b_k = \tfrac{1}{2}\hat{T}_{ij}\mathcal{L}_\beta\, T_k\,, \tag{6.37w}$$

$$\tfrac{1}{2}\xi^d_k\beta^k\hat{\xi}^d_{ij} = -2\hat{\kappa}^m_{ji}\,, \tag{6.37x}$$

$$\tfrac{1}{2}\hat{\zeta}^d_{ij}\zeta^d_k\beta^k\partial_\nu\hat{\theta}^i V^\mu = -\nabla_\nu\left(\hat{\chi}^g_{ij}\partial^\mu\hat{\theta}^i\right)\,. \tag{6.37y}$$

The sum (6.31) multiplied by $[\sigma', \Box\sigma]$ is set to zero by the following relations.

$$\beta^{\mathscr{C}} = -\tfrac{1}{8}\beta^{\mathscr{D}} + \tfrac{1}{4}\mathcal{L}_\beta\, d\,, \tag{6.38a}$$

$$\mathcal{L}_\beta\, V_{ij} = -\chi^b_{ijk}\beta^k \quad \text{and} \quad \chi^f_{ij} = \chi^g_{ij}\,, \tag{6.38b}$$

$$\frac{\delta\beta^{\mathscr{D}}}{\delta\lambda^i} = 4\chi^e_i\,, \tag{6.38c}$$

$$\chi^a_{ij}\beta^i = -\mathcal{L}_\beta\, U_j\,, \tag{6.38d}$$

$$\kappa^m_{ij}\beta^j = 4\kappa^n_i\,, \tag{6.38e}$$

$$6v^c = \tfrac{1}{2}\kappa^p_i\beta^i\,, \tag{6.38f}$$

$$v^e = \tfrac{1}{2}\kappa^o_i\beta^i - v^d\,, \tag{6.38g}$$

$$v^f V_\mu = -\kappa^q_i\partial_\mu\lambda^i\,, \tag{6.38h}$$

$$\hat{\chi}^g_{ij} = \hat{\chi}^f_{ij} - \tfrac{1}{2}\omega^b_k\beta^k\hat{\chi}^b_{ij}\,. \tag{6.38i}$$

The sum (6.32) multiplied by $[\partial_\mu\sigma', \partial_\nu\sigma]$ is set to zero by the following relations.

$$\kappa^r_i\beta^i V^{\mu\nu} = -2S_{[ij]}\ \partial^\mu\lambda^i\partial^\nu\lambda^j\,, \tag{6.39a}$$

$$v^o V^\nu = -2w_i\partial^\nu\lambda^i\,, \tag{6.39b}$$

$$0 = \kappa^v_i - \kappa^u_i + \kappa^q_i + \kappa^y_{ij}\beta^j - 2\kappa^w_{ij}\beta^j\,, \tag{6.39c}$$

$$\hat{S}_{[ij]} = -\tfrac{1}{4}t_k\beta^k\hat{t}_{ji}\,. \tag{6.39d}$$

The sum (6.33) multiplied by $[\partial_\mu \sigma', \Box\sigma]$ are set to zero by the following relations.

$$Y_i = U_i \quad \text{and} \quad \chi_i^e = -\frac{\delta U_l}{\delta \lambda^i}\beta^l - U_l \frac{\delta \beta^l}{\delta \lambda^i}\,, \tag{6.40a}$$

$$S_{ij} = -V_{(ij)} \quad , \tag{6.40b}$$

$$\tfrac{1}{2}\kappa_i^u \beta^i = 6v^n\,. \tag{6.40c}$$

This solution allows for writing down the functional derivative of the coefficient $\beta^{\mathscr{D}}$ multiplying the exact anomaly $\Box R$. We use equation (6.37d) as well as the relations $\chi_{ij}^f = \chi_{ij}^g = -2\chi_{ij}^a$ and $U_i = Y_i$ in equation (6.38c) and obtain the functional derivative

$$\frac{\delta \beta^{\mathscr{D}}}{\delta \lambda^i} = -4\chi_{ij}^a \beta^j - 2\mathcal{L}_\beta\, U_i\,. \tag{6.41}$$

In section 6.4 we use this derivative in the function F.

Now I discuss whether this particular solution of the Wess–Zumino consistency condition is algebraically consistent. As a first check we can build up chains of equations by inserting the solutions into each other. For example we combine (6.38c) and (6.37b) to obtain

$$\chi_i^e = -\tfrac{1}{3}\beta^k \frac{\delta \chi_k^e}{\delta \lambda^i}\,, \tag{6.42}$$

which is a restriction imposed on the coefficient χ_i^e. The results of the perturbative calculations are needed to check whether this differential equation possesses a solution.

We have split the integral into four sums $\mathcal{S}_I$. For those coefficients appearing in more than one sum, it is possible that they to "run into a multiple of themselves," therefore producing a wrong statement. The best situation would be that all possible chains starting from a coefficient "run into the coefficient itself" like $\kappa_i^c = \ldots = \kappa_i^c$. This has to be checked for each coefficient. When starting it, one quickly figures that this is a tedious algebraic procedure. So it is advisable to employ a computer algebra system.

The iterative process of partial integration, choosing and checking the solution is entirely deterministic. I plan to use the source code of this document as input for a computer program doing this job. If in the future our solution should turn out to be inconsistent, we can find another solution with the desired properties by partial integration of the variations.

What are the physical properties that characterize a consistent set of anomalies? At the fixed points, only the gravitational anomalies should appear in the Ward identity. I expect that this can be checked by inserting back the solutions to the local renormalization group equation. It should then consist of local anomaly terms with the following properties.

a) A beta function appears in the anomaly term, so that the local anomaly vanishes at the fixed points.

b) A subset of tensor monomials adds up to a total derivative. We can then use Stokes theorem and the physical argument that the integral vanishes at infinity.

c) A subset of the local anomalies has the same form as the conformally covariant Riegert operator. I discuss now briefly the paper of Riegert [50]. He found a functional Γ_R that yields the trace anomaly (5.6) under metric variation and is expandable around flat space. In our context, his discussion is valid at the fixed points. In order to make the functional Γ_R covariant, Riegert introduced the auxiliary field σ as a Lagrange multiplier. He interpreted it as "definer of" the renormalization scale for the expansion around flat metric. The effective action giving rise to the anomalies in (5.6), with exception of R^2, is then given by

$$\Gamma_R\left[g^{\mu\nu},\sigma\right] = \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\left(D_R\sigma\right)+\sigma aF+\sigma b\tfrac{1}{4}\left(G+\tfrac{2}{3}\Box R\right)\right)\,. \qquad (6.43)$$

The numerical values of the coefficients a and b are given in [50]. The first term is the Riegert operator,

$$D_R\sigma = \Box\Box\sigma - 2R^{\mu\nu}\nabla_\mu\nabla_\nu\sigma + \tfrac{2}{3}R\Box\sigma - \tfrac{1}{3}\left(\nabla^\mu R\right)\nabla_\mu\sigma\,. \qquad (6.44)$$

Now we consider the behavior of the various terms under an infinitesimal conformal transformation. The scalar field σ has conformal weight zero. The Riegert operator acting on σ is the unique conformally invariant operator of fourth order in four dimensions, $\delta_\sigma(\sqrt{g}D_R\sigma)=0$. The kernel of the Green's function $\sqrt{g}D_RG(x,y)=\delta(x-y)$ is given by $\mathcal{K}=\frac{1}{4}\sqrt{g}(G+\frac{2}{3}\Box R)$. It yields the Riegert operator acting on the Weyl parameter under variation, $\delta_\sigma\mathcal{K}=\sqrt{g}D_R\sigma$. In the conformal structure (5.12), we have the relations

$$\sqrt{g}D_R\sigma=\sqrt{\bar{g}}\bar{D}_R\sigma\,,\quad \mathcal{K}=\bar{\mathcal{K}}+\sqrt{\bar{g}}\bar{D}_R\sigma\,,\quad \sqrt{g}F=\sqrt{\bar{g}}\bar{F}\,, \qquad (6.45)$$

$$\Gamma_R\left[g\right] = \int\sqrt{g}\,\mathrm{d}^4x\,\left(-D_R\sigma+F+\mathcal{K}\right)\,, \qquad (6.46)$$

$$\Gamma_R\left[\bar{g}\right] = \int\sqrt{\bar{g}}\,\mathrm{d}^4x\,\left(\bar{F}+\bar{\mathcal{K}}\right)\,. \qquad (6.47)$$

The functional Γ_R is constructed just in the right way to yield the gravitational anomalies. It corresponds to a classical field theory in curved space.

The formalism of local couplings allows to study the quantum theory and to investigate the renormalization group flow away from the fixed points. In the conformal local renormalization group equation (5.2), the local anomalies corresponding to the Riegert operator appear with different coefficients,

$$\tfrac{1}{2}\chi^a_{ij}\Box\lambda^i\Box\lambda^j\,,\quad \tfrac{1}{2}\chi^g_{ij}G^{\mu\nu}\nabla_\mu\lambda^i\nabla_\nu\lambda^j\,,\quad \tfrac{1}{6}\chi^f_{ij}Rg^{\mu\nu}\nabla_\mu\lambda^i\nabla_\nu\lambda^j\,,\quad \tfrac{1}{3}\chi^e_j\left(\nabla^\mu R\right)\nabla_\mu\lambda^j\,. \qquad (6.48)$$

At a fixed point $\beta^i = 0$, the Wess–Zumino consistency condition results in the relations $\chi^f_{ij} = \chi^g_{ij} = -2\chi^a_{ij}$ and $\chi^e_j = 0$. It is then interesting to recognize the Riegert operator terms inside the local renormalization group equation.

In summary, we expect the consistent Ward identity to have the form

$$\begin{aligned}\Delta_\sigma W = \int \sqrt{g}\, \mathrm{d}^4x \quad &\text{gravitational anomalies} \quad + \\ &\text{Riegert operator} \quad + \text{ local anomalies of zero Weyl weight} \quad + \\ &\text{local anomalies with beta functions in the coefficient.}\end{aligned} \tag{6.49}$$

These properties should ensure that only the gravitational anomalies appear in the Ward identity at the fixed points.

It is conceivable that the consistency relations discussed here can provide a consistency check for perturbative calculations in quantum field theories coupled to curved background space.

6.3 Implementation of the Supersymmetry Constraints?

In this section I discuss how the constraint of conserved global supersymmetry stated at the end of the last chapter could be realized in our component approach to the supersymmetric anomalies. A generic tensor structure is added to the local Ward identity of R-symmetry. The resulting consistency relations are put back to the Ward identity, and the additional anomalies add up to a total derivative.

We consider an antisymmetric tensor composed of the R-vector and a generic vector,

$$\Psi^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} t_\alpha V_\beta \, . \tag{6.50}$$

We form tensor monomials of the same structure as in the R-symmetry Ward identity, except that the Hodge dual of the field strength is replaced by the generic tensor,

$$\mathcal{E}_\Psi = \mathcal{E} + \left(\tfrac{1}{2}\hat{\xi}^a V_{\mu\nu} + \xi^b_i \hat{\xi}^b_j \partial_\mu \lambda^i \partial_\nu \hat{\theta}^j + \hat{\xi}^c_i \partial_\mu \hat{\theta}^i V_\nu\right) \Psi^{\mu\nu} + \hat{\xi}^d_i \partial_\mu \hat{\theta}^i \nabla_\nu \Psi^{\mu\nu} \,, \tag{6.51a}$$

$$\mathcal{J}^\mu_\Psi = \mathcal{J}^\mu + \left(\hat{\xi}^e_i \partial_\nu \hat{\theta}^i + \hat{\xi}^f V_\nu + \partial_\nu \hat{\xi}^g\right) \Psi^{\mu\nu} \,. \tag{6.51b}$$

We add these monomials to the local Ward identity of R-symmetry (6.3a),

$$\Delta_\rho W = \int \sqrt{g}\, \mathrm{d}^4 x \, \left(-\rho \mathcal{E}_\Psi + \partial_\mu \rho \mathcal{J}^\mu_\Psi\right) \,. \tag{6.52}$$

We calculate the Wess–Zumino consistency condition for this local Ward identity under the assumption that the generic vector is inert under R-symmetry,

$$\delta_\rho \Psi^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} t_\alpha \partial_\beta \rho \,. \tag{6.53}$$

The calculation is done in the same way as for the R-anomalies in appendix A.4. Since the new tensor monomials are linearly independent from the local R-anomalies in $\mathcal{E}$ and $\mathcal{J}^\mu$, the consistency condition amounts to relations between the newly introduced coefficients.

$$\xi^b_i \hat{\xi}^b_j \partial_\mu \lambda^i \partial_\nu \hat{\theta}^j = \nabla_\nu \left(\hat{\xi}^d_i \partial_\mu \hat{\theta}^i\right) \,, \tag{6.54a}$$

$$\hat{\xi}^c_i \partial_\mu \hat{\theta}^i V_\nu = \tfrac{1}{2} \mathcal{L}_{\hat{\beta}} \hat{\xi}^g V_{\mu\nu} - \tfrac{1}{2} \hat{\xi}^a V_{\mu\nu} \,, \tag{6.54b}$$

$$\hat{\xi}^e_i \partial_\nu \hat{\theta}^i = \hat{\xi}^d_i \partial_\nu \hat{\theta}^i - \hat{\xi}^f V_\nu - \partial_\nu \hat{\xi}^g \,, \tag{6.54c}$$

$$\mathcal{L}_{\hat{\beta}} \hat{\xi}^g = \hat{\xi}^f - \hat{\xi}^e_i \hat{\beta}^i \,. \tag{6.54d}$$

We insert these relations into the Ward identity and see that the additional anomalies sum up to a total derivative, except for the first monomial,

$$\begin{aligned}\Delta_\rho W = &\int\sqrt{g}\,\mathrm{d}^4x\,\rho\left(-\mathcal{E}+\tfrac{1}{2}\left(\hat{\xi}^f-\hat{\xi}^e_i\hat{\beta}^i\right)V_{\mu\nu}\Psi^{\mu\nu}\right)\\ &-\int\sqrt{g}\,\mathrm{d}^4x\,\nabla_\nu\left(\rho\hat{\xi}^d_i\partial_\mu\hat{\theta}^i\Psi^{\mu\nu}\right)+\int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\rho\mathcal{J}^\mu\,.\end{aligned} \tag{6.55}$$

We can use Stokes theorem to argue that the divergence vanishes. The coefficient of the remaining anomaly is altered to the the form of a central charge equation,

$$\hat{\xi}^a=\tfrac{1}{2}\left(\hat{\xi}^f-\hat{\xi}^e_i\hat{\beta}^i\right)\,. \tag{6.56}$$

In order to investigate whether the coefficient $\hat{\xi}^f$ is related to a quantity that measures degrees of freedom, we have to specify the generic tensor ansatz.

Gauge invariance under $U(1)$ R-symmetry restricts any bosonic ansatz to the form $\epsilon^{\mu\nu\alpha\beta}\partial_\alpha V_\beta$. The Wess–Zumino consistency condition corresponding to this choice is trivially fulfilled and gives no new coefficient relations.

Alternatively we can search for a set of fermionic local anomalies, that preserves global supersymmetry (5.36), but breaks superconformal symmetry. We have already considered the graviphoton V_μ and the graviton $g_{\mu\nu}$ as external sources. In the supergravity multiplet (5.33) we still have the gravitino available. We make the ansatz for the "gravitino vector,"

$${e_b}^\mu t_\mu={\psi_a}^\alpha{\sigma^a}_{\alpha\dot{\alpha}}\,{\bar{\psi}_b}{}^{\dot{\alpha}}\,, \tag{6.57}$$

so the tensor consists of a bilinear of gravitinos and the graviphoton,

$$\Psi^{\alpha\beta}=\epsilon^{\alpha\beta\mu\nu}\left(\bar{\psi}\psi\right)_\mu V_\nu\,. \tag{6.58}$$

We have already shown that such a tensor structure can be added consistently to the R-symmetry Ward identity. Local anomalies built from this monomial may constitute a Ward identity for the supersymmetry current.

The important task is then to verify whether the Ward identities for R-symmetry and Weyl symmetry are the components of the conservation equations of global and local supersymmetry.

6.4 A Candidate for a Central Function

In this section we outline the recipe of Osborn [21] leading to a function F of the couplings that has the algebraic form (1.2) necessary for a central function.

We need a function that coincides with one of the gravitational anomaly coefficients at the fixed points, and its renormalization group derivative should have definite sign. The formulas indicating these properties can be obtained with the following recipe.

a) We take the functional derivatives of the conformal Ward identity with respect to the metric and the lambda couplings to obtain the $\langle TT \rangle$ correlator expressed in terms of anomaly coefficients. This way we obtain the candidate function F.

b) At the fixed points we obtain the relation

$$\langle T(x)\,T(0)\rangle = \text{ anomaly coefficient } \cdot \text{ derivative of } \delta^{(4)}(x)$$

between the 2-point function of the energy-momentum tensor trace and the anomaly coefficients.

c) We compare this relation to the general form of the 2-point function $\langle TT\rangle$. The aim is to relate the gravitational coefficient to a quantity proportional to physical degrees of freedom. In two dimensions we were able to compare the corresponding relation to the operator product expansion and read off the central charge. In four dimensions, there is no central charge and the operator product expansion is not closed, infinitely many operators appear on the right hand side.

d) The conformal Wess–Zumino consistency condition gives the functional derivative of the coefficient $\beta^{\mathscr{D}}$ with respect to the couplings (6.41). Comparing it to the renormalization group derivative of the candidate function gives a slope equation.

The Ward identity $\Delta_\sigma W$ can be seen as the renormalization group equation for the one point functions of the dimension four operators. By taking the functional derivative of the Ward identity with respect to the external sources we get connected correlation functions of the currents, for instance

$$\begin{aligned}\frac{\delta}{\delta\lambda^j(0)}\Delta_\sigma W = &\int\sqrt{g}\,\mathrm{d}^4x\,\sigma(x)\,\frac{\delta}{\delta\lambda^j(0)}2g^{\mu\nu}(x)\,\frac{\delta W}{\delta g^{\mu\nu}(x)} \\ &-\int\sqrt{g}\,\mathrm{d}^4x\,\sigma(x)\,\frac{\delta}{\delta\lambda^j(0)}\beta^i(x)\,\frac{\delta W}{\delta\lambda^i(x)}\,.\end{aligned} \tag{6.59}$$

We assume here that the vacuum expectation value of the local composite operators vanishes, $\langle[\mathcal{O}^i]\rangle = 0$, so no spontaneous symmetry breaking takes place. The 2-point function of the energy-momentum tensor and the local composite operators are then

$$\frac{\delta}{\delta\lambda^j(0)}\Delta_\sigma W = \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\langle[T(x)][\mathcal{O}_j(0)]\rangle - \beta^i(x)\,\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\rangle\right), \tag{6.60a}$$

$$2\frac{\delta}{\delta g^{\mu\nu}(0)}\Delta_\sigma W = \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\langle T(x)\,T_{\mu\nu}(0)\rangle - \beta^i(x)\,\langle\mathcal{O}_i(x)\,T_{\mu\nu}(0)\rangle\right). \tag{6.60b}$$

Then we have to take the functional derivative of all the anomalies on the right hand side of the Ward identity with respect to the external sources, and restrict to flat space, constant couplings and constant rescalings afterwards.

Most of the anomalies have then a vanishing functional derivative, for a number of reasons. The curvature tensors vanish in flat space, and the derivatives on sigma vanish. Most importantly, all derivatives of couplings vanish, so only the terms where the derivative acts on the single coupling are left over, for instance

$$\begin{aligned}\frac{\delta}{\delta\lambda^j(0)}\int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\partial^\mu\left(U_i\Box\lambda^i\right) &= -\int\sqrt{g}\,\mathrm{d}^4x\,\sigma(x)\,\Box\left(U^i(x)\,\Box\frac{\delta\lambda^i(x)}{\delta\lambda^j(0)}\right)\\ &= -\int\sqrt{g}\,\mathrm{d}^4x\,\sigma(x)\left(\Box U_j(x)\,\Box\delta^{(4)}(x)\right.\\ &\qquad\left.+\,U_j(x)\,\Box\,\Box\,\delta^{(4)}(x)\right)\,. \end{aligned} \tag{6.61}$$

At the fixed points of the renormalization group flow, the restriction to constant parameters reduces the expression to

$$\sigma\int\mathrm{d}^4x\,\left(-U_j\right)\Box\,\Box\,\delta^{(4)}(x)\,. \tag{6.62}$$

We take this expression as an example. The lambda derivative of the Ward identity then gives rise to the relation

$$\langle[T(x)][\mathcal{O}_j(0)]\rangle-\beta^i(x)\,\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\rangle=-U_j\Box\,\Box\,\delta^{(4)}(x)\,. \tag{6.63}$$

Now we have to take the functional derivative with respect to the metric, equation (6.60b). We take as an instance the anomaly

$$2\frac{\delta}{\delta g^{\mu\nu}(0)}\int\sqrt{g}\,\mathrm{d}^4x\,\tfrac{1}{12}\sigma\beta^{\mathscr{D}}\Box R=\tfrac{1}{6}\int\mathrm{d}^4x\,\frac{\delta}{\delta g^{\mu\nu}(0)}\left(\sqrt{g(x)}R(x)\,\Box\left(\sigma(x)\,\beta^{\mathscr{D}}(x)\right)\right)\,. \tag{6.64}$$

We use the generic variation formula

$$\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{g\left(x\right)}R\left(x\right)f\left(x\right)\right)=\sqrt{g\left(x\right)}\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu-G_{\mu\nu}\right)f\left(x\right) \tag{6.65}$$

to obtain

$$2\frac{\delta}{\delta g^{\mu\nu}\left(0\right)}\int\sqrt{g}\,\mathrm{d}^4x\ \tfrac{1}{12}\sigma\beta^{\mathscr{D}}\Box R=\tfrac{1}{6}\int\sqrt{g}\,\mathrm{d}^4x\ \left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu-G_{\mu\nu}\right)\sigma\left(x\right)\beta^{\mathscr{D}}\left(x\right)\Box\delta^{(4)}\left(x\right)\ . \tag{6.66}$$

We restrict to constant sources and rescalings. We use the projection operator $\pi_{\mu\nu}=\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu$. At the fixed points of the renormalization group flow we are left with the expression

$$\tfrac{1}{6}\sigma\beta^{\mathscr{D}}\int\mathrm{d}^4x\ \pi_{\mu\nu}\Box\delta^{(4)}\left(x\right)\ . \tag{6.67}$$

We take the trace with the metric. Using $\pi^\alpha{}_\alpha=3\Box$ we arrive at the relation

$$\left\langle T\left(x\right)T\left(0\right)\right\rangle=\beta^i\left(x\right)\left\langle\mathcal{O}_i\left(x\right)T\left(0\right)\right\rangle-\tfrac{1}{2}\beta^{\mathscr{D}}\Box\Box\,\delta^{(4)}\left(x\right)\ . \tag{6.68}$$

The final step is to replace $\left\langle\mathcal{O}_i\left(x\right)T\left(0\right)\right\rangle$ by (6.63). This gives the 2-point function of the energy-momentum tensor trace,

$$\left\langle T\left(x\right)T\left(0\right)\right\rangle=\beta^j\beta^i\left(x\right)\left\langle[\mathcal{O}_i\left(x\right)][\mathcal{O}_j\left(0\right)]\right\rangle-U_j\beta^j\Box\Box\,\delta^{(4)}\left(x\right)-\tfrac{1}{2}\beta^{\mathscr{D}}\Box\Box\,\delta^{(4)}\left(x\right)\ . \tag{6.69}$$

At the fixed points with vanishing beta functions the correlator is proportional to anomaly coefficients,

$$\left\langle T\left(x\right)T\left(0\right)\right\rangle\sim\left(-\tfrac{1}{2}\beta^{\mathscr{D}}-U_j\beta^j\right)\Box\Box\,\delta^{(4)}\left(x\right)\ . \tag{6.70}$$

Our candidate function coincides with the gravitational coefficient $\beta^{\mathscr{D}}$ of the exact anomaly $\Box R$ at the fixed points,

$$F = -\tfrac{1}{2}\beta^{\mathscr{D}} - U_k\beta^k\,. \tag{6.71}$$

The most important open question is whether the gravitational coefficients can be related to physical degrees of freedom. The most general form of the energy-momentum tensor 2-point function consistent with rotational invariance, regularity and reflection positivity was given by Cardy [18],

$$\begin{aligned}\langle T_{\mu\nu}\,(x)\,T_{\alpha\beta}\,(0)\,\rangle = {} & \frac{\mathrm{A}}{x^{12}}\,(x_\mu x_\nu x_\alpha x_\beta) + \frac{\mathrm{B}}{x^{10}}\,(x_\mu x_\nu \delta_{\alpha\beta} + \delta_{\mu\nu} x_\alpha x_\beta) \\ & + \frac{\mathrm{C}}{x^{10}}\,(x_\mu x_\alpha \delta_{\beta\nu} + x_\nu x_\alpha \delta_{\beta\mu} + x_\mu x_\beta \delta_{\alpha\nu} + x_\nu x_\beta \delta_{\alpha\mu}) \\ & + \frac{\mathrm{D}}{x^8}\delta_{\mu\nu}\delta_{\alpha\beta} + \frac{\mathrm{E}}{x^8}\,(\delta_{\alpha\mu}\delta_{\nu\beta} + \delta_{\alpha\nu}\delta_{\mu\beta})\;.\end{aligned} \tag{6.72}$$

The powers of x in the denominator are proportional to derivatives of delta functions. The energy-momentum 2-point function has to fulfill the conservation condition

$$\partial^\mu \langle T_{\mu\nu}\,(x)\,T_{\alpha\beta}\,(0)\,\rangle = 0\,, \tag{6.73}$$

which holds for any value of x different from zero. This restricts the 2-point function to the form

$$\langle T_{\mu\nu}\,(x)\,T_{\alpha\beta}\,(0)\,\rangle \sim -\Pi_{\mu\nu\alpha\beta}\,\frac{h\,(\lambda^i)}{x^4} + \pi_{\mu\nu}\,\pi_{\alpha\beta}\,\frac{f\,(\lambda^i,t)}{x^4}\,, \tag{6.74a}$$

using the projection operator onto symmetric traceless tensors,

$$\Pi_{\mu\nu\alpha\beta} = \tfrac{1}{3}\pi_{\mu\nu}\,\pi_{\alpha\beta} - \tfrac{1}{2}\pi_{\mu(\alpha}\,\pi_{\beta)\nu}\,. \tag{6.74b}$$

This form defines a primary central charge [52], but the corresponding operator product expansion is not closed, so new secondary central charges appear.

The second property of a central function can be obtained using the Wess–Zumino consistency condition. We take as an example the solution written down in section 6.2, which assumes a specific quantum field theory. The consistency relations yield the lambda derivative of the coefficient,

$$\frac{\delta \beta^{\mathscr{D}}}{\delta \lambda^i} = -4\chi^a_{ij}\beta^j - 2\mathcal{L}_\beta\, U_i\,. \tag{6.75}$$

Any central function is an invariant of the renormalization group flow, $\delta_\sigma C\left(\lambda^i, \mu\right) = 0$. With the variation rules $\delta_\sigma \lambda^i = \sigma\beta^i$ and $\delta_\sigma \mu = \sigma\mu$ we observe

$$\beta^i \frac{\delta}{\delta \lambda^i} C = -M \frac{\partial}{\partial M} C\,. \tag{6.76}$$

By taking the coupling derivative of our central function candidate,

$$\beta^i \frac{\delta F}{\delta \lambda^i} = -\tfrac{1}{2}\beta^i \frac{\delta \beta^{\mathscr{D}}}{\delta \lambda^i} - \beta^i \frac{\delta}{\delta \lambda^i}\left(U_k \beta^k\right)\,, \tag{6.77a}$$

and using the lambda derivative of the coefficient,

$$-M\frac{\partial}{\partial M} F = 2\chi^a_{ij}\beta^i\beta^j + \beta^i\mathcal{L}_\beta\, U_i - \beta^i\frac{\delta}{\delta \lambda^i}\left(U_k\beta^k\right)\,, \tag{6.77b}$$

we obtain the renormalization group flow slope equation

$$M\frac{\partial}{\partial M} F = -2\chi^a_{ij}\beta^i\beta^j \tag{6.78}$$

for this exemplary central function candidate. With this recipe due to Osborn [21] it is in principle possible to obtain functions of the desired algebraic form.

For our work it is important to extend this analysis to the 2-point functions of the R-current. The main open question is the value of the gravitational coefficients at the fixed points of the renormalization group flow. For this, the relations between the coefficients of the linearly independent forms of the energy-momentum tensor 3-point function and the gravitational anomaly coefficients obtained in [66] will be very useful. It is also important to show that χ^a_{ij} has definite sign, so the function F is monotonic decreasing along the renormalization group flow. Related work has been done in [46].

The central functions of supersymmetric gauge theories are discussed in [52]. It would be interesting to have monotonic central functions along the renormalization group flow that reproduce the relations between central charges found there. The Weyl coefficient $\beta^{\mathscr{A}}$ and Euler coefficient $\beta^{\mathscr{B}}$ are relevant for this.

Chapter 7

Conclusions and Perspectives

In the first part of this thesis, we presented an effective string theory action that we expect to be of relevance for phenomenological models of hadronic physics. Our configuration is an instance of the duality between ten dimensional gravity theories and four dimensional gauge theories. We investigated a D-brane probe embedded in a curved background with anti de Sitter geometry.

The interesting ingredient of our setup is a worldvolume flux on the probe brane sourced by a constant antisymmetric Kalb–Ramond B-field. Via the Seiberg–Witten map, this B-field gives rise to a quantization of the D-brane coordinates. The noncommutative nature of the quantized coordinates is reflected in a potential for the B-field. The minimum of this potential distinguishes a vacuum expectation value of the B-field, see figures 4.3 and 4.4.

Our results for the effective action (2.35) and the potential provide the basis for future research in the context of the AdS/CFT correspondence. For this we interpret our configuration of D-branes and the B-field as being on the gravity side of the duality. The following properties of our setup are important.

- We embed a single $D7$-brane probe. The abelian gauge theory of the open strings on the worldvolume of the probe brane is translated into a global $U(1)$ phase for the single massless quark on the field theory side.

- The B-field breaks the rotation symmetry of the background space (2.44). The residual symmetries of the system are translated to a subgroup of the rotation symmetry of the fermionic coordinates on the field theory side. The dual field theory of our setup is discussed in [13].

- The B-field gives rise to a dissolved $D3$-brane within the $\overline{D7}$-brane. This can be considered as a bound state between D-branes. The corresponding interaction potential vanishes [11]. In addition, there is no interaction potential between the background generating stack of N_C $D3$-branes and the dissolved $D3$-brane, confer table 2.2.

  We have presented here an alternative potential (4.30) for the B-field. We interpret its existence as follows. As discussed in section 4.1, the quantization of the y_m coordinates along the B-field is equivalent to an uncertainty relation of the schematic form
$$\Delta y_m \Delta y^m \geq \zeta \,. \tag{7.1}$$

  This leads to vacuum expectation values for the coordinates along the B-field. So they may give rise to a mechanism of spontaneous symmetry breaking leading to our potential. This would then be an alternative to the standard Higgs mechanism for D-branes [23, 38].

- The Seiberg–Witten map relates the closed string coupling B_{MN} to the open string coupling Θ_{MN}, see table 3.1. This gives rise to a correspondence between the vacuum expectation value of the B-field and the fundamental noncommutativity scale ζ. If existing, this scale is a constant of nature. Any measurement of it would give strong restrictions on our model.

It is important to clarify the validity of our assumption about the Seiberg–Witten map for integrals (4.12).

- We perform calculations in perturbative string theory, and the AdS/CFT correspondence is based on the diagrammatic equivalence between Riemann surfaces and 't Hooft double line graphs visualized in table 1.1. So, besides the postulate of quantized spacetime, our results rely on the postulate that the fundamental degrees of freedom of the underlying theory are one dimensional extended objects. The mutual consistency of these two concepts is the topic of an active research area [67].

In view of the application of our results to phenomenological gauge/gravity models, I regard the following lines of investigation as promising. It could be interesting to study fluctuations of the flavor gauge field around the static instanton configuration (4.34). The equations of motion of the fluctuations can be interpreted in terms of meson spectra [39].

In order to come closer to QCD, it is necessary to break the supersymmetry on the field theory side. The symmetries of the field theory are encoded in the geometry of the configuration on the gravity side. Deformations of the gravity background metric can give rise to a renormalization group flow on the field theory side [68]. Such deformations can be caused by the backreaction of a nonconstant B-field on the metric [44, 45]. I discuss this at the end of chapter 4.

I regard it as an interesting alternative to investigate whether the quantization of the D-brane coordinates influences our background metric (2.8). The noncommutativity scale ζ may then serve as a deformation parameter. A possible ansatz for the deformed metric could be

$$g_{MN}\,\mathrm{d}X^M\mathrm{d}X^N = \frac{\eta_{\mu\nu}}{\sqrt{H}}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + \sqrt{H}\left(f\left(\zeta,p\right)\delta_{mn}\mathrm{d}y^m\mathrm{d}y^n + \delta_{ij}\mathrm{d}z^i\mathrm{d}z^j\right)\,. \tag{7.2}$$

This ansatz describes a metric that is deformed in the directions along the B-field by a dimensionless function f of the noncommutativity scale and another dimensionful generic parameter p.

⋆ ⋆ ⋆

In the second part of this thesis I wrote down the local renormalization group of $\mathcal{N}=1$ supersymmetric gauge theories.

Inspired by Osborn [21], we use the method of local couplings in the functional integral formalism. We introduce the local theta couplings as external sources for local composite operators and the local R-vector as a source for the R-current, summarized in table 5.1. Insertions into connected correlation functions are generated by the vacuum energy functional,

$$\langle[R^\mu\left(x\right)]\rangle = -\frac{1}{\sqrt{g\left(x\right)}}\frac{\delta W}{\delta V_\mu\left(x\right)}\,, \tag{7.3a}$$

$$\langle[\hat{\mathcal{O}}_i\left(x\right)]\rangle = \frac{1}{\sqrt{g\left(x\right)}}\frac{\delta W}{\delta\hat{\theta}^i\left(x\right)}\,. \tag{7.3b}$$

The main results are the Ward identity for the local $U(1)$ R-symmetry (6.3a) and the local renormalization group equation for conformal symmetry (6.20). We found a basis of independent tensor monomials built up from spacetime curvature tensors and derivatives of external sources. We evaluated the Wess–Zumino consistency conditions for the local anomalies of R-symmetry (6.6),

$$\left[\Delta'_{\rho}\,,\Delta_{\rho}\right] W = 0\,, \tag{7.4}$$

and conformal symmetry (6.25),

$$\left[\Delta'_{\sigma}\,,\Delta_{\sigma}\right] W = 0\,. \tag{7.5}$$

Our aim is to find monotonic central functions along the local renormalization group flow. A central function interpolates between the values of the central charges at the fixed points. This would allow for an entropy interpretation.

In section 6.4 I gave an outline of Osborn's recipe leading to candidates for central functions. The functional derivative of the Ward identity is compared to the general form of the energy-momentum tensor 2-point function $< TT >$. The recipe allows for constructing candidate functions that coincide with a gravitational coefficient at the fixed points, for instance

$$F = -\tfrac{1}{2}\beta^{\mathscr{D}} - U_i\beta^i\,. \tag{7.6}$$

This function is just an example, because I have not taken into account possible contributions from other anomalies. Furthermore, in four dimensions it is unknown which central charges count the physical degrees of freedom [18]. Relations between gravitational coefficients and independent parameters of energy-momentum tensor correlation functions are given in [66].

By assuming a special solution to the Wess–Zumino consistency condition, I obtained the slope of the function along the renormalization group flow,

$$M\frac{\partial}{\partial M}F = -2\chi^a_{ij}\beta^i\beta^j\,. \tag{7.7}$$

Once such a candidate is found, it has to be shown that this function is monotonic decreasing by relating the coefficient χ^a_{ij} to connected correlation functions of local operators. This calculation was done in [46] for the case without supersymmetry.

I found a special solution to the Wess–Zumino consistency conditions that gives rise to these two equations. For obtaining a general solution, more constraints on the anomalies are necessary. Therefore we restrict the quantum field theories under investigation to supersymmetric gauge theories coupled to a classical supergravity background.

We demand that global supersymmetry is preserved, but superconformal symmetry is broken by the local anomalies presented in [43]. In order to quantify this restriction, it is important to verify the mutual consistency of R-symmetry and Weyl symmetry via the mixed consistency condition. We treat the Weyl transformations of the background metric and lambda couplings as being independent from the rotations of the fermionic superspace coordinates. This is imposed via the consistency condition

$$\left[\Delta_\sigma \,, \Delta_\rho\right] W = 0\,. \tag{7.8}$$

The most important next step is to perform this calculation. The necessary variations are done already and listed in sections A.2 and A.3.

Our formalism is compatible with spontaneous supersymmetry breaking, since we only describe the properties of the renormalization group in between two critical points.

The method of local couplings relies only on the basic assumptions of quantum field theory. It allows to construct candidates for central functions. $\mathcal{N} = 1$ supersymmetric gauge theories are expected to give enough restrictions on the form of these candidate functions, so that an entropy functional can be constructed explicitly. The application of our results to thermodynamic systems possessing two critical points is conceivable.

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Appendix A

Variation of Tensor Monomials

Here I write down the explicit variations of the anomaly terms used for our work.

A.1 Weyl Variation of the Conformal Anomalies

I use the infinitesimal Weyl transformations of the external fields,

$$\delta_\sigma \lambda^i = \sigma \beta^i \, , \quad \delta_\sigma \hat{\theta}^i = 0 \, , \quad \delta_\sigma g^{\mu\nu} = +2\sigma g^{\mu\nu} \, , \quad \delta_\sigma V_\mu = 0 \, , \tag{A.1}$$

as well as the conformal variations of the curvature tensors given in (B.16). The derivative operator is

$$\Delta'_\sigma = \int \mathrm{d}^4 y \, \sigma' \left(2 \frac{\delta}{\delta g^{\mu\nu}(y)} g^{\mu\nu} - \beta^i \frac{\delta}{\delta \lambda^i(y)} \right) . \tag{A.2}$$

The variation at different positions gives a delta function that is used to carry out the integration over y,

$$\frac{\delta \lambda^i(x)}{\delta \lambda^j(y)} = \delta^i_j \delta^{(4)}(x) \, , \tag{A.3a}$$

$$\frac{\delta V_\nu(x)}{\delta V_\mu(y)} = \delta^\mu_\nu \delta^{(4)}(x) \, . \tag{A.3b}$$

I only write down those variations proportional to nonvanishing sigma commutators, so I leave out the terms with symmetric combinations of sigma derivatives. In this section,

$\cong$ means "equal up to terms with vanishing sigma commutator."

The Weyl variations of the conformal local anomalies in (6.20) are listed here.

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\beta^{\mathscr{B}}G &\cong \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\sigma\nabla_\mu\nabla_\nu\sigma')\,8\beta^{\mathscr{B}}G^{\mu\nu} \\ &\cong \int\!\sqrt{g}\,\mathrm{d}^4x\ (\sigma\partial_\nu\sigma')\,8\left(\frac{\delta\beta^{\mathscr{B}}}{\delta\lambda^i}\partial_\mu\lambda^i G^{\mu\nu} + \beta^{\mathscr{B}}\nabla_\mu G^{\mu\nu}\right)\end{aligned} \tag{A.4}$$

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{12}\beta^{\mathscr{D}}\Box R &\cong \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\sigma\partial_\mu\sigma')\,\tfrac{1}{6}\beta^{\mathscr{D}}\partial^\mu R \\ &+ \int\!\sqrt{g}\,\mathrm{d}^4x\ (\sigma\Box\sigma')\,\tfrac{1}{6}\beta^{\mathscr{D}}R + \int\!\sqrt{g}\,\mathrm{d}^4x\ (\sigma\Box\,\Box\,\sigma')\,\tfrac{1}{2}\beta^{\mathscr{D}}\end{aligned} \tag{A.5}$$

We integrate the last line partially. This gives a symmetric term and a contribution to the second line.

$$\begin{aligned}&\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{12}\beta^{\mathscr{D}}\Box R \\ &\cong \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\sigma\partial_\mu\sigma')\,\tfrac{1}{6}\beta^{\mathscr{D}}\partial^\mu R + \int\!\sqrt{g}\,\mathrm{d}^4x\ (\sigma\Box\sigma')\,\tfrac{1}{2}\left(\tfrac{1}{3}\beta^{\mathscr{D}}R + \Box\beta^{\mathscr{D}}\right)\end{aligned} \tag{A.6}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{9}\beta^{\mathscr{C}}R^2 \cong \int\!\sqrt{g}\,\mathrm{d}^4x\ (\sigma\Box\sigma')\,\tfrac{4}{3}\beta^{\mathscr{C}}R \tag{A.7}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{9}\beta^{\mathscr{C}}R^2 \cong \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\sigma\partial_\mu\sigma')\,\tfrac{4}{3}\left(\beta^{\mathscr{C}}\partial^\mu R + \partial^\mu\beta^{\mathscr{C}}R\right) \tag{A.8}$$

The coefficients χ^a_{ij} and $\hat{\chi}^a_{ij}$ are symmetric in ij.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\tfrac{1}{2}\chi^a_{ij}\Box\lambda^i\Box\lambda^j\right) \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,2\chi^a_{ij}\partial^\mu\lambda^i\Box\lambda^j + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,\chi^a_{ij}\Box\lambda^i\beta^j \tag{A.9a}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\tfrac{1}{2}\hat{\chi}^a_{ij}\Box\hat{\theta}^i\Box\hat{\theta}^j\right) \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,2\hat{\chi}^a_{ij}\partial^\mu\hat{\theta}^i\Box\hat{\theta}^j \tag{A.9b}$$

The coefficients $\chi^b_{ijk}(\lambda)$ and $\hat{\chi}^b_{ij}(\hat{\theta})$ are symmetric in ij.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\tfrac{1}{2}\chi^b_{ijk}\Lambda^{ij}\Box\lambda^k\right) \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,\chi^b_{ijk}\left(\Lambda^{ij}\partial^\mu\lambda^k + \beta^i\partial^\mu\lambda^j\Box\lambda^k\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,\tfrac{1}{2}\chi^b_{ijk}\Lambda^{ij}\beta^k \tag{A.10}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\chi}^b_{ij}\omega^b_k\hat{\Lambda}^{ij}\Box\lambda^k \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,\hat{\chi}^b_{ij}\omega^b_k\hat{\Lambda}^{ij}\partial^\mu\lambda^k + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,\tfrac{1}{2}\hat{\chi}^b_{ij}\omega^b_k\hat{\Lambda}^{ij}\beta^k \tag{A.11}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\omega^d_i\hat{\chi}^d_{jk}\partial_\mu\lambda^i\partial^\mu\hat{\theta}^j\Box\hat{\theta}^k \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,\omega^d_i\hat{\chi}^d_{jk}\Big(\partial_\nu\lambda^i\partial^\nu\hat{\theta}^j\partial^\mu\hat{\theta}^k + \tfrac{1}{2}\beta^i\partial^\mu\hat{\theta}^j\Box\hat{\theta}^k\Big) \tag{A.12}$$

The coefficients $\chi^c_{ijkl}(\lambda)$, $\chi^h_{ij}(\lambda)$, $\hat{\chi}^h_{ij}(\hat{\theta})$, $\hat{\chi}^c_{ijkl}(\hat{\theta})$ are symmetric in all their indices.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\tfrac{1}{4}\chi^c_{ijkl}\Lambda^{ij}\Lambda^{kl}\right) \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,\chi^c_{ijkl}\Lambda^{ij}\beta^k\partial^\mu\lambda^l \tag{A.13a}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{4}\chi^h_{ij}\hat{\chi}^h_{kl}\Lambda^{ij}\hat{\Lambda}^{kl} \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,\tfrac{1}{2}\chi^h_{ij}\hat{\chi}^h_{kl}\beta^i\partial^\mu\lambda^j\hat{\Lambda}^{kl} \tag{A.13b}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{4}\hat{\chi}^c_{ijkl}\hat{\Lambda}^{ij}\hat{\Lambda}^{kl} \cong 0 \tag{A.13c}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\tfrac{1}{3}\chi^e_i\partial_\mu\lambda^i\partial^\mu R\right) \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\left(-\tfrac{2}{3}\chi^e_i\partial^\mu\lambda^i R\right) \tag{A.14}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{3}\chi^e_i\beta^i\partial^\mu R \tag{A.15}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\Box\sigma'\right)2\chi^e_i\partial^\mu\lambda^i \tag{A.16}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\Box\sigma'\right)2\nabla_\mu\left(\chi^e_i\partial^\mu\lambda^i\right) \tag{A.17}$$

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\tfrac{1}{6}\chi^f_{ij}\Lambda^{ij}R\right) \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\Box\sigma'\right)\chi^f_{ij}\Lambda^{ij}\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{3}\chi^f_{ij}\beta^i\partial^\mu\lambda^j R\end{aligned} \tag{A.18a}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{6}\hat{\chi}^f_{ij}\hat{\Lambda}^{ij}R \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\Box\sigma'\right)\hat{\chi}^f_{ij}\hat{\Lambda}^{ij} \tag{A.18b}$$

The coefficients χ^g_{ij} and $\hat{\chi}^g_{ij}$ are symmetric in ij.

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\left(\tfrac{1}{2}\chi^g_{ij}\partial^\mu\lambda^i\partial^\nu\lambda^j G_{\mu\nu}\right) \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\Box\sigma'\right)\chi^g_{ij}\Lambda^{ij}\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\big(\chi^g_{ij}\partial^\mu\lambda^i\Box\lambda^j\\ &\quad+\nabla_\nu\left(\chi^g_{ij}\partial^\mu\lambda^i\right)\partial^\nu\lambda^j+\chi^g_{ij}\beta^i\partial_\nu\lambda^j G^{\mu\nu}\big)\end{aligned} \tag{A.19a}$$

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\chi}^g_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j G^{\mu\nu} \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\Box\sigma'\right)\hat{\chi}^g_{ij}\hat{\Lambda}^{ij}\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\left(\nabla_\nu\left(\hat{\chi}^g_{ij}\partial^\mu\hat{\theta}^i\partial^\nu\hat{\theta}^j\right)\right)\end{aligned} \tag{A.19b}$$

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial^\mu\sigma w_i\partial^\nu\lambda^i G_{\mu\nu} \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\partial^\mu\sigma\nabla_\mu\partial_\nu\sigma'\right)2w_i\partial_\nu\lambda^i\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial^\mu\sigma\Box\sigma'\right)2w_i\partial_\mu\lambda^i\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\mathcal{L}_\beta\,w_i\partial_\nu\lambda^i G^{\mu\nu}\end{aligned} \tag{A.20}$$

It is also possible to partially integrate the ∇_μ away from the first line. This cancels the second line. In the consistency conditions I used the following variation.

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial^\mu\sigma w_i\partial^\nu\lambda^i G_{\mu\nu} \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial^\mu\sigma\partial_\nu\sigma')\,2\nabla_\mu\left(w_i\partial_\nu\lambda^i\right) \\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\partial_\mu\sigma')\,\mathcal{L}_\beta\,w_i\partial_\nu\lambda^i G^{\mu\nu}\end{aligned} \tag{A.21}$$

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\tfrac{1}{3}\partial_\mu\sigma\partial^\mu(dR) \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\;(\partial_\mu\sigma\Box\sigma')\,2\partial^\mu d+\int\!\sqrt{g}\,\mathrm{d}^4x\;(\partial_\mu\sigma\partial^\mu\Box\sigma')\,2\,d \\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\partial_\mu\sigma')\,\tfrac{1}{3}\partial^\mu\left(\frac{\delta d}{\delta\lambda^i}\beta^i R\right) \\ \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,\tfrac{1}{3}R\mathcal{L}_\beta\,d\end{aligned} \tag{A.22}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\partial^\mu\left(U_i\Box\lambda^i\right) \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\mu\sigma\nabla^\mu\nabla^\nu\sigma')\,2U_i\partial_\nu\lambda^i \tag{A.23}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\mu\sigma\partial_\nu\sigma')\,2\nabla^\mu\left(U_i\partial^\nu\lambda^i\right) \tag{A.24}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\partial_\mu\sigma')\,\partial^\mu\left(\mathcal{L}_\beta\,U_i\Box\lambda^i\right) \tag{A.25}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\mu\sigma\Box\sigma')\,\partial^\mu\left(U_i\beta^i\right) \tag{A.26}$$

I partially integrate the third line, and also the first line with respect to ∇^μ. This cancels the second line and gives a contribution to the last line. For the consistency conditions I use the variation

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\partial^\mu\left(U_i\Box\lambda^i\right) \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,\mathcal{L}_\beta\,U_i\Box\lambda^i \\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\mu\sigma\Box\sigma')\left(\partial^\mu\left(U_i\beta^i\right)+2U_i\partial^\mu\lambda^i\right)\,.\end{aligned} \tag{A.27}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\tfrac{1}{3}\partial_\mu\sigma Y_i\,\partial^\mu\lambda^i R \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\partial_\mu\sigma\Box\sigma'\right)2Y_i\partial^\mu\lambda^i + \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\tfrac{1}{3}\left(\mathcal{L}_\beta\,Y_i\right)\partial^\mu\lambda^i R \tag{A.28}$$

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma S_{ij}\partial^\mu\lambda^i\Box\lambda^j \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial_\nu\sigma'\right)2S_{[ij]}\;\partial^\mu\lambda^i\partial^\nu\lambda^j \\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\mathcal{L}_\beta\,S_{ij}\partial^\mu\lambda^i\Box\lambda^j \\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\Box\sigma'\right)S_{ij}\partial^\mu\lambda^i\beta^j\end{aligned} \tag{A.29a}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\hat{S}_{ij}\partial^\mu\hat{\theta}^i\Box\hat{\theta}^j \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial_\nu\sigma'\right)2\hat{S}_{[ij]}\;\partial^\mu\hat{\theta}^i\partial^\nu\hat{\theta}^j \tag{A.29b}$$

The coefficients $V_{ij}(\lambda)$ and $\hat{V}_{ij}(\hat{\theta})$ do not fulfill the tensor transformation law. Therefore we do not assign a symmetry property to them.

$$\begin{aligned}\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\left(V_{ij}\Lambda^{ij}\right) \cong &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\Box\sigma'\right)\tfrac{1}{2}\mathcal{L}_\beta\,V_{ij}\Lambda^{ij} \\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\Box\sigma'\right)V_{(ij)}\;\beta^i\partial^\mu\lambda^j\end{aligned} \tag{A.30a}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\left(\hat{V}_{ij}\hat{\Lambda}^{ij}\right) \cong 0 \tag{A.30b}$$

The coefficients T_{ijk} and $\hat{T}_{ij}$ are symmetric in all indices.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\tfrac{1}{2}\partial_\mu\sigma T_{ijk}\Lambda^{ij}\partial^\mu\lambda^k \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\mathcal{L}_\beta\,T_{ijk}\Lambda^{ij}\partial^\mu\lambda^k \tag{A.31}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}\hat{T}_{ij}T_k\hat{\Lambda}^{ij}\partial^\mu\lambda^k \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma'\partial_\mu\sigma\right)\tfrac{1}{2}\hat{T}_{ij}\mathcal{L}_\beta\,T_k\hat{\Lambda}^{ij}\partial^\mu\lambda^k \tag{A.32}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}t_i\hat{t}_{jk}\partial_\nu\lambda^i\partial^\nu\hat{\theta}^j\partial^\mu\hat{\theta}^k \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,(-\sigma'\partial_\mu\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,t_i\hat{t}_{jk}\partial_\nu\lambda^i\partial^\nu\hat{\theta}^j\partial^\mu\hat{\theta}^k + \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\nu\sigma'\partial_\mu\sigma\right)\tfrac{1}{2}t_i\hat{t}_{jk}\beta^i\partial^\nu\hat{\theta}^j\partial^\mu\hat{\theta}^k \tag{A.33}$$

Now we list the variations of the terms $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{Y}^\mu$ involving the external vector field in order of appearance.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{4}\beta^{\mathscr{V}}V^{\mu\nu}V_{\mu\nu} \cong \Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{4}v^aV^4 \cong 0 \tag{A.34}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}v^b\,(\nabla\cdot V)\,V^2 \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)v^bV^\mu V^2 \tag{A.35}$$

The coefficients κ^a_{ij} and $\hat{\kappa}^a_{ij}$ are antisymmetric.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^a_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^jV^{\mu\nu} \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^a_{ij}\beta^i\partial_\nu\lambda^jV^{\mu\nu} \tag{A.36}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\kappa}^a_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^jV^{\mu\nu} \cong 0 \tag{A.37}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{3}\kappa^b_i\partial_\mu\lambda^i\nabla_\nu V^{\mu\nu} \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{3}\kappa^b_i\beta^i\nabla_\nu V^{\mu\nu} \tag{A.38}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{3}\kappa^c_i\partial_\mu\lambda^iV_\nu V^{\mu\nu} \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{3}\kappa^c_i\beta^iV_\nu V^{\mu\nu} \tag{A.39}$$

The coefficients $\kappa^d_{ijk}(\lambda)$, $\hat{\kappa}^d_{ij}(\hat{\theta})$ are symmetric in ij.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^d_{ijk}\Lambda^{ij}\left(\partial\lambda^k\cdot V\right) \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^d_{ijk}\beta^i\partial^\mu\lambda^j\left(\partial\lambda^k\cdot V\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\kappa^d_{ijk}\Lambda^{ij}\beta^kV^\mu \tag{A.40}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\zeta}^d_{ij}\zeta^d_k\hat{\Lambda}^{ij}\left(\partial\lambda^k\cdot V\right) \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\hat{\zeta}^d_{ij}\zeta^d_k\tfrac{1}{2}\hat{\Lambda}^{ij}\beta^k V^\mu \quad \text{(A.41)}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\xi^d_i\hat{\xi}^d_{jk}\left(\partial\lambda^i\cdot\partial\hat{\theta}^j\right)\left(\partial\hat{\theta}^k\cdot V\right) \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\xi^d_i\hat{\xi}^d_{jk}\beta^i\partial^\mu\hat{\theta}^j\left(\partial\hat{\theta}^k\cdot V\right) \quad \text{(A.42)}$$

The coefficients κ^e_{ij} and $\hat{\kappa}^e_{ij}$ are symmetric.

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^e_{ij}\Lambda^{ij}V^2 \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^e_{ij}\beta^i\partial^\mu\lambda^j V^2 \quad \text{(A.43)}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\kappa}^e_{ij}\hat{\Lambda}^{ij}V^2 \cong 0 \quad \text{(A.44)}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^f_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j V^\mu V^\nu \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^f_{(ij)}\beta^i\partial_\nu\lambda^j V^\mu V^\nu \quad \text{(A.45a)}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\kappa}^f_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j V^\mu V^\nu \cong 0 \quad \text{(A.45b)}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\kappa^g_i\left(\partial\lambda^i\cdot V\right)V^2 \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^g_i\beta^i V^\mu V^2 \quad \text{(A.46)}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^h_i\left(\partial\lambda^i\cdot\partial\right)V^2 \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\kappa^h_i\beta^i\partial^\mu V^2 \quad \text{(A.47)}$$

Now we vary the terms in $\mathcal{D}$.

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^m_{ij}\left(\partial\lambda^i\cdot V\right)\Box\lambda^j \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^m_{ij}\left(\partial\lambda^i\cdot V\right)\partial^\mu\lambda^j \quad \text{(A.48)}$$

$$+\int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\kappa^m_{ij}\beta^i V^\mu\Box\lambda^j \quad \text{(A.49)}$$

$$+\int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\Box\sigma'\right)\tfrac{1}{2}\kappa^m_{ij}\left(\partial\lambda^i\cdot V\right)\beta^j \quad \text{(A.50)}$$

$$\Delta'_{\sigma}\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\hat{\kappa}^m_{ij}\left(\partial\hat{\theta}^i\cdot V\right)\Box\hat{\theta}^j \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\hat{\kappa}^m_{ij}2\left(\partial\hat{\theta}^i\cdot V\right)\partial^\mu\hat{\theta}^j \quad \text{(A.51)}$$

$$\Delta'_{\sigma}\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{3}\kappa^n_i\left(\partial\lambda^i\cdot V\right)R \;\cong\; \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\Box\sigma'\right)2\kappa^n_i\left(\partial\lambda^i\cdot V\right) \quad \text{(A.52)}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\tfrac{1}{3}\kappa^n_i\beta^iV^\mu R \quad \text{(A.53)}$$

$$\Delta'_{\sigma}\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma v^c\left(\nabla\cdot V\right)R \;\cong\; \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)2v^cV^\mu R \quad \text{(A.54)}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\Box\sigma'\right)6v^c\left(\nabla\cdot V\right) \quad \text{(A.55)}$$

$$\Delta'_{\sigma}\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^o_i\Box\lambda^iV^2 \;\cong\; \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^o_i\partial^\mu\lambda^iV^2 \quad \text{(A.56)}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\Box\sigma'\right)\tfrac{1}{2}\kappa^o_i\beta^iV^2 \quad \text{(A.57)}$$

$$\Delta'_{\sigma}\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}v^d\Box V^2 \;\cong\; \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)v^d\partial^\mu V^2 \quad \text{(A.58)}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\Box\sigma'\right)v^dV^2 \quad \text{(A.59)}$$

$$\Delta'_{\sigma}\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{6}v^eV^2R \cong \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\Box\sigma'\right)v^eV^2 \quad \text{(A.60)}$$

With the variation rule (B.13) we have

$$\Delta'_{\sigma}\int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^p_i\Box\lambda^i\left(\nabla\cdot V\right) \;\cong\; \int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^p_i\partial^\mu\lambda^i\left(\nabla\cdot V\right) \quad \text{(A.61)}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\sigma'\right)\kappa^p_i\Box\lambda^iV^\mu \quad \text{(A.62)}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\Box\sigma'\right)\tfrac{1}{2}\kappa^p_i\beta^i\left(\nabla\cdot V\right) \quad \text{(A.63)}$$

$$\begin{aligned}
\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa_i^q\partial^\mu\lambda^i V^\nu G_{\mu\nu} &\cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\nabla_\mu\partial_\nu\sigma')\,\kappa_i^q\partial^\mu\lambda^i V^\nu \\
&\quad+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,\kappa_i^q\partial_\mu\lambda^i V^\mu \\
&\quad+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,\tfrac{1}{2}\kappa_i^q\beta^i V_\nu G^{\mu\nu} \\
&\cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\mu\sigma\partial_\nu\sigma')\,\kappa_i^q\partial^\mu\lambda^i V^\nu \\
&\quad+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\nu\sigma')\,\nabla_\mu\left(\kappa_i^q\partial^\mu\lambda^i V^\nu\right) \\
&\quad+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,\kappa_i^q\partial_\mu\lambda^i V^\mu \\
&\quad+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\sigma')\,\tfrac{1}{2}\kappa_i^q\beta^i V_\nu G^{\mu\nu}
\end{aligned} \tag{A.64}$$

$$\begin{aligned}
\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}v^f V^\mu V^\nu G_{\mu\nu} &\cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\nabla_\mu\partial_\nu\sigma')\,v^f V^\mu V^\nu + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,v^f V^2 \\
&\cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\nu\sigma')\,\nabla_\mu\left(v^f V^\mu V^\nu\right) \\
&\quad+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\sigma')\,v^f V^2
\end{aligned} \tag{A.65}$$

Here is the Weyl variation of the current $\mathcal{Y}^\mu$.

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\kappa_i^r\partial_\nu\lambda^i V^{\mu\nu} \;\cong\; \int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\partial_\mu\sigma')\,\mathcal{L}_\beta\,\kappa_i^r\partial_\nu\lambda^i V^{\mu\nu} \tag{A.66}$$

$$+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\mu\sigma\partial_\nu\sigma')\,\kappa_i^r\beta^i V^{\mu\nu} \tag{A.67}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma v^g\nabla_\nu V^{\mu\nu} \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\partial_\mu\sigma')\,\mathcal{L}_\beta\,v^g\nabla_\nu V^{\mu\nu} \tag{A.68}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma v^h V_\nu V^{\mu\nu} \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\partial_\mu\sigma')\,\mathcal{L}_\beta\,v^h V_\nu V^{\mu\nu} \tag{A.69}$$

$$\Delta'_\sigma \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma v^m V^\mu V^2 \cong \int\!\sqrt{g}\,\mathrm{d}^4x\;(\sigma\partial_\mu\sigma')\,\mathcal{L}_\beta\,v^m V^\mu V^2 \tag{A.70}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}\kappa_i^s\partial^\mu\lambda^i V^2 \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\mathcal{L}_\beta\,\kappa_i^s\partial^\mu\lambda^i V^2 \tag{A.71}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\kappa_i^t\left(\partial\lambda^i\cdot V\right)V^\mu \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\mathcal{L}_\beta\,\kappa_i^t\left(\partial\lambda^i\cdot V\right)V^\mu \tag{A.72}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}\kappa_i^u\Box\lambda^i V^\mu \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial_\nu\sigma'\right)\kappa_i^u\partial^\nu\lambda^i V^\mu \tag{A.73}$$

$$+\int\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\mathcal{L}_\beta\,\kappa_i^u\Box\lambda^i V^\mu \tag{A.74}$$

$$+\int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\Box\sigma'\right)\tfrac{1}{2}\kappa_i^u\beta^i V^\mu \tag{A.75}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}\kappa_i^v\partial^\mu\lambda^i\left(\nabla\cdot V\right) \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial_\nu\sigma'\right)\kappa_i^v\partial^\mu\lambda^i V^\nu \tag{A.76}$$

$$+\int\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\tfrac{1}{2}\mathcal{L}_\beta\,\kappa_i^v\partial^\mu\lambda^i\left(\nabla\cdot V\right) \tag{A.77}$$

The coefficients κ_{ij}^w and κ_{ij}^w are symmetric in ij.

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\kappa_{ij}^w\Lambda^{ij}V^\mu \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\mathcal{L}_\beta\,\kappa_{ij}^w\Lambda^{ij}V^\mu \tag{A.78}$$

$$+\int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial_\nu\sigma'\right)2\kappa_{ij}^w\beta^i\partial^\nu\lambda^j V^\mu \tag{A.79}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\hat{\kappa}_{ij}^w\hat{\Lambda}^{ij}V^\mu \cong 0 \tag{A.80}$$

The coefficients κ_{ij}^y and κ_{ij}^y are symmetric in ij.

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\kappa_{ij}^y\partial^\mu\lambda^i\left(\partial\lambda^j\cdot V\right) \cong \int\sqrt{g}\,\mathrm{d}^4x\,\left(\sigma\partial_\mu\sigma'\right)\mathcal{L}_\beta\,\kappa_{ij}^y\partial^\mu\lambda^i\left(\partial\lambda^j\cdot V\right) \tag{A.81}$$

$$+\int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial_\nu\sigma'\right)\kappa_{ij}^y\partial^\mu\lambda^i\beta^j V^\nu \tag{A.82}$$

$$\Delta'_\sigma \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\hat{\kappa}_{ij}^y\partial^\mu\hat{\theta}^i\left(\partial\hat{\theta}^j\cdot V\right) \cong 0 \tag{A.83}$$

$$\Delta'_{\sigma} \int \sqrt{g}\, \mathrm{d}^4x\, \partial_\mu \sigma v^n V^\mu R \;\cong\; \int \sqrt{g}\, \mathrm{d}^4x\, \left(\partial_\mu \sigma \Box \sigma'\right) 6 v^n V^\mu \tag{A.84}$$

$$+ \int \sqrt{g}\, \mathrm{d}^4x\, \left(\sigma \partial_\mu \sigma'\right) \mathcal{L}_\beta\, v^n V^\mu R \tag{A.85}$$

$$\Delta'_{\sigma} \int \sqrt{g}\, \mathrm{d}^4x\, \partial^\mu \sigma \tfrac{1}{2} v^o V^\nu G_{\mu\nu} \cong \int \sqrt{g}\, \mathrm{d}^4x\, \left(\partial^\mu \sigma \nabla_\mu \partial_\nu \sigma'\right) v^o V^\nu + \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\partial^\mu \sigma \Box \sigma'\right) v^o V_\mu$$
$$+ \int \sqrt{g}\, \mathrm{d}^4x\, \left(\sigma \partial_\mu \sigma'\right) \tfrac{1}{2} \mathcal{L}_\beta\, v^o V_\nu G^{\mu\nu} \tag{A.86}$$

I partially integrate the first line. In the consistency conditions I use the variation

$$\Delta'_{\sigma} \int \sqrt{g}\, \mathrm{d}^4x\, \partial^\mu \sigma \tfrac{1}{2} v^o V^\nu G_{\mu\nu} \cong \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\partial_\mu \sigma \partial_\nu \sigma'\right) \nabla^\mu \left(v^o V^\nu\right)$$
$$+ \int \sqrt{g}\, \mathrm{d}^4x\, \left(\sigma \partial_\mu \sigma'\right) \tfrac{1}{2} \mathcal{L}_\beta\, v^o V_\nu G^{\mu\nu}\,. \tag{A.87}$$

A.2 R-Variation of the Conformal Anomalies

The infinitesimal R-transformations of the external fields are

$$\delta_\rho \lambda^i = 0\,, \quad \delta_\rho \hat{\theta}^i = \rho \hat{\beta}^i\,, \quad \delta_\rho g^{\mu\nu} = 0\,, \quad \delta_\rho V_\mu = \partial_\mu \rho\,. \tag{A.88}$$

The variation operator is

$$\Delta_\rho = \int \mathrm{d}^4y\, \left(\partial_\mu \rho\,(y)\, \frac{\delta}{\delta V_\mu\,(y)} - \rho\,(y)\, \hat{\beta}^i\,(y)\, \frac{\delta}{\delta \hat{\theta}^i\,(y)}\right). \tag{A.89}$$

The variations of the anomalies in (6.20) are listed. I leave in the Lie derivatives $\mathcal{L}_\beta$ with respect to the lambda coupling vector fields.

$$\Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \beta^{\mathscr{A}} F = \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \mathcal{L}_\beta\, \beta^{\mathscr{A}} F \tag{A.90a}$$

$$\Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \beta^{\mathscr{B}} G = \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \mathcal{L}_\beta\, \beta^{\mathscr{B}} G \tag{A.90b}$$

$$\Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \tfrac{1}{9} \beta^{\mathscr{C}} R^2 = \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \tfrac{1}{9} \mathcal{L}_\beta\, \beta^{\mathscr{C}} R^2 \tag{A.90c}$$

$$\Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \tfrac{1}{12} \beta^{\mathscr{D}} \Box R = \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \tfrac{1}{12} \mathcal{L}_\beta\, \beta^{\mathscr{D}} \Box R \tag{A.90d}$$

$$\begin{aligned} \Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \tfrac{1}{2} \chi^a_{ij} \Box\lambda^i \Box\lambda^j = & \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \tfrac{1}{2} \mathcal{L}_\beta\, \chi^a_{ij} \Box\lambda^i \Box\lambda^j \\ & + \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\sigma\Box\rho\right) \chi^a_{ij} \beta^i \Box\lambda^j \end{aligned} \tag{A.90e}$$

$$\begin{aligned} \Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \tfrac{1}{2} \hat{\chi}^a_{ij} \Box\hat{\theta}^i \Box\hat{\theta}^j = & \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \tfrac{1}{2} \mathcal{L}_{\hat{\beta}}\, \hat{\chi}^a_{ij} \Box\hat{\theta}^i \Box\hat{\theta}^j \\ & + \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\sigma\Box\rho\right) \hat{\chi}^a_{ij} \hat{\beta}^i \Box\hat{\theta}^j \end{aligned} \tag{A.90f}$$

The coefficients $\chi^b_{ijk}(\lambda)$ and $\hat{\chi}^b_{ij}(\hat{\theta})$ are symmetric in ij.

$$\begin{aligned} \Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \tfrac{1}{2} \chi^b_{ijk} \Lambda^{ij} \Box\lambda^k = & \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \tfrac{1}{2} \mathcal{L}_\beta\, \chi^b_{ijk} \Lambda^{ij} \Box\lambda^k \\ & + \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\sigma\partial_\mu\rho\right) \chi^b_{ijk} \beta^i \partial^\mu \lambda^j \Box\lambda^k \\ & + \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\sigma\Box\rho\right) \tfrac{1}{2} \chi^b_{ijk} \Lambda^{ij} \beta^k \end{aligned} \tag{A.91}$$

$$\begin{aligned} \Delta_\rho \int \sqrt{g}\, \mathrm{d}^4x\, \sigma \tfrac{1}{2} \hat{\chi}^b_{ij} \zeta^b_k \hat{\Lambda}^{ij} \Box\lambda^k = & \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\rho\sigma\right) \tfrac{1}{2} \left(\zeta^b_k \mathcal{L}_{\hat{\beta}}\, \hat{\chi}^b_{ij} + \hat{\chi}^b_{ij} \mathcal{L}_\beta\, \zeta^b_k \right) \hat{\Lambda}^{ij} \Box\lambda^k \\ & + \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\sigma\partial_\mu\rho\right) \hat{\chi}^b_{ij} \zeta^b_k \hat{\beta}^i \partial^\mu \hat{\theta}^j \Box\lambda^k \\ & + \int \sqrt{g}\, \mathrm{d}^4x\, \left(-\sigma\Box\rho\right) \tfrac{1}{2} \hat{\chi}^b_{ij} \zeta^b_k \hat{\Lambda}^{ij} \beta^k \end{aligned} \tag{A.92}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}\zeta^b_i\hat{\chi}^b_{jk}\partial_\mu\lambda^i\partial^\mu\hat{\theta}^j\Box\hat{\theta}^k = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\Big(\zeta^b_i\mathcal{L}_{\hat{\beta}}\,\hat{\chi}^b_{jk} + \hat{\chi}^b_{jk}\mathcal{L}_\beta\,\zeta^b_i\Big)\,\partial_\mu\lambda^i\partial^\mu\hat{\theta}^j\Box\hat{\theta}^k + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\tfrac{1}{2}\zeta^b_i\hat{\chi}^b_{jk}\Big(\beta^i\partial^\mu\hat{\theta}^j + \partial^\mu\lambda^i\hat{\beta}^j\Big)\Box\hat{\theta}^k + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\rho)\,\tfrac{1}{2}\zeta^b_i\hat{\chi}^b_{jk}\partial_\mu\lambda^i\partial^\mu\hat{\theta}^j\hat{\beta}^k \tag{A.93}$$

The coefficients $\chi^c_{ijkl}(\lambda)$, $\zeta^c_{ij}(\lambda)$, $\hat{\zeta}^c_{ij}(\hat{\theta})$, $\hat{\chi}^c_{ijkl}(\hat{\theta})$ are symmetric in all their indices.

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{4}\chi^c_{ijkl}\Lambda^{ij}\Lambda^{kl} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{4}\mathcal{L}_\beta\,\chi^c_{ijkl}\Lambda^{ij}\Lambda^{kl} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\chi^c_{ijkl}\beta^i\partial^\mu\lambda^j\Lambda^{kl} \tag{A.94a}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{4}\zeta^c_{ij}\hat{\zeta}^c_{kl}\Lambda^{ij}\hat{\Lambda}^{kl} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{4}\Big(\zeta^c_{ij}\mathcal{L}_{\hat{\beta}}\,\hat{\zeta}^c_{kl} + \mathcal{L}_\beta\,\zeta^c_{ij}\hat{\zeta}^c_{kl}\Big)\,\Lambda^{ij}\hat{\Lambda}^{kl} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\tfrac{1}{2}\zeta^c_{ij}\beta^i\partial^\mu\lambda^j\hat{\zeta}^c_{kl}\hat{\Lambda}^{kl} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\tfrac{1}{2}\zeta^c_{ij}\Lambda^{ij}\hat{\zeta}^c_{kl}\hat{\beta}^k\partial^\mu\hat{\theta}^j \tag{A.94b}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{4}\hat{\chi}^c_{ijkl}\hat{\Lambda}^{ij}\hat{\Lambda}^{kl} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{4}\mathcal{L}_{\hat{\beta}}\,\hat{\chi}^c_{ijkl}\hat{\Lambda}^{ij}\hat{\Lambda}^{kl} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\hat{\chi}^c_{ijkl}\hat{\beta}^i\partial^\mu\hat{\theta}^j\hat{\Lambda}^{kl} \tag{A.94c}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{3}\chi^e_i\partial_\mu\lambda^i\partial^\mu R = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{3}\mathcal{L}_\beta\,\chi^e_i\partial_\mu\lambda^i\partial^\mu R + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\chi^e_i\beta^i\partial^\mu R \tag{A.94d}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{6}\chi^f_{ij}\Lambda^{ij}R = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{6}\mathcal{L}_\beta\,\chi^f_{ij}\Lambda^{ij}R + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\tfrac{1}{3}\chi^f_{ij}\beta^i\partial^\mu\lambda^j R \tag{A.94e}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{6}\hat{\chi}^f_{ij}\hat{\Lambda}^{ij}R = \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\sigma\right)\tfrac{1}{6}\mathcal{L}_\beta\,\hat{\chi}^f_{ij}\hat{\Lambda}^{ij}R + \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\rho\right)\tfrac{1}{3}\hat{\chi}^f_{ij}\hat{\beta}^i\partial^\mu\hat{\theta}^j R \tag{A.94f}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\chi^g_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j G^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\sigma\right)\tfrac{1}{2}\mathcal{L}_\beta\,\chi^g_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j G^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\rho\right)\chi^g_{ij}\beta^i\partial_\nu\lambda^j G^{\mu\nu} \tag{A.94g}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\chi}^g_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j G^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\sigma\right)\tfrac{1}{2}\mathcal{L}_\beta\,\hat{\chi}^g_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j G^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\rho\right)\hat{\chi}^g_{ij}\hat{\beta}^i\partial_\nu\hat{\theta}^j G^{\mu\nu} \tag{A.94h}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma w_i\partial_\nu\lambda^i G^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\mathcal{L}_\beta\,w_i\partial_\nu\lambda^i G^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial_\nu\rho\right)w_i\beta^i G^{\mu\nu} \tag{A.94i}$$

This variation is included for completeness.

$$\begin{aligned}\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{3}\partial^\mu\left(d\,R\right) &= \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\tfrac{1}{3}\partial^\mu\left(\mathcal{L}_\beta\,d\,R\right) \\ &\quad+ \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial^\mu\rho\right)\tfrac{1}{3}\mathcal{L}_\beta\,d\,R \\ &= \int\sqrt{g}\,\mathrm{d}^4x\,\left(+\rho\Box\sigma\right)\tfrac{1}{3}\mathcal{L}_\beta\,d\,R\end{aligned} \tag{A.94j}$$

$$\begin{aligned}\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\partial^\mu\left(U_i\Box\lambda^i\right) &= \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\partial^\mu\left(\left(\mathcal{L}_\beta\,U_i\right)\Box\lambda^i\right) \\ &\quad+ \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\partial^\mu\rho\right)\left(\mathcal{L}_\beta\,U_i\right)\Box\lambda^i \\ &\quad+ \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\Box\sigma\Box\rho\right)U_i\beta^i \\ &= \int\sqrt{g}\,\mathrm{d}^4x\,\left(+\rho\Box\sigma\right)\left(\mathcal{L}_\beta\,U_i\right)\Box\lambda^i \\ &\quad+ \int\sqrt{g}\,\mathrm{d}^4x\,\left(-\Box\sigma\Box\rho\right)U_i\beta^i\end{aligned} \tag{A.94k}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\, \partial_\mu\sigma \tfrac{1}{2}\partial^\mu \left(V_{ij}\Lambda^{ij}\right) = \int\!\sqrt{g}\,\mathrm{d}^4x\ (+\rho\Box\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\, V_{ij}\Lambda^{ij} + \int\!\sqrt{g}\,\mathrm{d}^4x\ (+\partial_\mu\rho\Box\sigma)\, V_{ij}\beta^i\partial^\mu\lambda^j \tag{A.94l}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\, \partial_\mu\sigma \tfrac{1}{2}\partial^\mu \left(\hat{V}_{ij}\hat{\Lambda}^{ij}\right) = \int\!\sqrt{g}\,\mathrm{d}^4x\ (+\rho\Box\sigma)\,\tfrac{1}{2}\mathcal{L}_{\hat\beta}\, \hat{V}_{ij}\hat{\Lambda}^{ij} + \int\!\sqrt{g}\,\mathrm{d}^4x\ (+\partial_\mu\rho\Box\sigma)\, \hat{V}_{ij}\hat{\beta}^i\partial^\mu\hat{\theta}^j \tag{A.94m}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\, \partial_\mu\sigma \tfrac{1}{3}Y_i\partial^\mu\lambda^i R = \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\rho\partial_\mu\sigma)\,\tfrac{1}{3}\mathcal{L}_\beta\, Y_i\partial^\mu\lambda^i R + \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\partial_\mu\sigma\partial^\mu\rho)\,\tfrac{1}{3}Y_i\beta^i R \tag{A.94n}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\, \partial_\mu\sigma S_{ij}\partial^\mu\lambda^i\Box\lambda^j = &\int\!\sqrt{g}\,\mathrm{d}^4x\ (-\rho\partial_\mu\sigma)\,\mathcal{L}_\beta\, S_{ij}\partial^\mu\lambda^i\Box\lambda^j\\ &+ \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\partial_\mu\sigma\partial^\mu\rho)\, S_{ij}\beta^i\Box\lambda^j\\ &+ \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\partial_\mu\sigma\Box\rho)\, S_{ij}\partial^\mu\lambda^i\beta^j\end{aligned} \tag{A.94o}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\, \partial_\mu\sigma \hat{S}_{ij}\partial^\mu\hat{\theta}^i\Box\hat{\theta}^j = &\int\!\sqrt{g}\,\mathrm{d}^4x\ (-\rho\partial_\mu\sigma)\,\mathcal{L}_{\hat\beta}\, \hat{S}_{ij}\partial^\mu\hat{\theta}^i\Box\hat{\theta}^j\\ &+ \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\partial_\mu\sigma\partial^\mu\rho)\, \hat{S}_{ij}\hat{\beta}^i\Box\hat{\theta}^j\\ &+ \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\partial_\mu\sigma\Box\rho)\, \hat{S}_{ij}\partial^\mu\hat{\theta}^i\hat{\beta}^j\end{aligned} \tag{A.94p}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\, \partial_\mu\sigma \tfrac{1}{2}T_{ijk}\Lambda^{ij}\partial^\mu\lambda^k = &\int\!\sqrt{g}\,\mathrm{d}^4x\ (-\rho\partial_\mu\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\, T_{ijk}\Lambda^{ij}\partial^\mu\lambda^k\\ &+ \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\partial_\nu\rho\partial_\mu\sigma)\, T_{ijk}\beta^i\partial^\nu\lambda^j\partial^\mu\lambda^k\\ &+ \int\!\sqrt{g}\,\mathrm{d}^4x\ (-\partial^\mu\rho\partial_\mu\sigma)\,\tfrac{1}{2}T_{ijk}\Lambda^{ij}\beta^k\end{aligned} \tag{A.95}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma\tfrac{1}{2}\hat{T}_{ij}\zeta^T_k\hat{\Lambda}^{ij}\partial^\mu\lambda^k = &\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\partial_\mu\sigma)\,\tfrac{1}{2}\left(\zeta^T_k\mathcal{L}_{\hat{\beta}}\,\hat{T}_{ij}+\hat{T}_{ij}\mathcal{L}_\beta\,\zeta^T_k\right)\hat{\Lambda}^{ij}\partial^\mu\lambda^k\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\nu\rho\partial_\mu\sigma)\,\hat{T}_{ij}\zeta^T_k\hat{\beta}^i\partial^\nu\hat{\theta}^j\partial^\mu\lambda^k\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial^\mu\rho\partial_\mu\sigma)\,\tfrac{1}{2}\hat{T}_{ij}\zeta^T_k\hat{\Lambda}^{ij}\beta^k\end{aligned}\tag{A.96}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma\tfrac{1}{2}\zeta^T_i\hat{T}_{jk}\partial_\nu\lambda^i\partial^\nu\hat{\theta}^j\partial^\mu\hat{\theta}^k = &\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\partial_\mu\sigma)\,\tfrac{1}{2}\Big(\zeta^T_i\mathcal{L}_{\hat{\beta}}\,\hat{T}_{jk}\\ &\qquad+\mathcal{L}_\beta\,\zeta^T_i\hat{T}_{jk}\Big)\,\partial_\nu\lambda^i\partial^\nu\hat{\theta}^j\partial^\mu\hat{\theta}^k\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\nu\rho\partial_\mu\sigma)\,\tfrac{1}{2}\zeta^T_i\hat{T}_{jk}\Big(\beta^i\partial^\nu\hat{\theta}^j\\ &\qquad+\partial^\nu\lambda^i\hat{\beta}^j\Big)\,\partial^\mu\hat{\theta}^k\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial^\mu\rho\partial_\mu\sigma)\,\tfrac{1}{2}\zeta^T_i\hat{T}_{jk}\partial_\nu\lambda^i\partial^\nu\hat{\theta}^j\hat{\beta}^k\end{aligned}\tag{A.97}$$

The terms involving R-vectors are

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{4}\beta^{\mathscr{V}}V^{\mu\nu}V_{\mu\nu} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{4}\mathcal{L}_\beta\,\beta^{\mathscr{V}}V^{\mu\nu}V_{\mu\nu}\tag{A.98}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}v^b\,(\nabla\cdot V)\,V^2 = &\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,v^b\,(\nabla\cdot V)\,V^2\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\Box\rho)\,\tfrac{1}{2}v^bV^2\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,v^b\,(\nabla\cdot V)\,V^\mu\end{aligned}\tag{A.99}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{4}v^bV^4 = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{4}\mathcal{L}_\beta\,v^bV^4+\int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,v^bV^\mu V^2\tag{A.100}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^a_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j V^{\mu\nu} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\sigma\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^a_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^j V^{\mu\nu} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\kappa^a_{ij}\beta^i\partial_\nu\lambda^j V^{\mu\nu} \tag{A.101}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat\kappa^a_{ij}\partial_\mu\hat\theta^i\partial_\nu\hat\theta^j V^{\mu\nu} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\sigma\tfrac{1}{2}\mathcal{L}_{\hat\beta}\,\hat\kappa^a_{ij}\partial_\mu\hat\theta^i\partial_\nu\hat\theta^j V^{\mu\nu} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\hat\kappa^a_{ij}\hat\beta^i\partial_\nu\hat\theta^j V^{\mu\nu} \tag{A.102}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{3}\kappa^b_i\partial_\mu\lambda^i\nabla_\nu V^{\mu\nu} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{3}\mathcal{L}_\beta\,\kappa^b_i\partial_\mu\lambda^i\nabla_\nu V^{\mu\nu} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\tfrac{1}{3}\kappa^b_i\beta^i\nabla_\nu V^{\mu\nu} \tag{A.103}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{3}\kappa^c_i\partial_\mu\lambda^i V_\nu V^{\mu\nu} = \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{3}\mathcal{L}_\beta\,\kappa^c_i\partial_\mu\lambda^i V_\nu V^{\mu\nu} + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\tfrac{1}{3}\kappa^c_i\beta^i V_\nu V^{\mu\nu} \tag{A.104}$$

The coefficients $\kappa^d_{ijk}(\lambda)$, $\hat\kappa^d_{ij}(\hat\theta)$ are symmetric in ij.

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^d_{ijk}\Lambda^{ij}\left(\partial\lambda^k\cdot V\right) = &\int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{2}\kappa^d_{ijk}\Lambda^{ij}\left(\partial^\mu\lambda^k - \beta^k V^\mu\right)\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^d_{ijk}\Lambda^{ij}\left(\partial\lambda^k\cdot V\right)\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\kappa^d_{ijk}\beta^i\partial^\mu\lambda^j\left(\partial\lambda^k\cdot V\right)\end{aligned} \tag{A.105}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\kappa}^d_{ij}\zeta^d_k\hat{\Lambda}^{ij}\left(\partial\lambda^k\cdot V\right) = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{2}\hat{\kappa}^d_{ij}\zeta^d_k\hat{\Lambda}^{ij}\left(\partial^\mu\lambda^k-\beta^kV^\mu\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\left(\mathcal{L}_{\hat{\beta}}\,\hat{\kappa}^d_{ij}\zeta^d_k+\hat{\kappa}^d_{ij}\mathcal{L}_\beta\,\zeta^d_k\right)\hat{\Lambda}^{ij}\left(\partial\lambda^k\cdot V\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\hat{\kappa}^d_{ij}\zeta^d_k\hat{\beta}^i\partial^\mu\hat{\theta}^j\left(\partial\lambda^k\cdot V\right) \tag{A.106}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\zeta^d_i\hat{\kappa}^d_{jk}\left(\partial\lambda^i\cdot\partial\hat{\theta}^j\right)\left(\partial\hat{\theta}^k\cdot V\right) = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{2}\zeta^d_i\hat{\kappa}^d_{jk}\left(\partial\lambda^i\cdot\partial\hat{\theta}^j\right)\left(\partial^\mu\hat{\theta}^k-\hat{\beta}^kV^\mu\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\left(\zeta^d_i\mathcal{L}_{\hat{\beta}}\,\hat{\kappa}^d_{jk}+\hat{\kappa}^d_{jk}\mathcal{L}_\beta\,\zeta^d_i\right)\left(\partial\lambda^i\cdot\partial\hat{\theta}^j\right)\left(\partial\hat{\theta}^k\cdot V\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\partial_\mu\rho)\,\tfrac{1}{2}\zeta^d_i\hat{\kappa}^d_{jk}\left(\beta^i\partial^\mu\hat{\theta}^j+\partial^\mu\lambda^i\hat{\beta}^j\right)\left(\partial\lambda^k\cdot V\right) \tag{A.107}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^e_{ij}\Lambda^{ij}V^2 = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\kappa^e_{ij}\left(\Lambda^{ij}V^\mu-\beta^i\partial^\mu\lambda^j\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^e_{ij}\Lambda^{ij}V^2 \tag{A.108}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\hat{\kappa}^e_{ij}\hat{\Lambda}^{ij}V^2 = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\hat{\kappa}^e_{ij}\left(\hat{\Lambda}^{ij}V^\mu-\hat{\beta}^i\partial^\mu\hat{\theta}^j\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_{\hat{\beta}}\,\hat{\kappa}^e_{ij}\hat{\Lambda}^{ij}V^2 \tag{A.109}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa^f_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^jV^\mu V^\nu = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\kappa^f_{ij}\left(\partial_\mu\lambda^i-\beta^iV^\mu\right)\partial_\nu\lambda^jV^\nu + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^f_{ij}\partial_\mu\lambda^i\partial_\nu\lambda^jV^\mu V^\nu \tag{A.110}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}\hat{\kappa}^f_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j V^\mu V^\nu = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\hat{\kappa}^f_{ij}\left(\partial_\mu\hat{\theta}^i-\hat{\beta}^i V^\mu\right)\partial_\nu\hat{\theta}^j V^\nu \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_{\hat{\beta}}\,\hat{\kappa}^f_{ij}\partial_\mu\hat{\theta}^i\partial_\nu\hat{\theta}^j V^\mu V^\nu \tag{A.111}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\kappa^g_i\left(\partial\lambda^i\cdot V\right)V^2 = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\kappa^g_i\left(\partial^\mu\lambda^i V^2+2\left(\partial\lambda^i\cdot V\right)V^\mu-\beta^i V^\mu V^2\right) \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\mathcal{L}_\beta\,\kappa^g_i\left(\partial\lambda^i\cdot V\right)V^2 \tag{A.112}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}\kappa^h_i\left(\partial\lambda^i\cdot\partial\right)V^2 = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\kappa^h_i\left(\partial_\nu\lambda^i\partial^\nu V^\mu-\tfrac{1}{2}\beta^i\partial^\mu V^2\right) \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\nabla_\mu\nabla_\nu\rho)\,\kappa^h_i\partial^\mu\lambda^i V^\nu \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^h_i\left(\partial\lambda^i\cdot\partial\right)V^2 \tag{A.113}$$

Now the variation of the terms in $\mathcal{D}$.

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}\kappa^m_{ij}\left(\partial\lambda^i\cdot V\right)\Box\lambda^j = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{2}\kappa^m_{ij}\left(\partial^\mu\lambda^i-\beta^i V^\mu\right)\Box\lambda^j \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^m_{ij}\left(\partial\lambda^i\cdot V\right)\Box\lambda^j \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\rho)\,\tfrac{1}{2}\kappa^m_{ij}\left(\partial\lambda^i\cdot V\right)\beta^j \tag{A.114}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}\hat{\kappa}^m_{ij}\left(\partial\hat{\theta}^i\cdot V\right)\Box\hat{\theta}^j = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{2}\hat{\kappa}^m_{ij}\left(\partial^\mu\hat{\theta}^i-\hat{\beta}^i V^\mu\right)\Box\hat{\theta}^j \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_{\hat{\beta}}\,\hat{\kappa}^m_{ij}\left(\partial\hat{\theta}^i\cdot V\right)\Box\hat{\theta}^j \\ +\int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\rho)\,\tfrac{1}{2}\hat{\kappa}^m_{ij}\left(\partial\hat{\theta}^i\cdot V\right)\hat{\beta}^j \tag{A.115}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{3}\kappa_i^n\left(\partial\lambda^i\cdot V\right)R = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{3}\kappa_i^n\left(\partial^\mu\lambda^i-\beta^iV^\mu\right)R + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{3}\mathcal{L}_\beta\,\kappa_i^n\left(\partial\lambda^i\cdot V\right)R \tag{A.116}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma v^c\,(\nabla\cdot V)\,R = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\Box\rho)\,v^cR + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\mathcal{L}_\beta\,v^c\,(\nabla\cdot V)\,R \tag{A.117}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa_i^o\Box\lambda^iV^2 = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\kappa_i^o\Box\lambda^iV^\mu + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa_i^o\Box\lambda^iV^2 + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\rho)\,\tfrac{1}{2}\kappa_i^o\beta^iV^2 \tag{A.118}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}v^d\Box V^2 = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,v^d\Box V^\mu + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\mu\sigma\Box\rho)\,v^dV^\mu + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\sigma\Box\rho)\,\partial_\mu\left(v^dV^\mu\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,v^d\Box V^2 \tag{A.119}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{6}v^eV^2R = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{3}v^eV^\mu R + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{6}\mathcal{L}_\beta\,v^eV^2R \tag{A.120}$$

$$\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\sigma\tfrac{1}{2}\kappa_i^p\Box\lambda^i\,(\nabla\cdot V) = \int\!\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\Box\rho)\,\tfrac{1}{2}\kappa_i^p\left(\Box\lambda^i-\beta^i\,(\nabla\cdot V)\right) + \int\!\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa_i^p\Box\lambda^i\,(\nabla\cdot V) \tag{A.121}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}\kappa^q_i\partial_\mu\lambda^i V_\nu G^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,\tfrac{1}{2}\kappa^q_i\left(\partial_\nu\lambda^i - \beta^i V_\nu\right)G^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^q_i\partial_\mu\lambda^i V_\nu G^{\mu\nu} \tag{A.122}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\sigma\tfrac{1}{2}v^f V_\mu V_\nu G^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\;(+\sigma\partial_\mu\rho)\,v^f V_\nu G^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\;(-\rho\sigma)\,\tfrac{1}{2}\mathcal{L}_\beta\,v^f V_\mu V_\nu G^{\mu\nu} \tag{A.123}$$

Now the variation of the terms in the current $\mathcal{Y}^\mu$.

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma\kappa^r_i\partial_\nu\lambda^i V^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\;(-\rho\partial_\mu\sigma)\,\mathcal{L}_\beta\,\kappa^r_i\partial_\nu\lambda^i V^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\;(-\partial_\nu\rho\partial_\mu\sigma)\,\kappa^r_i\beta^i V^{\mu\nu} \tag{A.124}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma v^g\nabla_\nu V^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\;(-\rho\partial_\mu\sigma)\,\mathcal{L}_\beta\,v^g\nabla_\nu V^{\mu\nu} \tag{A.125}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma v^h V_\nu V^{\mu\nu} = \int\sqrt{g}\,\mathrm{d}^4x\;(+\partial_\nu\rho\partial_\mu\sigma)\,v^h V^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\;(-\rho\partial_\mu\sigma)\,\mathcal{L}_\beta\,v^h V_\nu V^{\mu\nu} \tag{A.126}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma v^m V^\mu V^2 = \int\sqrt{g}\,\mathrm{d}^4x\;(+\partial^\mu\rho\partial_\mu\sigma)\,v^m V^2 + \int\sqrt{g}\,\mathrm{d}^4x\;(+\partial_\nu\rho\partial_\mu\sigma)\,2v^m V^\mu V^\nu + \int\sqrt{g}\,\mathrm{d}^4x\;(-\rho\partial_\mu\sigma)\,\mathcal{L}_\beta\,v^m V^\mu V^2 \tag{A.127}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}\kappa^s_i\partial^\mu\lambda^i V^2 = &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(+\partial_\nu\rho\partial_\mu\sigma\right)\kappa^s_i\partial^\mu\lambda^i V^\nu\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^s_i\partial^\mu\lambda^i V^2\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial^\mu\rho\partial_\mu\sigma\right)\tfrac{1}{2}\kappa^s_i\beta^i V^2\end{aligned} \tag{A.128}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\kappa^t_i\left(\partial\lambda^i\cdot V\right)V^\mu = &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(+\partial^\mu\rho\partial_\mu\sigma\right)\kappa^t_i\left(\partial\lambda^i\cdot V\right)\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(+\partial_\nu\rho\partial_\mu\sigma\right)\kappa^t_i\left(\partial^\nu\lambda^i-\beta^i V^\nu\right)V^\mu\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\mathcal{L}_\beta\,\kappa^t_i\left(\partial\lambda^i\cdot V\right)V^\mu\end{aligned} \tag{A.129}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}\kappa^u_i\Box\lambda^i V^\mu = &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(+\partial^\mu\rho\partial_\mu\sigma\right)\tfrac{1}{2}\kappa^u_i\Box\lambda^i\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^u_i\Box\lambda^i V^\mu\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\mu\sigma\Box\rho\right)\tfrac{1}{2}\kappa^u_i\beta^i V^\mu\end{aligned} \tag{A.130}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\tfrac{1}{2}\kappa^v_i\partial^\mu\lambda^i\left(\nabla\cdot V\right) = &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(+\partial_\mu\sigma\Box\rho\right)\tfrac{1}{2}\kappa^v_i\partial^\mu\lambda^i\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\tfrac{1}{2}\mathcal{L}_\beta\,\kappa^v_i\partial^\mu\lambda^i\left(\nabla\cdot V\right)\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial^\mu\rho\partial_\mu\sigma\right)\tfrac{1}{2}\kappa^v_i\beta^i\left(\nabla\cdot V\right)\end{aligned} \tag{A.131}$$

$$\begin{aligned}\Delta_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\sigma\kappa^w_{ij}\Lambda^{ij}V^\mu = &\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(+\partial^\mu\rho\partial_\mu\sigma\right)\kappa^w_{ij}\Lambda^{ij}+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\mu\sigma\right)\mathcal{L}_\beta\,\kappa^w_{ij}\Lambda^{ij}V^\mu\\ &+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\nu\rho\partial_\mu\sigma\right)2\kappa^w_{ij}\beta^i\partial^\nu\lambda^j V^\mu\end{aligned} \tag{A.132}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma\hat{\kappa}^w_{ij}\hat{\Lambda}^{ij}V^\mu = \int\sqrt{g}\,\mathrm{d}^4x\;\left(+\partial^\mu\rho\partial_\mu\sigma\right)\hat{\kappa}^w_{ij}\hat{\Lambda}^{ij} + \int\sqrt{g}\,\mathrm{d}^4x\;\left(-\rho\partial_\mu\sigma\right)\mathcal{L}_{\hat{\beta}}\,\hat{\kappa}^w_{ij}\hat{\Lambda}^{ij}V^\mu$$
$$+\int\sqrt{g}\,\mathrm{d}^4x\;\left(-\partial_\nu\rho\partial_\mu\sigma\right)2\hat{\kappa}^w_{ij}\hat{\beta}^i\partial^\nu\hat{\theta}^jV^\mu \tag{A.133}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma\kappa^y_{ij}\partial^\mu\lambda^i\left(\partial\lambda^j\cdot V\right) = \int\sqrt{g}\,\mathrm{d}^4x\;\left(+\partial_\nu\rho\partial_\mu\sigma\right)\kappa^y_{ij}\partial^\mu\lambda^i\left(\partial^\nu\lambda^j-\beta^jV^\nu\right)$$
$$+\int\sqrt{g}\,\mathrm{d}^4x\;\left(-\rho\partial_\mu\sigma\right)\mathcal{L}_\beta\,\kappa^y_{ij}\partial^\mu\lambda^i\left(\partial\lambda^j\cdot V\right)$$
$$+\int\sqrt{g}\,\mathrm{d}^4x\;\left(-\partial^\mu\rho\partial_\mu\sigma\right)\kappa^y_{ij}\beta^i\left(\partial\lambda^j\cdot V\right) \tag{A.134}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma\hat{\kappa}^y_{ij}\partial^\mu\hat{\theta}^i\left(\partial\hat{\theta}^j\cdot V\right) = \int\sqrt{g}\,\mathrm{d}^4x\;\left(+\partial_\nu\rho\partial_\mu\sigma\right)\hat{\kappa}^y_{ij}\partial^\mu\hat{\theta}^i\left(\partial^\nu\hat{\theta}^j-\hat{\beta}^jV^\nu\right)$$
$$+\int\sqrt{g}\,\mathrm{d}^4x\;\left(-\rho\partial_\mu\sigma\right)\mathcal{L}_{\hat{\beta}}\,\hat{\kappa}^y_{ij}\partial^\mu\hat{\theta}^i\left(\partial\hat{\theta}^j\cdot V\right)$$
$$+\int\sqrt{g}\,\mathrm{d}^4x\;\left(-\partial^\mu\rho\partial_\mu\sigma\right)\hat{\kappa}^y_{ij}\hat{\beta}^i\left(\partial\hat{\theta}^j\cdot V\right) \tag{A.135}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma v^nRV^\mu = \int\sqrt{g}\,\mathrm{d}^4x\;\left(+\partial^\mu\rho\partial_\mu\sigma\right)v^nR + \int\sqrt{g}\,\mathrm{d}^4x\;\left(-\rho\partial_\mu\sigma\right)\mathcal{L}_\beta\,v^nRV^\mu \tag{A.136}$$

$$\Delta_\rho \int\sqrt{g}\,\mathrm{d}^4x\;\partial_\mu\sigma\tfrac{1}{2}v^oV_\nu G^{\mu\nu} \tag{A.137}$$
$$= \int\sqrt{g}\,\mathrm{d}^4x\;\left(+\partial_\nu\rho\partial_\mu\sigma\right)\tfrac{1}{2}v^oG^{\mu\nu} + \int\sqrt{g}\,\mathrm{d}^4x\;\left(-\rho\partial_\mu\sigma\right)\tfrac{1}{2}\mathcal{L}_\beta\,v^oV_\nu G^{\mu\nu}$$

A.3 Weyl Variation of the R-Anomalies

The infinitesimal Weyl transformations of the external fields are

$$\delta_\sigma \lambda^i = \sigma \beta^i\,, \quad \delta_\sigma \hat{\theta}^i = \sigma \hat{\beta}^i\,, \quad \delta_\sigma g^{\mu\nu} = +2\sigma g^{\mu\nu}\,, \quad \delta_\sigma V_\mu = 0\,. \tag{A.138}$$

The derivative operator is

$$\Delta_\sigma = \int \mathrm{d}^4 y\, \sigma \left(2 \frac{\delta}{\delta g^{\mu\nu}(y)} g^{\mu\nu} - \beta^i \frac{\delta}{\delta \lambda^i(y)} \right)\,. \tag{A.139}$$

The infinitesimal Weyl variation of the Riemann tensor can be found in (B.9).

The variations of the anomalies in (6.3a) are listed. I leave in the Lie derivatives $\mathcal{L}_{\hat{\beta}}$ with respect to the theta coupling vector fields.

$$\begin{aligned} \Delta_\sigma \int \sqrt{g}\, \mathrm{d}^4 x \left(-\rho \hat{\beta}^{\mathscr{R}} R^{\mu}{}_{\nu\alpha\beta}\ {}^*R_{\mu}{}^{\nu\alpha\beta} \right) = & \int \sqrt{g}\, \mathrm{d}^4 x\ (-\rho)\,(\text{to be done}) \\ & + \int \sqrt{g}\, \mathrm{d}^4 x\ (\rho\sigma)\, \mathcal{L}_{\hat{\beta}}\, \hat{\beta}^{\mathscr{R}} R^{\mu}{}_{\nu\alpha\beta}\ {}^*R_{\mu}{}^{\nu\alpha\beta} \end{aligned} \tag{A.140}$$

$$\Delta_\sigma \int \sqrt{g}\, \mathrm{d}^4 x \left(-\rho \tfrac{1}{2} \hat{\beta}^{\mathscr{V}} V_{\mu\nu}\, {}^*V^{\mu\nu} \right) = \int \sqrt{g}\, \mathrm{d}^4 x\ (\rho\sigma)\, \tfrac{1}{2} \mathcal{L}_{\hat{\beta}}\, \hat{\beta}^{\mathscr{V}} V_{\mu\nu}\, {}^*V^{\mu\nu} \tag{A.141}$$

$$\begin{aligned} \Delta_\sigma \int \sqrt{g}\, \mathrm{d}^4 x \left(-\rho a_i \hat{a}_j \partial_\mu \lambda^i \partial_\nu \hat{\theta}^j\, {}^*V^{\mu\nu} \right) = & \int \sqrt{g}\, \mathrm{d}^4 x\ (\rho\sigma) \left(\mathcal{L}_{\beta}\, a_i \hat{a}_j + a_i \mathcal{L}_{\hat{\beta}}\, \hat{a}_j \right) \partial_\mu \lambda^i \partial_\nu \hat{\theta}^j\, {}^*V^{\mu\nu} \\ & + \int \sqrt{g}\, \mathrm{d}^4 x\ (\rho \partial_\mu \sigma)\, a_i \hat{a}_j \left(\beta^i \partial_\nu \hat{\theta}^j - \partial_\nu \lambda^i \hat{\beta}^j \right) {}^*V^{\mu\nu} \end{aligned} \tag{A.142}$$

$$\Delta_\sigma \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\hat{b}_i\partial_\mu\hat{\theta}^i V_\nu\,{}^*V^{\mu\nu}\right) = \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\sigma\right)\mathcal{L}_{\hat{\beta}}\,\hat{b}_i\partial_\mu\hat{\theta}^i V_\nu\,{}^*V^{\mu\nu} + \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\partial_\mu\sigma\right)\hat{b}_i\hat{\beta}^i V_\nu\,{}^*V^{\mu\nu} \tag{A.143}$$

$$\Delta_\sigma \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\hat{c}_i\partial_\mu\hat{\theta}^i \nabla_\nu\,{}^*V^{\mu\nu}\right) = \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\sigma\right)\mathcal{L}_{\hat{\beta}}\,\hat{c}_i\partial_\mu\hat{\theta}^i \nabla_\nu\,{}^*V^{\mu\nu} + \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\partial_\mu\sigma\right)\hat{c}_i\hat{\beta}^i \nabla_\nu\,{}^*V^{\mu\nu} \tag{A.144}$$

$$\Delta_\sigma \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\rho\hat{d}_i\partial_\nu\hat{\theta}^i\,{}^*V^{\mu\nu} = \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\rho\right)\mathcal{L}_{\hat{\beta}}\,\hat{d}_i\partial_\nu\hat{\theta}^i\,{}^*V^{\mu\nu} + \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\nu\sigma\partial_\mu\rho\right)\hat{d}_i\hat{\beta}^i\,{}^*V^{\mu\nu} \tag{A.145}$$

$$\Delta_\sigma \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\rho\left(\partial_\nu\hat{f}\right)\,{}^*V^{\mu\nu} = \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\rho\right)\partial_\nu\left(\mathcal{L}_{\hat{\beta}}\,\hat{f}\right)\,{}^*V^{\mu\nu} + \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\nu\sigma\partial_\mu\rho\right)\mathcal{L}_{\hat{\beta}}\,\hat{f}\,{}^*V^{\mu\nu} \tag{A.146}$$

$$\Delta_\sigma \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\partial_\mu\rho\hat{e}V_\nu\,{}^*V^{\mu\nu} = \int\!\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\sigma\partial_\mu\rho\right)\mathcal{L}_{\hat{\beta}}\,\hat{e}V_\nu\,{}^*V^{\mu\nu} \tag{A.147}$$

A.4 R-Variation of the Psi Monomials

In this section I write down the variation of the tensor structure (6.50) discussed in section 6.3. The infinitesimal variation is

$$\delta_\rho \Psi^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} t_\alpha \partial_\beta \rho \,. \tag{A.148}$$

The derivative operator is given by

$$\Delta'_\rho = \int \mathrm{d}^4 y \left(\partial_\mu \rho' \frac{\delta}{\delta V_\mu(y)} - \rho' \hat{\beta}^i \frac{\delta}{\delta \hat{\theta}^i(y)} \right) . \tag{A.149}$$

The infinitesimal R-transformations of the external fields are

$$\delta_\rho \lambda^i = 0\,, \quad \delta_\rho \hat{\theta}^i = \rho \hat{\beta}^i\,, \quad \delta_\rho g^{\mu\nu} = 0\,, \quad \delta_\rho V_\mu = \partial_\mu \rho\,. \tag{A.150}$$

I write down the local anomalies that are to be added to the R-symmetry Ward-identity (6.3a) in the naming scheme defined by

$$\mathcal{E}_\Psi = \mathcal{E} + \left(\tfrac{1}{2} a V_{\mu\nu} + \xi_i^d \hat{\xi}_j^d \partial_\mu \lambda^i \partial_\nu \hat{\theta}^j + \hat{\xi}_i^a \partial_\mu \hat{\theta}^i V_\nu \right) \Psi^{\mu\nu} + \hat{\xi}_i^b \partial_\mu \hat{\theta}^i \nabla_\nu \Psi^{\mu\nu}\,, \tag{A.151}$$

$$\mathcal{J}_\Psi^\mu = \mathcal{J}^\mu + \left(\hat{\xi}_i^c \partial_\nu \hat{\theta}^i + c V_\nu + \partial_\nu b \right) \Psi^{\mu\nu} \,. \tag{A.152}$$

I leave out the terms with symmetric combinations of rho parameters. In this section,

$\cong$ means "equal up to terms with vanishing rho commutator."

The variations of the individual terms are

$$\Delta'_\rho \int \sqrt{g}\, \mathrm{d}^4 x \, \left(-\rho \tfrac{1}{2} a V_{\mu\nu} \Psi^{\mu\nu} \right) \cong \int \sqrt{g}\, \mathrm{d}^4 x \, \left(-\rho \partial_\beta \rho' \right) t_\alpha a \, {}^*V^{\mu\nu} \tag{A.153}$$

$$\Delta'_{\rho}\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\xi_i^d\hat{\xi}_j^d\partial_\mu\lambda^i\partial_\nu\hat{\theta}^j\Psi^{\mu\nu}\right)=\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\beta\rho'\right)t_\alpha\epsilon^{\mu\nu\alpha\beta}\xi_i^d\hat{\xi}_j^d\partial_\mu\lambda^i\partial_\nu\hat{\theta}^j$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,(\rho\rho')\left(\mathcal{L}_\beta\,\xi_i^d\hat{\xi}_j^d+\xi_i^d\mathcal{L}_{\hat{\beta}}\,\hat{\xi}_j^d\right)\partial_\mu\lambda^i\partial_\nu\hat{\theta}^j\Psi^{\mu\nu}$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\partial_\mu\rho'\right)\xi_i^d\hat{\xi}_j^d\left(\beta^i\partial_\nu\hat{\theta}^j-\partial_\nu\lambda^i\hat{\beta}^j\right)\Psi^{\mu\nu} \tag{A.154a}$$

$$\Delta'_{\rho}\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\hat{\xi}_i^a\partial_\mu\hat{\theta}^iV_\nu\Psi^{\mu\nu}\right)=\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\nu\rho'\right)\hat{\xi}_i^a\partial_\mu\hat{\theta}^i\Psi^{\mu\nu} \tag{A.154b}$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\partial_\beta\rho'\right)t_\alpha\epsilon^{\mu\nu\alpha\beta}\hat{\xi}_i^a\partial_\mu\hat{\theta}^iV_\nu$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\rho'\right)\mathcal{L}_\beta\,\hat{\xi}_i^a\partial_\mu\hat{\theta}^iV_\nu\Psi^{\mu\nu}$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\partial_\mu\rho'\right)\hat{\xi}_i^a\hat{\beta}^iV_\nu\Psi^{\mu\nu}$$

$$\Delta'_{\rho}\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\hat{\xi}_i^b\partial_\mu\hat{\theta}^i\nabla_\nu\Psi^{\mu\nu}\right)=\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho\nabla_\nu\nabla_\alpha\nabla_\beta\rho'\right)\epsilon^{\mu\nu\alpha\beta}\hat{\xi}_i^b\partial_\mu\hat{\theta}^i \tag{A.154c}$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\rho'\right)\mathcal{L}_{\hat{\beta}}\,\hat{\xi}_i^b\partial_\mu\hat{\theta}^i\nabla_\nu\Psi^{\mu\nu}$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\rho\partial_\mu\rho'\right)\hat{\xi}_i^b\hat{\beta}^i\nabla_\nu\Psi^{\mu\nu}$$

$$\Delta'_{\rho}\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\partial_\mu\rho\hat{\xi}_i^c\partial_\nu\hat{\theta}^i\Psi^{\mu\nu}\right)=\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(\partial_\mu\rho\partial_\beta\rho'\right)t_\alpha\epsilon^{\mu\nu\alpha\beta}\hat{\xi}_i^c\partial_\nu\hat{\theta}^i \tag{A.154d}$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\rho'\partial_\mu\rho\right)\mathcal{L}_{\hat{\beta}}\,\hat{\xi}_i^c\partial_\nu\hat{\theta}^i\Psi^{\mu\nu}$$
$$+\int\!\sqrt{g}\,\mathrm{d}^4x\,\left(-\partial_\nu\rho'\partial_\mu\rho\right)\hat{\xi}_i^c\hat{\beta}^i\Psi^{\mu\nu}$$

$$
\begin{aligned}
{\Delta'}_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\partial_\mu\rho\,(\partial_\nu b)\,\Psi^{\mu\nu}\right) &= \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\partial_\mu\rho\partial_\beta\rho'\right) t_\alpha\epsilon^{\mu\nu\alpha\beta}\partial_\nu b \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(-\partial_\nu\rho'\partial_\mu\rho\right)\mathcal{L}_{\hat\beta}\,b\Psi^{\mu\nu} \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(-\rho'\partial_\mu\rho\right)\partial_\nu\left(\mathcal{L}_{\hat\beta}\,b\right)\Psi^{\mu\nu} \\
&= \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\partial_\mu\rho\partial_\beta\rho'\right) t_\alpha\epsilon^{\mu\nu\alpha\beta}\partial_\nu b \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(+\rho'\nabla_\nu\nabla_\mu\rho\right)\mathcal{L}_{\hat\beta}\,b\Psi^{\mu\nu} \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(+\rho'\partial_\mu\rho\right)\mathcal{L}_{\hat\beta}\,b\nabla_\nu\Psi^{\mu\nu}
\end{aligned}
\tag{A.154e}
$$

$$
\begin{aligned}
{\Delta'}_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\ \partial_\mu\rho\,(\partial_\nu b)\,\Psi^{\mu\nu} &\cong \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(-\partial_\nu\rho'\partial_\mu\rho\right)\mathcal{L}_{\hat\beta}\,b\,\Psi^{\mu\nu} \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(-\rho'\partial_\mu\rho\right)\partial_\nu\left(\mathcal{L}_{\hat\beta}\,b\right)\Psi^{\mu\nu} \\
&\cong \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\rho'\nabla_\nu\partial_\mu\rho\right)\mathcal{L}_{\hat\beta}\,b\,\Psi^{\mu\nu} \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\rho'\partial_\mu\rho\right)\mathcal{L}_{\hat\beta}\,b\,\nabla_\nu\Psi^{\mu\nu}
\end{aligned}
\tag{A.154f}
$$

$$
\begin{aligned}
{\Delta'}_\rho \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\partial_\mu\rho c V_\nu\Psi^{\mu\nu}\right) &= \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\partial_\mu\rho\partial_\nu\rho'\right) c\Psi^{\mu\nu} \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(\partial_\mu\rho\partial_\beta\rho'\right) t_\alpha\epsilon^{\mu\nu\alpha\beta} c V_\nu \\
&\quad + \int\!\sqrt{g}\,\mathrm{d}^4x\ \left(-\rho'\partial_\mu\rho\right)\mathcal{L}_{\hat\beta}\,c V_\nu\Psi^{\mu\nu}
\end{aligned}
\tag{A.154g}
$$

I write down the consistency condition in the naming scheme defined in equations (6.51). The commutator $\mathcal{F}'_1 = [-\rho, \partial_\beta\rho']$ multiplies the sum

$$\mathcal{S}'_1 =$$

$$\left(-\nabla_\nu\left(\hat\xi^d_i\partial_\mu\hat\theta^i\right) + \xi^b_i\hat\xi^b_j\partial_\mu\lambda^i\partial_\nu\hat\theta^j\right) t_\alpha\epsilon^{\mu\nu\alpha\beta} \tag{A.155a}$$

$$+\hat\xi^c_i\partial_\mu\hat\theta^i V_\nu t_\alpha\epsilon^{\mu\nu\alpha\beta} \tag{A.155b}$$

$$+\left(-\tfrac{1}{2}\mathcal{L}_{\hat\beta}\,\hat\xi^g + \tfrac{1}{2}\hat\xi^a\right) V_{\mu\nu} t_\alpha\epsilon^{\mu\nu\alpha\beta}\,. \tag{A.155c}$$

The commutator $\mathcal{F}_1 = [\rho, \partial_\mu \rho']$ multiplies the sum

$$\mathcal{S}_1 =$$

$$\left(\mathcal{L}_{\hat{\beta}}\,\hat{\xi}_i^e + \hat{\xi}_i^c\right) \partial_\nu \hat{\theta}^i \Psi^{\mu\nu} \tag{A.156a}$$

$$+ \left(\xi_i^b \hat{\xi}_j^b \beta^i \partial_\nu \hat{\theta}^j - \xi_i^b \hat{\xi}_j^b \hat{\beta}^j \partial_\nu \lambda^i\right) \Psi^{\mu\nu} \tag{A.156b}$$

$$+ \left(\mathcal{L}_{\hat{\beta}}\,\hat{\xi}^f + \hat{\xi}_i^c \hat{\beta}^i\right) V_\nu \Psi^{\mu\nu} \tag{A.156c}$$

$$+ \left(-\mathcal{L}_{\hat{\beta}}\,\hat{\xi}^g + \hat{\xi}_i^d \hat{\beta}^i\right) \nabla_\nu \Psi^{\mu\nu}\,. \tag{A.156d}$$

The commutator $\mathcal{F}_2 = [\partial_\nu \rho, \partial_\mu \rho']$ multiplies the sum

$$\mathcal{S}_2 =$$

$$\left(-\hat{\xi}^f + \hat{\xi}_i^e \hat{\beta}^i + \mathcal{L}_{\hat{\beta}}\,\hat{\xi}^g\right) \Psi^{\mu\nu} \tag{A.157a}$$

$$- \left(\hat{\xi}_i^d \partial_\beta \hat{\theta}^i - \hat{\xi}_i^e \partial_\beta \hat{\theta}^i - \partial_\beta \hat{\xi}^g - \hat{\xi}^f V_\beta\right) t_\alpha \epsilon^{\mu\nu\alpha\beta}\,. \tag{A.157b}$$

The solution is divided in two types of relations. The first type consists of relations that can be put back into the Ward identity. These relations are

$$\nabla_\nu \left(\hat{\xi}_i^d \partial_\mu \hat{\theta}^i\right) = \xi_i^b \hat{\xi}_j^b \partial_\mu \lambda^i \partial_\nu \hat{\theta}^j\,, \tag{A.158a}$$

$$\tfrac{1}{2}\hat{\xi}^a V_{\mu\nu} + \hat{\xi}_i^c \partial_\mu \hat{\theta}^i V_\nu = \tfrac{1}{2}\mathcal{L}_{\hat{\beta}}\,\hat{\xi}^g V_{\mu\nu}\,, \tag{A.158b}$$

which are to be put in $\mathcal{A}$. The relation

$$\hat{\xi}_i^e \partial_\nu \hat{\theta}^i = \hat{\xi}_i^d \partial_\nu \hat{\theta}^i - \partial_\nu \hat{\xi}^g - \hat{\xi}^f V_\nu \tag{A.159}$$

is put in the current $\mathcal{J}^\mu$, in order to replace $\hat{\xi}_i^c$. Finally we can use the relations

$$\mathcal{L}_{\hat{\beta}}\,\hat{\xi}^g = \hat{\xi}^f - \hat{\xi}_i^e \hat{\beta}^i\,, \tag{A.160}$$

$$\hat{\xi}_i^a = \hat{\xi}^f - \hat{\xi}_i^c \hat{\beta}^i \,. \tag{A.161}$$

The second subset of coefficient relations can not be put back into the Ward identity. These relations arise out of the variation with respect to the couplings. They contain beta functions and are nontrivial only away from the fixed points.

$$\mathcal{L}_{\hat{\beta}} \, \hat{\xi}_i^c = -\hat{\xi}_i^a \,, \tag{A.162a}$$

$$\xi_i^d \hat{\xi}_j^d \beta^i \partial_\nu \hat{\theta}^j = \xi_i^d \hat{\xi}_j^d \hat{\beta}^j \partial_\nu \lambda^i \,, \tag{A.162b}$$

$$\mathcal{L}_{\hat{\beta}} \, c = -\hat{\xi}_i^a \hat{\beta}^i \,, \tag{A.162c}$$

$$\mathcal{L}_{\hat{\beta}} \, b = \hat{\xi}_i^b \hat{\beta}^i \,. \tag{A.162d}$$

Appendix B

Miscellaneous Useful Formulas

B.1 Weyl Variation of Curvature Tensors

The first variation with parameter σ of the generic functional $F[J]$ is

$$\begin{aligned} \delta F\,[J] &= \int \sqrt{g}\,\mathrm{d}^4x \;\; \frac{\delta F}{\delta \sigma} \\ &= \int \sqrt{g}\,\mathrm{d}^4x \;\; \frac{\delta F}{\delta J}\frac{\delta J}{\delta \sigma}\,. \end{aligned} \tag{B.1}$$

In particular we define the derivative of the currents with respect to the Weyl parameter as the "small deviation" δ_σ,

$$\frac{\delta J}{\delta \sigma} = \delta_\sigma J\,, \quad \bar{J} \approx J + \delta_\sigma J\,, \tag{B.2}$$

which amounts to

$$\delta F\,[J] = \int \sqrt{g}\,\mathrm{d}^4x \;\; \frac{\delta F}{\delta J}\delta_\sigma J\,. \tag{B.3}$$

The Christoffel Symbol is given by

$$\Gamma^\kappa{}_{\mu\nu} = \tfrac{1}{2} g^{\kappa\lambda} \left(g_{\lambda\nu,\mu} \; + g_{\lambda\mu,\nu} \; - g_{\mu\nu,\lambda} \; \right)\,. \tag{B.4}$$

The coordinates of the Riemann tensor are

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\beta,\alpha} - \Gamma^{\mu}{}_{\nu\alpha,\beta} + \Gamma^{\mu}{}_{\alpha\gamma}\Gamma^{\gamma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\gamma}{}_{\nu\alpha} \ . \tag{B.5}$$

The dual Riemann tensor is given by

$$^{*}R_{\mu}{}^{\nu\alpha\beta} = g_{\mu\gamma}\, g^{\nu\varepsilon}\, \epsilon^{\kappa\lambda\alpha\beta}\, R^{\gamma}{}_{\varepsilon\kappa\lambda} \ . \tag{B.6}$$

The coordinates of the Ricci tensor are

$$R_{\nu\beta} = R^{\mu}{}_{\nu\mu\beta} = \Gamma^{\mu}{}_{\nu\beta,\mu} - \Gamma^{\mu}{}_{\nu\mu,\beta} + \Gamma^{\mu}{}_{\mu\gamma}\Gamma^{\gamma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\gamma}{}_{\nu\mu} \ . \tag{B.7}$$

I used the textbook [69]. The Christoffel Symbol is varied as

$$\delta_{\sigma}\Gamma^{\mu}{}_{\alpha\beta} = -\delta^{\mu}_{\beta}\,\sigma_{,\alpha} - \delta^{\mu}_{\alpha}\,\sigma_{,\beta} + g_{\alpha\beta}\,\sigma^{,\mu} \ . \tag{B.8}$$

The infinitesimal variation of the Riemann tensor is

$$\delta_{\sigma}R^{\mu}{}_{\nu\alpha\beta} = \delta^{\mu}_{\beta}\sigma_{\nu\alpha} - \delta^{\mu}_{\alpha}\sigma_{\nu\beta} + g^{\mu\gamma}\left(g_{\nu\alpha}\,\sigma_{\gamma\beta} - g_{\nu\beta}\,\sigma_{\gamma\alpha}\right) + \left(\delta^{\mu}_{\beta}g_{\nu\alpha} - \delta^{\mu}_{\alpha}g_{\nu\beta}\right)\sigma_{,\varepsilon}\,\sigma^{,\varepsilon} \ , \tag{B.9}$$

$$\sigma_{\mu\nu} = \sigma_{,\mu,\nu} - \sigma_{,\mu}\,\sigma_{,\nu} \ . \tag{B.10}$$

We also need the variation of covariant derivatives of the R-vectors,

$$\nabla_{\alpha}V_{\mu} = \partial_{\alpha}V_{\mu} - \Gamma^{\kappa}{}_{\alpha\mu}V_{\kappa} \ , \tag{B.11}$$

and the antisymmetric R-field strength,

$$\nabla_{\alpha}V_{\mu\nu} = \partial_{\alpha}V_{\mu\nu} - \Gamma^{\kappa}{}_{\alpha\mu}V_{\kappa\nu} - \Gamma^{\kappa}{}_{\alpha\nu}V_{\mu\kappa} \ . \tag{B.12}$$

Both tensors have Weyl weight zero, which simplifies the calculations. With the variation (B.8) of the connection coefficients we obtain

$$\delta_\sigma \left(\nabla_\alpha V_\mu\right) = V_\mu \, \sigma_{,\alpha} + V_\alpha \, \sigma_{,\mu} - g_{\alpha\mu} \, V_\kappa \, \sigma^{,\kappa} \ , \quad g^{\alpha\mu} \, \delta_\sigma \left(\nabla_\alpha V_\mu\right) = -2\sigma_{,\kappa} \, V^\kappa \, , \quad \text{(B.13)}$$

$$\delta_\sigma \left(\nabla_\alpha V_{\mu\nu}\right) = 2V_{\mu\nu}\sigma_{,\alpha} + V_{\alpha\nu}\sigma_{,\mu} + V_{\mu\alpha}\sigma_{,\nu} - g_{\alpha\mu}V_{\kappa\nu}\sigma^{,\kappa} - g_{\alpha\nu}V_{\mu\kappa}\sigma^{,\kappa} \ , \quad \text{(B.14)}$$

$$g^{\alpha\nu}\delta_\sigma \left(\nabla_\alpha V_{\mu\nu}\right) = 0 \, . \quad \text{(B.15)}$$

We use the infinitesimal variations for the curvature tensors from Osborn [21] formula (3.6),

$$\delta_\sigma F = 4\sigma F \, , \quad \text{(B.16a)}$$

$$\delta_\sigma G = 4\sigma G - 8G^{\mu\nu}\nabla_\mu \partial_\nu \sigma \, , \quad \text{(B.16b)}$$

$$\delta_\sigma R = 2\sigma R + 6\Box\sigma \, , \quad \text{(B.16c)}$$

$$\delta_\sigma G_{\mu\nu} = 2\nabla_\mu \partial_\nu \sigma - 2g_{\mu\nu}\Box\sigma \, . \quad \text{(B.16d)}$$

B.2 Matrices and Sums

We write down some symmetry properties of matrices. We denote

$$M_{(ij)} = \tfrac{1}{2}(M_{ij} + M_{ji}) \, , \quad \text{(B.17a)}$$

$$M_{[ij]} = \tfrac{1}{2}(M_{ij} - M_{ji}) \, . \quad \text{(B.17b)}$$

The symmetric matrix S is equal to its transpose, $S = S^T$. The skew-symmetric matrix A is equal to minus its transpose, $A = -A^T$. We write a square matrix M in the form

$$M = \tfrac{1}{2}\left(M + M^T\right) + \tfrac{1}{2}\left(M - M^T\right) \, . \quad \text{(B.18)}$$

The first summand on the right hand side is symmetric, $S = \frac{1}{2}\left(M + M^T\right)$ and the second summand antisymmetric, $A = \frac{1}{2}\left(M - M^T\right)$. So we can decompose M into a symmetric and a skew symmetric matrix,

$$M = S + A\,. \tag{B.19}$$

Now we do the same decomposition for the inverse matrix M^{-1},

$$\begin{array}{ll} M^{-1} = \frac{1}{2}\left(M^{-1} + (M^{-1})^T\right) + \frac{1}{2}\left(M^{-1} - (M^{-1})^T\right)\,, \\ M^{-1} = \qquad \left(\frac{1}{S+A}\right)_S \qquad + \qquad \left(\frac{1}{S+A}\right)_A\,. \end{array} \tag{B.20}$$

In terms of S and A we have

$$\left(M^{-1}\right)^T = \frac{1}{(S+A)^T} = \frac{1}{S-A}\,. \tag{B.21}$$

Since matrix inversion and matrix transposition are interchangeable, we have

$$\left(\frac{1}{S+A}\right)_S = \frac{1}{2}\left(\frac{1}{S+A} + \frac{1}{S-A}\right)\,, \tag{B.22}$$

$$\left(\frac{1}{S+A}\right)_A = \frac{1}{2}\left(\frac{1}{S+A} - \frac{1}{S-A}\right)\,. \tag{B.23}$$

We expand these expressions,

$$\left(\frac{1}{S+A}\right)_{S,A} = \frac{1}{2}\left(\frac{1}{S+A} \pm \frac{1}{S-A}\right) \tag{B.24}$$

$$= \frac{1}{2}\frac{1}{S+A}\left((S-A) \pm (S+A)\right)\frac{1}{S-A} \tag{B.25}$$

$$= \frac{1}{S+A}\begin{pmatrix} S \\ -A \end{pmatrix}\frac{1}{S-A}\,. \tag{B.26}$$

A useful sum formula for the inverse of a sum of matrices is

$$\frac{1}{S+A} = S^{-1}\sum_{n=0}^{\infty}\left(-AS^{-1}\right)^n \tag{B.27}$$

$$= S^{-1} - S^{-1}AS^{-1} + S^{-1}AS^{-1}AS^{-1} - \ldots \tag{B.28}$$

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}\,, \quad |q| < 1\,, \tag{B.29}$$

$$\sum_{N=1}^{\infty}\frac{N!}{(N+p)\,!} = \frac{1}{p!\,p}\,. \tag{B.30}$$

B.3 The Field Strength Tensor

The Hodge dual of an antisymmetric 2-form is given in coordinates by

$$^*F_{mn} = \tfrac{1}{2}\epsilon_{mnrs}F_{rs}\,. \tag{B.31}$$

We make use of the index expressions

$$\tfrac{1}{2}F_{mn}F_{mn} = F_{45}^2 + F_{46}^2 + F_{47}^2 + F_{56}^2 + F_{57}^2 + F_{67}^2\,, \tag{B.32}$$

$$\tfrac{1}{4}\epsilon_{mnkl}F_{mn}F_{kl} = 2\left(F_{45}F_{67} - F_{46}F_{57} + F_{47}F_{56}\right)\,. \tag{B.33}$$

There are two sets of rules for calculating with antisymmetric forms in coordinates. The first one is the "increasing sum" convention. The Faraday tensor reads then

$$F = \sum_{m<n} F_{mn}\,\mathrm{d}y^m \wedge \mathrm{d}y^n\,. \tag{B.34}$$

The Hodge dual is

$$^*F = \sum_{m<n} F_{mn}\,{}^*(\mathrm{d}y^m \wedge \mathrm{d}y^n)\ . \tag{B.35}$$

The rules are:

- take the sum *only* over increasing indices,
- don't use the anti commutativity $F_{mn} = -F_{nm}$,
- don't use factors of $p!^{-1}$,
- don't use ϵ-tensors.

I call the second convention the "ϵ tensor rule." The Faraday tensor reads then

$$F = \tfrac{1}{2}F_{mn}\mathrm{d}y^m \wedge \mathrm{d}y^n\,. \tag{B.36}$$

The rules are:

- take the sum over *all* indices,
- use the anti commutativity $F_{mn} = -F_{nm}$,
- use factors of $p!^{-1}$,
- use ϵ-tensors.

Bibliography

[1] Blumenhagen, Ralph and Körs, Boris and Lüst, Dieter and Stieberger, Stephan, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," *Phys. Rept.* **445** (2007) 1–193, hep-th/0610327.

[2] M. P. Hertzberg, M. Tegmark, S. Kachru, J. Shelton, and O. Ozcan, "Searching for Inflation in Simple String Theory Models: An Astrophysical Perspective," *Phys. Rev.* **D76** (2007) 103521, 0709.0002.

[3] J. Erdmenger, N. Evans, I. Kirsch, and E. Threlfall, "Mesons in Gauge/Gravity Duals - A Review," 0711.4467.

[4] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, hep-th/9711200.

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, hep-th/9905111.

[6] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," *Nucl. Phys.* **B72** (1974) 461.

[7] Jänich, Klaus, *Topologie.* Springer, 1999. 239 p.

[8] A. Karch and E. Katz, "Adding flavor to AdS/CFT," *JHEP* **06** (2002) 043, `hep-th/0205236`.

[9] J. Erdmenger, J. Grosse, and Z. Guralnik, "Spectral flow on the Higgs branch and AdS/CFT duality," *JHEP* **06** (2005) 052, `hep-th/0502224`.

[10] G. T. Horowitz and A. Strominger, "Black strings and p-branes," *Nucl. Phys.* **B360** (1991) 197–209.

[11] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **09** (1999) 032, `hep-th/9908142`.

[12] N. Nekrasov and A. S. Schwarz, "Instantons on noncommutative R**4 and (2,0) superconformal six dimensional theory," *Commun. Math. Phys.* **198** (1998) 689–703, `hep-th/9802068`.

[13] Ammon, M. and Erdmenger, J. and Höhne, S. and Lüst, D. and Meyer, R., "Fayet–Iliopoulos terms in AdS/CFT with flavour," *JHEP* **07** (2008) 068, `0805.1917`.

[14] H. S. Snyder, "Quantized Space-Time," *Phys. Rev.* **71** (Jan, 1947) 38–41.

[15] M. R. Douglas and N. A. Nekrasov, "Noncommutative field theory," *Rev. Mod. Phys.* **73** (2001) 977–1029, `hep-th/0106048`.

[16] A. Connes, *Noncommutative geometry.* Academic Press, Inc, 1994. 661 p.

[17] K. G. Wilson and J. B. Kogut, "The renormalization group and the ε expansion," *Phys. Rept.* **12** (1974) 75–200.

[18] J. L. Cardy, "Is there a c-theorem in four dimensions?," *Phys. Lett.* **B215** (1988) 749–752.

[19] L. Bonora, P. Cotta-Ramusino, and C. Reina, "Conformal Anomaly and Cohomology," *Phys. Lett.* **B126** (1983) 305.

[20] L. Bonora, P. Pasti, and M. Bregola, "Weyl Cocycles," *Class. Quant. Grav.* **3** (1986) 635.

[21] H. Osborn, "Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories," *Nucl. Phys.* **B363** (1991) 486–526.

[22] J. Wess and B. Zumino, "Consequences of anomalous Ward identities," *Phys. Lett.* **B37** (1971) 95.

[23] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge, UK: Univ. Pr., 1998. 531 p.

[24] I. Bakas and C. Sourdis, "Dirichlet sigma models and mean curvature flow," *JHEP* **06** (2007) 057, `0704.3985`.

[25] P. G. O. Freund and M. A. Rubin, "Dynamics of Dimensional Reduction," *Phys. Lett.* **B97** (1980) 233–235.

[26] J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations: A graduate course for physicists.* Cambridge University Press, 1997. 438 p.

[27] R. Slansky, "Group Theory for Unified Model Building," *Phys. Rept.* **79** (1981) 1–128.

[28] L. Alvarez-Gaume and D. Z. Freedman, "Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model," *Commun. Math. Phys.* **80** (1981) 443.

[29] K. Furuuchi, "Topological charge of U(1) instantons on noncommutative R**4," *Prog. Theor. Phys. Suppl.* **144** (2001) 79–91, hep-th/0010006.

[30] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge, UK: Univ. Pr., 1998. 402 p.

[31] J. Callan, Curtis G., E. J. Martinec, M. J. Perry, and D. Friedan, "Strings in Background Fields," *Nucl. Phys.* **B262** (1985) 593.

[32] M. Kalb and P. Ramond, "Classical direct interstring action," *Phys. Rev.* **D9** (1974) 2273.

[33] D. Friedan, "Nonlinear Models in $2 + \epsilon$ Dimensions," *Phys. Rev. Lett.* **45** (1980) 1057.

[34] D. H. Friedan, "Nonlinear Models in $2 + \epsilon$ Dimensions," *Ann. Phys.* **163** (1985) 318.

[35] A. M. Polyakov, "Quantum geometry of fermionic strings," *Phys. Lett.* **B103** (1981) 211–213.

[36] B. Zwiebach, *A first course in string theory.* Cambridge University Press, 2004. 558 p.

[37] A. Abouelsaood, J. Callan, Curtis G., C. R. Nappi, and S. A. Yost, "Open Strings in Background Gauge Fields," *Nucl. Phys.* **B280** (1987) 599.

[38] C. V. Johnson, *D-branes.* Cambridge, USA: Univ. Pr., 2003. 548 p.

[39] R. Apreda, J. Erdmenger, N. Evans, and Z. Guralnik, "Strong coupling effective Higgs potential and a first order thermal phase transition from AdS/CFT duality," *Phys. Rev.* **D71** (2005) 126002, hep-th/0504151.

[40] C.-S. Chu, V. V. Khoze, and G. Travaglini, "Notes on noncommutative instantons," *Nucl. Phys.* **B621** (2002) 101–130, `hep-th/0108007`.

[41] R. Apéry, "Irrationalité de $\zeta(2)$ et $\zeta(3)$," *Astérisque* **61** (1979) 11–13.

[42] Haack, Michael and Krefl, Daniel and Lüst, Dieter and Van Proeyen, Antoine and Zagermann, Marco, "Gaugino condensates and D-terms from D7-branes," *JHEP* **01** (2007) 078, `hep-th/0609211`.

[43] J. Große, *Quantum Field Theories Coupled to Supergravity: AdS/CFT and Local Couplings.* PhD thesis, Ludwig–Maximilians–Universität München, 2006. `arXiv:0711.0444 [hep-th]`.

[44] J. Polchinski and M. J. Strassler, "The string dual of a confining four-dimensional gauge theory," `hep-th/0003136`.

[45] R. Apreda, J. Erdmenger, D. Lust, and C. Sieg, "Adding flavour to the Polchinski-Strassler background," *JHEP* **01** (2007) 079, `hep-th/0610276`.

[46] S. Höhne, "Local Couplings in Quantum Field Theory." Diploma thesis, Humboldt–Universität zu Berlin, 2005.

[47] L. S. Brown and J. C. Collins, "Dimensional Renormalization of Scalar Field Theory in Curved Space-Time," *Ann. Phys.* **130** (1980) 215.

[48] S. J. Hathrell, "Trace Anomalies and QED in Curved Space," *Ann. Phys.* **142** (1982) 34.

[49] S. J. Hathrell, "Trace Anomalies and $\lambda\phi^4$ Theory in Curved Space," *Ann. Phys.* **139** (1982) 136.

[50] R. J. Riegert, "A Nonlocal Action for the Trace Anomaly," *Phys. Lett.* **B134** (1984) 56–60.

[51] M. Nakahara, *Geometry, Topology and Physics.* Institute of Physics Publishing, Bristol and Philadelphia, 2003.

[52] D. Anselmi, D. Z. Freedman, M. T. Grisaru, and A. A. Johansen, "Nonperturbative formulas for central functions of supersymmetric gauge theories," *Nucl. Phys.* **B526** (1998) 543–571, `hep-th/9708042`.

[53] E. Witten, "Dyons of Charge $e\theta/2\pi$," *Phys. Lett.* **B86** (1979) 283–287.

[54] C. Montonen and D. I. Olive, "Magnetic Monopoles as Gauge Particles?," *Phys. Lett.* **B72** (1977) 117.

[55] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's Functions," *Commun. Math. Phys.* **31** (1973) 83–112.

[56] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's Functions. 2," *Commun. Math. Phys.* **42** (1975) 281.

[57] M. E. Peskin, "Duality in supersymmetric Yang-Mills theory," `hep-th/9702094`.

[58] H. Osborn, "Local couplings and Sl(2,R) invariance for gauge theories at one loop," *Phys. Lett.* **B561** (2003) 174–182, `hep-th/0302119`.

[59] S. L. Adler and W. A. Bardeen, "Absence of higher order corrections in the anomalous axial vector divergence equation," *Phys. Rev.* **182** (1969) 1517–1536.

[60] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl. Phys.* **B435** (1995) 129–146, `hep-th/9411149`.

[61] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric- magnetic duality," *Nucl. Phys. Proc. Suppl.* **45BC** (1996) 1–28, `hep-th/9509066`.

[62] I. L. Buchbinder and S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity: A Walk through superspace.* Bristol, UK: IOP 640 p, 1995.

[63] J. Erdmenger and C. Rupp, "Superconformal Ward identities for Green functions with multiple supercurrent insertions," *Annals Phys.* **276** (1999) 152–187, `hep-th/9811209`.

[64] F. Gieres, "Geometry of Supersymmetric Gauge Theories: Including an Introduction to BRS Differential Algebras and Anomalies," *Lect. Notes Phys.* **302** *1.* (1988).

[65] S. L. Adler, J. C. Collins, and A. Duncan, "Energy-momentum-tensor trace anomaly in spin-1/2 quantum electrodynamics," *Phys. Rev. D* **15** (Mar, 1977) 1712–1721.

[66] J. K. Erdmenger, *Implications of Conformal Invariance for Quantum Field Theories in General Dimensions.* PhD thesis, Trinity College and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, 1996.

[67] A. Connes, M. R. Douglas, and A. S. Schwarz, "Noncommutative geometry and matrix theory: Compactification on tori," *JHEP* **02** (1998) 003, `hep-th/9711162`.

[68] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," `hep-th/0201253`.

[69] Y. Choquet-Bruhat, C. D. Witt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds And Physics, Part I: Basics.* North-Holland, Amsterdam, revised edition ed., 1982.