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On symplectic 4-manifolds and contact 5-manifolds

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Abstract

In this thesis we prove some results on symplectic structures on 4-dimensional manifolds and contact structures on 5-dimensional manifolds. We begin by discussing the relation between holomorphic and symplectic minimality for Kähler surfaces and the irreducibility of minimal simply-connected symplectic 4-manifolds under connected sum. We also prove a result on the conformal systoles of symplectic 4-manifolds. For the generalized fibre sum construction of 4-manifolds we calculate the integral homology groups if the summation is along embedded surfaces with trivial normal bundle. In the symplectic case we derive a formula for the canonical class of the generalized fibre sum and give several applications, in particular to the geography of simply-connected symplectic 4-manifolds whose canonical class is divisible by a given integer. We also use branched coverings of complex surfaces of general type to construct simply-connected algebraic surfaces with divisible canonical class. In the second part of the thesis we show that these geography results together with the Boothby-Wang construction of contact structures on circle bundles over symplectic manifolds imply that certain simply-connected 5-manifolds admit inequivalent contact structures in the same (non-trivial) homotopy class of almost contact structures.

Zusammenfassung

In dieser Arbeit beweisen wir einige Aussagen über symplektische Strukturen auf 4-dimensionalen Mannigfaltigkeiten und Kontaktstrukturen auf 5-dimensionalen Mannigfaltigkeiten. Wir untersuchen zunächst den Zusammenhang zwischen dem symplektischen und dem holomorphen Minimalitätsbegriff für Kählerflächen. Außerdem beweisen wir ein Ergebnis über die Irreduzibilität minimaler, einfach-zusammenhängender symplektischer 4- Mannigfaltigkeiten unter zusammenhängender Summe und eine Aussage über die konformen Systolen symplektischer 4-Mannigfaltigkeiten. Als nächstes betrachten wir die Konstruktion von differenzierbaren 4-dimensionalen Mannigfaltigkeiten durch die verallgemeinerte Fasersumme. Für den Fall, dass die Summation entlang eingebetteter Flächen mit trivialem Normalenbündel erfolgt, berechnen wir die ganzzahligen Homologiegruppen und im symplektischen Fall auch die kanonische Klasse der Fasersumme. Wir betrachten verschiedene Anwendungen, insbesondere hinsichtlich der Geographie einfach-zusammenhängender symplektischer 4-Mannigfaltigkeiten deren kanonische Klasse durch eine vorgegebene natürliche Zahl teilbar ist. Wir zeigen auch, dass man mit geeigneten verzweigten Überlagerungen von komplexen Flächen vom allgemeinen Typ einfach-zusammenhängende algebraische Flächen konstruieren kann, deren kanonische Klasse eine vorgegebene Teilbarkeit besitzt. Im zweiten Teil der Arbeit betrachten wir die Boothby-Wang Konstruktion von Kontaktstrukturen auf Kreisbündeln über symplektischen Mannigfaltigkeiten. Zusammen mit den Resultaten über Geographie aus dem ersten Teil der Arbeit zeigen wir, dass es auf bestimmten einfach-zusammenhängenden 5-Mannigfaltigkeiten Kontaktstrukturen gibt, die nicht äquivalent sind, aber die in derselben (nicht-trivialen) Homotopieklasse von Fast-Kontaktstrukturen liegen.

Contents

Chapter I

Introduction

In this thesis we are interested in symplectic structures on closed 4-dimensional manifolds and contact structures on closed 5-dimensional manifolds. A particularly interesting case is when the manifolds are simply-connected, because simply-connected 4-manifolds can be classified up to *homeomorphism* by a theorem of M. H. Freedman [45] and simply-connected 5-manifolds can be classified up to *diffeomorphism* by a theorem of D. Barden [6]. It follows from Barden's classification theorem that two simply-connected smooth closed 5-manifolds are diffeomorphic if and only if they are homeomorphic. This does not hold for simply-connected 4-manifolds because of the existence of many "exotic" 4-manifolds and explains to some extent why a corresponding classification for simply-connected 4-manifolds up to *diffeomorphism* is not known. We now briefly describe the background and then summarize the content of each chapter.

It is a basic question in the theory of 4-manifolds to determine whether a given differentiable 4-manifold admits a symplectic structure or not. Historically, the first examples of symplectic 4-manifolds were Kähler surfaces, because the Kähler form is always a symplectic form. In particular, all complex algebraic surfaces have a symplectic structure. The first example of a symplectic 4-manifold which cannot be Kähler is due to K. Kodaira and W. P. Thurston [137]. This manifold is a torus bundle over the torus and has first Betti number equal to 3. It admits a symplectic structure by an explicit construction. However, since the first Betti number of Kähler surfaces is always even by Hodge theory, it follows that the manifold cannot be Kähler.

In addition to the construction for surface bundles by Thurston, there are several ways to construct new symplectic 4-manifolds. A very useful construction is the *generalized fibre sum* due to R. E. Gompf [52] and J. D. McCarthy and J. G. Wolfson [91]. This construction works in arbitrary even dimensions. In particular, it can be applied to symplectic 4-manifolds which contain symplectic surfaces with trivial normal bundle: Given two symplectic 4-manifolds M and N and embedded symplectic surfaces Σ_M and Σ_N of the same genus and with self-intersection zero, there exists a new symplectic 4-manifold $X = M\#_{\Sigma_M=\Sigma_N}N$ obtained by "summing" M and N along the embedded surfaces. This construction also works for differentiable 4-manifolds and embedded surfaces without symplectic structures and in this way yields new differentiable and often exotic 4-manifolds.

Another construction, related to the generalized fibre sum, is called *knot surgery* and is due to R. Fintushel and R. J. Stern [38]. Given a 4-manifold X which contains an embedded torus T of self-intersection zero and an arbitrary knot K in S^3, a new 4-manifold X_K can be constructed with the following properties: If the manifold X and the complement of the torus in X are simply-connected, then the knot surgery manifold X_K is again simply-connected and homeomorphic to X for every knot K. Moreover, if the manifold X is symplectic, the torus T symplectically embedded and the knot K *fibred*, then the manifold X_K also admits a symplectic structure.

With these and several other constructions (in particular, the rational blow-down construction [37] and Luttinger surgery [2]) it is possible to construct many new simply-connected symplectic and non-symplectic 4-manifolds. To mention some examples, one can find symplectic 4-manifolds which are simply-connected and cannot be Kähler, generalizing the result of Thurston to simply-connected manifolds. In some cases the manifolds cannot be homeomorphic to a Kähler surface because of the Kodaira-Enriques classification of complex algebraic surfaces, in particular the existence of the Noether inequality $c_1^2 \geq 2\chi_h - 6$ for minimal surfaces of general type. In other cases the symplectic 4-manifolds are homeomorphic to Kähler surfaces but still do not admit a Kähler structure. There are also constructions of simply-connected 4-manifolds which cannot admit a symplectic structure at all, even though there exists a 4-manifold homeomorphic to it which does admit a symplectic structure. This shows that the existence of symplectic structures on 4-manifolds depends in a subtle way on the differentiable structure of the 4-manifold.

To distinguish symplectic 4-manifolds from Kähler surfaces and from non-symplectic 4-manifolds often requires the invariants derived from the theories of S. K. Donaldson [30, 31] and N. Seiberg and E. Witten [145], which have their origin in theoretical physics. In particular, there are several theorems of C. H. Taubes on the Seiberg-Witten invariants of symplectic 4-manifolds [131, 132, 133, 134] and extensions by T.-J. Li and A.-K. Liu to the exceptional case of $b_2^+ = 1$ [86, 87, 88, 90]. The Seiberg-Witten invariants for the constructions mentioned above can be calculated by theorems of several authors [38, 103, 104, 109, 136].

It is also possible to give (at least partial) answers to the so-called *geography question* for symplectic manifolds: Suppose a lattice point (x, y) in $\mathbb{Z} \times \mathbb{Z}$ is given. Then the geography question asks whether there exists a simply-connected symplectic 4-manifold M such that the Euler characteristic $e(M)$ is equal to x and the signature $\sigma(M)$ is equal to y. In other words, which coordinate points in the plane can be realized by the topological invariants of simply-connected symplectic 4-manifolds? A similar question can be asked for simply-connected complex surfaces of general type. There are several parts and sectors of the plane that have been filled for both geography questions, in some cases under the additional assumption that the manifolds are spin.

Another interesting question, sometimes called *botany*, tries to determine whether a given lattice point can be realized by several different 4-manifolds. For example, the constructions above imply that many lattice points can be realized by infinitely many homeomorphic but pairwise non-diffeomorphic simply-connected symplectic 4-manifolds. One can also consider the botany question for symplectic structures on a given differentiable 4-manifold, i.e. whether a fixed differentiable simply-connected 4-manifold admits several inequivalent symplectic structures. Some results for this question in the case of homotopy elliptic surfaces can be found in articles by C. T. McMullen and C. H. Taubes [97], I. Smith [126] and S. Vidussi [140]. A (non-exhaustive) list of references for the geography results and the constructions of symplectic 4-manifolds mentioned above, in addition to the references already cited, is [26, 35, 39, 41, 44, 60, 81, 85, 106, 107, 108, 110, 111, 112, 113, 114, 115, 116, 128, 130, 141].

The second part of this thesis concerns contact structures on 5-manifolds. By a construction of W. M. Boothby and H. C. Wang [13], it is possible to associate to every symplectic manifold a contact structure on a certain circle bundle over this manifold. In particular, one can associate to every simply-connected symplectic 4-manifold M a simply-connected 5-manifold X which is a circle bundle over M and admits a contact structure related to the symplectic structure on M. This is the connection between the manifolds of dimension 4 and 5 in our thesis.

The existence question for contact structures on simply-connected 5-manifolds in general (which is the analogue of the geography question for simply-connected contact 5-manifolds) has been solved by H. Geiges [51]: A simply-connected 5-manifold X admits a contact structure if and only if the third integral Stiefel-Whitney class $W_3(X) \in H^3(X; \mathbb{Z})$ vanishes. The proof of this theorem relies on the fact that simply-connected 5-manifolds can be classified up to diffeomorphism by the theorem of D. Barden mentioned above.

However, there still remains the question concerning uniqueness or non-uniqueness of contact structures on simply-connected 5-manifolds (corresponding to the botany question). There are several ways in which contact structures on the same manifold can be "equivalent": contact structures can be deformed into each other through contact structures or there could exist a self-diffeomorphism of the manifold which maps one contact structure to the other contact structure. By a theorem of J. W. Gray [57] the first case is actually contained in the second. In any of these cases, we call the contact structures *equivalent*. One can also consider a different form of deformation between contact structures, where one does not assume that the deformation is through contact structures but only the symplectic structure on the contact distribution, given by the definition of contact structures, is carried along in the deformation. In this case the contact structures are deformed through so-called almost contact structures. One can similarly define an *equivalence of almost contact structures* by allowing combinations of deformations and arbitrary self-diffeomorphisms of the manifold.

If two contact structures are equivalent then they are also equivalent as almost contact structures, but the converse is not always true. The existence theorem of Geiges mentioned above provides a contact structure in every equivalence class of almost contact structures on simply-connected 5-manifolds. One can think of contact structures in different equivalence classes of almost contact structures as being "trivially" different for topological reasons. The interesting question is then to find contact structures which are equivalent as almost contact structures but *not as contact structures*.

To distinguish such inequivalent contact structures there exists a theory called *contact homology*, invented by Y. Eliashberg, A. Givental and H. Hofer [33]. Using invariants derived from this theory inequivalent contact structures which are equivalent as almost contact structures have been found on several 5-manifolds: on the sphere S^5 by I. Ustilovsky [139], on $T^2 \times S^3$ and T^5 by F. Bourgeois [15] and on many simply-connected 5-manifolds by O. van Koert [74]. The constructions in these cases yield infinitely many inequivalent contact structures in the same homotopy class of almost contact structures. However, the examples are all in the trivial homotopy class whose first Chern class is zero. As far as we know, inequivalent contact structures on 5-manifolds have only been found in this homotopy class. In Chapter X we construct some examples of inequivalent contact structures in homotopy classes with non-vanishing Chern class.

We now describe the content of each chapter separately. **Chapter II** collects some basic preliminaries on 4-manifolds, in particular on the intersection form and on complex algebraic surfaces.

Chapter III was published together with D. Kotschick under the same title in *Int. Math. Res. Notices 2006, Art. ID 35032, 1-13.* We only made some very minor adaptations for inclusion in this thesis. The first part of the chapter concerns the difference between two notions of minimality for Kähler surfaces, *symplectic* and *holomorphic minimality*, where the first one is defined by the non-existence of a symplectic embedded (-1)-sphere and the second one by the non-existence of a holomorphic embedded (-1)-sphere. It is not clear that both notions agree. We will prove that they are identical for all Kähler surfaces except the non-spin Hirzebruch surfaces X_n for $n > 1$ odd, cf. Theorem 3.2. The second part of Chapter 3 concerns the irreducibility of symplectic 4-manifolds. The main theorem 3.3 was proved by D. Kotschick for the case $b_2^+ \geq 2$ in [79], cf. also [80]. It is extended here to the case $b_2^+ = 1$ which is exceptional because the Seiberg-Witten invariants are not completely independent on the choice of parameters but depend on certain chambers. The theorem implies that minimal simply-connected symplectic 4-manifolds X are *irreducible*, meaning that in any connected sum decomposition $X = X_1 \# X_2$ one summand has to be homeomorphic to S^4.

Chapter IV has been published under the same title in *Manuscripta math.* **121**, *417-424 (2006).* I have only made minor modifications for inclusion here. The main result, Corollary 4.4, is an extension of a theorem of M. Katz [70] on the so-called *conformal systoles* for blow-ups of the projective plane to a larger class of manifolds. The proof uses some results derived from the Seiberg-Witten theory for symplectic 4-manifolds.

Chapter V on the generalized fibre sum is a cornerstone of this thesis, because many constructions in Chapter VI use fibre sums. In the first part of the chapter we calculate the integral homology of the generalized fibre sum $X = M\#_{\Sigma_M=\Sigma_N}N$ of two differentiable 4-manifolds M, N (without symplectic structures) along embedded surfaces Σ_M, Σ_N with trivial normal bundles. The first homology is determined in Theorem 5.11, the first cohomology in Proposition 5.15 and an exact sequence for the second homology in Theorem 5.36. If the cohomology of M, N and X is torsion free and the classes represented by the surfaces Σ_M and Σ_N are indivisible, a formula for the intersection form of X is determined in Theorem 5.37. Such formulas are known in many special cases and are often derived in applications using the generalized fibre sum *ad hoc*. However, as far as we know, they have not appeared in complete generality. The second part of Chapter V concerns the canonical class of the symplectic generalized fibre sum X of two symplectic 4-manifolds along symplectic surfaces. In Theorem 5.55 a formula for the canonical class of X is derived under the assumptions of Theorem 5.37 describing the intersection form. This is also one of the reasons why we calculated the cohomology of X in detail, because this is necessary to identify the terms giving a contribution to the canonical class. A formula for the canonical class is known in the case that the generalized fibre sum is along tori (there is also a more general formula by E.-N. Ionel and T. H. Parker [69]). However, also for the case of tori we did not find a complete proof in the literature, in particular taking care of the existence of rim tori.[1] We compare the formula in Theorem 5.55 with some of the formulas used in the literature and give some applications: In Section V.6.1 we consider the generalized fibre sum of elliptic surfaces $E(n)$ and $E(m)$ which are not glued together by a fibre preserving diffeomorphism but with a "twisting" and determine the rim tori contribution to the canonical class in this case. In Section V.6.2 a variation of an idea of I. Smith [126] is described for the construction of inequivalent symplectic forms on the same 4-manifold if a symplectic 4-manifold admits certain Lagrangian tori of self-intersection zero. The construction uses that, given a Lagrangian torus which represents an essential homology class in a symplectic 4-manifold M, one can deform the symplectic structure on the manifold such that it induces either a negative volume form, the zero form or a positive volume form on the torus while the canonical class remains unchanged.

Chapter VI concerns the geography of simply-connected symplectic 4-manifolds whose canonical class is divisible by a given integer $d > 1$. This version of the geography question has not been considered before, as far as we know, except for the case $d = 2$ which corresponds to spin manifolds. The examples which are constructed can be used in Chapter X to find inequivalent contact structures on certain simply-connected 5-manifolds. Following some general remarks in Section VI.1, we apply in Section VI.2 the calculations in Chapter V on the generalized fibre sum. First we consider the case that the simply-connected 4-manifold has $c_1^2 = 0$ (hence is a homotopy elliptic surface) and later the case $c_1^2 > 0$. The case $c_1^2 < 0$ is very simple if one uses the results of C. H. Taubes [134] and A. K. Liu [90].

The main existence result for symplectic structures with divisible canonical class in the case of homotopy elliptic surfaces is Theorem 6.11. The idea of the construction is to first raise the divisibility of the canonical class of an elliptic surface by doing a knot surgery along the fibre and then "breaking" the divisibility to the appropriate divisor by doing a further knot surgery on a rim torus. Using a refinement of this construction and the results from Section V.6.2, we show that one can also realize on the *same* homotopy elliptic surface several symplectic structures whose canonical classes have

[1] There is however an indirect proof using the calculation of the Seiberg-Witten invariants for gluing along T^3, cf. [109] and [136].

different divisibilities by breaking the divisibility in several different ways, cf. Proposition 6.14, Theorem 6.16 and Corollary 6.18. Hence these symplectic structures are inequivalent, which generalizes the work of McMullen-Taubes [97], Smith [126] and Vidussi [140] mentioned above, who also found inequivalent symplectic structures on homotopy elliptic surfaces. The construction uses the existence of several independent triples of Lagrangian tori (as rim tori) in elliptic surfaces, which are needed for the construction from Section V.6.2.

In the next subsection some of these results are generalized to the case where $c_1^2 > 0$. The construction uses a form of "generalized knot surgery" along surfaces of higher genus [41]. In this way one can increase c_1^2 while keeping the signature of the manifold and the divisibility of the canonical class fixed. The symplectic surfaces of higher genus which we use arise from the knot surgery construction. In particular, Theorem 6.20 solves the existence question for simply-connected symplectic manifolds with $c_1^2 > 0$ and negative signature whose canonical class is divisible by a given even integer $d \geq 2$. We also have some results for odd divisibility, cf. Theorem 6.27 and Proposition 6.32. However, we do not have as complete an answer as for the case of even divisibility, because in the even case the signature is constrained by Rochlin's theorem which does not hold in the odd case. Using the construction from the previous subsection it is possible to find inequivalent symplectic structures on some of these manifolds, cf. Theorem 6.22 and Theorem 6.29 (explicit examples of this type on simply-connected closed 4-manifolds with $c_1^2 > 0$ do not appear in the literature, though their existence is implicitly clear by [126]).

In the following sections of Chapter VI a second, independent way is described to construct simply-connected symplectic 4-manifolds with divisible canonical class. This construction uses branched coverings over pluricanonical divisors on algebraic surfaces of general type. Hence the examples will again be surfaces of general type. In Section VI.3 we define branched coverings and give a criterion when a branched covering over a simply-connected complex surface is again simply-connected, cf. Theorem 6.45 and Corollary 6.47. The proof uses a theorem of M. V. Nori [105] on the fundamental group of the complement of a complex curve in a complex surface. Section VI.4 contains a description of some results on the geography of simply-connected surfaces of general type, in particular those due to U. Persson, C. Peters and G. Xiao [115, 116]. In the following section these geography results and the existence of base point free pluricanonical divisors (summarized in Section II.3.7) are used to construct the branched coverings with divisible canonical class.

In **Chapter VII** we summarize the classification of simply-connected 5-manifolds by D. Barden [6] and S. Smale [125], including the topological invariants of simply-connected 5-manifolds X used for the classification, in particular the linking form on the torsion subgroup of $H_2(X;\mathbb{Z})$ which gives rise to the so-called i-invariant. Also some details for the construction of the irreducible building blocks of simply-connected 5-manifolds are given in Section VII.5 and a proof for the theorem on the connected sum decomposition in Section VII.6.

Chapter VIII recalls some basic facts about contact structures and we define the notion of equivalence of contact structures in Definition 8.10. In Theorem 8.18 we show that two almost contact structures on a 5-manifold X whose $H^2(X;\mathbb{Z})$ does not contain 2-torsion are homotopic as almost contact structures if and only if they have the same first Chern class. This extends a theorem of H. Geiges [51] who proved the same result under the assumption that X is simply-connected. In Theorem 8.20 and Corollary 8.22 this result is combined with the classification theorem for simply-connected 5-manifolds to deduce that two almost contact structures on a simply-connected 5-manifold X are equivalent if and only if their first Chern classes have the same divisibility as elements in $H^2(X;\mathbb{Z})$. The proof uses that certain automorphisms of $H^2(X;\mathbb{Z})$ can be realized by orientation preserving self-diffeomorphisms of X. We call the divisibility of the first Chern class of an almost contact structure ξ on X its *level*. It follows that two almost contact structures on X are equivalent if and only if they lie on the same level.

In **Chapter IX** we collect and prove some results on the topology of circle bundles. In particular, Lemma 9.8 shows that the total space of a circle bundle is simply-connected if and only if the base manifold M is simply-connected and the Euler class is indivisible as an element in $H^2(M;\mathbb{Z})$. In the case where M is a simply-connected 4-manifold and the Euler class e is indivisible Barden's classification theorem of simply-connected 5-manifolds from Chapter VII can be used to determine the total space X up to diffeomorphism. It turns out that X is diffeomorphic to a connected sum of several copies of $S^2 \times S^3$ if X is spin. If X is not spin there is one additional summand given by the non-trivial S^3-bundle over S^2. The total number of summands in both cases is equal to $b_2(M) - 1$, cf. Theorem 9.12 (this has also been proved in [32]). These manifolds are, up to diffeomorphism, precisely the simply-connected 5-manifolds X with torsion free $H_2(X;\mathbb{Z})$. In Section IX.3 we describe the Boothby-Wang construction of contact structures on circle bundles. Together with the diffeomorphism classification above, it follows that one can realize the same abstract simply-connected 5-manifold X with torsion free $H_2(X;\mathbb{Z})$ as a Boothby-Wang total space over different simply-connected symplectic 4-manifold with the same second Betti number. In this way one can construct numerous contact structures on a given simply-connected 5-manifold with torsion free second homology.

In **Chapter X** we show that some of these contact structures are inequivalent using a version of contact homology for the Morse-Bott case [15, 33]. Let ξ_1 and ξ_2 be two contact structures on an abstract simply-connected 5-manifold X with torsion free $H_2(X;\mathbb{Z})$ which are on the same level (hence both are equivalent as almost contact structures). Suppose that both contact structures can be realized as Boothby-Wang contact structures over two different simply-connected symplectic 4-manifolds M_1 and M_2:

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (M_1,\omega_1) & & (M_2,\omega_2) \end{tikzcd}$$

We prove essentially that if the divisibilities of the canonical classes of the symplectic structures ω_1, ω_2 on M_1 and M_2 are different, then the contact structures on X are inequivalent, cf. Corollary 10.18. In this way the existence of inequivalent contact structures on simply-connected 5-manifolds with torsion free $H_2(X;\mathbb{Z})$ is related to the geography of symplectic 4-manifolds with divisible canonical class as in Chapter VI. In the second part of the chapter some explicit examples will be given, in particular on non-zero levels corresponding to non-vanishing first Chern class.

The **Appendix** finally contains some calculations for the complement of a submanifold F of dimension $n-2$ in a manifold M of dimension n which are used in several places in Chapters V and VI.

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Chapter II

Preliminaries on 4-manifolds

Contents

In this chapter we collect some results and formulas on differentiable 4-manifolds which will be used throughout the thesis. We give some references at the beginning of each section where the proofs for the statements can be found (or in the references therein). *The manifolds we consider in this thesis are all smoothly differentiable.*

II.1 Differentiable 4-manifolds

General references for this section are the books by Freedman-Quinn [46] and Gompf-Stipsicz [56].

II.1.1 The intersection form

Let M be a closed, oriented 4-manifold. By Poincaré duality and the Universal Coefficient Theorem, the torsion subgroups of all homology and cohomology groups are determined by $\mathrm{Tor}H_1(M;\mathbb{Z})$:

$$\begin{aligned}\mathrm{Tor}H_1(M;\mathbb{Z}) &\cong \mathrm{Tor}H^2(M;\mathbb{Z})\\ &\cong \mathrm{Tor}H_2(M;\mathbb{Z})\\ &\cong \mathrm{Tor}H^3(M;\mathbb{Z}).\end{aligned}$$

All other torsion groups vanish. The *intersection form*,

$$Q_M\colon H^2(M;\mathbb{Z})\times H^2(M;\mathbb{Z})\longrightarrow \mathbb{Z},$$

is defined by $Q_M(\alpha,\beta)=\langle\alpha\cup\beta,[M]\rangle$, where $[M]\in H^4(M;\mathbb{Z})$ denotes the fundamental class given by the orientation. Via Poincaré duality we get an equivalent form on $H_2(M;\mathbb{Z})$, which we also denote by Q_M. One often writes

$$a\cdot b = Q_M(PD(a),PD(b)).$$

The intersection form Q_M is a symmetric and bilinear form. If α is a torsion element of $H^2(M;\mathbb{Z})$, then $Q_M(\alpha,x)=0$ for all $x\in H^2(M;\mathbb{Z})$. Hence the intersection form induces a symmetric and bilinear form on $H^2(M;\mathbb{Z})/\mathrm{Tor}$. By Poincaré duality

$$Q_M(\alpha,\beta)=\langle\alpha,PD(\beta)\rangle,$$

and the Universal Coefficient Theorem

$$H^2(M;\mathbb{Z})/\mathrm{Tor}\cong \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}).$$

It follows that the intersection form on $H^2(M;\mathbb{Z})/\mathrm{Tor}$ is non-degenerate. A homotopy equivalence between closed, oriented 4-manifolds induces an isomorphism of intersection forms.

Q_M is called *even* if $Q_M(\alpha,\alpha)\equiv 0 \bmod 2$ for all $\alpha\in H^2(X;\mathbb{Z})$ and *odd* otherwise. This is called the *type* of Q_M. A *characteristic element* for Q_M is an element $\beta\in H^2(M;\mathbb{Z})$ such that

$$Q_M(\beta,\alpha)\equiv Q_M(\alpha,\alpha) \bmod 2,\quad \text{for all } \alpha\in H^2(M;\mathbb{Z}).$$

There also exists a corresponding intersection form on $H^2(M;\mathbb{R})$. We can choose a basis of the vector space $H^2(M;\mathbb{R})$ such that this form is represented by a diagonal matrix of type

$$diag(+1,+1,\ldots,+1,-1,-1,\ldots,-1).$$

In other words, Q_M is always diagonalizable over $\mathbb{R}$. The number of $+1$ and -1 entries are denoted by $b_2^+(M)$ and $b_2^-(M)$. These numbers do not depend on the choice of basis for $H^2(M;\mathbb{R})$ and are homotopy invariants of M. The intersection form Q_M is called

positive definite if $b_2^-(M)=0$,

negative definite if $b_2^+(M)=0$,

definite in either case and *indefinite* if both $b_2^\pm(M)\geq 1$.

The *signature* $\sigma(M)$ is defined as

$$\sigma(M)=b_2^+(M)-b_2^-(M).$$

One can show that

$$Q_M(x,x)\equiv\sigma(M) \bmod 8 \tag{2.1}$$

for every characteristic element x of $H^2(M;\mathbb{Z})$, cf. [56, Lemma 1.2.20]. Note that 0 is a characteristic element if Q_M is even. Hence in this case the signature $\sigma(M)$ is divisible by 8.

We consider in particular the non-degenerate, symmetric, bilinear forms Q, determined by the following matrices:

$Q = (1)$ on $\mathbb{Z}$, with $Q(e,e) = 1$ on the basis element.

$Q = (-1)$ on $\mathbb{Z}$, with $Q(e,e) = -1$ on the basis element.

$Q = H$ on $\mathbb{Z}^2$, given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$Q = E_8$ on $\mathbb{Z}^8$, given by

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

The forms (1) and (-1) are odd and the forms H (indefinite of signature 0) and E_8 (positive definite of signature 8) are even.

Indefinite, non-degenerate, symmetric, bilinear forms Q of rank b and signature σ can be classified as follows (up to isomorphism) [99]:

- If Q is odd, then Q is isomorphic to

$$b_2^+(1) \oplus b_2^-(-1).$$

- If Q is even, then Q is isomorphic to

$$\tfrac{\sigma}{8}E_8 \oplus \tfrac{b-|\sigma|}{2}H.$$

Definite forms are not classified in general. However, by *Donaldson's theorem* [29, 31], if Q is the intersection form Q_M of a smooth, closed, oriented 4-manifold M and Q_M is definite, then Q_M is isomorphic to

$Q_M = b_2(1) = (1) \oplus \ldots \oplus (1)$ if Q_M is positive definite.

$Q_M = b_2(-1) = (-1) \oplus \ldots \oplus (-1)$ if Q_M is negative definite.

Hence in this case Q_M *is diagonalizable over* $\mathbb{Z}$. The classification of indefinite forms above, together with Donaldson's theorem for the definite case, imply that the intersection form Q_M of a smooth, closed, oriented 4-manifold is determined by $b_2(M)$, $\sigma(M)$ and the type.

II.1.2 The second Stiefel-Whitney class

Let M be a closed, oriented 4-manifold and $w_2(M) \in H^2(M;\mathbb{Z}_2)$ the second Stiefel-Whitney class of M. The 4-manifold M is spin if and only if $w_2(M) = 0$. By the *Wu formula*

$$\langle w_2(M), a\rangle \equiv Q_M(a,a) \bmod 2, \quad \text{for all } a \in H_2(M;\mathbb{Z}).$$

Hence if $c \in H^2(M;\mathbb{Z})$ is a class with

$$c \equiv w_2(M) \bmod 2,$$

then c is a characteristic element for Q_M. Since every closed, oriented 4-manifold is Spin^c, such classes always exist.

Suppose that M is spin. It follows that

$$Q_M(a,a) \equiv 0 \bmod 2, \quad \text{for all } a \in H_2(M;\mathbb{Z}),$$

hence Q_M is an even form. By equation (2.1) this implies that $\sigma(M)$ is divisible by 8. Note that this holds already for topological 4-manifolds. If a closed spin 4-manifold M is smooth, *Rochlin's theorem* [119] implies that the signature $\sigma(M)$ is in fact divisible by 16.

Conversely, suppose that Q_M is even. Then

$$\langle w_2(M), a\rangle \equiv 0 \bmod 2$$

for all $a \in H_2(M;\mathbb{Z})$. By the following exact sequence, coming from the Universal Coefficient Theorem,

$$0 \rightarrow \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2) \xrightarrow{i} H^2(M;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}_2) \rightarrow 0,$$

the class $w_2(M)$ is in the image of the homomorphism i. The group $\mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2)$ vanishes for example if M is simply-connected. Hence if M is a simply-connected, closed, oriented 4-manifold, then M is spin if and only if Q_M is even (the other direction follows from the previous paragraph). The following theorem is due to Freedman [45, 46].

Theorem 2.1. *Let M, N be simply-connected, closed, smooth 4-manifolds. Suppose $\theta\colon H_2(M;\mathbb{Z}) \rightarrow H_2(N;\mathbb{Z})$ is an isomorphism preserving intersection forms. Then there exists a homeomorphism $f\colon M \rightarrow N$, unique up to isotopy, such that $f_* = \theta$.*

We denote the Euler characteristic of closed, oriented 4-manifolds M by $e(M)$. Suppose M is simply-connected. Then

$$e(M) = 2 + b_2(M).$$

Hence $e(M)$ determines $b_2(M)$ and vice versa. If M is simply-connected then the invariants $e(M)$, $\sigma(M)$ and whether M is spin or not spin determine the intersection form Q_M by Section II.1.1 up to isomorphism and by Freedman's theorem the 4-manifold M up to homeomorphism.

II.2 Symplectic manifolds

General references for this section are the books by Gompf-Stipsicz [56] and McDuff-Salamon [96].

II.2.1 Almost complex structures

Let M be a smooth manifold and $E \rightarrow M$ a smooth $\mathbb{R}$-vector bundle of rank $2n$. A *complex structure* on the vector bundle E is a smooth bundle isomorphism $J\colon E \rightarrow E$ (fibrewise linear and covering the identity of M) such that $J^2 = -\mathrm{Id}_E$. Given such an endomorphism J, the vector bundle E becomes a $\mathbb{C}$-vector bundle of rank n with multiplication

$$\mathbb{C} \times E \rightarrow E, \quad (a+ib)\cdot v = av + bJ(v) \quad (a,b \in \mathbb{R}).$$

In particular, the Chern classes $c_i(E, J)$ of E, for $i \geq 0$, are well-defined.

An *almost complex structure* on a smooth manifold M of even dimensions $2n$ is a complex structure on the vector bundle TM. Let M be a closed, oriented 4-manifold with an almost complex structure J. Then the following always holds, cf. [56, Theorem 1.4.15]:

$$c_1(M, J) \equiv w_2(M) \bmod 2$$

$$c_1^2(M, J) = 2e(M) + 3\sigma(M)$$

$$c_2(M, J) = e(M) \in H^4(M; \mathbb{Z}).$$

Note in particular that the mod 2 reduction of $c_1(M, J)$ and the integers $c_1^2(M, J)$ and $c_2(M, J)$ are purely topological invariants of M which do not depend on the almost complex structure J. The existence question for almost complex structures on oriented 4-manifolds is solved by *Wu's theorem* [146, 64]: suppose that M is a closed, oriented 4-manifold and $c \in H^2(X; \mathbb{Z})$ a class with

$$c \equiv w_2(M) \bmod 2, \quad c^2 = 2e(M) + 3\sigma(M).$$

Then there exists an almost complex structure J on M such that $c_1(M, J) = c$.

Definition 2.2. Let M be an arbitrary closed, oriented 4-manifold. We define the integers

$$c_1^2(M) = 2e(M) + 3\sigma(M)$$
$$c_2(M) = e(M).$$

Hence if M admits an almost complex structure J then $c_1^2(M, J) = c_1^2(M)$ and $c_2(M, J) = c_2(M)$.

II.2.2 Symplectic structures

A *symplectic structure* on a real vector space V is by definition a non-degenerate, bilinear skew-symmetric form $\omega\colon V \times V \to \mathbb{R}$. Non-degeneracy here means that for every non-zero vector $v \in V$ there exists a vector $w \in V$ with $\omega(v, w) \neq 0$. A symplectic form exists on a vector space V if and only if the dimension of V is even. A symplectic structure on a real vector bundle $E \to M$ is by definition a family of symplectic structures on each fibre E_p which varies smoothly with the base point p. If M is an even-dimensional manifold, one can consider symplectic structures in this sense on the tangent bundle TM. They correspond to non-degenerate 2-forms on M. A symplectic structure *on a manifold*, however, is a non-degenerate 2-form ω on M which satisfies in addition $d\omega = 0$.

Suppose $E \to M$ is a vector bundle with a symplectic structure σ. A complex structure J on E is called *compatible* with σ if $\sigma(v, Jv) > 0$ for all non-zero v in E and $\sigma(Jv, Jw) = \sigma(v, w)$ for all $v, w \in V$. This implies that $g(v, w) := \sigma(v, Jw)$ defines a metric on E (an inner product) such that J becomes skew-adjoint. Every symplectic vector bundle admits a compatible complex structures and the space of such structures for fixed σ is contractible. Hence the Chern classes of symplectic vector bundles are well-defined, independent of the choice of compatible complex structure. In particular, every symplectic manifold (M, ω) admits a compatible almost complex structure. The *canonical class* K of ω is by definition $-c_1(TX, J)$, where J is an almost complex structure compatible with ω.

II.3 Complex manifolds

Some general references for this section are the books by Barth-Peters-Van de Ven [8], Friedman [47], Gompf-Stipsicz [56], Griffiths-Harris [58], Harris [61] and Hartshorne [62].

II.3.1 Divisors

Let M be a smooth compact complex manifold of dimension n. A *divisor* D on M is by definition a locally finite linear combination (over $\mathbb{Z}$) of irreducible complex hypersurfaces,

$$D = \sum a_i V_i.$$

The divisor D is called effective if all $a_i \geq 0$ and not all a_i vanish. Every divisor D defines a holomorphic line bundle denoted by $\mathcal{O}(D) \to M$. The Chern class of $\mathcal{O}(D)$ is given by

$$c_1(\mathcal{O}(D)) = \sum a_i PD[V_i] \in H^2(M;\mathbb{Z}).$$

Two divisors are called linearly equivalent if they define isomorphic holomorphic line bundles. The linear system $|D|$ defined by a divisor D is the set of all effective divisors linearly equivalent to D and the zero divisor. Let $L \to M$ be a holomorphic line bundle. Then the following holds:

- If L has a global non-trivial meromorphic section s, then the locus of singularities and zeroes of s defines a divisor $D = (s)$ with $\mathcal{O}(D) \cong L$.
- If D is any divisor such that $\mathcal{O}(D) \cong L$, then there exists a meromorphic section s of L with $(s) = D$. Hence L is isomorphic to $\mathcal{O}(D)$ for some divisor D if and only if L has a global non-trivial meromorphic section and L is isomorphic to $\mathcal{O}(D)$ for some *effective* divisor D if and only if L has a global non-trivial holomorphic section.
- The linear system $|D|$ defined by D consists of the zero loci of all holomorphic sections of $\mathcal{O}(D)$ and there is an identification $|D| \cong \mathbb{P}H^0(M, \mathcal{O}(D))$.
- Finally, if M is algebraic, then every holomorphic line bundle $L \to M$ has a non-trivial meromorphic section.

II.3.2 Representing line bundles by non-singular curves

If M^n is a smooth (real) manifold then every class in $H_{n-2}(M;\mathbb{Z})$ can be represented by a smooth submanifold $F^{n-2} \subset M$ of codimension 2 and each class in $H^2(M;\mathbb{Z})$ can be represented as the first Chern class $c_1(L)$ of a complex line bundle L. The relation between the two is that the zero set of a smooth section of L, which is transverse to the zero section, is a smooth codimension 2 submanifold in M which represents the Poincaré dual of $c_1(L)$.

We want to do a similar construction for complex manifolds. Let M be a smooth complex algebraic manifold and $L \to M$ a holomorphic line bundle. We would like to represent the Poincaré dual of $c_1(L)$ by a smooth complex hypersurface.

By definition, a *base point* of L (or the linear system $|L|$) is a point $p \in M$ where all holomorphic sections of L vanish. Equivalently, the point is contained in each element of $|L|$. Suppose L has no base points. In particular, L has non-trivial holomorphic sections. Then we can define a holomorphic map

$$f_L \colon M \to \mathbb{C}P^N, \quad N = h^0(M, \mathcal{O}(L)) - 1,$$

in the following way: let $s_0, ..., s_N$ be a basis of the finite dimensional complex vector space $H^0(M, \mathcal{O}(L))$ of holomorphic sections of L. Then f_L is given by

$$f_L(p) = [s_0(p) \colon \, \ldots : s_N(p)].$$

In this situation, the zero set of holomorphic sections of L are precisely the preimages of hyperplanes $H \cong \mathbb{C}P^{N-1} \subset \mathbb{C}P^N$. By Bertini's Theorem (cf. [61, Theorem 17.16]), the preimage is a smooth hypersurface for a generic hyperplane H. Hence L has a holomorphic section with zero set D, which is a smooth hypersurface with $c_1(L) = PD([D])$.

A line bundle L without base points is called *ample* if there exists an $n \geq 1$ such that the map $f_{L^{\otimes n}}$ defined by the line bundle $L^{\otimes n}$ is an embedding. By the *Nakai-Moishezon Criterion* (cf. [62, Chapter V, Theorem 1.10]) a line bundle L on a complex algebraic surface M is ample if and only if $L^2 > 0$ and $L \cdot C > 0$ for all irreducible curves C on M.

II.3.3 Invariants of complex surfaces

Let M be a compact complex surface, i.e. a smooth compact complex manifold of dimension 2. The *canonical line bundle* K of M is the bundle of holomorphic 2-forms on M. The *canonical class* is the first Chern class of the canonical bundle, also denoted by K. It is related to the first Chern class of the tangent bundle by $c_1(M) = c_1(TM, J) = -K$. We denote the trivial line bundle on M by $\mathcal{O}$. The following invariants are defined for M:

The *irregularity*

$$q(M) = h^{0,1}(M) = \dim H^1(M, \mathcal{O}).$$

The *geometric genus*

$$p_g(M) = h^{0,2}(M) = \dim H^2(M, \mathcal{O}).$$

The *plurigenera*

$$P_m(M) = \dim H^0(M, \mathcal{O}(mK)).$$

The *holomorphic Euler characteristic*

$$\chi_h(M) = \chi(\mathcal{O}) = 1 - q(M) + p_g(M).$$

Some of them can be related to topological invariants of the closed, oriented 4-manifold M:

- By the *Noether formula*, which is the Riemann-Roch formula for the holomorphic tangent bundle of M:

$$\begin{aligned}\chi_h(M) &= \tfrac{1}{12}(c_1^2(M) + c_2(M)) \\ &= \tfrac{1}{8}(c_1^2(M) - \sigma(M)) \\ &= \tfrac{1}{4}(e(M) + \sigma(M)).\end{aligned}$$

- For complex surfaces in general we have $b_1(M) = h^{1,0}(M) + q(M)$.

- If $b_1(M)$ is even, which is always the case for Kähler surfaces, $b_1(M) = 2q(M)$ and $b_2^+(M) = 2p_g(M) + 1$.

Definition 2.3. Let M be an arbitrary closed, oriented 4-manifold. We define the number

$$\chi_h(M) = \tfrac{1}{4}(e(M) + \sigma(M)).$$

If M admits the structure of a compact complex surface, then $\chi_h(M)$ is equal to the holomorphic Euler characteristic by the Noether formula. In the general case of an arbitrary closed oriented 4-manifold we can calculate $\chi_h(M)$ as

$$\chi_h(M) = \tfrac{1}{2}(1 - b_1(M) + b_2^+(M)).$$

Hence $\chi_h(M)$ is an integer if and only if $b_2^+(M) - b_1(M)$ is odd. On compact complex surfaces, $\chi_h(M)$ is by definition an integer. One can prove that the number $\chi_h(M)$ is also an integer if M admits an almost complex structure: Since $c_1(M,J) \equiv w_2(M)$ mod 2, the class $c_1(M,J)$ is characteristic. This implies that $c_1^2 \equiv \sigma(M)$ mod 8 by equation (2.1), hence $e(M) + \sigma(M) \equiv 0$ mod 4.

If M is a closed, *spin* 4-manifold, then

$$c_1^2(M) \equiv 8\chi_h(M) \text{ mod } 16.$$

This follows because $\sigma(M) = c_1^2(M) - 8\chi_h(M)$ and $\sigma(M) \equiv 0$ mod 16 by Rochlin's theorem. If M is a closed, *spin* 4-manifold which admits in addition an *almost complex structure*, then

$$c_1^2(M) \equiv 0 \text{ mod } 8.$$

This follows because $\chi_h(M)$ is in this case an integer.

II.3.4 Kodaira-Enriques classification

Let M be a compact complex surface. The *Kodaira dimension* $\kappa(M)$ of M can be defined as follows (see [47, 56]):

$$\kappa(M) = \min\{k \in \mathbb{Z} \mid P_n(M)/n^k \text{ is a bounded function of } n \geq 1\},$$

where $P_n(M)$ denote the plurigenera of M. This implies:

$\kappa(M) = -\infty$ if $P_n(M) = 0$ for all n.

$\kappa(M) = 0$ if some $P_n(M)$ is non-zero and $\{P_n(M)\}$ is a bounded sequence.

$\kappa(M) = 1$ if $\{P_n(M)\}$ is unbounded but $\{P_n(M)/n\}$ is bounded.

$\kappa(M) = 2$ if $\{P_n(M)/n\}$ is unbounded.

By definition, a surface of *general type* is a complex surface M with $\kappa(M) = 2$. In the remaining cases the following is known by the Kodaira-Enriques classification:

- If M is a minimal complex surface with $\kappa(M) = -\infty$ then M is either $\mathbb{C}P^2$, *geometrically ruled* or of *Class VII*. A geometrically ruled surface is by definition a holomorphic $\mathbb{C}P^1$-bundle over a Riemann surface and a surface of Class VII is by definition a complex surface with $\kappa(M) = -\infty$ and $b_1(M) = 1$.

- If M is a simply-connected minimal complex surface with $\kappa(M) = 0$ then M is a $K3$-*surface*. A $K3$-surface is by definition a complex surface M with trivial canonical bundle and $b_1(M) = 0$. Every $K3$-surface is simply-connected and Kähler. Any two $K3$-surfaces are diffeomorphic.

- If M is a minimal surface with $\kappa(M) = 1$ then M is an *elliptic surface*. An elliptic surface is by definition a complex surface M with a holomorphic projection $\pi\colon M \to C$ onto a compact complex curve, such that the generic fibres of π are elliptic curves. Note that there are elliptic surfaces with $\kappa(M) = -\infty$ or 0 (e.g. $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ or $K3$-surfaces).

II.3.5 Elliptic surfaces

Additional references for this section are [48, 53]. Let M be an elliptic surface with elliptic fibration $\pi\colon M \to C$. We will only consider the case where M is smooth and usually $C = \mathbb{C}P^1$. In particular, an elliptic fibration $\pi\colon M \to S^2$ is a singular T^2-fibration. A *relatively minimal elliptic surface* is an elliptic surface, which is not the blow-up of another elliptic surface. One can give a complete list of relatively minimal simply-connected elliptic surfaces:

- There exist simply-connected elliptic surfaces without multiple fibres, denoted by $E(n)$ for $n \geq 1$, with invariants

  $$\begin{array}{llll} b_2(E(n)) = 12n-2, & b_2^+(E(n)) = 2n-1, & p_g(E(n)) = n-1. & \\ e(E(n)) = 12n, & \sigma(E(n)) = -8n, & c_1^2(E(n)) = 0, & \chi_h(E(n)) = n. \end{array}$$

  In particular, $E(1) \cong \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ and $E(2)$ is a $K3$-surface. The elliptic surface $E(n)$ is spin if and only if n is even. The canonical class of $E(n)$ is given by

  $$K = (n-2)F$$

  where F denotes the class of a general fibre.

- There exist simply-connected elliptic surfaces with multiple fibres, denoted by $E(n)_{p,q}$ with $n \geq 1$ and p, q coprime. The surfaces $E(n)_{p,q}$ have the same Betti numbers and Chern invariants as $E(n)$ above and $E(n)_{1,1} = E(n)$. If n is odd, then all $E(n)_{p,q}$ are non-spin. If n is even, then $E(n)_{p,q}$ is spin if and only if pq is odd. The class of a general fibre F is divisible by pq. Let f denote the homology class $\frac{1}{pq}F$. Then f is indivisible in homology and the canonical class of $E(n)_{p,q}$ is given by

  $$K = (npq - p - q)f.$$

These surfaces can be classified up to diffeomorphism as follows, cf. [56, Section 3.3]: If n is ≥ 2 then $E(n)_{p,q}$ and $E(n)_{p',q'}$ are diffeomorphic if and only if $\{p, q\} = \{p', q'\}$ as unordered pairs. The surfaces $E(1)_{p,q}$ are called *Dolgachev surfaces*. For $p \geq 1$, the surfaces $E(1)_{1,p}$ are all diffeomorphic to $E(1)$. If p, q, p', q' are ≥ 2 then $E(1)_{p,q}$ is diffeomorphic to $E(1)_{p',q'}$ if and only if $\{p, q\} = \{p', q'\}$. These surfaces are never diffeomorphic to $E(1)$.

II.3.6 Surfaces of general type

Let M be a smooth minimal surface of general type. Every complex surface of general type is algebraic. There are a number of important inequalities, which the invariants of M have to satisfy:

$c_1^2(M) > 0$ and $c_2(M) = e(M) > 0$

$c_1^2(M) \geq 2p_g(M) - 4$ (Noether's inequality)

$c_1^2(M) \leq 3c_2(M)$ (Bogomolov-Miyaoka-Yau inequality)

If M is a minimal surface of general type and C an irreducible complex curve on M, then $K_M C \geq 0$ with equality if and only if C is a smooth rational curve of self-intersection -2. Hence by the Nakai-Moishezon Criterion (cf. Section II.3.2) the canonical bundle K_M is ample if and only if M does not contain rational (-2)-curves.

II.3.7 Pluricanonical divisors

Let M be a minimal smooth complex algebraic surface of general type. We consider the multiples $L = nK = K^{\otimes n}$ of the canonical line bundle of M. By a theorem of Bombieri ([12], [8]), all divisors in the linear system $|nK|$ are connected. If $|nK|$ is base point free, then we can find a non-singular divisor representing nK by subsection II.3.2. The question of existence of base points in pluricanonical systems of the form $|nK|$ has been studied in great detail. We summarize what is known in the following theorem.

Theorem 2.4. *Let M be a minimal smooth complex algebraic surface of general type. Then the pluricanonical system $|nK|$ has no base points in the following cases:*

- $n \geq 4$
- $n = 3$ *and* $K^2 \geq 2$
- $n = 2$ *and* $K^2 \geq 5$ *or* $p_g \geq 1$.

For the proofs and references see [11, 12, 23, 73, 98, 118]. The case $n \geq 4$ has been proved by Kodaira who also proved the case $n = 3, K^2 \geq 2$ for $p_g > 1$; in this case the claim for $p_g = 0, 1$ has been proved by Bombieri. Reider reproved these results and the case $n = 2, K^2 \geq 5$. The case $n = 2, K^2 \leq 4, p_g \geq 1$ has been proved more recently.

Remaining cases: We describe what is known in the cases with $n \geq 2$ not covered by Theorem 2.4. Suppose $\mathbf{n = 3, K^2 = 1}$: By Noether's inequality $K^2 = 1$ implies $p_q \leq 2$. We discuss each case $p_g = 0, 1, 2$ separately.

(1.) A *numerical Godeaux surface* is by definition a minimal surface M of general type with $K^2 = 1, p_g = 0$. The number b of base points of $|3K|$ on such a surface is determined by $\mathrm{Tor} H_2(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z})$ in the following way (see [101]):

$$b = \tfrac{1}{2} \left| \{ t \in H_1(M;\mathbb{Z}) \mid t \neq -t \} \right| .$$

For numerical Godeaux surfaces $H_1(M;\mathbb{Z})$ can only be a cyclic group of order ≤ 5. All these cases occur [117]. In particular, $|3K|$ is base point free if $H_1(M;\mathbb{Z}) = 0$ or $\mathbb{Z}_2$, e.g. if M is simply-connected.

(2.) On surfaces with $K^2 = 1, p_g = 1$, the linear system $|3K|$ is always base point free [19].

(3.) If $K^2 = 1, p_g = 2$, then $|3K|$ always has a base point [11].

Suppose $\mathbf{n = 2, p_g = 0}$ and $\mathbf{1 \leq K^2 \leq 4}$:

(1.) If M is a numerical Godeaux surfaces ($K^2 = 1$) then $|2K|$ always has base points.

(2.) No example is known of a surface with $p_g = 0$ and $2 \leq K^2 \leq 4$ such that $|2K|$ has base points [98]. This includes *numerical Campedelli surfaces*, i.e. minimal surfaces of general type with $K^2 = 2, p_g = 0$. For $K^2 = 4$ it is known that $|2K|$ is base point free under certain assumptions on the fundamental group of M, in particular if $\pi_1(M)$ is cyclic or of odd order [23, 77].

Chapter III

Minimality and irreducibility of symplectic 4-manifolds

Contents

We prove that all minimal symplectic four-manifolds are essentially irreducible. We also clarify the relationship between holomorphic and symplectic minimality of Kähler surfaces. This leads to a new proof of the deformation-invariance of holomorphic minimality for complex surfaces with even first Betti number which are not Hirzebruch surfaces.[1]

III.1 Introduction and statement of results

In this chapter we discuss certain geometric and topological properties of symplectic four-manifolds. Our main concern is the notion of minimality and its topological consequences. We shall extend to manifolds with $b_2^+ = 1$ the irreducibility result proved in [79, 80] for the case that $b_2^+ > 1$. We also show that holomorphic and symplectic minimality are equivalent precisely for those Kähler surfaces which are not Hirzebruch surfaces. Together with work of Buchdahl [17], this yields a new proof of the deformation-invariance of holomorphic minimality for complex surfaces with even first Betti number, again with the exception of Hirzebruch surfaces.

III.1.1 Minimality

A complex surface is said to be minimal if it contains no holomorphic sphere of selfintersection -1, see for example [8]. A symplectic four-manifold is usually considered to be minimal if it contains no symplectically embedded sphere of selfintersection -1, see for example [92, 52]. In the case of a Kähler surface both notions of minimality can be considered, but it is not at all obvious whether they agree. In

[1] This chapter has been published under the same title with D. Kotschick in *Int. Math. Res. Notices 2006, Art. ID 35032, 1-13*.

the recent literature on symplectic four-manifolds there are frequent references to (symplectic) minimality, and often Kähler surfaces are considered as examples, but we have found no explicit discussion of the relationship between the two definitions in print, compare e. g. [92, 93, 94, 122, 52, 80, 54].

An embedded holomorphic curve in a Kähler manifold is a symplectic submanifold. Therefore, for Kähler surfaces symplectic minimality implies holomorphic minimality. The following counterexample to the converse should be well known:

Example 3.1. Let $X_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ be the n^{th} Hirzebruch surface. If n is odd and $n > 1$, then X_n is holomorphically minimal but not symplectically minimal.

In Section III.2 below we explain this example in detail, and then we prove that there are no other counterexamples:

Theorem 3.2. *A Kähler surface that is not a Hirzebruch surface X_n with n odd and $n > 1$ is holomorphically minimal if and only if it is symplectically minimal.*

A proof can be given using the known calculations of Seiberg–Witten invariants of Kähler surfaces. Using Seiberg–Witten theory, it turns out that for non-ruled Kähler surfaces symplectic and holomorphic minimality coincide because they are both equivalent to smooth minimality, that is, the absence of smoothly embedded (-1)-spheres. The case of irrational ruled surfaces is elementary.

Such a proof is not satisfying conceptually, because the basic notions of symplectic topology should be well-defined without appeal to results in gauge theory. Therefore, in Section III.2 we give a proof of Theorem 3.2 within the framework of symplectic topology, using Gromov's theory of J-holomorphic curves. We shall use results of McDuff [92] for which Gromov's compactness theorem is crucial. Essentially the same argument can be used to show that symplectic minimality is a deformation-invariant property, see Theorem 3.6. This natural result is lurking under the surface of McDuff's papers [92, 93, 94], and is made explicit in [98], compare also [121, 122]. Of course this result is also a corollary of Taubes's deep work in [132, 134, 135, 80], where he showed, among other things, that if there is a smoothly embedded (-1)-sphere, then there is also a symplectically embedded one.

In Section III.2 we shall also prove that for compact complex surfaces with even first Betti number which are not Hirzebruch surfaces holomorphic minimality is preserved under deformations of the complex structure. This result is known, and is traditionally proved using the Kodaira classification, cf. [8]. The proof we give is intrinsic and independent of the classification. Instead, we combine the result of Buchdahl [17] with the deformation invariance of symplectic minimality and Theorem 3.2.

III.1.2 Irreducibility

Recall that an embedded (-1)-sphere in a four-manifold gives rise to a connected sum decomposition where one of the summands is a copy of $\overline{\mathbb{C}P^2}$. For symplectic manifolds no other non-trivial decompositions are known. Gompf [52] conjectured that minimal symplectic four-manifolds are irreducible, meaning that in any smooth connected sum decomposition one of the summands has to be a homotopy sphere. In Section III.3 below we shall prove the following result in this direction:

Theorem 3.3. *Let X be a minimal symplectic 4-manifold with $b_2^+ = 1$. If X splits as a smooth connected sum $X = X_1 \# X_2$, then one of the X_i is an integral homology sphere whose fundamental group has no non-trivial finite quotient.*

For manifolds with $b_2^+ > 1$ the corresponding result was first proved in [79] and published in [80]. As an immediate consequence of these results we verify Gompf's irreducibility conjecture in many cases:

Corollary 3.4. *Minimal symplectic* 4*-manifolds with residually finite fundamental groups are irreducible.*

To prove Theorem 3.3 we shall follow the strategy of the proof for $b_2^+ > 1$ in [79, 80]. In particular we shall use the deep work of Taubes [132, 134, 135], which produces symplectic submanifolds from information about Seiberg–Witten invariants. What is different in the case $b_2^+ = 1$, is that the Seiberg–Witten invariants depend on chambers, and one has to keep track of the chambers one is working in.

In addition to conjecturing the irreducibility of minimal symplectic four-manifolds, Gompf [52] also raised the question whether minimal non-ruled symplectic four-manifolds satisfy $K^2 \geq 0$, where K is the canonical class. For manifolds with $b_2^+ > 1$ this was proved by Taubes [132, 134], compare also [80, 135]. The case $b_2^+ = 1$ was then treated by Liu [90], who refers to this question as "Gompf's conjecture". Liu [90] also proved that minimal symplectic four-manifolds which are not rational or ruled satisfy $K \cdot \omega \geq 0$. We shall use Liu's inequalities to keep track of the chambers in our argument. Although the results of Liu [90], and also those of Li–Liu [88, 89], are related to Theorem 3.3, this theorem does not appear there, or anywhere else in the literature that we are aware of.

III.2 Notions of minimality

First we discuss the Hirzebruch surfaces $X_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$, with n odd and > 1, in order to justify the assertions made in Example 3.1 in the Introduction.

If $n = 2k + 1$, consider the union of a holomorphic section S of X_n of selfintersection $-n$ and of k disjoint parallel copies of the fibre F. This reducible holomorphic curve can be turned into a symplectically embedded sphere E by replacing each of the transverse intersections of S and F by a symplectically embedded annulus. Then

$$E \cdot E = (S + kF)^2 = S \cdot S + 2k\, S \cdot F = -n + 2k = -1 \ .$$

This shows that X_n is not symplectically minimal. To see that it is holomorphically minimal, note that a homology class E containing a smooth holomorphic (-1)-sphere would satisfy $E^2 = K \cdot E = -1$, and would therefore be $S + kF$, as above. However, this class has intersection number

$$E \cdot S = (S + kF) \cdot S = -n + k = -k - 1 < 0$$

with the smooth irreducible holomorphic curve S. Therefore, E can only contain a smooth irreducible holomorphic curve if $E = S$, in which case $k = 0$ and $n = 1$.

Next we prove that for all other Kähler surfaces symplectic and holomorphic minimality are equivalent.

Proof of Theorem 3.2. In view of the discussion in III.1.1 above, we only have to prove that if (X, ω) is a Kähler surface which is not a Hirzebruch surface X_n with n odd and $n > 1$, then holomorphic minimality implies symplectic minimality.

We start by assuming that (X, ω) is not symplectically minimal, so that it contains a smoothly embedded (-1)-sphere $E \subset X$ with $\omega|_E \neq 0$. Orient E so that $\omega|_E > 0$, and denote by $[E] \in H_2(X; \mathbb{Z})$ the corresponding homology class. The almost complex structures J compatible with ω are all homotopic to the given integrable J_∞; in particular their canonical classes agree with the canonical class K of the Kähler structure. It is elementary to find a compatible J for which the sphere E with the chosen orientation is J-holomorphic. Therefore E satisfies the adjunction formula

$$g(E) = 1 + \tfrac{1}{2}(E^2 + K \cdot E) \ .$$

We conclude that $K \cdot E = -1$. (Note that the orientation of E is essential here.) This implies in particular that the expected dimension of the moduli space of J-holomorphic curves in the homology class $[E]$ vanishes.

Let $\mathcal{J}$ be the completion – with respect to a suitable Sobolov norm – of the space of C^∞ almost complex structures compatible with ω, cf. [95]. McDuff has proved that, for almost complex structures J from an everywhere dense subset in $\mathcal{J}$, there is a unique smooth J-holomorphic sphere C in the homology class $[E]$, see Lemma 3.1 in [92].

The uniqueness implies that the curve C varies smoothly with J. One then uses Gromov's compactness theorem for a family of almost complex structures to conclude that for all J, not necessarily generic, there is a unique J-holomorphic representative of the homology class $[E]$ which, if it is not a smooth curve, is a reducible curve $C = \sum_i C_i$ such that each C_i is a smooth J-holomorphic sphere. Compare again Lemma 3.1 in [92] and [95]. (In these references reducible J-holomorphic curves are called cusp curves.)

Let J_j be a sequence of generic almost complex structures in $\mathcal{J}$ which converges to the integrable J_∞ as $j \to \infty$. For each J_j there is a smooth J_j-holomorphic sphere E_j in the homology class $[E]$. As $j \to \infty$, the E_j converge weakly to a possibly reducible J_∞-holomorphic curve E_∞. If E_∞ is irreducible, then it is a holomorphic (-1)-sphere, showing that (X, J_∞) is not holomorphically minimal. If E_∞ is reducible, let

$$E_\infty = \sum_{i=1}^{k} m_i C_i$$

be the decomposition into irreducible components. The multiplicities m_i are positive integers. Each C_i is an embedded sphere, and therefore the adjunction formula implies

$$C_i^2 + K \cdot C_i = -2 \ .$$

Multiplying by m_i and summing over i we obtain

$$\sum_{i=1}^{k} m_i C_i^2 + K \cdot \sum_{i=1}^{k} m_i C_i = -2 \sum_{i=1}^{k} m_i \ .$$

Now the second term on the left hand side equals $K \cdot E = -1$, so that we have

$$\sum_{i=1}^{k} m_i C_i^2 = 1 - 2 \sum_{i=1}^{k} m_i \ .$$

It follows that there is an index i such that $C_i^2 \geq -1$. If $C_i^2 = -1$ for some i, then we again conclude that (X, J_∞) is not holomorphically minimal. If $C_i^2 \geq 0$ for some i, then (X, J_∞) is birationally ruled or is rational, cf. Proposition 4.3 in Chapter V of [8]. Thus, if it is holomorphically minimal, it is either a minimal ruled surface or $\mathbb{C}P^2$, but the latter is excluded by our assumption that (X, J_∞) is not symplectically minimal. If (X, J_∞) were ruled over a surface of positive genus, $X \xrightarrow{\pi} B$, then the embedding of the (-1)-sphere E would be homotopic to a map with image in a fibre, because $\pi|_E \colon E \to B$ would be homotopic to a constant. But this would contradict the fact that E has non-zero selfintersection.

Thus we finally reach the conclusion that (X, J_∞) is ruled over $\mathbb{C}P^1$. If it is holomorphically minimal, then it is a Hirzebruch surface X_n with n odd and $n > 1$, because X_1 is not holomorphically minimal, and X_{2k} has even intersection form and is therefore symplectically minimal.

This concludes the proof of Theorem 3.2. $\square$

Remark 3.5. We have used that the existence of a rational holomorphic curve of non-negative selfintersection in a complex surface implies that the surface is rational or ruled. Such a statement also holds in the symplectic category, cf. [92], but we do not need that here.

The exposition of the proof of Theorem 3.2 can be shortened considerably if one simply uses McDuff's Lemma 3.1 from [92] as a black box. We have chosen to include some of the details so that the reader can see that the degeneration of the J_j-holomorphic curves E_j as $j \to \infty$ is the exact inverse of the regeneration used in the discussion of Example 3.1.

The following theorem, Proposition 2.3.A in [98], can be proved by essentially the same argument, allowing the symplectic form to vary smoothly, compare also [92, 122]:

Theorem 3.6 ([98]). *Symplectic minimality is a deformation-invariant property of compact symplectic four-manifolds.*

Note that holomorphic minimality of complex surfaces is not invariant under deformations of the complex structure. In the Kähler case the Hirzebruch surfaces X_n with n odd are all deformation-equivalent, but are non-minimal for $n = 1$ and minimal for $n > 1$. In the non-Kähler case there are other examples among the so-called Class VII surfaces.

For complex surfaces of non-negative Kodaira dimension it is true that holomorphic minimality is deformation-invariant, but the traditional proofs for this are exceedingly cumbersome, see for example [8], section 7 of Chapter VI, where it is deduced from the Kodaira classification and a whole array of additional results. For the case of even first Betti number we now give a direct proof, which does not use the classification.

Theorem 3.7. *Let X be a holomorphically minimal compact complex surface with even first Betti number, which is not a Hirzebruch surface X_n with n odd. Then any surface deformation equivalent to X is also holomorphically minimal.*

Proof. Let X_t with $t \in [0, 1]$ be a smoothly varying family of complex surfaces such that $X_0 = X$. Buchdahl [17] has proved that every compact complex surface with even first Betti number is Kählerian, without appealing to any classification results. Thus, each X_t is Kählerian, and we would like to choose Kähler forms ω_0 and ω_1 on X_0 and X_1 respectively, which can be joined by a smooth family of symplectic forms ω_t. There are two ways to see that this is possible.

On the one hand, Buchdahl [17] characterizes the Kähler classes, and one can check that one can choose a smoothly varying family of Kähler classes for X_t, which can then be realized by a smoothly varying family of Kähler metrics. On the other hand, we could just apply Buchdahl's result for each value of the parameter t separately, without worrying about smooth variation of the Kähler form with the parameter, and then construct a smooth family ω_t of symplectic not necessarily Kähler forms from this, cf. [122] Proposition 2.1. In detail, start with arbitrary Kähler forms ω_t on X_t. As the complex structure depends smoothly on t, there is an open neighbourhood of each $t_0 \in [0, 1]$ such that ω_{t_0} is a compatible symplectic form for all X_s with s in this neighbourhood of t_0. By compactness of $[0, 1]$, we only need finitely many such open sets to cover $[0, 1]$. On the overlaps we can deform these forms by linear interpolation, because the space of compatible symplectic forms is convex. In this way we obtain a smoothly varying family of symplectic forms.

Now $X = X_0$ was assumed to be holomorphically minimal and not a Hirzebruch surface X_n with odd n. Therefore, Theorem 3.2 shows that X_0 is symplectically minimal, and Theorem 3.6 then implies that X_1 is also symplectically minimal. The easy direction of Theorem 3.2 shows that X_1 is holomorphically minimal. $\square$

Let us stress once more that this result is not new, but its proof is. The above proof does not use the Kodaira classification. The only result we have used from the traditional theory of complex surfaces is that a surface containing a holomorphic sphere of positive square is rational, which entered in the proof of Theorem 3.2. We have not used the generalization of this result to symplectic manifolds, and we have not used any Seiberg–Witten theory either. Our proof does depend in an essential way on the work of Buchdahl [17]. Until that work, the proof that complex surfaces with even first Betti numbers are Kählerian depended on the Kodaira classification.

III.3 Connected sum decompositions of minimal symplectic 4-manifolds

In this section we prove restrictions on the possible connected sum decompositions of a minimal symplectic four-manifold with $b_2^+ = 1$, leading to a proof of Theorem 3.3. To do this we have to leave the realm of symplectic topology and use Seiberg–Witten gauge theory.

Let X be a closed oriented smooth 4-manifold with $b_2^+(X) = 1$. We fix a Spin^c structure $\mathfrak{s}$ and a metric g on X and consider the Seiberg–Witten equations for a positive spinor ϕ and a Spin^c connection A:

$$\begin{aligned} D_A^+\phi &= 0 \\ F_{\hat{A}}^+ &= \sigma(\phi,\phi) + \eta \, , \end{aligned}$$

where the parameter η is an imaginary-valued g-self-dual 2-form. Here $\hat{A}$ denotes the $U(1)$-connection on the determinant line bundle induced from A, so that $F_{\hat{A}}$ is an imaginary-valued 2-form. A reducible solution of the Seiberg–Witten equations is a solution with $\phi = 0$.

For every Riemannian metric g there exists a g-self-dual harmonic 2-form ω_g with $[\omega_g]^2 = 1$. Because $b_2^+(X) = 1$, this 2-form is determined by g up to a sign. We choose a forward cone, i. e. one of the two connected components of $\{\alpha \in H^2(X;\mathbb{R}) \mid \alpha^2 > 0\}$. Then we fix ω_g by taking the form whose cohomology class lies in the forward cone.

Let L be the determinant line bundle of the Spin^c structure $\mathfrak{s}$. The curvature F_A represents $\frac{2\pi}{i}c_1(L)$ in cohomology, and every form which represents this class can be realized as the curvature of $\hat{A}$ for a Spin^c connection A. For given (g,η) there exists a reducible solution of the Seiberg–Witten equations if and only if there is a Spin^c connection A such that $F_{\hat{A}}^+ = \eta$, equivalently $(c_1(L) - \frac{i}{2\pi}\eta)\cdot\omega_g = 0$. Define the discriminant of the parameters (g,η) by

$$\Delta_L(g,\eta) = (c_1(L) - \tfrac{i}{2\pi}\eta)\cdot\omega_g \, .$$

One divides the space of parameters (g,η) for which there are no reducible solutions into the plus and minus chambers according to the sign of the discriminant. Two pairs of parameters (g_1,η_1) and (g_2,η_2) can be connected by a path avoiding reducible solutions if and only if their discriminants have the same sign, i. e. if and only if they lie in the same chamber. A cobordism argument then shows that the Seiberg–Witten invariant is the same for all parameters in the same chamber. In this way we get the invariants $SW_+(X,\mathfrak{s})$, $SW_-(X,\mathfrak{s})$ which are constant on the corresponding chambers.

Suppose now that X has a symplectic structure ω. Then ω determines an orientation of X and a forward cone in $H^2(X;\mathbb{R})$. We will take the chambers with respect to this choice. Moreover, ω determines a canonical class K and a Spin^c structure $\mathfrak{s}_{K^{-1}}$ with determinant K^{-1}. One can obtain every other Spin^c structure by twisting $\mathfrak{s}_{K^{-1}}$ with a line bundle E, to obtain $\mathfrak{s}_{K^{-1}} \otimes E$. This Spin^c structure has determinant $K^{-1} \otimes E^2$.

The *Taubes chamber* is the chamber determined by parameters (g, η) with g chosen such that it is almost Kähler with $\omega_g = \omega$ and

$$\eta = F^+_{A_0} - \tfrac{i}{4} r\omega \quad \text{with} \quad r \gg 0\,,$$

where A_0 is a canonical connection on K^{-1}. We have the following:

Lemma 3.8. *The Taubes chamber is the minus chamber, for the choice of forward cone as above.*

Proof. We have

$$\begin{aligned}(c_1(-K) - \tfrac{i}{2\pi}\eta)\cdot\omega_g &= (\tfrac{i}{2\pi}F_{\hat{A}_0} - \tfrac{i}{2\pi}F^+_{\hat{A}_0} - \tfrac{1}{8\pi}r\omega)\cdot\omega \\ &= (\tfrac{i}{2\pi}F^-_{\hat{A}_0} - \tfrac{1}{8\pi}r\omega)\cdot\omega \\ &= -\tfrac{1}{8\pi}r\omega^2 < 0\,,\end{aligned}$$

because the wedge product of a self-dual and an anti-self-dual two-form vanishes. □

The following theorem is due to Taubes [131, 132, 134], compare [88, 89] for the case $b_2^+ = 1$.

Theorem 3.9. *The Seiberg–Witten invariant in the minus chamber for the canonical* Spin^c *structure is non-zero. More precisely, $SW_-(X, \mathfrak{s}_{K^{-1}}) = \pm 1$. Moreover, if $SW_-(X, \mathfrak{s}_{K^{-1}} \otimes E)$ is non-zero and $E \neq 0$, then for a generic ω-compatible almost complex structure J, the Poincaré dual of the Chern class of E can be represented by a smooth J-holomorphic curve $\Sigma \subset X$.*

We have the following more precise version of the second part of Theorem 3.9, which is also due to Taubes.

Proposition 3.10. *Suppose $SW_-(X, \mathfrak{s}_{K^{-1}} \otimes E)$ is non-zero, and $E \neq 0$. Then for a generic almost complex structure J compatible with ω there exist disjoint embedded J-holomorphic curves C_i in X such that*

$$PD(c_1(E)) = \sum_{i=1}^{n} m_i [C_i],$$

where each C_i satisfies $K \cdot C_i \leq C_i \cdot C_i$ and each multiplicity m_i is equal to 1, except possibly for those i for which C_i is a torus with self-intersection zero.

This depends on a transversality result for J-holomorphic curves, see Proposition 7.1 in [134] and also [135, 80]. Proposition 3.10 immediately implies the following:

Corollary 3.11. *If $SW_-(X, \mathfrak{s}_{K^{-1}} \otimes E) \neq 0$ with $E^2 < 0$ then X contains an embedded symplectic (-1)-sphere Σ.*

Proof. Choose a generic compatible almost complex structure J as in Proposition 3.10, and consider $E = \sum_i m_i C_i$. Then $E^2 = \sum_i m_i^2 C_i^2$ because the C_i are disjoint, hence $C_j^2 < 0$ for some j. We can compute the genus of C_j from the adjunction formula:

$$g(C_j) = 1 + \tfrac{1}{2}(C_j \cdot C_j + K \cdot C_j) \leq 1 + C_j \cdot C_j \leq 0.$$

Hence $\Sigma = C_j$ is a sphere with self-intersection number -1. □

After these preparations we can now prove Theorem 3.3.

Proof of Theorem 3.3. Let (X, ω) be a closed symplectic 4-manifold with $b_2^+ = 1$. We denote by K both the first Chern class of any compatible almost complex structure, and the complex line bundle with this Chern class.

First, suppose that (X, ω) is symplectically minimal and rational or ruled. Then, by the classification of ruled symplectic four-manifolds, X is diffeomorphic either to $\mathbb{C}P^2$, to an even Hirzebruch surface, or to a geometrically ruled Kähler surface over a complex curve of positive genus, compare e. g. [95]. These manifolds are all irreducible for purely topological reasons. This is clear for $\mathbb{C}P^2$ and for the even Hirzebruch surfaces, because the latter are diffeomorphic to $S^2 \times S^2$. For the irrational ruled surfaces note that the fundamental group is indecomposable as a free product. Therefore, in any connected sum decomposition one of the summands is simply connected. If this summand were not a homotopy sphere, then the other summand would be a smooth four-manifold with the same fundamental group but with strictly smaller Euler characteristic than the ruled surface. This is impossible, because the irrational ruled surfaces realize the smallest possible Euler characteristic for their fundamental groups, compare [78].

Thus, we may assume that (X, ω) is not only symplectically minimal, but also not rational or ruled. Then Liu's results in [90] tell us that $K^2 \geq 0$ and $K \cdot \omega \geq 0$.

If X decomposes as a connected sum $X = M \# N$ then one of the summands, say N, has negative definite intersection form. Moreover, the fundamental group of N has no non-trivial finite quotients, by Proposition 1 of [81]. In particular $H_1(N; \mathbb{Z}) = 0$, and hence the homology and cohomology of N are torsion-free. If N is an integral homology sphere, then there is nothing more to prove.

Suppose N is not an integral homology sphere. By Donaldson's theorem [31], the intersection form of N is diagonalizable over $\mathbb{Z}$. Thus there is a basis $e_1, ..., e_n$ of $H^2(N; \mathbb{Z})$ consisting of elements with square -1 which are pairwise orthogonal. Write

$$K = K_M + \sum_{i=1}^{n} a_i e_i \ ,$$

with $K_M \in H^2(M; \mathbb{Z})$. The $a_i \in \mathbb{Z}$ are odd, because K is a characteristic vector. This shows in particular that K is not a torsion class. Its orthogonal complement $K^\perp$ in $H^2(X; \mathbb{R})$ is then a hyperplane. As $K^2 \geq 0$ and $b_2^+(X) = 1$, the hyperplane $K^\perp$ does not meet the positive cone. Thus Liu's inequality $K \cdot \omega \geq 0$ must be strict: $K \cdot \omega > 0$.

Now we know $SW_-(X, \mathfrak{s}_{K^{-1}}) = \pm 1$ from Taubes's result, where $\mathfrak{s}_{K^{-1}}$ is the Spin^c structure with determinant K^{-1} induced by the symplectic form ω. The inequality $(-K) \cdot \omega < 0$ shows that a pair $(g, 0)$ is in the negative, i. e. the Taubes chamber, whenever g is almost Kähler with fundamental two-form ω. As $K^\perp$ does not meet the positive cone, all pairs $(g, 0)$ are in the negative chamber, for all Riemannian metrics g. We choose a family of Riemannian metrics g_r on X which pinches the neck connecting M and N down to a point as $r \to \infty$. For r large we may assume that g_r converges to metrics on the (punctured) M and N, which we denote by g_M and g_N.

Lemma 3.12. *If we choose the forward cone for M to be such that it induces on X the forward cone determined by the symplectic structure, then for every Riemannian metric g' on M, the point $(g', 0)$ is in the negative chamber of M with respect to the Spin^c structure $\mathfrak{s}_M$ on M obtained by restriction of $\mathfrak{s}_{K^{-1}}$.*

Proof. The chamber is determined by the sign of $c_1(\mathfrak{s}) \cdot \omega_g$. We have

$$\begin{aligned} 0 > (-K) \cdot \omega_{g_r} = c_1(\mathfrak{s}_{K^{-1}}) \cdot \omega_{g_r} &= c_1(\mathfrak{s}_M) \cdot \omega_{g_r} + c_1(\mathfrak{s}_N) \cdot \omega_{g_r} \\ &\longrightarrow c_1(\mathfrak{s}_M) \cdot \omega_{g_M} + c_1(\mathfrak{s}_N) \cdot \omega_{g_N}, \text{ as } r \to \infty \ . \end{aligned}$$

We know that ω_{g_N} is self-dual harmonic with respect to g_N, and hence vanishes because $b_2^+(N) = 0$. This implies that $c_1(\mathfrak{s}_{K^{-1}}) \cdot \omega_{g_r}$ converges to $c_1(\mathfrak{s}_M) \cdot \omega_{g_M}$ for $r \to \infty$. Thus

$$c_1(\mathfrak{s}_M) \cdot \omega_{g_M} \leq 0 \ .$$

However, we have $c_1(\mathfrak{s}_M) = K_M^{-1}$, and

$$K_M^2 = K^2 + \sum_{i=1}^{n} a_i^2 \geq K^2 + n \geq n \geq 1 \ ,$$

showing that $K_M^{\perp}$ does not meet the positive cone of M. Thus $c_1(\mathfrak{s}_M) \cdot \omega_{g_M} < 0$. Again because $K_M^{\perp}$ does not meet the positive cone of M, this inequality holds for all metrics g' on M. □

The degeneration of the g_r as r goes to infinity takes place in the negative chamber for $\mathfrak{s}_{K^{-1}}$, where the Seiberg–Witten invariant is ± 1, and by the Lemma g_M is in the negative chamber for $\mathfrak{s}_M$. It follows that $SW_-(M, \mathfrak{s}_M) = \pm 1$.

We now reverse the metric degeneration, but use a different Spin^c structure on N. Instead of using $\mathfrak{s}_N$ with $c_1(\mathfrak{s}_N) = -\sum_{i=1}^n a_i e_i$, we use the unique Spin^c structure $\mathfrak{s}'_N$ with $c_1(\mathfrak{s}'_N) = a_1 e_1 - \sum_{i=2}^n a_i e_i$. For every metric on N there is a unique reducible solution of the Seiberg–Witten equations for this Spin^c structure with $\eta = 0$. Gluing this solution to the solutions on M given by the invariant $SW_-(M, \mathfrak{s}_M)$, we find $SW_-(X, \mathfrak{s}') = \pm 1$, where $\mathfrak{s}'$ is the Spin^c structure on X obtained from $\mathfrak{s}_M$ and $\mathfrak{s}'_N$, compare Proposition 2 of [81]. We have $\mathfrak{s}' = \mathfrak{s}_{K^{-1}} \otimes E$, with $E = a_1 e_1$. Therefore $E^2 = -a_1^2 \leq -1$, and Corollary 3.11 shows that X is not minimal. This completes the proof of Theorem 3.3. □

Chapter IV

On the conformal systoles of 4-manifolds

Contents

We extend a result of M. Katz on the conformal systoles for blow-ups of the projective plane to all four-manifolds with $b_2^+ = 1$ and odd intersection form of type $(+1) \oplus n(-1)$. The same result holds for all four-manifolds with $b_2^+ = 1$ with even intersection form of type $-nE_8 \oplus H$ for $n \geq 0$ and which are symplectic or satisfy the so-called $\frac{5}{4}$-conjecture.[1]

IV.1 Introduction

There are several notions of systolic invariants for Riemannian manifolds, which were introduced by M. Berger and M. Gromov (see [59] and [9, 28] for an overview). The most basic concept is the k-systole $\mathrm{sys}_k(X, g)$ of a Riemannian manifold X, defined as the infimum over the volumes of all cycles representing non-zero classes in $H_k(X; \mathbb{Z})$. In this note we discuss a different systole, namely the *conformal systole*, which depends only on the conformal class of the Riemannian metric. We briefly review its definition (see Section IV.2 for details).

Let (X^{2n}, g) be a closed oriented even dimensional Riemannian manifold. The Riemannian metric defines an L^2-norm on the space of harmonic n-forms on X and hence induces a norm on the middle-dimensional cohomology $H^n(X; \mathbb{R})$. The *conformal n-systole* $\mathrm{confsys}_n(X, g)$ is the smallest norm of a non-zero element in the integer lattice $H^n(X; \mathbb{Z})_{\mathbb{R}}$ in $H^n(X; \mathbb{R})$. It is known that for a fixed manifold X the conformal n-systoles are bounded from above as g varies over all Riemannian metrics. Hence the supremum $CS(X) = \sup_g \mathrm{confsys}_n(X, g)$ of the conformal systoles over all metrics g is a finite number, which is *a priori* a diffeomorphism invariant of X.

The interest in the literature has been to find bounds for $CS(X)$ that depend only on the topology of X, e.g. the Euler characteristic of X, where X runs over some class of manifolds. In [18] P. Buser

[1] This chapter has been published under the same title in *Manuscripta math.* **121**, *417-424 (2006).*

and P. Sarnak proved the following inequalities for the closed orientable surfaces Σ_s of genus s: there exists a constant $C > 0$ independent of s such that

$$C^{-1} \log s < CS(\Sigma_s)^2 < C \log s, \quad \forall s \geq 2. \tag{4.1}$$

In dimension 4, M. Katz [70] proved a similar inequality for the conformal 2-systole of blow-ups of the complex projective plane $\mathbb{C}P^2$: there exists a constant $C > 0$ independent of n such that

$$C^{-1}\sqrt{n} < CS(\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2})^2 < Cn, \quad \forall n > 0. \tag{4.2}$$

In his proof, M. Katz used a conjecture on the *period map* of 4-manifolds X with $b_2^+ = 1$. The period map is defined as the map taking a Riemannian metric g to the point in the projectivization of the positive cone in $H^2(X;\mathbb{R})$ given by the g-selfdual direction (see Section IV.2). The conjecture, which is still open, claims that this map is surjective. However, an inspection of the proof of M. Katz shows that this surjectivity conjecture in full strength is not needed and that in fact his theorem holds in much greater generality.

In Section IV.3, we first remark that the following proposition holds as a consequence of recent work of D. T. Gay and R. Kirby [50].

Proposition 4.1. *The period map for all closed 4-manifolds with $b_2^+ = 1$ has dense image.*

Using the argument of M. Katz, this implies the following theorem.

Theorem 4.2. *There exists a universal constant C independent of X and $n = b_2(X)$ such that*

$$C^{-1}\sqrt{n} < CS(X)^2 < Cn, \tag{4.3}$$

for all closed 4-manifolds X with $b_2^+ = 1$ which have odd intersection form.

Another consequence of Proposition 4.1 is the following theorem.

Theorem 4.3. *Let X, X' be closed 4-manifolds with $b_2^+ = 1$ which have isomorphic intersection forms. Then $CS(X) = CS(X')$.*

This shows that in dimension 4 the invariant CS is much coarser than a diffeomorphism invariant. Theorem 4.3 can be compared to a result of I. K. Babenko ([5], Theorem 8.1.), who showed that a certain 1-dimensional systolic invariant for manifolds of arbitrary dimension is a homotopy-invariant.

Theorem 4.3 enables us to deal with even intersection forms. Suppose X is a closed 4-manifold with $b_2^+ = 1$ and even intersection form. By the classification of indefinite even quadratic forms, the intersection form of X is isomorphic to $H \oplus (-k)E_8$ for some $k \geq 0$. In particular, for each $r \in \mathbb{N}$ there are only finitely many possible even intersection forms of rank less or equal than r. Hence by Theorem 4.3, the invariant CS takes only finitely many values on all 4-manifolds with even intersection form, $b_2^+ = 1$ and $b_2 \leq r$. We will show that *symplectic* 4-manifolds X with $b_2^+(X) = 1$ and even intersection form necessarily have $b_2(X) \leq 10$ (see Section IV.4). The same bound holds if X satisfies the so-called $\frac{5}{4}$-*conjecture* (see Section IV.5). Hence together with Theorem 4.2, we get the following corollary, which possibly covers all 4-manifolds with $b_2^+ = 1$.

Corollary 4.4. *There exists a universal constant C independent of X and $n = b_2(X)$ such that*

$$C^{-1}\sqrt{n} < CS(X)^2 < Cn, \tag{4.4}$$

for all closed 4-manifolds X with $b_2^+ = 1$ which are symplectic or have odd intersection form Q or satisfy the $\frac{5}{4}$-conjecture if Q is even.

IV.2 Definitions

Let (X^{2n}, g) be a closed oriented Riemannian manifold. We denote the space of g-harmonic n-forms on X by $\mathcal{H}^n(X)$. The Riemannian metric defines an L^2-norm on $\mathcal{H}^n(X)$ by

$$|\alpha|^2_{L^2} = \int_X \alpha \wedge *\alpha, \quad \alpha \in \mathcal{H}^n(X), \tag{4.5}$$

where $*$ is the Hodge operator.

Given the unique representation of cohomology classes by harmonic forms, we obtain an induced norm $|\cdot|_g$, which we call the *g-norm*, on the middle-dimensional cohomology $H^n(X;\mathbb{R})$. The conformal n-systole is defined by

$$\text{confsys}_n(X, g) = \min\{|\alpha|_g \mid \alpha \in H^n(X;\mathbb{Z})_{\mathbb{R}} \setminus \{0\}\}, \tag{4.6}$$

where $H^n(X;\mathbb{Z})_{\mathbb{R}}$ denotes the integer lattice in $H^n(X;\mathbb{R})$. More generally, if L is any lattice with a norm $|\cdot|$, we define

$$\lambda_1(L, |\cdot|) = \min\{|v| \mid v \in L \setminus \{0\}\}, \tag{4.7}$$

hence $\text{confsys}_n(X, g) = \lambda_1(H^n(X;\mathbb{Z})_{\mathbb{R}}, |\cdot|_g)$. The conformal systole depends only on the conformal class of g since the Hodge star operator in the middle dimension is invariant under conformal changes of the metric.

The conformal systoles satisfy the following universal bound (see [70] equation (4.3)):

$$\text{confsys}_n(X, g)^2 < \frac{2}{3} b_n(X), \quad \text{for } b_n(X) \geq 2. \tag{4.8}$$

Clearly, there is also a bound for $b_n(X) = 1$, since the Hodge operator on harmonic forms is up to a sign the identity in this case, hence $\text{confsys}_n(X, g) = 1$. Therefore, the supremum

$$CS(X) = \sup_g \text{confsys}_n(X, g) \tag{4.9}$$

is well-defined for all closed orientable manifolds X^{2n}.

We now consider the case of 4-manifolds, $n = 2$. In this case the g-norm on $H^2(X;\mathbb{R})$ is related to the intersection form Q by the following formula:

$$|\alpha|^2_g = Q(\alpha^+, \alpha^+) - Q(\alpha^-, \alpha^-), \tag{4.10}$$

where $\alpha = \alpha^+ + \alpha^-$ is the decomposition given by the splitting $H^2(X;\mathbb{R}) = H^+ \oplus H^-$ into the subspaces represented by g-selfdual and anti-selfdual harmonic forms. We abbreviate this formula to

$$|\cdot|^2_g = SR(Q, H^-), \tag{4.11}$$

where SR denotes sign-reversal. Since H^+ is the Q-orthogonal complement of H^-, we conclude that the norm $|\cdot|_g$ is completely determined by the intersection form and the g-anti-selfdual subspace H^-.

In particular, let X be a closed oriented 4-manifold with $b_2^+ = \dim H^+ = 1$. The map which takes a Riemannian metric to the selfdual line H^+ in the cone $\mathcal{P}$ of elements of positive square in $H^2(X;\mathbb{R})$ (or to the point in the projectivization $\mathbb{P}(\mathcal{P})$ of this cone) is called the period map. In the proof of his theorem, M. Katz used the following conjecture, which is still open, in the case of blow-ups of $\mathbb{C}P^2$.

Conjecture 1. *The period map is surjective for all closed oriented 4-manifolds with $b_2^+ = 1$.*

If X is a 4-manifold with $b_2^+ = 1$, we switch the orientation (this does not change the g-norm on $H^2(X;\mathbb{R})$) to obtain $b_2^- = 1$. Then the g-norm is completely determined by the intersection form and the selfdual line, which in the new orientation is H^-.

Lemma 4.5. *Let $\bar{X}$ be a 4-manifold with $b_2^- = 1$ and intersection form $\bar{Q}$ and let L be the integer lattice in $H^2(\bar{X};\mathbb{R})$. Then $\lambda_1(L, SR(\bar{Q}, V)^{1/2})$ depends continuously on the anti-selfdual line V.*

This follows because the vector space norm $SR(\bar{Q}, V)^{1/2}$ depends continuously on V and the minimum in λ_1 cannot jump (cf. Remark 9.1. in [70]).

IV.3 Proofs of the theorems on conformal systoles

The following theorem is a corollary to [50, Theorem 1] of D. T. Gay and R. Kirby (compare also [3]).

Theorem 4.6. *If X is a closed oriented 4-manifold and $\alpha \in H^2(X;\mathbb{Z})_{\mathbb{R}}$ a class of positive square, then there exists a Riemannian metric on X such that the harmonic representative of α is selfdual.*

In fact, in the cited theorem it is shown that there exists a closed 2-form ω (with certain properties) representing α and a Riemannian metric g such that ω is g-selfdual and hence harmonic. Theorem 4.6 implies Proposition 4.1, because the set of points given by the lines through integral classes in $H^2(X;\mathbb{R})$ form a dense subset of $\mathbb{P}(\mathcal{P})$. We can now prove Theorem 4.3.

Proof. Let $\bar{X}$ be X with the opposite orientation, L be the integer lattice in $H^2(\bar{X};\mathbb{R})$ and $\bar{Q} = -Q$. We have

$$CS(X) \leq \sup_V \lambda_1(L, SR(\bar{Q}, V)^{1/2}), \tag{4.12}$$

where the supremum extends over all negative definite lines V in $H^2(\bar{X};\mathbb{R})$. This inequality is an equality because the image of the period map is dense and because of Lemma 4.5. The right-hand side depends only on the intersection form. ☐

We now prove Theorem 4.2.

Proof. Let X be a closed 4-manifold with intersection form $Q \cong (1) \oplus n(-1)$ for some $n > 0$. It is enough to prove inequalities of the form $A\sqrt{n} < CS(X)^2 < Bn$ for some constants $A, B > 0$, since we can then take $C = \max\{A^{-1}, B\}$. The inequality on the right-hand side follows from equation (4.8). We are going to prove the inequality on the left, following the proof of M. Katz.

Lemma 4.7. *There exists a constant $k(n) > 0$ (which depends only on n and is asymptotic to $n/2\pi e$ for large n) such that*

$$CS(X) \geq k(n)^{1/4}. \tag{4.13}$$

Proof. It is more convenient to work with $\bar{X}$, which is X with the opposite orientation. We identify $H^2(\bar{X};\mathbb{R}) = \mathbb{R}^{n,1} = \mathbb{R}^{n+1}$ with quadratic form $q_{n,1}$ given by $q_{n,1}(x) = x_1^2 + ... + x_n^2 - x_{n+1}^2$. If g is a metric on $\bar{X}$ then the g-norm is given by $|\cdot|_g^2 = SR(q_{n,1}, v)$ where $*v = -v$ and SR means sign reversal in the direction of v.

Let $L = I_{n,1} = \mathbb{Z}^{n,1} \subset \mathbb{R}^{n,1}$ be the integer lattice. According to Conway-Thompson (see [99], Ch. II, Theorem 9.5), there exists a positive definite odd integer lattice CT_n of rank n with

$$\min_{x \in CT_n \setminus \{0\}} x \cdot x \geq k(n), \tag{4.14}$$

where $k(n)$ is asymptotic to $n/2\pi e$ for $n \to \infty$. By the classification of odd indefinite unimodular forms, $CT_n \oplus I_{0,1} \cong I_{n,1}$, hence there exists a vector $v \in \mathbb{Z}^{n,1}$ with $q_{n,1}(v) = -1$ such that $v^\perp \cong CT_n$.

According to M. Katz, there exists an isometry A of $(\mathbb{R}^{n,1}, q_{n,1})$ such that

$$\lambda_1(L, SR(q_{n,1}, Av)^{1/2}) \geq k(n)^{1/4}. \tag{4.15}$$

By Proposition 4.1, there exists a sequence of Riemannian metrics g_i on X whose selfdual lines converge to the line through Av. Lemma 4.5 implies

$$\mathrm{confsys}_n(X, g_i) \overset{i\to\infty}{\longrightarrow} \lambda_1(L, SR(q_{n,1}, Av)^{1/2}). \tag{4.16}$$

Hence $CS(X) \geq k(n)^{1/4}$. ☐

Lemma 4.7 finishes the proof of Theorem 4.2. ☐

IV.4 Symplectic manifolds

We now show that symplectic 4-manifolds with $b_2^+ = 1$ necessarily have $b_2 \leq 10$, as stated in the introduction (note that we always assume symplectic forms to be compatible with the orientation, i.e. of positive square).

Lemma 4.8. *Let X be a closed symplectic 4-manifold with $b_2^+ = 1$ and even intersection form Q. Then $Q \cong H$ or $Q \cong H \oplus (-E_8)$.*

Here H denotes the bilinear form given by $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is a positive-definite, even form of rank 8 associated to the Dynkin diagram of the Lie group E_8 (see [56]).

Proof. If $b_2^- = 0$ then $Q = (1)$. If $b_2^- > 0$ then Q is indefinite and hence of the form $Q = H \oplus (-k)E_8$, since Q is even. It follows that X is minimal because the intersection form does not split off a (-1). If $K^2 < 0$ then according to a theorem of A.-K. Liu [90], X is an irrational ruled surface and hence has intersection form $Q = H$ (or $Q = (1) \oplus (-1)$, which is odd). If $K^2 = 2\chi + 3\sigma \geq 0$ then $4b_1 + b_2^- \leq 9$ and $b_1 = 0$ or $b_1 = 2$, because $1 - b_1(X) + b_2^+(X)$ is an even number for every almost complex 4-manifold X. If $b_1 = 0$ then $b_2^- \leq 9$, hence $Q = H$ or $H \oplus (-E_8)$. If $b_1 = 2$ then $b_2^- \leq 1$, hence $Q = H$. ☐

Remark 4.9. It is possible to give a different proof of Proposition 4.1 for symplectic manifolds, which relies on a theorem of T.-J. Li and A.-K. Liu ([89], Theorem 4). This theorem implies that on a closed 4-manifold X with $b_2^+ = 1$ which admits a symplectic structure, the set of classes in $H^2(X;\mathbb{R})$ represented by symplectic forms is dense in the positive cone, because it is the complement of at most countably many hyperplanes. If a closed symplectic 4-manifold with $b_2^+ = 1$ is *minimal* (i.e. there are no symplectic (-1)-spheres), then the period map is in fact surjective.

IV.5 The $\frac{5}{4}$-conjecture and some examples

The $\frac{5}{4}$-conjecture is a (weak) analogue of the $\frac{11}{8}$-conjecture which relates the signature and second Betti number of spin 4-manifolds. The main result in this direction is a theorem of M. Furuta [49] that all closed oriented spin 4-manifolds X with $b_2(X) > 0$ satisfy the inequality

$$\tfrac{5}{4}|\sigma(X)| + 2 \leq b_2(X), \tag{4.17}$$

where $\sigma(X)$ denotes the signature. This generalizes work of S. K. Donaldson [30, 31]. C. Bohr [10] then proved a (slightly weaker) inequality $\frac{5}{4}|\sigma(X)| \leq b_2(X)$ for all 4-manifolds with even intersection form and certain fundamental groups, including all finite and all abelian groups. These are special instances of the following general $\frac{5}{4}$-conjecture.

Conjecture 2. *If X is a closed oriented even 4-manifold, then*

$$\frac{5}{4}|\sigma(X)| \leq b_2(X), \tag{4.18}$$

where $\sigma(X)$ denotes the signature.

Here we call a 4-manifold *even*, if it has even intersection form.[2]

Lemma 4.10. *If X is an even 4-manifold with $b_2^+ = 1$, then the $\frac{5}{4}$-conjecture holds for X if and only if $Q \cong H$ or $Q \cong H \oplus (-E_8)$.*

Proof. If X is an even 4-manifold with $b_2^+ = 1$, then $Q \cong H \oplus (-k)E_8$ for some $k \geq 0$. The $\frac{5}{4}$-conjecture is equivalent to $k \leq 1$. $\square$

In particular, by Lemma 4.8, the $\frac{5}{4}$-conjecture holds for all even symplectic 4-manifolds which satisfy $b_2^+ = 1$.

There are many examples of 4-manifolds with $b_2^+ = 1$ where Theorem 4.3 applies, e.g. the infinite family of simply-connected pairwise non-diffeomorphic Dolgachev surfaces which are all homeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ (see [56]). These 4-manifolds are Kähler, hence symplectic. There are also recent constructions of infinite families of non-symplectic and pairwise non-diffeomorphic 4-manifolds homeomorphic to $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for $n \geq 5$ (see [43, 114]). If we take multiple blow-ups of these manifolds, the blow-up formula for the Seiberg-Witten invariants [36] shows that the resulting manifolds stay pairwise non-diffeomorphic. Hence we obtain infinite families of symplectic and non-symplectic 4-manifolds X with $n = b_2(X) \to \infty$, where Theorem 4.2 applies.

[2] J.-H. Kim [71] has proposed a proof of the $\frac{5}{4}$-conjecture. However, some doubts have been raised about the validity of the proof. Hence we have chosen to state the result still as a conjecture.

Chapter V

The generalized fibre sum of 4-manifolds

Contents

In this chapter we describe a construction of 4-manifolds known as the *generalized fibre sum* which is due to R. E. Gompf [52] and J. D. McCarthy and J. G. Wolfson [91]. This construction can be applied to find new 4-manifolds. It can also be done symplectically and yields new examples of symplectic 4-manifolds.

In Section V.1 we define the generalized fibre sum for the case of two closed oriented 4-manifolds M and N which contain closed embedded surfaces Σ_M, Σ_N of the same genus g. We only consider the case when both surfaces have trivial normal bundle, i.e. their self-intersection numbers Σ_M^2 and

Σ_N^2 vanish. Let Σ denote some fixed surface of genus g. We consider the surfaces Σ_M and Σ_N as coming from embeddings $i_M\colon \Sigma \rightarrow M$ and $i_N\colon \Sigma \rightarrow N$ and also choose a trivialization for the normal bundle of both surfaces, i.e. a *framing*. We then delete an open tubular neighbourhood of each surface in the corresponding 4-manifold and glue the manifolds together along their boundaries, which are diffeomorphic to $\Sigma \times S^1$. The gluing diffeomorphism ϕ is chosen such that it preserves the natural S^1-fibration on the boundaries of the tubular neighbourhoods, given by the *meridians* to the surfaces. The resulting 4-manifold is denoted by $X = M\#_{\Sigma_M=\Sigma_N}N$ and can depend on the choice of gluing diffeomorphism ϕ.

In Sections V.2 and V.3 we calculate the homology groups of X using the Mayer-Vietoris sequence and give some applications in V.4, in particular we review some constructions using the generalized fibre sum. In Section V.5 we consider the symplectic version of this construction and derive a formula for the canonical class K_X of a symplectic generalized fibre sum $X = M\#_{\Sigma_M=\Sigma_N}N$. We will give some applications in Section V.6 and compare the formula to some other formulas which can be found in the literature on this subject. In the final subsection we derive a theorem, following an idea of I. Smith [126], which shows how one can find inequivalent symplectic structures on a simply-connected 4-manifold if there exists a simply-connected symplectic 4-manifold which contains a certain triple of Lagrangian tori. The formula for the canonical class and the construction of inequivalent symplectic structures will be applied in Chapter VI.

V.1 Definition of the generalized fibre sum

Let M and N be closed, oriented, connected 4-manifolds. Suppose that Σ_M and Σ_N are closed, oriented, connected embedded surfaces in M and N of the same genus g. Let $\nu\Sigma_M$ and $\nu\Sigma_N$ denote the normal bundles of Σ_M and Σ_N. The normal bundle of the surface Σ_M is trivial if and only if the self-intersection number Σ_M^2 is zero. This follows because the Euler class of the normal bundle is given by $e(\nu\Sigma_M) = i^*PD[\Sigma_M]$, where $i\colon \Sigma_M \rightarrow M$ denotes the inclusion and the evaluation of $PD[\Sigma_M]$ on $[\Sigma_M]$ is given by $\Sigma_M \cdot \Sigma_M$. From now on we will assume that Σ_M and Σ_N have zero self-intersection.

For the construction of the generalized fibre sum we choose a closed oriented surface Σ of genus g and smooth embeddings

$$\begin{aligned} i_M\colon \Sigma &\longrightarrow M \\ i_N\colon \Sigma &\longrightarrow N, \end{aligned}$$

with images Σ_M and Σ_N. We assume that the orientation induced by the embeddings on Σ_M and Σ_N is the given one.

Since the normal bundles of Σ_M and Σ_N are trivial, there exist D^2-bundles $\nu\Sigma_M$ and $\nu\Sigma_N$ embedded in M and N which form tubular neighbourhoods for Σ_M and Σ_N. We fix once and for all embeddings

$$\begin{aligned} \tau_M\colon \Sigma \times S^1 &\longrightarrow M \\ \tau_N\colon \Sigma \times S^1 &\longrightarrow N, \end{aligned}$$

with images $\partial\nu\Sigma_M$ and $\partial\nu\Sigma_N$, which commute with the embeddings i_M and i_N above and the natural projections $\Sigma \times S^1 \rightarrow \Sigma$, $\partial\nu\Sigma_M \rightarrow \Sigma_M$ and $\partial\nu\Sigma_N \rightarrow \Sigma_N$. The maps τ_M and τ_N form fixed reference trivialisations which we call *framings* for the normal bundles of the embedded surfaces Σ_M and Σ_N. We can think of the framings τ_M and τ_N as giving sections of the S^1-bundles $\partial\nu\Sigma_M$ and $\partial\nu\Sigma_N$.

They correspond to "push-offs" of Σ_M and Σ_N into the boundary of the tubular neighbourhoods. In fact, since trivializations of vector bundles are linear, the framings are completely determined by such push-offs.

Definition 5.1. Let Σ^M and Σ^N denote push-offs of Σ_M and Σ_N into $\partial\nu\Sigma_M$ and $\partial\nu\Sigma_N$ given by the framings τ_M and τ_N.

We set $M' = M \setminus \operatorname{int}\nu\Sigma_M$ and $N' = N \setminus \operatorname{int}\nu\Sigma_N$, which are compact, oriented 4-manifolds with boundary. We choose the orientations as follows: On $\Sigma \times D^2$ choose the orientation of Σ followed by the standard orientation of D^2 given by $dx \wedge dy$. We can assume that the framings τ_M and τ_N induce orientation preserving embeddings of $\Sigma \times D^2$ into M and N as tubular neighbourhoods. We define the orientation on $\Sigma \times S^1$ to be the orientation of Σ followed by the orientation of S^1. This determines orientations on $\partial M'$ and $\partial N'$. Both conventions together imply that the orientation on $\partial M'$ followed by the orientation of the normal direction pointing *out of* M' is the orientation on M. Similarly for N.

We want to glue M' and N' together using diffeomorphisms between the boundaries which preserve the fibres of the S^1-bundles $\partial\nu\Sigma_M$ and $\partial\nu\Sigma_N$. Note that $Diff^+(S^1)$ retracts onto $SO(2)$. Hence by an isotopy we can assume that the gluing diffeomorphism is linear on the fibres of $\nu\Sigma_M$ and $\nu\Sigma_N$. The gluing diffeomorphism then corresponds to a bundle isomorphism covering the diffeomorphism $i_N \circ i_M^{-1}$. The "trivial" diffeomorphism will correspond to the diffeomorphism which identifies the push-offs of Σ_M and Σ_N in the boundary of the normal bundles.

Suppose $E = \Sigma \times \mathbb{R}^2$ is a trivialized, oriented $\mathbb{R}^2$-vector bundle over Σ. Every bundle isomorphism $E \to E$ covering the identity of Σ and preserving the orientation on the fibres is given by a map of the form

$$\begin{aligned} F\colon \Sigma \times \mathbb{R}^2 &\to \Sigma \times \mathbb{R}^2 \\ (x, v) &\mapsto (x, A(x) \cdot v) \end{aligned}$$

where A is a smooth map $A\colon \Sigma \to GL^+(2, \mathbb{R})$ with values in the 2×2-matrices with positive determinant. We can isotop this bundle isomorphism to a new one such that A maps to $SO(2)$. If we restrict to the unit circle bundle in E, the map is of the form

$$\begin{aligned} F\colon \Sigma \times S^1 &\to \Sigma \times S^1, \\ (x, \alpha) &\mapsto (x, C(x) \cdot \alpha), \end{aligned} \tag{5.1}$$

where $C\colon \Sigma \to S^1$ is a map and multiplication is in the group S^1. Every smooth map C of this kind defines an orientation preserving bundle isomorphism. Let r denote the orientation reversing diffeomorphism

$$r\colon \Sigma \times S^1 \to \Sigma \times S^1, (x, \alpha) \mapsto (x, \overline{\alpha}),$$

where $S^1 \subset \mathbb{C}$ is embedded in the standard way and $\overline{\alpha}$ denotes complex conjugation. Then the diffeomorphism

$$\begin{aligned} \rho = F \circ r\colon \Sigma \times S^1 &\to \Sigma \times S^1, \\ (x, \alpha) &\mapsto (x, C(x)\overline{\alpha}) \end{aligned}$$

is orientation reversing. We define

$$\phi = \phi(C) = \tau_N \circ \rho \circ \tau_M^{-1}. \tag{5.2}$$

Then ϕ is an orientation reversing diffeomorphism $\phi\colon \partial\nu\Sigma_M \to \partial\nu\Sigma_N$, preserving fibres. If C is a constant map then ϕ is a diffeomorphism which identifies the push-offs of Σ_M and Σ_N.

Definition 5.2. Let M and N be closed, oriented, connected 4-manifolds M and N with embedded oriented surfaces Σ_M and Σ_N of genus g and self-intersection 0. The *generalized fibre sum* of M and N along Σ_M and Σ_N, determined by the diffeomorphism ϕ, is given by

$$X(\phi) = M' \cup_\phi N'.$$

$X(\phi)$ is again a differentiable, closed, oriented, connected 4-manifold.

See [52] and [91] for the original construction. The generalized fibre sum is often denoted by $M\#_{\Sigma_M=\Sigma_N}N$ or $M\#_\Sigma N$ and is also called the Gompf sum or the normal connected sum. By a construction of Gompf (cf. Section V.5) the generalized fibre sum $M\#_{\Sigma_M=\Sigma_N}N$ admits a symplectic structure if (M, ω_M) and (N, ω_N) are symplectic 4-manifolds and Σ_M, Σ_N symplectically embedded surfaces.

In the general case, the differentiable structure on X is defined in the following way: We identify the interior of slightly larger tubular neighbourhoods $\nu\Sigma'_M$ and $\nu\Sigma'_N$ via the framings τ_M and τ_N with $\Sigma \times D$ where D is an open disk of radius 1. We think of $\partial M'$ and $\partial N'$ to be $\Sigma \times S$, where S denotes the circle of radius $1/\sqrt{2}$. Hence the tubular neighbourhoods $\nu\Sigma_M$ and $\nu\Sigma_N$ above have in this convention radius $1/\sqrt{2}$. We also choose polar coordinates r, θ on D. The manifolds $M \setminus \Sigma_M$ and $N \setminus \Sigma_N$ are glued together along $\operatorname{int}\nu\Sigma'_M \setminus \Sigma_M$ and $\operatorname{int}\nu\Sigma'_N \setminus \Sigma_N$ by the diffeomorphism

$$\begin{aligned} \Phi\colon \Sigma \times (D \setminus \{0\}) &\to \Sigma \times (D \setminus \{0\}) \\ (x, r, \theta) &\mapsto (x, \sqrt{1-r^2}, C(x) - \theta). \end{aligned} \tag{5.3}$$

This diffeomorphism is orientation *preserving* because it reverses on the disk the orientation on the boundary circle and the inside-outside direction. It is also fibre preserving and identifies $\partial M'$ and $\partial N'$ via ϕ. It is literally an extension of ρ and hence should be denoted by R. We nevertheless denote it by Φ since we will only use this diffeomorphism if the trivializations τ_M, τ_N are fixed, hence its meaning is unambiguous.

Definition 5.3. Let Σ_X denote the genus g surface in X given by the image of the push-off Σ^M under the inclusion $M' \to X$. Similarly, let Σ'_X denote the genus g surface in X given by the image of the push-off Σ^N under the inclusion $N' \to X$.

In general (depending on the diffeomorphism ϕ and the homology of X) the surfaces Σ_X and Σ'_X do not represent the same homology class in X.

V.1.1 Basic notations and definitions

We now collect some additional basic definitions and notations which will be used in the following sections. Their meaning and interpretation will be given later at the appropriate place.

Let M and N be again two closed, oriented 4-manifolds with embedded closed oriented surfaces Σ_M and Σ_N of genus g and $X = M\#_{\Sigma_M=\Sigma_N}N$ the generalized fibre sum. In general, we often denote homology classes of degree 2 on M, N and X and their Poincaré duals by the same symbol. The symbols $H_*(Y)$ and $H^*(Y)$ denote the homology and cohomology groups with $\mathbb{Z}$-coefficients of a topological space Y. If a definition involves an index M there will be a corresponding definition for N.

(1.) **Embeddings** We fix the following notation for some embeddings of manifolds into other manifolds. We denote the maps induced by them on homology by the same symbol:

$i_M\colon \Sigma \to M$
$\rho_M\colon M' \to M$
$\eta_M\colon M' \to X$
$\mu_M\colon \partial\nu\Sigma_M \to M'$

There is also a projection

$p\colon \Sigma \times S^1 \to S^1$

and induced projections

$p_M\colon \partial\nu\Sigma_M \to \Sigma,\, p_N\colon \partial\nu\Sigma_N \to \Sigma$

defined via the framings τ_M and τ_N.

(2.) **Basis for homology** We define bases for the homology of the boundary of M' and N' in the following way: Any given basis of $H_1(\Sigma)$ can be represented by oriented embedded loops $\gamma_1, \ldots, \gamma_{2g}$ in Σ.

(a) Denote the loops $\gamma_i \times \{*\}$ in $\Sigma \times S^1$ also by γ_i for all $i = 1, \ldots, 2g$. Let σ denote the loop $\{*\} \times S^1$ in $\Sigma \times S^1$. Then the loops

$$\gamma_1, \ldots, \gamma_{2g}, \sigma,$$

represent homology classes (denoted by the same symbols) which determine a basis for $H_1(\Sigma \times S^1) \cong \mathbb{Z}^{2g+1}$. The bases for $H_1(\partial\nu\Sigma_M)$ and $H_1(\partial\nu\Sigma_N)$ are chosen as follows:

$$\begin{aligned} \gamma_i^M &= \tau_{M*}\gamma_i, \quad \sigma^M = \tau_{M*}\sigma \\ \gamma_i^N &= \tau_{N*}\gamma_i, \quad \sigma^N = \tau_{N*}\sigma. \end{aligned}$$

The classes σ^M, σ^N represented by the circle fibres in the boundary of the tubular neighbourhoods are called the *meridians* to the surfaces Σ_M and Σ_N in M', N'.

(b) Let $\gamma_1^*, \ldots, \gamma_{2g}^*, \sigma^* \in H^1(\Sigma \times S^1) = \mathrm{Hom}(H_1(\Sigma \times S^1), \mathbb{Z})$ denote the dual basis. By Poincaré duality this determines a basis

$$\begin{aligned} \Gamma_i &= PD(\gamma_i^*), \quad i = 1, \ldots, 2g, \\ \Sigma &= PD(\sigma^*) \end{aligned}$$

for $H_2(\Sigma \times S^1)$. The bases for $H_2(\partial\nu\Sigma_M)$ and $H_2(\partial\nu\Sigma_N)$ are chosen as follows:

$$\begin{aligned} \Gamma_i^M &= \tau_{M*}\Gamma_i, \quad \Sigma^M = \tau_{M*}\Sigma \\ \Gamma_i^N &= \tau_{N*}\Gamma_i, \quad \Sigma^N = \tau_{N*}\Sigma. \end{aligned}$$

The surfaces representing Σ^M and Σ^N are the push-offs of Σ_M and Σ_N given by the framings τ_M and τ_N.

(3.) **Cohomology class C** The map $C\colon \Sigma \to S^1$ in equation (5.1) which was used to define the gluing diffeomorphism ϕ determines a cohomology class in the following way:

(a) Let $[C] \in H^1(\Sigma; \mathbb{Z})$ denote the cohomology class given by pulling back the standard generator of $H^1(S^1; \mathbb{Z})$. We sometimes denote $[C]$ by C if a confusion is not possible.

(b) We also define the following integers: For $i = 1, \ldots, 2g$, let a_i be the integer

$$\begin{aligned} a_i &= \deg(C \circ \gamma_i \colon S^1 \to S^1) \\ &= \langle [C], \gamma_i \rangle = \langle C, \gamma_i \rangle \in \mathbb{Z}. \end{aligned}$$

The integers a_i together determine the cohomology class $[C]$. Since the map C can be chosen arbitrarily, the integers a_i can (independently) take any possible value.

(4.) **Divisibilities $\mathbf{k_M}, \mathbf{k_N}$** We define integers k_M, k_N as follows:

(a) We denote the homology and cohomology classes defined by Σ_M and Σ_N in M and N by the same symbol.

(b) The image of the homomorphism

$$\begin{aligned} H_2(M; \mathbb{Z}) &\longrightarrow \mathbb{Z}, \\ \alpha &\mapsto \langle PD(\Sigma_M), \alpha \rangle \end{aligned}$$

is a subgroup of the form $k_M\mathbb{Z}$ with $k_M \geq 0$. We define $k_N \geq 0$ for Σ_N similarly and denote the greatest common divisor of k_M and k_N by n_{MN}.

(5.) **Homology classes $\mathbf{A_M}, \mathbf{A_N}$ and $\mathbf{B_M}, \mathbf{B_N}$** We make two additional assumptions:

(a) We assume that Σ_M and Σ_N are non-torsion homology classes. Then $k_M, k_N > 0$.

(b) We also assume that there exist classes $A_M \in H_2(M; \mathbb{Z})$ and $A_N \in H_2(N; \mathbb{Z})$ such that $\Sigma_M = k_M A_M$ and $\Sigma_N = k_N A_N$.

We then choose classes $B_M \in H_2(M)$ and $B_N \in H_2(N)$ which have intersection numbers $B_M \cdot A_M = 1$ and $B_N \cdot A_N = 1$. These classes exist because A_M, A_N are non-torsion and indivisible.

(6.) **Perpendicular classes** The group of perpendicular classes is defined as follows:

(a) Let $P(M) = (\mathbb{Z}A_M \oplus \mathbb{Z}B_M)^\perp$ be the orthogonal complement of the subgroup $\mathbb{Z}A_M \oplus \mathbb{Z}B_M$ in $H_2(M)$ with respect to the intersection form Q_M. We call $P(M)$ the group of *perpendicular classes*. It contains in particular all torsion elements in $H_2(M)$ and has rank $b_2(M) - 2$. Similarly for N.

(b) There is a splitting $H_2(M) = \mathbb{Z}A_M \oplus \mathbb{Z}B_M \oplus P(M)$. Under this splitting, an element $\alpha \in H_2(M)$ decomposes as

$$\alpha = (\alpha \cdot B_M - B_M^2(\alpha \cdot A_M))A_M + (\alpha \cdot A_M)B_M + \overline{\alpha},$$

where $\overline{\alpha} = \alpha - (\alpha \cdot A_M)B_M - (\alpha \cdot B_M - B_M^2(\alpha \cdot A_M))A_M \in P(M)$.

(7.) **Homomorphisms $\mathbf{i_M} \oplus \mathbf{i_N}$ and $\mathbf{i_M^*} + \mathbf{i_N^*}$** The following homomorphisms will occur several times:

$$\begin{aligned} i_M \oplus i_N \colon H_1(\Sigma; \mathbb{Z}) &\longrightarrow H_1(M; \mathbb{Z}) \oplus H_1(N; \mathbb{Z}) \\ \lambda &\mapsto (i_M(\lambda), i_N(\lambda)), \end{aligned}$$

and

$$\begin{aligned} i_M^* + i_N^* \colon H^1(M; \mathbb{Z}) \oplus H^1(N; \mathbb{Z}) &\longrightarrow H^1(\Sigma; \mathbb{Z}) \\ (\alpha, \beta) &\mapsto i_M^*\alpha + i_N^*\beta. \end{aligned}$$

(8.) **Rim tori** The groups $R(M')$, $R(N')$ and $R(X)$ of *rim tori* in M', N' and X are defined as the image of $H^1(\Sigma;\mathbb{Z})$ under the homomorphisms

$$\begin{aligned} \mu_M \circ PD \circ p_M^* \colon H^1(\Sigma;\mathbb{Z}) &\to H_2(M';\mathbb{Z}) \\ \mu_N \circ PD \circ p_N^* \colon H^1(\Sigma;\mathbb{Z}) &\to H_2(N';\mathbb{Z}) \\ \eta_M \circ \mu_M \circ PD \circ p_M^* \colon H^1(\Sigma;\mathbb{Z}) &\to H_2(X;\mathbb{Z}). \end{aligned}$$

By Proposition 5.25 there are isomorphisms

$$\begin{aligned} \text{Coker}\, i_M^* &\xrightarrow{\cong} R(M') \\ \text{Coker}\, i_N^* &\xrightarrow{\cong} R(N') \\ \text{Coker}\, (i_M^* + i_N^*) &\xrightarrow{\cong} R(X). \end{aligned}$$

(9.) **Split classes** The group $S(X)$ of *split classes* (or *vanishing classes*) of X is defined as $S(X) = \ker f$, where

$$\begin{aligned} f \colon \mathbb{Z}B_M \oplus \mathbb{Z}B_N \oplus \ker\,(i_M \oplus i_N) &\longrightarrow \mathbb{Z} \\ (x_M B_M, x_N B_N, \alpha) &\mapsto x_M k_M + x_N k_N - \langle C, \alpha \rangle. \end{aligned}$$

(10.) **Dimension d** We also consider the homomorphisms $i_M \oplus i_N$ and $i_M^* + i_N^*$ for homology and cohomology with $\mathbb{R}$-coefficients.

(a) We denote by d the dimension of the kernel of the linear map

$$\begin{aligned} i_M \oplus i_N \colon H_1(\Sigma;\mathbb{R}) &\longrightarrow H_1(M;\mathbb{R}) \oplus H_1(N;\mathbb{R}) \\ \lambda &\mapsto (i_M\lambda, i_N\lambda). \end{aligned}$$

(b) In Lemma 5.8 we show that

$$\begin{aligned} \text{dim Ker}\,(i_M^* + i_N^*) &= b_1(M) + b_1(N) - 2g + d = \text{dim Coker}\,(i_M \oplus i_N) \\ \text{dim Coker}\,(i_M^* + i_N^*) &= d = \text{dim Ker}\,(i_M \oplus i_N), \end{aligned}$$

This implies that the rank of $R(X)$ is equal to d and the rank of $S(X)$ equal to $d+1$.

(11.) **Special surfaces in X** We define the following elements in the homology of X:

(a) The surfaces in X determined by the push-offs of Σ_M, Σ_N under inclusion:

$$\Sigma_X = \eta_M \circ \mu_M \Sigma^M, \quad \Sigma_X' = \eta_N \circ \mu_N \Sigma^N \in H_2(X).$$

(b) A class in X sewed together from the classes $\frac{k_N}{n_{MN}} B_M$ and $\frac{k_M}{n_{MN}} B_N$ which bound in M' and N' the $\frac{k_M k_N}{n_{MN}}$-fold multiple of the meridians σ^M and σ^N:

$$B_X = \tfrac{1}{n_{MN}}(k_N B_M - k_M B_M) \in S(X).$$

(c) A rim torus in X determined by the diffeomorphism ϕ:

$$R_C = \eta_M \circ \mu_M(-\sum_{i=1}^{2g} a_i \Gamma_i^M) \in R(X),$$

where the coefficients $a_i = \langle C, \gamma_i \rangle$ are defined as above. By Lemma 5.6 we have

$$\Sigma_X' - \Sigma_X = R_C, \quad \text{and}$$
$$R_C = \eta_N \circ \mu_N(\sum_{i=1}^{2g} a_i \Gamma_i^N).$$

(12.) **Mayer-Vietoris sequences** We use the following Mayer-Vietoris sequences for X, M and N:

(a) For $M = M' \cup \nu\Sigma_M$:

$$\ldots \to H_k(\partial M') \to H_k(M') \oplus H_k(\Sigma) \to H_k(M) \to H_{k-1}(\partial M') \to \ldots$$

with homomorphisms

$$H_k(\partial M') \to H_k(M') \oplus H_k(\Sigma), \quad \alpha \mapsto (\mu_M \alpha^M, p_M \alpha)$$
$$H_k(M') \oplus H_k(\Sigma) \to H_k(M), \quad (x, y) \mapsto \rho_M x - i_M y.$$

(b) For $N = N' \cup \nu\Sigma_N$:

$$\ldots \to H_k(\partial N') \to H_k(N') \oplus H_k(\Sigma) \to H_k(N) \to H_{k-1}(\partial N') \to \ldots$$

with homomorphisms

$$H_k(\partial N') \to H_k(N') \oplus H_k(\Sigma), \quad \alpha \mapsto (\mu_N \alpha^N, p_N \alpha)$$
$$H_k(N') \oplus H_k(\Sigma) \to H_k(N), \quad (x, y) \mapsto \rho_N x - i_N y.$$

(c) For $X = M' \cup N'$:

$$\ldots \to H_k(\partial M') \stackrel{\psi_k}{\to} H_k(M') \oplus H_k(N') \to H_k(X) \to H_{k-1}(\partial M') \to \ldots$$

with homomorphisms

$$\psi_k : H_k(\partial M') \to H_k(M') \oplus H_k(N'), \quad \alpha \mapsto (\mu_M \alpha, \mu_N \phi_* \alpha)$$
$$H_k(M') \oplus H_k(N') \to H_k(X), \quad (x, y) \mapsto \eta_M x - \eta_N y.$$

We will also consider the Mayer-Vietoris sequences for cohomology.

V.1.2 Action of the gluing diffeomorphism on the basis for homology

Recall that the generalized fibre sum is defined as $X = X(\phi) = M' \cup_\phi N'$ where $\phi\colon \partial\nu\Sigma_M \to \partial\nu\Sigma_N$ is a diffeomorphism preserving the meridians and covering the diffeomorphism $i_N \circ i_M^{-1}$. In general, different choices of diffeomorphisms ϕ can give non-diffeomorphic manifolds $X(\phi)$. However, if ϕ and ϕ' are isotopic, then $X(\phi)$ and $X(\phi')$ are diffeomorphic. We want to determine how many different isotopy classes of diffeomorphisms ϕ of the form above exist: Suppose that

$$C' \colon \Sigma \to S^1,$$

is any other smooth map. Then C' determines a self-diffeomorphism ρ' of $\Sigma \times S^1$ and a diffeomorphism $\phi' \colon \partial\nu\Sigma_M \to \partial\nu\Sigma_N$ as before.

Proposition 5.4. *The diffeomorphisms $\phi, \phi' \colon \partial\nu\Sigma_M \longrightarrow \partial\nu\Sigma_N$ are smoothly isotopic if and only if $[C] = [C'] \in H^1(\Sigma)$. In particular, if $[C] = [C']$, then the generalized fibre sums $X(\phi)$ and $X(\phi')$ are diffeomorphic.*

Proof. Suppose that ϕ and ϕ' are isotopic. Since

$$\rho = \tau_N^{-1} \circ \phi \circ \tau_M,$$

this implies that the diffeomorphisms ρ, ρ' are isotopic, hence homotopic. The maps C, C' can be written as

$$C = pr \circ \rho \circ \iota, \quad C' = pr \circ \rho' \circ \iota,$$

where $\iota \colon \Sigma \to \Sigma \times S^1$ denotes the inclusion $x \mapsto (x, 1)$ and pr denotes the projection onto the second factor in $\Sigma \times S^1$. This implies that C and C' are homotopic, hence the cohomology classes $[C]$ and $[C']$ coincide.

Conversely, if $[C] = [C']$ then C and C' are homotopic maps. We can choose a smooth homotopy

$$\begin{aligned} \Delta \colon \Sigma \times [0,1] &\longrightarrow S^1, \\ (x,t) &\mapsto \Delta(x,t) \end{aligned}$$

with $\Delta_0 = C$ and $\Delta_1 = C'$. Define the map

$$\begin{aligned} R \colon (\Sigma \times S^1) \times [0,1] &\longrightarrow \Sigma \times S^1, \\ (x, \alpha, t) &\mapsto R_t(x, \alpha), \end{aligned}$$

where

$$R_t(x,\alpha) = (x, \Delta(x,t) \cdot \overline{\alpha}).$$

Then R is a homotopy between ρ and ρ'. Note that the maps $R_t \colon \Sigma \times S^1 \to \Sigma \times S^1$ are diffeomorphisms with inverse

$$(y, \beta) \mapsto (y, \overline{\Delta(y,t)^{-1} \cdot \beta}),$$

where $\Delta(y,t)^{-1}$ denotes the inverse as a group element in S^1. Hence R is an isotopy between ρ and ρ' which defines via the trivializations τ_M, τ_N an isotopy between ϕ, ϕ'. □

We now determine the action of the gluing diffeomorphism $\phi \colon \partial M' \to \partial N'$ for a generalized fibre sum $X = X(\phi)$ on the homology of the boundaries $\partial M'$ and $\partial N'$. We use the given framings to describe this action in bases for the homology groups chosen above. This calculation will be needed later because the induced map ϕ_* on homology appears in the Mayer-Vietoris sequences for the calculation of the homology groups of X.

Lemma 5.5. *The map $\phi_* \colon H_1(\partial\nu\Sigma_M) \to H_1(\partial\nu\Sigma_N)$ is given by*

$$\begin{aligned} \phi_*\gamma_i^M &= \gamma_i^N + a_i\sigma^N, \quad i = 1, \ldots, 2g \\ \phi_*\sigma^M &= -\sigma^N. \end{aligned}$$

Proof. We have

$$\rho(\gamma_i(t), *) = (\gamma_i(t), (C \circ \gamma_i)(t) \cdot \overline{*}),$$

which implies $\rho_*\gamma_i = \gamma_i + a_i\sigma$ for all $i = 1, \ldots, 2g$. Similarly,

$$\rho(*, t) = (*, C(*) \cdot \overline{t}),$$

which implies $\rho_*\sigma = -\sigma$. The claim follows from these equations and equation (5.2). □

Note that ρ_* is given in the basis $\gamma_1, \ldots, \gamma_{2g}, \sigma$ by the following matrix in $GL(2g+1, \mathbb{Z})$ with determinant equal to -1:

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ a_1 & a_2 & \ldots & a_{2g} & -1 \end{pmatrix}$$

Lemma 5.6. *The map $\phi_*\colon H_2(\partial\nu\Sigma_M) \to H_2(\partial\nu\Sigma_N)$ is given by*

$$\phi_*\Gamma_i^M = -\Gamma_i^N, \quad i = 1, \ldots, 2g$$
$$\phi_*\Sigma^M = -\left(\sum_{i=1}^{2g} a_i\Gamma_i^N\right) + \Sigma^N.$$

Proof. We first compute the action of ρ on the first cohomology of $\Sigma \times S^1$. By the proof of Lemma 5.5,

$$(\rho^{-1})_*\gamma_i = \gamma_i + a_i\sigma, \quad i = 1, \ldots, 2g$$
$$(\rho^{-1})_*\sigma = -\sigma.$$

We claim that

$$(\rho^{-1})^*(\gamma_i^*) = \gamma_i^*, \quad i = 1, \ldots, 2g,$$
$$(\rho^{-1})^*(\sigma^*) = \left(\sum_{i=1}^{2g} a_i\gamma_i^*\right) - \sigma^*.$$

This is easy to check by evaluating both sides on the given basis of $H_1(\Sigma\times S^1)$ and using $\langle(\rho^{-1})^*\mu, v\rangle = \langle\mu, (\rho^{-1})_*v\rangle$. By the formula

$$\lambda_*(\lambda^*\alpha \cap \beta) = \alpha \cap \lambda_*\beta, \tag{5.4}$$

for continuous maps λ between topological spaces, homology classes β and cohomology classes α (see [16], Chapter VI. Theorem 5.2.), we get for all $\mu \in H^*(\Sigma \times S^1)$,

$$\begin{aligned} \rho_*PD(\rho^*\mu) &= \rho_*(\rho^*\mu \cap [\Sigma \times S^1]) \\ &= \mu \cap \rho_*[\Sigma \times S^1] \\ &= -\mu \cap [\Sigma \times S^1] \\ &= -PD(\mu). \end{aligned} \tag{5.5}$$

since ρ is orientation reversing. This implies $\rho_*PD(\mu) = -PD((\rho^{-1})^*\mu)$ and hence

$$\rho_*\Gamma_i = -\Gamma_i, \quad i = 1, \ldots, 2g,$$
$$\rho_*\Sigma = -\left(\sum_{i=1}^{2g} a_i\Gamma_i\right) + \Sigma.$$

The claim follows from this. □

Proposition 5.7. *The diffeomorphism ϕ is determined up to isotopy by the difference of the homology classes $\phi_*\Sigma^M$ and Σ^N in $\partial\nu\Sigma_N$.*

This follows because by the formula in Lemma 5.6 above, the difference determines the coefficients a_i. Hence it determines the class $[C]$ and by Proposition 5.4 the diffeomorphism ϕ up to isotopy. An interpretation of the difference $\phi_*\Sigma^M - \Sigma^N = -\sum_{i=1}^{2g} a_i\Gamma_i^N$ will be given in Section V.3.1.

V.1.3 Calculation of the dimension d

Recall that we defined homomorphisms

$$\begin{aligned} i_M \oplus i_N \colon H_1(\Sigma;\mathbb{Z}) &\longrightarrow H_1(M;\mathbb{Z}) \oplus H_1(N;\mathbb{Z}) \\ \lambda &\mapsto (i_M(\lambda), i_N(\lambda)), \end{aligned}$$

and

$$\begin{aligned} i_M^* + i_N^* \colon H^1(M;\mathbb{Z}) \oplus H^1(N;\mathbb{Z}) &\longrightarrow H^1(\Sigma;\mathbb{Z}) \\ (\alpha,\beta) &\mapsto i_M^*\alpha + i_N^*\beta. \end{aligned}$$

The kernels of $i_M \oplus i_N$ and $i_M^* + i_N^*$ are free abelian groups, but the cokernels can have torsion. We can also consider both homomorphisms for homology and cohomology with $\mathbb{R}$-coefficients.

Lemma 5.8. *Consider the homomorphisms $i_M \oplus i_N$ and $i_M^* + i_N^*$ for homology and cohomology with $\mathbb{R}$-coefficients. Let $d =$ dim Ker $(i_M \oplus i_N)$. Then*

$$\begin{aligned} \textit{dim Ker}\,(i_M^* + i_N^*) &= b_1(M) + b_1(N) - 2g + d = \textit{dim Coker}\,(i_M \oplus i_N) \\ \textit{dim Coker}\,(i_M^* + i_N^*) &= d = \textit{dim Ker}\,(i_M \oplus i_N), \end{aligned}$$

where g denotes the genus of the surface Σ.

Proof. By general linear algebra, $i_M^* + i_N^*$ is the dual homomorphism to $i_M \oplus i_N$ under the identification of cohomology with the dual vector space of homology with $\mathbb{R}$-coefficients. Moreover,

$$\begin{aligned} \text{dim Coker}\,(i_M \oplus i_N) &= b_1(M) + b_1(N) - \text{dim Im}\,(i_M \oplus i_N) \\ &= b_1(M) + b_1(N) - (2g - \text{dim Ker}\,(i_M \oplus i_N)) \\ &= b_1(M) + b_1(N) - 2g + d. \end{aligned}$$

This implies

$$\begin{aligned} \text{dim Ker}\,(i_M^* + i_N^*) &= \text{dim Coker}\,(i_M \oplus i_N) = b_1(M) + b_1(N) - 2g + d \\ \text{dim Coker}\,(i_M^* + i_N^*) &= \text{dim Ker}\,(i_M \oplus i_N) = d. \end{aligned}$$

□

V.1.4 Choice of framings

In this subsection, we define certain specific reference trivializations τ_M, τ_N, which are adapted to the splitting of $H_1(M')$ into $H_1(M)$ and the torsion group determined by the meridian of Σ_M in $\partial M'$. This is a slightly "technical" issue which will make the calculations much easier. We use the results from the Appendix.

By subsection A.4 there exist certain classes

$$\mathcal{A}_M \in H^1(M';\mathbb{Z}_{k_M}), \quad \mathcal{A}_N \in H^1(N';\mathbb{Z}_{k_N})$$

which determine splittings

$$\begin{aligned} s_{\mathcal{A}_M} \colon H_1(M';\mathbb{Z}) &\longrightarrow H_1(M;\mathbb{Z}) \oplus \mathbb{Z}_{k_M} \\ \alpha &\mapsto (\rho_M\alpha, \langle \mathcal{A}_M, \alpha\rangle), \end{aligned}$$

and similarly for N. We want the framings τ_M and τ_N to be compatible with these splittings in the following way: The exact sequence

$$H_1(\partial M') \to H_1(M') \oplus H_1(\Sigma) \to H_1(M),$$

coming from the Mayer-Vietoris sequence for M, maps

$$\begin{aligned} \gamma_i^M &\mapsto (\mu_M \gamma_i^M, \gamma_i) \mapsto \rho_M \mu_M \gamma_i^M - i_M \gamma_i \\ \sigma^M &\mapsto (\mu_M \sigma^M, 0) \ \mapsto \rho_M \mu_M \sigma^M. \end{aligned}$$

By exactness, $\rho_M \mu_M \gamma_i^M = i_M \gamma_i$ and $\rho_M \mu_M \sigma^M = 0$, where γ_i^M is determined by γ_i via the trivialization τ_M. Let

$$s_{\mathcal{A}_M} : H_1(M') \to H_1(M) \oplus \mathbb{Z}_{k_M}$$

be the splitting map above. This maps

$$\begin{aligned} \mu_M \gamma_i^M &\mapsto (\rho_M \mu_M \gamma_i^M, \langle \mathcal{A}_M, \mu_M \gamma_i^M \rangle) = (i_M \gamma_i, \langle \mathcal{A}_M, \mu_M \gamma_i^M \rangle) \\ \mu_M \sigma^M &\mapsto (0, 1). \end{aligned}$$

Let $[c_i^M] = \langle \mathcal{A}_M, \mu_M \gamma_i^M \rangle \in \mathbb{Z}_{k_M}$. It follows that the composition

$$H_1(\partial M') \stackrel{\mu_M}{\to} H_1(M') \stackrel{s_{\mathcal{A}_M}}{\to} H_1(M) \oplus \mathbb{Z}_{k_M} \tag{5.6}$$

is given on generators by

$$\begin{aligned} \gamma_i^M &\mapsto (i_M \gamma_i, [c_i^M]) \\ \sigma^M &\mapsto (0, 1). \end{aligned}$$

We can change the reference trivialization τ_M to a new trivialization τ_M' such that γ_i^M changes to

$$\gamma_i^{M\prime} = \gamma_i^M - c_i^M \sigma^M,$$

for all $i = 1, \ldots, 2g$ and σ^M stays the same. This follows as in Lemma 5.5. The composition in equation (5.6) is now given by

$$\begin{aligned} \gamma_i^{M\prime} &\mapsto (i_M \gamma_i, 0) \\ \sigma^M &\mapsto (0, 1). \end{aligned}$$

Lemma 5.9. *There exists a trivialization τ_M of the normal bundle of Σ_M in M, such that the composition*

$$H_1(\partial M') \stackrel{\mu_M}{\to} H_1(M') \stackrel{s_{\mathcal{A}_M}}{\to} H_1(M) \oplus \mathbb{Z}_{k_M}$$

is given by

$$\begin{aligned} \gamma_i^M &\mapsto (i_M \gamma_i, 0), \quad i = 1, \ldots, 2g \\ \sigma^M &\mapsto (0, 1). \end{aligned}$$

There exists a similar trivialization τ_N for the normal bundle of Σ_N.

Definition 5.10. We call such framings for the normal bundles of Σ_M and Σ_N *allowed*. They depend on the choice of $\mathcal{A}_M$ and $\mathcal{A}_N$.

From now on we only work with a fixed, allowed framing for the normal bundles of both Σ_M and Σ_N.

V.2 Calculation of the first integral homology

V.2.1 Calculation of $H_1(X;\mathbb{Z})$

In the case that the greatest common divisor n_{MN} of k_M, k_N is not equal to 1, the formula for $H_1(X;\mathbb{Z})$ will involve a certain torsion term. Let r denote the homomorphism defined by

$$\begin{aligned} r\colon H_1(\Sigma;\mathbb{Z}) &\longrightarrow \mathbb{Z}_{n_{MN}}, \\ \lambda &\mapsto \langle C, \lambda\rangle \mod n_{MN}. \end{aligned}$$

We then have the following formula for $H_1(X;\mathbb{Z})$:

Theorem 5.11. *Consider the homomorphism*

$$\begin{aligned} H_1(\Sigma;\mathbb{Z}) &\stackrel{i_M\oplus i_N\oplus r}{\longrightarrow} H_1(M;\mathbb{Z}) \oplus H_1(N;\mathbb{Z}) \oplus \mathbb{Z}_{n_{MN}}, \\ \lambda &\mapsto (i_M\lambda, i_N\lambda, r(\lambda)). \end{aligned}$$

Then $H_1(X;\mathbb{Z}) \cong$ Coker$(i_M \oplus i_N \oplus r)$.

In the proof we use a small algebraic lemma which can be formulated as follows. Let H and G be abelian groups and $f\colon H \to G$ and $h\colon H \to \mathbb{Z}$ homomorphisms. Let k_M, k_N be positive integers with greatest common divisor n_{MN}. Consider the (well-defined) map

$$\begin{aligned} p\colon \mathbb{Z}_{k_M} \oplus \mathbb{Z}_{k_N} &\to \mathbb{Z}_{n_{MN}}, \\ ([x],[y]) &\mapsto [x+y]. \end{aligned}$$

Lemma 5.12. *The homomorphisms*

$$\begin{aligned} \psi\colon H\oplus\mathbb{Z} &\to G \oplus \mathbb{Z}_{k_M} \oplus \mathbb{Z}_{k_N} \\ (x,a) &\mapsto (f(x), a \textit{ mod } k_M, h(x) - a \textit{ mod } k_N), \end{aligned}$$

and

$$\begin{aligned} \psi'\colon H &\to G \oplus \mathbb{Z}_{n_{MN}} \\ x &\mapsto (f(x), h(x) \textit{ mod } n_{MN}) \end{aligned}$$

have isomorphic cokernels. The isomorphism is induced by $Id \oplus p$.

Proof. The map $Id \oplus p$ is a surjection, hence it induces a surjection

$$P\colon G \oplus \mathbb{Z}_{k_M} \oplus \mathbb{Z}_{k_N} \to \text{Coker}\,\psi'.$$

We compute the kernel of P and show that it is equal to the image of ψ. This will prove the lemma. Suppose an element is in the image of ψ. Then it is of the form $(f(x), a \text{ mod } k_M, h(x) - a \text{ mod } k_N)$. The image under P of this element is $(f(x), h(x) \text{ mod } n_{MN})$, hence in the image of ψ'. Conversely, let $(g, u \text{ mod } k_M, v \text{ mod } k_N)$ be an element in the kernel of P. The element maps under $Id \oplus p$ to $(g, u+v \text{ mod } n_{MN})$, hence there exists an element $x \in H$ such that $g = f(x)$ and $u + v \equiv h(x)$ mod n_{MN}. We can choose integers c, d, e such that the following equations hold:

$$u + v - h(x) = cn_{MN} = dk_M + ek_N.$$

Define an integer $a = u - dk_M$. Then:

$$\begin{aligned} u &\equiv a \text{ mod } k_M \\ v &\equiv h(x) - a + ek_N = h(x) - a \text{ mod } k_N. \end{aligned}$$

Hence $(g, u \text{ mod } k_M, v \text{ mod } k_N) = \psi(x, a)$ and the element is in the image of ψ. □

We now prove Theorem 5.11.

Proof. Since $\partial M'$, M' and N' are connected, the Mayer-Vietoris sequence for X shows that

$$H_1(X) \cong \text{Coker}(\psi\colon H_1(\partial M') \to H_1(M') \oplus H_1(N')).$$

The homomorphism ψ_1 is given on the standard basis by

$$\begin{aligned} \gamma_i^M &\mapsto (\mu_M \gamma_i^M, \mu_N \gamma_i^N + a_i \mu_N \sigma^N) \\ \sigma^M &\mapsto (\mu_M \sigma^M, -\mu_N \sigma^N). \end{aligned}$$

We want to replace $H_1(M')$ by $H_1(M) \oplus \mathbb{Z}_{k_M}$ and $H_1(N')$ by $H_1(N) \oplus \mathbb{Z}_{k_N}$, as in Proposition A.2. We choose splittings as in subsection V.1.4. Since we are working with an allowed framing, the composition

$$H_1(\partial M') \overset{\mu_M}{\to} H_1(M') \overset{s_{\mathcal{A}_M}}{\to} H_1(M) \oplus \mathbb{Z}_{k_M} \tag{5.7}$$

is given on generators by

$$\begin{aligned} \gamma_i^M &\mapsto (i_M \gamma_i, 0) \\ \sigma^M &\mapsto (0, 1), \end{aligned}$$

as before. Similarly, the composition

$$H_1(\partial N') \overset{\mu_N}{\to} H_1(N') \overset{s_{\mathcal{A}_N}}{\to} H_1(N) \oplus \mathbb{Z}_{k_N} \tag{5.8}$$

is given by

$$\begin{aligned} \gamma_i^N &\mapsto (i_N \gamma_i, 0) \\ \sigma^N &\mapsto (0, 1). \end{aligned}$$

If we add these maps together, we can replace ψ_1 by

$$\begin{aligned} H_1(\partial M') &\to H_1(M) \oplus \mathbb{Z}_{k_M} \oplus H_1(N) \oplus \mathbb{Z}_{k_N}, \\ \gamma_i^M &\mapsto (i_M \gamma_i, 0, i_N \gamma_i, a_i) \\ \sigma^M &\mapsto (0, 1, 0, -1). \end{aligned}$$

Using the isomorphism $H_1(\Sigma \times S^1) \cong H_1(\Sigma) \oplus \mathbb{Z} \to H_1(\partial M')$ given by τ_M, we get the map

$$\begin{aligned} H_1(\Sigma) \oplus \mathbb{Z} &\to H_1(M) \oplus \mathbb{Z}_{k_M} \oplus H_1(N) \oplus \mathbb{Z}_{k_N}, \\ (\lambda, \alpha) &\mapsto (i_M \lambda, \alpha \bmod k_M, i_N \lambda, \langle C, \lambda \rangle - \alpha \bmod k_N), \end{aligned} \tag{5.9}$$

which we call again ψ_1. To finish the proof, we have to show that this map has the same cokernel as the map

$$\begin{aligned} i_M \oplus i_N \oplus r\colon H_1(\Sigma) &\to H_1(M) \oplus H_1(N) \oplus \mathbb{Z}_{n_{MN}}, \\ \lambda &\mapsto (i_M \lambda, i_N \lambda, \langle C, \lambda \rangle \bmod n_{MN}). \end{aligned}$$

This follows from Lemma 5.12. $\square$

An immediate corollary is the following.

Corollary 5.13. *If the greatest common divisor of k_M and k_N is equal to 1, then $H_1(X;\mathbb{Z}) \cong$ Coker$(i_M \oplus i_N)$. In particular, $H_1(X;\mathbb{Z})$ does not depend on $[C]$ in this case (up to isomorphism).*

V.2.2 Calculation of the Betti numbers of X

As a corollary to Theorem 5.11 we can compute the Betti numbers of X.

Corollary 5.14. *The Betti numbers of a generalized fibre sum $X = M\#_{\Sigma_M=\Sigma_N}N$ along surfaces Σ_M and Σ_N of genus g and self-intersection 0 are given by*

$$\begin{aligned}
b_0(X) &= b_4(X) = 1 \\
b_1(X) &= b_3(X) = b_1(M) + b_1(N) - 2g + d \\
b_2(X) &= b_2(M) + b_2(N) - 2 + 2d \\
b_2^+(X) &= b_2^+(M) + b_2^+(N) - 1 + d \\
b_2^-(X) &= b_2^-(M) + b_2^-(N) - 1 + d,
\end{aligned}$$

where d is the integer from Lemma 5.8.

Proof. The formula for $b_1(X)$ follows from Theorem 5.11 and Lemma 5.8, since we can leave away all torsion terms to calculate $b_1(X)$. To determine the formula for $b_2(X)$, we use the formula for the Euler characteristic of a space decomposed into two pieces A, B:

$$e(A \cup B) = e(A) + e(B) - e(A \cap B),$$

For $M = M' \cup \nu\Sigma_M$, with $M' \cap \nu\Sigma_M \cong \Sigma \times S^1$, we get

$$\begin{aligned}
e(M) &= e(M') + e(\nu\Sigma_M) - e(\Sigma \times S^1) \\
&= e(M') + 2 - 2g,
\end{aligned}$$

since $\nu\Sigma_M$ is homotopy equivalent to Σ_M and $\Sigma \times S^1$ is a 3-manifold, hence has zero Euler characteristic. This implies

$$e(M') = e(M) + 2g - 2, \quad \text{and similarly} \quad e(N') = e(N) + 2g - 2.$$

For $X = M' \cup N'$, with $M' \cap N' \cong \Sigma \times S^1$, we then get

$$\begin{aligned}
e(X) &= e(M') + e(N') \\
&= e(M) + e(N) + 4g - 4.
\end{aligned}$$

Together with the formula for $b_1(X) = b_3(X)$ above, this implies

$$\begin{aligned}
b_2(X) &= -2 + 2(b_1(M) + b_1(N) - 2g + d) + 2 - 2b_1(M) + b_2(M) \\
&\quad + 2 - 2b_1(N) + b_2(N) + 4g - 4 \\
&= b_2(M) + b_2(N) - 2 + 2d.
\end{aligned}$$

It remains to prove the formulas for $b_2^\pm(X)$. By Novikov additivity for the signature [56, Remark 9.1.7],

$$\sigma(X) = \sigma(M) + \sigma(N),$$

we get by adding $b_2(X)$ on both sides,

$$2b_2^+(X) = 2b_2^+(M) + 2b_2^+(N) - 2 + 2d,$$

hence $b_2^+(X) = b_2^+(M) + b_2^+(N) - 1 + d$. This also implies the formula for $b_2^-(X)$. □

A direct computation of $b_2(X)$ as the rank of $H_2(X; \mathbb{Z})$ will be given in Section V.3.4.

V.2.3 Calculation of $H^1(X;\mathbb{Z})$

Since $H^1(X;\mathbb{Z})$ is torsion free it is determined up to isomorphism completely by its rank, given by the first Betti number $b_1(X)$ from Corollary 5.14. We nevertheless give an explicit calculation of $H^1(X;\mathbb{Z})$ because this will be useful later on.

Consider the following part of the Mayer-Vietoris sequence in cohomology:

$$0 \to H^1(X) \xrightarrow{\eta_M^* \ominus \eta_N^*} H^1(M') \oplus H^1(N') \xrightarrow{\psi_1^*} H^1(\partial M').$$

Since $\eta_M^* - \eta_N^*$ is injective, $H^1(X)$ is isomorphic to the kernel of $\psi_1^* = \mu_M^* + \phi^*\mu_N^*$. We want to calculate this kernel. Consider the map

$$\mu_M^* \colon H^1(M') \to H^1(\partial M').$$

By the proof of Proposition A.2, the map $\rho_M^* \colon H^1(M) \to H^1(M')$ is an isomorphism. Via the framing τ_M we can identify

$$H^1(\partial M') \cong H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^M).$$

Note that $\sigma^{M*} = PD(\Sigma^M)$ in $\partial M'$.

We want to calculate the composition

$$H^1(M) \cong H^1(M') \xrightarrow{\mu_M^*} H^1(\partial M') \cong H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^M).$$

Let $\alpha \in H^1(M)$. Then

$$\begin{aligned}\langle \mu_M^* \rho_M^* \alpha, \gamma_i^M \rangle &= \langle \alpha, \rho_M \mu_M \gamma_i^M \rangle \\ &= \langle \alpha, i_M \gamma_i \rangle \\ &= \langle i_M^* \alpha, \gamma_i \rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \mu_M^* \rho_M^* \alpha, \sigma_i^M \rangle &= \langle \alpha, \rho_M \mu_M \sigma^M \rangle \\ &= 0.\end{aligned}$$

Hence the composition $H^1(M) \to H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^M)$ is given by $i_M^* \oplus 0$. Similarly, the composition $H^1(N) \to H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^N)$ is given by $i_N^* \oplus 0$. We now consider the composition

$$H^1(M) \oplus H^1(N) \cong H^1(M') \oplus H^1(N') \xrightarrow{\psi_1^*} H^1(\partial M') \cong H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^M).$$

The map ψ_1^* is given by $\mu_M^* + \phi^*\mu_N^*$. Since $\phi^*\gamma_i^{N*} = \gamma_i^{M*}$ for all $i = 1, \ldots, 2g$, we see that this composition is given by

$$(i_M^* + i_N^*) \oplus 0 \colon H^1(M) \oplus H^1(N) \to H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^M). \tag{5.10}$$

In particular, we get:

Proposition 5.15. *The first cohomology $H^1(X;\mathbb{Z})$ is isomorphic to the kernel of*

$$i_M^* + i_N^* \colon H^1(M;\mathbb{Z}) \oplus H^1(N;\mathbb{Z}) \to H^1(\Sigma;\mathbb{Z}).$$

V.3 Calculation of the second integral cohomology

We consider the following part of the Mayer-Vietoris sequence:

$$H^1(M') \oplus H^1(N') \xrightarrow{\psi_1^*} H^1(\partial M') \longrightarrow H^2(X) \xrightarrow{\eta_M^* \ominus \eta_N^*} H^2(M') \oplus H^2(N') \xrightarrow{\psi_2^*} H^2(\partial M').$$

This implies a short exact sequence

$$0 \longrightarrow \text{Coker}\, \psi_1^* \longrightarrow H^2(X) \longrightarrow \text{Ker}\, \psi_2^* \longrightarrow 0. \tag{5.11}$$

By equation (5.10),

$$\text{Coker}\, \psi_1^* \cong \text{Coker}\,(i_M^* + i_N^*) \oplus \mathbb{Z}PD(\Sigma^M). \tag{5.12}$$

We calculate Coker $(i_M^* + i_N^*)$ in the next section, which is related to the notion of *rim tori*.

V.3.1 Rim tori in $H_2(X;\mathbb{Z})$

We consider the following part of the Mayer-Vietoris sequence for M:

$$H_2(\partial M') \xrightarrow{\mu_M \oplus p_M} H_2(M') \oplus H_2(\Sigma) \xrightarrow{\rho_M - i_M} H_2(M) \xrightarrow{\partial} H_1(\partial M'). \tag{5.13}$$

The subgroup ker p_M in $H_2(\partial M')$ is generated by the classes Γ_i^M, for $i = 1, \ldots, 2g$.

Lemma 5.16. *The kernel of* $\rho_M \colon H_2(M') \to H_2(M)$ *is equal to the image of* ker p_M *under the homomorphism* μ_M.

Proof. Suppose α is an element in ker p_M. Then $\rho_M \mu_M \alpha = i_M p_M \alpha = 0$. Conversely, suppose β is an element in $H_2(M')$ with $\rho_M \beta = 0$. Then $0 = \rho_M \beta - i_M(0)$, hence by exactness of the Mayer-Vietoris sequence there exists an $\alpha \in H_2(\partial M')$ with $\beta = \mu_M \alpha, 0 = p_M \alpha$. This implies that Im μ_M = ker ρ_M. $\square$

Note that there exists an isomorphism $PD \circ p_M^* \colon H^1(\Sigma;\mathbb{Z}) \to$ ker p_M, where

$$p_M^* \colon H^1(\Sigma) \to H^1(\partial M')$$

and

$$PD \colon H^1(\partial M') \to H_2(\partial M')$$

is Poincaré duality. In our standard basis, this isomorphism is given by

$$\begin{aligned} H^1(\Sigma;\mathbb{Z}) &\to \text{ker}\, p_M \\ \sum c_i \gamma_i^* &\mapsto \sum c_i \Gamma_i^M. \end{aligned} \tag{5.14}$$

Lemma 5.17. *Every element in the kernel of* ρ_M *can be represented by a smoothly embedded torus in the interior of* M'.

Proof. Note that the classes $\Gamma_i^M \subset H_2(\partial M')$ are of the form $\chi_i^M \times \sigma^M$ where χ_i^M is a curve on Σ_M. Hence every element $T \in$ ker p_M is represented by a surface of the form $c^M \times \sigma^M$, where c^M is a closed, oriented curve on Σ_M with transverse self-intersections. A collar of $\partial M' = \partial\nu\Sigma_M$ in M' is of the form $\Sigma_M \times S^1 \times I$. We can eliminate the self-intersection points of the curve c^M in $\Sigma_M \times I$, without changing the homology class. If we then cross with σ^M, we see that $\mu_M(T) = c^M \times \sigma^M$ can be represented by a smoothly embedded torus in M'. $\square$

We have the following definition, see e.g. [34, 42, 68].

Definition 5.18. We call $\mu_M(T) \in H_2(M')$ for an element $T \in \ker p_M$ the *rim torus* in M' associated to T. Equivalently, we can consider rim tori as being associated to elements in $\alpha \in H^1(\Sigma;\mathbb{Z})$ via $\mu_M \circ PD \circ p_M^*(\alpha)$. We denote by $R(M')$ the group of all rim tori, i.e. the image of the homomorphism

$$\mu_M \circ PD \circ p_M^*\colon H^1(\Sigma) \rightarrow H_2(M').$$

Rim tori are already "virtually" in the manifold M as embedded null-homologous tori. Some of them can become non-zero homology classes if the tubular neighbourhood $\nu\Sigma_M$ is deleted. There is an analogous construction for N.

We want to discuss orientations and intersection numbers of rim tori and related surfaces: Note that by definition $\Gamma_i = PD(\gamma_i^*)$. This implies

$$\langle PD(\Gamma_i), \gamma_j\rangle = \delta_{ij},$$

hence the surfaces Γ_i have intersection δ_{ij} with the curves γ_j. Similarly, suppose that a torus T is associated to an element $\sum_{i=1}^{2g} c_i\gamma_i^*$. Then $PD(T) = \sum_{i=1}^{2g} c_i\Gamma_i$ and

$$T \cdot \gamma_j = c_j.$$

Suppose that T is given by $a \times \sigma$ where a is an oriented curve on Σ. Give T the orientation determined by the orientation of a followed by the orientation of σ. Then

$$T \cdot \gamma_j = -(a \cdot \gamma_j),$$

because the orientation of $\Sigma \times S^1$ is the orientation of Σ followed by the orientation of S^1. These relations also hold on $\partial M'$ and $\partial N'$. Finally, suppose that e is another oriented curve on Σ. We view e as a curve on the push-off Σ^M in M'. The curve e defines a small annulus E_M in a collar of $\partial M' = \Sigma \times S^1 \times I$ given by $E_M = e \times I$ where I is an interval pointing radially outwards along the D^2 factor in $\nu\Sigma_M = \Sigma \times D^2$. Give E_M the orientation of e followed by the orientation of I pointing *into* M'. Denote by T_M the rim torus in M' associated to $T = a \times \sigma$. We then have

$$\begin{aligned} T_M \cdot E_M &= (a \times \sigma) \cdot (e \times I) \\ &= a \cdot e, \end{aligned}$$

because the orientation of a collar $\Sigma \times S^1 \times I$ is given by the orientation of Σ followed by the orientation of S^1 and followed by the orientation of I pointing *out of* M', cf. Section V.1.

Lemma 5.19. *With our orientation conventions, the algebraic intersection number of a rim torus T_M and an annulus E_M as above is given by $T_M \cdot E_M = a \cdot e$.*

We can map every rim torus in M' under the inclusion $\eta_M\colon M' \rightarrow X$ to a homology class in X.

Definition 5.20. We call $\eta_M \circ \mu_M(\alpha)$ the rim torus in X associated to the element $\alpha \in H^1(\Sigma)$. The group of rim tori in X is defined as the image of the homomorphism

$$\eta_M \circ \mu_M \circ PD \circ p_M^*\colon H^1(\Sigma) \rightarrow H_2(X).$$

We can also map every rim torus in N' via the inclusion $\eta_N\colon N' \rightarrow X$ to an embedded torus in X. This torus is related to the rim torus coming via M' in the following way:

Lemma 5.21. *Let α be a class in $H^1(\Sigma)$. Then $\eta_M \circ \mu_M \circ PD \circ p_M^* \alpha = -\eta_N \circ \mu_N \circ PD \circ p_N^* \alpha$. Hence for the same element $\alpha \in H^1(\Sigma)$ the rim torus in X coming via N' is minus the rim torus coming via M'.*

Proof. The action of the gluing diffeomorphism ϕ on second homology is given by $\phi_* \Gamma_i^M = -\Gamma_i^N$. Let $\alpha \in H^1(\Sigma)$ be a fixed class,

$$\alpha = \sum_{i=1}^{2g} c_i \gamma_i^*.$$

The rim tori in M' and N' associated to α are given by

$$a_M = \sum_{i=1}^{2g} c_i \mu_M \Gamma_i^M, \quad a_N = \sum_{i=1}^{2g} c_i \mu_N \Gamma_i^N = -\sum_{i=1}^{2g} c_i \mu_N \phi_* \Gamma_i^M.$$

In X we get

$$\begin{aligned} \eta_M a_M + \eta_N a_N &= \sum_{i=1}^{2g} c_i (\eta_M \mu_M - \eta_N \mu_N \phi_*) \Gamma_i^M \\ &= 0, \end{aligned}$$

by the Mayer-Vietoris sequence for X. Hence if a_M and a_N are rim tori in M' and N' associated to the same element $\alpha \in H^1(\Sigma; \mathbb{Z})$ then $\eta_M a_M = -\eta_N a_N$. □

Definition 5.22. Let R_C denote the rim torus in X determined by the class $-\sum_{i=1}^{2g} a_i \Gamma_i^M \in H_2(\partial M')$ under the inclusion of $\partial M'$ in X as in Definition 5.18. This class is equal to the image of the class $\sum_{i=1}^{2g} a_i \Gamma_i^N \in H_2(\partial N')$ under the inclusion of $\partial N'$ in X.

Recall that Σ_X is the class in X which is the image of the push-off Σ^M under the inclusion $M' \to X$. Similarly, Σ_X' is the image of the push-off Σ^N under the inclusion $N' \to X$.

Lemma 5.23. *The classes Σ_X' and Σ_X in X differ by*

$$\Sigma_X' - \Sigma_X = R_C.$$

This follows since by Lemma 5.6,

$$\phi_* \Sigma^M = -\left(\sum_{i=1}^{2g} a_i \Gamma_i^N \right) + \Sigma^N.$$

The difference is due to the fact that the diffeomorphism ϕ does not necessarily match the classes Σ^M and Σ^N.

Recall that the embedding $i_M \colon \Sigma \to M$ defines a homomorphism

$$i_M^* \colon H^1(M) \to H^1(\Sigma).$$

We now determine the set of elements in $H^1(\Sigma)$ which map to null-homologous rim tori in M'.

Proposition 5.24. *The kernel of the map $\mu_M \circ PD \circ p_M^*$ is equal to the image of i_M^*.*

Proof. Consider the following sequence coming from the long exact sequence for the pair $(M', \partial M')$:

$$H_3(M', \partial M') \xrightarrow{\partial} H_2(\partial M') \xrightarrow{\mu_M} H_2(M').$$

Under Poincaré duality

$$\begin{array}{ccc} H_3(M', \partial M') & \xrightarrow{\partial} & H_2(\partial M') \\ \cong\big\downarrow & & \cong\big\downarrow \\ H^1(M') & \xrightarrow{\mu_M^*} & H^1(\partial M') \end{array} \tag{5.15}$$

This shows that the kernel of $\mu_M \circ PD \circ p_M^*$ is equal to the set of all elements $c \in H^1(\Sigma)$ such that $p_M^* c$ is in the image of μ_M^*. Note that the embedding $\rho_M \colon M' \to M$ induces an isomorphism $\rho_M^* \colon H^1(M) \to H^1(M')$ by the proof of Proposition A.2. Hence the image of μ_M^* is equal to the image of $\mu_M^* \rho_M^*$.

Suppose $c \in H^1(\Sigma)$ is an element such that $p_M^* c$ is in the image of $\mu_M^* \rho_M^*$. Hence we can write

$$p_M^* c = \mu_M^* \rho_M^* \beta,$$

for some $\beta \in H^1(M)$. We have $p_M^* c = \sum_{j=1}^{2g} c_j \gamma_j^*$ for certain coefficients c_j. The coefficients can be determined as follows:

$$\begin{aligned} c_i &= \langle p_M^* c, \gamma_i \rangle \\ &= \langle \mu_M^* \rho_M^* \beta, \gamma_i \rangle \\ &= \langle \beta, i_M \gamma_i \rangle \\ &= \langle i_M^* \beta, \gamma_i \rangle. \end{aligned}$$

In the third step we have used the Mayer-Vietors sequence (5.13). We have also denoted classes γ_i on Σ and $\Sigma \times S^1$, which correspond under the projection p, by the same symbol. This implies

$$\begin{aligned} p_M^* c &= \sum_{i=1}^{2g} \langle i_M^* \beta, \gamma_i \rangle \gamma_i^* \\ &= p_M^* i_M^* \beta. \end{aligned}$$

Since p_M^* is injective it follows that $c = i_M^* \beta$. Hence c is in the image of i_M^*.

Conversely, the same calculation backwards shows that every class in the image of $i_M^* \colon H^1(M) \to H^1(\Sigma)$ gives under p_M^* a class in $H^1(\partial M')$ in the image of $\mu_M^* \rho_M^*$. □

We can now prove the main theorem in this subsection.

Theorem 5.25. *Let i_M^*, i_N^* denote the homomorphisms*

$$i_M^* \colon H^1(M; \mathbb{Z}) \to H^1(\Sigma; \mathbb{Z}), \quad \textit{and} \quad i_N^* \colon H^1(N; \mathbb{Z}) \to H^1(\Sigma; \mathbb{Z}).$$

Then the defining maps in Definitions 5.18 and 5.20 for the groups of rim tori in M', N' and X' induce isomorphisms

$$\begin{aligned} \textit{Coker}\, i_M^* &\xrightarrow{\cong} R(M') \\ \textit{Coker}\, i_N^* &\xrightarrow{\cong} R(N') \\ \textit{Coker}\, (i_M^* + i_N^*) &\xrightarrow{\cong} R(X). \end{aligned}$$

Proof. The statement about $R(M')$ and $R(N')$ follows immediately from Lemma 5.24. It remains to prove the statement about $R(X)$. Consider the following diagram:

$$\begin{array}{ccccc} H_3(X,M') & \xrightarrow{\partial} & H_2(\partial M') & \xrightarrow{\eta_M} & H_2(X) \\ \cong\big\uparrow & & \mu_M\phi_*^{-1}\big\uparrow & & \eta_N\big\uparrow \\ H_3(N',\partial N') & \xrightarrow{\partial} & H_2(\partial N') & \longrightarrow & H_2(N') \end{array}$$

The horizontal parts come from the long exact sequences of pairs, the vertical parts come from inclusion. The isomorphism on the left is by excision. Hence the kernel of η_M is given by the image of

$$\mu_M \circ \phi_*^{-1} \circ \partial \colon H_3(N',\partial N') \to H_2(M').$$

We claim that this is equal to the image of

$$\mu_M \circ PD \circ p_M^* \circ i_N^* \colon H^1(N) \to H_2(M').$$

This follows in three steps: First, by equation (5.15) and the remark following it, the image of ∂ is equal to the image of $PD \circ \mu_N^* \circ \rho_N^*$. By the Mayer-Vietoris sequence for N

$$H^1(N) \overset{\rho_N^* \ominus i_N^*}{\longrightarrow} H^1(N') \oplus H^1(\Sigma) \overset{\mu_N^* + p_N^*}{\longrightarrow} H^1(\partial N')$$

we have $\mu_N^* \circ \rho_N^* = p_N^* \circ i_N^*$. Finally, we use the identity

$$\phi_*^{-1} \circ PD \circ p_N^* = -PD \circ p_M^*,$$

which is equivalent to the known identity $\phi_*\Gamma_i^M = -\Gamma_i^N$, for all $i = 1, \ldots, 2g$.

Suppose that $\alpha \in H^1(\Sigma)$ is in the kernel of $\eta_M \circ \mu_M \circ PD \circ p_M^*$. This happens if and only if $\mu_M \circ PD \circ p_M^* \alpha$ is in the kernel of η_M. By the argument above this is equivalent to the existence of a class $\beta_N \in H^1(N)$ with

$$\mu_M \circ PD \circ p_M^* \alpha = \mu_M \circ PD \circ p_M^* \circ i_N^* \beta_N.$$

By Lemma 5.24, this is equivalent to the existence of a class $\beta_M \in H^1(M)$ with

$$\alpha = i_M^* \beta_M + i_N^* \beta_N.$$

This shows that the kernel of $\eta_M \circ \mu_M \circ PD \circ p_M^*$ is equal to the image of $i_M^* + i_N^*$ and proves the claim. $\square$

Corollary 5.26. *The rank of the abelian subgroup $R(X)$ of rim tori in $H_2(X;\mathbb{Z})$ is equal to the integer d, defined in Lemma 5.8.*

V.3.2 Perpendicular classes

For the calculation of $H^2(X;\mathbb{Z})$ it remains to calculate the kernel of

$$\psi_2^* \colon H^2(M') \oplus H^2(N') \to H^2(\partial M'),$$

where $\psi_2^* = \mu_M^* + \phi^* \mu_N^*$, as in equation (5.11). Consider the homomorphism

$$\mu_M^* \colon H^2(M') \to H^2(\partial M').$$

By Lemma A.1 there exists a short exact sequence

$$0 \longrightarrow \frac{H^2(M)}{\mathbb{Z}\Sigma_M} \xrightarrow{\rho_M^*} H^2(M') \xrightarrow{\partial} \ker i_M \longrightarrow 0, \tag{5.16}$$

where $\ker i_M$ is the kernel of $i_M\colon H_1(\Sigma) \to H_1(M)$. This sequence splits because $\ker i_M$ is free abelian and we can write

$$H^2(M') \cong \frac{H^2(M)}{\mathbb{Z}\Sigma_M} \oplus \ker i_M. \tag{5.17}$$

A splitting can be defined as follows: The images of loops representing the classes in $\ker i_M$ under the embedding $i_M\colon \Sigma \to M$ bound surfaces in M. Using the trivialization τ_M we can think of these loops to be on the push-off Σ^M and the surfaces they bound in M'. In this way the elements in $\ker i_M$ determine classes in $H^2(M') \cong H_2(M', \partial M')$.

We also use the trivialization τ_M to identify

$$H^2(\partial M') \cong H_1(\Sigma \times S^1) \cong \mathbb{Z} \oplus H_1(\Sigma),$$

where the $\mathbb{Z}$ summand is spanned by $PD(\sigma^M)$. We can then consider the composition

$$\frac{H^2(M)}{\mathbb{Z}\Sigma_M} \oplus \ker i_M \cong H^2(M') \xrightarrow{\mu_M^*} H^2(\partial M') \cong \mathbb{Z} \oplus H_1(\Sigma).$$

Proposition 5.27. *The composition*

$$\mu_M^*\colon \frac{H^2(M)}{\mathbb{Z}\Sigma_M} \oplus \mathit{ker}\, i_M \to \mathbb{Z} \oplus H_1(\Sigma) \tag{5.18}$$

is given by

$$([A], \alpha) \mapsto (A \cdot \Sigma_M, \alpha).$$

Note that this map is well-defined in the first variable since $\Sigma_M^2 = 0$. The map in the second variable is inclusion.

Proof. The proof is in two steps. To show that μ_M^* is the identity on the second summand note that by Poincaré duality

$$\begin{array}{ccc} H_2(M', \partial M') & \xrightarrow{\partial} & H_1(\partial M') \\ \cong\downarrow & & \cong\downarrow \\ H^2(M') & \xrightarrow{\mu_M^*} & H^2(\partial M') \end{array}$$

as in equation (5.15). This implies the claim by our choice of splitting. It remains to prove that

$$\mu_M^* \rho_M^* [A] = (A \cdot \Sigma_M) PD(\sigma^M)$$

Note that $\mu_M^* \rho_M^* A = p_M^* i_M^* A$ by the Mayer-Vietoris sequence for $M = M' \cup \nu\Sigma_M$. Since

$$\langle i_M^* A, \Sigma \rangle = \langle A, \Sigma_M \rangle = A \cdot \Sigma_M$$

the class $i_M^* A$ is equal to $(A \cdot \Sigma_M)1$, where 1 denotes the generator of $H^2(\Sigma)$, Poincaré dual to a point. Since $p_M^*(1)$ is the Poincaré dual of a fibre in $\partial M' = \partial \nu \Sigma_M$ the claim follows. $\square$

Note that the map

$$\mu_M^* \rho_M^* \colon \frac{H^2(M)}{\mathbb{Z}\Sigma_M} \to \mathbb{Z}$$

is given by intersection with Σ_M. Hence it can take values only in $k_M\mathbb{Z}$ because Σ_M is divisible by k_M.

Definition 5.28. We choose a class $B_M \in H^2(M;\mathbb{Z})$ with $B_M \cdot A_M = 1$. Such a class exists because A_M is indivisible. We denote the image of this class in $H^2(M') \cong H_2(M', \partial M')$ by B_M'.

Then the equation $B_M \cdot \Sigma_M = k_M$ implies that the map given by intersection with Σ_M is surjective onto $k_M\mathbb{Z}$.

Lemma 5.29. *The class $B_M' \in H_2(M', \partial M')$ bounds the k_M-fold multiple of the meridian in $\partial M'$.*

Proof. Under Poincaré duality the sequence

$$H^2(M) \xrightarrow{\rho_M^*} H^2(M') \xrightarrow{\mu_M^*} H^2(\partial M')$$

corresponds to

$$H_2(M) \longrightarrow H_2(M', \partial M') \xrightarrow{\partial} H_1(\partial M'),$$

where the first map is $H_2(M) \to H_2(M, \Sigma_M) \cong H_2(M', \partial M')$. Hence the class $B_M' \in H_2(M', \partial M')$ maps to $k_M \sigma^M$. □

Consider the subgroup in $H^2(M;\mathbb{Z})$ generated by the classes B_M and A_M.[1] Since $A_M^2 = 0$ and $A_M \cdot B_M = 1$, the intersection form on these (indivisible) elements looks like

$$\begin{pmatrix} B_M^2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 5.30. Let $P(M) = (\mathbb{Z}B_M \oplus \mathbb{Z}A_M)^\perp$ denote the orthogonal complement in $H^2(M)$ with respect to the intersection form. The elements in $P(M)$ are called *perpendicular classes*.

Since the restriction of the intersection form to $(\mathbb{Z}B_M \oplus \mathbb{Z}A_M)$ is unimodular (it is equivalent to H if B_M^2 is even and to $(+1) \oplus (-1)$ if B_M^2 is odd) it follows that there exists a direct sum decomposition

$$H^2(M) = \mathbb{Z}B_M \oplus \mathbb{Z}A_M \oplus P(M). \tag{5.19}$$

The restriction of the intersection form to $P(M)/\mathrm{Tor}$ is again unimodular (see [56, Lemma 1.2.12]). This implies also that the rank of $P(M)$ is $b_2(M) - 2$.

Lemma 5.31. *For every element $\alpha \in H^2(M)$ there exists a decomposition of the form*

$$\alpha = (\alpha \cdot A_M)B_M + (\alpha \cdot B_M - B_M^2(\alpha \cdot A_M))A_M + \overline{\alpha}, \tag{5.20}$$

where

$$\overline{\alpha} = \alpha - (\alpha \cdot A_M)B_M - (\alpha \cdot B_M - B_M^2(\alpha \cdot A_M))A_M$$

is an element in $P(M)$, hence orthogonal to both A_M and B_M.

[1] This subgroup corresponds to the Gompf nucleus in elliptic surfaces defined as a regular neighbourhood of a cusp fibre and a section, cf. [53], [56, Section 3.1].

This follows by writing $p(\alpha) = \alpha + aA_M + bB_M$. The equations $p(\alpha) \cdot A_M = 0 = p(\alpha) \cdot B_M$ determine the coefficients a, b.

Since $\Sigma_M = k_M A_M$, we can now write

$$\frac{H^2(M)}{\mathbb{Z}\Sigma_M} \cong \mathbb{Z}_{k_M} A_M \oplus \mathbb{Z}B_M \oplus P(M).$$

Definition 5.32. We define $P(M)_{A_M} = \mathbb{Z}_{k_M} A_M \oplus P(M)$.

Note that all constructions and definitions in this section can be done for the manifold N as well.

V.3.3 Split classes in $H_2(X;\mathbb{Z})$

In this section we calculate the kernel of the homomorphism

$$\psi_2^* \colon H^2(M') \oplus H^2(N') \to H^2(\partial M').$$

Consider the following map:

$$\begin{aligned} f \colon \mathbb{Z}B_M \oplus \mathbb{Z}B_N \oplus \ker(i_M \oplus i_N) &\longrightarrow \mathbb{Z} \\ (x_M B_M, x_N B_N, \alpha) &\mapsto x_M k_M + x_N k_N - \langle C, \alpha \rangle. \end{aligned}$$

Here B_M, B_N are just formal terms which could be left away.

Definition 5.33. Let $\ker f = S(X)$. We call $S(X)$ the group of *split classes* of X. It is a free abelian group of rank $d+1$ since $\ker(i_M \oplus i_N)$ has rank d by Lemma 5.8.

The elements in $S(X)$ have the following interpretation:

Lemma 5.34. *The elements $(x_M B_M, x_N B_N, \alpha)$ in $S(X)$ are precisely those elements in $\mathbb{Z}B_M \oplus \mathbb{Z}B_N \oplus \ker(i_M \oplus i_N)$ such that $\alpha^M + x_M k_M \sigma^M$ bounds in M', $\alpha^N + x_N k_N \sigma^N$ bounds in N', and both elements get identified under the gluing diffeomorphism ϕ.*

Proof. Suppose that an element

$$\alpha^M + r\sigma^M = \tau_M \alpha + r\sigma^M \in H_1(\partial M'),$$

with $\alpha \in H_1(\Sigma)$, is null-homologous in M'. By the proof of Theorem 5.11 this happens if and only if $i_M \alpha = 0 \in H_1(M)$ and r is divisible by k_M, hence $r = x_M k_M$ for some $x_M \in \mathbb{Z}$. In this case it bounds a surface in M'. The class $\alpha^M + r\sigma^M$ maps under ϕ to the class

$$\alpha^N + \langle C, \alpha \rangle \sigma^N - r\sigma^N.$$

This class is null-homologous in N' if and only if $i_N \alpha = 0$ and $\langle C, \alpha \rangle - r = \langle C, \alpha \rangle - x_M k_M$ is divisible by k_N, hence

$$\langle C, \alpha \rangle - x_M k_M = x_N k_N.$$

□

We can now prove:

Theorem 5.35. *The kernel of the homomorphism*

$$\psi_2^* \colon H^2(M') \oplus H^2(N') \to H^2(\partial M')$$

is isomorphic to $S(X) \oplus P(M)_{A_M} \oplus P(N)_{A_N}$.

Proof. By equation 5.18, the map μ_M^* is given by

$$\begin{aligned} P(M)_{A_M} \oplus \mathbb{Z}B_M \oplus \ker i_M &\longrightarrow k_M \mathbb{Z} PD(\sigma^M) \oplus H_1(\Sigma) \\ (c_M, x_M, \alpha_M) &\mapsto (x_M k_M, \alpha_M). \end{aligned}$$

We can replace μ_N^* by a similar map

$$\begin{aligned} P(N)_{A_N} \oplus \mathbb{Z}B_N \oplus \ker i_N &\longrightarrow k_N \mathbb{Z} PD(\sigma^N) \oplus H_1(\Sigma) \\ (c_N, x_N, \alpha_N) &\mapsto (x_N k_N, \alpha_N). \end{aligned}$$

Under the identifications

$$\begin{aligned} H^2(\partial M') \cong H_1(\partial M') \cong \mathbb{Z} \oplus H_1(\Sigma), \quad \text{and} \\ H^2(\partial N') \cong H_1(\partial N') \cong \mathbb{Z} \oplus H_1(\Sigma) \end{aligned}$$

given by the framings τ_M, τ_N, we can calculate the map

$$\phi^* \colon H^2(\partial N') \to H^2(\partial M')$$

as follows: By equation (5.5), we have

$$\begin{aligned} \phi_2^* PD(\sigma^N) &= -(\phi^{-1})_* \sigma^N = \sigma^M \\ \phi_2^* PD(\gamma_i^N) &= -(\phi^{-1})_* \gamma_i^N = -\gamma_i^M - a_i \sigma^M. \end{aligned}$$

Hence ϕ^* is given by

$$\begin{aligned} \mathbb{Z} \oplus H_1(\Sigma) &\longrightarrow \mathbb{Z} \oplus H_1(\Sigma) \\ (x, y) &\mapsto (x - \langle C, y \rangle, -y). \end{aligned}$$

Therefore, we can replace the map $\psi_2^* = \mu_M^* + \phi^* \mu_N^*$ by the following homomorphism:

$$P(M)_{A_M} \oplus P(N)_{A_N} \oplus \mathbb{Z}B_M \oplus \mathbb{Z}B_N \oplus \ker i_M \oplus \ker i_N \longrightarrow \mathbb{Z} \oplus H_1(\Sigma)$$

given by

$$(c_M, c_N, x_M, x_N, \alpha_M, \alpha_N) \mapsto (x_M k_M + x_N k_N - \langle C, \alpha_N \rangle, \alpha_M - \alpha_N).$$

Elements in the kernel must satisfy $\alpha_M = \alpha_N$. In particular, both elements are in $\ker i_M \cap \ker i_N = \ker (i_M \oplus i_N)$. Hence the kernel of the replaced ψ_2^* is given by

$$S(X) \oplus P(M)_{A_M} \oplus P(N)_{A_N}.$$

□

V.3.4 Calculation of $H^2(X; \mathbb{Z})$

We can now write the short exact sequence (5.11) in the following form, using the calculations in equation (5.12), Theorem 5.25 and Theorem 5.35:

Theorem 5.36. *There exists a short exact sequence*

$$0 \to R(X) \oplus \mathbb{Z}\Sigma_X \to H^2(X; \mathbb{Z}) \to S(X) \oplus P(M)_{A_M} \oplus P(N)_{A_N} \to 0. \tag{5.21}$$

Note that Σ_X is the class (or its Poincaré dual) coming from the push-off Σ^M under the inclusion $M' \to X$. We can check the second Betti number given by the exact sequence for $H^2(X;\mathbb{Z})$ in Theorem 5.36: together with our previous calculation of the ranks for the corresponding groups we get

$$\begin{aligned} b_2(X) &= d+1+(d+1)+(b_2(M)-2)+(b_2(N)-2) \\ &= b_2(M)+b_2(N)-2+2d. \end{aligned}$$

This is the same number as in Corollary 5.14.

V.3.5 The intersection form of X

The group of split classes $S(X)$ always contains the element

$$B_X = \tfrac{1}{n_{MN}}(k_N B_M - k_M B_N).$$

In particular, if Σ_M and Σ_N represent indivisible classes we have $B_X = B_M - B_N$. Suppose in addition that the cohomologies of M, N and X are torsion free. This is equivalent to H^2 or H_1 being torsion free. To check whether $H_1(X)$ is torsion free one can use Theorem 5.11 and Corollary 5.13. We want to prove that we can choose d elements $S_1,\ldots,S_d$ in $S(X)$, which form a basis for $S(X)$ together with the clas B_X, and a basis $R_1,\ldots,R_d$ for the group of rim tori $R(X)$ such that the following holds:

Theorem 5.37. *Let $X = M\#_{\Sigma_M=\Sigma_N}N$ be a generalized fibre sum of closed oriented 4-manifolds M and N along embedded surfaces Σ_M, Σ_N of genus g which represent indivisible homology classes. Suppose that the cohomology of M, N and X is torsion free. Then there exists a splitting*

$$H^2(X;\mathbb{Z}) = P(M)\oplus P(N)\oplus (S'(X)\oplus R(X))\oplus(\mathbb{Z}B_X\oplus\mathbb{Z}\Sigma_X),$$

where

$$(S'(X)\oplus R(X)) = (\mathbb{Z}S_1\oplus\mathbb{Z}R_1)\oplus\ldots\oplus(\mathbb{Z}S_d\oplus\mathbb{Z}R_d).$$

The direct sums are all orthogonal, except the direct sums inside the brackets. In this direct sum, the restriction of the intersection form Q_X to $P(M)$ and $P(N)$ is equal to the intersection form induced from M and N and has the structure

$$\begin{pmatrix} B_M^2+B_N^2 & 1 \\ 1 & 0 \end{pmatrix}$$

on $\mathbb{Z}B_X\oplus\mathbb{Z}\Sigma_X$ and the structure

$$\begin{pmatrix} S_i^2 & 1 \\ 1 & 0 \end{pmatrix}$$

on each summand $\mathbb{Z}S_i\oplus\mathbb{Z}R_i$.

The construction of the surfaces representing $S_1,\ldots,S_d$ is rather lengthy and will be done step by step.

Choose a basis $\alpha_1,\ldots,\alpha_d$ for the subgroup of ker$(i_M\oplus i_N)$ of those elements α such that $\langle C,\alpha\rangle$ is divisible by n_{MN}. We then get a basis of $S(X)$ consisting of the element

$$B_X = \tfrac{1}{n_{MN}}(k_N B_M - k_M B_N)$$

and d further elements of the form

$$S_i = x_M(\alpha_i)B_M + x_N(\alpha_i)B_N + \alpha_i, \quad 1\le i\le d,$$

where $x_M(\alpha_i), x_N(\alpha_i)$ are coefficients with

$$x_M(\alpha_i)k_M + x_N(\alpha_i)k_N = \langle C, \alpha_i \rangle.$$

The class B_X is sewed together from surfaces in M' and N' which represent the classes $\frac{k_N}{n_{MN}}B'_M$ and $\frac{k_M}{n_{MN}}B'_N$ that bound the $\frac{k_M k_N}{n_{MN}}$-fold multiple of the meridians σ^M and σ^N in $\partial M'$ and $\partial N'$.

The (immersed) surfaces representing S_i are constructed as follows (see Lemma 5.34): The images of the loops α_i on Σ under the embeddings i_M, i_N bound in M and N surfaces D_i^M and D_i^N. We can consider the images of the α_i to be curves α_i^M and α_i^N on the push-offs Σ^M and Σ^N on the boundary of tubular neighbourhoods $\nu\Sigma'_M$ and $\nu\Sigma'_N$. The surfaces D_i^M and D_i^N can be taken disjoint from the interior of the tubular neighbourhoods and can be considered as elements in $H_2(M', \partial M')$ and $H_2(N', \partial N')$. On the boundary of $\nu\Sigma'_M$ and $\nu\Sigma'_N$ we consider $x_M(\alpha_i)k_M$ and $x_N(\alpha_i)k_N$ parallel copies of the fibre σ^M and σ^N which are disjoint from the curves α_i^M and α_i^N. They bound surfaces in M' and N' homologous to $x_M(\alpha_i)B'_M$ and $x_N(\alpha_i)B'_N$. We can connect the disjoint union of these curves on the boundaries of $\nu\Sigma'_M$ and $\nu\Sigma'_N$ by homologies Q_i^M and Q_i^N to connected curves c_i^M and c_i^N on the boundary of tubular neighbourhoods $\nu\Sigma_M$ and $\nu\Sigma_N$ of slightly smaller radius where we think the gluing of M' and N' via ϕ to take place. We can achieve that $\phi \circ c_i^M = c_i^N$. Then the surfaces

$$S_i^M = D_i^M \cup x_M(\alpha_i)B'_M \cup Q_i^M$$
$$S_i^N = D_i^N \cup x_N(\alpha_i)B'_N \cup Q_i^N$$

sew together to give the split classes S_i in X.

We have to choose the orientations carefully to get oriented surfaces B_X and S_i: The surfaces Σ_M and Σ_N are oriented by the embeddings i_M, i_N from a fixed oriented surface Σ. The surfaces B_M and B_N are oriented such that $\Sigma_M B_M = +k_M$ and $\Sigma_N B_N = +k_N$. The extension Φ of the gluing diffeomorphism ϕ (see equation (5.3)) inverts on D the inside-outside direction and the direction along the boundary ∂D. Hence with the orientation induced from B_M and B_N, the punctured surfaces representing $\frac{k_N}{n_{MN}}B'_M$ and $\frac{k_M}{n_{MN}}B'_N$ sew together to give an oriented surface B_X in X.

We orient the surfaces S_i^M and S_i^N in the following way: The curves c_i^M and c_i^N are oriented so that they represent the classes $\alpha_i^M + x_M(\alpha_i)k_M\sigma^M$ and $\alpha_i^N + x_N(\alpha_i)k_N\sigma^N$. The surfaces S_i^M and S_i^N are in a collar $\Sigma \times S^1 \times I$ of $\partial M'$ and $\partial N'$ of the form $c_i^M \times I$ and $c_i^N \times I$. We can choose the surfaces S_i^M and S_i^N connected. We define the orientation on S_i^M to be induced from the orientation of c_i^M followed by the orientation of I pointing into M'. Exactly in the same way the orientation of S_i^N is induced from the orientation of c_i^N followed by the orientation of I pointing into N'.

In this case the orientation of I is inverted by Φ but $\phi_* c_i^M = c_i^N$. This implies that the surface S_i^M with its given orientation and the surface S_i^N with the *opposite* orientation sew together to give an oriented surface S_i in X.

Lemma 5.38. *With this choice of orientations we have*

$$B_X \cdot \Sigma_X = (k_M k_N)/n_{MN}$$
$$S_i \cdot \Sigma_X = x_M(\alpha_i)k_M = \langle C, \alpha_i \rangle - x_N(\alpha_i)k_N.$$

Proof. We can calculate the intersection numbers either on the M side or the N side and check that the results are the same. Note that by Lemma 5.23

$$\Sigma_X = \Sigma'_X - R_C.$$

Since $B_M \cdot \Sigma_M = k_M$ we get on the M side

$$B_X \cdot \Sigma_X = (k_N/n_{MN})B_M \cdot \Sigma_M = (k_N k_M)/n_{MN}.$$

On the N side we have

$$B_X \cdot \Sigma_X = (k_M/n_{MN}) B_N \cdot (\Sigma_N - \sum_{j=1}^{d} a_j \Gamma_j^N) = (k_M k_N)/n_{MN},$$

since we can assume that the surface B_N is disjoint from the rim tori induced by Γ_j^N in N. Similarly we get for $S_i \cdot \Sigma_X$ on the M side

$$S_i \cdot \Sigma_X = x_M(\alpha_i) B_M \cdot \Sigma_M = x_M(\alpha_i) k_M.$$

On the N side we have with our orientation convention

$$S_i \cdot \Sigma_X = -x_N(\alpha_i) B_N \cdot \Sigma_N - D_i^N \cdot (-\sum_{j=1}^{d} a_j \Gamma_j^N) = -x_N(\alpha_i) k_N + \langle C, \alpha_i \rangle.$$

□

We can also calculate the intersection of certains classes with rim tori: Let R_T^M be a rim torus in M' induced from an element $T \in H^1(\Sigma)$. Then R_T^M is the image of

$$\sum_{j=1}^{2g} \langle T, \gamma_j \rangle \Gamma_j^M$$

under the inclusion of $\partial M'$ in M'. The rim torus R_T^M induces under the inclusion $M' \to X$ a rim torus in X which we denote by R_T. The class $T \in H^1(\Sigma)$ also induces a rim torus R_T^N in N' which is the image of $\sum_{j=1}^{2g} \langle T, \gamma_j \rangle \Gamma_j^N$ in N'. Under the inclusion $N' \to X$ the class R_T^N maps to $-R_T$, cf. Lemma 5.21.

Lemma 5.39. *The rim tori R_T^M and R_T^N do not intersect with Σ_M and Σ_N. They also do not intersect with themselves or other rim tori. We can also assume that they do not intersect with B_M, B_N. Hence*

$$R_T \cdot \Sigma_X = 0, \quad R_T \cdot B_X = 0, \quad R_T \cdot R_{T'} = 0.$$

This follows because the rim tori can be moved away from all of the surfaces mentioned in the lemma. We want to calculate the intersection of rim tori with the split classes S_i: We can assume that R_T^M intersects S_i^M only in D_i^M. Let α_i^M denote the curves on the push-off Σ^M determined by the curves $\alpha_i \in \ker(i_M \oplus i_N)$ above. We expand $\alpha_i^M = \sum_{k=1}^{2g} \alpha_{ik} \gamma_k^M$. Then by Lemma 5.19

$$\begin{aligned} R_T^M \cdot S_i^M &= R_T^M \cdot D_i^M \\ &= \sum_{j=1}^{2g} \langle T, \gamma_j \rangle \alpha_{ij} \\ &= \langle T, \alpha_i \rangle. \end{aligned}$$

Similarly we get

$$R_T^N \cdot S_i^N = \langle T, \alpha_i \rangle.$$

Lemma 5.40. *Let R_T denote the rim torus in X which is the image of R_T^M under the inclusion $M' \to X$. Then $R_T \cdot S_i = \langle T, \alpha_i \rangle$.*

Proof. This can be calculated again on the M side or the N side: On the M side we have

$$R_T \cdot S_i = R_T^M \cdot S_i^M = \langle T, \alpha_i \rangle.$$

On the N side we have

$$R_T \cdot S_i = (-R_T^N) \cdot (-S_i^N) = \langle T, \alpha_i \rangle,$$

because we know that R_T^N induces via the inclusion $N' \to X$ the rim torus $-R_T$ and we have to change the orientation on S_i^N by the argument above. □

We now assume that the divisibilities k_M, k_N are equal to 1 and the cohomologies of M, N and X torsion free. For the following arguments it is useful to choose a basis for $P(M)$ consisting of pairwise transverse surfaces $P_1, \ldots, P_n$ embedded in M. The surfaces $P_1, \ldots, P_n$ can be chosen disjoint from Σ_M. We choose similar surfaces in N which give a basis for $P(N)$.

We simplify the surfaces S_i as follows: We can connect the surface D_i^M to any other closed surface in M in the complement of $\nu\Sigma_M'$ to get a new surface which still bounds the same loop α_i^M. We can consider the surface $D = D_i^M$ to be transverse to the surfaces $P_1, \ldots, P_n$ and disjoint from their intersections. Let δ_j be the algebraic intersection number of the surface D with the surface P_j. We want to add closed surfaces to D to make the intersection numbers δ_j for all $j = 1, \ldots, n$ zero. The new surface D' then does not intersect algebraically the surfaces giving a basis for the free part of $P(M)$.

Let β denote the matrix with entries $\beta_{kj} = P_k \cdot P_j$ for $k, j = 1, \ldots, n$, determined by the intersection form of M. This matrix is invertible over $\mathbb{Z}$ since the restriction of the intersection form to $P(M)$ is unimodular. Hence there exists a unique vector $r \in \mathbb{Z}^n$ such that

$$\sum_{k=1}^{n} r_k \beta_{kj} = -\delta_j.$$

Let $D' = D + \sum_{k=1}^{n} r_k P_k$. Then

$$D' \cdot P_j = \delta_j + \sum_{k=1}^{n} r_k \beta_{kj} = 0.$$

We can also add some copies of Σ_M to $x_M(\alpha_i)B_M$ to get a surface which has zero intersection with B_M. This can be done for each index $i = 1, \ldots, d$ to change the surfaces S_i^M to new surfaces in M (denoted by the same symbol) which still bound c_i^M in $\partial M'$ and do not intersect (algebraically) with the surfaces in $P(M)$ and the surface B_M.

A similar construction can be done for N to get new surfaces S_i^N which do not intersect with surfaces defining a basis for the free part of $P(N)$ and the surface B_N. Since their boundaries get identified under the diffeomorphism ϕ they sew together pairwise to give new split classes S_i in X which form a basis for $S(X)$ together with the class B_X. Thus we have proved:

Lemma 5.41. *There exists a basis $B_X, S_1, \ldots, S_d$ of $S(X)$, where the split surfaces $S_1, \ldots, S_d$ are sewed together from surfaces $S_i^M \in H_2(M', \partial M')$ and $S_i^N \in H_2(N', \partial N')$ which do not intersect algebraically with the surfaces B_M and B_N and the surfaces giving a basis for $P(M)$ and $P(N)$.*

By our assumption $k_M = k_N = 1$ we have $B_X = B_M - B_N$ and we can add suitable multiples of B_X to the elements S_i to get new basis elements of the form $S_i = x_N(\alpha_i)B_N + \alpha_i$ where

$$x_N(\alpha_i) = \langle C, \alpha_i \rangle,$$

The surface S_i is now sewed together from surfaces

$$\begin{aligned} S_i^M &= D_i^M \\ S_i^N &= D_i^N \cup Q_i^N \cup U_i^{N\prime}. \end{aligned}$$

The surface $U_i^{N\prime}$ is a punctured surface constructed from the surface

$$U_i^N = x_N(\alpha_i)B_N \cup -x_N(\alpha_i)B_N^2\Sigma^N$$

by smoothing double points and deleting the part in $\nu\Sigma_N$. This surface represents the class $x_N(\alpha_i)(B_N - B_N^2\Sigma_N)$ in N. We have added $-x_N(\alpha_i)B_N^2$ parallel copies of Σ^N outside of $\nu\Sigma_N$ to make the intersection number of S_i^N with B_N zero.

By the calculation in Lemma 5.38 above, the elements S_i have zero intersection with Σ_X while $B_X\Sigma_X = 1$. Moreover, $B_X^2 = B_M^2 + B_N^2$. Let $S'(X)$ be the subgroup generated by the elements $S_1, \ldots, S_d$ such that $S(X) = \mathbb{Z}B_X \oplus S'(X)$. By our assumption $k_M = k_N = 1$, the sequence (5.21) simplifies to

$$0 \to \mathbb{Z}\Sigma_X \oplus R(X) \to H^2(X) \to S(X) \oplus P(M) \oplus P(N) \to 0.$$

Since $S(X)$ is free abelian, we can lift this group to a direct summand of $H^2(X;\mathbb{Z})$. Since we also assumed that the cohomology of M, N and X is torsion free, the whole sequence splits and we can write

$$H^2(X) = P(M) \oplus P(N) \oplus S(X) \oplus R(X) \oplus \mathbb{Z}\Sigma_X.$$

Different splittings of this form are possible: We can add elements in $R(X) \oplus \mathbb{Z}\Sigma_X$ to the lift of basis elements of $P(M)$ and $P(N)$ to get a new lift. However, we can specify a lift by declaring that the elements in the lifted $P(M)$ and $P(N)$ are orthogonal to the classes in $S(X)$.

Lemma 5.42. *There exists a splitting*

$$H^2(X) = P(M) \oplus P(N) \oplus (S'(X) \oplus R(X)) \oplus (\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X),$$

where the direct sums are all orthogonal, except the two direct sums inside the brackets. In this direct sum, the restriction of the intersection form Q_X to $P(M)$ and $P(N)$ is the intersection form induced from M and N, it vanishes on $R(X)$ and has the structure

$$\begin{pmatrix} B_M^2 + B_N^2 & 1 \\ 1 & 0 \end{pmatrix}$$

on $\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X$.

We now simplify the intersection form on $S'(X) \oplus R(X)$. This will complete the proof of Theorem 5.37. Note that for every non-zero element in $R(X)$ there has to exist an element in $S'(X)$ such that both have non-zero intersection because the intersection form Q_X is non-degenerate.

Lemma 5.43. *The subgroup ker$(i_M \oplus i_N)$ is a direct summand of $H_1(\Sigma)$.*

Proof. Suppose that $\alpha \in$ ker$(i_M \oplus i_N)$ is divisible by an integer $c > 1$ so that $\alpha = c\alpha'$ with $\alpha' \in H_1(\Sigma)$. Then $ci_M\alpha' = 0 = ci_N\alpha'$. Since $H_1(M)$ and $H_1(N)$ are torsion free this implies that $\alpha' \in$ ker$(i_M \oplus i_N)$. Hence ker$(i_M \oplus i_N)$ is a direct summand. $\square$

By this lemma we can complete the basis $\alpha_1, \ldots, \alpha_d$ for ker $(i_M \oplus i_N)$ by elements $\beta_{d+1}, \ldots, \beta_{2g} \in H_1(\Sigma)$ to a basis of $H_1(\Sigma)$. Since the basis elements are indivisible, we can represent them by closed, embedded, oriented, connected curves in Σ. In particular, the surfaces S_i^M and S_i^N can be chosen as embedded surfaces.

Let $\alpha_1^*, \ldots, \alpha_d^*, \beta_{d+1}^*, \ldots, \beta_{2g}^*$ denote the dual basis of $H^1(\Sigma)$ and $R_1, \ldots, R_{2g}$ the corresponding rim tori in $H^2(X)$. Then

$$\begin{aligned} S_i \cdot R_j &= \delta_{ij}, \quad \text{for } 1 \le j \le d \\ S_i \cdot R_j &= 0, \quad \text{for } d+1 \le j \le 2g. \end{aligned}$$

This implies that $R_1, \ldots, R_d$ are a basis of $R(X)$ and $R_{d+1}, \ldots, R_{2g}$ vanish. We simplify the surfaces S_i as follows: Let $r_{ij} = S_i \cdot S_j$ for $i, j = 1, \ldots, d$ denote the intersection matrix for the chosen basis of $S'(X)$. Let

$$S_i' = S_i - \sum_{k>i} r_{ik} R_k.$$

The surfaces S_i' are tubed together from the surfaces S_i and certain rim tori. They can still be considered as split classes sewed together from surfaces in M' and N' bounding the loops α_i and still have intersection $S_i' \cdot R_j = \delta_{ij}$. However, the intersection numbers $S_i' \cdot S_j'$ for $i \neq j$ simplify to (where w.l.o.g. $j > i$)

$$\begin{aligned} S_i' \cdot S_j' &= (S_i - \sum_{k>i} r_{ik} R_k) \cdot (S_j - \sum_{l>j} r_{jl} R_l) \\ &= S_i \cdot S_j - r_{ij} \\ &= 0. \end{aligned}$$

Denote these new split classes again by $S_1, \ldots, S_d$ and the subgroup spanned by them in $S'(X) \oplus R(X)$ again by $S'(X)$. The intersection form on $S'(X) \oplus R(X)$ now has the form as in Theorem 5.37 and completes the proof.

Remark 5.44. Note that we can choose the basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(\Sigma)$ we started with in Section V.1.1 as

$$\begin{aligned} \gamma_i &= \alpha_i, \quad \text{for } 1 \le i \le d \\ \gamma_i &= \beta_i, \quad \text{for } d+1 \le i \le 2g. \end{aligned}$$

This choice does not depend on the choice of C since $\alpha_1, \ldots, \alpha_d$ are merely a basis for ker$(i_M \oplus i_N)$. Then the rim tori $R_1, \ldots, R_d$ are given by the image of the classes $\Gamma_1^M, \ldots, \Gamma_d^M$ under the inclusion $\partial M' \to M' \to X$ and the rim tori determined by $\Gamma_{d+1}^M, \ldots, \Gamma_{2g}^M$ are null-homologous in X. In this basis the rim torus R_C in X is given by

$$R_C = -\sum_{i=1}^{d} a_i R_i.$$

The split classes S_i are sewed together from certain surfaces S_i^M, S_i^N bounding loops c_i^M in $\partial M'$ and c_i^N in $\partial N'$ which represent the classes

$$\begin{gathered} \gamma_i^M \text{ in } \partial M' \\ \gamma_i^N + a_i \sigma^N \text{ in } \partial N', \end{gathered}$$

and get identified under the diffeomorphism ϕ, where $i = 1, \ldots, d$. The surfaces S_i^M and S_i^N are of the form

$$S_i^M = D_i^M$$
$$S_i^N = D_i^N \cup Q_i^N \cup U_i^{N\prime},$$

where $U_i^{N\prime}$ is a punctured surface constructed from a surface U_i^N representing $a_i(B_N - B_N^2\Sigma_N)$ in N.

In particular, under our assumptions $H^2(X;\mathbb{Z})$ does not depend as an abelian group on the choice of C. However, the self-intersection numbers S_i^2 and hence the intersection form Q_X might depend on the choice of C.

Remark 5.45. Under the assumptions in Theorem 5.37 there exists a group monomorphism $H^2(M;\mathbb{Z}) \rightarrow H^2(X;\mathbb{Z})$ given by

$$\Sigma_M \mapsto \Sigma_X$$
$$B_M \mapsto B_X$$
$$Id : P(M) \rightarrow P(M).$$

Here we have used the decomposition of $H^2(M)$ given by equation (5.19). A class $\alpha \in H^2(M)$ maps under this homomorphism to

$$\overline{\alpha} + (\alpha\Sigma_M)B_X + (\alpha B_M - B_M^2(\alpha\Sigma_M))\Sigma_X \in H^2(X)$$

by equation (5.20), where $\overline{\alpha} \in P(M)$. In this way, the free abelian group $H^2(M)$ can be realized as a direct summand of $H^2(X)$. There exists a similar monomorphism $H^2(N;\mathbb{Z}) \rightarrow H^2(X;\mathbb{Z})$ given by

$$\Sigma_N \mapsto \Sigma_X' = \Sigma_X + R_C$$
$$B_N \mapsto B_X$$
$$Id : P(N) \rightarrow P(N).$$

For the first line cf. Lemma 5.23. Hence $H^2(N)$ can also be realized as a direct summand of $H^2(X)$. Note that in general the embeddings do *not* preserve the intersection form, the images of both embeddings have non-trivial intersection and in general do not span $H^2(X)$.

V.4 Applications

The formula in Theorem 5.37 is well-known in the case of a fibre sum of elliptic surfaces, see e.g. [56, Section 3.1]: We begin with the fibre sum of two copies of the elliptic surface $E(1)$ along a regular fibre, giving the $K3$-surface $E(2)$. The elliptic fibration on $M = E(1) \rightarrow S^2$ determines a normal bundle of a regular fibre $\Sigma_M = F$ by taking the preimage of a small disk in S^2. This also determines a canonical push-off given by a nearby fibre and hence a trivialization of the normal bundle. A section of the elliptic fibration is a sphere B_M of self-intersection -1. Since $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ the group $P(M)$ is free abelian of rank 8. In [56] it is shown that the intersection form Q_M restricted to $P(M)$ is isomorphic to $-E_8$.

Take a second copy N of $E(1)$ and a regular fibre Σ_N. Let C be an arbitrary class in $H^1(T^2)$ and ϕ a corresponding gluing diffeomorphism. We form the generalized fibre sum $X(\phi) = E(1)\#_{F=F}E(1)$. In this case the resulting manifold X does not depend up to diffeomorphism on the choice of C since every orientation preserving self-diffeomorphism of $\partial\nu F$ extends over $E(1) \setminus \operatorname{int}\nu F$ [56, Theorem

8.3.11]. Hence we can choose ϕ as the identity. Then ϕ identifies the fibres in the boundary of the normal bundles and we get an elliptic fibration of $X = E(2) = K3$ over S^2.

The spheres B_M and B_N sew together to a sphere B_X in X of self-intersection -2. Since $E(1)$ is simply-connected, Ker $(i_M \oplus i_N) = H^1(T^2;\mathbb{Z})$, hence $d = 2$. This implies that $S(X)$ is a free abelian group of rank 3 and $R(X) \cong H^1(T^2;\mathbb{Z})$ is free abelian of rank 2. Since $E(1)$ admits an elliptic fibration with a cusp fibre, one can show that there exists an identification of the fibre F with $T^2 = S^1 \times S^1$ such that the simple closed loops given by $S^1 \times 1$ and $1 \times S^1$ bound in $E(1) \setminus \operatorname{int} \nu F$ disks D_1 and D_2 of self-intersection -1 ([56]). Take copies of these disks D_1^M, D_2^M and D_1^N, D_2^N in M and N. Since ϕ is the identity, these disks sew together to give split classes S_1 and S_2 in X which are spheres of self-intersection -2.

By Theorem 5.37 we have

$$H^2(E(2);\mathbb{Z}) = -E_8 \oplus -E_8 \oplus \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

The last term is the intersection form on $\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X$. Since

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H$$

as quadratic forms over $\mathbb{Z}$, we get for the intersection form of $K3$ the well-known formula $-2E_8 \oplus 3H$.

This can be extended inductively to the elliptic surfaces $E(n) = E(1)\#_{F=F}E(n-1)$. For $E(3)$ we have

$$H^2(E(3);\mathbb{Z}) = P(E(1)) \oplus P(E(2)) \oplus \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}.$$

The fibre sum has been done along the fibre Σ_X in $X = E(2)$ and used the surface B_X constructed above which sews together with the section of $E(1)$ to give a sphere in $E(3)$ of self-intersection -3. This accounts for the last summand. We have again two split classes S_1 and S_2 represented by spheres of self-intersection -2. We can read off $P(E(2))$ from the calculation above and get

$$H^2(E(3);\mathbb{Z}) = -3E_8 \oplus 4 \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix} \cong (+1) \oplus (-1)$$

as integral quadratic forms, the intersection form of $E(3)$ is isomorphic to $5(+1) \oplus 29(-1)$. For $E(4)$ we get

$$H^2(E(4);\mathbb{Z}) = P(E(1)) \oplus P(E(3)) \oplus \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $P(E(3))$ is isomorphic to $-3E_8 \oplus 4H$ we see that the intersection form of $E(4)$ is isomorphic to $-4E_8 \oplus 7H$, and so on.

V.4.1 Knot surgery

The following construction is due to Fintushel and Stern [38]. Let K be a knot in S^3. Denote a tubular neighbourhood of K by $\nu K \cong S^1 \times D^2$. Let m be a fibre of the circle bundle $\partial\nu K \rightarrow K$ and use an oriented Seifert surface for K to define a section $l\colon K \rightarrow \partial\nu K$. The circles m and l are called the *meridian* and the *longitude* of K. Let M_K be the closed 3-manifold obtained by 0-Dehn surgery on K. M_K is constructed in the following way: Consider $S^3 \setminus \operatorname{int}\nu(K)$ and let

$$f\colon \partial(S^1 \times D^2) \rightarrow \partial(S^3 \setminus \operatorname{int}\nu(K))$$

be a diffeomorphism which maps in homology the circle ∂D^2 onto l. Then one defines

$$M_K = (S^3 \setminus \operatorname{int}\nu(K)) \cup_f (S^1 \times D^2).$$

The manifold M_K is determined by this construction uniquely up to diffeomorphism. One can show that it has the same integral homology as $S^2 \times S^1$. The meridian m, which bounds the fibre in the normal bundle to K in S^3, becomes non-zero in the homology of M_K and defines a generator in $H_1(M_K;\mathbb{Z})$. The longitude l is null-homotopic in M_K since it bounds one of the D^2-fibres glued in. This copy of D^2 determines together with the Seifert surface of K a closed, oriented surface in M_K which intersects m once and generates $H_2(M_K;\mathbb{Z})$.

Consider the closed, oriented 4-manifold $M_K \times S^1$. It contains a torus $T_M = m \times S^1$ of self-intersection 0. Let X be an arbitrary closed, oriented 4-manifold, which contains an embedded torus T_X of self-intersection 0 representing an indivisible homology class. Then the result of *knot surgery* on X is given by the generalized fibre sum

$$X_K = X\#_{T_X=T_M}(M_K \times S^1).$$

The 4-manifold X_K may depend on the choice of gluing diffeomorphism, which is not specified. The 4-manifold M_K has the same integral homology as $S^2 \times T^2$. The surface constructed from the Seifert surface for K intersects T_M precisely once. We can use this surface as B_M. We also choose a class B_X intersecting T_X once. The embedding i_M of the torus T_M in $M_K \times S^1$ is an isomorphism on first homology and we can write

$$\begin{aligned} i_M \oplus i_X\colon \mathbb{Z}^2 &\rightarrow \mathbb{Z}^2 \oplus H_1(X;\mathbb{Z}) \\ a &\mapsto (a, i_X a). \end{aligned}$$

In particular, the map

$$\begin{aligned} \mathbb{Z}^2 \oplus H_1(X;\mathbb{Z}) &\rightarrow H_1(X;\mathbb{Z}) \\ (x,y) &\mapsto y - i_X x, \end{aligned}$$

determines an isomorphism between $H_1(X_K;\mathbb{Z}) = \mathrm{Coker}(i_M \oplus i_X)$ and $H_1(X;\mathbb{Z})$. Moreover, $\ker(i_M \oplus i_X) = 0$ and the group of split classes $S(X_K) \cong \mathbb{Z}$ is generated by $B_{X_K} = B_M - B_X$. Since i_M^* is an isomorphism, there are no non-zero rim tori in X_K. The group $P(M_K \times S^1)$ is also zero and we get a short exact sequence

$$0 \rightarrow \mathbb{Z}T_{X_K} \rightarrow H^2(X_K;\mathbb{Z}) \rightarrow \mathbb{Z}B_{X_K} \oplus P(X) \rightarrow 0.$$

Since $B_{X_K} \cdot T_{X_K} = 1$, the classes T_{X_K} and B_{X_K} define indivisible elements in $H^2(X_K)$ and the sequence splits, so we can write

$$H^2(X_K;\mathbb{Z}) \cong \mathbb{Z}T_{X_K} \oplus \mathbb{Z}B_{X_K} \oplus P(X). \tag{5.22}$$

(Note that we do not have to assume that the cohomology of X is torsion free as in Theorem 5.37.) There is a similar splitting

$$H^2(X;\mathbb{Z}) \cong \mathbb{Z}T_X \oplus \mathbb{Z}B_X \oplus P(X).$$

Hence we can define an isomorphism

$$H^2(X;\mathbb{Z}) \cong H^2(X_K;\mathbb{Z}) \tag{5.23}$$

of abelian groups, given by

$$\begin{aligned} T_X &\mapsto T_{X_K} \\ B_X &\mapsto B_{X_K} \\ Id : P(X) &\to P(X), \end{aligned} \tag{5.24}$$

cf. Remark 5.45. The class T_{X_K} has zero intersection with the classes in $P(X)$ since they can be moved away from the boundary. The class B_{X_K} also has zero intersection with the elements in $P(X)$ since this holds for B_X. The self-intersection number of B_{X_K} is equal to the self-intersection number of B_X, because the class B_M has zero self-intersection (it can be moved away in the S^1 direction). Hence the isomorphism $H^2(X_K;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$ also holds on the level of intersection forms.

Assume in addition that X and $X' = X \setminus T_X$ are simply-connected. Then X_K is again simply-connected and by Freedman's theorem [45], X and X_K are homeomorphic. However, one can show with Seiberg-Witten theory that X and X_K are in many cases not diffeomorphic [38].

Suppose that K is a *fibred knot*, i.e. there exists a fibration

$$\begin{array}{ccc} S^3 \setminus \mathrm{int}\,\nu(K) & \longleftarrow & \Sigma_h' \\ \big\downarrow & & \\ S^1 & & \end{array}$$

over the circle, where Σ_h' are punctured surfaces of genus h, forming Seifert surfaces for K. Then M_K is fibred by closed surfaces B_M of genus h. This induces a fibre bundle

$$\begin{array}{ccc} M_K \times S^1 & \longleftarrow & \Sigma_h \\ \big\downarrow & & \\ T^2 & & \end{array}$$

and the torus $T_M = m \times S^1$ is a section of this bundle. By a theorem of Thurston [137], $M_K \times S^1$ admits a symplectic form such that T_M and the fibres are symplectic. This construction can be used to do symplectic generalized fibre sums along T_M, cf. Section V.5. The canonical class of $M_K \times S^1$ can be calculated by the adjunction inequality, because the fibres B_M and the torus T_M are symplectic surfaces and form a basis of $H_2(M_K \times S^1;\mathbb{Z})$. We get:

$$K_{M_K \times S^1} = (2h-2)T_M. \tag{5.25}$$

V.4.2 Lefschetz fibrations

For the following discussion see [1], [4], [56, Chapter 8] and [84]. Let (M,ω) be a closed, symplectic 4-manifold. For every point $p\in M$ we can choose smooth coordinate charts

$$\psi=(z_1,z_2)\colon U\rightarrow\mathbb{C}^2\cong\mathbb{R}^4,$$

defined on an open neighbourhood $U\subset M$ of p such that $\psi(p)=0$. We call a coordinate chart of this kind *adapted* to the symplectic structure if the complex lines in the local coordinates are symplectic with respect to ω.

Definition 5.46. A symplectic Lefschetz pencil on (M,ω) consists of the following data:

(1.) A non-empty set of points $B\subset M$, called the set of *base points*.

(2.) A smooth, surjective map $\pi\colon M\setminus B\rightarrow\mathbb{C}P^1$.

(3.) A finite set of points $\Delta\subset M\setminus B$, called the set of *critical points*, away from which the map π is a submersion.

In addition, the data have to satisfy the following local models:

(1.) For every point $p\in B$ there exists an adapted chart (z_1,z_2) such that $\pi(z_1,z_2)=z_2/z_1$.

(2.) For every point $p\in\Delta$ there exists an adapted chart (z_1,z_2) in which $\pi(z_1,z_2)=z_1^2+z_2^2+c$ for some constant $c\in\mathbb{C}P^1$.

For $x\in\mathbb{C}P^1$ the *fibre* F_x of the pencil is defined as $\pi^{-1}(x)\cup B\subset M$. Let $n=|B|$ denote the number of base points. The local model around the base points implies that one can blow up the set B to get a symplectic 4-manifold $N=M\#n\overline{\mathbb{C}P^2}$ and a smooth, surjective map

$$\pi_N\colon N\rightarrow\mathbb{C}P^1,$$

which is a submersion away from the set of critical points $\Delta\subset N$ and still has the local form $\pi_N(z_1,z_2)=z_1^2+z_2^2+c$ at every $p\in\Delta$. In particular, $\pi_N\colon N\rightarrow\mathbb{C}P^1$ is a singular fibration with symplectic fibres Σ_x which are the proper transforms of F_x for every $x\in\mathbb{C}P^1$. The fibration $N\rightarrow\mathbb{C}P^1$ is called a *symplectic Lefschetz fibration*. By a perturbation one can assume that each fibre contains at most one critical point.

The classical construction of these fibrations for complex algebraic surfaces, due to Lefschetz, is as follows: Let $M\subset\mathbb{C}P^D$ be an algebraic surface, embedded in some projective space of dimension D. Let $A\cong\mathbb{C}P^{D-2}$ be a generic linear subspace of $\mathbb{C}P^D$ of codimension 2 which intersects M in a number of points B. Consider the set of all hyperplanes $H_x\cong\mathbb{C}P^{D-1}$ of $\mathbb{C}P^D$ which contain A. This set is called a *pencil* and is parametrized by $x\in\mathbb{C}P^1$. Every point in $M\setminus B$ is contained in a unique hyperplane H_x. This defines a holomorphic map $\pi\colon M\setminus B\rightarrow\mathbb{C}P^1$. One can show that π satisfies the local model of a symplectic Lefschetz pencil as above with fibres $F_x=\pi^{-1}(x)\cup B$ given by the hyperplane sections $M\cap H_x$.

The hyperplane sections $M\cap H_x$ intersect pairwise precisely in B. They are all homologous and have self-intersection n, where $n=|B|$. The proper transforms Σ_x in $N=M\#n\overline{\mathbb{C}P^2}$ are complex curves of genus g (hence symplectic surfaces with respect to the Kähler form) of self-intersection 0, all but finitely many of which are smooth.

By the Lefschetz Hyperplane theorem, the homomorphism

$$i_N\colon H_1(\Sigma_N;\mathbb{Z})\rightarrow H_1(N;\mathbb{Z}),$$

induced by inclusion is a surjection and the kernel is generated by the set of *vanishing cycles*. The vanishing cycles bound disks in N, called *Lefschetz thimbles* or *vanishing disks*, which intersect Σ_N only in the vanishing cycle and contain precisely one critical point $p \in \Delta$. For each critical point there is a corresponding vanishing cycle and a thimble. One can construct the thimbles in such a way that they are Lagrangian disks [4]: By our assumption on Lefschetz fibrations, a singular fibre contains only one critical point. Let x denote the parameter index in $\mathbb{C}P^1$ of the smooth fibre Σ_N and let x_1 be the parameter index of a singular fibre Σ_{x_1}. Connect x and x_1 by a path γ in $\mathbb{C}P^1$, which avoids all other critical values. The symplectic Kähler form induces a natural horizontal distribution on $N \setminus \Delta$, given by the symplectic complement to the tangent space along the fibres. The parallel transport of a vanishing cycle in Σ_N along the curve γ then converges to the critical point in the fibre above x_1 and defines the Lagrangian vanishing disk.

We can assume that all critical values of π_N are contained in a small neighbourhood of x in $\mathbb{C}P^1$. This implies that we can assume that the Lefschetz thimbles are disjoint from the surfaces representing the classes in $P(N)$, which can be moved away from Σ_N. Similarly, by using a homotopy, we can assume that the point where the exceptional sphere B_N intersects the fibre Σ_N does not lie on any of the vanishing cycles. This implies that the thimbles can be made disjoint from B_N as well.

Suppose that N has torsion free cohomology and consider the generalized fibre sum

$$X = N\#_{\Sigma_N=\Sigma_N}N.$$

The Lefschetz fibration $N \to \mathbb{C}P^1$ defines a natural tubular neighbourhood of Σ_N with a canonical trivialization given by a push-off into a nearby fibre. If we take the gluing diffeomorphism ϕ which is the identity with respect to this trivialization, it follows that X also admits a Lefschetz fibration in genus g curves over $\mathbb{C}P^1$. Suppose that the group of vanishing cycles in $H_1(\Sigma_N;\mathbb{Z})$ has rank d and choose a basis $\delta_1,\ldots,\delta_d$. Then the corresponding Lefschetz thimbles for both copies of N sew together to give basis elements $S_1,\ldots,S_d$ for the group of split classes, represented by 2-spheres of self-intersection -2. Since the thimbles are Lagrangian disks, we can assume that these two spheres are Lagrangian if the fibre sum is done symplectically as in Section V.5. Two copies of the exceptional sphere B_N give a (symplectic) sphere B_X of self-intersection -2 in X. The second cohomology of X can be calculated by Theorem 5.37:

$$H^2(X;\mathbb{Z}) = 2P(N) \oplus (d+1)\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

This generalizes the formula for the fibre sum $E(1)\#_{F=F}E(1) = E(2)$ above. We can also add further copies of N, cf. Section VI.2.4.

V.5 A formula for the canonical class

In this section we recall the definition of the symplectic generalized fibre sum by the construction of Gompf [52]. Let (M,ω_M) and (N,ω_N) be closed, symplectic 4-manifolds and Σ_M,Σ_N embedded symplectic surfaces of genus g. Denote the symplectic generalized fibre sum by $X = M\#_{\Sigma_M=\Sigma_N}N$. We want to determine a formula for the canonical class K_X in terms of M and N.

The symplectic generalized fibre sum is constructed using the following lemma. Recall that we have a fixed trivialization of tubular neighbourhoods $\nu\Sigma_M$ and $\nu\Sigma_N$ by τ_M and τ_N. Hence we can identify them with $\Sigma \times D$, where D denotes the open disk of radius 1 in $\mathbb{R}^2$.

Lemma 5.47. *The symplectic structures ω_M and ω_N can be deformed by rescaling and isotopies such that both restrict on the tubular neighbourhoods $\nu\Sigma_M$ and $\nu\Sigma_N$ to the same symplectic form*

$$\omega = \omega_\Sigma + \omega_D,$$

where ω_D is the standard symplectic structure $\omega_D = dx \wedge dy$ on the open unit disk D and ω_Σ is a symplectic form on Σ.

Proof. We follow the proof in [52]. Choose an arbitrary symplectic form ω on Σ and rescale ω_M and ω_N such that

$$\int_{\Sigma_M} \omega_M = \int_{\Sigma_N} \omega_M = \int_\Sigma \omega.$$

We can then isotop the embeddings $i_M\colon \Sigma \to M$ and $i_N\colon \Sigma \to N$ without changing the images, such that both become symplectomorphisms onto Σ_M and Σ_N. The isotopies can be realized by taking fixed embeddings i_M, i_N and composing them with isotopies of self-diffeomorphisms of M and N (because M and N are closed manifolds). Hence we can consider the embeddings to be fixed and instead change the symplectic forms ω_M and ω_N by pulling them back under isotopies of self-diffeomorphisms.

The embeddings $\tau_M\colon \Sigma \times D \to M$ and $\tau_N\colon \Sigma \times D \to N$ are symplectic on the submanifold $\Sigma \times 0$. We can isotop both embeddings to new embeddings which are symplectic on small neighbourhoods of $\Sigma \times 0$ with respect to the symplectic form $\omega + \omega_D$ on $\Sigma \times D$. Since Σ is compact, we can assume that both are symplectic on $\Sigma \times D_\epsilon$ where D_ϵ denotes the disk with radius $\epsilon < 1$. Again the isotopies can be achieved by considering τ_M and τ_N unchanged and pulling back the symplectic forms on M and N under isotopies of self-diffeomorphisms.

It is easier to work with disks of radius 1: We rescale the symplectic forms ω_M, ω_N and $\omega + \omega_D$ by the factor $1/\epsilon^2$. Then we compose the symplectic embeddings τ_M and τ_N on $(\Sigma \times D_\epsilon, (1/\epsilon^2)(\omega + \omega_D))$ with the symplectomorphism

$$\begin{aligned} \Sigma \times D &\to \Sigma \times D_\epsilon \\ (p, (x, y)) &\mapsto (p, (\epsilon x, \epsilon y)), \end{aligned}$$

where $\Sigma \times D$ has the symplectic form $(1/\epsilon^2)\omega + \omega_D$. We then define $\omega_\Sigma = (1/\epsilon^2)\omega$ to get the statement we want to prove. $\square$

It is useful to choose polar coordinates (r, θ) on D such that

$$\begin{aligned} dx &= dr \cos\theta - r \sin\theta d\theta \\ dy &= dr \sin\theta + r \cos\theta d\theta. \end{aligned}$$

Then $\omega_D = r dr \wedge d\theta$. The manifolds $M \setminus \Sigma_M$ and $N \setminus \Sigma_N$ are glued together along $\operatorname{int} \nu\Sigma_M \setminus \Sigma_M$ and $\operatorname{int} \nu\Sigma_N \setminus \Sigma_N$ by the orientation preserving and fibre preserving diffeomorphism

$$\begin{aligned} \Phi\colon (D \setminus \{0\}) \times \Sigma &\to (D \setminus \{0\}) \times \Sigma \\ (r, \theta, x) &\mapsto (\sqrt{1 - r^2}, C(x) - \theta, x). \end{aligned} \tag{5.26}$$

The action of Φ on the 1-forms dr and $d\theta$ is given by

$$\begin{aligned} \Phi^* dr &= d(r \circ \Phi) = d\sqrt{1 - r^2} = \frac{-r}{\sqrt{1 - r^2}} dr \\ \Phi^* d\theta &= d(\theta \circ \Phi) = dC - d\theta. \end{aligned}$$

This implies that $\Phi^*\omega_D = \omega_D - r dr \wedge dC$. We can think of the gluing of M' and N' along their boundaries to take place along $S \times \Sigma$, where S denotes the circle of radius $\frac{1}{\sqrt{2}}$. Let Ann denote the annulus in D between radius $1/\sqrt{2}$ and 1. On the N side we take the standard symplectic structure ω_D on $Ann \times \Sigma$ which extends over the rest of N. On the boundary $\partial N'$ given by $S \times \Sigma$ this form pulls back to the form $\Phi^*\omega_D = \omega_D - rdr \wedge dC$ on $\partial M'$. The S^1-valued function C has the same differential as a certain function $f\colon \Sigma \to \mathbb{R}$. Let ρ be a smooth cut-off function on Ann which is identical to 1 near $r = 1/\sqrt{2}$, identical to 0 near $r = 1$ and depends only on the radius r. Consider the following closed 2-form on $\Sigma \times Ann$:

$$\begin{aligned}\omega_D - rdr \wedge d(\rho f) &= \omega_D - rdr \wedge (fd\rho + \rho dC)\\ &= \omega_D - \rho rdr \wedge dC.\end{aligned}$$

Since this form is non-degenerate at every point over the annulus it follows that we can deform the symplectic structure at radius $1/\sqrt{2}$ through a symplectic structure on $Ann \times \Sigma$ on the M side such that it coincides with the standard form ω_D at $r = 1$. From here it can be extended over the rest of M. In this way we define a symplectic structure ω_X on X.

Remark 5.48. Note that the Gompf construction for the symplectic generalized fibre sum can only be done if (after a rescaling) the symplectic structures ω_M and ω_N have the same volume on Σ_M and Σ_N:

$$\int_{\Sigma_M} \omega_M = \int_{\Sigma_N} \omega_N.$$

To calculate this number both Σ_M and Σ_N have to be oriented, which we have assumed *a priori*. It is not necessary that this number is positive, the construction also works with negative volume. In the first case the orientation induced by the symplectic forms coincides with the given orientation on Σ_M and Σ_N and is the opposite orientation in the second case.

We will also need compatible almost complex structures: We choose the standard almost complex structure J_D on D which maps $dx \circ J_D = -dy$ and $dy \circ J_D = dx$. In polar coordinates

$$\begin{aligned}dr \circ J_D &= -rd\theta\\ rd\theta \circ J_D &= dr.\end{aligned}$$

We also choose a compatible almost complex structure on Σ. The almost complex structure $J_D + J_\Sigma$ extends to compatible almost complex structures on M and N.

Recall that the smooth sections of the canonical bundle K_M are 2-forms on M which are "holomorphic", i.e. complex linear. We choose the holomorphic 1-form $\Omega_D = dx + idy$ on D, which is in polar coordinates

$$\Omega_D = (dr + ird\theta)e^{i\theta}. \tag{5.27}$$

This form satisfies $\Omega_D \circ J_D = i\Omega_D$. We also choose a holomorphic 1-form Ω_Σ on Σ. We can choose this form such that it has precisely $2g - 2$ different zeroes $p_1, \ldots, p_{2g-2}$ of index $+1$. We can assume that all zeroes are contained in a small disk D_Σ around a point q disjoint from the zeroes. The form $\Omega_D \wedge \Omega_\Sigma$ is then a holomorphic 2-form on $D \times \Sigma$ which has transverse zero set consisting of $2g - 2$ parallel copies of D. This 2-form can be extended to holomorphic 2-forms on M and N as sections of the canonical bundles.

Note that J_D and Ω_D are not invariant under Φ, even if $C = 0$: On $S \times \Sigma$ we have

$$\begin{aligned}\Phi^* dr &= -dr\\ \Phi^* d\theta &= dC - d\theta.\end{aligned}$$

Hence

$$\begin{aligned}\Phi^*\Omega_D &= -(dr + ir(d\theta - dC))e^{-i\theta + iC}\\ dr \circ \Phi^* J_D &= -r(d\theta - dC)\\ r(d\theta - dC)\Phi^* J_D &= dr,\end{aligned}$$

at $r = 1/\sqrt{2}$. By a similar argument as above, we can deform $\Phi^* J_D$ through an almost complex structure on $Ann \times \Sigma$ on the M side such that it coincides with the standard J_D at $r = 1$. We can do this in such a way that the almost complex structure is compatible with the symplectic structure on $Ann \times \Sigma$ above. We can also deform $\Phi^*\Omega_D$ on $Ann \times \Sigma$ through a nowhere vanishing 1-form which is holomorphic for this almost complex structure such that it becomes at $r = 1$ equal to

$$\Phi^*\Omega_D = -(dr + ird\theta)e^{-i\theta + iC}.$$

Then

$$\Phi^*(\Omega_D \wedge \Omega_\Sigma) = -\Omega_D e^{-2i\theta} \wedge \Omega_\Sigma e^{iC}.$$

We now construct a section Ω_X of K_X in the following way: Choose a holomorphic 2-form on the tubular neighbourhood $\nu\Sigma_M$ of radius 1 of the form $\Omega_D^M \wedge \Omega_\Sigma$ where

$$\Omega_D^M = (dr + ird\theta)e^{i\theta},$$

as in equation (5.27). Also choose a holomorphic 2-form Ω_N on the normal bundle $\nu\Sigma_N$ of radius 1 of the form $\Omega_D^N \wedge \Omega_\Sigma$ where

$$\Omega_D^N = -(dr + ird\theta)e^{i\theta}.$$

We think of M' and N' as being glued together along $S \times \Sigma$ where S is the circle of radius $1/\sqrt{2}$. On the N side we have on $S \times \Sigma$ the holomorphic 2-form

$$-\Omega_D^N \wedge \Omega_\Sigma.$$

It pulls back under Φ to a holomorphic 2-form on $S \times \Sigma$ on the M side. By the argument above it can be deformed on Ann to the holomorphic 2-form

$$\Omega_D^M e^{-2i\theta} \wedge \Omega_\Sigma e^{iC}$$

at $r = 1$. The almost complex structure coming from N under Φ can be deformed similarly such that it becomes the standard J_D at $r = 1$. Let Ann' denote the annulus between radius 1 and 2. We now want to change the form $\Omega_D^M e^{-2i\theta} \wedge \Omega_\Sigma e^{iC}$ over $Ann' \times \Sigma$ through holomorphic 2-forms to the form $\Omega_D^M \wedge \Omega_\Sigma$ at $r = 2$. We will always extend the almost complex structure by the standard one if we extend over annuli.

The change will be done by changing the function $e^{-2i\theta+iC}$ at $r = 1$ over $Ann' \times \Sigma$ to the constant function with value 1 at $r = 2$. This is not possible if we consider the functions as having image in S^1, because they represent different cohomology classes on $S^1 \times \Sigma$. Hence we consider $S^1 \subset \mathbb{C}$ and the change will involve crossings of zero. We choose a smooth function $f\colon Ann' \times \Sigma \to \mathbb{C}$ which is transverse to 0 and satisfies $f_1 = e^{-2i\theta+iC}$ and $f_2 \equiv 1$. The Poincaré dual of the zero set of f is then the cohomology class of $S^1 \times \Sigma$ determined by the S^1-valued function $e^{2i\theta - iC}$.

Let $\gamma_1^{M*}, \ldots, \gamma_{2g}^{M*}, \sigma^{M*}$ be a basis of $H^1(S^1 \times \Sigma_M; \mathbb{Z})$ as in Section V.1.2. Then the cohomology class determined by $e^{2i\theta - iC}$ is equal to $-\sum_{i=1}^{2g} a_i \gamma_i^{M*} + 2\sigma^{M*}$. The Poincaré dual of this class is

$$-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M.$$

Proposition 5.49. *There exists a 2-form Ω' on $Ann' \times \Sigma_M$ which is holomorphic for $J_D + J_\Sigma$ and satisfies:*

- $\Omega' = \Omega_D^M e^{-2i\theta} \wedge \Omega_\Sigma e^{iC}$ *at* $r = 1$ *and* $\Omega' = \Omega_D^M \wedge \Omega_\Sigma$ *at* $r = 2$.

- *The zeroes of the form Ω' are all transverse and the zero set represents the class $-\sum_{i=1}^{2g} a_i\Gamma_i^M + 2\Sigma^M$ in the interior of $Ann' \times \Sigma_M$ and $2g-2$ parallel copies of Ann'.*

The appearance of the zero set $-\sum_{i=1}^{2g} a_i\Gamma_i^M + 2\Sigma^M$ can be seen as the obstruction to extending the S^1-valued function on the boundary of $Ann' \times \Sigma$ given by f_1 at $r=1$ and f_2 at $r=2$ into the interior (cf. Section VIII.3). Under inclusion in X, this class becomes $R_C + 2\Sigma_X$, cf. Definition 5.22. We get the following corollary:

Corollary 5.50. *There exists a symplectic form ω_X with compatible almost complex structure J_X and holomorphic 2-form Ω_X on X such that:*

- *On the boundary $\partial\nu\Sigma_N$ of the tubular neighbourhood of Σ_N in N of radius 2 the symplectic form and the almost complex structure are $\omega_X = \omega_D + \omega_\Sigma$ and $J_X = J_D + J_\Sigma$ while $\Omega_X = -\Omega_D \wedge \Omega_\Sigma$.*
- *On the boundary $\partial\nu\Sigma_M$ of the tubular neighbourhood of Σ_M in M of radius 2 the symplectic form and the almost complex structure are $\omega_X = \omega_D + \omega_\Sigma$ and $J_X = J_D + J_\Sigma$ while $\Omega_X = \Omega_D \wedge \Omega_\Sigma$.*
- *On the subset of $\nu\Sigma_N$ between radius $1/\sqrt{2}$ and 2, which is an annulus times Σ_N, the zero set of Ω_X consists of $2g-2$ parallel copies of the annulus.*
- *On the subset of $\nu\Sigma_M$ between radius $1/\sqrt{2}$ and 2, which is an annulus times Σ_M, the zero set of Ω_X consists of $2g-2$ parallel copies of the annulus and a surface in the interior representing $-\sum_{i=1}^{2g} a_i\Gamma_i^M + 2\Sigma^M$.*

We now assume that $k_M = k_N = 1$ and that the cohomology groups of M, N and X are torsion free, so that we can use Theorem 5.37. Split the canonical class K_X as

$$K_X = p_M + p_N + \sum_{i=1}^{d} s_i S_i + \sum_{i=1}^{d} r_i R_i + b_X B_X + \sigma_X \Sigma_X,$$

where $p_M \in P(M)$ and $p_N \in P(N)$.[2] The coefficients can be determined by using intersections:

$$\begin{aligned} K_X \cdot S_j &= s_j S_j^2 + r_j \\ K_X \cdot R_j &= s_j \\ K_X \cdot B_X &= b_X(B_M^2 + B_N^2) + \sigma_X \\ K_X \cdot \Sigma_X &= b_X. \end{aligned}$$

Similarly, the coefficients p_M and p_N can be determined by intersecting K_X with classes in $P(M)$ and $P(N)$. We assume that Σ_M and Σ_N are oriented by the symplectic forms ω_M and ω_N. Then Σ_X is a symplectic surface in X of genus g and self-intersection 0, oriented by the symplectic form ω_X. This implies by the adjunction formula

$$b_X = K_X \cdot \Sigma_X = 2g - 2,$$

hence

$$\sigma_X = K_X \cdot B_X - (2g-2)(B_M^2 + B_N^2).$$

[2]In the proof of [39, Theorem 3.2.] a similar formula is used to compute the SW-basic classes for a certain generalized fibre sum.

Similarly, note that every rim torus R_j is of the form $c_j \times \sigma^M$ in $\partial M' \subset X$ for some closed oriented curve c_j on Σ_M. By writing c_j as a linear combination of closed curves on Σ_M without self-intersections and placing the corresponding rim tori into different layers $\Sigma_M \times S^1 \times t_i$ in a collar $\Sigma_M \times S^1 \times I$ of $\partial M'$, we see that R_j is a linear combination of embedded Lagrangian tori of self-intersection 0 in X. Since the adjunction formula holds for each one of them,

$$s_j = 0, \quad \text{for all } j = 1, \ldots, d$$

hence also

$$r_j = K_X \cdot S_j.$$

It remains to determine $p_M, p_N, K_X \cdot B_X$ and $K_X \cdot S_j$. To determine p_M note that $\eta_M^* K_X = K_{M'} = \rho_M^* K_M$. This implies that the intersection of a class in $P(M)$ with K_X is equal to its intersection with K_M. Recall that we have by equation (5.19) a decomposition

$$H^2(M;\mathbb{Z}) = P(M) \oplus \mathbb{Z}\Sigma_M \oplus \mathbb{Z}B_M. \tag{5.28}$$

By our choice of orientation for Σ_M, the adjunction formula holds and we have $K_M \Sigma_M = 2g-2$. By equation 5.20 we can decompose K_M in the direct sum (5.28) as

$$K_M = \overline{K_M} + (K_M B_M - (2g-2)B_M^2)\Sigma_M + (2g-2)B_M, \tag{5.29}$$

where we have set

$$\overline{K_M} = K_M - (2g-2)B_M - (K_M B_M - (2g-2)B_M^2)\Sigma_M \in P(M).$$

It is then clear that

$$p_M = \overline{K_M}.$$

Similarly,

$$K_N = \overline{K_N} + (K_N B_N - (2g-2)B_N^2)\Sigma_N + (2g-2)B_N$$

with

$$\overline{K_N} = K_N - (2g-2)B_N - (K_N B_N - (2g-2)B_N^2)\Sigma_N \in P(N)$$

and we have

$$p_N = \overline{K_N}.$$

We now calculate $K_X \cdot B_X$. Our choice of orientation for Σ_M and Σ_N and the fact that $\Sigma_M B_M = +1 = \Sigma_N B_N$ determines an orientation of B_M and B_N and hence an orientation for B_X.

Lemma 5.51. *With this choice of orientation, we have $K_X B_X = K_M B_M + K_N B_N + 2$.*

Proof. We extend the holomorphic 2-form $\Omega_D \wedge \Omega_\Sigma$ on the boundary $\partial\nu\Sigma_M$ of the tubular neighbourhood of Σ_M in M of radius 2 to the holomorphic 2-form on $\nu\Sigma_M$ given by the same formula and then to a holomorphic 2-form on $M \setminus \nu\Sigma_M$. The zero set of the resulting holomorphic 2-form Ω_M restricted to $\nu\Sigma_M = D_M \times \Sigma_M$ consists of $2g-2$ parallel copies of D_M. We can choose the surface B_M such that it is parallel but disjoint from these copies of D_M inside $\nu\Sigma_M$ and intersects the zero set of Ω_M outside transverse. The zero set on B_M then consists of a set of points which count algebraically as $K_M B_M$. We can do a similar construction for N. We think of the surface B_X as being glued together from the surfaces B_M and B_N by deleting in each a disk of radius $1/\sqrt{2}$ in D_M and D_N around 0. On the M side we get two additional positive zeroes coming from the intersection with the class $-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M$ in Corollary 5.50 over the annulus in D_M between radius $1/\sqrt{2}$ and 2. Adding these terms proves the claim. □

It remains to determine the intersections $K_X \cdot S_i$ which give the rim tori contribution to the canonical class. This is not possible in general and depends on the situation. We make the following definition.

Definition 5.52. Let Ω_Σ be a given 1-form on Σ with $2g-2$ transverse zeroes, holomorphic with respect to a given almost complex structure J_D. Under the embedding i_M and the trivialization τ_M of the normal bundle equip the tubular neighbourhood $\nu\Sigma_M$ of radius 2 with the almost complex structure $J_D + J_\Sigma$ and the holomorphic 2-form $\Omega_D \wedge \Omega_\Sigma$. Let S^M be a closed oriented surface in $M' = M \setminus \nu\Sigma_M$ which bounds a closed curve α^M on $\partial\nu\Sigma_M$ which is disjoint from the zeroes of $\Omega_D \wedge \Omega_\Sigma$ on the boundary. Then $K_M S^M$ denotes the obstruction to extend the given section of K_M on α^M over the whole surface S^M. This is the number of zeroes one encounters when trying to extend the non-vanishing section of K_M on ∂S^M over all of S^M. There is an exactly analogous definition for N with almost complex structure $J_D + J_\Sigma$ and holomorphic 2-form $\Omega_D \wedge \Omega_\Sigma$ on the tubular neighbourhood $\nu\Sigma_N$ of radius 2.

In particular, there are numbers $K_M S_i^M$ and $K_N S_i^N$ for the surfaces bounding loops c_i^M in $\partial M'$ and c_i^N in $\partial N'$ which represent the classes

$$\begin{aligned} &\gamma_i^M \text{ in } \partial M' \\ &\gamma_i^N + a_i \sigma^N \text{ in } \partial N', \end{aligned}$$

and get identified under the diffeomorphism ϕ. We choose the basis for $H_1(\Sigma)$, the rim tori R_i, the curves c_i^M, c_i^N and the surfaces S_i^M, S_i^N as described in Section V.3.4 and Remark 5.44.

Lemma 5.53. *With the choice of orientation as in Section V.3.4, we have* $K_X S_i = K_M S_i^M - K_N S_i^N - a_i$.

Proof. The proof is the similar to the proof for Lemma 5.51. The minus sign in front of $K_N S_i^N$ comes in because we have to change the orientation on S_i^N if we want to sew it to S_i^M to get the surface S_i in X. This time the non-zero intersections over the annulus in D_M between radius $1/\sqrt{2}$ and 2 come from the intersection of the annulus

$$\gamma_i^M \times [1/\sqrt{2}, 2]$$

and the class

$$-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M = -\sum_{i=1}^{d} a_i R_i + 2\Sigma^M,$$

giving $-a_i$. $\square$

We can evaluate this term further because we have chosen

$$\begin{aligned} S_i^M &= D_i^M \\ S_i^N &= Q_i^N \cup D_i^N \cup U_i^{N'}, \end{aligned}$$

where $U_i^{N'}$ is constructed from a surface U_i^N representing $a_i(B_N - B_N^2 \Sigma_N)$ by deleting the part in $\nu\Sigma_N$. There are additional rim tori terms in the definition of the S_i used to separate S_i and S_j for $i \neq j$ which we can ignore here because the canonical class evaluates to zero on them. We think of the surface Q_i^N as being constructed in the annulus between radius 2 and 3 times Σ_N. We extend the almost complex structure and the holomorphic 2-form over this annulus without change. Hence there are no zeroes of Ω_N on Q_i^N. The surface D_i^N contributes $K_N D_i^N$ to the number $K_N S_i^N$ and the surface $U_i^{N'}$ contributes

$$\begin{aligned} K_N U_i^{N'} &= a_i K_N (B_N - (B_N^2)\Sigma_N) \\ &= a_i (K_N B_N - (2g-2)B_N^2). \end{aligned}$$

Hence we get:

Lemma 5.54. *With our choice of the surfaces S_i^M and S_i^N, we have*

$$K_X S_i = K_M D_i^M - K_N D_i^N - a_i (K_N B_N - (2g-2)B_N^2 + 1).$$

This formula has the advantage that the first two terms are independent of the choice of the diffeomorphism ϕ. By collecting our calculations we get:

Theorem 5.55. *Let $X = M\#_{\Sigma_M=\Sigma_N}N$ be a symplectic generalized fibre sum of closed oriented symplectic 4-manifolds M and N along embedded symplectic surfaces Σ_M, Σ_N of genus g which represent indivisible homology classes and are oriented by the symplectic forms. Suppose that the cohomology of M, N and X is torsion free. Choose a basis for $H^2(X;\mathbb{Z})$ as in Theorem 5.37, where the split classes are constructed from surfaces S_i^M, S_i^N as in Section V.3.4 and Remark 5.44. Then the canonical class of X is given by*

$$K_X = \overline{K_M} + \overline{K_N} + \sum_{i=1}^{d} r_i R_i + b_X B_X + \sigma_X \Sigma_X,$$

where

$$\begin{aligned} \overline{K_M} &= K_M - (2g-2)B_M - (K_M B_M - (2g-2)B_M^2)\Sigma_M \in P(M) \\ \overline{K_N} &= K_N - (2g-2)B_N - (K_N B_N - (2g-2)B_N^2)\Sigma_N \in P(N) \\ r_i &= K_X S_i = K_M D_i^M - K_N D_i^N - a_i (K_N B_N - (2g-2)B_N^2 + 1) \\ b_X &= 2g-2 \\ \sigma_X &= K_M B_M + K_N B_N + 2 - (2g-2)(B_M^2 + B_N^2). \end{aligned}$$

Note that K_X depends in this formula on the diffeomorphism ϕ through the term

$$-a_i (K_N B_N - (2g-2)B_N^2 + 1)$$

which gives the contribution

$$(K_N B_N - (2g-2)B_N^2 + 1)R_C = -\sum_{i=1}^{d} a_i (K_N B_N - (2g-2)B_N^2 + 1)R_i$$

to the canonical class.

Remark 5.56. The apparent asymmetry between M and N in the rim tori contribution to K_X is related to the asymmetry in defining Σ_X to come from Σ^M and not from Σ^N. To write the formula in a symmetric way note that $-\sum_{i=1}^{d} a_i R_i$ is precisely the rim torus R_C in X determined by the gluing diffeomorphism ϕ, cf. Definition 5.22 and Remark 5.44. By Lemma 5.23 we have $R_C = \Sigma'_X - \Sigma_X$. Hence we can write

$$K_X = \overline{K_M} + \overline{K_N} + \sum_{i=1}^{d} t_i R_i + b_X B_X + \eta_X \Sigma_X + \eta'_X \Sigma'_X,$$

where

$$\begin{aligned}
\overline{K_M} &= K_M - (2g-2)B_M - (K_M B_M - (2g-2)B_M^2)\Sigma_M \in P(M) \\
\overline{K_N} &= K_N - (2g-2)B_N - (K_N B_N - (2g-2)B_N^2)\Sigma_N \in P(N) \\
t_i &= K_M D_i^M - K_N D_i^N \\
b_X &= 2g-2 \\
\eta_X &= K_M B_M + 1 - (2g-2)B_M^2 \\
\eta'_X &= K_N B_N + 1 - (2g-2)B_N^2.
\end{aligned}$$

Note that under the embeddings of $H^2(M)$ and $H^2(N)$ into $H^2(X)$ given by Remark 5.45, the canonical classes of M and N map to

$$\begin{aligned}
K_M &\mapsto \overline{K_M} + (2g-2)B_X + (K_M B_M - (2g-2)B_M^2)\Sigma_X \\
K_N &\mapsto \overline{K_N} + (2g-2)B_X + (K_N B_N - (2g-2)B_N^2)\Sigma'_X.
\end{aligned}$$

This implies with Remark 5.56:

Corollary 5.57. *Under the assumptions in Theorem 5.55 and the embeddings of $H^2(M)$ and $H^2(N)$ into $H^2(X)$ given by Remark 5.45, the canonical class of $X = M\#_{\Sigma_M=\Sigma_N} N$ is given by*

$$K_X = K_M + K_N + \Sigma_X + \Sigma'_X - (2g-2)B_X + \sum_{i=1}^{d} t_i R_i,$$

where $t_i = K_M D_i^M - K_N D_i^N$.

For example, suppose that $g = 1$, the coefficients $t_1, \ldots t_d$ vanish and $\Sigma_X = \Sigma'_X$. Then we get the classical formula for the generalized fibre sum along tori

$$K_X = K_M + K_N + 2\Sigma_X,$$

which can be found in the literatur, e.g. [126]. See Section V.6.1 for more applications in the torus case.

V.6 Examples and applications

To check the formula for the canonical class given by Theorem 5.55 we calculate the square $K_X^2 = Q_X(K_X, K_X)$ and compare it with the classical formula

$$c_1(X)^2 = c_1(M)^2 + c_1^2(N) + (8g-8), \tag{5.30}$$

which can be derived independently using the formulas for the Euler characteristic and the signature of a generalized fibre sum (see the proof of Corollary 5.14) and the formula $c_1^2 = 2e + 3\sigma$. We do this step by step. We have (cf. Theorem 5.37):

$$\begin{aligned} Q_X(\overline{K_M}, \overline{K_M}) &= Q_M(\overline{K_M}, \overline{K_M}) \\ &= Q_M(\overline{K_M}, K_M) \\ &= K_M^2 - (2g-2)K_M B_M - (2g-2)(K_M B_M - (2g-2)B_M^2) \\ &= K_M^2 - (4g-4)K_M B_M + (2g-2)^2 B_M^2. \end{aligned}$$

The second step in this calculation follows since by definition $\overline{K_M}$ is orthogonal to B_M and Σ_M. Similarly

$$Q_X(\overline{K_N}, \overline{K_N}) = K_N^2 - (4g-4)K_N B_N + (2g-2)^2 B_N^2.$$

The rim torus term $\sum_{i=1}^d r_i R_i$ has zero intersection with itself and all other terms in K_X. We have

$$Q_X(b_X B_X, b_X B_X) = (2g-2)^2(B_M^2 + B_N^2),$$

and

$$2Q_X(b_X B_X, \sigma_X \Sigma_X) = 2(2g-2)(K_M B_M + K_N B_N + 2 - (2g-2)(B_M^2 + B_N^2)).$$

The self-intersection of Σ_X is zero. Adding these terms together, we get

$$\begin{aligned} K_X^2 &= K_M^2 - (4g-4)K_M B_M + (2g-2)^2 B_M^2 + K_N^2 - (4g-4)K_N B_N + (2g-2)^2 B_N^2 \\ &\quad + (2g-2)^2(B_M^2 + B_N^2) + 2(2g-2)(K_M B_M + K_N B_N + 2 - (2g-2)(B_M^2 + B_N^2)) \\ &= K_M^2 + K_N^2 + (2g-2)^2(B_M^2 + B_M^2) + (2g-2)^2(B_M^2 + B_N^2) \\ &\quad - 2(2g-2)^2(B_M^2 + B_N^2) + (8g-8) \\ &= K_M^2 + K_N^2 + (8g-8). \end{aligned}$$

This is the expected result in equation (5.30).

As another check we compare the formula for K_X in Theorem 5.55 with a formula of Ionel and Parker which determines the intersection of K_X with certain homology classes for symplectic generalized fibre sums in arbitrary dimension and without the assumption of trivial normal bundles of Σ_M and Σ_N (see [69, Lemma 2.4] and an application in [138]). For dimension 4 with surfaces of genus g and self-intersection zero the formula can be written (in our notation for the cohomology of X):

$$\begin{aligned} K_X C &= K_M C \quad \text{for } C \in P(M) \\ K_X C &= K_N C \quad \text{for } C \in P(N) \\ K_X \Sigma_X &= K_M \Sigma_M = K_N \Sigma_N \\ &= 2g-2 \quad \text{(by the adjunction formula)} \\ K_X R &= 0 \quad \text{for all elements in } R(X) \\ K_X B_X &= K_M B_M + K_N B_N + 2(B_M \Sigma_M = B_N \Sigma_N) \\ &= K_M B_M + K_N B_N + 2. \end{aligned}$$

There is no statement about the intersection with classes in $S'(X)$ that have a non-zero component in ker $(i_M \oplus i_N)$. We calculate the corresponding intersections with the formula for K_X in Theorem 5.55. For $C \in P(M)$ we have

$$\begin{aligned} K_X \cdot C &= \overline{K_M} \cdot C \\ &= K_M \cdot C, \end{aligned}$$

where the second line follows because the terms in the formula for $\overline{K_M}$ involving B_M and Σ_M have zero intersection with C, being a perpendicular element. A similar equation holds for N. The intersection with Σ_X is given by

$$\begin{aligned} K_X \cdot \Sigma_X &= (2g-2)B_X \cdot \Sigma_X \\ &= 2g-2. \end{aligned}$$

The intersection with rim tori is zero and

$$\begin{aligned} K_X \cdot B_X &= b_X B_X^2 + \sigma_X \\ &= (2g-2)(B_M^2 + B_N^2) + K_M B_M + K_N B_N + 2 - (2g-2)(B_M^2 + B_N^2) \\ &= K_M B_M + K_N B_N + 2, \end{aligned}$$

which also follows by Lemma 5.51. Hence with the formula in Theorem 5.55 we get the same result as with the formula of Ionel and Parker.

The following corollary gives a criterion when the canonical class K_X is divisible by d as an element in $H^2(X;\mathbb{Z})$.

Corollary 5.58. *Let X be a symplectic generalized fibre sum $M\#_{\Sigma_M=\Sigma_N}N$ as in Theorem 5.55. If K_X is divisible by an integer $d \geq 0$ then*

- *the integers $2g-2$ and $K_M B_M + K_N B_N + 2$ are divisible by d, and*
- *the cohomology classes $K_M - (K_M B_M)\Sigma_M$ in $H^2(M;\mathbb{Z})$ and $K_N - (K_N B_N)\Sigma_N$ in $H^2(N;\mathbb{Z})$ are divisible by d.*

Conversely, if all r_i vanish, then these conditions are also sufficient for K_X being divisible by d.

The proof is immediate by the formula for the canonical class K_X since B_X and Σ_X are indivisible. The following proposition gives a criterion which excludes the existence of non-zero rim tori in the cohomology of X.

Proposition 5.59. *Let M, N be closed 4-manifolds with embedded surfaces Σ_M and Σ_N of genus g. Suppose that the first homology of M and N is torsion free and the map $i_M \oplus i_N \colon H_1(\Sigma;\mathbb{Z}) \to H_1(M;\mathbb{Z}) \oplus H_1(N;\mathbb{Z})$ is injective with torsion free cokernel. Then the cohomology of the generalized fibre sum $X = M\#_{\Sigma_M=\Sigma_N}N$ does not contain non-zero rim tori. This holds, in particular, if one of the maps i_M, i_N is injective with torsion free cokernel.*

Proof. Under the assumptions, there is a splitting $H_1(M) \oplus H_1(N) = \mathrm{Im}\,(i_M \oplus i_N) \oplus \mathrm{Coker}\,(i_M \oplus i_N)$. We can find a basis $e_1, \ldots, e_{2g}$ of $H_1(\Sigma;\mathbb{Z})$ consisting of elements whose images $(v_i, w_i) = (i_M e_i, i_N e_i)$ for $i = 1, \ldots, 2g$ can be completed to a basis of $H_1(M) \oplus H_1(N)$ by elements

$$(v_{2g+1}, w_{2g+1}), \ldots, (v_N, w_N).$$

Take the dual basis (α_i, β_i). Then $\alpha_i \in H^1(M)$ and $\beta_i \in H^1(N)$. We have

$$\begin{aligned} \langle i_M^*\alpha_i + i_N^*\beta_i, e_j\rangle &= \langle \alpha_i, v_j\rangle + \langle \beta_i, w_j\rangle \\ &= \delta_{ij}. \end{aligned}$$

Hence the images $\{i_M^*\alpha_i + i_N^*\beta_i\}$, with $i = 1, \ldots, 2g$, form a dual basis to $\{e_i\}$ for $H^1(\Sigma)$. In particular, $i_M^* + i_N^*$ is surjective and $R(X) = 0$. If one of the maps i_M, i_N satisfies the condition, then clearly $i_M \oplus i_N$ is injective. A torsion element in the cokernel is also a torsion element in the cokernel of both maps i_M and i_N. This proves the claim. $\square$

Consider, for example, the manifold $M = M_K \times S^1$ used in the knot surgery construction from Section V.4.1. The first homology of M is generated by the image of the torus $T_M = m \times S^1$. Hence $i_M\colon T^2 \to M$ induces an isomorphism on H_1 and the knot surgery manifolds X_K do not contain rim tori, for arbitrary closed 4-manifolds X.

We can calculate the canonical class in the following way: Recall that $M_K \times S^1$ fibres over T^2 with fibre a surface Σ_h of genus h. The generalized fibre sum is done along a section T_M and the canonical class of $M_K \times S^1$ is $(2h-2)T_M$. We will use the fibre Σ_h as B_M. We have $B_M^2 = 0$ and $B_M K_M = 2h-2$. This implies that $\overline{K_M} = 0$.

Corollary 5.60. *Let X be a closed, symplectic 4-manifold with torsion free cohomology. Suppose that X contains a symplectic torus of self-intersection 0 oriented by the symplectic form. Let K be a fibred knot and X_K the result of knot surgery along T_X. Then the canonical class of X_K is given by*

$$K_{X_K} = \overline{K_X} + (2h + K_X B_X) T_{X_K},$$

where $\overline{K_X} = K_X - (K_X B_X) T_X$.

The proof is immediate by the formula for the canonical class in Theorem 5.55. We want to compare this formula to the formula given by Fintushel and Stern in [38, Corollary 1.7]:

$$K_{X_K} = K_X + 2hT_X. \tag{5.31}$$

This formula involves the identification $H^2(X;\mathbb{Z}) \cong H^2(X_K;\mathbb{Z})$ in equation (5.23) which sends

$$\begin{aligned} T_X &\mapsto T_{X_K} \\ B_X &\mapsto B_{X_K} \\ Id : P(X) &\to P(X). \end{aligned}$$

We can split the class $K_X \in H^2(X;\mathbb{Z})$ as before into $K_X = \overline{K_X} + (K_X B_X) T_X$ where $\overline{K_X} \in P(X)$. Then the class $K_X + 2hT_X$ maps under this isomorphism to $\overline{K_X} + (2h + K_X B_X) T_{X_K}$, which is identical to our formula. See also Corollary 5.57.

V.6.1 Generalized fibre sums along tori

We consider some further applications of Theorem 5.55. Let M and N be closed symplectic 4-manifolds which contain symplectically embedded tori T_M and T_N of self-intersection 0, representing indivisible classes. Suppose that M and N have torsion free homology and both tori are contained in cusp neighbourhoods. Then each torus has two vanishing cycles coming from the cusp. We choose identifications of both T_M and T_N with T^2 such that the vanishing cycles are given by the simple closed loops $\gamma_1 = S^1 \times 1$ and $\gamma_2 = 1 \times S^1$. The loops bound embedded vanishing disks in M and N of self-intersection -1 which we denote by (D_1^M, D_2^M) and (D_1^N, D_2^N). The existence of the vanishing disks shows that the embeddings $T_M \to M$ and $T_N \to N$ induce the zero map on the fundamental group.

We choose for both tori trivializations of the normal bundles and corresponding push-offs T^M and T^N. By choosing the trivializations appropriately we can assume that the vanishing disks bound the vanishing cycles on these push-offs and are contained in $M \setminus \operatorname{int} \nu T_M$ and $N \setminus \operatorname{int} \nu T_N$. We consider the symplectic generalized fibre sum $X = X(\phi) = M \#_{T_M = T_N} N$ for a gluing diffeomorphism

$$\phi\colon \partial(M \setminus \operatorname{int} \nu T_M) \to \partial(N \setminus \operatorname{int} \nu T_N).$$

The vanishing cycles on both tori give a basis for $H_1(T^2;\mathbb{Z})$ and we can describe the cohomology class C with respect to this basis: If $a_i = \langle C, \gamma_i\rangle$ and σ denotes the meridians to T_M in M and T_N in N then $\phi\colon \partial\nu T_M \to \partial\nu T_N$ maps in homology

$$\begin{aligned}\gamma_1 &\mapsto \gamma_1 + a_1\sigma\\ \gamma_2 &\mapsto \gamma_2 + a_2\sigma\\ \sigma &\mapsto -\sigma\end{aligned}$$

by Lemma 5.5. Note that $H_1(X(\phi)) \cong H_1(M)\oplus H_1(N)$ by Theorem 5.11. Hence under our assumptions the homology of $X(\phi)$ is torsion free. The group of rim tori is $R(X) = \mathrm{Coker}(i_M^* + i_N^*) \cong \mathbb{Z}^2$. Let R_1, R_2 denote a basis for $R(X)$. We calculate the canonical class of $X = X(\phi)$ by Theorem 5.55: Let B_M and B_N denote surfaces in M and N which intersect T_M and T_N transversely once. Then the canonical class is given by

$$K_X = \overline{K_M} + \overline{K_N} + (r_1R_1 + r_2R_2) + b_XB_X + \sigma_XT_X,$$

where

$$\begin{aligned}\overline{K_M} &= K_M - (K_MB_M)T_M \in P(M)\\ \overline{K_N} &= K_N - (K_NB_N)T_N \in P(N)\\ r_i &= K_XS_i = K_MD_i^M - K_ND_i^N - a_i(K_NB_N + 1)\\ b_X &= 2g-2 = 0\\ \sigma_X &= K_MB_M + K_NB_N + 2.\end{aligned}$$

Lemma 5.61. *In the situation above we have $K_MD_i^M - K_ND_i^N = 0$ for $i = 1,2$.*

Proof. Note that the pairs (D_1^M, D_1^N) and (D_2^M, D_2^N) sew together in the generalized fibre sum $X_0 = X(Id)$ to give embedded spheres S_1, S_2 of self-intersection -2. We claim that

$$K_{X_0}S_i = K_MD_i^M - K_ND_i^N = 0, \quad i = 1,2.$$

This is clear by the adjunction formula if the spheres are symplectic or Lagrangian. In the general case, note that in X_0 there are rim tori R_1, R_2 which are dual to the spheres S_1, S_2 and which can be assumed Lagrangian by the Gompf construction. Consider the pair R_1 and S_1: By the adjunction formula we have $K_{X_0}R_1 = 0$. The sphere S_1 and the torus R_1 intersect once. By smoothing the intersection point we get a smooth torus of self-intersection zero in X_0 representing $R_1 + S_1$. Note that K_{X_0} is a Seiberg-Witten basic class. The adjunction inequality [104] implies that $K_{X_0}(R_1 + S_1) = 0$ which shows that $K_{X_0}S_1 = 0$. In a similar way it follows that $K_{X_0}S_2 = 0$. ☐

This implies:

Proposition 5.62. *Let M, N be closed symplectic 4-manifolds with torsion free homology. Suppose that T_M and T_N are embedded symplectic tori of self-intersection 0 which are contained in cusp neighbourhoods in M and N. Then the canonical class of the symplectic generalized fibre sum $X = X(\phi) = M\#_{T_M=T_N}N$ is given by*

$$\begin{aligned}K_X &= \overline{K_M} + \overline{K_N} + (r_1R_1 + r_2R_2) + \sigma_XT_X\\ &= K_M + K_N + T_X + T_X',\end{aligned}$$

where

$$\begin{aligned}\overline{K_M} &= K_M - (K_M B_M)T_M \in P(M)\\ \overline{K_N} &= K_N - (K_N B_N)T_N \in P(N)\\ r_i &= -a_i(K_N B_N + 1)\\ \sigma_X &= K_M B_M + K_N B_N + 2.\end{aligned}$$

The second line in the formula for K_X holds by Corollary 5.57 under the embeddings of $H^2(M)$ and $H^2(N)$ in $H^2(X)$.

Here T_X is the torus in X determined by the push-off T^M and T'_X is determined by the push-off T^N. As a special case suppose that the tori T_M and T_N are contained in smoothly embedded nuclei $N(m) \subset M$ and $N(n) \subset N$ which are by definition diffeomorphic to neighbourhoods of a cusp fibre and a section in the elliptic surfaces $E(m)$ and $E(n)$, cf. [53], [56]. The surfaces B_M and B_N can now be chosen as the spheres S_M, S_N inside the nuclei corresponding to the sections. The spheres have self-intersection $-m$ and $-n$ respectively. If the sphere S_M is symplectic or Lagrangian in M we get by the adjunction formula

$$K_M S_M = m - 2.$$

If $m = 2$ this holds by an argument similar to the one in Lemma 5.61 also without the assumption that S_M is symplectic or Lagrangian. With Proposition 5.62 we get:

Corollary 5.63. *Let M, N be closed symplectic 4-manifolds with torsion free homology. Suppose that T_M and T_N are embedded symplectic tori of self-intersection 0 which are contained in embedded nuclei $N(m) \subset M$ and $N(n) \subset N$. Suppose that $m = 2$ or the sphere S_M is symplectic or Lagrangian. Similarly, suppose that $n = 2$ or the sphere S_N is symplectic or Lagrangian. Then the canonical class of the symplectic generalized fibre sum $X = X(\phi) = M\#_{T_M=T_N}N$ is given by*

$$\begin{aligned}K_X &= \overline{K_M} + \overline{K_N} - (n-1)(a_1R_1 + a_2R_2) + (m+n-2)T_X\\ &= \overline{K_M} + \overline{K_N} + (m-1)T_X + (n-1)T'_X,\end{aligned}$$

where

$$\begin{aligned}\overline{K_M} &= K_M - (m-2)T_M \in P(M)\\ \overline{K_N} &= K_N - (n-2)T_N \in P(N).\end{aligned}$$

For the second line in this formula for K_X see Remark 5.56. Note that the class $-(a_1R_1 + a_2R_2)$ is equal to the rim torus R_C in X which satisfies $R_C = T'_X - T_X$. We consider two examples:

Example 5.64. Suppose that M is an arbitrary closed symplectic 4-manifold with torsion free homology and T_M is contained in a nucleus $N(2) \subset M$. Suppose that N is the elliptic surface $E(n)$ with general fibre T_N. Since $K_{E(n)} = (n-2)T_N$ we get $\overline{K_N} = 0$. Hence the canonical class of $X = X(\phi) = M\#_{T_M=T_N}E(n)$ is given by

$$K_X = K_M - (n-1)(a_1R_1 + a_2R_2) + nT_X.$$

Note that $K_M = \overline{K_M} \in P(M)$ in this case. If both a_1 and a_2 vanish (hence the vanishing cycles in the generalized fibre sum are identified) we get $K_X = K_M + nT_X$. This can be compared to the classical formula $K_X = K_M + K_N + 2T_X$ which can be found in the literature, e.g. [126]. If M is simply-connected then X is again simply-connected: This follows because $N(2) \setminus T_M$ and $E(n) \setminus T_N$ are simply-connected (the meridians bound punctured disks given by the sections).

Example 5.65. Suppose that $M = E(m)$ and $N = E(n)$ with general fibres T_M and T_N. Then the canonical class of $X = X(\phi) = E(m)\#_{T_M=T_N}E(n)$ is given by[3]

$$\begin{aligned} K_X &= -(n-1)(a_1R_1 + a_2R_2) + (m+n-2)T_X \\ &= (m-1)T_X + (n-1)T'_X. \end{aligned} \tag{5.32}$$

If both coefficients a_1 and a_2 vanish, we get the standard formula $K_X = (m+n-2)T_X$ for the fibre sum $E(m+n) = E(m)\#_{T_M=T_N}E(n)$. If $n = 1$ we see from the first line that there is no rim tori contribution, independent of the gluing diffeomorphism ϕ. The canonical class is always given by $(m-1)T_X$. This can be explained because every orientation-preserving self-diffeomorphism of $\partial(E(1) \setminus \mathrm{int}\,\nu F)$ extends over $E(1) \setminus \mathrm{int}\,\nu F$ where F denotes a general fibre. Hence all generalized fibre sums $X(\phi)$ are diffeomorphic to the elliptic surface $E(m+1)$ in this case (see [56, Theorem 8.3.11]). If $n \neq 1$ but $m = 1$ a similar argument holds by the second line.

If both $m, n \neq 1$, then there can exist a non-trivial rim tori contribution. For example if $m = n = 2$ and we consider the generalized fibre sum $X = X(\phi) = E(2)\#_{T_M=T_N}E(2)$ of two $K3$-surfaces $E(2)$ then

$$\begin{aligned} K_X &= -(a_1R_1 + a_2R_2) + 2T_X \\ &= T_X + T'_X \end{aligned}$$

If the greatest common divisor of a_1 and a_2 is odd then K_X is indivisible (because there exist certain split classes in X dual to the rim tori R_1 and R_2). In this case the manifold X is no longer spin, hence cannot be homeomorphic to the spin manifold $E(4)$.

We return to the general case of closed 4-manifolds M and N which contain tori T_M and T_N of self-intersection 0, lying in cusp neighbourhoods. For the following lemma we do not have to assume that the manifolds and the tori are symplectic. By varying the parameters a_1, a_2 which determine the gluing diffeomorphism ϕ up to isotopy, we get a $\mathbb{Z} \oplus \mathbb{Z}$ family of closed 4-manifolds

$$X(a_1, a_2) = M\#_{T_M=T_N}N.$$

Using the existence of a cusp one can show that this reduces to a $\mathbb{N}_0$ family up to diffeomorphism:

Lemma 5.66. *The manifold $X(a_1, a_2)$ is diffeomorphic to $X(p, 0)$ where $p \geq 0$ denotes the greatest common divisor of a_1, a_2.*

Proof. In the basis $\gamma_1, \gamma_2, \sigma$ for $H_1(T^2 \times S^1)$ the gluing diffeomorphism ϕ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_1 & a_2 & -1 \end{pmatrix}.$$

Every automorphism $A \in SL(2, \mathbb{Z})$ acting on (γ_1, γ_2) can be realized by an orientation preserving self-diffeomorphism of T_M. Since T_M is contained in a cusp neighbourhood this diffeomorphism can be extended (using the monodromy around the cusp) to an orientation preserving self-diffeomorphism of M which maps T_M to itself and has support in the cusp neighbourhood, cf. [53], [56, Lemma 8.3.6]. Similarly, any automorphism in $SL(3, \mathbb{Z})$ of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[3] This formula can also be derived from a gluing formula for the Seiberg-Witten invariants along T^3, cf. [109, Corollary 22].

acting on $(\gamma_1, \gamma_2, \sigma)$ can be realized by an orientation preserving self-diffeomorphism ψ_M of $\partial\nu T_M$ which preserves the push-off T^M (as a set) and the meridian σ^M. This diffeomorphism can be extended to an orientation preserving self-diffeomorphism of $M \setminus \operatorname{int}\nu T_M$. A similar result holds for automorphisms in $SL(3,\mathbb{Z})$ realized by diffeomorphisms ψ_N acting on $\partial\nu T_N$ since T_N is also contained in a cusp neighbourhood.

We can choose integers r_1, r_2 such that

$$r_1\tfrac{a_1}{p} + r_2\tfrac{a_2}{p} = 1.$$

Let ψ_N be a diffeomorphism corresponding to the matrix

$$\begin{pmatrix} \frac{a_1}{p} & \frac{a_2}{p} & 0 \\ -r_2 & r_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3,\mathbb{Z})$$

and ψ_M a diffeomorphism corresponding to the inverse matrix

$$\begin{pmatrix} r_1 & -\frac{a_2}{p} & 0 \\ r_2 & \frac{a_1}{p} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3,\mathbb{Z}).$$

Consider the diffeomorphism

$$\phi' = \psi_N \circ \phi \circ \psi_M : \partial\nu T_M \to \partial\nu T_N$$

Multiplying matrices one can check that ϕ' is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & 0 & -1 \end{pmatrix}.$$

In particular, ϕ' can be realized as a gluing diffeomorphism and since ψ_M, ψ_N extend over the complements of the tubular neighbourhoods in M and N, it follows that the manifolds $X(\phi')$ and $X(\phi)$ are diffeomorphic. This proves the claim. $\square$

In particular, for $M = E(m)$ and $N = E(n)$ with general fibres T_M, T_N we get a family of simply-connected symplectic 4-manifolds

$$X(m,n,p) = E(m)\#_{T_M=T_N}E(n), \quad p \in \mathbb{Z}.$$

Note that $X(m,n,p)$ has the same characteristic numbers c_1^2 and σ as the elliptic surface $E(m+n)$. The manifolds $X(m,n,p)$ and $X(m,n,-p)$ are diffeomorphic and $X(m,n,0)$ is diffeomorphic to $E(m+n)$. The canonical class of $X = X(m,n,p)$ can be calculated by the formula in Example 5.65:

$$K_X = -(n-1)pR_1 + (m+n-2)T_X.$$

This implies:

Proposition 5.67. *If $(m+n-2)$ does not divide $(n-1)p$ then $X(m,n,p)$ is not diffeomorphic to the elliptic surface $E(m+n)$.*

Proof. If $X(m,n,p)$ is diffeomorphic to $E(m+n)$ and $(m+n-2)$ does not divide $(n-1)p$ then we have constructed a symplectic structure on $E(m+n)$ whose canonical class K_X is not divisible by $m+n-2$. Note that $E(m+n)$ has $b_2^+ \geq 3$ under our assumptions. The canonical class K_X is a Seiberg-Witten basic class on $E(m+n)$. The Seiberg-Witten basic classes of the smooth manifold underlying $E(m+n)$ are known. They are of the form kF where F is a general fibre and k is an integer with $k \equiv m+n \mod 2$ and $|k| \leq m+n-2$, cf. [48], [82]. However, a theorem of Taubes [133] shows that the only basic classes on $E(m+n)$ which can be the canonical class of a symplectic structure are $\pm(m+n-2)F$. This is a contradiction. □

As a corollary, we get a new proof of the following known result, cf. [56, Theorem 8.3.11]:

Corollary 5.68. *Let $n \geq 2$, $p \in \mathbb{Z}$ and F a general fibre in the elliptic surface $E(n)$ with fibred tubular neighbourhood νF. Suppose that ψ is an orientation preserving self-diffeomorphism of $\partial\nu F$ realizing*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & 0 & 1 \end{pmatrix} \in SL(3,\mathbb{Z})$$

on $H_1(\partial\nu F)$. Then ψ extends to an orientation preserving self-diffeomorphism of $E(n) \setminus \mathrm{int}\,\nu F$ if and only if $p = 0$.

Proof. Suppose that $p \neq 0$. If ψ extends to a self-diffeomorphism of $E(n) \setminus \mathrm{int}\,\nu F$, then $X(m,n,p)$ is diffeomorphic to $E(m+n)$ for all $m \geq 1$. Since $n \neq 1$ we can choose m large enough such that $(m+n-2)$ does not divide $(n-1)p$. This is a contradiction to Proposition 5.67. □

Note that the diffeomorphism ψ *does* extend in the case of $E(1)$ for all integers $p \in \mathbb{Z}$ by [56, Theorem 8.3.11].

V.6.2 Inequivalent symplectic structures

In this section we will prove a theorem similar to a result of I. Smith [126, Theorem 1.5] which can be used to show that certain 4-manifold X admit inequivalent symplectic structures, where "equivalence" is defined in the following way (cf. [140]):

Definition 5.69. Two symplectic forms on a closed oriented 4-manifold M are called *equivalent*, if they can be made identical by a combination of deformations through symplectic forms and orientation preserving self-diffeomorphisms of M.

Note that the canonical classes of equivalent symplectic forms have the same (maximal) divisibility as elements of $H^2(M;\mathbb{Z})$. This follows because deformations do not change the canonical class and the application of an orientation preserving self-diffeomorphism does not change the divisibility.

We will use the following lemma.

Lemma 5.70. *Let (M,ω) be a symplectic 4-manifold with canonical class K. Then the symplectic structure $-\omega$ has canonical class $-K$.*

Proof. Let J be an almost complex structure on M, compatible with ω. Then $-J$ is an almost complex structure compatible with $-\omega$. The complex vector bundle $(TX,-J)$ is the conjugate bundle to (TX,J). By [100], this implies that $c_1(TX,-J) = -c_1(TX,J)$. Since the canonical class is minus the first Chern class of the tangent bundle the claim follows. □

Let $M_K \times S^1$ be a 4-manifold used in the knot surgery construction where K is a fibred knot of genus h. Let T_K be a section of the fibre bundle

$$\begin{array}{ccc} M_K \times S^1 & \longleftarrow & \Sigma_h \\ \downarrow & & \\ T^2 & & \end{array}$$

and B_K a fibre. We fix an orientation on T_K and choose the orientation on B_K such that $T_K \cdot B_K = +1$. There exist symplectic structures on $M_K \times S^1$ such that both the fibre and the section are symplectic. We can choose such a symplectic structure ω^+ which restricts to both T_K and B_K as a positive volume form with respect to the orientations. It has canonical class

$$K^+ = (2h-2)T_K$$

by the adjunction formula. We also define the symplectic form $\omega^- = -\omega^+$. It restricts to a negative volume form on T_K and B_K. By Lemma 5.70, the canonical class of this symplectic structure is

$$K^- = -(2h-2)T_K.$$

Let X be a closed oriented 4-manifold with torsion free cohomology which contains an embedded oriented torus T_X of self-intersection 0. We form the oriented 4-manifold

$$X_K = X \#_{T_X = T_K} (M_K \times S^1),$$

by doing the generalized fibre sum along the pair (T_X, T_K) of oriented tori. Suppose that X has a symplectic structure ω_X such that T_X is symplectic. We consider two cases: If the symplectic form ω_X restricts to a positive volume form on T_X we can glue this symplectic form to the symplectic form ω^+ on $M_K \times S^1$ to get a symplectic structure $\omega^+_{X_K}$ on X_K. The canonical class of this symplectic structure is

$$K^+_{X_K} = K_X + 2hT_X,$$

as seen above, cf. equation (5.31).

Lemma 5.71. *Suppose that ω_X restricts to a negative volume form on T_X. We can glue this symplectic form to the symplectic form ω^- on $M_K \times S^1$ to get a symplectic structure $\omega^-_{X_K}$ on X_K. The canonical class of this symplectic structure is*

$$K^-_{X_K} = K_X - 2hT_X.$$

Proof. We use Lemma 5.70 twice: The symplectic form $-\omega_X$ restricts to a positive volume form on T_X. We can glue this symplectic form to the symplectic form ω^+ on $M_K \times S^1$ which also restricts to a positive volume form on T_K. Then we can use the standard formula (5.31) to get for the canonical class of the resulting symplectic form on X_K

$$K = -K_X + 2hT_X.$$

The symplectic form $\omega^-_{X_K}$ we want to consider is *minus* the symplectic form we have just constructed. Hence its canonical class is $K^-_{X_K} = K_X - 2hT_X$. □

Lemma 5.72. *Let (M, ω) be a closed symplectic 4-manifold with canonical class K_M. Suppose that M contains pairwise disjoint embedded oriented Lagrangian surfaces $T_1, \ldots, T_{r+1}$ ($r \geq 1$) with the following properties:*

- *The classes of the surfaces $T_1, \ldots, T_r$ are linearly independent in $H_2(M; \mathbb{R})$.*
- *The surface T_{r+1} is homologous to $a_1T_1 + \ldots + a_rT_r$, where all coefficients $a_1, \ldots, a_r$ are positive integers.*

Then for every non-empty subset $S \subset \{T_1, \ldots, T_r\}$ there exists a symplectic form ω_S on M with the following properties:

- *All surfaces $T_1, \ldots, T_{r+1}$ are symplectic.*
- *The symplectic form ω_S induces on the surfaces in S and the surface T_{r+1} a positive volume form and on the remaining surfaces in $\{T_1, \ldots, T_r\} \setminus S$ a negative volume form.*

Moreover, the canonical classes of the symplectic structures ω_S are all equal to K_M. We can also assume that any given closed oriented surface in M, disjoint from the surfaces $T_1, \ldots, T_{r+1}$, which is symplectic with respect to ω, stays symplectic for ω_S with the same sign of the induced volume form.

Proof. The proof is similar to the proof of [52, Lemma 1.6]. We can assume that $S = \{T_{s+1}, \ldots, T_r\}$ with $s + 1 \leq r$. Let

$$c = \sum_{i=1}^{s} a_i, \quad c' = \sum_{i=s+1}^{r-1} a_i.$$

Since the classes of the surfaces $T_1, \ldots, T_r$ are linearly independent in $H_2(M; \mathbb{R})$ and $H^2_{DR}(M)$ is the dual space of $H_2(M; \mathbb{R})$ there exists a closed 2-form η on M with the following properties:

$$\begin{aligned} \int_{T_1} \eta = -1, \ldots, \int_{T_s} \eta = -1 \\ \int_{T_{s+1}} \eta = +1, \ldots, \int_{T_{r-1}} \eta = +1 \\ \int_{T_r} \eta = \tfrac{1}{a_r}(c+1) \\ \int_{T_{r+1}} \eta = c' + 1. \end{aligned}$$

Note that we can choose the value of η on $T_1, \ldots, T_r$ arbitrarily. The value on T_{r+1} is then determined by $T_{r+1} = a_1T_1 + \ldots + a_rT_r$. We can choose symplectic forms ω_i on each T_i such that

$$\int_{T_i} \omega_i = \int_{T_i} \eta, \quad \text{for all } i = 1, \ldots, r+1.$$

The symplectic ω_i induces on T_i a negative volume form if $i \leq s$ and a positive volume form if $i \geq s+1$. The difference $\omega_i - j_i^*\eta$, where $j_i \colon T_i \to M$ is the embedding, has vanishing integral and hence is an exact 2-form on T_i of the form $d\alpha_i$. We can extend each α_i to a small tubular neighbourhood of T_i in M, cut it off differentiably in a slightly larger tubular neighbourhood and extend by 0 to all of M. We can do this such that the tubular neighbourhoods of $T_1, \ldots, T_{r+1}$ are pairwise disjoint. Define the closed 2-form $\eta' = \eta + \sum_{i=1}^{r+1} d\alpha_i$ on M. Then

$$j_i^*\eta' = j_i^*\eta + d\alpha_i = \omega_i.$$

The closed 2-form $\omega' = \omega + t\eta'$ is for small values of t symplectic. Since $j_i^*\omega = 0$ we get that $j_i^*\omega' = t\omega_i$. Hence ω' is for small values $t > 0$ a symplectic form on M which induces a volume form on T_i of the same sign as ω_i for all $i = 1, \dots, r+1$. The claim about the canonical class follows because the symplectic structures ω_S are constructed by a deformation of ω. We can also choose $t > 0$ small enough such that ω' still restricts to a symplectic form on any given symplectic surface disjoint from the tori without changing the sign of the induced volume form on this surface. □

We consider the construction in Lemma 5.72 on triples of Lagrangian tori. Recall that the nucleus $N(2)$ is the smooth manifold with boundary defined as a regular neighbourhood of a cusp fibre and a section in the $K3$-surface $E(2)$. It contains an embedded torus given by a regular fibre homologous to the cusp. It also contains two embedded disks of self-intersection -1 which bound vanishing cycles on the torus. The vanishing cycles are the simple-closed loops given by the factors in $T^2 = S^1 \times S^1$.

Suppose that (M, ω) is a simply-connected symplectic 4-manifold which contains pairwise disjoint oriented Lagrangian tori T_1, T_2, R of self-intersection zero which represent indivisible classes such that T_1, T_2 are linearly independent and R is homologous to $aT_1 + T_2$ for some integer $a \geq 1$. We assume that R is contained as the torus coming from a general fibre in an embedded nucleus $N(2) \subset M$. In $N(2)$ there exists an oriented embedded sphere S of self-intersection -2 intersecting R transversely and positively once. We assume that T_1 is disjoint from $N(2)$ and that there exists a further embedded sphere S_1 in M which is disjoint from $N(2)$ and intersects T_1 transversely and positively once. We also assume that S intersects the torus T_2 transversely once.

Example 5.73. Let M be the elliptic surface $E(n)$ with $n \geq 2$. In this example we show that there exist $n-1$ triples of Lagrangian tori (T_1^i, T_2^i, R^i) as above where R^i is homologous to $a_i T_1^i + T_2^i$, for $i = 1, \dots, n-1$. The integers $a_i > 0$ can be chosen arbitrarily and for each triple independently. All tori T_1^i and R^i are contained in disjoint embedded nuclei $N(2)$. Together with their dual 2-spheres they realize $2(n-1)$ H-summands in the intersection form of $E(n)$. In particular, the tori in different triples are linearly independent. The tori are constructed as rim tori, where T_1^i and R^i are standard rim tori coming from the factors in $F = T^2 = S^1 \times S^1$ and T_2^i is realized by taking the product of a torus knot on the fibre F with the meridian. We can also achieve that all Lagrangian tori and the 2-spheres that intersect them once are disjoint from the nucleus $N(n) \subset E(n)$ defined as a regular neighbourhood of a cusp fibre and a section in $E(n)$.

The construction is quite clear by [55, Section 2]. We nevertheless give the explicit construction here. The proof is by induction: Suppose the Lagrangian triples are already constructed for $E(n)$ and consider a splitting of $E(n+1)$ as a fibre sum $E(n+1) = E(n)\#_{F=F}E(1)$ along a general fibre. Choose a general fibre F in both $E(n)$ and $E(1)$ with fibred tubular neighbourhoods. The boundaries of the tubular neighbourhoods can be identified with $T^3 = S^1 \times S^1 \times S^1$. Consider $E(1)$ and a collar $S^1 \times S^1 \times S^1 \times I$ for the boundary of $E(1) \setminus \operatorname{int} \nu F$. In this collar we consider three disjoint tori given by

$$\begin{aligned} V_0 &= S^1 \times S^1 \times 1 \times r_0 \\ V_1 &= S^1 \times 1 \times S^1 \times r_1 \\ V_2 &= 1 \times S^1 \times S^1 \times r_2, \end{aligned}$$

where $0 < r_0 < r_1 < r_2 < 1$ and the numbers in the interval I increase towards the interior of $E(1) \setminus \operatorname{int} \nu F$. The torus V_0 is a push-off of the fibre F. Similarly, we consider in a collar for $E(n) \setminus \operatorname{int} \nu F$ three disjoint tori given by

$$\begin{aligned} V_0 &= S^1 \times S^1 \times 1 \times s_0 \\ V_1 &= S^1 \times 1 \times S^1 \times s_1 \\ V_2 &= 1 \times S^1 \times S^1 \times s_2, \end{aligned}$$

where $s_0 > s_1 > s_2$ are chosen such that the tori get identified pairwise in the fibre sum. We can assume that V_0 is symplectic while V_1, V_2 are Lagrangian (note that ϕ is the identity in this case).

We can choose elliptic fibrations such that near the general fibre F there exist two cusp fibres in $E(1)$ and three cusp fibres in $E(n)$ (note that $E(m)$ has an elliptic fibration with $6m$ cusp fibres for all m, cf. [56, Corollary 7.3.23]).

In $E(1)$ there exist three disjoint sections for the elliptic fibration. We can assume that they are parallel inside the collar to $1 \times 1 \times S^1 \times I$ and intersect the fibre F in three distinct points $\{a_0 \times a_0', a_1 \times 1, 1 \times a_2\}$, where all $a_0, a_0', a_1, a_2 \neq 1$. Hence the circles

$$
\begin{aligned}
A_0 &= a_0 \times a_0' \times S^1 \times r_0 \\
A_1 &= a_1 \times 1 \times S^1 \times r_1 \\
A_2 &= 1 \times a_2 \times S^1 \times r_2,
\end{aligned}
$$

bound three disjoint disks of self-intersection -1 in $E(1) \setminus F$. Since the numbers r_i are ordered increasingly, it follows that the disk bounding A_i only intersects the torus V_i for $i = 0, 1, 2$. In particular, the disk bounding A_0 intersects V_0 in a single point and the disks bounding A_1, A_2 intersect V_1, V_2 in a circle.

The two cusp fibres in $E(1)$ determine four disjoint vanishing cycles on $\partial\nu F = S^1 \times S^1 \times S^1$. We can assume that they lie at some parameter $r_i \in I$. We can choose the following three out of them: There is one cycle of the form

$$B_2 = S^1 \times b_2 \times b_3 \times r_2,$$

and two parallel cycles of the form

$$
\begin{aligned}
C_1 &= c_1 \times S^1 \times c_2 \times r_1 \\
C_2 &= 1 \times S^1 \times c_3 \times r_2.
\end{aligned}
$$

Here B_2 and C_2 correspond to the second cusp and we have ignored one vanishing cycle for the first cusp. The three vanishing cycles bound disks of self-intersection -1 which are the cores of certain 2-handles attached to these circles. We can assume that all b_i, c_i are pairwise different and different from 1 and a_0, a_0', a_1, a_2. Then all B_i, C_i are pairwise disjoint and disjoint from A_0. The only intersection with A_1, A_2 is between A_2, C_2 in one point. We can also assume that the vanishing disks are inside the collar of the form $\gamma \times I$ where the curve γ is given by B_2, C_1 or C_2. Note that the disks bounding B_2, C_2 are disjoint from V_0, V_1 because they start at radius r_2. The disk bounding C_1 is disjoint from V_0 for the same reason and from V_2 because $c_1 \neq 1$.

Similarly, on the $E(n)$ side we have a section which determines a disk of self-intersection $-n$ that bounds the circle

$$A_0 = a_0 \times a_0' \times S^1 \times s_0,$$

intersects V_0 in one point and is disjoint from V_1, V_2. We also have six vanishing cycles coming from the three cusps and choose the following five: There are three parallel cycles of the form

$$
\begin{aligned}
D_0 &= S^1 \times d_0 \times 1 \times s_0 \\
D_1 &= S^1 \times 1 \times d_1 \times s_1 \\
B_2 &= S^1 \times b_2 \times b_3 \times s_2,
\end{aligned}
$$

and two parallel cycles of the form

$$
\begin{aligned}
E_0 &= e_0 \times S^1 \times 1 \times s_0 \\
C_1 &= c_1 \times S^1 \times c_2 \times s_1.
\end{aligned}
$$

We have ignored one vanishing cycle for the third cusp. We can assume that d_1, d_2, e_0 are pairwise different and different from 1, the b_i, c_i and a_0, a_0'. Note that the disks bounding D_0, E_0 are disjoint from V_1, V_2 on the $E(n)$ side because s_0 is the largest parameter. Also the disks bounding D_1, C_1 are disjoint from V_0 because $d_1, c_2 \neq 1$ and the disk bounding B_2 is disjoint from V_0, V_1 because $b_2, b_3 \neq 1$. We can also assume that all disks defined so far are disjoint if they have different indices.

We can now define the nuclei: The nucleus $N(n+1)$ containing V_0 has dual sphere sewed together from the disks bounding A_0 and vanishing disks bounding D_0 and E_0. The nucleus $N(2)$ containing V_1 has dual sphere sewed together from the disks bounding C_1 and vanishing disks bounding A_1 and D_1. Finally, the nucleus $N(2)$ containing V_2 has dual sphere sewed together from the disks bounding B_2 and vanishing disks bounding A_2 and C_2.

To define the Lagrangian triple (T_1, T_2, R) let $T_1 = V_1$ and $R = V_2$. Denote by $c_a \colon S^1 \to S^1 \times S^1$ the embedded curve given by the $(-a, 1)$-torus knot and let T_2 denote the Lagrangian rim torus

$$T_2 = c_a \times S^1 \times r_3$$

in the collar above. Then T_2 represents the class $-aT_1 + R$, hence $R = aT_1 + T_2$. This torus has one positive transverse intersection with the sphere coming from B_2 and a negative transverse intersections with the sphere coming from C_1. This finally proves the claim about the existence of $n-1$ triples of Lagrangian tori in $E(n)$.

Remark 5.74. Since the elliptic nucleus $N(n) \subset E(n)$ is disjoint from the nuclei containing the Lagrangian tori it follows that the knot surgery manifold $E(n)\#_{F=T_K}(M_K \times S^1)$ for any fibred knot K still contains $n-1$ triples of Lagrangian tori as above.

Remark 5.75. Suppose that Y is an arbitrary closed symplectic 4-manifold which contains an embedded symplectic torus T_Y of self-intersection 0, representing an indivisible class. Then the symplectic generalized fibre sum[4] $Y\#_{T_Y=F}E(n)$ contains $n-1$ triples of Lagrangian tori. By the previous remark this is also true for $V = Y\#_{T_Y=F}E(n)\#_{F=T_K}(M_K \times S^1)$ where K is an arbitrary fibred knot. Suppose that the homology of Y is torsion free, T_Y is contained in a cusp neighbourhood in Y and the fibre sum with $E(n)$ is done such that the vanishing cycles on the tori get identified, cf. Section V.6.1. Let g denote the genus of the knot K and let B_Y be a surface in Y which intersects T_Y once. Then the formulas in Proposition 5.62 and equation (5.31) imply that the canonical class of V is given by

$$K_V = \overline{K_Y} + (K_Y B_Y + n + 2g)T_V$$

where $\overline{K_Y} = K_Y - (K_Y B_Y)T_Y$. If $\pi_1(Y) = \pi_1(Y \setminus T_Y) = 1$ then V is again simply-connected. In this way one can construct simply-connected symplectic manifolds not homeomorphic to elliptic surfaces which contain triples of Lagrangian tori.

We return to the general case of a simply-connected symplectic 4-manifold (M, ω) which contains a triple of Lagrangian tori T_1, T_2, R as above. By Lemma 5.72 there exist two symplectic structures ω_+, ω_- on M with the same canonical class K_M as ω such that

- The tori T_1, T_2, R are symplectic with respect to both symplectic forms.
- The form ω_+ induces on T_1, T_2, R a positive volume form.
- The form ω_- induces on T_1 a negative volume form and on T_2, R a positive volume form.

[4]See [35, Section 8 and 9] for a related construction.

We can also achieve that S is symplectic with positive volume form in both cases.

Let K_1 and K_2 be fibred knots of genus h_1, h_2 to be fixed later. Consider the associated oriented 4-manifolds $M_{K_i} \times S^1$ as in the knot surgery construction, for $i = 1, 2$. We denote sections of the fibre bundles

$$\begin{tikzcd} M_{K_i} \times S^1 \arrow[d] & \Sigma_{h_i} \arrow[l] \\ T^2 & \end{tikzcd}$$

by T_{K_i} which are tori of self-intersection 0. Choose an orientation on each torus T_{K_i}. Note that the Lagrangian tori T_1, T_2 in M are oriented *a priori*.

We construct a smooth oriented 4-manifold X as follows: For an integer $m \geq 1$ consider the elliptic surface $E(m)$ and denote an oriented general fibre by F. Let M_0 denote the smooth manifold obtained by the generalized fibre sum of the pairs (M, R) and $(E(m), F)$:

$$M_0 = E(m) \#_{F=R} M.$$

The gluing diffeomorphism is chosen as follows: We choose the natural trivializations for the normal bundles of R in $N(2) \subset M$ and $F \subset E(m)$ given by the fibrations. Consider the push-off R' of R into the boundary of the tubular neighbourhood νR. The vanishing cycles on R' bound two disks of self-intersection -1 in $N(2) \setminus \operatorname{int} \nu R$. There are similar vanishing cycles on the push-off F' into $\partial \nu F$ coming from a cusp in an elliptic fibration of $E(m)$. We choose the gluing such that the push-offs and the vanishing cycles get identified. The disks then sew together pairwise to give two embedded spheres S', S'' in M_0 of self-intersection -2. By choosing two different push-offs given by the same trivializations we can assume that S' and S'' are disjoint. Note that the sphere S which intersects R once and a section for the elliptic fibration on $E(m)$ sew together to give an embedded sphere S_2 in M_0 of self-intersection $-(m+2)$. The sphere S also ensures that M_0 is simply-connected since M is simply-connected.

We denote the torus in M_0 coming from the push-off R' by R_0. There exist two disjoint tori in M_0, which we still denote by T_1, T_2, such that R_0 is homologous to $aT_1 + T_2$ in M_0. Note that the sphere S_1 was disjoint from R and is still contained in M_0. Hence we have the following surfaces in M_0:

- Embedded oriented tori T_1, T_2, R_0 of self-intersection 0 such that R_0 is homologous to $aT_1 + T_2$.
- Disjoint embedded spheres S_1, S_2 where S_2 has self-intersection $S_2^2 = -(m+2)$.
- The sphere S_1 intersects T_1 transversely once, has intersection $-a$ with T_2 and is disjoint from R_0.
- The sphere S_2 intersects R_0, T_2 transversely once and is disjoint from T_1.

We do knot surgery with the fibred knot K_1 along the torus T_1 to get the oriented 4-manifold

$$M_1 = M_0 \#_{T_1 = T_{K_1}} (M_{K_1} \times S^1).$$

Since the manifold M_0 is simply-connected and contains a sphere S_1 intersecting T_1 once, we see that M_1 is simply-connected.

The manifold M_1 contains a torus which we still denote by T_2. We do knot surgery with the fibred knot K_2 along the torus T_2 to get the oriented 4-manifold

$$X = M_1 \#_{T_2 = T_{K_2}} (M_{K_2} \times S^1).$$

Note that the sphere S_2 in M_0 is disjoint from T_1, hence it is still contained in M_1 and intersects T_2 once. This shows that the manifold X is simply-connected.

Lemma 5.76. *The closed oriented 4-manifold*

$$X = E(m)\#_{F=R}M\#_{T_1=T_{K_1}}(M_{K_1}\times S^1)\#_{T_2=T_{K_2}}(M_{K_2}\times S^1)$$

is simply-connected.

Definition 5.77. There exists a surface in X which we call C_2 sewed together from the sphere S_2 and a Seifert surface for K_2. The surface has genus h_2 and self-intersection $-(m+2)$. It intersects the tori T_2 and R_0 in X transversely and positively once and is disjoint from T_1.

Lemma 5.78. *There exists a surface C_1 in X which has intersections $C_1R_0 = a$, $C_1T_1 = 1$ and is disjoint from T_2. We can also assume that $C_1C_2 = 0$.*

Proof. We can construct the surface C_1 explicitly as follows: Note that in $E(m)$ there exists a surface of some genus homologous to a copies of a section which intersects the fibre F transversely a times. A similar surface exists in the nucleus $N(2)\subset E(2)$. These surfaces glue together to give a surface A in M_0 which has intersections $AR_0 = AT_2 = a$ and is disjoint from T_1. We tube this surface to the sphere S_1 which is disjoint from R_0 and has intersections $S_1T_1 = 1$ and $S_1T_2 = -a$. We get a surface B in M_0 with $BR_0 = a$, $BT_1 = 1$ and $BT_2 = 0$. By increasing the genus if necessary we can assume that B is disjoint from T_2. Sewing the surface B to a Seifert surface for K_1 we get a surface C_1' in X with $C_1'R_0 = a$, $C_1'T_1 = 1$. The surface is disjoint from T_2. Suppose that $C_1'C_2$ is non-zero. Then by adding suitable many copies of R_0 to C_1' we get a surface C_1 which has $C_1C_2 = 0$ while the intersections with R_0 and T_1 do not change. □

We now define two symplectic forms ω_X^+ and ω_X^- on X. On the elliptic surface $E(m)$ we can choose a symplectic (Kähler) form ω_E which restricts to a positive volume form on the oriented fibre F. It has canonical class

$$K_E = (m-2)F.$$

Note that the oriented torus R in M is symplectic for ω_+ and ω_- such that both forms induce positive volume forms on R. Hence we can glue both symplectic forms to the symplectic form ω_E to get two symplectic forms ω_0^+,ω_0^- on M_0. By Example 5.64, the canonical class for both symplectic forms on M_0 is

$$K_{M_0} = K_M + mR_0.$$

We now extend the symplectic forms to X. We choose in each fibre bundle $M_{K_i}\times S^1$ for $i = 1,2$ a fibre B_{K_i} and orient the surface B_{K_i} such that $T_{K_i}\cdot B_{K_i} = +1$ with the chosen orientation on T_{K_i}. There exist symplectic structures on the closed 4-manifolds $M_{K_i}\times S^1$ such that both the section and the fibre are symplectic. We choose a symplectic structure ω_2 on $M_{K_2}\times S^1$ which induces a positive volume form on T_{K_2} and B_{K_2}. The canonical class is given by

$$K_2 = (2h_2-2)T_{K_2}.$$

On $M_{K_1}\times S^1$ we define two symplectic forms ω_1^+ and ω_1^-. The form ω_1^+ induces again a positive volume form on T_{K_1}, B_{K_1}. It has canonical class

$$K_1^+ = (2h_1-2)T_{K_1}.$$

The form ω_1^- is given by $-\omega_1^+$. It induces a negative volume form on both T_{K_1} and B_{K_1} and has canonical class

$$K_1^- = -(2h_1-2)T_{K_1}.$$

The oriented torus T_1 in M_0 is symplectic for both forms ω_0^+, ω_0^- such that ω_0^+ induces a positive volume form and ω_0^- a negative volume form. Consider $M_1 = M_0 \#_{T_1 = T_{K_1}} (M_{K_1} \times S^1)$. This closed oriented 4-manifold has two symplectic structures with canonical classes

$$
\begin{aligned}
K_{M_1}^+ &= K_M + mR_0 + 2h_1T_1 \\
K_{M_1}^- &= K_M + mR_0 - 2h_1T_1
\end{aligned}
$$

by Lemma 5.71. The symplectic forms are glued together from the pairs (ω_0^+, ω_1^+) and (ω_0^-, ω_1^-).

The torus T_2 can be considered as a symplectic torus in M_1 such that both symplectic structures induce positive volume forms on it since we can assume that the symplectic forms on M_1 are still of the form ω_0^+ and ω_0^- in a neighbourhood of T_2. Hence on the generalized fibre sum $X = M_1 \#_{T_2 = T_{K_2}} M_{K_2} \times S^1$ we can glue each of the two symplectic forms on M_1 to the symplectic form ω_2 on $M_{K_2} \times S^1$. We get two symplectic structures on X with canonical classes

$$
\begin{aligned}
K_X^+ &= K_M + mR_0 + 2h_1T_1 + 2h_2T_2 \\
K_X^- &= K_M + mR_0 - 2h_1T_1 + 2h_2T_2.
\end{aligned}
$$

This can be written using $R_0 = aT_1 + T_2$ as

$$
\begin{aligned}
K_X^+ &= K_M + (2h_1 + am)T_1 + (2h_2 + m)T_2 \\
K_X^- &= K_M + (-2h_1 + am)T_1 + (2h_2 + m)T_2.
\end{aligned}
$$

Theorem 5.79. *Let (M, ω) be a simply-connected symplectic 4-manifold which contains pairwise disjoint oriented Lagrangian tori T_1, T_2, R of self-intersection zero, representing indivisible classes such that T_1, T_2 are linearly independent in $H_2(M; \mathbb{R})$ and R is homologous to $aT_1 + T_2$ for some integer $a \geq 1$. Suppose that R is contained in a nucleus $N(2) \subset M$, the sphere S in $N(2)$ intersects T_2 transversely once and T_1 is disjoint from $N(2)$. Suppose also that there exists an embedded sphere S_1 in M which is disjoint from $N(2)$ and intersects T_1 transversely once. Then the closed oriented 4-manifold*

$$
X = E(m) \#_{F=R} M \#_{T_1 = T_{K_1}} (M_{K_1} \times S^1) \#_{T_2 = T_{K_2}} (M_{K_2} \times S^1)
$$

is simply-connected and admits two symplectic structures ω_X^+, ω_X^- with canonical classes

$$
\begin{aligned}
K_X^+ &= K_M + (2h_1 + am)T_1 + (2h_2 + m)T_2 \\
K_X^- &= K_M + (-2h_1 + am)T_1 + (2h_2 + m)T_2.
\end{aligned}
$$

For example, suppose that K_M is divisible as a cohomology class by an integer $d \geq 2$. Choose fibred knots K_1, K_2 of genus $h_1 = 1$ and $h_2 = d - 1$ and take $m = 2$ and $a = 1$. Then the 4-manifold X admits two symplectic structures with canonical classes

$$
\begin{aligned}
K_X^+ &= K_M + 4T_1 + 2dT_2 \\
K_X^- &= K_M + 2dT_2.
\end{aligned}
$$

Suppose that d does not divide 4. Note that the class T_1 is indivisible and linearly independent from T_2, cf. Lemma 5.78. Hence the second canonical class is divisible by d while the first canonical class is not divisible by d. Therefore the symplectic structures ω_X^+ and ω_X^- are inequivalent. The manifold X is simply-connected and has invariants

$$
\begin{aligned}
c_1^2(X) &= c_1^2(M) \\
e(X) &= e(M) + 24 \\
\sigma(X) &= \sigma(M) - 16.
\end{aligned}
$$

We can replace h_2 by $rd - 1$ where $r \geq 1$ is an arbitrary integer to get the same divisibility result. By choosing different knots K_2 the formula for the Seiberg-Witten invariants in [38] show that we can find an infinite family $(X_k)_{k \in \mathbb{N}}$ of simply-connected 4-manifolds homeomorphic to X and pairwise non-diffeomorphic such that each X_k admits a pair of inequivalent symplectic structures.

Remark 5.80. If the sphere S which intersects R is symplectic in M then we can assume that the surface C_2 in X of genus h_2 and self-intersection $-(m+2)$ is symplectic for both symplectic structures ω_X^+ and ω_X^- on X.

Remark 5.81. Instead of doing the generalized fibre sum with $E(m)$ in the first step of the construction we could also do a knot surgery with a fibred knot K_0 of genus $h_0 \geq 1$. This has the advantage that both c_1^2 and the signature do not change under the construction. However, the sphere S_2 in M_0 is then replaced by a surface of genus h_0, sewed together from the sphere S in M and a Seifert surface for K_0. Hence we do not have a natural candidate in the last step to show that $M_1 \setminus T_2$ and hence X are simply-connected.

Chapter VI

Geography and the canonical class of symplectic 4-manifolds

Contents

In this chapter we derive some geography results for simply-connected symplectic 4-manifolds and for surfaces of general type whose canonical classes are divisible as a cohomology class by a given integer $d > 1$. Recall that *geography* tries to find for any given pair of integers (x, y) in $\mathbb{Z} \times \mathbb{Z}$ a 4-manifold M with some specified properties such that the Euler characteristic $e(M)$ equals x and the signature $\sigma(M)$ equals y. Note that this can be expressed in an equivalent way in terms of other invariants, since any two invariants out of the set $\{e, \sigma, c_1^2, \chi_h\}$ determine the remaining two. If the 4-manifold is simply-connected then any two invariants together with the *type* of the manifold, i.e. whether it is spin or not, determine the manifold up to homeomorphism by Freedman's theorem [45], cf. Chapter II.

The general geography question for simply-connected symplectic 4-manifolds and for surfaces of general type has been studied by several authors (references can be found in Chapter I). In particular, with the intention to cover a large geographical area, the spin and non-spin case for simply-connected symplectic 4-manifolds has been considered by R. E. Gompf [52], J. Park [111, 112] and B. D. Park and Z. Szabó [110]. The spin and non-spin case for simply-connected complex surfaces of general type has been considered by Z. Chen, U. Persson, C. Peters and G. Xiao in [26, 115, 116]. The geography question for simply-connected symplectic 4-manifolds whose canonical class is divisible by a given integer $d > 1$ has not been considered systematically, as far as we know, except the case $d = 2$ which corresponds to the general case of symplectic spin 4-manifolds.

In Section VI.2 we construct several families of simply-connected symplectic 4-manifolds with divisible canonical class using the generalized fibre sum construction from Chapter V. In particular, in the case of homotopy elliptic surfaces ($c_1^2 = 0$) a complete answer to the geography question is possible, cf. Theorem 6.11. We can also answer the question in the case of simply-connected symplectic 4-manifolds with $c_1^2 > 0$, negative signature and even divisibility (Theorem 6.20) and have some partial results for the corresponding case of odd divisibility in Section VI.2.3. The emphasis of the construction here is to find examples which are as small as possible in terms of c_1^2, the Euler characterstic e and the signature σ. We will also show, by the construction in Section V.6.2, that some of these manifolds have several inequivalent symplectic structures, whose canonical classes have different divisibilities. This can be viewed as a *botany* result for symplectic structures on a given differentiable 4-manifold. Similar examples have been found on homotopy elliptic surfaces by C. T. McMullen and C. H. Taubes [97], I. Smith [126] and S. Vidussi [140]. We did not try to find simply-connected symplectic 4-manifolds with canonical class divisible by an integer $d \geq 3$ and *non-negative* signature, since even without a restriction on the divisibility of the canonical class such 4-manifolds are notoriously difficult to construct.

In the remaining part of this chapter, starting from Section VI.3, we will show that simply-connected complex surfaces of general type with divisible canonical class can be constructed by using branched coverings over smooth curves in pluricanonical linear systems $|nK|$. The main results can be found in Section VI.5. Some of these algebraic surfaces are because of their topological invariants (c_1^2, e and the parity of the divisibility of K) and Freedman's theorem homeomorphic to some of the simply-connected symplectic 4-manifolds constructed with the generalized fibre sum. However, it is quite clear from the construction that these symplectic 4-manifolds have several Seiberg-Witten basic classes. In particular, they can not be diffeomorphic to any minimal surface of general type, cf. Theorem 6.4.

VI.1 General restrictions on the divisibility of the canonical class

We begin by deriving some general restrictions for symplectic 4-manifolds which admit a symplectic structure whose canonical class is divisible by an integer $d \neq 1$.

Let M be a closed, symplectic 4-manifold with canonical class K. Since M admits an almost complex structure, the number

$$\chi_h(M) = \tfrac{1}{4}(e(M) + \sigma(M))$$

has to be an integer. If $b_1(M) = 0$, this number is $\frac{1}{2}(1 + b_2^+(M))$. In particular, in this case, $b_2^+(M)$ has to be an odd integer and $\chi_h(M) > 0$. As explained in Chapter II, there are two further constraints if M is spin:

$$c_1^2(M) \equiv 0 \bmod 8 \quad \text{and} \quad c_1^2(M) \equiv 8\chi_h(M) \bmod 16,$$

where $c_1^2(M) = 2e(M) + 3\sigma(M)$. We say that K is divisible by an integer d if there exists a cohomology class $A \in H^2(M;\mathbb{Z})$ with $K = dA$.

Lemma 6.1. *Let (M, ω) be a closed symplectic 4-manifold. Suppose that the canonical class K is divisible by an integer d. Then $c_1^2(M)$ is divisible by d^2 if d is odd and by $2d^2$ if d is even.*

Proof. If K is divisible by d we can write $K = dA$, where $A \in H^2(M;\mathbb{Z})$. The equation $c_1^2(M) = K^2 = d^2A^2$ implies that $c_1^2(M)$ is divisible by d^2 in any case. If d is even, then $w_2(M) \equiv K \equiv 0$ mod 2, hence M is spin and the intersection form Q_M is even. This implies that A^2 is divisible by 2, hence $c_1^2(M)$ is divisible by $2d^2$. □

Note that the case $c_1^2(M) = 0$ is special, since there are no restrictions from this lemma (see Section VI.2.1). For the general case of spin symplectic 4-manifolds ($d = 2$) we recover the constraint that c_1^2 is divisible by 8.

Further restrictions come from the adjunction inequality

$$2g - 2 = K \cdot C + C \cdot C,$$

where C is an embedded symplectic surface of genus g, oriented by the symplectic form.

Lemma 6.2. *Let (M, ω) be a closed symplectic 4-manifold. Suppose that the canonical class K is divisible by an integer d.*

- *If M contains a symplectic surface of genus g and self-intersection 0, then d divides $2g - 2$.*
- *If $d \neq 1$ then M is minimal. If the manifold M is in addition simply-connected, then it is irreducible.*

Proof. The first part follows immediately by the adjunction formula. If M is not minimal (see Chapter III) then it contains a symplectically embedded sphere S of self-intersection (-1). The adjunction formula can be applied and yields $K \cdot S = -1$, hence K has to be indivisible. The claim about irreducibility follows from Corollary 3.4 in Chapter III. □

The canonical class of a 4-manifold M with $b_2^+ \geq 2$ is a Seiberg-Witten basic class, i.e. it has non-vanishing Seiberg-Witten invariant. Hence only finitely many classes in $H^2(M;\mathbb{Z})$ can be the canonical class of a symplectic structure on M.

The following is proved in [89].

Theorem 6.3. *Let M be a (smoothly) minimal closed 4-manifold with $b_2^+ = 1$ which admits a symplectic structure. Then the canonical classes of all symplectic structures on M are equal up to sign.*

If M is a Kähler surface, we can also consider the canonical class of the Kähler form.

Theorem 6.4. *Suppose that M is a minimal Kähler surface with $b_2^+ > 1$.*

- *If M is of general type, then $\pm K_M$ are the only Seiberg-Witten basic classes of M.*
- *If N is another minimal Kähler surface with $b_2^+ > 1$ and $\phi\colon M \to N$ a diffeomorphism, then $\phi^* K_N = \pm K_M$.*

For the proofs see [48], [102] and [145]. Note that the second part of this theorem is not true in general for the canonical class of *symplectic structures* on 4-manifolds with $b_2^+ > 1$: there are examples of 4-manifolds M which admit several symplectic structures whose canonical classes are different elements in $H^2(M;\mathbb{Z})$ and lie in disjoint orbits for the action of the group of orientation preserving self-diffeomorphisms on $H^2(M;\mathbb{Z})$ [97]. In some cases the canonical classes have different divisibilities and for that reason can not be permuted by a diffeomorphism, cf. [126], [140] and examples in the following sections.

It is useful to define the (maximal) divisibility of the canonical class, at least in the case that $H^2(M;\mathbb{Z})$ is torsion free.

Definition 6.5. Suppose H is a finitely generated free abelian group. For $a \in H$ let

$$d(a) = \max\{k \in \mathbb{N}_0 \mid \text{there exists an element } b \in H,\, b \neq 0, \text{ with } a = kb\}.$$

We call $d(a)$ the *divisibility* of a (or sometimes, to emphasize, the *maximal* divisibility). With this definition the divisibility of a is 0 if and only $a = 0$. We call a *indivisible* if $d(a) = 1$.

In particular, if M is a simply-connected manifold, the cohomology group $H^2(M;\mathbb{Z})$ is torsion free and the divisibility of the canonical class $K \in H^2(M;\mathbb{Z})$ is well-defined.

Proposition 6.6. *Suppose that M is a simply-connected closed 4-manifold which admits at least two symplectic structures whose canonical classes have different divisibilities. Then M is not diffeomorphic to a complex surface.*

Proof. The assumptions imply that M has a symplectic structure whose canonical class has divisibility > 1. By Lemma 6.2, the manifold M is (smoothly) minimal. Suppose that M is diffeomorphic to a complex surface. The Kodaira-Enriques classification implies that M is diffeomorphic to $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ an elliptic surface $E(n)_{p,q}$, with $n \geq 1$ and p, q coprime, or to a surface of general type.

Note that M cannot be diffeomorphic to $\mathbb{C}P^2$ because the structure of the cup product on cohomology and $c_1^2 = 9$ imply that the canonical class of every symplectic structure has divisibility 3. A similar argument applies to $\mathbb{C}P^1 \times \mathbb{C}P^1$. The SW-basic classes of $E(n)_{p,q}$ are known [37]. They consist of the set of classes of the form kf where f denotes the indivisible class $f = F/pq$ and k is an integer

$$k \equiv npq - p - q \bmod 2, \quad |k| \leq npq - p - q.$$

By a theorem of Taubes [133] it follows that the canonical class of any symplectic structure on $E(n)_{p,q}$ is given by $\pm(npq - p - q)f$. Hence there is only one possible divisibility. This follows for surfaces of general type by Theorem 6.4. ☐

VI.2 Constructions using the generalized fibre sum

We begin with the case $c_1^2 < 0$. The following theorem is due to C. H. Taubes [134] in the case $b_2^+ \geq 2$ and to A. K. Liu [90] in the case $b_2^+ = 1$.

Theorem 6.7. *Let M be a closed, symplectic 4-manifold. Suppose that M is minimal.*

- *If $b_2^+(M) \geq 2$, then $K^2 \geq 0$.*
- *If $b_2^+(M) = 1$ and $K^2 < 0$, then M is a ruled surface, i.e. an S^2-bundle over a surface (of genus ≥ 2).*

Since ruled surfaces over irrational curves are not simply-connected, any simply-connected, symplectic 4-manifold M with $K^2 < 0$ is not minimal. By Lemma 6.2 this implies that K is indivisible, $d(K) = 1$.

Let $(\chi_h, c_1^2) = (n, -r)$ be any lattice point, with $n, r \geq 1$ and M a simply-connected symplectic 4-manifold with these invariants. Since M is not minimal, we can blow down a (-1)-sphere in M to get a symplectic manifold M' such that there exists a diffeomorphism

$$M = M' \# \overline{\mathbb{C}P^2}.$$

Since

$$e(M') = e(M) - 1$$
$$\sigma(M') = \sigma(M) + 1,$$

the manifold M' has invariants

$$(\chi_h, c_1^2) = (n, -r + 1).$$

Hence by blowing down r spheres in M of self-intersection -1 we get a simply-connected symplectic 4-manifold N with $M = N \# r\overline{\mathbb{C}P^2}$ and invariants $(\chi_h, c_1^2) = (n, 0)$.

Conversely, consider the manifold

$$M = E(n) \# r\overline{\mathbb{C}P^2}.$$

Then M is a simply-connected symplectic 4-manifold with indivisible K. Since $\chi_h(E(n)) = n$ and $c_1^2(E(n)) = 0$, this implies

$$(\chi_h(M), c_1^2(M)) = (n, -r).$$

Hence the point $(n, -r)$ can be realized by a simply-connected symplectic 4-manifold.

VI.2.1 Homotopy elliptic surfaces

We now consider the case $c_1^2 = 0$.

Definition 6.8. A closed, simply-connected 4-manifold M is called a *homotopy elliptic surface* if M is homeomorphic to a relatively minimal, simply-connected elliptic surface, i.e. to a surface of the form $E(n)_{p,q}$ with p, q coprime, cf. Section II.3.5.

Note that by definition homotopy elliptic surfaces M are simply-connected and have invariants

$$c_1^2(M) = 0$$
$$e(M) = 12n$$
$$\sigma(M) = -8n.$$

The integer n is equal to $\chi_h(M)$. In particular, symplectic homotopy elliptic surfaces have $K^2 = 0$. We want to prove the following converse.

Lemma 6.9. *Let M be a closed, simply-connected, symplectic 4-manifold with $K^2 = 0$. Then M is a homotopy elliptic surface.*

Proof. Since M is almost complex, the number $\chi_h(M)$ is an integer. The Noether formula

$$\chi_h(M) = \tfrac{1}{12}(K^2 + e(M)) = \tfrac{1}{12} e(M)$$

implies that $e(M)$ is divisible by 12, hence $e(M) = 12k$ for some $k > 0$. Together with the equation

$$0 = K^2 = 2e(M) + 3\sigma(M),$$

it follows that $\sigma(M) = -8k$. Suppose that M is non-spin. If k is odd, then M has the same Euler characteristic, signature and type as $E(k)$. If k is even, then M has the same Euler characteristic, signature and type as the non-spin manifold $E(k)_2$. Since M is simply-connected, M is homeomorphic to the corresponding elliptic surface by Freedman's theorem [45].

Suppose that M is spin. Then the signature is divisible by 16, due to Rochlin's theorem. Hence the integer k above has to be even. Then M has the same Euler characteristic, signature and type as the spin manifold $E(k)$. Again by Freedman's theorem, M is homeomorphic to $E(k)$. □

Lemma 6.10. *Suppose that M is a symplectic homotopy elliptic surface such that the divisibility of K is even. Then $\chi_h(M)$ is even.*

Proof. The assumption implies that M is spin. The Noether formula then shows that $\chi_h(M)$ is even, since $K^2 = 0$ and $\sigma(M)$ is divisible by 16. □

The next theorem shows that this is the only restriction on the divisibility of the canonical class K for symplectic homotopy elliptic surfaces.

Theorem 6.11. *Let n and d be positive integers. If d is even, suppose in addition that n is even. Then there exists a symplectic homotopy elliptic surface (M, ω) with $\chi_h(M) = n$ whose canonical class K has divisibility equal to d.*

Proof. If n is 1 or 2, the symplectic manifold can be realized as an elliptic surface. Recall from Section II.3.3 that the canonical class of an elliptic surface $E(n)_{p,q}$ with p, q coprime is given by

$$K = (npq - p - q)f,$$

where f is indivisible and $F = pqf$ denotes the class of a generic fibre. For $n = 1$ and d odd we can take the surface $E(1)_{d+2,2}$, since

$$(d + 2)2 - (d + 2) - 2 = d.$$

For $n = 2$ and d arbitrary we can take $E(2)_{d+1} = E(2)_{d+1,1}$, since

$$2(d + 1) - (d + 1) - 1 = d.$$

We now consider the case $n \geq 1$ in general. We separate the proof into several cases. Suppose that $\mathbf{d = 2k}$ and $\mathbf{n = 2m}$ are even, with $k, m \geq 1$. Consider the elliptic surface $E(n)$. It contains a general fibre F which is a symplectic torus of self-intersection 0. In addition, it contains a rim torus R which arises from a decomposition of $E(n)$ as a fibre sum $E(n) = E(n-1)\#_F E(1)$. The rim torus R has self-intersection 0 and a dual (Lagrangian) 2-sphere S, which has intersection $RS = 1$. We can assume that R and S are disjoint from the fibre F. The rim torus is in a natural way Lagrangian. By a perturbation of the symplectic form we can assume that it becomes symplectic. We give R the orientation induced by the symplectic form. The proof consists in doing knot surgery along the fibre F and the rim torus R (see Section V.4.1).[1]

Let K_1 be a fibred knot of genus $g_1 = m(k - 1) + 1$. We do knot surgery along F with the knot K_1 to get a new symplectic 4-manifold M_1. The elliptic fibration $E(n) \to \mathbb{C}P^1$ has a section which shows that the meridian of F, which is the S^1-fibre of $\partial\nu F \to F$, bounds a disk in $E(n) \setminus \operatorname{int} \nu F$. This implies that the complement of F in $E(n)$ is simply-connected (see Corollary A.4), hence the manifold M_1 is again simply-connected. By the knot surgery construction, the manifold M_1 is homeomorphic to $E(n)$. The canonical class is given by formula (5.31):

$$\begin{aligned} K_{M_1} &= (n-2)F + 2g_1 F \\ &= (2m - 2 + 2mk - 2m + 2)F \\ &= 2mkF. \end{aligned}$$

Here we have identified the cohomology of M_1 and $E(n)$ as explained in connection with formula (5.31). Note that the rim torus R is still an embedded oriented symplectic torus in M_1 and has a dual

[1] Generalized fibre sums along rim tori have been considered e.g. in [35], [40], [52], [60] and [142].

2-sphere S, because we can assume that the knot surgery takes place in a small neighbourhood of F disjoint from R and S. In particular, the complement of R in M_1 is simply-connected. Let K_2 be a fibred knot of genus $g_2 = k$ and M the result of knot surgery on M_1 along R. Then M is a simply-connected symplectic 4-manifold homeomorphic to $E(n)$. The canonical class is given by

$$K = 2mkF + 2kR.$$

The class K is divisible by $2k$. The sphere S sews together with a Seifert surface for K_2 to give a surface C in M with $C \cdot R = 1$ and $C \cdot F = 0$, hence $C \cdot K = 2k$. This implies that the divisibility of K is precisely $d = 2k$.

Suppose that $\mathbf{d = 2k + 1}$ and $\mathbf{n = 2m + 1}$ are odd, with $k \geq 0$ and $m \geq 1$. We consider the elliptic surface $E(n)$ and do a similar construction. Let K_1 be a fibred knot of genus $g_1 = 2km + k + 1$ and do knot surgery along F as above. We get a simply-connected symplectic 4-manifold M_1 with canonical class

$$\begin{aligned}
K_{M_1} &= (n-2)F + 2g_1F \\
&= (2m + 1 - 2 + 4km + 2k + 2)F \\
&= (4km + 2k + 2m + 1)F \\
&= (2m + 1)(2k + 1)F.
\end{aligned}$$

Next we consider a fibred knot K_2 of genus $g_2 = 2k+1$ and do knot surgery along the rim torus R. The result is a simply-connected symplectic 4-manifold M homeomorphic to $E(n)$ with canonical class

$$K = (2m + 1)(2k + 1)F + 2(2k + 1)R.$$

The class K is divisible by $(2k + 1)$. The same argument as above shows that there exists a surface C in M with $C \cdot K = 2(2k + 1)$. We claim that the divisibility of K is precisely $(2k + 1)$: Note that M is still homeomorphic to $E(n)$ by the knot surgery construction. Since n is odd, the manifold M is not spin and this implies that 2 does not divide K (an explicit surface with odd intersection number can be constructed from a section of $E(n)$ and a Seifert surface for the knot K_1. This surface has self-intersection number $-n$ and intersection number $(2m + 1)(2k + 1)$ with K.)

To cover the case $m = 0$ (corresponding to $n = 1$) we can do knot surgery on the elliptic surface $E(1)$ along a general fibre F with a knot K_1 of genus $g_1 = k + 1$. The resulting manifold M_1 has canonical class

$$K_{M_1} = -F + (2k + 2)F = (2k + 1)F.$$

Suppose that $\mathbf{d = 2k + 1}$ is odd and $\mathbf{n = 2m}$ is even, with $k \geq 0$ and $m \geq 1$. We consider the elliptic surface $E(n)$ and perform a logarithmic transformation along F of index 2. Let f denote the multiple fibre such that F is homologous to $2f$. There exists a 2-sphere in $E(n)_2$ which intersects f in a single point (for a proof see the following lemma). In particular, the complement of f in $E(n)_2$ is simply-connected. The canonical class of $E(n)_2 = E(n)_{2,1}$ is given by

$$K = (2n - 3)f.$$

We can assume that the torus f is symplectic (e.g. by considering the logarithmic transformation to be done on the complex surface $E(n)$ to get the complex surface $E(n)_2$). Let K_1 be a fibred knot of genus $g_1 = 4km + k + 2$. We do knot surgery along f with K_1 as above. The result is a simply-connected symplectic 4-manifold homeomorphic to $E(n)_2$. The canonical class is given by

$$\begin{aligned} K_{M_1} &= (2n-3)f + 2g_1 f \\ &= (4m - 3 + 8km + 2k + 4)f \\ &= (8km + 4m + 2k + 1)f \\ &= (4m+1)(2k+1)f. \end{aligned}$$

We now consider a fibred knot K_2 of genus $g_2 = 2k+1$ and do knot surgery along the rim torus R. We get a simply-connected symplectic 4-manifold M homeomorphic to $E(n)_2$ with canonical class

$$K = (4m+1)(2k+1)f + 2(2k+1)R.$$

A similar argument as above shows that the divisibility of K is $d = 2k+1$. □

Lemma 6.12. *Let $p \geq 1$ be an integer and f the multiple fibre in $E(n)_p$. Then there exists a sphere in $E(n)_p$ which intersects f transversely in one point.*

Proof. We can think of the logarithmic transformation as gluing $T^2 \times D^2$ into $E(n) \setminus \operatorname{int} \nu F$ by a certain diffeomorphism $\phi\colon T^2 \times S^1 \to \partial\nu F$. The fibre f corresponds to $T^2 \times \{0\}$. Consider a disk of the form $\{*\} \times D^2$. It intersects f once and its boundary maps under ϕ to a certain simple closed curve on $\partial\nu F$. Since $E(n) \setminus \operatorname{int} \nu F$ is simply-connected, this curve bounds a disk in $E(n) \setminus \operatorname{int} \nu F$. The union of this disk and the disk $\{*\} \times D^2$ is a sphere in $E(n)_p$ which intersects f once. □

Remark 6.13. In Theorem 6.11, and similarly in the following theorems, it is possible to construct *infinitely* many homeomorphic homotopy elliptic surfaces $(M_r)_{r\in\mathbb{N}}$ with $\chi_h(M_r) = n$ and the following properties:

(1.) The 4-manifolds $(M_r)_{r\in\mathbb{N}}$ are pairwise non-diffeomorphic.

(2.) For every index $r \in \mathbb{N}$ the manifold M_r admits a symplectic structure whose canonical class has divisibility equal to d.

This follows because we can vary in each case the knot K_1 and its genus g_1. For example in the first case in the proof above (d and n even) we can choose $h = bmk - m + 1$ where $b \geq 1$ is arbitrary to get the same divisibility. The claim then follows by the formula for the Seiberg-Witten invariants of knot surgery manifolds [38].

We can give another construction of homotopy elliptic surfaces as in Theorem 6.11 that also yields a second inequivalent symplectic structure on the same manifold. Let $n \geq 3$ and d be positive integers. If d is even, assume that n is even. We consider two cases. Suppose that $\mathbf{d = 2k+1 \geq 3}$ is odd. Consider the elliptic surface $E(n-1)$. By Example 5.73 the 4-manifold $E(n-1)$ has two disjoint embedded nuclei $N(2)$, each of which contains an oriented Lagrangian rim torus R and T_1 coming from a splitting $E(n-1) = E(n-2)\#_{F=F}E(1)$. There also exists a (connected) oriented Lagrangian rim torus T_2 representing $R - T_1$ in homology. We then use the construction for Theorem 5.79: Let K_1, K_2 be fibred knots of genus $h_1 = h_2 = k$. We first do a generalized fibre sum along R with an elliptic surface $E(1)$ (along a general fibre in $E(1)$) and then knot surgeries along the tori T_1, T_2. We get a simply-connected 4-manifold

$$X = E(1)\#_{F=R}E(n-1)\#_{T_1=T_{K_1}}(M_{K_1} \times S^1)\#_{T_2=T_{K_2}}(M_{K_2} \times S^1).$$

There exist two symplectic structures ω_X^+, ω_X^- on the smooth manifold X whose canonical classes are given by

$$K_X^+ = (n-3)F + dT_1 + dT_2$$
$$K_X^- = (n-3)F + (-d+2)T_1 + dT_2.$$

The manifold X has invariants

$$c_1^2(X) = 0$$
$$e(X) = 12n$$
$$\sigma(X) = -8n.$$

Note that the general fibre F of $E(n-1)$ is still an oriented embedded torus in X of self-intersection 0. We can assume that F is symplectic with respect to the symplectic forms ω_X^+, ω_X^- on X, both inducing a positive volume form. The sphere giving a section for an elliptic fibration of $E(n-1)$ is also still contained in X. Consider the even integer $n(d-1)+(d+3)$ and a fibred knot K_3 of genus h_3 with $2h_3 = n(d-1)+(d+3)$. We can do knot surgery with this knot along the general fibre to get a simply-connected 4-manifold W. It has two symplectic structures with canonical classes

$$K_W^+ = d(n+1)F + dT_1 + dT_2$$
$$K_W^- = d(n+1)F + (-d+2)T_1 + dT_2.$$

There exists a surface C_1 in W which intersects T_1 once and is disjoint from T_2 and F, cf. the construction in Lemma 5.78. Since $d \geq 3$ the first canonical class is divisible by d while the second is not. Note that W is because of its invariants and Lemma 6.9 a homotopy elliptic surface with $\chi_h(W) = n$.

Similarly suppose that $\mathbf{d = 2k \geq 6}$ and $n \geq 4$ are even. We do the same construction is above: This time we start with $E(n-2)$. Let K_1, K_2 be fibred knots of genus $h_1 = h_2 = k-1$. We first do a Gompf sum on $E(m-2)$ along the rim torus R with the elliptic surface $E(2)$ and then knot surgeries along the tori T_1, T_2. We get a simply-connected 4-manifold

$$X = E(2)\#_{F=R}E(n-2)\#_{T_1=T_{K_1}}(M_{K_1}\times S^1)\#_{T_2=T_{K_2}}(M_{K_2}\times S^1)$$

with two symplectic structures ω_X^+, ω_X^-, whose canonical classes are

$$K_X^+ = (n-4)F + dT_1 + dT_2$$
$$K_X^- = (n-4)F + (-d+4)T_1 + dT_2.$$

Consider the even integer $n(d-1)+4$ (note that n is even) and a fibred knot K_3 of genus h_3 with $2h_3 = n(d-1)+4$. We do knot surgery along the symplectic torus F in X with this knot to get a simply-connected 4-manifold W. It has two symplectic structures with canonical classes

$$K_W^+ = dnF + dT_1 + dT_2$$
$$K_W^- = dnF + (-d+4)T_1 + dT_2.$$

Since $d \geq 6$ the first canonical class is divisible by d while the second is not, again by the surface from Lemma 5.78. The manifold W is a homotopy elliptic surface with $\chi_h(W) = n$.

Proposition 6.14. *Let $n \geq 3$ and d be positive integers with $d \neq 1, 2, 4$. If d is even, suppose in addition that n is even. Then there exists a homotopy elliptic surface W with $\chi_h(W) = n$ which admits at least two inequivalent symplectic structures ω_1, ω_2. The canonical class of ω_1 has divisibility d while the canonical class of ω_2 is not divisible by d.*

This construction can be generalized since the elliptic surface $E(N+1)$ contains N pairs of nuclei $N(2)$ as above which come from iterated splittings $E(N+1) = E(N)\#_F E(1)$, $E(N) = E(N-1)\#_F E(1)$, etc. (see Example 5.73). These nuclei generate $2N$ summands of the form

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$$

in the intersection form of $E(N+1)$. The construction can be done on each pair of nuclei $N(2)$ separately by a mild generalization of Lemma 5.72 (note that the construction in this lemma changes the symplectic structure only in small tubular neighbourhood of the Lagrangian surfaces). Thus on the same homotopy elliptic surface Y possibly more divisors of d can be realized as the divisibility of a canonical class. We make the following definition:

Definition 6.15 (Definition of the set Q). Let $N \geq 0$, $d \geq 1$ be integers and $d_0, \ldots, d_N$ positive integers dividing d, where $d = d_0$. If d is even, assume that all $d_1, \ldots, d_N$ are even. We define a set Q of positive integers as follows:

- If d is either odd or not divisible by 4, let Q be the set consisting of the greatest common divisors of all (non-empty) subsets of $\{d_0, \ldots, d_N\}$.
- If d is divisible by 4 we can assume by reordering that $d_1, \ldots, d_s$ are those elements such that d_i is divisible by 4 while $d_{s+1}, \ldots, d_N$ are those elements such that d_i is not divisible by 4, where $s \geq 0$ is some integer. Then Q is defined as the set of integers consisting of the greatest common divisors of all (non-empty) subsets of $\{d_0, \ldots, d_s, 2d_{s+1}, \ldots, 2d_N\}$.

We can now formulate the main theorem on the existence of inequivalent symplectic structures on homotopy elliptic surfaces:

Theorem 6.16. *Let $N, d \geq 1$ be integers and $d_0, \ldots, d_N$ positive integers dividing d, as in Definition 6.15. Let Q be the associated set of greatest common divisors. Choose an integer $n \geq 3$ as follows:*

- *If d is odd let n be an arbitrary integer with $n \geq 2N + 1$.*
- *If d is even let n be an even integer with $n \geq 3N + 1$.*

Then there exists a homotopy elliptic surface W with $\chi_h(W) = n$ and the following property: For each integer $q \in Q$ the manifold W admits a symplectic structure whose canonical class K has divisibility equal to q. Hence W admits at least $|Q|$ many inequivalent symplectic structures.

Proof. Suppose that **d is odd**. Then all divisors $d_1, \ldots, d_N$ are odd. Let a_i, h_i and h be the integers defined by

$$\begin{aligned} a_i &= d + d_i \\ 2h_i &= d - d_i \\ 2h &= d - 1, \end{aligned}$$

for every $1 \leq i \leq N$. Let l be an integer $\geq N + 1$ and consider the elliptic surface $E(l)$. It contains N pairs of disjoint nuclei $N(2)$ where each pair contains Lagrangian rim tori T_1^i and R^i, representing indivisible classes, which arise by splitting off an $E(1)$ summand, cf. Example 5.73. There also exists for each pair a third disjoint Lagrangian rim torus T_2^i representing $R^i - a_i T_1^i$.

We do the construction from Section V.6.2 on each triple T_1^i, T_2^i, R^i in $E(l)$ $(1 \leq i \leq N)$: We first do a generalized fibre sum of $E(l)$ with $E(1)$ along R^i and then knot surgeries along T_1^i and T_2^i with fibred knots of genus h_i and h, respectively. We get a (simply-connected) homotopy elliptic surface X with $\chi_h(X) = l + N$. By Theorem 5.79 the 4-manifold X has 2^N symplectic structures with canonical classes

$$\begin{aligned} K_X &= (l-2)F + \sum_{i=1}^{N} \left((\pm 2h_i + a_i)T_1^i + (2h+1)T_2^i\right) \\ &= (l-2)F + \sum_{i=1}^{N} \left((\pm(d-d_i) + d + d_i)T_1^i + dT_2^i\right). \end{aligned}$$

Here F denotes the torus in X coming from a general fibre in $E(l)$ and the $\pm$-signs in each summand can be varied independently. We can assume that F is symplectic with positive induced volume form for all 2^N symplectic structures on X. Consider the even integer $l(d-1)+2$ and let K be a fibred knot of genus g with $2g = l(d-1)+2$. We do knot surgery with K along the symplectic torus F to get a homotopy elliptic surface W with $\chi_h(W) = l + N$ which has symplectic structures whose canonical classes are

$$\begin{aligned} K_W &= (l-2+2g)F + \sum_{i=1}^{N} \left((\pm(d-d_i) + d + d_i)T_1^i + dT_2^i\right) \\ &= dlF + \sum_{i=1}^{N} \left((\pm(d-d_i) + d + d_i)T_1^i + dT_2^i\right). \end{aligned}$$

Suppose that $q \in Q$ is the greatest common divisor of certain elements $\{d_i\}_{i\in I}$, where I is a non-empty subset of $\{0,\ldots,N\}$. Let J be the complement of I in $\{0,\ldots,N\}$. We choose the minus sign for each i in I and the plus sign for each j in J to get a symplectic structure ω_q on W. It has canonical class

$$K_W = dlF + \sum_{i\in I}(2d_iT_1^i + dT_2^i) + \sum_{j\in J}(2dT_1^j + dT_2^j).$$

Note that 2 does not divide d because d is odd. Considering the surfaces from Definition 5.77 and Lemma 5.78 for each Lagrangian pair (T_1^i, T_2^i) implies that the canonical class K_W of ω_q has divisibility equal to q.

Suppose that **d is even but not divisible by 4**. We can write $d = 2k$ and $d_i = 2k_i$ for all $i = 1,\ldots,N$. The assumption implies that all integers k, k_i are odd. Let a_i, h_i and h be the integers defined by

$$\begin{aligned} 2a_i &= k + k_i \\ 2h_i &= k - k_i \\ h &= k - 1. \end{aligned}$$

Let l be an even integer $\geq N+1$. For each of the N pairs of nuclei $N(2)$ in $E(l)$ we consider a triple of Lagrangian tori with $T_2^i = R^i - a_iT_1^i$. We do the following construction on each triple T_1^i, T_2^i, R^i, with $1 \leq i \leq N$ in $E(l)$: We first do a generalized fibre sum of $E(l)$ with $E(2)$ along R^i and then knot surgeries along T_1^i and T_2^i with fibred knots of genus h_i and h. We get a homotopy elliptic surface X

with $\chi_h(X) = l + 2N$. The 4-manifold X has 2^N symplectic structures with canonical classes

$$\begin{aligned} K_X &= (l-2)F + \sum_{i=1}^{N} \left((\pm 2h_i + 2a_i)T_1^i + (2h+2)T_2^i\right) \\ &= (l-2)F + \sum_{i=1}^{N} \left((\pm(k-k_i) + k + k_i)T_1^i + dT_2^i\right). \end{aligned}$$

Consider a fibred knot K of genus g where $2g = l(d-1)+2$ (note that l is even). Doing knot surgery with K along the symplectic torus F in X we get a homotopy elliptic surface W with $\chi_h(W) = l+2N$ which has symplectic structures whose canonical classes are

$$\begin{aligned} K_W &= (l-2+2g)F + \sum_{i=1}^{N} \left((\pm(k-k_i) + k + k_i)T_1^i + dT_2^i\right) \\ &= dlF + \sum_{i=1}^{N} \left((\pm(k-k_i) + k + k_i)T_1^i + dT_2^i\right). \end{aligned}$$

Let $q \in Q$ be the greatest common divisor of elements d_i where $i \in I$ for some non-empty index set I with complement J in $\{0, \ldots, N\}$. Choosing the plus and minus signs as before, we get a symplectic structure ω_q on W with canonical class

$$K_W = dlF + \sum_{i \in I}(d_i T_1^i + dT_2^i) + \sum_{j \in J}(dT_1^i + dT_2^i). \tag{6.1}$$

As above, the canonical class of ω_q has divisibility equal to q.

Finally we consider the case that **d is divisible by** 4. We can write $d = 2k$ and $d_i = 2k_i$ for all $i = 1, \ldots, N$. We can assume that the divisors are ordered as in Definition 6.15, i.e. $d_1, \ldots, d_s$ are those elements such that d_i is divisible by 4 while $d_{s+1}, \ldots, d_N$ are those elements such that d_i is not divisible by 4. This is equivalent to $k_1, \ldots, k_s$ being even and $k_{s+1}, \ldots, k_N$ odd. Let a_i and h_i be the integers defined by

$$\begin{aligned} 2a_i &= k + k_i \\ 2h_i &= k - k_i, \end{aligned}$$

for $i = 1, \ldots, s$ and

$$\begin{aligned} 2a_i &= k + 2k_i \\ 2h_i &= k - 2k_i, \end{aligned}$$

for $i = s+1, \ldots, N$. We also define $h = k-1$. Let l be an even integer $\geq N+1$. We consider the same construction as above starting from $E(l)$ to get a homotopy elliptic surface X with $\chi_h(X) = l + 2N$ that has 2^N symplectic structures with canonical classes

$$\begin{aligned} K_X &= (l-2)F + \sum_{i=1}^{N} \left((\pm 2h_i + 2a_i)T_1^i + (2h+2)T_2^i\right) \\ &= (l-2)F + \sum_{i=1}^{s} \left((\pm(k-k_i) + k + k_i)T_1^i + dT_2^i\right) + \sum_{i=s+1}^{N} \left((\pm(k-2k_i) + k + 2k_i)T_1^i + dT_2^i\right). \end{aligned}$$

We then do knot surgery with a fibred knot K of genus g with $2g = l(d-1)+2$ along the symplectic torus F in X to get a homotopy elliptic surface W with $\chi_h(W) = l + 2N$ which has symplectic structures whose canonical classes are

$$\begin{aligned} K_W &= (l-2+2g)F + \sum_{i=1}^{N} \left((\pm(k-k_i)+k+k_i)T_1^i + dT_2^i\right) \\ &= dlF + \sum_{i=1}^{s} \left((\pm(k-k_i)+k+k_i)T_1^i + dT_2^i\right) + \sum_{i=s+1}^{N} \left((\pm(k-2k_i)+k+2k_i)T_1^i + dT_2^i\right). \end{aligned} \tag{6.2}$$

Let q be an element in Q. Note that this time

$$\begin{aligned} (k-k_i)+(k_i+k) &= d \\ -(k-k_i)+(k_i+k) &= d_i \end{aligned}$$

for $i \leq s$ while

$$\begin{aligned} (k-2k_i)+(k+2k_i) &= d \\ -(k-2k_i)+(k+2k_i) &= 2d_i \end{aligned}$$

for $i \geq s+1$. Since q is the greatest common divisor of certain elements d_i for $i \leq s$ and $2d_i$ for $i \geq s+1$ this shows that we can choose the plus and minus signs appropriately to get a symplectic structure ω_q on W whose canonical class has divisibility equal to q. □

Example 6.17. Suppose that $d = 45$ and choose $d_0 = 45, d_1 = 15, d_2 = 9, d_3 = 5$. Then $Q = \{45, 15, 9, 5, 3, 1\}$ and for every integer $n \geq 7$ there exists a homotopy elliptic surfaces W with $\chi_h(W) = n$ that admits at least 6 inequivalent symplectic structures whose canonical classes have divisibility equal to the elements in Q. One can also find an infinite family of homeomorphic but non-diffeomorphic manifolds of this kind.

Corollary 6.18. *Let $m \geq 1$ be an arbitrary integer.*

- *There exist simply-connected non-spin 4-manifolds W homeomorphic to the elliptic surfaces $E(2m+1)$ and $E(2m+2)_2$ which admit at least 2^m inequivalent symplectic structures.*
- *There exist simply-connected spin 4-manifolds W homeomorphic to $E(6m-2)$ and $E(6m)$ which admit at least 2^{2m-1} inequivalent symplectic structures and spin manifolds homeomorphic to $E(6m+2)$ which admit at least 2^{2m} inequivalent symplectic structures.*

Proof. Choose N pairwise different odd prime numbers $p_1, \ldots, p_N$. Let $d = d_0 = p_1 \cdot \ldots \cdot p_N$ and consider the integers

$$\begin{aligned} d_1 &= p_2 \cdot p_3 \cdot \ldots \cdot p_N \\ d_2 &= p_1 \cdot p_3 \cdot \ldots \cdot p_N \\ &\vdots \\ d_N &= p_1 \cdot \ldots \cdot p_{N-1}, \end{aligned}$$

obtained by deleting the corresponding prime in d. Then the associated set Q of greatest common divisors consists of all products of the p_i where each prime occurs at most once: If such a product x does not contain precisely the primes $p_{i_1}, \ldots, p_{i_r}$ then x is the greatest common divisor of $d_{i_1}, \ldots, d_{i_r}$. The set Q has 2^N elements.

Let $m \geq 1$ be an arbitary integer. Setting $N = m$ there exists by Theorem 6.16 for every integer $n \geq 2N + 1 = 2m + 1$ a homotopy elliptic surface W with $\chi_h(W) = n$ which has 2^m symplectic structures realizing all elements in Q as the divisibility of their canonical classes. Since d is odd, the 4-manifolds W are non-spin.

Setting $N = 2m - 1$ there exists for every even integer $n \geq 3N + 1 = 6m - 2$ a homotopy elliptic surface W with $\chi_h(W) = n$ which has 2^{2m-1} symplectic structures realizing all elements in Q multiplied by 2 as the divisibility of their canonical classes. Since all divisibilities are even, the manifold W is spin. Setting $N = 2m$ we can choose $n = 6m + 2$ to get a spin homotopy elliptic surface W with $\chi_h(W) = 6m + 2$ and 2^{2m} inequivalent symplectic structures. $\square$

VI.2.2 Spin symplectic 4-manifolds with $c_1^2 > 0$ and negative signature

Symplectic manifolds with $c_1^2 > 0$ and divisible canonical class can be constructed with a version of knot surgery for higher genus surfaces described in [41]. Let $K = K_h$ denote the $(2h+1, -2)$-torus knot. It is a fibred knot of genus h. Consider the manifold $M_K \times S^1$ from the knot surgery construction, cf. Section V.4.1. This manifold has the structure of a Σ_h-bundle over T^2:

$$\begin{array}{ccc} M_K \times S^1 & \longleftarrow & \Sigma_h \\ \downarrow & & \\ T^2 & & \end{array}$$

We denote a fibre of this bundle by Σ_F. The fibration defines a trivialization of the normal bundle $\nu\Sigma_F$. We form g consecutive generalized fibre sums along the fibres Σ_F to get

$$Y_{g,h} = (M_K \times S^1)\#_{\Sigma_F=\Sigma_F}\#\ldots\#_{\Sigma_F=\Sigma_F}(M_K \times S^1).$$

The gluing diffeomorphism is chosen such that it identifies the Σ_h fibres in the boundary of the tubular neighbourhoods. This implies that $Y_{g,h}$ is a Σ_h-bundle over Σ_g:

$$\begin{array}{ccc} Y_{g,h} & \longleftarrow & \Sigma_h \\ \downarrow & & \\ \Sigma_g & & \end{array}$$

We denote the fibre again by Σ_F. The fibre bundle has a section Σ_S sewed together from g torus sections of $M_K \times S^1$. Since the knot K is a fibred knot, the manifold $M_K \times S^1$ admits a symplectic structure such that the fibre and the section are symplectic. By the Gompf construction this is then also true for $Y_{g,h}$.

The invariants can be calculated by the standard formulas:

$$\begin{aligned} c_1^2(Y_{g,h}) &= 8(g-1)(h-1) \\ e(Y_{g,h}) &= 4(g-1)(h-1) \\ \sigma(Y_{g,h}) &= 0. \end{aligned}$$

By induction on g one can show that $\pi_1(Y_{g,h})$ is normally generated by the image of $\pi_1(\Sigma_S)$ under inclusion [41, Proposition 2]. This fact, together with the exact sequence

$$H_1(\Sigma_F) \to H_1(Y_{g,h}) \to H_1(\Sigma_g) \to 0$$

coming from the long exact homotopy sequence for the fibration $\Sigma_F \to Y_{g,h} \to \Sigma_g$ via Lemma A.5, shows that the inclusion $\Sigma_S \to Y_{g,h}$ induces an isomorphism on H_1 and the inclusion $\Sigma_F \to Y_{g,h}$ induces the zero map. In particular, $H_1(Y_{g,h};\mathbb{Z})$ is free abelian of rank

$$b_1(Y_{g,h}) = gb_1(M_K \times S^1) = 2g,$$

cf. also Theorem 5.11 and Corollary 5.14. This implies with the formula for the Euler characteristic

$$b_2(Y_{g,h}) = 4h(g-1) + 2.$$

The summand $4h(g-1)$ results from $2h$ split classes together with $2h$ dual rim tori which are created in each fibre sum. Fintushel and Stern show that there exists a basis for the group of split classes (or *vanishing classes*) consisting of $2h(g-1)$ disjoint surfaces of genus 2 and self-intersection 2. This implies

$$H^2(Y_{g,h};\mathbb{Z}) = 2h(g-1)\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where the last summand is the intersection form on $(\mathbb{Z}\Sigma_S \oplus \mathbb{Z}\Sigma_F)$. They also show that the canonical class of $Y_{g,h}$ is given by

$$K_Y = (2h-2)\Sigma_S + (2g-2)\Sigma_F, \tag{6.3}$$

where Σ_S and Σ_F are oriented by the symplectic form. This can be proved inductively with the formula for the canonical class in Theorem 5.55: The case $g = 1$ is clear by the general formula for standard knot surgery, cf. equation (5.25). Suppose the formula is proved for $N = Y_{g,h}$ and we want to prove it for $Y = Y_{g+1,h} = N\#_{\Sigma_F=\Sigma_F} M$ where $M = M_K \times S^1$. If we use for B_M and B_N the surfaces given by a section for the fibration, we can see that

$$\begin{aligned}
\overline{K_M} &= 0\\
\overline{K_N} &= 0\\
b_Y &= 2h-2\\
\sigma_Y &= 2g-2+2 = 2(g+1)-2.
\end{aligned}$$

Since B_Y corresponds to the section Σ_S in $Y_{g+1,h}$ and Σ_Y to the fibre Σ_F, the claim in equation (6.3) follows if the rim tori coefficients $r_i = K_Y S_i$ vanish. This can be proved with the adjunction inequality [104] for Seiberg-Witten basic classes, because S_i are surfaces of genus 2 and self-intersection 2.

Suppose that M is a closed symplectic 4-manifold which contains a symplectic surface Σ_M of genus g and self-intersection 0, oriented by the symplectic form. We can then form the symplectic generalized fibre sum

$$X = M\#_{\Sigma_M=\Sigma_S} Y_{g,h}.$$

If $\pi_1(M) = \pi_1(M \setminus \Sigma_M) = 1$, then X is again simply-connected because the fundamental group of $Y_{g,h}$ is normally generated by the image of $\pi_1(\Sigma_S)$. Since the inclusion $\Sigma_S \to Y_{g,h}$ is an isomorphism on H_1 no rim tori occur in this generalized fibre sum. Hence we can write by Theorem 5.37

$$H^2(X;\mathbb{Z}) = P(M) \oplus P(Y_{g,h}) \oplus (\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X).$$

The surface B_X is sewed together from a surface B_M in M with $B_M\Sigma_M = 1$ and the fibre Σ_F. In particular, $B_X^2 = B_M^2$ because $\Sigma_F^2 = 0$. This implies that the embedding $H^2(M;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ given by

$$\begin{aligned} \Sigma_M &\mapsto \Sigma_X, \\ B_M &\mapsto B_X \\ Id : P(M) &\to P(M) \end{aligned} \tag{6.4}$$

preserves the intersection form. Therefore we can write

$$H^2(X;\mathbb{Z}) = H^2(M;\mathbb{Z}) \oplus P(Y_{g,h}) \tag{6.5}$$

with intersection form

$$Q_X = Q_M \oplus 2h(g-1)\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

This generalizes equation (5.23). The invariants of X are given by

$$\begin{aligned} c_1^2(X) &= c_1^2(M) + 8h(g-1) \\ e(X) &= e(M) + 4h(g-1) \\ \sigma(X) &= \sigma(M). \end{aligned}$$

We calculate the canonical class of X: Since no rim tori occur in the Gompf sum, the formula in Theorem 5.55 simplifies to

$$K_X = \overline{K_M} + \overline{K_Y} + b_X B_X + \sigma_X \Sigma_X,$$

where

$$\begin{aligned} \overline{K_M} &= K_M - (2g-2)B_M - (K_M B_M - (2g-2)B_M^2)\Sigma_M \in P(M) \\ \overline{K_Y} &= K_Y - (2g-2)\Sigma_F - (K_Y\Sigma_F - (2g-2)\Sigma_F^2)\Sigma_S \\ &= K_Y - (2g-2)\Sigma_F - (2h-2)\Sigma_S \\ &= 0 \\ b_X &= 2g-2 \\ \sigma_X &= K_M B_M + K_Y \Sigma_F + 2 - (2g-2)(B_M^2 + \Sigma_F^2) \\ &= K_M B_M + 2h - (2g-2)B_M^2. \end{aligned}$$

This implies

$$K_X = \overline{K_M} + (2g-2)B_X + (K_M B_M + 2h - (2g-2)B_M^2)\Sigma_X.$$

Note that the class

$$K_M = \overline{K_M} + (2g-2)B_M + (K_M B_M - (2g-2)B_M^2)\Sigma_M$$

maps under the embedding $H^2(M) \to H^2(X)$ in equation (6.4) to the class

$$\overline{K_M} + (2g-2)B_X + (K_M B_M - (2g-2)B_M^2)\Sigma_X$$

and $2h\Sigma_M$ maps to $2h\Sigma_X$. Therefore we can write under the isomorphism in equation (6.5)

$$K_X = K_M + 2h\Sigma_M, \tag{6.6}$$

where the class on the right is an element in the subgroup $H^2(M;\mathbb{Z})$ of $H^2(M;\mathbb{Z}) \oplus P(Y_{g,h})$. This follows also by Corollary 5.57. Formula (6.6) generalizes (5.31). In particular we get:

Proposition 6.19. *Let M be a closed, symplectic 4-manifold which contains a symplectic surface Σ_M of genus $g > 1$ and self-intersection 0. Suppose that $\pi_1(M) = \pi_1(M \setminus \Sigma_M) = 1$ and that the canonical class of M is divisible by d.*

- *If d is odd there exists for every integer $t \geq 1$ a simply-connected symplectic 4-manifold X with invariants*

$$\begin{aligned} c_1^2(X) &= c_1^2(M) + 8td(g-1) \\ e(X) &= e(M) + 4td(g-1) \\ \sigma(X) &= \sigma(M) \end{aligned}$$

  and canonical class divisible by d.

- *If d is even there exists for every integer $t \geq 1$ a simply-connected symplectic 4-manifold X with invariants*

$$\begin{aligned} c_1^2(X) &= c_1^2(M) + 4td(g-1) \\ e(X) &= e(M) + 2td(g-1) \\ \sigma(X) &= \sigma(M) \end{aligned}$$

  and canonical class divisible by d.

This follows from the construction above by taking the genus of the torus knot $h = td$ if d is odd and $2h = td$ if d is even. Hence if a symplectic surface Σ_M of genus $g > 1$ and self-intersection 0 exists in M we can raise c_1^2 without changing the signature or the divisibility of the canonical class. Note that by Lemma 6.2 the integer d necessarily divides $g-1$ if d is odd and d divides $2g-2$ if d is even.

We can apply this construction to the symplectic homotopy elliptic surfaces constructed in Theorem 6.11. In this section we consider the case of even divisibility d and in the following section the case of odd d.

Recall that we constructed a simply-connected symplectic 4-manifold M from the elliptic surface $E(2m)$ by doing knot surgery along a general fibre F with a fibred knot K_1 of genus $g_1 = (k-1)m+1$ and a further knot surgery along a rim torus R with a fibred knot K_2 of genus $g_2 = k$. Here $2m \geq 2$ and $d = 2k \geq 2$ are arbitrary even integers. The canonical class is given by

$$K_M = 2mkF + 2kR = mdF + dR.$$

The manifold M is still homeomorphic to $E(2m)$. There exists an embedded 2-sphere S in $E(2m)$ of self-intersection -2 which intersects the rim torus R once. We can assume that S is disjoint from the fibre F and that the symplectic structure on $E(2m)$ we began with was chosen such that the regular fibre F, the rim torus R and the dual 2-sphere S are all symplectic and the symplectic form induces a positive volume form on each of them.

The 2-sphere S minus a disk sews together with a Seifert surface for K_2 to give in M a symplectic surface C of genus k and self-intersection -2 which intersects the rim torus R once. By smoothing the double point we get a symplectic surface Σ_M in M of genus $g = k+1$ and self-intersection 0 which represents $C + R$.

The complement $M \setminus \Sigma_M$ is simply-connected: This follows because we can assume that $R \cup S$ is contained in a nucleus $N(2)$, cf. [55], [56] and Example 5.73. Inside $N(2)$ there exists a cusp which is homologous to R and disjoint from it. The cusp is still contained in M and intersects the surface Σ_M

once. Since M is simply-connected and the cusp homeomorphic to S^2, the claim $\pi_1(M \setminus \Sigma_M) = 1$ follows.

Let $t \geq 1$ be an arbitrary integer and K_3 the $(2h+1, -2)$-torus knot of genus $h = tk$. We consider the generalized fibre sum

$$X = M\#_{\Sigma_M=\Sigma_S} Y_{g,h}$$

where $g = k + 1$. Then X is a simply-connected symplectic 4-manifold with invariants

$$\begin{aligned} c_1^2(X) &= 8tk^2 = 2td^2 \\ e(X) &= 24m + 4tk^2 = 24m + td^2 \\ \sigma(X) &= -16m. \end{aligned}$$

The canonical class is given by

$$\begin{aligned} K_X &= K_M + 2tk\Sigma_M \\ &= d(mF + R + t\Sigma_M). \end{aligned}$$

Hence K_X has divisibility d, since the class $mF + R + t\Sigma_M$ has intersection 1 with Σ_M. We get:

Theorem 6.20. *Let $d \geq 2$ be an even integer. Then for every pair of positive integers m, t there exists a simply-connected closed spin symplectic 4-manifold X with invariants*

$$\begin{aligned} c_1^2(X) &= 2td^2 \\ e(X) &= td^2 + 24m \\ \sigma(X) &= -16m, \end{aligned}$$

such that the canonical class K_X has divisibility d.

Note that this solves by Lemma 6.1 and Rochlin's theorem the existence question for simply-connected 4-manifolds with canonical class divisible by an even integer and negative signature. In particular (for $d = 2$), every possible lattice point with $c_1^2 > 0$ and $\sigma < 0$ can be realized by a simply-connected spin symplectic 4-manifolds with this construction (the existence of such 4-manifolds has also been proved in [110] in a similar way).

Example 6.21. To identify the homeomorphism type of some of these manifolds let $d = 2k$. We then have

$$\begin{aligned} c_1^2(X) &= 8tk^2 \\ \chi_h(X) &= tk^2 + 2m. \end{aligned}$$

We want to determine when the invariants are on the Noether line $c_1^2 = 2\chi_h - 6$: This is the case if and only if

$$6tk^2 = 4m - 6$$

hence $2m = 3tk^2 + 3$, which has a solution if and only if both t and k are odd. Hence for every pair of odd integers $t, k \geq 1$ there exists a simply-connected symplectic 4-manifold X with invariants

$$\begin{aligned} c_1^2(X) &= 8tk^2 \\ \chi_h(X) &= 4tk^2 + 3 \end{aligned}$$

such that the divisibility of K_X is $2k$. Recall that for every odd integer $r \geq 1$ there exists by a construction of Horikawa [65] a simply-connected spin complex algebraic surface M on the Noether line with invariants

$$\begin{aligned} c_1^2(M) &= 8r \\ \chi_h(M) &= 4r + 3. \end{aligned}$$

See also Theorem 6.53 in Section VI.4 and [56, Theorem 7.4.20] where this surface is called $U(3, r+1)$. For every given odd integer $r \geq 1$ a symplectic 4-manifold homeomorphic to such a spin Horikawa surface can be realized by the construction above with $k = 1$ and $t = r$. For odd integers $k \geq 3, t \geq 1$ we also get spin homotopy Horikawa surfaces X whose canonical class K_X is divisible by $2k$. These manifolds cannot be diffeomorphic to Horikawa surfaces: Since $b_2^+ > 1$ the canonical class K_X is a Seiberg-Witten basic class on X. It is proved in [65] that all Horikawa surfaces M have a fibration in genus 2 curves, hence by Lemma 6.2 the divisibility of K_M is at most 2 and in the spin case the divisibility is equal to 2. Since Horikawa surfaces are minimal surfaces of general type, they have a unique Seiberg-Witten basic class up to sign, given by the canonical class K_M, cf. Theorem 6.4. Since the divisibilities of K_M and K_X do not match, this proves the claim.

Returning to the general case of Theorem 6.20 we can extend the construction in the proof of Theorem 6.16 to show:

Theorem 6.22. *Let $N \geq 1$ be an integer. Suppose that $d \geq 2$ is an even integer and $d_0, \ldots, d_N$ are positive even integers dividing d, as in Definition 6.15. Let Q be the associated set of greatest common divisors. Let m be an integer such that $2m \geq 3N + 2$ and $t \geq 1$ an arbitrary integer. Then there exists a simply-connected closed spin 4-manifold W with invariants*

$$\begin{aligned} c_1^2(W) &= 2td^2 \\ e(W) &= td^2 + 24m \\ \sigma(W) &= -16m, \end{aligned}$$

and the following property: For each integer $q \in Q$ the manifold W admits a symplectic structure whose canonical class K has divisibility equal to q.

Proof. Let $l = 2m - 2N$. By the construction of Theorem 6.20 there exists a simply-connected symplectic spin 4-manifold X with invariants

$$\begin{aligned} c_1^2(X) &= 2td^2 \\ e(X) &= td^2 + 12l \\ \sigma(X) &= -8l \\ K_X &= d(mF + R + t\Sigma_M). \end{aligned}$$

In particular, the canonical class of X has divisibility d. In the construction of X starting from the elliptic surface $E(l)$ we have used only one Lagrangian rim torus. Hence $l - 2$ of the $l - 1$ triples of Lagrangian rim tori in $E(l)$ (cf. Example 5.73) remain unchanged. Note that $l - 2 \geq N$ by our assumptions. Since the symplectic form on $E(l)$ in a neighbourhood of these tori does not change in the construction of X by the Gompf fibre sum, we can assume that X contains at least N triples of Lagrangian tori as in the proof of Theorem 6.16. We can now use the same construction as in this theorem on the N triples of Lagrangian tori in X to get a simply-connected spin 4-manifold W with invariants

$$\begin{aligned} c_1^2(W) &= 2td^2 \\ e(W) &= td^2 + 12l + 24N = td^2 + 24m \\ \sigma(W) &= -8l - 16N = -16m. \end{aligned}$$

For each $q \in Q$ the manifold W admits a symplectic structure ω_q whose canonical class is given by the formulas in equation (6.1) and (6.2) where the term dlF is replaced by $K_X = d(mF + R + t\Sigma_M)$. It follows again that the canonical class of ω_q has divisibility precisely equal to q. □

Corollary 6.23. *Let $d \geq 6$ be an even integer and $t \geq 1, m \geq 3$ arbitrary integers. Then there exists a simply-connected closed spin 4-manifold W with invariants*

$$\begin{aligned} c_1^2(W) &= 2td^2 \\ e(W) &= td^2 + 24m \\ \sigma(W) &= -16m, \end{aligned}$$

and W admits at least two inequivalent symplectic structures.

This follows with $N = 1$ and choosing $d_0 = d$ and $d_1 = 2$, since in this case Q contains two elements.

Example 6.24. We consider the case $N = 1$ of Theorem 6.22 for the spin homotopy Horikawa surfaces in Example 6.21. Let $k \geq 3$ be an arbitrary odd integer and $d = d_0 = 2k, d_1 = 2$. Note that the assumption $2m \geq 3N + 2 = 5$ is always satisfied because $2m = 3tk^2 + 3$ in this case by the calculation above. Since $d = 2k$ is not divisible by 4, the set Q is equal to $\{2k, 2\}$ by Definition 6.15. By Theorem 6.22 there exists for every odd integer $t \geq 1$ a spin homotopy Horikawa surface X on the Noether line with invariants

$$\begin{aligned} c_1^2(X) &= 8tk^2 \\ \chi_h(X) &= 4tk^2 + 3, \end{aligned}$$

which admits two inequivalent symplectic structures: the canonical class of the first symplectic structure is divisible by $2k$ while the canonical class of the second symplectic structure is divisible only by 2.

Remark 6.25. With more care it is possible to do the construction in Theorem 6.22 starting from $E(l)$ in the case that n is even and $l = N + 1$. Thus the same theorem can be proved for integers m with $2m = 3N + 1$.

We start with the same construction as in the proof of Theorem 6.16 where $l = N + 1$ and we use N triples of rim tori. We have now used up all available triples. Note that the tori T_2^i in Y are by construction symplectic for all symplectic structures ω_q with positive induced volume form. We can consider for instance the torus T_2^1. We use the existence of a surface C_2 in Y, cf. Definition 5.77 and Remark 5.80. The surface C_2 intersects the tori T_2^1 and R_1 transversely and positively once and is disjoint from T_1^1 and all other rim tori in the construction from Theorem 6.16. It is also disjoint from the torus F. The surface C_2 has genus $k - 1$ and self-intersection -4. Since we can assume that the sphere S in the nucleus containing R_1 was symplectic in $E(l)$ it follows that C_2 can be assumed symplectic (with positive induced volume form) for all symplectic structures ω_q on Y.

By adding two parallel copies of the torus T_2^1 to C_2 and smoothing the two double points we get a symplectic surface Σ_Y in Y of genus $g' = k+1$ and self-intersection 0 representing the class $2T_2^1 + C_2$. The complement $Y \setminus \Sigma_Y$ is simply-connected: This follows because the surface Σ_Y came from the sphere S in the nucleus $N(2)$ containing the torus R_1. Hence Σ_Y is still intersected once by a cusp homologous to R_1. We can now do the same construction as before (raising c_1^2 by a generalized knot surgery with the $(2h+1,-2)$-torus knot of genus $h = tk$ on the surface Σ_M). To show that the canonical class $K_W(q)$ of the resulting manifold W has divisibility q one has to use the explicit formulas in (6.1) and (6.2) and the surfaces C_1 from Lemma 5.78.

VI.2.3 Non-spin symplectic 4-manifolds with $c_1^2 > 0$ and negative signature

In this section we construct some families of simply-connected symplectic 4-manifolds with $c_1^2 > 0$ such that the divisibility of K is a given odd integer $d > 1$. However, we do not have a complete existence result as in Theorem 6.20.

We first consider the case that the canonical class K_X is divisible by an odd integer d and the signature $\sigma(X)$ is divisible by 8.

Lemma 6.26. *Let X be a closed simply-connected symplectic 4-manifold such that K_X is divisible by an odd integer $d \geq 1$ and $\sigma(X)$ is divisible by 8. Then $c_1^2(X)$ is divisible by $8d^2$.*

Proof. Suppose that $\sigma(X) = 8m$ for some integer $m \in \mathbb{Z}$. Then $b_2^-(X) = b_2^+(X) - 8m$ hence $b_2(X) = 2b_2^+(X) - 8m$. This implies

$$e(X) = 2b_2^+(X) + 2 - 8m.$$

Since X is symplectic, the integer $b_2^+(X)$ is odd, so we can write $b_2^+(X) = 2k+1$ for some $k \geq 0$. This implies

$$e(X) = 4k + 4 - 8m,$$

hence $e(X)$ is divisible by 4. The equation $c_1^2(X) = 2e(X) + 3\sigma(X)$ shows that $c_1^2(X)$ is divisible by 8. Since $c_1^2(X)$ is also divisible by the odd integer d^2 the claim follows. □

The following theorem covers the case that K_X has odd divisibility and the signature is negative, divisible by 8 and ≤ -16:

Theorem 6.27. *Let $d \geq 1$ be an odd integer. Then for every pair of positive integers n, t with $n \geq 2$ there exists a simply-connected closed non-spin symplectic 4-manifold X with invariants*

$$\begin{aligned} c_1^2(X) &= 8td^2 \\ e(X) &= 4td^2 + 12n \\ \sigma(X) &= -8n \end{aligned}$$

such that the canonical class K_X has divisibility d.

Proof. The proof is similar to the proof of Theorem 6.20. We can write $d = 2k+1$ with $k \geq 0$. Suppose that $\mathbf{n} = \mathbf{2m} + \mathbf{1}$ is odd where $m \geq 1$. In the proof of Theorem 6.11 we constructed a homotopy elliptic surface M with $\chi_h(M) = n$ from the elliptic surface $E(n)$ by doing knot surgery along a general fibre F with a fibred knot K_1 of genus $g_1 = 2km + k + 1$ and a further knot surgery along a rim torus R with a fibred knot K_2 of genus $g_2 = 2k + 1 = d$. The canonical class is given by

$$\begin{aligned} K_M &= (2m+1)(2k+1)F + 2(2k+1)R \\ &= (2m+1)dF + 2dR. \end{aligned}$$

There exist a symplectically embedded 2-sphere S in $E(n)$ of self-intersection -2 which sews together with a Seifert surface for K_2 to give in M a symplectic surface C of genus d and self-intersection -2 which intersects the rim torus R once. By smoothing the double point we get a symplectic surface Σ_M in M of genus $g = d+1$ and self-intersection 0 which represents $C + R$. Using a cusp which intersects Σ_M once, it follows as above that the complement $M \setminus \Sigma_M$ is simply-connected.

Let $t \geq 1$ be an arbitrary integer and K_3 the $(2h+1, -2)$-torus knot of genus $h = td$. We consider the generalized fibre sum

$$X = M \#_{\Sigma_M = \Sigma_S} Y_{g,h}$$

where $g = d + 1$. Then X is a simply-connected symplectic 4-manifold with invariants

$$\begin{aligned} c_1^2(X) &= 8td^2 \\ e(X) &= 4td^2 + 12n \\ \sigma(X) &= -8n. \end{aligned}$$

The canonical class is given by

$$\begin{aligned} K_X &= K_M + 2td\Sigma_M \\ &= d((2m+1)F + 2R + 2t\Sigma_M). \end{aligned}$$

Hence K_X has divisibility d, since the class $(2m+1)F + 2R + 2t\Sigma_M$ has intersection 2 with Σ_M and intersection $(2m+1)$ with a surface coming from a section of $E(n)$ and a Seifert surface for K_1.

The case that $\mathbf{n = 2m}$ is even where $m \geq 1$ can be proved similarly. By doing a logarithmic transform on the fibre F in $E(n)$ and two further knot surgeries with a fibred knot K_1 of genus $g_1 = 4km + k + 2$ on the multiple fibre f and with a fibred knot K_2 of genus $g_2 = 2k + 1 = d$ along a rim torus R, we get a homotopy elliptic surface X with $\chi_h(X) = n$ and canonical class

$$K_X = (4m+1)df + 2dR.$$

The same construction as above yields a simply-connected symplectic 4-manifold X with invariants

$$\begin{aligned} c_1^2(X) &= 8td^2 \\ e(X) &= 4td^2 + 24n \\ \sigma(X) &= -8n. \end{aligned}$$

The canonical class is given by

$$\begin{aligned} K_X &= K_M + 2td\Sigma_M \\ &= d((4m+1)f + 2R + 2t\Sigma_M). \end{aligned}$$

Hence K_X has again divisibility d. $\square$

Example 6.28. The manifolds in Theorem 6.27 have invariants

$$\begin{aligned} c_1^2(X) &= 8td^2 \\ \chi_h(X) &= td^2 + n. \end{aligned}$$

In a similar way to Example 6.21, this implies that for every pair of positive integers $d, t \geq 1$ with d odd and t arbitrary there exists a non-spin symplectic homotopy Horikawa surface X on the Noether line $c_1^2 = 2\chi_h - 6$ with invariants

$$\begin{aligned} c_1^2(X) &= 8td^2 \\ \chi_h(X) &= 4td^2 + 3, \end{aligned}$$

whose canonical class has divisibility d. Note that for every integer $s \geq 1$ there exists a non-spin Horikawa surface M [65] with invariants

$$\begin{aligned} c_1^2(M) &= 8s \\ \chi_h(M) &= 4s + 3. \end{aligned}$$

If s is odd there exists only one deformation type of such surfaces, denoted by $X(3, 2s+2)$ in [56, Theorem 7.4.20]. If s is even there exist two deformation types given by the homeomorphic manifolds $X(3, 2s+2)$ and $U(3, s+1)$. For every given integer $s \geq 1$ a simply-connected symplectic 4-manifold homeomorphic to such a non-spin Horikawa surface can be realized by the construction above with $d = 1$ and $t = s$ (note that $n \geq 2$ holds automatically in this case). If $d \geq 3$ is an odd integer and $t \geq 1$ an arbitrary integer we also get non-spin homotopy Horikawa surfaces whose canonical class has divisibility d. By the same argument as before in Example 6.21, these 4-manifolds cannot be diffeomorphic to a Horikawa surface.

In the general case, one can prove the following as in Theorem 6.22.

Theorem 6.29. *Let $N \geq 1$ be an integer. Suppose that $d \geq 3$ is an odd integer and $d_0, \ldots, d_N$ positive integers dividing d, as in Definition 6.15. Let Q be the associated set of greatest common divisors. Let m be an integer such that $m \geq 2N + 2$ and $t \geq 1$ an arbitrary integer. Then there exists a simply-connected closed non-spin 4-manifold W with invariants*

$$\begin{aligned} c_1^2(W) &= 8td^2 \\ e(W) &= 4td^2 + 12m \\ \sigma(W) &= -8m, \end{aligned}$$

and the following property: For each integer $q \in Q$ the manifold W admits a symplectic structure whose canonical class K has divisibility equal to q.

Proof. The proof is analogous to the proof of Theorem 6.22. Let $l = m - N$. By the construction of Theorem 6.27 there exists a simply-connected non-spin symplectic 4-manifold X with invariants

$$\begin{aligned} c_1^2(X) &= 8td^2 \\ e(X) &= 4td^2 + 12l \\ \sigma(X) &= -8l, \end{aligned}$$

whose canonical class K_X has divisibility d. The manifold X contains $l-2$ triples of Lagrangian tori. By our assumptions $l-2 \geq N$. Hence we can do the construction in Theorem 6.16 (for d odd) to get a simply-connected non-spin 4-manifold W with invariants

$$\begin{aligned} c_1^2(W) &= 8td^2 \\ e(W) &= 4td^2 + 12l + 12N = 4td^2 + 12m \\ \sigma(X) &= -8l - 8N = -8m. \end{aligned}$$

The 4-manifold W admits for every integer $q \in Q$ a symplectic structure whose canonical class has divisibility equal to q. □

Choosing $N = 1$, $d_0 = d$ and $d_1 = 1$, the set Q contains two elements. This implies:

Corollary 6.30. *Let $d \geq 3$ be an odd integer and $t \geq 1, m \geq 4$ arbitrary integers. Then there exists a simply-connected closed non-spin 4-manifold W with invariants*

$$\begin{aligned} c_1^2(W) &= 8td^2 \\ e(W) &= 4td^2 + 12m \\ \sigma(W) &= -8m, \end{aligned}$$

and W admits at least two inequivalent symplectic structures.

Remark 6.31. Let Y be an arbitrary closed symplectic 4-manifold which contains an embedded symplectic torus T_Y of self-intersection 0. Suppose that T_Y is contained in a cusp neighbourhood and represents an indivisible class. Consider the symplectic generalized fibre sum

$$V = Y \#_{T_Y = F} E(n) \#_{F = T_K} (M_K \times S^1)$$

where K is an arbitrary fibred knot. The manifold V has invariants

$$\begin{aligned} c_1^2(V) &= c_1^2(Y) \\ e(V) &= e(Y) + 12n \\ \sigma(V) &= \sigma(Y) - 8n. \end{aligned}$$

By Remark 5.75 the symplectic manifold V contains $n-1$ triples of Lagrangian tori. If the canonical class K_Y has a suitable divisibility and the genus of the knot K is chosen appropriately one can find inequivalent symplectic structures by starting from the smooth manifold V.

We want to describe a second example that yields for every odd integer $d \geq 1$ a simply-connected symplectic 4-manifold W whose canonical class has divisibility d and $c_1^2(W) = 2d^2$.

The first building block V is constructed as follows: Consider the product of two closed surfaces $U = \Sigma_g \times \Sigma_h$ of genus g and h. In U we have the singular surface given by the one point union $\Sigma_g \vee \Sigma_h$. We can smooth the intersection point to get a symplectic surface Σ_U of genus $g+h$ and self-intersection 2, blow up two points on $\Sigma_U$and let Σ_V denote the proper transform in $V = U \# 2\overline{\mathbb{C}P^2}$. Then Σ_V is a symplectic surface of self-intersection 0. The Euler characteristic of U is

$$e(U) = e(\Sigma_g)e(\Sigma_h) = 4(g-1)(h-1).$$

Since $\sigma(U) = 0$, the manifold V has invariants

$$\begin{aligned} c_1^2(V) &= 8(g-1)(h-1) - 2 \\ e(V) &= 4(g-1)(h-1) + 2 \\ \sigma(V) &= -2. \end{aligned}$$

Note that the inclusion induces a surjection $\pi_1(\Sigma_V) \to \pi_1(V)$ and an isomorphism $H_1(\Sigma_V;\mathbb{Z}) \to H_1(V;\mathbb{Z})$ (compare to the building block Q_1 in [52, Section 5]).

The second building block X consists of the simply-connected symplectic 4-manifold $X(n,1)$ defined in [56, Chapter 7], see also Section VI.5.3. It is diffeomorphic to $\mathbb{C}P^2\#(4n+1)\overline{\mathbb{C}P^2}$ and has invariants

$$c_1^2(X) = -4n + 8$$

$$e(X) = 4n + 4$$

$$\sigma(X) = -4n$$

The manifold X has two fibrations over $\mathbb{C}P^1$: one of them has a fibre F_1 of genus 0 and the other one has a fibre F_2 of genus $n-1$. We define $\Sigma_X = F_2$. Both fibrations have a section; in particular, the complement $X \setminus \Sigma_X$ is simply-connected.

Suppose that $n > 1$ is an arbitrary integer and let g be any integer with $1 \le g \le n-1$. Define $h = n - 1 - g$ and consider the manifold V as above. Then the genus of Σ_V is equal to the genus of Σ_X and we can construct the symplectic generalized fibre sum

$$W = X\#_{\Sigma_X=\Sigma_V}V.$$

Since $X \setminus \Sigma_X$ is simply-connected and $\pi_1(\Sigma_V) \to \pi_1(V)$ is a surjection we see that W is simply-connected. Note that the inclusion induced isomorphism $H_1(\Sigma_V) \to H_1(V)$ implies by Proposition 5.59 that the generalized fibre sum W does not contain rim tori. Hence there are no rim tori contributions to the formula for the canonical class. We can use Corollary 5.58 to determine the divisibility d of the canonical class K_W of W:

The canonical class of V is given by

$$K_V = (2h-2)\Sigma_g + (2g-2)\Sigma_h + E_1 + E_2,$$

where E_1, E_2 denote the exceptional spheres. Let B_V be one of the exceptional spheres. Then $B_V^2 = K_V B_V = -1$. Since Σ_V represents

$$\Sigma_V = \Sigma_g + \Sigma_h - E_1 - E_2$$

it follows that

$$K_V - (K_V B_V)\Sigma_Y = (2h-1)\Sigma_g + (2g-1)\Sigma_h.$$

The canonical class of X is (cf. Section VI.5.3)

$$K_X = (n-2)F_1 - F_2.$$

The fibration with fibre $F_2 = \Sigma_X$ has a section which is a symplectic sphere B_X of self-intersection $B_X^2 = -1$. We have again $K_X B_X = -1$ and

$$K_X - (K_X B_X)\Sigma_X = (n-2)F_1.$$

By Corollary 5.58 the divisibility of K_W is the greatest common divisor of the integers

$$n-2,\ 2g-1,\ 2h-1,\ 2(g+h)-2.$$

By our choice $g+h=n-1$ we have $2(g+h)-2=2(n-2)$. Hence we can leave the last term away to calculate the greatest common divisor. Moreover,

$$2h-1=2n-2-2g-1=2(n-2)-(2g-1).$$

Hence the divisibility of K_W is the greatest common divisor of $n-2$ and $2g-1$.

Proposition 6.32. *Let $W = X\#_{\Sigma_X=\Sigma_V}V$ be the generalized fibre sum above where $g+h=n-1$. Then W is a simply-connected symplectic 4-manifold with invariants*

$$c_1^2(W) = 8g(n-1-g)-4n+6$$

$$e(W) = 4g(n-1-g)+4n+6$$

$$\sigma(W) = -4n-2$$

The divisibility of K_W is the greatest common divisor of $n-2$ and $2g-1$.

The formulas for the invariants of W follow by the standard formulas (cf. Corollary 5.14 and equation (5.30)):

$$\begin{aligned} c_1^2(W) &= c_1^2(X)+c_1^2(V)+8(g+h)-8 \\ e(W) &= e(X)+e(V)+4(g+h)-4 \\ \sigma(W) &= \sigma(X)+\sigma(V). \end{aligned}$$

To get a particular example choose $g=h\geq 1$ arbitrarily. Then $n=2g+1$ and $n-2=2g-1$. The manifold W has invariants

$$c_1^2(W) = 8g^2-8g+2$$

$$e(W) = 4g^2+8g+10$$

$$\sigma(W) = -8g-6$$

and K_W has divisibility $d=2g-1$. We can write the invariants also in terms of d and get:

Corollary 6.33. *For every odd integer $d\geq 1$ there exists a simply-connected symplectic 4-manifold W with invariants*

$$\begin{aligned} c_1^2(W) &= 2d^2 \\ e(W) &= d^2+6d+15 \\ \sigma(W) &= -4d-10, \end{aligned}$$

such that the canonical class K_W has divisibility d.

For example, for $g=2$, we get a symplectic manifold W with $c_1^2=18, e=42, \sigma=-22$ such that K_W has divisibility 3. The manifold W is homeomorphic to $9\mathbb{C}P^2\#31\overline{\mathbb{C}P^2}$. For $g=3$ we get a symplectic manifold W with $c_1^2=50, e=70, \sigma=-30$ such that K_W has divisibility 5. This manifold is homeomorphic to $19\mathbb{C}P^2\#49\overline{\mathbb{C}P^2}$. The Chern number c_1^2 and the Euler characteristic for a given divisibility are smaller than the ones in Theorem 6.27.

In general, we could not answer the following question:

Question 1. *For a given odd integer $d>1$ find a simply-connected symplectic 4-manifold M with $c_1^2(M)=d^2$ whose canonical class has divisibility d.*

Note that there is a trivial example for $d=3$, namely $\mathbb{C}P^2$.

VI.2.4 Constructions using Lefschetz fibrations

Let $M = M'\#r\overline{\mathbb{C}P^2} \rightarrow \mathbb{C}P^1$ be a holomorphic Lefschetz fibration with fibres Σ_M of genus g. The fibration defines a natural trivialization of the normal bundle of Σ_M in M. We take the generalized fibre sum of two copies of M along Σ_M such that the gluing diffeomorphism ϕ on $\partial\nu\Sigma_M$ is the identity with respect to the natural trivialization given by the fibration:

$$X = M\#_{\Sigma_M=\Sigma_M}M.$$

Then X is the fibre sum of two copies of M and has an induced Lefschetz fibration over $\mathbb{C}P^1$ in genus g curves Σ_X. Suppose that M is simply-connected. Then X is simply-connected because the exceptional spheres in M intersect the surface Σ_M once. By our choice of gluing diffeomorphism the vanishing disks for Σ_M in M sew together pairwise to give Lagrangian 2-spheres $S_1, \ldots, S_{2g}$ in X of self-intersection -2 which determine a basis of the subgroup $S'(X) \subset H^2(X)$, cf. Theorem 5.37 and Section V.4.2. The group of rim tori $R(X)$ in X is free abelian of rank $2g$. We choose a basis $R_1, \ldots, R_{2g}$ dual to the basis for $S'(X)$.

The fibre summing can be iterated:

$$M(n) = M\#_{\Sigma_M=\Sigma_M}M\# \ldots \#_{\Sigma_M=\Sigma_M}M.$$

Then $M(n)$ is a simply-connected Lefschetz fibration over $\mathbb{C}P^1$ in genus g curves Σ_X.

Proposition 6.34. *The canonical class of $X = M(n)$ is given by*

$$K_X = \sum_{i=1}^{n} \overline{K_{M_i}} + (2g-2)B_X + ((n-2) + (2g-2)n)\Sigma_X,$$

where

$$\overline{K_{M_i}} = (K_M + \Sigma_M) - (2g-2)(B_M + \Sigma_M)$$

for all $i = 1, \ldots, n$.

This formula should be interpreted such that the classes $\overline{K_{M_i}}$ lie in different copies $P(M_i)$ of $P(M)$, each of which is a direct summand of $H^2(X;\mathbb{Z})$.

Proof. The proof is by induction, cf. the formula in Theorem 5.55. We first check the case $n = 1$. We have:

$$\begin{aligned} K_X &= (K_M + \Sigma_M) - (2g-2)(B_M + \Sigma_M) + (2g-2)B_M + (-1 + (2g-2))\Sigma_M \\ &= K_M. \end{aligned}$$

Suppose that $n \geq 2$ and the formula is correct for $n-1$. Write $N = M(n-1)$ and consider the fibre sum $X = M\#_{\Sigma_M=\Sigma_M}N$. We use for B_M an exceptional sphere in M and for B_N the symplectic sphere of self-intersection $-(n-1)$ from the previous step. Using the adjunction formula we have

$$\begin{aligned} K_M B_M &= -1 \\ \overline{K_M} &= K_M - (2g-2)B_M - (-1 + (2g-2))\Sigma_M \\ &= (K_M + \Sigma_M) - (2g-2)(B_M + \Sigma_M) \\ &= \overline{K_{M_n}}, \end{aligned}$$

and similarly

$$\begin{aligned}
K_N B_N &= n-3\\
\overline{K_N} &= K_N - (2g-2)B_N - ((n-3)+(2g-2)(n-1))\Sigma_N\\
&= \sum_{i=1}^{n-1}\overline{K_{M_i}} + (2g-2)B_N + ((n-3)+(2g-2)(n-1))\Sigma_N\\
&\quad - (2g-2)B_N - ((n-3)+(2g-2)(n-1))\Sigma_N\\
&= \sum_{i=1}^{n-1}\overline{K_{M_i}}.
\end{aligned}$$

We also have

$$\begin{aligned}
b_X &= 2g-2\\
\sigma_X &= -1+(n-3)+2-(2g-2)(-1-(n-1))\\
&= (n-2)-(2g-2)n.
\end{aligned}$$

Note that all coefficients a_i vanish by our choice for the trivialization and the gluing diffeomorphism and

$$K_X S_i = K_M D_i^M - K_N D_i^N = 0,$$

since S_i is a Lagrangian sphere of self-intersection -2. Hence all rim tori coefficients r_i are zero. Adding the terms above proves the proposition. □

Remark 6.35. One can also derive a formula for the canonical class of a twisted fibre sum of some $M(n)$ and $M(m)$, as in Section V.6.1. This could have applications as in Corollary 5.68.

Note that for $g=1$ and $M=E(1)$ with general fibre F we have $K_M+\Sigma_M=-F+F=0$. Hence we get again the formula $K_X=(n-2)F$ for the canonical class of $X=E(n)$. In the general case we have:

Corollary 6.36. *Let $X=M(n)$ be the n-fold fibre sum of simply-connected holomorphic Lefschetz fibrations. Then the divisibility of K_X is the greatest common divisor of $n-2$ and the divisibility of the class $K_M+\Sigma_M\in H^2(M;\mathbb{Z})$.*

Proof. The greatest common divisor of $n-2$ and the divisibility of $K_M+\Sigma_M$ divides K_X: This follows because this number also divides $2g-2=(K_M+\Sigma_M)\Sigma_M$ by the adjunction formula. The number then divides all terms in the formula in Proposition 6.34.

Conversely, let d denote the divisibility of K_X. It is clear that d divides $2g-2$ since $K_X\Sigma_X=2g-2$ by the adjunction formula or the formula above. We have

$$K_X B_X = (2g-2)B_X^2 + ((n-2)+(2g-2)n).$$

This implies that d divides also $n-2$. The integer d also has to divide every term $\overline{K_{M_i}}$. This shows that it divides the class $K_M+\Sigma_M$, proving the claim. □

Remark 6.37. Since the complex curve Σ_M on the blow-up $M=M'\#r\overline{\mathbb{C}P^2}\rightarrow\mathbb{C}P^1$ is the proper transform of a curve $\Sigma_{M'}$ in M', the divisibility of $K_M+\Sigma_M$ is equal to the divisibility of $K_{M'}+\Sigma_{M'}$. This follows because the canonical class and the class of the proper transform are given by

$$\begin{aligned}
\Sigma_M &= \Sigma_{M'} - E_1 - \ldots - E_r\\
K_M &= K_{M'} + E_1 + \ldots + E_r,
\end{aligned}$$

where E_i denotes the exceptional spheres.

Remark 6.38. If $g > 1$, we can use the construction in Proposition 6.19 on the genus g surface Σ_X to increase $c_1^2(X)$ while keeping the signature and the divisibility of K_X fixed. Note that $\pi_1(X \setminus \Sigma_X) = 1$ since X is simply-connected and the sphere B_X sewed together from exceptional spheres in both copies of M intersects Σ_X once. Hence the 4-manifold we obtain is again simply-connected (cf. [41, Section 3] for a related construction).

Remark 6.39. In principle it should also be possible to do the construction with triples of Lagrangian rim tori from Theorem 5.79 like in the previous sections to find inequivalent symplectic structures on simply-connected 4-manifolds, starting from an n-fold fibre sum $M(n)$. Note that every fibre sum contributes $2g$ rim tori out of which we can form g Lagrangian triples. One can probably extend Example 5.73 to show that some of these rim tori are contained in nuclei $N(2)$. In particular, this should work for the fibrations $X(m, n)$ in Section VI.5.3.

VI.3 Branched coverings

In the following sections we will describe another construction of simply-connected symplectic 4-manifolds with divisible canonical class. This construction uses branched coverings of algebraic surfaces. We will first define the notion of branched coverings and give a criterion in Corollary 6.47 which ensures that the branched coverings we use are simply-connected if we start with a simply-connected manifold.

VI.3.1 Definition

Let M^n be a closed, oriented manifold and F^{n-2} a closed, oriented submanifold of codimension 2. Suppose that the fundamental class $[F] \in H_{n-2}(M; \mathbb{Z})$ is divisible by some integer $m > 1$. Choose a class $B \in H_{n-2}(M; \mathbb{Z})$ such that $[F] = mB$. Let L_F, L_B denote the complex line bundles with Chern classes

$$c_1(L_F) = PD[F], \quad c_1(L_B) = PD(B).$$

Since $c_1(L_F) = mc_1(L_B)$, there exists an isomorphism

$$L_B^{\otimes m} \cong L_F.$$

We consider the following map

$$\begin{aligned} \phi\colon L_B &\to L_B^{\otimes m}, \\ x &\mapsto x \otimes \cdots \otimes x \quad (m \text{ factors}). \end{aligned}$$

On each fibre, this map is given by

$$\mathbb{C} \to \mathbb{C}^{\otimes m}, z \mapsto z \otimes \cdots \otimes z.$$

Let e be a basis vector of the $\mathbb{C}$-vector space $\mathbb{C}$. Then $e \otimes \cdots \otimes e$ is a basis of $\mathbb{C}^{\otimes m}$ which induces an isomorphism

$$\begin{aligned} \mathbb{C}^{\otimes m} &\to \mathbb{C} \\ z_1 e \otimes \cdots \otimes z_m e &\mapsto (z_1 \cdot \ldots \cdot z_m)e. \end{aligned}$$

The composition

$$\mathbb{C} \to \mathbb{C}^{\otimes m} \to \mathbb{C}$$

is then the map $z \mapsto z^m$. On the unit circle, this is an m-fold covering. Hence we get

Lemma 6.40. *Let $L_B, L_F \to M$ be complex line bundles with $c_1(L_F) = mc_1(L_B)$ and denote the associated circle bundles by E_F, E_B. Then the map*

$$\phi\colon L_B \to L_B^{\otimes m} \cong L_F$$

induces a fibrewise m-fold covering $E_B \to E_F$.

Let $s\colon M \to L_F$ be a section which vanishes along F, is non-zero on $M' = M \setminus F$ and is transverse to the zero section.

Theorem 6.41. *Consider the set*

$$X = \phi^{-1}(s(M)) \subset L_B.$$

Then X is again a smooth manifold of dimension n. Let $\pi\colon X \to M$ denote the restriction of the projection $L_B \to M$.

- *Over M', the map $\phi\colon \phi^{-1}(M') \to M'$ is an m-fold cyclic covering.*
- *The intersection of X with the zero section of L_B is a smooth submanifold $\overline{F}$ of X and π maps $\overline{F}$ diffeomorphically onto F.*
- *Let $\nu(\overline{F})$ denote a tubular neighbourhood of $\overline{F}$ in X. The projection π maps $\nu(\overline{F})$ onto a tubular neighbourhood $\nu(F)$ of F in M. Under the identification $\overline{F} = F$ via π, there is a vector bundle isomorphism $\nu(F) = \nu(\overline{F})^{\otimes m}$ and the map π corresponds to the map ϕ above. In other words, there are local coordinates of the form $U \times D^2_X \subset \nu(\overline{F})$ and $U \times D^2_M \subset \nu(F)$, with $U \subset \overline{F} \cong F$ such that π has the form*

$$U \times D^2_X \to U \times D^2_M, (x, z) \mapsto (x, z^m).$$

For a proof, see [63].

Definition 6.42. The *m-fold branched (or ramified) covering $M(F, B, m)$ of M branched over F and determined by B* is defined as

$$M(F, B, m) = \phi^{-1}(s(M)) \subset L_B.$$

Suppose M is a smooth complex algebraic surface and $D \subset M$ a smooth connected complex curve. If $m > 0$ is an integer that divides $[D]$ and $B \in H_2(M; \mathbb{Z})$ a homology class such that $[D] = mB$, then there exists a branched covering $M(D, B, m)$. Since the divisor D has an associated holomorphic line bundle, one can show that the line bundle L_B in the previous section can be chosen as a holomorphic line bundle as well (see [63]). This implies that the branched covering admits the structure of an algebraic surface. The invariants of M can be calculated by the following proposition.

Proposition 6.43. *Let D be a smooth connected complex curve in a complex surface M such that $[D] = mB$. Let $\phi\colon M(D, B, m) \to M$ be the branched covering. Then the invariants of $N := M(D, B, m)$ are given by:*

(a) $K_N = \phi^*(K_M + (m-1)B)$

(b) $c_1^2(N) = m(K_M + (m-1)B)^2$

(c) $e(N) = me(M) - (m-1)e(D)$,

where $e(D) = 2 - 2g(D) = -(K_M \cdot D + D^2)$ *by the adjunction formula.*

Proof. The formula for $e(N)$ can be calculated by the well-known formula for the Euler characteristic of a space decomposed into two pieces (which we used already in the proof of Corollary 5.14) and the formula for standard, unramified coverings:

$$\begin{aligned} e(N) &= e(N \setminus \overline{D}) + e(\nu(\overline{D})) - e(\partial\nu(\overline{D})) \\ &= me(M \setminus D) + e(D) = m(e(M) - e(D)) + e(D) \\ &= me(M) - (m-1)e(D). \end{aligned}$$

Here $\overline{D}$ denotes the complex curve in N over the branching divisor D as in Theorem 6.41. The formula for $c_1^2(N)$ follows then by the signature formula of Hirzebruch [63]:

$$\sigma(N) = m\sigma(M) - \frac{m^2-1}{3m} D^2.$$

The formula for K_N can be found in [8, Chapter I, Lemma 17.1]. □

We will consider the particular case that the complex curve D is in the linear system $|nK_M|$ and hence represents in homology a multiple nK_M of the canonical class of M. Let $m > 0$ be an integer dividing n and write $n = ma$.

Lemma 6.44. *Let* D *be a smooth connected complex curve in a complex surface* M *with* $[D] = nK_M$*. Then the invariants of the* m*-fold ramified cover* $\phi\colon M(D, aK_M, m) \to M$ *branched over* D *are given by:*

(a) $K_N = (n+1-a)\phi^* K_M$
(b) $c_1^2(N) = m(n+1-a)^2 c_1^2(M)$
(c) $e(N) = me(M) + (m-1)n(n+1)c_1^2(M)$

Proof. We have $[D] = nK_M$ and $B = aK_M$. Hence we can calculate:

$$\begin{aligned} K_M + (m-1)B &= (1 + ma - a)K_M = (n+1-a)K_M \\ e(D) &= -(K_M \cdot D + D^2) \\ &= -(n+n^2)c_1^2(M) = -n(n+1)c_1^2(M) \end{aligned}$$

This implies the formulas. □

VI.3.2 The fundamental group of branched covers

Let M^n be a closed oriented manifold and F^{n-2} a closed oriented submanifold. Suppose that $[F] = mB$ and consider the branched covering $\overline{M} = M(F, B, m)$. Even if the base manifold M is simply-connected the fundamental group of $\overline{M}$ is in general non-trivial. The following theorem can be used to ensure that the branched covers are simply-connected. Let $M' = M \setminus F$ denote the complement of F.

Theorem 6.45. *Let* M^n *be a closed oriented manifold and* F^{n-2} *a closed oriented submanifold such that* $[F]$ *is a non-torsion class in* $H_{n-2}(M;\mathbb{Z})$*. Suppose in addition that the fundamental group of* M' *is abelian. Then for all* m *and* B *with* $[F] = mB$ *there exists an isomorphism*

$$\pi_1(M(F, B, m)) \cong \pi_1(M).$$

Proof. Let $k > 0$ denote the maximal integer dividing $[F]$. Then m divides k and we can write $k = ma$ with $a > 0$. Let $\overline{M'} = M(F, B, m) \setminus \overline{F}$. Denote the meridian to $\overline{F}$ in $\overline{M'}$, i.e. the class of a fibre in $\partial\nu(\overline{F})$, by $\overline{\sigma}$. By Proposition A.3 we get

$$\pi_1(M(F, B, m)) \cong \pi_1(\overline{M'})/N(\overline{\sigma}).$$

We have an exact sequence

$$0 \to \pi_1(\overline{M'}) \xrightarrow{\pi_*} \pi_1(M') \to \mathbb{Z}_m \to 0,$$

since $\pi\colon \overline{M'} \to M'$ is an m-fold cyclic covering. The assumption that $\pi_1(M')$ is abelian implies that $\pi_1(\overline{M'})$ is also abelian. Therefore, the normal subgroups generated by the fibres in these groups are cyclic and we get an exact sequence of subgroups

$$0 \to \mathbb{Z}_a\overline{\sigma} \xrightarrow{m\cdot} \mathbb{Z}_{ma}\sigma \to \mathbb{Z}_m \to 0,$$

where σ is the meridian of F in M'. The surjection $\mathbb{Z}_{ma}\sigma \to \mathbb{Z}_m$ implies that for each element $\alpha \in \pi_1(M')$ there is an integer $r \in \mathbb{Z}$ such that $\alpha + r\sigma$ maps to zero in $\mathbb{Z}_m$ and hence is in the image of π_*. In other words, the induced map

$$\pi_*\colon \pi_1(\overline{M'}) \longrightarrow \pi_1(M')/\langle\sigma\rangle$$

is surjective. The kernel of this map is $\langle\overline{\sigma}\rangle$, hence

$$\pi_1(\overline{M'})/\langle\overline{\sigma}\rangle \xrightarrow{\cong} \pi_1(M')/\langle\sigma\rangle.$$

Again by Proposition A.3, this implies $\pi_1(M(F, B, m)) \cong \pi_1(M)$. □

We will use this theorem in the case where M is a 4-manifold and F an embedded surface. In general, the complement of a 2-dimensional submanifold in a 4-manifold does not have abelian fundamental group even if M is simply-connected. However, this is sometimes the case if we consider complex curves in complex manifolds. The following theorem is due to Nori ([105], Proposition 3.27).

Theorem 6.46. *Let M be a smooth complex algebraic surface and $D, E \subset M$ smooth complex curves which intersect transversely. Assume that $D'^2 > 0$ for every connected component $D' \subset D$. Then the kernel of $\pi_1(M \setminus (D \cup E)) \to \pi_1(M \setminus E)$ is a finitely generated abelian group.*

In particular, for $E = \emptyset$, this implies that the kernel of

$$\pi_1(M') \to \pi_1(M)$$

is a finitely generated abelian group if D is connected and $D^2 > 0$, where M' denotes $M \setminus D$. If M is simply-connected it follows that $\pi_1(M')$ is abelian. Together with Theorem 6.45 we get the following corollary to Nori's theorem.

Corollary 6.47. *Let M be a simply-connected, smooth complex algebraic surface and $D \subset M$ a smooth connected complex curve with $D^2 > 0$. Let $\overline{M}$ be a cyclic ramified cover of M branched over D. Then $\overline{M}$ is also simply-connected.*

If the divisor not only satisfies $D^2 > 0$ but is ample, there is a more general theorem by Cornalba [27]:

Theorem 6.48. *Let M be an n-dimensional smooth complex algebraic manifold and $D \subset M$ a smooth ample divisor. Let $\overline{M}$ be a ramified cover of M branched over D. Then*

$$\pi_k(\overline{M}) \cong \pi_k(M), \quad 0 \leq k \leq n-1,$$

and $\pi_n(\overline{M})$ surjects onto $\pi_n(M)$.

In particular, we get in the case of complex surfaces ($n = 2$):

Corollary 6.49. *Let M be a smooth complex algebraic surface and $D \subset M$ a smooth ample divisor. Let $\overline{M}$ be a ramified cover of M branched over D. Then $\pi_1(\overline{M}) \cong \pi_1(M)$.*

In a different situation, Catanese [20] has also used restrictions on divisors to ensure that the complement of a curve in a surface and certain ramified coverings are simply-connected.

Example 6.50. Let $M = \mathbb{C}P^2$ and D a smooth complex curve of degree $n > 0$ representing $nH \in H_2(M;\mathbb{Z})$, where $H = [\mathbb{C}P^1]$ denotes the class of a hyperplane. The canonical class of $\mathbb{C}P^2$ is $K = -3PD(H)$. By the adjunction formula,

$$g(D) = 1 + \tfrac{1}{2}(K \cdot D + D^2)$$

we can compute the genus of D: $g(D) = 1 + \frac{1}{2}n(n-3)$. Since $D^2 > 0$ and $\mathbb{C}P^2$ is simply-connected, the complement $\mathbb{C}P^2 \setminus D$ has abelian fundamental group by Nori's theorem. This implies that

$$\pi_1(\mathbb{C}P^2 \setminus D) \cong H_1(\mathbb{C}P^2 \setminus D;\mathbb{Z}) \cong \mathbb{Z}_n,$$

which has been proved by Zariski in 1929 [147]. We can also consider the n-fold cyclic branched covering

$$\phi\colon \overline{M} = M(D,H,n) \to M.$$

By Corollary 6.47 the complex algebraic surface $\overline{M}$ is simply-connected. The invariants are given by the formulas in Proposition 6.43:

$$K_{\overline{M}} = (n-4)\phi^* H$$

$$c_1^2(\overline{M}) = n(n-4)^2$$

$$c_2(\overline{M}) = 3n + (n-1)n(n-3)$$

since $c_2(\mathbb{C}P^2) = 3$ and $e(D) = -n(n-3)$. The calculation

$$\begin{aligned} c_1^2(\overline{M}) - 2c_2(\overline{M}) &= n(n^2 - 8n + 16) - n(6 + 2n^2 - 8n + 6) \\ &= n(-n^2 + 4), \end{aligned}$$

implies

$$\sigma(\overline{M}) = -\tfrac{1}{3}(n^2-4)n$$

Note that $\overline{M}$ is a simply-connected 4-manifold such that $K_{\overline{M}}$ is divisible by $d = n-4$. However, $c_1^2(\overline{M})$ grows with the third power of d and is rather larger. One can show that $\overline{M}$ is diffeomorphic to a complex hypersurface in $\mathbb{C}P^3$ of degree n (cf. [56, Exercise 7.1.6]).

VI.4 Geography of simply-connected surfaces of general type

In this section, we collect some results on the geography of simply-connected surfaces of general type. We consider branched coverings of some of these surfaces over pluricanonical divisors in $|nK|$ in the next section. The surfaces we obtain will then have a canonical class divisible by a certain integer $d > 1$. We begin with the following result due to Persson [115, Proposition 3.23] which is the main geography result we will use for our constructions.

Theorem 6.51. *Let x, y be positive integers such that*

$$2x - 6 \leq y \leq 4x - 8.$$

Then there exists a simply-connected minimal complex surface M of general type such that $\chi_h(M) = x$ and $c_1^2(M) = y$. Furthermore, M can be chosen as a genus 2 fibration.

The smallest integer x to get an inequality which can be realized with $y > 0$ is $x = 3$. Since $\chi_h(X) = p_g(X) + 1$ for simply-connected surfaces, this corresponds to surfaces with $p_g = 2$. Hence we get minimal simply-connected complex surfaces M with

$$p_g = 2 \text{ and } K^2 = 1, 2, 3, 4.$$

Similarly, for $x = 4$, we get surfaces with

$$p_g = 3 \text{ and } K^2 = 2, \ldots, 8.$$

We consider surfaces of general type with $K^2 = 1$ and $K^2 = 2$ in general.

Proposition 6.52. *For $K^2 = 1$ and $K^2 = 2$ all possible values for p_g given by the Noether inequality can be realized by simply-connected minimal complex surfaces of general type.*

Proof. By the Noether inequality, only the following values for p_g are possible:

$$\begin{aligned} K^2 = 1: &\quad p_g = 0, 1, 2 \\ K^2 = 2: &\quad p_g = 0, 1, 2, 3. \end{aligned}$$

The cases $K^2 = 1, p_g = 2$ and $K^2 = 2, p_g = 2, 3$ are covered by Persson's theorem. The surfaces with $K^2 = 1, p_g = 2$ and $K^2 = 2, p_g = 3$ are Horikawa surfaces, as described in [65], [66]. The remaining cases can also be covered: The Barlow surface, constructed in [7], is a simply-connected minimal complex surface of general type with $K^2 = 1, p_g = 0$, hence it is a numerical Godeaux surface. Minimal surfaces of general type with $K^2 = 1, p_g = 1$ exist by constructions due to Enriques. They are described in [19]: they are all simply-connected and deformation equivalent, in particular diffeomorphic. Simply-connected minimal surfaces with $K^2 = 2, p_g = 1$ have also been constructed by Enriques (see [25], [22]). Finally, Lee and Park have recently constructed in [85] a simply-connected minimal surface of general type with $K^2 = 2, p_g = 0$. It is a numerical Campedelli surface. □

We now consider the case of surfaces of general type which are spin (the following two theorems are from [116]). Recall that spin complex surfaces necessarily have

$$c_1^2(M) \equiv 0 \bmod 8 \quad \text{and} \quad c_1^2(M) \equiv 8\chi_h(M) \bmod 16.$$

The first theorem shows that not all lattice points which satisfy these congruences can be realized by a simply-connected minimal complex surface of general type.

Theorem 6.53. *Suppose M is a simply-connected spin surface of general type with*

$$2\chi_h(M) - 6 \leq c_1^2(M) < 3(\chi_h(M) - 5).$$

Then M admits a fibration in genus 2 or genus 3 curves and the invariants are either

- $c_1^2(M) = 2\chi_h(M) - 6$*, where* $c_1^2(M)$ *is an odd multiple of* 8*, or*
- $c_1^2(M) = \frac{8}{3}(\chi_h(M) - 4)$*, with* $\chi_h(M) \equiv 1$ *mod* 3.

All possible points with these constraints can be realized by simply-connected spin complex surfaces of general type.

Note that $2\chi_h(M) - 6 \leq c_1^2(M)$ holds automatically by the Noether inequality. The first case (spin Horikawa surfaces on the Noether line) occurs if and only if there exists an integer $n \geq 0$, such that

$$\begin{aligned} c_1^2(M) &= 8(1 + 2n), \\ \chi_h(M) &= 7 + 8n. \end{aligned}$$

This implies that

$$e(M) = 76 + 80n, \quad \sigma(M) = -48 - 48n.$$

The second case occurs if and only if there exists an integer $n \geq 0$, such that

$$\begin{aligned} c_1^2(M) &= 8(1 + n), \\ \chi_h(M) &= 7 + 3n. \end{aligned}$$

This implies

$$e(M) = 76 + 28n, \quad \sigma(M) = -48 - 16n.$$

In [116] also an area with $c_1^2 \geq 3(\chi_h - 5)$ is covered. The congruences $c_1^2 \equiv 0 \bmod 8$ and $c_1^2 \equiv 8\chi_h \bmod 16$ imply

$$\frac{c_1^2}{8} \equiv \chi_h \bmod 2.$$

This congruence can be split in two cases:

$$\frac{c_1^2}{8} + \chi_h \equiv 0 \bmod 4, \quad \text{and} \quad \frac{c_1^2}{8} + \chi_h \equiv 2 \bmod 4.$$

The following theorem covers a sector for the second case.

Theorem 6.54. *Suppose that x, y are positive integers with $y \equiv 0$ mod 8 and $\frac{y}{8} + x \equiv 2$ mod 4. If*

$$3(x - 5) \leq y < \tfrac{16}{5}(x - 4),$$

then there exists a simply-connected spin surface M of general type, such that $\chi_h(M) = x$ and $c_1^2(M) = y$. The surface M can be realized as a fibration in genus 4 curves.

The surfaces of general type in this section all have $c_1^2 < 4\chi_h$ which is equivalent to $\sigma < -c_1^2$. There are also geography results for simply-connected surfaces of general type closer to the $\sigma = 0$ line ($c_1^2 = 8\chi_h$) or with positive signature [26, 115, 116]. In the simply-connected case, all surfaces have to lie below the line $c_1^2 = 9\chi_h$, which is given by the Bogomolov-Miyaoka-Yau inequality.

VI.5 Branched covering construction of algebraic surfaces with divisible canonical class

In this section we construct simply-connected complex algebraic surfaces as branched coverings such that the canonical class is divisible by a given integer $d > 0$. In subsections VI.5.1 and VI.5.2 we consider coverings branched over a smooth curve in the pluricanonical linear system $|nK_M|$, where M is a surface of general type. In subsection VI.5.3 we consider an example where the curve is singular, not a multiple of the canonical divisor and the surface M is not of general type.

We begin with the first case. Suppose that M is a simply-connected minimal complex surface of general type. Let $m, d \geq 2$ be integers such that $m-1$ divides $d-1$. We can write $a = \frac{d-1}{m-1}$ and define $n = ma$. Then $d = n + 1 - a$ and the assumptions imply that $n \geq 2$. We assume that nK_M can be represented by a smooth complex connected curve D in M (see Sections II.3.2 and II.3.7). Let $\overline{M} = M(D, aK_M, m)$ denote the associated m-fold branched cover over the curve D. We have

$$D^2 = n^2 K_M^2 > 0,$$

hence $\overline{M}$ is a simply-connected complex surface by Corollary 6.47. We can calculate the invariants by Lemma 6.44.

Theorem 6.55. *Let M be a simply-connected minimal surface of general type and $m, d \geq 2$ integers such that $d-1$ is divisible by $m-1$ with quotient a. Suppose that D is a smooth connected curve in the linear system $|nK_M|$ where $n = ma$. Then the m-fold cover of M, branched over D, is a simply-connected complex surface $\overline{M}$ of general type with invariants*

- $K_{\overline{M}} = d\phi^* K_M$
- $c_1^2(\overline{M}) = md^2 c_1^2(M)$
- $e(\overline{M}) = m(e(M) + (d-1)(d+a)c_1^2(M))$
- $\chi_h(\overline{M}) = m\chi_h(M) + \frac{1}{12}m(d-1)(2d+a+1)c_1^2(M)$
- $\sigma(\overline{M}) = -\frac{1}{3}m(2e(M) + (d(d-2) + 2a(d-1))c_1^2(M))$.

In particular, the canonical class $K_{\overline{M}}$ is divisible by d and $\overline{M}$ is minimal.

The surface $\overline{M}$ is of general type because $c_1^2(\overline{M}) > 0$ and $\overline{M}$ cannot be rational or ruled. The claim about minimality follows because the divisibility of $K_{\overline{M}}$ is at least $d \geq 2$, cf. Lemma 6.2. The formula for $\chi_h(\overline{M})$ follows by writing $e(\overline{M})$ in terms of $\chi_h(M), c_1^2(M)$,

- $e(\overline{M}) = 12m\chi_h(M) + m((d-1)(d+a) - 1)c_1^2(M)$

and calculating

$$\chi_h(\overline{M}) = m\chi_h(M) + \frac{1}{12}m((d-1)(d+a) + d^2 - 1)c_1^2(M),$$

which gives the formula above. Note also that $\sigma(\overline{M})$ is always negative. Hence we cannot construct surfaces with positive signature in this way, even if we start with surfaces of positive signature.

VI.5.1 General results

We want to compute the image of the sector of surfaces of general type from Theorem 6.51 for the transformation $\Phi\colon (e(M), c_1^2(M)) \mapsto (e(\overline{M}), c_1^2(\overline{M}))$ given by the formulas in Theorem 6.55. We use the following equivalent formulation of Persson's theorem:

Corollary 6.56. *Let e, c be positive integers such that $c \geq 36 - e$ and $e + c \equiv 0$ mod 12. If*

$$\tfrac{1}{5}(e-36) \leq c \leq \tfrac{1}{2}(e-24),$$

then there exists a simply-connected minimal surface M of general type with invariants $e(M) = e$ and $c_1^2(M) = c$.

Proof. Under the linear transformation $\chi_h = \frac{1}{12}(c_1^2 + e)$, the Noether line $c_1^2 = 2\chi_h - 6$ maps to

$$c_1^2 = \tfrac{1}{6}(c_1^2 + e) - 6,$$

hence $c_1^2 = \frac{1}{5}(e-36)$. Similarly, the line $c_1^2 = 4\chi_h - 8$ maps to $c_1^2 = \frac{1}{2}(e-24)$. Persson's theorem gives reasonable points (x, y) only for $x \geq 3$. The line $\chi_h = 3, c_1^2 = t$ for $t \geq 0$ maps to $c_1^2 = 36 - e$. The points we consider in the (e, c) plane have to be to the right from this line, hence $c \geq 36 - e$. Conversely, if (e, c) is an integral point in the sector defined by these three lines and satisfies the condition $e + c \equiv 0$ mod 12 coming from the Noether formula, we can compute the integer $\chi_h = \frac{1}{12}(c_1^2 + e)$ and see that (e, c) is the image of a point in the sector in Persson's theorem. $\square$

Let m, a, d be integers as above. We can write the transformation as

$$\begin{pmatrix} e(\overline{M}) \\ c_1^2(\overline{M}) \end{pmatrix} = m \begin{pmatrix} 1 & \Delta \\ 0 & d^2 \end{pmatrix} \begin{pmatrix} e(M) \\ c_1^2(M) \end{pmatrix},$$

where we have made the abbreviation $\Delta = (d-1)(d+a)$. Φ is a linear map, which is invertible over $\mathbb{R}$ and maps the quadrant where both coordinates have non-negative entries into the same quadrant. The inverse of Φ is given by

$$\begin{pmatrix} e(M) \\ c_1^2(M) \end{pmatrix} = \tfrac{1}{m} \begin{pmatrix} 1 & -\Delta/d^2 \\ 0 & 1/d^2 \end{pmatrix} \begin{pmatrix} e(\overline{M}) \\ c_1^2(\overline{M}) \end{pmatrix},$$

Since $e(M)$ and $c_1^2(M)$ are integers with $e(M) + c_1^2(M) \equiv$ mod 12, we see that a point $(x, y) = (e', c') \in \mathbb{Z} \times \mathbb{Z}$ is in the image of the map Φ, if and only if c' is divisible by md^2, e' is divisible by m and $\frac{1}{m}e' + \frac{1-\Delta}{md^2}c' \equiv 0$ mod 12.

We want to calculate the image of the line $c = \frac{1}{5}(e-36)$, which appears in the version of Persson's theorem above. Let $e = t, c = \frac{1}{5}(t-36)$, for $t \geq 0$. Then

$$\Phi \begin{pmatrix} t \\ \frac{1}{5}(t-36) \end{pmatrix} = m \begin{pmatrix} t + (t-36)\Delta/5 \\ d^2(t-36)/5 \end{pmatrix}.$$

This implies

$$\begin{aligned} e(\overline{M}) &= mt(1 + \tfrac{1}{5}\Delta) - \tfrac{36}{5}m\Delta \\ c_1^2(\overline{M}) &= \tfrac{1}{5}md^2t - \tfrac{36}{5}md^2. \end{aligned}$$

We can solve the first equation for t and replace t in the second equation. We get:

$$c_1^2(\overline{M}) = \frac{1}{(1 + \frac{1}{5}\Delta)} \left(\tfrac{1}{5}d^2 e(\overline{M}) - \tfrac{36}{5}md^2\right),$$

hence

$$c_1^2(\overline{M}) = \frac{d^2}{(5+\Delta)}(e(\overline{M}) - 36m). \tag{6.7}$$

Similarly the line $c_1^2 = \frac{1}{2}(e-24)$ maps to

$$c_1^2(\overline{M}) = \frac{d^2}{(2+\Delta)}(e(\overline{M}) - 24m). \tag{6.8}$$

The points given by Persson's theorem have to satisfy the constraint $c \geq 36 - e$. The image of the line $c_1^2 = 36 - e$ is

$$c_1^2(\overline{M}) = -\frac{d^2}{(1-\Delta)}(e(\overline{M}) - 36m). \tag{6.9}$$

Summarizing the calculation, we see that the image of the lattice points in the sector $\frac{1}{5}(e-36) \leq c \leq \frac{1}{2}(e-24)$, with $c \geq 36-e$ and $e+c \equiv 0$ mod 12, is given precisely by the points in the sector between the lines (6.7) and (6.8), which are to the right of the line (6.9) and satisfy $e(\overline{M}) \equiv 0$ mod m, $c_1^2(\overline{M}) \equiv$ mod md^2 and

$$\tfrac{1}{m}e(\overline{M}) + \tfrac{1-\Delta}{md^2}c_1^2(\overline{M}) \equiv 0 \text{ mod } 12.$$

The surfaces in Persson's theorem 6.51 have $p_g \geq 2$ and $K^2 \geq 1$. By section II.3.2, the linear system $|nK|$, for $n \geq 2$, on these surfaces has no base points, except in the case $p_g = 2, K^2 = 1$ and $n = 3$. Since $n = ma$ and $m \geq 2$, this occurs only for $m = 3, a = 1$ and $d = 3$. The corresponding image under Φ has invariants $(e, c_1^2) = (129, 27)$. This exception is always understood in the following.

In all other cases we can consider the branched covering construction from this section to get minimal surfaces of general type with the invariants above, such that the canonical class is divisible by d. We can summarize this as follows: Consider integers m, a, d as above, with $m, d \geq 2, a \geq 1$ and $\Delta = (d-1)(d+a)$.

Theorem 6.57. *Let x, y be positive integers such that $y(1-\Delta) \geq 36 - x$ and $x + (1-\Delta)y \equiv 0$ mod 12. If*

$$\frac{1}{(5+\Delta)}(x-36) \leq y \leq \frac{1}{(2+\Delta)}(x-24),$$

then there exists a simply-connected minimal complex surface M of general type with invariants $e(M) = mx$ and $c_1^2(M) = md^2y$, such that the canonical class of M is divisible by d.

Note that the sector in Persson's theorem 6.51 intersects non-trivially with the lines and sectors for spin surfaces, given by 6.53 and 6.54. In this case, a point in the (χ_h, c_1^2) plane can be realized by a spin surface and the formula for the canonical class of the branched covering shows that K is then already divisible by $2d$. We have calculated some examples for small values of d and m, see Table VI.1.

VI.5.2 Examples

In this section we calculate some further examples for the branched covering construction given by Theorem 6.57 and for some surfaces not covered by Persson's theorem. Note that for any $d \geq 2$, we can choose $m = 2$ and $a = d-1$ corresponding to 2-fold covers branched over $(2d-2)K$. The formulas for the invariants simplify to

- $c_1^2(\overline{M}) = 2d^2c_1^2(M)$
- $e(\overline{M}) = 24\chi_h(M) + 2d(2d-3)c_1^2(M)$

d	m	Δ	Invariants $e(M), c_1^2(M)$ with the corresponding point (x,y) underneath				
3	2	10	90, 18 $(45,1)$	108, 36 $(54,2)$	132, 36 $(66,2)$	126, 54 $(63,3)$	150, 54 $(75,3)$
3	3	8	$--$ $(43,1)$	150, 54 $(50,2)$	186, 54 $(62,2)$	171, 81 $(57,3)$	207, 81 $(69,3)$
4	2	21	112, 32 $(56,1)$	154, 64 $(76,2)$	176, 64 $(88,2)$	192, 96 $(96,3)$	216, 96 $(108,3)$
4	4	15	200, 64 $(50,1)$	256, 128 $(64,2)$	304, 128 $(76,2)$	312, 192 $(78,3)$	360, 192 $(90,3)$
5	2	36	142, 50 $(71,1)$	212, 100 $(106,2)$	236, 100 $(118,2)$	282, 150 $(141,3)$	306, 150 $(153,3)$
6	2	55	180, 72 $(90,1)$	288, 144 $(144,2)$	312, 144 $(156,2)$	396, 216 $(198,3)$	420, 216 $(210,3)$

Table VI.1: Ramified coverings of surfaces from Persson's theorem 6.51 with divisible K.

- $\chi_h(\overline{M}) = 2\chi_h(M) + \frac{1}{2}d(d-1)c_1^2(M)$.

The first two examples are double coverings with $m = 2$, the third example uses coverings of higher degree. Note that some of the surfaces are because of their invariants (c_1^2, e and the parity of the divisibility of K) homeomorphic to some of the simply-connected symplectic 4-manifolds constructed in Sections VI.2.2 and VI.2.3.

Example 6.58. We consider the Horikawa surfaces [65] on the Noether line $c_1^2 = 2\chi_h - 6$, which exist for every $\chi_h \geq 4$ and are also given by Persson's theorem 6.51. We have $p_g \geq 3$ and $c_1^2 \geq 2$. Hence by Theorem 2.4 the linear system $|nK|$ for $n \geq 2$ on these surfaces has no base points. The Noether line corresponds in the version of Persson's theorem in Corollary 6.56 to the line $c_1^2 = \frac{1}{5}(e-36)$. We take $m = 2$ and $a = d-1$. It is easier in this case to calculate the points in the image of the Noether line under Φ directly. The equation $c_1^2(M) = 2\chi_h(M) - 6$ implies

$$\chi_h(\overline{M}) = 6 + (1 + \tfrac{1}{2}d(d-1))c_1^2(M),$$

by the formulas above.

Proposition 6.59. *Let M be a Horikawa surface on the Noether line $c_1^2 = 2\chi_h - 6$ where $\chi_h = 4 + l$ for $l \geq 0$. Then the 2-fold cover $\overline{M}$ of the surface M, branched over $(2d-2)K_M$ for an integer $d \geq 2$, has invariants*

$$c_1^2(\overline{M}) = 4d^2(l+1)$$

$$\chi_h(\overline{M}) = 6 + (2 + d(d-1))(l+1)$$

$$e(\overline{M}) = 72 + 4(l+1)(6 + 2d^2 - 3d)$$

$$\sigma(\overline{M}) = -48 - 4(l+1)(4 + d^2 - 2d).$$

The canonical class $K_{\overline{M}}$ is divisible by d.

For d even, the integer $d^2-2d=d(d-2)$ is divisible by 4, hence σ is indeed divisible by 16, which is necessary by Rochlin's theorem. Since there exist spin Horikawa surfaces for $c_1^2(M)=8(1+2k)$ with $k\geq 0$, the canonical class on the branched covers with $l=8k+3$ are divisible by $2d$. The invariants are on the line

$$c_1^2(\overline{M})=\frac{4d^2}{2+d(d-1)}(\chi_h(\overline{M})-6), \tag{6.10}$$

which has inclination close to 4 for d very large. Moreover, we have

$$c_1^2(\overline{M})=\frac{d^2}{6+2d^2-3d}(e(\overline{M})-72).$$

Since $\Delta=(d-1)(2d+1)=1+2d^2-3d$, this is exactly the line

$$y=\frac{1}{(5+\Delta)}(x-36),$$

given by Theorem 6.57, for $c_1^2=2d^2y, e=2x$.

Example 6.60. We calculate the invariants for the branched covers with $m=2$ and integers $d\geq 3$ for the surfaces given by Proposition 6.52. Since $n=ma\geq 4$ in this case, Theorem 2.4 shows that the linear system $|nK|$ has no base points and we can use the branched covering construction.

Proposition 6.61. *Let M be a minimal complex surface of general type with $K^2=1$ or 2. Then the 2-fold cover $\overline{M}$ of the surface M, branched over $(2d-2)K_M$ for an integer $d\geq 3$, has invariants*

$$c_1^2(\overline{M})=2d^2$$

$$e(\overline{M})=24(p_g+1)+2d(2d-3)$$

$$\sigma(\overline{M})=-16(p_g+1)-2d(d-2), \quad \textit{if } K^2=1 \textit{ and } p_g=0,1,2.$$

$$c_1^2(\overline{M})=4d^2$$

$$e(\overline{M})=24(p_g+1)+4d(2d-3)$$

$$\sigma(\overline{M})=-16(p_g+1)-4d(d-2), \quad \textit{if } K^2=2 \textit{ and } p_g=0,1,2,3.$$

In both cases the canonical class $K_{\overline{M}}$ is divisible by d.

Example 6.62. Consider the Barlow surface M_B and the surface M_{LP} of Lee and Park that were mentioned in the proof of Proposition 6.52. They have invariants

$c_1^2(M_B)=1, \chi_h(M_B)=1$ and $c_2(M_B)=11$

$c_1^2(M_{LP})=2, \chi_h(M_{LP})=1$ and $c_2(M_{LP})=10.$

By section II.3.2, we can consider branched covers over both surfaces with $ma\geq 3$ (the Barlow surface is a simply-connected numerical Godeaux surface, hence $|3K|$ is base point free). See Tables VI.2 and VI.3 for a calculation of the invariants of $\overline{M}$ for small values of d and m. The 2-fold covering of the Barlow surface branched over $4K_M$ has the same invariants c_1^2, σ and divisibility of the canonical class ($d=3$) as a simply-connected symplectic 4-manifold obtained in Corollary 6.33. There is also a coincidence between the 4-fold cover of the Barlow surface branched over $4K_M$ and the 2-fold cover of the surface of Lee and Park branched over $6K_M$: Both have the same Chern invariants and the same divisibility $d=4$ of the canonical class. Hence the manifolds are homeomorphic, but it is unclear whether they are diffeomorphic. By Lemma 6.4, both branched coverings have the same Seiberg-Witten invariants.

d	m	ma	$(d-1)(d+a)$	$e(\overline{M_B})$	$c_1^2(\overline{M_B})$	$\chi_h(\overline{M_B})$	$b_2^+(\overline{M_B})$	$\sigma(\overline{M_B})$
3	2	4	10	42	18	5	9	-22
3	3	3	8	57	27	7	13	-29
4	2	6	21	64	32	8	15	-32
4	4	4	15	104	64	14	27	-48
5	2	8	36	94	50	12	23	-46
5	3	6	28	117	75	16	31	-53
5	5	5	24	175	125	25	49	-75
6	2	10	55	132	72	17	33	-64
6	6	6	35	276	216	41	81	-112

Table VI.2: Ramified coverings of the Barlow surface M_B of degree m branched over maK.

VI.5.3 Branched covers over singular curves

One can also construct examples of algebraic surfaces with divisible canonical class by taking branched covers over singular curves. It is also not necessary to start with surfaces of general type and branching divisors which are a multiple of the canonical class. The following example of such a covering is described in [56, Chapter 7]: Let $B_{n,m}$ denote the singular complex curve in $\mathbb{C}P^1 \times \mathbb{C}P^1$ which is the union of $2n$ parallel copies of the first factor and $2m$ parallel copies of the second factor. The curve $B_{n,m}$ represents in cohomology the class $2nS_1 + 2mS_2$, where $S_1 = [\mathbb{C}P^1 \times \{*\}]$ and $S_2 = [\{*\} \times \mathbb{C}P^1]$. Let $X'(n,m)$ denote the double covering of $\mathbb{C}P^1 \times \mathbb{C}P^1$ branched over $B_{n,m}$. It is a singular complex surface, which has a canonical resolution $X(n,m)$ (see [8, Chapter III]). As a smooth 4-manifold, $X(n,m)$ is diffeomorphic to the double cover of $\mathbb{C}P^1 \times \mathbb{C}P^1$ branched over the smooth curve $\widetilde{B}_{n,m}$ given by smoothing the double points. Hence we can calculate the topological invariants for $X = X(n,m)$ with the formulas from Proposition 6.43 and get:

$$c_1^2(X) = 4(n-2)(m-2)$$

$$e(X) = 6 + 2(2m-1)(2n-1)$$

$$\sigma(X) = -4mn$$

We write $X' = X'(n,m)$ and $M = \mathbb{C}P^1 \times \mathbb{C}P^1$. Let $\phi\colon X' \to M$ denote the double covering, $\pi\colon X \to X'$ the canonical resolution and $\psi = \phi \circ \pi$ the composition. Since all singularities of $B_{n,m}$ are ordinary double points we can calculate the canonical class of X by [8, Theorem 7.2, Chapter III]:

$$\begin{aligned} K_X &= \psi^*(K_M + \tfrac{1}{2}B_{m,n}) \\ &= \psi^*(-2S_1 - 2S_2 + nS_1 + mS_2) \\ &= \psi^*((n-2)S_1 + (m-2)S_2). \end{aligned}$$

One can give the following interpretation of this formula: The map $\psi\colon X \to \mathbb{C}P^1 \times \mathbb{C}P^1$ followed by the projection onto the first factor defines a fibration $X \to \mathbb{C}P^1$ whose fibres are the branched covers of the rational curves $\{p\} \times \mathbb{C}P^1$, where $p \in \mathbb{C}P^1$. The generic rational curve among them

d	m	ma	$(d-1)(d+a)$	$e(\overline{M_{LP}})$	$c_1^2(\overline{M_{LP}})$	$\chi_h(M_{LP})$	$b_2^+(\overline{M_{LP}})$	$\sigma(\overline{M_{LP}})$
3	2	4	10	60	36	8	15	-28
3	3	3	8	78	54	11	21	-34
4	2	6	21	104	64	14	27	-48
4	4	4	15	160	128	24	47	-64
5	2	8	36	164	100	22	43	-76
5	3	6	28	198	150	29	57	-82
5	5	5	24	290	250	45	89	-110
6	2	10	55	240	144	32	63	-112
6	6	6	35	480	432	76	151	-176

Table VI.3: Ramified coverings of the Lee-Park surface M_{LP} of degree m branched over maK.

is disjoint from the $2m$ curves in $B_{n,m}$ parallel to $\{*\} \times \mathbb{C}P^1$ and intersects the $2n$ curves parallel to $\mathbb{C}P^1 \times \{*\}$ in $2n$ points. This implies that the generic fibre F_2 of the fibration is a double branched cover of $\mathbb{C}P^1$ in $2n$ distinct points and hence a smooth complex curve of genus $n-1$. In the surface X it represents the class $\psi^* S_2$. Similarly, there is a fibration $X \to \mathbb{C}P^1$ in genus $m-1$ curves which represent $F_1 = \psi^* S_1$. Hence we can write

$$K_X = (n-2)F_1 + (m-2)F_2.$$

Since the rational curves given by the factors in $\mathbb{C}P^1 \times \mathbb{C}P^1$ intersect in one point, the fibres F_1 and F_2 will intersect on the resolution of the double covering in two points, hence $F_1F_2 = 2$. This implies again $c_1^2(X) = 4(n-2)(m-2)$.

One can show that all of the surfaces $X(n,m)$ are simply-connected [56, Exercise 7.3.16]. By varying n and m we can achieve all divisibilities, e.g. for $n = m = 6$ we get an algebraic surface $X(6,6)$ with invariants $c_1^2 = 64, e = 248, \sigma = -144$ and K_X divisible by 4. In general, one can show that $X(1,m)$ is diffeomorphic to $\mathbb{C}P^2\#(4m+1)\overline{\mathbb{C}P^2}$ (see [40], [56, Exercise 7.3.8]), $X(2,m)$ is diffeomorphic to the elliptic surface $E(m)$ and $X(3,m)$ is a Horikawa surface on the Noether line $c_1^2 = 2\chi_h - 6$.

Remark 6.63. Catanese and Wajnryb [24, 20, 21] have constructed surfaces via branched coverings over singular curves with the following properties: Suppose $a, b, c-1 \geq 2$ are integers. Then there exist simply-connected surfaces S of general type with invariants

$$\begin{aligned} c_1^2(S) &= 8(a+c-2)(2b-2) \\ \chi_h(S) &= (a+c-2)(2b-2) + 4b(a+c), \end{aligned}$$

and the divisibility of K_S is the greatest common divisor of $a+c-2$ and $2b-2$. Moreover, some of these surfaces are diffeomorphic but not deformation equivalent, thus giving counter-examples to a well-known conjecture.

Chapter VII

The classification of simply-connected 5-manifolds

Contents

In 1965, D. Barden gave a complete classification of simply-connected closed 5-manifolds [6]. The proof uses the theory of h-cobordisms developed by S. Smale [124] to conclude that two 5-manifolds which agree in certain topological invariants are diffeomorphic (Smale gave a classification for spin simply-connected 5-manifolds in 1962, cf. [125]).

In this chapter we describe the topological invariants of simply-connected 5-manifolds X used in the classification, in particular the linking form on the torsion subgroup of the second integral homology. The linking form gives rise to the so-called *i-invariant* which takes integer values in $\{0, 1, \ldots, \infty\}$. The i-invariant is also related to the second Stiefel-Whitney class and vanishes if X is spin. The result of Barden's theorem is that two simply-connected 5-manifolds are diffeomorphic if and only if they have isomorphic second homology and the i-invariants are the same.

Using Barden's theorem it is possible to determine all simply-connected 5-manifolds which are irreducible under connected sum. One can also show that every simply-connected 5-manifold X can be decomposed under connected sum into finitely many irreducible pieces. The splitting is unique if it contains at most one non-spin summand. As a corollary, we determine all simply-connected 5-manifolds with torsion free homology up to diffeomorphism. This will be used in Chapter IX to classify simply-connected 5-manifolds which can be obtained as circle bundles over simply-connected 4-manifolds.

The material in this chapter is not new, it is already contained in Barden's article [6]. However, we try to do some of the calculations in more detail, in particular in Sections VII.5 and VII.6 on the constructions of the irreducible building blocks of simply-connected 5-manifolds and the connected sum decomposition.

VII.1 Linking forms

VII.1.1 The topological linking form

Let X^n be a closed oriented n-dimensional manifold. Fix an element $\xi \in \mathrm{Tor}H_{n-q-1}(X;\mathbb{Z})$ and let $y = PD(\xi) \in \mathrm{Tor}H^{q+1}(X;\mathbb{Z})$ denote the Poincaré dual of ξ. We consider the long exact sequence in cohomology associated to the sequence of coefficient groups $0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z} \to 0$:

$$\ldots \longrightarrow H^q(X;\mathbb{Q}) \xrightarrow{p_*} H^q(X;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H^{q+1}(X;\mathbb{Z}) \xrightarrow{i_*} H^{q+1}(X;\mathbb{Q}) \longrightarrow \ldots$$

Here β denotes the associated Bockstein homomorphism. Since y is a torsion element, $i_* y = 0$. Hence there exists an $x \in H^q(X;\mathbb{Q}/\mathbb{Z})$ with $\beta(x) = y$.

Definition 7.1. Let $\xi \in \mathrm{Tor}H_{n-q-1}(X;\mathbb{Z})$ be as above and $\eta \in \mathrm{Tor}H_q(X;\mathbb{Z})$ an arbitrary element. The *linking number* of η and ξ is defined as

$$b(\eta, \xi) = \langle x, \eta \rangle \in \mathbb{Q}/\mathbb{Z}.$$

This number is well-defined, independent of the choice of x: if $x' \in H^q(X;\mathbb{Q}/\mathbb{Z})$ is another element with $\beta(x') = y$, then $x' - x = p_*\mu$, for some element $\mu \in H^q(X;\mathbb{Q})$. Since rational cohomology classes evaluate to zero on torsion homology classes, $\langle x', \eta \rangle = \langle x, \eta \rangle$.

The name "linking number" has the following interpretation: one can represent the homology classes η and ξ by cycles u and z. Since η is a torsion class, there exists a chain $c \in C_{q+1}(X)$ such that $\partial c = au$ for some $a \in \mathbb{Z}$. One can show that c has a well-defined intersection number with z, which is equal to $a \cdot b(\eta, \xi)$ (cf. [129]).

The following theorem summarizes the basic properties of linking numbers.

Theorem 7.2. *The linking numbers define a non-degenerate bilinear form*

$$b\colon \mathit{Tor}H_q(X;\mathbb{Z}) \times \mathit{Tor}H_{n-q-1}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

This form is called the linking form. *In different degrees, the linking forms are related by* $b(\eta, \xi) = (-1)^{nq+1} b(\xi, \eta)$ *for all* $\eta \in \mathit{Tor}H_q(X;\mathbb{Z})$ *and* $\xi \in \mathit{Tor}H_{n-q-1}(X;\mathbb{Z})$.

A proof can be found, e.g. in [129], Chapter 14.7 and 15.6.

Proposition 7.3. *If* $h\colon X \to Y$ *is a homotopy equivalence, then*

$$b_Y(h_*\eta, h_*\xi) = b_X(\eta, \xi).$$

Proof. Let $g\colon Y \longrightarrow X$ be a homotopy inverse to h and $\beta(x) = PD(\xi)$ as in the definition of the linking number. Then we have:

$$PD(h_*\xi) = g^* PD(\xi) = g^*\beta(x) = \beta(g^* x).$$

The claim now follows from

$$\langle g^* x, h_*\eta \rangle = \langle x, g_* h_* \eta \rangle = \langle x, \eta \rangle.$$

□

Suppose the dimension of X is odd, $n = 2q+1$. Then the linking numbers define a non-degenerate bilinear form

$$b\colon \mathrm{Tor}H_q(X;\mathbb{Z}) \times \mathrm{Tor}H_q(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

If q is even, then b is skew-symmetric by Theorem 7.2.

Definition 7.4. The linking form of a closed, oriented 5-manifold X is the non-degenerate skew-symmetric bilinear form given by

$$b\colon \mathrm{Tor}H_2(X;\mathbb{Z}) \times \mathrm{Tor}H_2(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

VII.1.2 Skew-symmetric bilinear forms

Let G be a finite abelian group and $b\colon G \times G \to \mathbb{Q}/\mathbb{Z}$ a non-degenerate skew-symmetric bilinear form. Then b defines a homomorphism $\overline{b}\colon G \to \mathbb{Z}_2$ in the following way: By skew-symmetry, we have

$$2b(x,x) = b(x,x) + b(x,x) = 0,$$

hence $b(x,x) \in \{0, \frac{1}{2}\}$ for all $x \in G$. We can then consider the map

$$\begin{aligned} \overline{b}\colon G &\longrightarrow \mathbb{Z}_2, \\ x &\mapsto 2b(x,x). \end{aligned}$$

This is a homomorphism:

$$\overline{b}(x+y) = 2b(x+y,x+y) = \overline{b}(x) + 2b(x,y) + 2b(y,x) + \overline{b}(y) = \overline{b}(x) + \overline{b}(y).$$

More generally, let H be a finitely generated abelian group and $\phi\colon H \to \mathbb{Z}_p$ a homomorphism, where p is a prime.

Definition 7.5. A basis for H as an abelian group, such that ϕ is non-zero on at most one basis element, is called a *ϕ-basis*.

Suppose $x \in H$ is an element with $\phi(x) \neq 0$. In particular $x \neq 0$. Let $1 \leq r \leq \infty$ denote the order of x. Then $r\phi(x) = 0$, hence r is divisible by p. This implies that the order of x is of the form $r = p^i$ with $1 \leq p \leq \infty$.

Lemma 7.6. *If H is a finitely generated abelian group and $\phi\colon H \to \mathbb{Z}_p$ a homomorphism, then H has a ϕ-basis such that all basis elements have prime power order.*

Proof. We follow the proof in [6]. Let $e_1, \ldots, e_a$ denote a basis of H such that all elements have prime power order (including, possibly, infinite order). If the order of e_i is not a power of p, then $\phi(e_i) = 0$. We can assume without loss of generality that the basis elements of H of order a power of p, on which ϕ is non-zero, are $e_1, \ldots, e_b$, where $0 \leq b \leq a$ and the order of e_i is at least the order of e_{i+1}, for all $0 \leq i \leq b-1$. The orders of e_1, e_2 are of the form p^r, p^s, with $1 \leq s \leq r \leq \infty$. Then

$$\phi(e_1) = kn, \;\; \phi(e_2) = n,$$

for some $n \in \mathbb{Z}_p$ and $k \in \mathbb{Z}$, not divisible by p. The elements $\{e_1 - ke_2, e_2\}$ form a basis of the subgroup H' generated by $\{e_1, e_2\}$: If $xe_1 + ye_2$ is an arbitrary element in this subgroup, then

$$xe_1 + ye_2 = x(e_1 - ke_2) + (y + kx)e_2.$$

Hence $\{e_1 - ke_2, e_2\}$ generate H'. Suppose $v(e_1 - ke_2) + we_2 = 0$ for integers $v, w \in \mathbb{Z}$. We get $ve_1 + (w - vk)e_2 = 0$, which implies $ve_1 = 0 = (w - vk)e_2$. Hence p^r divides v and p^s divides $w - vk$. Since $s \leq r$, the integer p^s also divides v, hence it divides w. Therefore, $we_2 = 0 = v(e_1 - ke_2)$.

Note that $\phi(e_1 - ke_2) = 0$. Hence we can change the basis elements $e_1, \ldots, e_b$ to new basis elements such that ϕ vanishes on one of them. In this way, we can change the basis inductively, until ϕ is non-zero on at most one basis element. □

Choose a ϕ-basis for H consisting of elements of prime power order. If $\phi \equiv 0$, set $i(\phi) = 0$. If ϕ is not identically zero, let p^i with $1 \leq i \leq \infty$ be the order of the basis element on which ϕ is non-zero. We set $i(\phi) = i$.

Definition 7.7. The integer $i(\phi) \in \{0, 1, \ldots, \infty\}$ is called the *i-invariant* of the homomorphism $\phi \colon H \to \mathbb{Z}_p$. One can show that $i(\phi)$ does not depend on the choice of ϕ-basis for H and $i(\phi) = i(\phi \circ \alpha)$ for any automorphism α of H (see [6]).

We now consider again a finite abelian group G and $b \colon G \times G \to \mathbb{Q}/\mathbb{Z}$ a non-degenerate skew-symmetric bilinear form. Then there is the homomorphism $\overline{b} \colon G \to \mathbb{Z}_2$ as above. The following theorem is proved in [6].

Theorem 7.8. *For a finite abelian group G as above, the form b is determined by the i-invariant $i(\overline{b})$ up to isomorphism.*

One can also give an explicit classification of non-degenerate skew-symmetric bilinear forms on finite abelian groups. Consider the following forms:

- A on $\mathbb{Z}_2$, given on the generator x by $b(x, x) = 1/2$.
- B_m on $\mathbb{Z}_m \oplus \mathbb{Z}_m$ for $m \geq 2$, given on the standard generators by

$$\begin{pmatrix} 0 & 1/m \\ -1/m & 0 \end{pmatrix}.$$

- C_m on $\mathbb{Z}_m \oplus \mathbb{Z}_m$ for $m \geq 2$ even, given on the standard generators by

$$\begin{pmatrix} 0 & 1/m \\ -1/m & 1/2 \end{pmatrix}.$$

One can prove the following theorem:

Theorem 7.9. *Let G be a finite abelian group and $b \colon G \times G \to \mathbb{Q}/\mathbb{Z}$ a non-degenerate skew-symmetric bilinear form. Then G has a basis such that b is given by a form of one of the following three types:*

- $B_{m_1} \oplus \ldots \oplus B_{m_k}$
- $B_{m_1} \oplus \ldots \oplus B_{m_{k-1}} \oplus A$
- $B_{m_1} \oplus \ldots \oplus B_{m_{k-1}} \oplus C_{2^r}$, $r \geq 1$.

For a proof, see [129]. Since the corresponding bases are $\overline{b}$-bases, we can read off the i-invariants: They are 0, 1 and r, with $r \geq 1$, respectively. Note that the second case and the third case for $r = 1$ are distinguished by the isomorphism type of the underlying groups. As a corollary, using the toplogical linking form from the first section in this chapter, we get:

Corollary 7.10. *If X is a closed, oriented manifold of dimension $n = 4q + 1$, then*

$$\mathit{Tor} H_{2q}(X; \mathbb{Z}) \cong H \oplus H \text{ or } H \oplus H \oplus \mathbb{Z}_2,$$

for some finite abelian group H. In the second case, the i-invariant of $\bar{b}$ has to be equal to 1.

In particular, this holds for $\mathrm{Tor} H_2(X; \mathbb{Z})$ for a closed, oriented 5-manifold.

VII.2 The Stiefel-Whitney classes

In this section we show that the Stiefel-Whitney classes of a closed differentiable manifold M depend only on the homotopy type of M. This will be needed later to prove that if two simply-connected closed 5-manifolds are homotopy equivalent, then they are already diffeomorphic. A reference for this section is [16, Chapter VI, Section 17.].

If X is a topological space, the *Steenrod squares* are certain homomorphisms

$$Sq^i \colon H^k(X; \mathbb{Z}_2) \longrightarrow H^{k+i}(X; \mathbb{Z}_2),$$

which exist for all $i, k \geq 0$ and are natural with respect to continuous maps $f \colon X \to Y$. Let M be a closed differentiable manifold of dimension n. We need not assume that M is oriented. In any case, it has a $\mathbb{Z}_2$-fundamental class $[M] \in H_n(M; \mathbb{Z}_2)$.

Lemma 7.11. *The homomorphism*

$$H^i(M; \mathbb{Z}_2) \longrightarrow \mathit{Hom}(H^{n-i}(M; \mathbb{Z}_2), \mathbb{Z}_2), a \mapsto \langle a \cup -, [M] \rangle,$$

is an isomorphism.

Proof. We have two isomorphisms:

(1.) $H^i(M; \mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_i(M; \mathbb{Z}_2), \mathbb{Z}_2), a \mapsto \langle a, - \rangle$, given by the Universal Coefficient Theorem since $\mathbb{Z}_2$ is a field, and

(2.) $H^{n-i}(M; \mathbb{Z}_2) \longrightarrow H_i(M; \mathbb{Z}_2), c \mapsto c \cap [M]$, given by Poincaré duality.

Both isomorphisms combine to the isomorphism in the statement of the lemma. □

Consider now the homomorphism

$$H^{n-i}(M; \mathbb{Z}_2) \to \mathbb{Z}_2, c \mapsto \langle Sq^i(c), [M] \rangle.$$

The Lemma implies that there exist unique classes $v_i(M) \in H^i(M; \mathbb{Z}_2)$, for $i \geq 0$, such that

$$\langle v_i(M) \cup c, [M] \rangle = \langle Sq^i(c), [M] \rangle \quad \forall c \in H^{n-i}(M; \mathbb{Z}_2). \tag{7.1}$$

The $v_i(M)$ are called the *Wu classes* of M. One can prove that they determine the Stiefel-Whitney classes in the following way:

$$w_k(M) = \sum_j Sq^{k-j} v_j(M).$$

See [16], Theorem 17.5 Chapter VI. From this we deduce

Proposition 7.12. *Suppose $h\colon M \to N$ is a homotopy equivalence between smooth closed manifolds. Then*
$$h^* w_k(N) = w_k(M) \quad \forall k \geq 0.$$

Proof. It is enough to show that $h^* v_i(N) = v_i(M)$ for all k. Let $g\colon N \longrightarrow M$ be a homotopy inverse to h. We get:
$$\begin{aligned}\langle h^* v_i(N) \cup c, [M]\rangle &= \langle v_i(N) \cup g^* c, g_*[M]\rangle = \langle v_i(N) \cup g^* c, [N]\rangle \\ &= \langle Sq^i(g^* c), [N]\rangle = \langle g^* Sq^i(c), [N]\rangle \\ &= \langle Sq^i(c), g_*[N]\rangle = \langle Sq^i(c), [M]\rangle, \quad \forall c \in\in H^{n-i}(M;\mathbb{Z}_2).\end{aligned}$$
By uniqueness, this implies $h^* v_i(N) = v_i(M)$. □

VII.3 The topological invariants of simply-connected 5-manifolds

In this section, let X be a closed, simply-connected, oriented 5-manifold. We want to describe the topological invariants of X.

VII.3.1 Homology and cohomology of X

Let $G = H_2(X;\mathbb{Z})$. Then the homology and cohomology groups of X are completely determined by G: This follows by Poincaré duality
$$H^k(X;\mathbb{Z}) \cong H_{5-k}(X;\mathbb{Z})$$
and the Universal Coefficient Theorem, which implies
$$\mathrm{Tor} H^k(X;\mathbb{Z}) \cong \mathrm{Tor} H_{k-1}(X;\mathbb{Z}) \quad \text{and} \quad H^k(X;\mathbb{Z})/\mathrm{Tor} \cong H_k(X;\mathbb{Z})/\mathrm{Tor}.$$

	0	1	2	3	4	5
$H_n(X;\mathbb{Z})$	$\mathbb{Z}$	0	G	$G/\mathrm{Tor}\, G$	0	$\mathbb{Z}$
$H^n(X;\mathbb{Z})$	$\mathbb{Z}$	0	$G/\mathrm{Tor}\, G$	G	0	$\mathbb{Z}$

Table VII.1: Integral homology and cohomology in degree n of simply-connected 5-manifolds X.

VII.3.2 The linking form

The linking numbers define a non-degenerate skew-symmetric bilinear form
$$b\colon \mathrm{Tor} H_2(X;\mathbb{Z}) \times \mathrm{Tor} H_2(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$
By Corollary 7.10, we have
$$\mathrm{Tor} H_2(X;\mathbb{Z}) \cong H \oplus H \text{ or } \cong H \oplus H \oplus \mathbb{Z}_2,$$
for some finite abelian group H. We also get the homomorphism
$$\bar{b}\colon \mathrm{Tor} H_2(X;\mathbb{Z}) \to \mathbb{Z}_2, x \mapsto 2b(x,x).$$

VII.3.3 The second Stiefel-Whitney class

Since $H_1(X;\mathbb{Z}) = 0$, the Universal Coefficient Theorem implies that

$$H^2(X;\mathbb{Z}_2) \cong \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}_2),$$

via evaluation of cohomology on homology classes. Hence we can think of the second Stiefel-Whitney class $w_2(X) \in H^2(X;\mathbb{Z}_2)$ as a homomorphism

$$w_2(X)\colon H_2(X;\mathbb{Z}) \to \mathbb{Z}_2.$$

This homomorphism has an i-invariant as in Definition 7.7.

Definition 7.13. We set $i(X) = i(w_2(X))$ and call this number in $\{0, ..., \infty\}$ the *i-invariant* of the closed, simply-connected 5-manifold X. By Definition 7.7 and Proposition 7.12, the integer $i(X)$ is a homotopy invariant of X.

The following proposition is due to Wall, cf. [143, Proposition 1 and 2].[1]

Proposition 7.14. *The homomorphisms $\bar{b}$ and w_2 are identical on the torsion subgroup of $H_2(X;\mathbb{Z})$, i.e.*

$$w_2(x) \equiv 2b(x,x) \textit{ mod } 2,$$

for all torsion elements $x \in \mathit{Tor} H_2(X;\mathbb{Z})$.

Theorem 7.15. *Let X, Y be closed, simply-connected, oriented 5-manifolds. Suppose that the second homology $H_2(X;\mathbb{Z})$ and $H_2(Y;\mathbb{Z})$ are isomorphic as abelian groups and $i(X) = i(Y)$. Then there exists an isomorphism $\theta\colon H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ which preserves the linking forms on the torsion subgroups and satisfies $w_2(Y) \circ \theta = w_2(X)$.*

Proof. By Theorem 7.8 we can find an isomorphism

$$\sigma\colon \mathrm{Tor}H_2(X;\mathbb{Z}) \to \mathrm{Tor}H_2(Y;\mathbb{Z})$$

which preserves the linking form. By Proposition 7.14,

$$w_2(Y)|_{Tor} \circ \sigma = w_2(X)|_{Tor}.$$

We fix $w_2(X)$- and $w_2(Y)$-bases for $H_2(X;\mathbb{Z})$ and $H_2(Y;\mathbb{Z})$. Then we get splittings

$$H_2(X;\mathbb{Z}) = F(X) \oplus \mathrm{Tor}H_2(X;\mathbb{Z}), \quad H_2(Y;\mathbb{Z}) = F(Y) \oplus \mathrm{Tor}H_2(Y;\mathbb{Z}),$$

where $F(X)$ and $F(Y)$ are isomorphic and free abelian groups. If $i(X) = i(Y) < \infty$, then $w_2(X)$ and $w_2(Y)$ vanish on the free parts of this splitting. Hence any isomorphism

$$\tau\colon F(X) \to F(Y),$$

gives an isomorphism $\theta = \tau \oplus \sigma$ that satisfies the condition of the theorem. If $i(X) = i(Y) = \infty$, then the second Stiefel Whitney classes are non-zero on precisely one basis element of the free parts of the splitting above. Choosing an isomorphism of the free parts mapping these basis elements into each other gives again an isomorphism θ, which satisfies the conditions of the theorem. ☐

[1]The formula in this proposition can be compared to the Wu formula $w_2(\alpha) \equiv Q(\alpha,\alpha)$ mod 2 for all $\alpha \in H_2(M;\mathbb{Z})$ on a closed oriented 4-manifold M with intersection form Q.

VII.4 Barden's classification theorem

The following theorem is the classification theorem for simply-connected 5-manifolds and was proved by D. Barden in [6] using surgery theory.

Theorem 7.16 (Barden). *Let X, Y be simply-connected, closed, oriented 5-manifolds. Suppose that $\theta\colon H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ is an isomorphism preserving the linking forms on the torsion subgroups and such that $w_2(Y)\circ\theta = w_2(X)$. Then there exists an orientation preserving diffeomorphism $f\colon X \to Y$ such that $f_* = \theta$.*

We sketch the proof. Since it involves the h-cobordism theorem, we briefly recall the notion of cobordisms.

Definition 7.17. A *cobordism* between closed manifolds X and Y is a compact manifold V with $\partial V = X \amalg Y$.

The manifolds V, X and Y need not be connected. The *trivial cobordism* is $X \times [0,1]$.

Definition 7.18. A cobordism V between X and Y is an h*-cobordism* if the inclusions $X \hookrightarrow V$ and $Y \hookrightarrow V$ are homotopy-equivalences.

Equivalently, both X and Y are (strong) deformation retracts of V. The following h-cobordism theorem for simply-connected h-cobordisms is due to Smale [124].

Theorem 7.19. *If V^n is a simply-connected h-cobordism of dimension $n \geq 6$, then V is diffeomorphic to the trivial cobordism.*

In particular, if the boundary of V is of the form $\partial V = X \amalg Y$ for connected manifolds X and Y, then X and Y are diffeomorphic.

Let X, Y be closed, simply-connected, oriented 5-manifolds and $\theta\colon H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ an isomorphism preserving the linking forms and such that $w_2(Y) \circ \theta = w_2(X)$. Barden first shows in his proof that there exists a simply-connected cobordism V between X and Y such that the inclusions $i\colon X \hookrightarrow V$ and $j\colon Y \hookrightarrow V$ induce isomorphisms $i_*\colon H_2(X;\mathbb{Z}) \to H_2(V;\mathbb{Z})$ and $j_*\colon H_2(Y;\mathbb{Z}) \to H_2(V;\mathbb{Z})$ on second homology, with $j_*^{-1} \circ i_* = \theta$. He then shows that V can be replaced by an h-cobordism, inducing the same isomorphism θ on the second homology groups of X and Y. By the h-cobordism theorem of Smale, there exists a diffeomorphism

$$F\colon V \to Y \times I.$$

This induces an orientation preserving diffeomorphism

$$f\colon X \to Y, f = pr_1 \circ F \circ i.$$

Since $pr_1 \circ F \circ j$ can be assumed to be the identity on Y, we see that

$$\theta = f_*\colon H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z}).$$

This is a rough sketch of the proof for Barden's theorem. With Theorem 7.15, we get the following corollary.

Corollary 7.20. *Let X, Y be closed, simply-connected 5-manifolds with isomorphic second homology $H_2(X;\mathbb{Z}) \cong H_2(Y;\mathbb{Z})$ and $i(X) = i(Y)$. Then X and Y are diffeomorphic.*

Hence $H_2(X;\mathbb{Z})$ and $i(X)$ form a complete set of invariants for closed, simply-connected 5-manifolds. Since the linking form and the second Stiefel-Whitney class are homotopy invariants, we get:

Corollary 7.21. *If two closed, simply-connected 5-manifolds X, Y are homotopy equivalent, then they are diffeomorphic.*

Proof. If $h\colon X \to Y$ is a homotopy equivalence, then $\theta = h_*$ preserves linking numbers (Proposition 7.3) and $w_2(Y) \circ \theta = w_2(X)$ (Proposition 7.12); hence there exists a diffeomorphism $f\colon X \to Y$ such that $f_* = h_*$. □

VII.5 Construction of building blocks

Recall the following definition:

Definition 7.22. A smooth manifold X^n of dimension n is called *irreducible* if in any connected sum decomposition $X = Y_1 \# Y_2$ one of the summands is diffeomorphic to S^n.

There is a different definition, used for example in Section III.1.2, where a smooth n-manifold is called irreducible if and only if in any connected sum decomposition one of the summands is *homeomorphic* to S^n. In the 5-dimensional case this difference is inessential by Corollary 7.21.

Note that a connected sum of two manifolds is simply-connected if and only if both summands are simply-connected. It is possible to give a complete list of all simply-connected, closed, irreducible 5-manifolds. They are constructed in [6] (see Table VII.2). There are three special manifolds (W, $S^2 \times S^3$, $S^2 \tilde{\times} S^3$) and several families: a family X_k where $k \in \mathbb{N} = \{1, 2, \dots\}$ and for every prime number p a family M_{p^k}, $k \in \mathbb{N}$. The manifold X_1 is exceptional in this list because it is diffeomorphic to $W \# W$, cf. Proposition 7.28. All other manifolds in Table VII.2 are irreducible.

	Manifold X	$H_2(X;\mathbb{Z})$	$w_2(X)$	$b(X)$	$i(X)$	$W_3(X)$
(1)	X_k, $k \in \mathbb{N}$	$\mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$	$\neq 0$	C_{2^k}	k	$\neq 0$
(2)	Wu-manifold W	$\mathbb{Z}_2$	$\neq 0$	A	1	$\neq 0$
(3)	M_{p^k}, p prime, $k \in \mathbb{N}$	$\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$	0	B_{p^k}	0	0
(4)	$S^2 \times S^3$	$\mathbb{Z}$	0	–	0	0
(5)	$S^2 \tilde{\times} S^3$	$\mathbb{Z}$	$\neq 0$	–	∞	0

Table VII.2: Building blocks of simply-connected 5-manifolds.

Here $S^2 \tilde{\times} S^3$ denotes the non-trivial S^3-bundle over S^2 (which is unique up to isomorphism, because $\pi_1(SO(5)) = \mathbb{Z}_2$) and $W_3(X) \in H^3(X;\mathbb{Z})$ denotes the third integral Stiefel-Whitney class, given by the image of $w_2(X)$ under the Bockstein homomorphism β,

$$\dots \longrightarrow H^2(X;\mathbb{Z}) \xrightarrow{p_*} H^2(X;\mathbb{Z}_2) \xrightarrow{\beta} H^3(X;\mathbb{Z}) \longrightarrow \dots$$

associated to the short exact sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_2 \to 0$. The manifolds in the table above are pairwise not homotopy equivalent, distinguished by their invariants.

We want to give an explicit construction of the manifolds in Table VII.2. The following theorem is a generalization of the Heegaard decomposition of 3-manifolds to manifolds of higher dimension (see [76, Chapter VIII, Cor. 6.3]).

Theorem 7.23. *A* $(k-1)$*-connected closed* $(2k+1)$*-dimensional manifold,* $k \geq 1$*, is obtained by identifying the boundaries of two manifolds, each of which is a connected sum along the boundary of a number of* $(k+1)$*-disc bundles over* S^k*.*

In particular for $k = 2$, all simply-connected closed 5-manifolds can be obtained in this way from D^3-bundles over S^2. We explicitly describe this decomposition for the manifolds in Table VII.2 and then prove that all simply-connected 5-manifolds can be obtained by connected sums of these building blocks.

Up to isomorphism, there are two D^3-bundles over S^2 (because $\pi_1(SO(3)) = \mathbb{Z}_2$): the trivial bundle $A = S^2 \times D^3$ and a non-trivial bundle $B = S^2 \tilde{\times} D^3$. The boundaries are $\partial A = S^2 \times S^2$ and $\partial B = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, since $\partial B = S^2 \tilde{\times} S^2$ is the non-trivial S^2-bundle over S^2. Let A', B' denote the boundary connected sums

$$A' = A \#_b A, \quad B' = B \#_b B.$$

Then A' and B' are simply-connected compact 5-manifolds with boundary

$$\partial A' = \partial A \# \partial A, \quad \partial B' = \partial B \# \partial B.$$

We want to show that all building blocks in Table VII.2 are constructed by taking two copies of a manifold of the same type A, A', B or B' and gluing them together along their boundaries via certain orientation reversing diffeomorphisms.

Since A and B are homotopy equivalent to S^2 they have homology only in degree 0 and 2. We denote the generator of $H_2(A;\mathbb{Z})$ by u and the generator of $H_2(B;\mathbb{Z})$ by v. Let x, y denote the standard generators of $H_2(\partial A;\mathbb{Z})$, corresponding to the S^2-factors, and p, q the standard generators of $H_2(\partial B;\mathbb{Z})$. If i denotes the inclusion of the boundary into the manifold, we have

$$i_*(x) = u, i_*(y) = 0, \quad \text{and} \quad i_*(p) = v = i_*(q).$$

The claim for B follows because p and q are the fundamental classes of the image of sections in $S^2 \tilde{\times} S^2$.

Similarly, $H_2(A';\mathbb{Z})$ has generators u_1, u_2 and $H_2(\partial A';\mathbb{Z})$ has generators x_1, y_1, x_2, y_2 such that

$$i_*(x_j) = u_j, i_*(y_j) = 0,$$

whereas $H_2(B';\mathbb{Z})$ has generators v_1, v_2 and $H_2(\partial B';\mathbb{Z})$ has generators p_1, q_1, p_2, q_2 such that

$$i_*(p_j) = v_j = i_*(q_j).$$

Let $A(k)$ and $B(n)$ for $1 \leq k, n < \infty$ denote the matrices

$$A(k) = \begin{pmatrix} 1 & 0 & 0 & -k \\ 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B(n) = \begin{pmatrix} 1 & n & -n & 0 \\ n & 1 & 0 & n \\ n & 0 & 1 & n \\ 0 & -n & n & 1 \end{pmatrix}.$$

We write $\phi_*(e_1, e_2, e_3, e_4)$ as a shorthand notation for $(\phi_* e_1, \phi_* e_2, \phi_* e_3, \phi_* e_4)$.

The 4-manifolds ∂A and ∂B have natural orientation reversing self-diffeomorphisms, given by an orientation reversing self-diffeomorphism on one S^2-factor and the identity on the other S^2-factor for ∂A and by interchanging the summands in ∂B. They induce orientation reversing self-diffeomorphisms on the connected sums $\partial A'$ and $\partial B'$ (see Lemma 2 in [144]). If ϕ is an orientation *preserving* self-diffeomorphism of one of the manifolds $\partial A, \partial B, \partial A', \partial B'$, we can compose it with this orientation *reversing* self-diffeomorphism to get an orientation *reversing* self-diffeomorphism, which we denote by $\overline{\phi}$.

We construct the following manifolds:

- $S^2\tilde{\times}S^3 = B \cup_{\overline{g_\infty}} B$, where
$$g_\infty\colon \partial B \longrightarrow \partial B$$
is an orientation preserving self-diffeomorphism realizing on second homology $(g_\infty)_*(p,q) = (p,q)$.

- $W = B \cup_{\overline{g_{-1}}} B$, where
$$g_{-1}\colon \partial B \longrightarrow \partial B$$
is a orientation preserving self-diffeomorphism realizing on second homology $(g_{-1})_*(p,q) = (p,-q)$.

- $M[A(k)] = A' \cup_{\overline{f_k}} A'$, where
$$f_k\colon \partial A' \longrightarrow \partial A'$$
is a orientation preserving self-diffeomorphism realizing on second homology
$$(f_k)_*(x_1,y_1,x_2,y_2) = (x_1,y_1,x_2,y_2)A(k).$$

- $X[B(n)] = B' \cup_{\overline{g_n}} B'$, where
$$g_n\colon \partial B' \longrightarrow \partial B'$$
is a orientation preserving self-diffeomorphism realizing on second homology
$$(g_n)_*(p_1,q_1,p_2,q_2) = (p_1,q_1,p_2,q_2)B(n).$$

Since the maps on homology above always preserve the intersection form, the existence of the corresponding diffeomorphisms follows from a theorem of Wall (see [144]):

Theorem 7.24. *Let M be a closed, simply-connected 4-manifold which is a connected sum of copies of $\mathbb{C}P^2, \overline{\mathbb{C}P^2}$ and $S^2 \times S^2$. For $b_2(M) > 10$ exclude the case that $b_2^+(M) = 1$ or $b_2^-(M) = 1$. Then any automorphism of the intersection form Q_M can be realized by an orientation preserving self-diffeomorphism of M.*

The corresponding building blocks in Table VII.2 are defined as $M_{p^k} = M[A(p^k)]$ and $X_k = X[B(2^{k-1})]$. Since the manifolds A, A', B, B' are simply-connected, the manifolds we have constructed are simply-connected closed oriented 5-manifolds. We now compute their homology, which can be reduced to computing $H_2(X;\mathbb{Z})$ by Section VII.3.1.

Proposition 7.25. *The second homology groups are given by:*

(1.) $H_2(S^2\tilde{\times}S^3;\mathbb{Z}) = \mathbb{Z}$

(2.) $H_2(W;\mathbb{Z}) = \mathbb{Z}_2$

(3.) $H_2(M[A(k)];\mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k$

(4.) $H_2(X[B(n)];\mathbb{Z}) = \mathbb{Z}_{2n} \oplus \mathbb{Z}_{2n}$

Proof. We need the following form of the Mayer-Vietoris sequence: Suppose U, V are manifolds with boundary and $X = U \cup_\phi V$ with a diffeomorphism $\phi\colon \partial U \longrightarrow \partial V$. Let i^U, i^V denote the inclusion of the boundary in the manifolds. Then there is the exact sequence, cf. Section V.1.1:

$$\ldots \longrightarrow H_{n+1}(X) \longrightarrow H_n(\partial U) \xrightarrow{\Psi} H_n(U) \oplus H_n(V) \longrightarrow H_n(X) \longrightarrow \ldots$$

with $\Psi(x) = (i^U_*(x), i^V_* \phi_*(x))$. In our situation we have

$$0 \longrightarrow H_3(X) \longrightarrow H_2(\partial U) \xrightarrow{\Psi} H_2(U) \oplus H_2(V) \longrightarrow H_2(X) \longrightarrow 0.$$

Hence $H_2(X)$ is isomorphic to the cokernel of Ψ. Since $U = V$ in our case, we denote the homology generators of the manifold V with a bar, like $\overline{v}$.

(1.) Ψ is given by

$$\begin{aligned} p &\mapsto v + \overline{v} \\ q &\mapsto v + \overline{v}. \end{aligned}$$

Hence $\mathrm{Im}\Psi = \mathbb{Z}(v + \overline{v})$. Since $v, v + \overline{v}$ is a basis for $H_2(B) \oplus H_2(B)$ we get $\mathrm{Coker}\,\Psi \cong \mathbb{Z}$.

(2.) Ψ is given by

$$\begin{aligned} p &\mapsto v + \overline{v} \\ q &\mapsto v - \overline{v}. \end{aligned}$$

Hence $\mathrm{Im}\Psi = 2\mathbb{Z}v \oplus \mathbb{Z}(v - \overline{v})$. With the same basis as in (a) this implies $\mathrm{Coker}\,\Psi \cong \mathbb{Z}_2$.

(3.) Ψ is given by

$$\begin{aligned} x_1 &\mapsto u_1 - \overline{u_1} \\ y_1 &\mapsto k\overline{u_2} \\ x_2 &\mapsto u_2 - \overline{u_2} \\ y_2 &\mapsto -k\overline{u_1}. \end{aligned}$$

We take the basis $\overline{u_1}, \overline{u_2}, u_1 - \overline{u_1}, u_2 - \overline{u_2}$ for $H_2(A') \oplus H_2(A')$. Then $\mathrm{Coker}\,\Psi \cong \mathbb{Z}_k \oplus \mathbb{Z}_k$.

(4.) Ψ is given by

$$\begin{aligned} p_1 &\mapsto v_1 + (n+1)\overline{v_1} + n\overline{v_2} \\ q_1 &\mapsto v_1 + (n+1)\overline{v_1} - n\overline{v_2} \\ p_2 &\mapsto v_2 - n\overline{v_1} + (n+1)\overline{v_2} \\ q_2 &\mapsto v_2 + n\overline{v_1} + (n+1)\overline{v_2}. \end{aligned}$$

Hence a basis for the image of Ψ is $2n\overline{v_1}, 2n\overline{v_2}, v_1 + (n+1)\overline{v_1} + n\overline{v_2}, v_2 + n\overline{v_1} + (n+1)\overline{v_2}$. For $H_2(B') \oplus H_2(B')$ we take as basis the last two elements of the basis of $\mathrm{Im}\Psi$ together with $\overline{v_1}, \overline{v_2}$. Then $\mathrm{Coker}\,\Psi \cong \mathbb{Z}_{2n} \oplus \mathbb{Z}_{2n}$.

□

We want to determine the i-invariant of the closed, simply-connected 5-manifolds constructed above. Because of Theorems 7.8 and 7.9 this will determine their linking forms. It is clear that

$$i(S^2 \times S^3) = 0, \text{ and}$$
$$i(W) = 1,$$

since $S^2 \times S^3$ is spin and the only possible linking form on $H_2(W;\mathbb{Z}) \cong \mathbb{Z}_2$ is of type A, which has i-invariant 1.

Lemma 7.26. *A connected sum $X = Y_1 \# Y_2$ of n-dimensional oriented manifolds is spin if and only if both Y_1 and Y_2 are spin. A similar result holds for boundary connected sums.*

Proof. To define the connected sum of Y_1 and Y_2 one chooses embedded disks D_1^n and D_2^n in Y_1, Y_2 and an orientation reversing diffeomorphism $\phi\colon D_1^n \to D_2^n$. Suppose Y_1 and Y_2 are spin and choose spin structures. Since there is only one spin structure on D^n up to homotopy, the image under ϕ of the induced spin structure on D_1^n and the induced spin structure on D_2^n are homotopic. This is also true for the induced spin structures on $\partial(Y_1 \setminus \operatorname{int} D_1^n)$ and $\partial(Y_2 \setminus \operatorname{int} D_2^n)$. Hence the image under ϕ of the induced spin structure on $\partial(Y_1 \setminus \operatorname{int} D_1^n)$ extends over $Y_2 \setminus D_2^n$ to give a spin structure on X.

Conversely, suppose that X is spin. We only prove the case $n \geq 3$. A spin structure on X induces spin structures on $Y_1 \setminus \operatorname{int} D_1^n$ and $Y_2 \setminus \operatorname{int} D_2^n$. Since $H^1(S^{n-1};\mathbb{Z}_2) = 0$ if $n \geq 3$, there is only one spin structure on S^{n-1}. It extends over D^n. Hence the spin structures on $\partial(Y_1 \setminus \operatorname{int} D_1^n)$ and $\partial(Y_2 \setminus \operatorname{int} D_1^n)$ extend over D_1^n and D_2^n to give spin structures on Y_1 and Y_2. $\square$

Lemma 7.27. *The manifolds $M[A(k)]$ are spin for all $k \geq 1$ and the manifolds $X[B(n)]$ are non-spin for all $n \geq 1$.*

Proof. Suppose a manifold of type $X[B(n)]$ is spin. A spin structure on $X[B(n)]$ induces a spin structure on B', which induces a spin structure on $\partial B' = 2\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. This is impossible, since $2\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ has odd intersection form.

We denote $M[A(k)]$ by $A_1' \cup_{\overline{f_k}} A_2'$. The manifold A is spin, since it is homotopy equivalent to S^2. By Lemma 7.26, A' is spin. Since $H^1(S^2 \times S^2 \# S^2 \times S^2;\mathbb{Z}_2) = 0$, there exists a unique spin structure on $\partial A'$, up to homotopy. Choose spin structures on A_1', A_2'. The image under $\overline{f_k}$ of the induced spin structure on $\partial A_1'$ is homotopic to the induced spin structure on $\partial A_2'$, hence extends over A_2' to give a spin structure on $M[A(k)]$. $\square$

In particular, X_k is not spin for $k \geq 1$ and M_{p^k} is spin for all primes p and integers $k \geq 1$. This implies that

$$i(M_{p^k}) = 0 \text{ for all primes } p \text{ and integers } k \geq 1.$$

On the other hand, $i(X_k) \neq 0$ for all $k \geq 1$. Since

$$H_2(X_k;\mathbb{Z}) = \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k},$$

it follows by Theorem 7.9 that $b(X_k) \cong C_{2^k}$. In particular,

$$i(X_k) = k \text{ for all integers } k \geq 1.$$

VII.6 Connected sum decomposition of simply-connected 5-manifolds

In this section we prove that all building blocks in Table VII.2 are irreducible, except X_1 which is diffeomorphic to $W\#W$. We also prove the existence and uniqueness of the connected sum decomposition.

Proposition 7.28. *The closed, simply-connected 5-manifolds in Table VII.2 are all irreducible, except X_1, which is diffeomorphic to $W\#W$.*

Proof. If $X = Y_1\#Y_2$ is a connected sum decomposition, then

$$H_2(X;\mathbb{Z}) \cong H_2(Y_1;\mathbb{Z}) \oplus H_2(Y_2;\mathbb{Z}).$$

Hence if X is one of the manifolds $W, S^2 \times S^3, S^2\tilde{\times}S^3$, then one of the summands – say Y_2 – has $H_2(Y_2;\mathbb{Z}) = 0$. By Theorem 7.16, Y_2 is diffeomorphic to S^5.

Suppose that $M_{p^k} = Y_1\#Y_2$ is a non-trivial connected sum decomposition. For any prime p and integer $k \geq 1$, an isomorphism of the form

$$\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k} \cong G \oplus G'$$

with $G, G' \neq 0$, implies $G = G' = \mathbb{Z}_{p^k}$, by writing G and G' as a direct sum of cyclic groups of prime power order and using the uniqueness of this decomposition. Hence

$$H_2(Y_1;\mathbb{Z}) \cong H_2(Y_2;\mathbb{Z}) \cong \mathbb{Z}_{p^k}.$$

By Corollary 7.10, this is possible only if $p = 2$ and $k = 1$. Since the linking forms of Y_1 and Y_2 are non-trivial, they have to be isomorphic to A, i.e. of the form

$$\begin{aligned} y_1 &\mapsto b(y_1, y_1) = 1/2 \\ y_2 &\mapsto b(y_2, y_2) = 1/2, \end{aligned}$$

where y_1, y_2 denote generators for the second homology of Y_1 and Y_2. By Corollary 7.20, the manifolds Y_1, Y_2 have to be diffeomorphic to the Wu-manifold W.

Similarly, if $X_k = Y_1\#Y_2$ is a non-trivial connected sum decomposition, then $k = 1$ and

$$H_2(Y_1;\mathbb{Z}) \cong H_2(Y_2;\mathbb{Z}) \cong \mathbb{Z}_2,$$

hence $Y_1 \cong Y_2 \cong W$.

We want to determine the connected sum $W\#W$: The elements $x_1 = y_1 + y_2, x_2 = y_2$ form a basis for $H_2(W\#W;\mathbb{Z})$ with

$$\begin{aligned} w_2(M_2)(x_1) &\equiv 2b(x_1, x_1) = 0 \mod 2 \\ w_2(M_2)(x_2) &\equiv 2b(x_2, x_2) = 1 \mod 2. \end{aligned}$$

Hence x_1, x_2 form a w_2-basis for $W\#W$ and it follows that $i(W\#W) = 1$. By Corollary 7.20, $W\#W$ is diffeomorphic to X_1, but not diffeomorphic to M_2. This proves the proposition. ☐

Theorem 7.29. *Every closed, simply-connected 5-manifold X is diffeomorphic to a unique (up to order) connected sum*

$$X \cong Q_1\#\dots\#Q_n\#P,$$

where

- *$Q_1, \ldots, Q_n$ are simply-connected irreducible spin 5-manifolds.*
- *If X is spin then P is S^5.*
- *If X is not spin then P is either W, $X_1 = W\#W$ or a simply-connected irreducible non-spin 5-manifold X_k with $k \geq 2$.*

Proof. We first prove **Uniqueness:** Suppose that there exists a diffeomorphism between closed, simply-connected 5-manifolds of the form

$$X \cong Q_1\# \ldots \#Q_n\#P \cong Q'_1\# \ldots \#Q'_m\#P'.$$

If X is spin, then all summands in X have to be spin. This implies $P = P' = S^5$. The manifolds Q_i, Q'_j are of the form M_{p^k} for primes p and integers $k \geq 1$ or $S^2 \times S^3$. Since $H_2(M_{p^k}; \mathbb{Z})$ is always torsion and $H_2(S^2 \times S^3; \mathbb{Z}) \cong \mathbb{Z}$, the number of $S^2 \times S^3$'s among the Q_i, Q_j must be equal to the rank of $H_2(X; \mathbb{Z})$. Writing the torsion subgroup of $H_2(X; \mathbb{Z})$ as a sum of cyclic groups of prime power order determines the M_{p^k} summands among the Q_i, Q_j uniquely. This proves the uniqueness claim if X is spin.

Suppose that X is not spin. We can find a w_2-basis for

$$H_2(X; \mathbb{Z}) = H_2(Q_1; \mathbb{Z}) \oplus \ldots \oplus H_2(Q_n; \mathbb{Z}) \oplus H_2(P; \mathbb{Z})$$

which is non-zero only on one basis element in $H_2(P; \mathbb{Z})$. Hence $i(X) = i(P)$. This determines P if $i(P) \geq 2$. If $i(P) = 1$, then P is diffeomorphic to W or X_2. The sum of the torsion subgroups of the second homology for $Q_1, \ldots, Q_n$ is of the form $H \oplus H$, where H is a direct sum of groups of prime power order. Hence

$$\text{Tor}H_2(X; \mathbb{Z}) \cong H \oplus H \text{ or } \cong H \oplus H \oplus \mathbb{Z}_2,$$

if $P = X_2$ or $P = W$, respectively. Therefore, $\text{Tor}H_2(X; \mathbb{Z})$ determines whether $P = X_2$ or $P = W$. This shows that the non-spin summand P is uniquely determined by X, which implies $P \cong P'$.

The number of $S^2 \times S^3$ is again equal to the rank of $H_2(X; \mathbb{Z})$, if $P \neq S^2 \tilde{\times} S^3$, and to the rank minus 1, if $P = S^2 \tilde{\times} S^3$. Since $\text{Tor}H_2(P; \mathbb{Z})$ is already determined, the remaining summands Q_i, Q_j of the form M_{p^k} are determined by $\text{Tor}H_2(X; \mathbb{Z})$. This proves uniqueness of the decomposition if X is non-spin.

We now prove **Existence:** Let X be a closed, simply-connected 5-manifold with linking form b. Suppose that $i(X) < \infty$. All possible linking forms given by Theorem 7.9 can be realized by a connected sum of manifolds of the type X_k, M_{p^k}, W, where only one X_k or W summand is needed. This follows because p can by any prime and $k \geq 1$ any integer. We get a closed, simply-connected 5-manifold X' with

$$H_2(X'; \mathbb{Z}) \cong \text{Tor}H_2(X; \mathbb{Z}), \quad i(X') = i(X).$$

Let $X'' = X\#rS^2 \times S^3$, where r denotes the rank of $H_2(X; \mathbb{Z})$. Then

$$H_2(X''; \mathbb{Z}) \cong H_2(X; \mathbb{Z}), \quad i(X'') = i(X).$$

The closed, simply-connected 5-manifold X'' is of the form as in the statement of the theorem and by Corollary 7.20, X and X'' are diffeomorphic.

Suppose that $i(X) = \infty$. By Corollary 7.10, the torsion subgroup of $H_2(X; \mathbb{Z})$ has to be of the form $H \oplus H$. We can realize this direct sum as the torsion subgroup of a connected sum of manifolds of type M_{p^k}. We add one $S^2 \tilde{\times} S^3$ summand to get a closed, simply-connected 5-manifold X' with

$$H_2(X'; \mathbb{Z}) \cong \text{Tor}H_2(X; \mathbb{Z}) \oplus \mathbb{Z}, \quad i(X') = i(X).$$

Let $X'' X\#(r-1)S^2 \times S^3$, where r denotes again the rank of $H_2(X;\mathbb{Z})$. Then

$$H_2(X'';\mathbb{Z}) \cong H_2(X;\mathbb{Z}), \quad i(X'') = i(X).$$

Hence X and X'' are diffeomorphic by Corollary 7.20. □

The following corollary will be used in Chapter IX.

Corollary 7.30. *Let X be a simply-connected closed oriented 5-manifold with $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^k$. Then X is diffeomorphic to*

- *$\#kS^2 \times S^3$ if X is spin, and*
- *$\#(k-1)S^2 \times S^3 \# S^2 \tilde{\times} S^3$ if X is not spin.*

Proof. This follows from Theorem 7.29 because $H_2(X;\mathbb{Z})$ is torsion free. □

Chapter VIII

Contact structures on 5-manifolds

Contents

In this chapter we recall the basic notions related to contact structures. We then focus on the 5-dimensional case and show that a theorem of H. Geiges [51] on the classification of almost contact structures up to homotopy on simply-connected 5-manifolds can be extended to all 5-manifolds X whose $H^2(X;\mathbb{Z})$ does not contain 2-torsion. In the last section, we show how to classify almost contact structures on simply-connected 5-manifolds X up to *equivalence*, where a combination of homotopies and orientation preserving self-diffeomorphisms is allowed. The proof uses Barden's classification theorem for simply-connected 5-manifolds from Chapter VII, in particular the possibility to realize certain automorphisms of $H_2(X;\mathbb{Z})$ by an orientation preserving self-diffeomorphism of X.

VIII.1 Basic definitions

Let X^{2n+1} be connected, oriented manifold of odd dimension. Suppose $\alpha \in \Omega^1(X)$ is a 1-form on X without zeroes. Then

$$\xi = \ker \alpha = \{(p, v) \in TX \mid \alpha_p(v) = 0\}$$

is a smooth distribution on X (a subbundle of TX) of rank $2n$, since α_p is a non-vanishing linear map $T_pX \to \mathbb{R}$ for all $p \in X$. We consider the 2-form $d\alpha$ on X. It defines a skew-symmetric bilinear form on each tangent space T_pX.

Definition 8.1. If the restriction $(d\alpha)|_\xi$ is symplectic (i.e. non-degenerate), then we call α a *contact form*. The hyperplane distribution ξ is called the *underlying contact structure*.

Every contact form induces an orientation on the contact structure ξ, as a vector bundle, through the symplectic form $(d\alpha)|_\xi$. Since TX is an oriented vector bundle by assumption, the quotient TX/ξ is an oriented real vector bundle of rank 1 and hence trivial. Therefore, we can write

$$TX = \underline{\mathbb{R}} \oplus \xi, \tag{8.1}$$

where $\underline{\mathbb{R}}$ denotes the trivial real vector bundle of rank 1, realized as a subbundle of TX. Since ξ is the kernel of α, the 1-form α is non-zero on each non-zero vector of $\underline{\mathbb{R}}$.

We have the following equivalent characterization of contact forms.

Lemma 8.2. *A 1-form α on X is a contact form if and only if $\alpha \wedge (d\alpha)^n$ is a volume form on X.*

Proof. Suppose α is a contact form. Then $(d\alpha)^n$ restricted to ξ is a volume form on each fibre. Choose a basis $e_1, \dots, e_{2n+1}$ of T_pX such that $\alpha(e_1) \neq 0$ and $e_2, \dots, e_{2n+1}$ define an oriented basis of ξ. Then

$$\alpha \wedge (d\alpha)^n(e_1, \dots, e_{2n+1}) = \alpha(e_1)(d\alpha)^n(e_2, \dots, e_{2n+1}) \neq 0.$$

Hence $\alpha \wedge (d\alpha)^n$ is non-zero at each point $p \in X$ and therefore a volume form on X.

Conversely, suppose that $\alpha \wedge (d\alpha)^n$ is a volume form. Let $e_1, e_2, \dots, e_{2n+1}$ be a basis of T_pX such that $\alpha(e_1) \neq 0$ and $e_2, \dots, e_{2n+1}$ form a basis of $\xi = \ker \alpha$. Since volume forms are always non-zero on bases, the calculation above shows that $(d\alpha)^n(e_2, \dots, e_{2n+1}) \neq 0$. This is equivalent to $(d\alpha)|_\xi$ being symplectic. $\square$

Since X was assumed oriented to start with, we can compare the orientation of X defined by $\alpha \wedge (d\alpha)^n$ with the given one.

Definition 8.3. A contact form α is called *positive* or *negative*, depending on whether the orientation of X coincides with the orientation defined by $\alpha \wedge (d\alpha)^n$.

If $f\colon X \to \mathbb{R}$ is a smooth, nowhere vanishing function and α a contact form on X, then $\alpha' := f\alpha$ and α have the same kernel ξ. Moreover,

$$d\alpha' = df \wedge \alpha + f d\alpha.$$

Hence $(d\alpha')|_\xi = f(d\alpha)|_\xi$ is symplectic and α' is also a contact form.

Conversely, suppose that two contact forms α, α' have the same underlying contact structure ξ. Then there exists a smooth, nowhere vanishing function $f\colon X \to \mathbb{R}$ such that $\alpha' = f\alpha$: We may choose a fixed complement $\underline{\mathbb{R}}$ of ξ in TX, such that α and α' are both non-zero on each non-zero vector in $\underline{\mathbb{R}}$. Since $\underline{\mathbb{R}}$ is trivial, we can choose a nowhere zero section v. Then

$$f(p) := \frac{\alpha'_p(v)}{\alpha_p(v)}$$

is a well defined smooth, nowhere vanishing function on X. This implies $\alpha' = f\alpha$, since this equation holds on the common kernel ξ and on the section v, spanning $\underline{\mathbb{R}}$. We conclude:

Lemma 8.4. *Two 1-forms α, α' are contact with the same underlying contact structure ξ if and only if there exists a smooth, nowhere vanishing function $f\colon X \to \mathbb{R}$ such that $\alpha' = f\alpha$. The symplectic structure induced by α' on ξ is of the form $(d\alpha')|_\xi = f(d\alpha)|_\xi$.*

Let α be a 1-form. We set

$$\ker d\alpha = \{(p, v) \in TX \mid d\alpha(v, x) = 0 \text{ for all } x \in T_pX\}.$$

Suppose that α is a contact form. Let $v \in T_pX$ be a vector in $\ker \alpha \cap \ker d\alpha$. Then

$$\alpha \wedge (d\alpha)^n(v, v_2, \dots, v_{2n+1}) = 0,$$

for all vectors $v_2, \dots, v_{2n+1}$ in T_pX. Since $\alpha \wedge (d\alpha)^n$ is a volume form, v has to be zero.

The 2-form $d\alpha$ cannot be symplectic on X, since X is of odd dimension. Hence the kernel of $d\alpha$ cannot be zero at any point $p \in X$. Since $\ker d\alpha \cap \ker \alpha = 0$ and $\ker \alpha$ has rank $2n$, the kernel of $d\alpha$ must be 1-dimensional. If R is a non-zero element in $\ker d\alpha$ then $\alpha(R) \neq 0$. Therefore we can make the following definition.

Definition 8.5. Let α be a contact form on X. Then there exists a unique vector field R_α on X with

$$d\alpha(R_\alpha) = 0, \quad \alpha(R_\alpha) = 1.$$

R_α is called the *Reeb vector field* of α.

The vector field R_α defines a splitting

$$TX = \mathbb{R}R_\alpha \oplus \xi,$$

as in equation 8.1. However, now ξ is not only the kernel of α, but $\mathbb{R}R_\alpha$ is the kernel of $d\alpha$.

By the Cartan formula,

$$\begin{aligned} L_{R_\alpha}\alpha &= di_{R_\alpha}\alpha + i_{R_\alpha}d\alpha \\ &= 0. \end{aligned}$$

Hence the flow of the Reeb vector field preserves the contact form and the contact structure.

Let ξ be a contact structure. We fix a splitting $TX = \underline{\mathbb{R}} \oplus \xi$ and a coorientation, i.e. an orientation on the line bundle $\underline{\mathbb{R}}$. We now only consider defining 1-forms α which evaluate positively on the vector defining the orientation on $\underline{\mathbb{R}}$. There exists a complex structure J on ξ compatible with the symplectic structure $(d\alpha)|_\xi$ on each fibre (see Section II.2 in the preliminaries). For fixed $(d\alpha)|_\xi$, the space of such J is contractible, hence we get well-defined Chern classes $c_k(\xi) \in H^{2k}(X;\mathbb{Z})$, independent of the choice of a compatible J. Moreover, if we choose a different defining form α' for ξ which evaluates positively on the orientation of $\underline{\mathbb{R}}$, then by Lemma 8.4 there exists a function $f\colon X \to \mathbb{R}$ which is everywhere positive and such that $\alpha' = f\alpha$. The function f can be deformed smoothly into the constant function with value 1, without ever crossing zero. This implies that the Chern classes do not depend on the choice of defining form α.

Definition 8.6. The Chern classes $c_k(\xi) \in H^{2k}(X;\mathbb{Z})$ of a cooriented contact structure ξ are well defined and independent of the choice of the defining form α, respecting the coorientation, and the almost complex structure J.

A contact structure determines, in particular, a symplectic subbundle of TX of corank 1. This is also known as an almost contact structure.

Definition 8.7. An *almost contact structure* on X^{2n+1} is a rank $2n$-distribution ξ with a symplectic structure σ on ξ.

Clearly, every contact structure determines an almost contact structure. The converse is true if and only if the symplectic structure σ on ξ is of the form $(d\alpha)|_\xi$ for a 1-form α on X defining ξ. For each almost contact structure ξ, we can choose again a compatible almost complex structure J. The space of such J is contractible, hence we get well-defined Chern classes. However, they will depend in general on the symplectic structure σ, not only on the distribution ξ as in the contact case. The first Chern class of ξ is related to the second Stiefel-Whitney class in a similar way as in the almost complex case:

Lemma 8.8. *Let ξ be an almost contact structure on X. Then $c_1(\xi) \equiv w_2(M)$ mod 2.*

Proof. By the Whitney sum formula for $TX = \xi \oplus \mathbb{R}$,

$$w_2(X) = w_2(\xi) \cup w_0(\mathbb{R}) = w_2(\xi).$$

Since $\xi \to X$ is a complex vector bundle, with complex structure compatible with σ, we have $w_2(\xi) \equiv c_1(\xi)$ mod 2. This implies the claim. □

Suppose that $\xi_t, t \in [0,1]$ is a smooth family of contact structures on a closed manifold X. We can choose a smooth family of 1-forms α_t defining ξ_t. Using the Moser technique, one can prove that there exists a smooth family ψ_t of self-diffeomorphisms of X with $\psi_0 = Id_X$ such that $\psi^*\alpha_t = f_t\alpha_0$, for smooth functions f_t on X [96]. This implies the following theorem of Gray [57].

Theorem 8.9. *Let $\xi_t, t \in [0,1]$ be a smooth family of contact structures on a closed manifold X. Then there exists an isotopy $\psi_t, t \in [0,1]$ of diffeomorphisms of X such that $\psi_t^*\xi_t = \xi_0$.*

Because of this theorem, we call contact structures ξ, ξ' which can be deformed into each other by a smooth family of contact structures *isotopic*. We call almost contact structures *homotopic*, if they can be connected by a smooth family of almost contact structures. The contact structures in an isotopy class or the almost contact structures in a homotopy class all have the same Chern classes. We can also consider (almost) contact structures ξ, ξ' which are permuted by an orientation-preserving self-diffeomorphism ψ of X, in the sense that $\psi^*\xi' = \xi$.

Definition 8.10. We call almost contact structures and contact structures on an oriented manifold X *equivalent*, if they can be made identical by a combination of deformations (homotopies, resp. isotopies) and by orientation-preserving self-diffeomorphisms of X.

See Vidussi's article [140] for a related definition for symplectic forms.

VIII.2 Almost contact structures as sections of a fibre bundle

For the homotopy classification of almost contact structures on 5-manifolds it is helpful to have a different description of almost contact structures as sections of some bundle, such that two almost contact structures are homotopic if and only if the corresponding sections are homotopic (through sections).

Let X be an oriented $2n+1$-dimensional manifold and $\mathrm{Fr}(X) \to X$ the frame bundle with fibre $SO(2n+1)$ for some auxiliary Riemannian metric on X. Fix an embedding of $SO(2n)$ in $SO(2n+1)$. Since $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we also have an embedding of $U(n)$ in $SO(2n)$. An almost contact structure on X is given by an hyperplane in the tangent bundle to X at each point together with a complex structure on this hyperplane. Hence an almost contact structure at a point of X is the same as an equivalence class of orthonormal frames under the action of $U(n)$ as a subgroup of $SO(2n+1)$ (see [57]).

Let Z denote the bundle $\mathrm{Fr}(X)/U(n)$. Then Z fibres over X with fibre $SO(2n+1)/U(n)$. An almost contact structure can be thought of as a section of Z. Two almost contact structures are homotopic if and only if the corresponding sections of Z are homotopic. We will need the following lemma.

Lemma 8.11. $SO(5)/U(2) \cong \mathbb{C}P^3$.

We will sketch a proof; details can be found in [123]. Let (M, g) be a Riemannian spin 4-manifold. Denote the positive spinor bundle over M by V_+, which is a vector bundle with fibre $\mathbb{C}^2$, and the bundle of self-dual 2-forms by Λ^2_+, which is a vector bundle with fibre $\mathbb{R}^3$. It is known that

$$\mathbb{P}_{\mathbb{C}}(V_+) \cong S(\Lambda^2_+), \tag{8.2}$$

as S^2 bundles, where $\mathbb{P}_{\mathbb{C}}(\cdot)$ denotes complex projectivization and $S(\cdot)$ the associated unit sphere bundle. It is also known that each element of $S(\Lambda^2_+)$ can be interpreted as a complex structure on the tangent space at the point of M below. In other words, $S(\Lambda^2_+)$ can be identified with the *twistor space* of M.

We now specialize to the case of $M = S^4$ with the standard metric. One can see that $SO(5)$ acts transitively on the twistor space of S^4, with stabilizer $U(2)$. This implies that

$$S(\Lambda^2_+) \cong SO(5)/U(2).$$

On the other hand, V_+ can be identified with the tautological quaternionic line bundle over $\mathbb{H}P^1 \cong S^4$. Hence, $V_+ \setminus \{\text{zero section}\} \cong \mathbb{H}^2 \setminus \{0\} \cong \mathbb{C}^4 \setminus \{0\}$. This implies that

$$\mathbb{P}_{\mathbb{C}}(V_+) \cong \mathbb{C}P^3.$$

The lemma now follows from equation (8.2).

VIII.3 Overview of obstruction theory

To classify almost contact structures on oriented 5-manifolds up to homotopy, we will use obstruction theory. We briefly recall the basic principles of this theory, following the exposition in Steenrod's book [127]. Let X be a CW-complex and $E \to X$ a fibre bundle. Obstruction theory tries to answer the questions whether there exist a section of E at all and, given two sections of E, whether there exist a homotopy between them.

We will begin by describing how to systematically answer the first question. Let $X^{(q)}$ denote the q-skeleton of X. Suppose a section $f\colon X^{(q)} \to E$ for some $q \geq 0$ is given. We want to extend it to the $(q+1)$-skeleton. This can be done if and only if it can be extended to the interior of every $(q+1)$-cell.

Let $\sigma = e^{q+1}$ be a $(q+1)$-cell with attaching map $\partial e^{q+1} \to X^{(q)}$. Pick some point $p \in \sigma$. By local triviality of the fibre bundle, $E|_\sigma \cong E_p \times \sigma$. Composing the section f with the attaching map and projecting on the first factor, we get a map $\bar{f}\colon \partial\sigma \to E_p$. It is not difficult to show that the section f on $\partial\sigma$ extends to all of σ if and only if $\bar{f}$ extends to a map from σ to E_p.

Since $\partial\sigma \cong S^q$, $\bar{f}$ determines an element $[\bar{f}] \in \pi_q(E_p)$. We will denote this element also by $c(f,\sigma) = c(f)(\sigma)$. We conclude that f extends to a section over $X^{(q+1)}$ if and only if $c(f)(\sigma) = 0$ for all $(q+1)$-cells σ.

We want to view $c(f)$ as a cellular $(q+1)$-cochain with values in the group $\pi_q(E)$. At the moment, $c(f)$ takes values in $\pi_q(E_p)$, where p depends on the cell σ. It is clear that $\pi_q(E_p) \cong \pi_q(E_{p'})$ for $p \neq p'$, however there is no canonical isomorphism. In the situation we are going to consider below, we can nevertheless make sense of $c(f)$ as a cochain with values in a fixed group $\pi_q(E)$. Hence $c(f) \in C^{q+1}(X; \pi_q(E))$.

Proposition 8.12. *A section f on $X^{(q)}$ extends over $X^{(q+1)}$ if and only if $c(f) = 0 \in C^{q+1}(X; \pi_q(E))$.*

One can prove that $c(f)$ is co-closed, $\delta c(f) = 0$, hence $c(f)$ defines a cohomology class in $H^{q+1}(X; \pi_q(E))$ which we denote by $\bar{c}(f)$. The vanishing of this cohomology class has the following interpretation.

Proposition 8.13. *A section f on $X^{(q)}$ can be changed to a section on $X^{(q)}$ extending over $X^{(q+1)}$, while leaving it unchanged on $X^{(q-1)}$, if and only if $\bar{c}(f) = 0 \in H^{q+1}(X; \pi_q(E))$.*

We now consider the second question above. Suppose we have sections f_0, f_1 of E over X and a homotopy K between $f_0|_{X^{(q-1)}}$ and $f_1|_{X^{(q-1)}}$ (note that K is a section of $E \times I$ on $X^{(q-1)} \times I$). Does K extend to a homotopy between f_0 and f_1 on $X^{(q)}$?

Let σ be a q-cell on X. This defines a $(q+1)$-cell $\sigma \times I$ on $X \times I$ with boundary

$$\partial(\sigma \times I) = \sigma \times \{0\} \cup \partial\sigma \times I \cup \sigma \times \{1\}.$$

Pick a point $p \in \sigma$. On $\partial(\sigma \times I)$ we have the map

$$f_0|_{\sigma\times\{0\}} \cup K|_{\partial\sigma\times I} \cup f_1|_{\sigma\times\{1\}} \colon \partial(\sigma \times I) \to E_p.$$

This map determines an element in $\pi_q(E_p)$ which we denote by $d(f_0, K.f_1)(\sigma)$. Again, we can view $d(f_0, K, f_1)$ as an element in $C^q(X; \pi_q(E))$.

Proposition 8.14. *A homotopy K between f_0 and f_1 on $X^{(q-1)}$ extends over $X^{(q)}$ if and only if $d(f_0, K, f_1) = 0 \in C^q(X; \pi_q(E))$.*

Again one can show that $d(f_0, K, f_1)$ is co-closed, hence $d(f_0, K, f_1)$ determines a cohomology class $\bar{d}(f_0, K, f_1) \in H^q(X; \pi_q(E))$. We then have a similar proposition as above for $\bar{c}(f)$.

Proposition 8.15. *A homotopy K between f_0 and f_1 on $X^{(q-1)}$ can be changed to a homotopy on $X^{(q-1)}$ extending over $X^{(q)}$, while leaving it unchanged on $X^{(q-2)}$, if and only if $\bar{d}(f_0, K, f_1) = 0 \in H^q(X; \pi_q(E))$.*

VIII.4 Homotopy classification of almost contact structures in dimension 5

Let X be a smooth manifold. We consider the long exact sequence

$$\ldots \longrightarrow H^2(X;\mathbb{Z}) \xrightarrow{p_*} H^2(X;\mathbb{Z}_2) \xrightarrow{\beta} H^3(X;\mathbb{Z}) \longrightarrow \ldots$$

associated to the short exact sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_2 \to 0$. The homomorphism β is the associated Bockstein homomorphism and $p_*\alpha \in H^2(X;\mathbb{Z}_2)$ for $\alpha \in H^2(X;\mathbb{Z})$ is called the mod 2 reduction of α. Let $E \to X$ be an $\mathbb{R}$-vector bundle. The image of the second Stiefel-Whitney class $w_2(E)$ under β is denoted by $W_3(E)$. In particular, $W_3(E) = 0$ if and only if $w_2(E)$ is the mod 2 reduction of an integral class.

The existence question for *almost contact structures* on 5-manifolds was settled by the following theorem of Gray [57].

Theorem 8.16. *Let X be a closed, orientable 5-manifold. Then X admits an almost contact structure if and only if $W_3(X) = 0$.*

The existence of *contact structures* on simply-connected 5-manifolds was proved by Geiges [51]. He also proved a classification theorem for almost contact structures on simply-connected 5-manifolds up to homotopy:

Theorem 8.17. *Let X be a simply-connected, closed 5-manifold.*

- *Every class in $H^2(X;\mathbb{Z})$ that reduces mod 2 to $w_2(X)$ arises as the first Chern class of an almost contact structure. Two almost contact structures ξ_0, ξ_1 are homotopic if and only if $c_1(\xi_0) = c_1(\xi_1)$.*
- *Every homotopy class of almost contact structures admits a contact structure.*

A different proof for the existence of contact structures on simply-connected 5-manifolds can be found in [74, 75]. We will prove the following generalization for the classification of almost contact structures:

Theorem 8.18. *Let X be a closed, oriented 5-manifold without 2-torsion in $H^2(X;\mathbb{Z})$. Then two almost contact structures ξ_0 and ξ_1 on X are homotopic if and only if $c_1(\xi_0) = c_1(\xi_1)$.*

One direction is clear: if two almost contact structures are homotopic, then they have the same first Chern classes. We now prove the other direction, which requires some preparations.

Let X be a closed, oriented 5-manifold and (ξ, J) an almost contact structure on X, where J is a compatible complex structure on ξ. Then ξ is the associated vector bundle of a principal $U(2)$ bundle over X that we denote, for simplicity, also by ξ.

There is a principal bundle

$$\begin{array}{ccc} Fr(X) & \longleftarrow & U(2) \\ \big\downarrow & & \\ Z & & \end{array}$$

which we call $\mathcal{E}$. Here Z denotes the manifold $\mathrm{Fr}(X)/U(2)$ as in Section VIII.2. As seen above, ξ can be thought of as a section f of the bundle

$$\begin{array}{ccc} Z & \longleftarrow & \mathbb{C}P^3 = SO(5)/U(2) \\ \big\downarrow & & \\ X & & \end{array}$$

In fact, $\xi = f^*\mathcal{E}$ as a $U(2)$-bundle.

We need to determine the first six homotopy groups of $\mathbb{C}P^3$. For this we consider the Hopf fibration

$$\begin{array}{ccc} S^7 & \longleftarrow & S^1 \\ \big\downarrow & & \\ \mathbb{C}P^3 & & \end{array}$$

and the following part of the long exact homotopy sequence for this fibration:

$$\begin{aligned} 0 \to 0 \to \pi_5(\mathbb{C}P^3) \to 0 \to 0 \to \pi_4(\mathbb{C}P^3) \to 0 \to 0 \to \pi_3(\mathbb{C}P^3) \to 0 \to \\ 0 \to \pi_2(\mathbb{C}P^3) \to \mathbb{Z} \to 0 \to \pi_1(\mathbb{C}P^3) \to 0 \to 0. \end{aligned}$$

From this we see that

$$\begin{aligned} \pi_2(\mathbb{C}P^3) &= \mathbb{Z} \\ \pi_i(\mathbb{C}P^3) &= 0 \quad i = 0,1,3,4,5 \end{aligned}$$

We now consider the following principal bundle

$$\begin{array}{ccc} SO(5) & \longleftarrow & U(2) \\ \big\downarrow & & \\ \mathbb{C}P^3 & & \end{array}$$

which we denote by E. Suppose $h\colon S^2 \to \mathbb{C}P^3$ is a continuous map. Let $[h]$ denote the integer given by $[h] \in \pi_2(\mathbb{C}P^3) \cong H_2(\mathbb{C}P^3;\mathbb{Z}) \cong \mathbb{Z}$. We want to prove the following relation:

$$\begin{aligned} 2[h] &= \langle c_1(E), h_*[S^2] \rangle \\ &= \langle c_1(h^*E), [S^2] \rangle. \end{aligned}$$

The following part of the long exact homotopy sequence for the bundle E

$$\pi_2(SO(5)) \to \pi_2(\mathbb{C}P^3) \xrightarrow{\partial} \pi_1(U(2)) \to \pi_1(SO(5)) \to \pi_1(\mathbb{C}P^3),$$

is given by

$$0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

This shows that $\partial\colon \pi_2(\mathbb{C}P^3) \to \pi_1(U(2))$ is multiplication by 2 in $\mathbb{Z}$. On the other hand it is known that

$$\partial h = \langle c_1(E), [h]\rangle$$

for all $[h] \in \pi_2(\mathbb{C}P^3)$, cf. Lemma 9.7. This implies the claim.

Let ξ_0, ξ_1 be two almost contact structures on X given by sections f_0, f_1 of the $\mathbb{C}P^3$ bundle $Z \to X$. We want to determine when f_0 and f_1 are homotopic as sections. Since $\pi_i(\mathbb{C}P^3)$ vanishes in all degrees less or equal than 5, except for $\pi_2(\mathbb{C}P^4) = \mathbb{Z}$, the only obstruction comes from degree 2. Hence we can assume that there exists a homotopy K between f_0 and f_1 on the 1-skeleton $X^{(1)}$ and have to see when we can find a homotopy between f_0 and f_1 on $X^{(2)}$. This happens if and only if the obstruction class $\bar{d}(f_0, K, f_1) \in H^2(X; \pi_2(\mathbb{C}P^3)) = H^2(X;\mathbb{Z})$ vanishes. The following lemma will therefore complete the proof of Theorem 8.18.

Lemma 8.19. *If* $c_1(\xi_0) = c_1(\xi_1)$, *then* $2\bar{d}(f_0, K, f_1) = 0$.

Proof. Let σ be a 2-cell from $X^{(2)}$. As explained above, we get a map

$$F_\sigma = f_0|_{\sigma\times\{0\}} \cup K|_{\partial\sigma\times I} \cup f_1|_{\sigma\times\{1\}}\colon \partial(\sigma \times I) \to \mathbb{C}P^3.$$

This map determines an element in $\pi_2(\mathbb{C}P^3)$ which we denoted by $d(f_0, K, f_1)(\sigma)$. Since $\pi_1(\mathbb{C}P^3) = 0$, we can homotop F_σ such that the domains of f_0, f_1 are shrunk to smaller 2-cells and K becomes constant. Hence we may assume that f_0 and f_1 are already identical and constant on $X^{(1)}$ and the homotopy K is a constant map.

The maps f_i on the 2-cell σ then induce maps $h_i^{\hat\sigma}$ on the 2-sphere $\hat\sigma = \sigma/\partial\sigma$, for $i = 0, 1$. We have

$$d(f_0, K, f_1)(\sigma) = [h_1^{\hat\sigma}] - [h_0^{\hat\sigma}].$$

These maps for all 2-cells in $X^{(2)}$ fit together to give a commutative diagram

$$\begin{tikzcd} X^{(2)} \arrow[r, "f_i"] \arrow[d, "p"'] & Z \\ X^{(2)}/X^{(1)} \arrow[ur, "h_i"'] & \end{tikzcd}$$

Now recall that we have the principal bundle $\mathcal{E}$

$$\begin{tikzcd} Fr(X) \arrow[d] & U(2) \arrow[l] \\ Z & \end{tikzcd}$$

We know that

$$c_1(\xi_i) = c_1(f_i^*\mathcal{E}) = p^* c_1(h_i^*\mathcal{E})$$

and $2[h_i^{\hat{\sigma}}] = \langle c_1(h_i^*\mathcal{E}), \hat{\sigma}\rangle$ by the relation above. Suppose $c_1(\xi_0) = c_1(\xi_1)$. We consider the long exact sequence in cohomology, associated to the pair $(X^{(2)}, X^{(1)})$:

$$\ldots \to H^1(X^{(1)};\mathbb{Z}) \xrightarrow{\delta} H^2(X^{(2)}/X^{(1)};\mathbb{Z}) \xrightarrow{p^*} H^2(X^{(2)};\mathbb{Z}) \xrightarrow{i^*} 0 \to \ldots$$

We see that $\ker p^* = \delta H^1(X^{(1)};\mathbb{Z})$. This implies $c_1(h_1^*\mathcal{E}) - c_1(h_0^*\mathcal{E}) = \delta\alpha$, for some $\alpha \in H^1(X^{(1)};\mathbb{Z})$. This in turn gives

$$2[h_0^{\hat{\sigma}}] - 2[h_1^{\hat{\sigma}}] = \langle \delta\alpha, \hat{\sigma}\rangle = \langle \alpha, \partial\hat{\sigma}\rangle = 0$$

for all 2-cells σ, since the $\hat{\sigma}$ are cycles. We finally get $2d(f_0, K, f_1)(\sigma) = 0$ for all σ, hence $2\bar{d}(f_0, K, f_1) = 0$. $\square$

VIII.5 The level structure of almost contact structures in dimension 5

Suppose X is a simply-connected 5-manifold. By the Universal Coefficient Theorem $H^2(X;\mathbb{Z})$ is torsion free. Hence the divisibility of elements $c \in H^2(X;\mathbb{Z})$ is well defined, cf. Definition 6.5. The classification of simply-connected 5-manifolds (see Chapter VII) implies the following theorem.

Theorem 8.20. *Suppose X is a simply-connected, closed, oriented 5-manifold. Let $c, c' \in H^2(X;\mathbb{Z})$ be classes with the same divisibility and whose mod 2 reduction is $w_2(X)$. Then there exists an orientation preserving self-diffeomorphism $\phi\colon X \to X$ such that $\phi^*c' = c$.*

The proof uses the following lemma.

Lemma 8.21. *Let G be a finitely generated free abelian group of rank n. Suppose $\alpha \in Hom(G,\mathbb{Z})$ is indivisible. Then there exists a basis $e_1, \ldots, e_n$ of G such that $\alpha(e_1) = 1$ and $\alpha(e_i) = 0$ for $i > 1$.*

Proof. The kernel of α is a free abelian subgroup of G of rank $n-1$. Let $e_2, \ldots, e_n$ be a basis of $\ker\alpha$. The image of α in $\mathbb{Z}$ is a subgroup, hence of the form $m\mathbb{Z}$. Since α is indivisible, $m = 1$, so there exists an $e_1 \in G$ such that $\alpha(e_1) = 1$. The set $e_1, \ldots, e_n$ is linearly independent. They also span G, because if $g \in G$ is some element, then $\alpha(g - \alpha(g)e_1) = 0$, hence $g = \alpha(g)e_1 + \sum_{i\geq 2}\lambda_i e_i$. $\square$

We can now prove Theorem 8.20.

Proof. By the Universal Coefficient Theorem, $H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ since X is simply-connected. Hence we can view c, c' as homomorphisms on $H_2(X;\mathbb{Z})$ with values in $\mathbb{Z}$. Let $p\colon \mathbb{Z} \longrightarrow \mathbb{Z}_2$ be mod 2 reduction. The assumption on c and c' is equivalent to

$$w_2(X) = p \circ c = p \circ c',$$

as homomorphisms on $H_2(X;\mathbb{Z})$ with values in $\mathbb{Z}_2$, cf. Section VII.3.3. Since c and c' have the same divisibility, we can write

$$c = k\alpha, \quad c' = k\alpha'$$

with $\alpha, \alpha' \in \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ indivisible. We can write $H_2(X;\mathbb{Z}) = G \oplus \mathrm{Tor}H_2(X;\mathbb{Z})$ with G free abelian. By Lemma 8.21 there exist bases $e_1, \ldots, e_n$ and $e_1', \ldots, e_n'$ of G such that

$$\alpha(e_1) = 1 = \alpha'(e_1'), \quad \alpha(e_k) = 0 = \alpha'(e_k') \quad \forall k > 1.$$

Let θ be the group automorphism of $H_2(X;\mathbb{Z})$ given by $\theta(e_i) = e'_i$ for all $i \geq 1$, and which is the identity on $\mathrm{Tor}H_2(X;\mathbb{Z})$. Then

$$(c' \circ \theta)(e_i) = c'(e'_i) = c(e_i) \quad \forall i \geq 1.$$

Hence $c' \circ \theta = c$ on G. This equality holds on all of $H_2(X;\mathbb{Z})$ since c and c' are homomorphism to $\mathbb{Z}$ and hence vanish on all torsion elements. By the assumption above, this implies that $w_2(X) \circ \theta = w_2(X)$. Moreover, since θ is the identity on $\mathrm{Tor}H_2(X;\mathbb{Z})$, it preserves the linking form. By Barden's theorem 7.16, the automorphism θ is induced by an orientation preserving self-diffeomorphism $\phi\colon X \longrightarrow X$ such that $\phi_* = \theta$. We have

$$c(\lambda) = c'(\phi_* \lambda) = (\phi^* c')(\lambda), \quad \text{for all } \lambda \in H_2(X;\mathbb{Z}).$$

Hence $\phi^* c' = c$. □

We can use Theorem 8.17 or 8.18 (cf. also Lemma 8.8 and Definition 8.10) to get the following corollary for almost contact structures.

Corollary 8.22. *Let X be a simply-connected, closed, oriented 5-manifold. Then two almost contact structures ξ_0 and ξ_1 on X are equivalent if and only if $c_1(\xi_0)$ and $c_1(\xi_1)$ have the same divisibility in integral cohomology.*

The other direction follows, because the divisibilities of elements in $H^2(X;\mathbb{Z})$ are preserved under automorphisms. Note that simply-connected manifolds have torsion free H^2 by the Universal Coefficient Theorem.

Definition 8.23. We denote the divisibility of $c_1(\xi)$ by $d(\xi)$, as in Definition 6.5.

We sometimes call $d(\xi)$ the *level* of ξ. By Corollary 8.22, almost contact structures and hence contact structures on a simply-connected 5-manifold X naturally form a "spectrum" consisting of levels which are indexed by the divisibility of the first Chern class. Two contact structures on X are equivalent as almost contact structures if and only if they lie on the same level. Note that simply-connected spin 5-manifolds have only even levels and non-spin 5-manifolds only odd levels, cf. Lemma 8.8. In Chapter X, we will use invariants from contact homology to investigate the "fine-structure" of each level in this spectrum. For instance, O. van Koert [74] has shown that for many simply-connected 5-manifolds the lowest level, given by divisibility 0, contains infinitely many inequivalent contact structures.

Chapter IX

Circle bundles over symplectic manifolds

Contents

In the first part of this chapter, we collect and prove some results on the topology of circle bundles over closed manifolds M. The results will be used in the case where the dimension of the base manifold is equal to 4 in Chapter X. In particular, we will show that the total space of a circle bundle is simply-connected if and only if the base manifold is simply-connected and the Euler class is indivisible. We also determine when the total space is spin. If M is a simply-connected 4-manifold and the Euler class of the circle bundle over M is indivisible, we can use the classification of simply-connected 5-manifolds from Chapter VII to determine the total space X up to diffeomorphism. It turns out that X is diffeomorphic to a connected sum of several copies of $S^2 \times S^3$ if X is spin. If X is not spin there is an additional summand of the form $S^2 \tilde{\times} S^3$.

The second part of this chapter describes the so-called Boothby-Wang construction: Suppose that ω is a symplectic form on a manifold M which represents an integral cohomology class and let X be the total space of the circle bundle over M with Euler class equal to $[\omega]$. The construction then associates to ω a contact structure on X. We will consider this construction in the Chapter X for symplectic 4-manifolds. By the classification of the total spaces X of circle bundles mentioned above, one can choose many different simply-connected symplectic 4-manifolds M which give diffeomorphic simply-connected 5-manifolds X and hence many contact structures on the same abstract 5-manifold. We will show that in some cases this gives rise to contact structures on simply-connected 5-manifolds, coming from different symplectic 4-manifolds, which are equivalent as almost contact structures but not equivalent as contact structures .

IX.1 Topology of circle bundles

Let M be a closed, connected, oriented n-manifold and $\pi\colon X \to M$ the total space of a circle bundle over M with Euler class $e \in H^2(M;\mathbb{Z})$, where $H^2(M;\mathbb{Z})$ might have torsion. We consider the map

$$\langle e, - \rangle\colon H_2(M;\mathbb{Z}) \to \mathbb{Z},$$

given by evaluation of the Euler class. We make the following generalization of definition 6.5 for this case:

Definition 9.1. We call e *indivisible* if $\langle e, -\rangle$ is surjective.

Clearly, if e is indivisible, e cannot be written as $e = kc$, with $k > 1$ and $c \in H^2(M;\mathbb{Z})$.

Lemma 9.2. *A class $e \in H^2(M;\mathbb{Z})$ is indivisible if and only if the map*

$$e\cup\colon H^{n-2}(M;\mathbb{Z}) \to H^n(M;\mathbb{Z}) \cong \mathbb{Z}$$

is surjective.

Proof. The map $e\cup$ on $H^{n-2}(M;\mathbb{Z})$ is surjective if and only if there exists an element $\alpha \in H^{n-2}(M;\mathbb{Z})$ such that

$$\langle e \cup \alpha, [M]\rangle = 1.$$

Via Poincaré duality ($c := \alpha \cap [M]$) this is equivalent to the existence of a class $c \in H_2(M;\mathbb{Z})$ such that

$$\langle e, c\rangle = 1,$$

which is equivalent to the map $\langle e, -\rangle$ being surjective. □

There is the following exact Gysin sequence for circle bundles [100]:

$$\ldots \xrightarrow{\pi^*} H^k(X) \xrightarrow{\pi_*} H^{k-1}(M) \xrightarrow{\cup e} H^{k+1}(M) \xrightarrow{\pi^*} H^{k+1}(X) \xrightarrow{\pi_*} \ldots$$

Lemma 9.3. *Integration along the fibre $\pi_*\colon H^{k+1}(X) \to H^k(M)$ is Poincaré dual to the map $\pi_*\colon H_{n-k}(X) \to H_{n-k}(M)$.*

Proof. Let $\pi\colon D \to M$ denote the disc bundle with Euler class e. Then $X \cong \partial D$ and integration along the fibre

$$\pi_*\colon H^{k+1}(\partial D) \to H^k(M)$$

is given by (see [100])

$$H^{k+1}(\partial D) \xrightarrow{\delta} H^{k+2}(D,\partial D) \xrightarrow{\tau^{-1}} H^k(D) \overset{\pi^*}{\cong} H^k(M).$$

Here δ denotes the connecting homomorphism in the long exact sequence of the pair $(D,\partial D)$ and τ^{-1} the inverse of the Thom isomorphism

$$\tau\colon H^k(D) \to H^{k+2}(D,\partial D),\ x \mapsto x \cup u,$$

where the Thom class $u \in H^2(D,\partial D)$ can be written as the Poincaré dual of the fundamental class of the zero section N in D. Under Poincaré duality, the connecting homomorphism δ corresponds to

$$i_*\colon H_{n-k}(\partial D) \to H_{n-k}(D),$$

where $i\colon \partial D \to D$ is the inclusion. We want to show that $\pi_* i_*\colon H_{n-k}(\partial D) \to H_{n-k}(M)$ is Poincaré dual to integration along the fibre. This is equivalent to

$$\pi_* \circ PD \circ \tau \circ \pi^*\colon H^k(M) \to H_{n-k}(M),$$

where $PD\colon H^{k+2}(D,\partial D) \to H_{n-k}(D)$ is Poincaré duality, being just Poincaré duality on M. Let $\alpha \in H^k(M)$. Then

$$\begin{aligned}\pi_* \circ PD \circ \tau \circ \pi^*(\alpha) &= \pi_*((\pi^*\alpha \cup u) \cap [D]) \\ &= \pi_*(\pi^*\alpha \cap (u \cap [D])) \\ &= \pi_*(\pi^*\alpha \cap [N]) = \alpha \cap \pi_*[N] \\ &= \alpha \cap [M].\end{aligned}$$

This proves the claim.

□

Lemma 9.4. *The image of* $\pi_*\colon H_2(X;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ *is the kernel of* $\langle e, -\rangle$.

Proof. We consider the following part of the Gysin sequence:

$$H^{n-1}(X) \xrightarrow{\pi_*} H^{n-2}(M) \xrightarrow{\cup e} H^n(M) \cong \mathbb{Z}.$$

A class $\alpha \in H^{n-2}(M;\mathbb{Z})$ is in the image of π_* if and only if $e \cup \alpha = 0$, which is the case if and only if the Poincaré dual $c = PD(\alpha) \in H_2(M;\mathbb{Z})$ satisfies $\langle e, c\rangle = 0$. Since integration along the fibre

$$\pi_*\colon H^{n-1}(X;\mathbb{Z}) \to H^{n-2}(M;\mathbb{Z})$$

is by Lemma 9.3 Poincaré dual to

$$\pi_*\colon H_2(X;\mathbb{Z}) \to H_2(M;\mathbb{Z}),$$

this proves the claim. □

We now consider the following part of the Gysin sequence:

$$\ldots \longrightarrow H^{n-2}(M) \xrightarrow{\cup e} H^n(M) \longrightarrow H^n(X) \xrightarrow{\pi_*} H^{n-1}(M) \longrightarrow 0.$$

This shows that e is indivisible if and only if $\pi_*\colon H^n(X;\mathbb{Z}) \to H^{n-1}(M;\mathbb{Z})$ is an isomorphism, in other words

$$\pi_*\colon H_1(X;\mathbb{Z}) \longrightarrow H_1(M;\mathbb{Z})$$

is an isomorphism. The long exact homotopy sequence of the fibration $S^1 \to X \to M$

$$\ldots \longrightarrow \pi_2(M) \xrightarrow{\partial} \pi_1(S^1) \longrightarrow \pi_1(X) \xrightarrow{\pi_*} \pi_1(M) \longrightarrow 1$$

induces via Lemma A.5 an exact sequence

$$H_1(S^1;\mathbb{Z}) \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow H_1(M;\mathbb{Z}) \longrightarrow 0.$$

Hence we see that e is indivisible if and only if the fibre $S^1 \subset X$ is *null-homologous*.

From the long exact homotopy sequence above, we see that the fibre is *null-homotopic* if and only if $\partial\colon \pi_2(M) \to \pi_1(S^1)$ is surjective. Both statements are equivalent to

$$\pi_*\colon \pi_1(X) \to \pi_1(M)$$

being an isomorphism.

Lemma 9.5. *The map* $\partial\colon \pi_2(M) \to \pi_1(S^1) \cong \mathbb{Z}$ *in the long exact homotopy sequence for fibre bundles is given by*

$$\pi_2(M) \xrightarrow{h} H_2(M;\mathbb{Z}) \xrightarrow{\langle e,-\rangle} \mathbb{Z}$$

where h *denotes the Hurewicz homomorphism.*

Proof. Let $f\colon S^2 \to M$ be a continous map and $E = f^*X$ the pull-back S^1-bundle over S^2. By naturality of the long exact homotopy sequence there is a commutative diagram

$$\begin{array}{ccccccccc}
\pi_2(X) & \longrightarrow & \pi_2(M) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(M) \\
\uparrow & & {\scriptstyle f_*}\uparrow & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow \\
\pi_2(E) & \longrightarrow & \pi_2(S^2) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(E) & \longrightarrow & 1
\end{array}$$

Since f can represent any element in $\pi_2(M)$ and the equation $f^*(e(X)) = e(E)$ holds by naturality of the Euler class it suffices to prove the claim for M equal to S^2. We then have to prove that the map $\partial\colon \pi_2(S^2) \to \pi_1(S^1)$ is multiplication $\mathbb{Z} \xrightarrow{a\cdot} \mathbb{Z}$ by the Euler number $a = \langle e(E), [S^2]\rangle$.

By the exact sequence above it follows that $\pi_1(S^1) = \mathbb{Z}$ maps surjectively onto $\pi_1(E)$. Hence $\pi_1(E)$ is a finite cyclic group, in particular abelian. Therefore we have to prove that $H^2(E) \cong H_1(E) \cong \pi_1(E)$ is equal to $\mathbb{Z}/a\mathbb{Z}$. This follows from the following part of the Gysin sequence:

$$H^0(S^2) \xrightarrow{\cup e} H^2(S^2) \longrightarrow H^2(E) \xrightarrow{\pi_*} H^1(S^2) = 0.$$

□

Lemma 9.5 implies that ∂ is surjective if and only if $\langle e, -\rangle$ is surjective on spherical classes.

Remark 9.6. More generally, let $X \to M$ be a $U(m)$-principal bundle. Using the clutching construction and a Mayer-Vietoris argument one can show that $\pi_1(E) = H_1(E) = \mathbb{Z}/a\mathbb{Z}$ for any principal bundle $U(m) \to E \to S^2$, where $a = \langle c_1(E), [S^2]\rangle$. This implies as above (this lemma has been used in the proof of Theorem 8.18):

Lemma 9.7. *Let* $X \to M$ *be a* $U(m)$*-principal bundle. Then the map* $\partial : \pi_2(M) \to \pi_1(U(m)) \cong \mathbb{Z}$ *in the long exact homotopy sequence is given by*

$$\pi_2(M) \xrightarrow{h} H_2(M;\mathbb{Z}) \xrightarrow{\langle c_1(X),-\rangle} \mathbb{Z}.$$

Lemma 9.8. X *is simply-connected if and only if* M *is simply-connected and* e *is indivisible.*

Proof. If X is simply-connected, the long exact homotopy sequence shows that $\pi_1(M) = 1$ and $\partial\colon \pi_2(M) \to \pi_1(S^1)$ is surjective. Hence M is simply-connected and the Hurewicz map $h\colon \pi_2(M) \to H_2(M;\mathbb{Z})$ is an isomorphism. The surjectivity of ∂ implies that e is indivisible. Conversely, suppose that M is simply-connected and e is indivisible. The same argument shows that ∂ is surjective. The long exact homotopy sequence then implies the exact sequence $1 \to \pi_1(X) \to 1$. Hence $\pi_1(X) = 1$. □

Lemma 9.9. *Suppose the first Betti number of* M *vanishes,* $b_1(M) = 0$*. Then the map* $\pi^*\colon H^2(M;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ *is surjective with kernel* $\mathbb{Z}\cdot e$*.*

Proof. We consider the following part of the Gysin sequence:

$$H^0(M) \xrightarrow{\cup e} H^2(M) \xrightarrow{\pi^*} H^2(X) \longrightarrow H^1(M).$$

By assumption, $H^1(M) = 0$. Hence $\pi^*\colon H^2(M) \to H^2(X)$ is surjective with kernel $H^0(M) \cup e = \mathbb{Z}\cdot e$. □

We now determine when the total space X is spin.

Lemma 9.10. *The total space X is spin if and only if $w_2(M) \equiv \alpha e$ mod 2 for some $\alpha \in \{0,1\}$, i.e. if and only if M is spin or $w_2(M) \equiv e$ mod 2.*

Proof. We claim that the following relation holds:

$$w_2(X) = \pi^* w_2(M).$$

This follows because the tangent bundle of X is given by $TX = \pi^*TM \oplus \mathbb{R}$ and the Whitney sum formula implies $w_2(TX) = w_2(\pi^*TM) \cup w_0(\mathbb{R}) = \pi^* w_2(TM)$. Hence X is spin if and only if $w_2(M)$ is in the kernel of π^*.

We consider the following part of the $\mathbb{Z}_2$-Gysin sequence:

$$H^0(M;\mathbb{Z}_2) \xrightarrow{\cup \overline{e}} H^2(M;\mathbb{Z}_2) \xrightarrow{\pi^*} H^2(X;\mathbb{Z}_2),$$

where $\overline{e}$ denotes the mod 2 reduction of e. We see that the kernel of π^* is $\{0, \overline{e}\}$. This implies the claim. □

We now specialize to the case where the dimension of M is equal to 4.

Theorem 9.11. *Let M be a simply-connected closed oriented 4-manifold and X the circle bundle over M with indivisible Euler class e. Then X is a simply-connected closed oriented 5-manifold and the homology and cohomology of X are torsion free. We have:*

- $H_0(X;\mathbb{Z}) \cong H_5(X;\mathbb{Z}) \cong \mathbb{Z}$
- $H_1(X;\mathbb{Z}) \cong H_4(X;\mathbb{Z}) \cong 0$
- $H_2(X;\mathbb{Z}) \cong H_3(X;\mathbb{Z}) \cong \mathbb{Z}^{b_2(M)-1}$.

Proof. We only have to prove that the cohomology of X is torsion free and the formula for $H_2(X;\mathbb{Z})$. The cohomology groups $H^0(X), H^1(X)$ and $H^5(X)$ are always torsion free for an oriented 5-manifold X. We have the following part of the Gysin sequence:

$$\ldots \longrightarrow H^3(M) \xrightarrow{\pi^*} H^3(X) \xrightarrow{\pi_*} H^2(M) \longrightarrow \ldots$$

By assumption, $H^3(M) = 0$. Therefore the homomorphism π_* injects $H^3(X)$ into $H^2(M)$, which is torsion free by the assumption that M is simply-connected. Hence $H^3(X;\mathbb{Z})$ is torsion free itself. It remains to consider $H^2(X)$ and $H^4(X)$. By the Universal Coefficient Theorem and Poincaré duality, $H^2(X)$ is torsion free if and only if $H_1(X)$ is torsion free, if and only if $H^4(X)$ is torsion free. Since $H_1(X) = 0$, we see that $H^2(X)$ and $H^4(X)$ are torsion free.

By Lemma 9.9 we have $H^2(X;\mathbb{Z}) \cong H^2(M;\mathbb{Z})/\mathbb{Z}\cdot e$. Since $H^2(M;\mathbb{Z})$ is torsion free and e is indivisible, $H^2(M;\mathbb{Z})/\mathbb{Z}\cdot e \cong \mathbb{Z}^{b_2(M)-1}$. This implies the formula for $H_2(X;\mathbb{Z}) \cong H_3(X;\mathbb{Z})$. □

By the classification theorem for simply-connected 5-manifolds, in particular Corollary 7.30, we get the following theorem (this has also been proved in [32]).

Theorem 9.12. *Let M be a simply-connected closed oriented 4-manifold and X the circle bundle over M with indivisible Euler class e. Then X is diffeomorphic to*

- $X = \#(b_2(M)-1)S^2 \times S^3$ *if X is spin, and*

- $X = \#(b_2(M)-2)S^2 \times S^3 \# S^2 \tilde{\times} S^3$ *if* X *is not spin.*

The first case occurs if and only if $w_2(M) \equiv \alpha e$ *mod* 2*, for some* $\alpha \in \{0,1\}$.

Since every closed oriented 4-manifold is Spin^c and hence $w_2(M)$ is the mod 2 reduction of an integral class, we conclude as a corollary that every closed simply-connected 4-manifold M is diffeomorphic to the quotient of a free and smooth S^1-action on $\#(b_2(M)-1)S^2 \times S^3$.

IX.2 Connections on circle bundles with prescribed curvature

In this section we give a proof for a theorem of Kobayashi [72] that every closed 2-form representing the Euler class of a circle bundle can be realized as the curvature form of a principal connection on this bundle. If the closed 2-form is a symplectic form, a multiple of the connection will be a contact form on the total space of the circle bundle. This will be shown in the next section.

We first discuss the relation between the Euler class of a principal S^1-bundle and the curvature of a connection on this bundle, see e.g. [14]. Let $\pi\colon P \to M$ be a principal S^1-bundle. We identify S^1 in the standard way with $U(1)$. Since $U(1) \cong SO(2)$, the principal bundle P has a first Chern class $c_1(P)$ and an Euler class $e(P)$ in $H^2(M;\mathbb{Z})$. Both are the same,

$$c_1(P) = e(P).$$

Hence it is enough to focus on the Euler class $e(P)$. We denote the natural image of this class in $H^2(M;\mathbb{R}) \cong H^2_{DR}(M)$ by $e(P)_{\mathbb{R}}$.

Let A be a $U(1)$-connection on P. This is a certain 1-form on P with values in $\mathfrak{u}(1) \cong i\mathbb{R}$ which is invariant under the S^1-action. The curvature F of A can be considered as a closed 2-form on M. Let R denote the fundamental vector field generated by the action of the element $2\pi i \in \mathfrak{u}(1)$. An orbit of R, topologically a fibre of P, has period 1. There are the following relations, coming from the definition of a connection on a principal bundle:

$$\begin{aligned} dA &= \pi^* F \\ A(R) &= 2\pi i. \end{aligned}$$

Finally, there is a formula for $e(P)_{\mathbb{R}}$ in terms of F:

$$e(P)_{\mathbb{R}} = \left[\tfrac{i}{2\pi} F\right] \in H^2_{DR}(M).$$

We now prove the following theorem of Kobayashi [72].

Theorem 9.13. *Let* M *be a smooth manifold and* $\pi\colon P \longrightarrow M$ *a principal* S^1*-bundle with Euler class* $e(P)_{\mathbb{R}} \in H^2_{DR}(M)$*. Let* ω *be a closed differential form representing* $e(P)_{\mathbb{R}}$*. Then there exists a connection* A *on* P *with curvature* F *equal to* $-2\pi i\omega$*.*

Proof. We choose an arbitrary connection $\tilde{A}$ on the principal S^1-bundle $P \longrightarrow M$. Then its curvature $\tilde{F}$ is an imaginary valued 2-form on M such that

$$\begin{aligned} d\tilde{A} &= \pi^* \tilde{F} \\ e(P)_{\mathbb{R}} &= \left[\frac{i}{2\pi}\tilde{F}\right]. \end{aligned}$$

As cohomology classes $[\frac{i}{2\pi}\tilde{F}] = [\omega]$, hence there exists a 1-form μ on M such that

$$\tfrac{i}{2\pi}\tilde{F} - \omega = d\mu.$$

We define a new connection

$$A = \tilde{A} + 2\pi i \pi^* \mu.$$

Then

$$\begin{aligned} dA &= d\tilde{A} + 2\pi i \pi^* d\mu \\ &= \pi^*\tilde{F} + 2\pi i (\tfrac{i}{2\pi}\pi^*\tilde{F} - \pi^*\omega) \\ &= -2\pi i \pi^* \omega. \end{aligned}$$

Hence the curvature F of A is $-2\pi i\omega$. □

IX.3 The Boothby-Wang construction

We want to construct circle bundles over symplectic manifolds M whose Euler class is represented by the symplectic form. Since the Euler class is an element of $H^2(M;\mathbb{Z})$ the symplectic form has to represent an integral cohomology class in $H^2(M;\mathbb{R})$, i.e. it has to lie in the image of the natural homomorphism

$$H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R}) \xrightarrow{\cong} H^2_{DR}(M).$$

The existence of such a symplectic form is guaranteed by the following argument (this argument is from [52, Observation 4.3]): Let (M,ω) be a closed symplectic manifold. For every Riemannian metric on M, there exists a small ϵ-ball B_ϵ around 0 in the space of harmonic 2-forms on M such that every element in $\omega + B_\epsilon$ is symplectic. Since the set of classes in $H^2(M;\mathbb{R})$ represented by these elements is open, there exists a symplectic form which represents a rational cohomology class. By multiplication with a suitable integer, we can find a symplectic form which represents an integral class. If we want, we can choose the integer such that the class is indivisible. Note also that all symplectic forms in $\omega + B_\epsilon$ can be connected to ω by a smooth path of symplectic forms. This implies that they all have the same canonical class K as ω.

We fix the following data:

- A closed connected symplectic manifold (M^{2n},ω) with symplectic form ω, representing an integral cohomology class in $H^2(M;\mathbb{R})$.
- An integral lift $[\omega]_\mathbb{Z} \in H^2(M;\mathbb{Z})$ of $[\omega] \in H^2_{DR}(M)$.

Let $\pi\colon X \to M$ be the principal circle bundle over M with Euler class $e(X) = [\omega]_\mathbb{Z}$. Choose a connection A on $X \longrightarrow M$ with curvature $-2\pi i\omega$, as in Theorem 9.13.

Proposition 9.14. *Define the real valued 1-form $\lambda = \frac{1}{2\pi i}A$ on X. Then λ is a contact form on X with*

$$\begin{aligned} d\lambda &= -\pi^*\omega \\ \lambda(R) &= 1. \end{aligned}$$

Proof. We have the relations

$$\begin{aligned} dA &= -2\pi i \pi^*\omega \\ A(R) &= 2\pi i. \end{aligned}$$

This implies the corresponding relations for λ. The tangent bundle of X splits as $TX \cong \mathbb{R} \oplus \pi^*TM$, where the trivial $\mathbb{R}$-summand is spanned by the vector field R. Fix a point of X and choose a basis $(R, v_1, \ldots, v_{2n})$ of its tangent space, where the v_i form an oriented basis of the kernel of λ. Then

$$\begin{aligned} \lambda \wedge (d\lambda)^n(R, v_1, \ldots, v_{2n}) &= (d\lambda)^n(v_1, \ldots, v_{2n}) \\ &= (-1)^n \omega^n(\pi_* v_1, \ldots, \pi_* v_{2n}) \\ &\neq 0. \end{aligned}$$

Hence $\lambda \wedge (d\lambda)^n$ is a volume form on X, and λ is contact. □

Remark 9.15. If we define the orientation on X via the splitting $TX \cong \mathbb{R} \oplus \pi^*TM$, where the trivial $\mathbb{R}$-summand is oriented by R and TM by ω, then λ is a positive contact form if n is even and negative otherwise.

Definition 9.16. The contact structure ξ on the closed oriented manifold X^{2n+1}, defined by the contact form λ above, is called the *Boothby-Wang contact structure* associated to the symplectic manifold (M, ω). Since $d\lambda(R) = 0$, the Reeb vector field of λ is given by the vector field R along the fibres.

For the original construction see [13].

Proposition 9.17. *If λ' is another contact form, defined by a different connection A' as above, then the associated contact structure ξ' is isotopic to ξ.*

Proof. The connection A' is an S^1-invariant 1-form on X with

$$\begin{aligned} dA' &= dA \\ A'(R) &= A(R). \end{aligned}$$

Hence $A' - A = \pi^*\alpha$ for some closed 1-form α on M. Define $A_t = A + \pi^* t\alpha$ for $t \in \mathbb{R}$. Then A_t is a connection on X with curvature $-2\pi i\omega$ for all t. Let $\lambda_t = \lambda + \pi^*(\frac{1}{2\pi i} t\alpha)$. Then λ_t is a contact form on X for all $t \in [0,1]$, with $\lambda_0 = \lambda$ and $\lambda_1 = \lambda'$. Therefore, ξ and ξ' are isotopic through the contact structures defined by λ_t. □

The Chern classes of ξ are given by the Chern classes of ω in the following way.

Lemma 9.18. *Let $X \to M$ be a Boothby-Wang fibration with contact structure ξ. Then $c_i(\xi) = \pi^* c_i(TM, \omega)$ for all $i \geq 0$. The manifold X is spin, if and only if*

$$c_1(M) \equiv \alpha[\omega]_{\mathbb{Z}} \textit{ mod } 2,$$

for some $\alpha \in \{0, 1\}$.

Proof. Let J be a compatible almost complex structure for ω on M. Then there exists a compatible complex structure J' for ξ on X such that $\pi^*(TM, J) \cong (\xi, J')$ as complex vector bundles. The naturality of characteristic classes proves the first claim. The second claim follows from Lemma 9.10 and $c_1(M) \equiv w_2(M)$ mod 2. □

Chapter X

Contact homology and Boothby-Wang fibrations

Contents

In this chapter we construct contact structures on certain simply-connected 5-manifolds X which are equivalent as almost contact structures but are not equivalent as contact structures. The contact structures arise on circle bundles over simply-connected symplectic 4-manifolds (M, ω) by the construction from Chapter IX. We will use the results form Chapter VIII to determine when they are equivalent as almost contact structures. In the first part of this chapter we show how the theory of contact homology implies that the divisibility of the canonical class K of the symplectic structure ω is an invariant of the contact structure on the total space of the Boothby-Wang circle bundle over M. We can then use the examples from Chapter VI to find examples of contact structures on simply-connected 5-manifolds X with torsion free $H_2(X;\mathbb{Z})$ which are equivalent as almost contact structures but not as contact structures. This adds examples of inequivalent contact structures with non-vanishing first Chern class to the (infinitely many) contact structures with first Chern class zero found by O. van Koert in [74] on these simply-connected 5-manifolds. Also I. Ustilovsky [139] found infinitely many contact structures on the sphere S^5 and F. Bourgeois [15] on $T^2 \times S^3$ and T^5, both in the case of vanishing first Chern class.

X.1 The construction for symplectic 4-manifolds

We fix the following data:

- A closed, simply-connected, symplectic 4-manifold (M, ω), with symplectic form ω representing an integral cohomology class in $H^2(M;\mathbb{R})$, cf. the argument at the beginning of Section IX.3. Since $H^2(M;\mathbb{Z})$ is torsion free, $[\omega]$ has a unique integral lift, which we also denote by $[\omega] \in H^2(M;\mathbb{Z})$. We assume that $[\omega]$ is indivisible.

- Let $\pi\colon X \longrightarrow M$ be the principal S^1-bundle over M with Euler class $e(X) = [\omega]$. Then X is a closed, simply-connected, oriented 5-manifold with torsion-free cohomology (see Theorem 9.11). We will often denote the class $[\omega]$ also by ω.

- Let λ be a Boothby-Wang contact form on X with associated contact structure ξ (Definition 9.16). By Proposition 9.17 the contact structure ξ does not depend on λ up to isotopy.

By Theorem 9.12, the 5-manifold X is diffeomorphic to

- $\#(b_2(M) - 1)S^2 \times S^3$ if X is spin, and

- $\#(b_2(M) - 2)S^2 \times S^3 \# S^2 \tilde{\times} S^3$ if X is not spin.

Hence the same abstract, closed, simply-connected 5-manifold X with torsion free homology can be realized in several different ways as a Boothby-Wang fibration over different simply-connected symplectic 4-manifolds M and therefore admits many, possibly non-equivalent, contact structures.

Definition 10.1. Let $d(\xi) \geq 0$ denote the divisibility of $c_1(\xi) \in H^2(X;\mathbb{Z})$, as in definition 6.5. Similarly, we denote the divisibility of the canonical class $K = -c_1(M) \in H^2(M;\mathbb{Z})$ of ω by $d(K)$.

Note that X is spin if and only if $d(\xi)$ is even by Lemma 8.8. With Corollary 8.22 we get:

Proposition 10.2. *Suppose that (M', ω') is another closed, simply-connected, symplectic 4-manifold with integral and indivisible symplectic form ω'. Denote the associated Boothby-Wang total space by (X', ξ').*

- *The simply-connected 5-manifolds X and X' are diffeomorphic if and only if $b_2(M) = b_2(M')$ and $d(\xi) \equiv d(\xi')$ mod 2.*

- *If $d(\xi) = d(\xi')$, then ξ and ξ' are equivalent as almost contact structures.*

The divisibility $d(\xi)$ can be calculated in the following way: By Lemma 9.9, the bundle projection π defines an isomorphism

$$\pi^*\colon H^2(M;\mathbb{Z})/\mathbb{Z}\omega \xrightarrow{\cong} H^2(X;\mathbb{Z}),$$

and by Lemma 9.18 we have

$$\pi^* c_1(M) = c_1(\xi).$$

Let $[c_1(M)]$ denote the image of $c_1(M)$ in the quotient $H^2(M;\mathbb{Z})/\mathbb{Z}\omega$, which is free abelian since ω is indivisible. Then $d(\xi)$ is also the divisibility of the class $[c_1(M)]$. We will use π^* to identify

$$H^2(X;\mathbb{Z}) = H^2(M;\mathbb{Z})/\mathbb{Z}\omega, \text{ and}$$
$$c_1(\xi) = [c_1(M)].$$

We then have:

Lemma 10.3. *The divisibility $d(\xi)$ is the maximal integer d such that*

$$c_1(M) = dR + \gamma\omega$$

where γ is some integer and $R \in H^2(M;\mathbb{Z})$ not a multiple of ω.

X.2 The Δ-invariant

Let $\pi\colon X \to M$ be a Boothby-Wang fibration as in the previous section. We can choose a class $A \in H_2(M;\mathbb{Z})$ such that $\omega(A) = \langle \omega, A\rangle = 1$ because we assumed that ω is indivisible. Consider the number

$$c_1(A) := \langle c_1(M), A\rangle.$$

We want to determine the set of all these integers. We fix one arbitrary element $A_0 \in H_2(M;\mathbb{Z})$ with $\omega(A_0) = 1$.

Lemma 10.4. *The set of integers*

$$\{c_1(A) \mid A \in H_2(M;\mathbb{Z}),\, \omega(A) = 1\}$$

is equal to $c_1(A_0) + d(\xi)\mathbb{Z}$. In particular, the reduction $c_1(A) \in \mathbb{Z}/d(\xi)\mathbb{Z}$ is independent of the choice of A.

Proof. By Lemma 9.4, the image of homomorphism $\pi_*\colon H_2(X;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ induced by the bundle projection is the kernel of $\langle \omega, -\rangle$. On the other hand, we know that $\pi^* c_1(M) = c_1(\xi)$. Suppose $A \in H_2(M;\mathbb{Z})$ is a class with $\omega(A) = 1$. Then $\omega(A - A_0) = 0$, hence $A - A_0 = \pi_* B$ for some $B \in H_2(X;\mathbb{Z})$. This implies

$$\begin{aligned} c_1(A) &= c_1(A_0) + \langle c_1(M), \pi_* B\rangle \\ &= c_1(A_0) + \langle c_1(\xi), B\rangle \\ &\in c_1(A_0) + d(\xi)\mathbb{Z}. \end{aligned}$$

Conversely, let $m \in \mathbb{Z}$ be an arbitrary integer. We can choose a class $B \in H_2(X;\mathbb{Z})$ with $\langle c_1(\xi), B\rangle = m d(\xi)$ since the divisibility of $c_1(\xi)$ is $d(\xi)$. Define the homology class $A := A_0 + \pi_* B$ on M. Then we have $\omega(A) = 1$ and $c_1(A) = c_1(A_0) + m d(\xi)$. This shows that all integers in the set $c_1(A_0) + d(\xi)\mathbb{Z}$ can be realized as $c_1(A)$ with $\omega(A) = 1$. $\square$

Definition 10.5. We call $c_1(A) \in \mathbb{Z}/d(\xi)\mathbb{Z}$ the Δ-*invariant* $\Delta(\omega)$ of the symplectic 4-manifold (M, ω).

The lemma implies that the set of all numbers $c_1(A)$, with $A \in H_2(M;\mathbb{Z})$ and $\omega(A) = 1$, is completely determined by $d(\xi) \in \mathbb{N}$ and $\Delta(\omega) \in \mathbb{Z}/d(\xi)\mathbb{Z}$. The following lemma describes some relations between these numbers.

Lemma 10.6. *The following relations hold:*

(1.) Let $c_1(M) = d(\xi)R + \gamma\omega$ for some class $R \in H^2(M;\mathbb{Z})$ and integer $\gamma \in \mathbb{Z}$. Then $\Delta(\omega) \equiv \gamma$ mod $d(\xi)$.

(2.) The integer $d(K)$ divides $d(\xi)$.

(3.) Let Δ be an integer reducing to $\Delta(\omega)$ modulo $d(\xi)$. Then $\gcd(\Delta, d(\xi)) = d(K)$.

Proof. (1.) This follows by the definition of $\Delta(\omega)$ if we evaluate both sides on $A \in H_2(M;\mathbb{Z})$ with $\omega(A) = 1$.

(2.) We can write $c_1(M) = d(K)W$ where W is a class in $H^2(M;\mathbb{Z})$. Then $[c_1(M)] = d(K)[W]$ in $H^2(M;\mathbb{Z})/\mathbb{Z}\omega$. Since $d(\xi)$ is the divisibility of $[c_1(M)]$, the integer $d(\xi)$ has to be a multiple of $d(K)$.

(3.) By part (a)

$$c_1(M)(A) = d(\xi)R(A) + \gamma,$$

where $A \in H_2(M;\mathbb{Z})$ is a class with $\omega(A) = 1$ and $\gamma \equiv \Delta \bmod d(\xi)$. Since $d(K)$ divides $c_1(M)$ and $d(\xi)$, it also divides divides γ and hence Δ. On the other hand, there exists a homology class $B \in H_2(M;\mathbb{Z})$ such that $d(K) = c_1(M)(B)$. Again by part (a)

$$d(K) = d(\xi)R(B) + \gamma\omega(B).$$

Hence there exist integers $x, y \in \mathbb{Z}$ such that $d(K) = xd(\xi) + y\Delta$. This proves the claim.

□

X.3 Contact homology

In this section we consider invariants derived from contact homology. We only take into account the *classical contact homology* $H_*^{cont}(X,\xi)$ which is a graded supercommutative algebra, defined using rational holomorphic curves with one positive puncture and several negative punctures in the symplectization of the contact manifold. We use a variant of this theory for the so-called Morse-Bott case, described in [15] and in Section 2.9.2. in [33].

We are going to associate to each Boothby-Wang fibration $\pi\colon X \to M$ as above a graded commutative algebra $\mathfrak{A}(X,M)$. Choose a basis $B_1,\dots,B_N$ of $H_2(X;\mathbb{Z})$, where $N = b_2(X) = b_2(M) - 1$ and let

$$A_n = \pi_* B_n \in H_2(M;\mathbb{Z}),\ 1 \le n \le N.$$

Note that

$$c_1(B_n) := \langle c_1(\xi), B_n\rangle = \langle c_1(M), A_n\rangle = c_1(A_n).$$

Choose a class $A_0 \in H_2(M;\mathbb{Z})$ such that

$$\omega(A_0) = 1.$$

The classes $A_0, A_1, \dots, A_N$ form a basis of $H_2(M;\mathbb{Z})$. We consider variables

$$z = (z_1,\dots,z_N), \text{ and}$$
$$q = \{q_{k,i}\}_{k\in\mathbb{N},\,0\le i\le a},$$

where $a = b_2(M) + 1$. They have degrees defined by

$$deg(z_n) = -2c_1(B_n)$$
$$deg(q_{k,i}) = deg\Delta_i - 2 + 2c_1(A_0)k,$$

where $deg\Delta_i$ is given by

$$deg\Delta_i = \begin{cases} 0 & \text{if } i = 0 \\ 2 & \text{if } i = 1,\dots,b_2(M) \\ 4 & \text{if } i = b_2(M)+1. \end{cases}$$

In our situation the degree of all variables is even (hence the algebra we are going to define is truly commutative, not only supercommutative).

Definition 10.7. We define the following algebras.

- $\mathfrak{L}(X) = \mathbb{C}[H_2(X;\mathbb{Z})]$ = the graded commutative ring of Laurent polynomials in the variables z with coefficients in $\mathbb{C}$.
- $\mathfrak{A}(X,M) = \bigoplus_{d\in\mathbb{Z}} \mathfrak{A}_d(X,M)$ = the graded commutative algebra of polynomials in the variables q with coefficients in $\mathfrak{L}(X)$.

A *homomorphism* ϕ of graded commutative algebras $\mathfrak{A}, \mathfrak{A}'$ over $\mathfrak{L}(X)$

$$\phi\colon \mathfrak{A} = \bigoplus_{d\in\mathbb{Z}} \mathfrak{A}_d \to \mathfrak{A}' = \bigoplus_{d\in\mathbb{Z}} \mathfrak{A}'_d$$

is a homomorphism of rings, which is the identity on $\mathfrak{L}(X)$ and such that $\phi(\mathfrak{A}_d) \subset \mathfrak{A}'_d$ for all $d \in \mathbb{Z}$.

Lemma 10.8.
- *Up to isomorphism, the ring $\mathfrak{L}(X)$ does not depend on the choice of basis $B_1, \dots, B_N$ for $H_2(X;\mathbb{Z})$.*
- *For fixed $\mathfrak{L}(X)$, the algebra $\mathfrak{A}(X,M)$ does not depend, up to isomorphism over $\mathfrak{L}(X)$, on the choice of the class $A_0 \in H_2(M;\mathbb{Z})$ as above.*

Proof. Let $\overline{B_1}, \dots, \overline{B_N}$ be another basis of $H_2(X;\mathbb{Z})$ and $\overline{\mathfrak{L}(X)}$ the associated ring, generated by variables $\overline{z}$. Then there exists matrix

$$(\beta_{mn}) \in SL(N,\mathbb{Z})$$

with $1 \le m,n \le N$, such that

$$\overline{B_m} = \sum_{n=1}^{N} \beta_{mn} B_n.$$

Define a homomorphism $\phi\colon \overline{\mathfrak{L}(X)} \to \mathfrak{L}(X)$ via

$$\overline{z_m} \mapsto \prod_{n=1}^{N} z_n^{\beta_{mn}},$$

for all $1 \le m \le N$. Then ϕ preserves degrees and is an isomorphism, since the matrix (β_{mn}) is invertible.

Let $\overline{A_0}$ be another element in $H_2(M;\mathbb{Z})$ such that $\omega(\overline{A_0}) = 1$ and $\overline{\mathfrak{A}(X,M)}$ the associated algebra, generated by variables $\overline{q}$. Then there exists a vector

$$(\alpha_n) \in \mathbb{Z}^N,$$

with $1 \le n \le N$, such that

$$\overline{A_0} = A_0 + \sum_{n=1}^{N} \alpha_n A_n.$$

Define a homomorphism $\psi\colon \overline{\mathfrak{A}(X,M)} \to \mathfrak{A}(X,M)$ via

$$\overline{q_{k,i}} \mapsto q_{k,i} \prod_{n=1}^{N} z_n^{-k\alpha_n}, \quad k \in \mathbb{N},\ 0 \le i \le a,$$

and which is the identity on $\mathfrak{L}(X)$. Then ψ preserves degrees and is invertible. □

We will now describe the relation of $\mathfrak{A}(X, M)$ to the Boothby-Wang contact structure ξ on X induced by ω and the Δ-invariant $\Delta(\omega)$. Let $d := d(\xi)$ and Δ an integral lift of $\Delta(\omega) \in \mathbb{Z}/d\mathbb{Z}$.

Definition 10.9. Suppose that $d \geq 1$. For $0 \leq b < d$ denote by Q_b the set of generators $\{q_{k,i}\}$ with

$$deg(q_{k,i}) \equiv 2b \bmod 2d.$$

Remark 10.10. If $c_1(\xi) \neq 0$, the variables $z_1, \ldots, z_n$ which generate the ring $\mathfrak{L}(X)$ do not all have degree zero. Hence $\mathfrak{B}(X, M) = \mathfrak{A}(X, M)/\mathfrak{L}(X)$, which is an algebra over $\mathbb{C}$, does not inherit a natural grading in this case. However, since the degrees of the variables z_n are all multiples of $2d$, the algebra $\mathfrak{B}(X, M)$ has a grading by elements in $\mathbb{Z}_{2d}$. The images of the generators $q_{k,i}$ form generators for this infinite polynomial algebra and Q_b is the set of generators of degree $2b$ mod $2d$ (I learnt this interpretation from K. Cieliebak).

Lemma 10.11. *Assume that $d \geq 1$. Then the set Q_b is infinite if $d(K)$ divides one of the integers $b-1, b, b+1$ and empty otherwise.*

Proof. Suppose $d(K) = \gcd(\Delta, d)$ divides one of the integers $b + \epsilon$, with $\epsilon \in \{-1, 0, 1\}$. Then the equation

$$b = -\epsilon + \Delta k + d\alpha$$

has infinitely many solutions $k \geq 1$ with $\alpha \in \mathbb{Z}$. Choose an integer $0 \leq i \leq a$ with $deg\Delta_i - 2 = -2\epsilon$. Since $\Delta \equiv c_1(A_0)$ mod d by Lemma 10.4,

$$deg(q_{k,i}) = -2\epsilon + 2c_1(A_0)k \equiv -2\epsilon + 2\Delta k \equiv 2b \bmod 2d$$

for infinitely many $k \geq 1$. Hence these $q_{k,i}$ are all in Q_b.

Conversely, suppose that $d(K)$ does not divide any of the integers $b + \epsilon$, with $\epsilon \in \{-1, 0, 1\}$. Suppose that Q_b contains an element $q_{l,j}$. We have $deg(q_{l,j}) = -2\epsilon + 2c_1(A_0)l$ for some $\epsilon \in \{-1, 0, 1\}$. By assumption,

$$deg(q_{l,j}) = -2\epsilon + 2c_1(A_0)l = 2b - 2d\alpha,$$

for some $\alpha \in \mathbb{Z}$. This implies

$$b + \epsilon = c_1(A_0)l + d\alpha = \Delta l + d\alpha',$$

for some integer $\alpha' \in \mathbb{Z}$. This is impossible, since $d(K)$ divides the right side, but not the left side. $\square$

Example 10.12. Suppose that $d \geq 1$. If $d(K) \in \{1, 2, 3\}$, then Lemma 10.11 implies that Q_b is infinite for all $b = 0, \ldots, d-1$. If $d(K) \geq 4$ (and hence $d \geq 4$ as well), then at least one of the Q_b is empty, e.g. Q_2 is always empty in this case.

We now make the following assumptions:

- The simply-connected 5-manifold X can be realized as the Boothby-Wang total space over another closed, simply-connected, symplectic 4-manifold (M', ω') where ω' represents an integral and indivisible class. This implies in particular that $b_2(M') = b_2(M) = a - 1$. Denote the canonical class of (M', ω') by K' and its divisibility by $d(K')$
- We assume that $d(\xi') = d(\xi) =: d$ and choose an integral lift Δ' of $\Delta(\omega') \in \mathbb{Z}/d\mathbb{Z}$.
- Let $\mathfrak{A}(X, M')$ denote the associated algebra over $\mathfrak{L}(X)$, generated by variables $\{q'_{l,j}\}$, with $l \in \mathbb{N}, 0 \leq j \leq a$.

If $d \geq 1$, denote by Q'_b as above the set of generators $\{q'_{l,j}\}$ of degree congruent to $2b$ modulo $2d$, for each $0 \leq b < d$.

Lemma 10.13. *Assume that $d \geq 4$ and at least one of the numbers $d(K), d(K')$ is ≥ 4. Then the following two statements are equivalent:*

- *There exists an integer $0 \leq b < d$ such that Q_b and Q'_b do not have the same cardinality (i.e. one of them is empty and the other infinite).*
- $d(K) \neq d(K')$.

Proof. Suppose that $d(K) = d(K')$. By Lemma 10.11, the sets Q_b and Q'_b have the same cardinality for all $0 \leq b < d$. Conversely, suppose that $d(K) \neq d(K')$; without loss of generality $d(K) < d(K')$. If $d(K) \in \{1, 2, 3\}$ let $b = 2$. Then Q_2 is infinite, while Q'_2 is empty (since $d(K') \geq 4$ by assumption). If $d(K) \geq 4$ let $b = d(K) - 1 \geq 3$. Then $d(K)$ divides $b + 1$, but $d(K')$ does not divide any of the integers $b - 1, b, b + 1$. Hence Q_b is infinite and Q'_b empty. □

Lemma 10.14. *Suppose that either (i) $d = 0$ or (ii) $d \geq 4$ and at least one of the numbers $d(K), d(K')$ is ≥ 4. If the $\mathbb{Z}_{2d}$-graded polynomial algebras $\mathfrak{B}(X, M)$ and $\mathfrak{B}(X, M')$ over $\mathbb{C}$ are isomorphic, then $d(K) = d(K')$.*

Proof. Suppose that $d = 0$ and that there exists an isomorphism $\phi\colon \mathfrak{B}(X, M) \to \mathfrak{B}(X, M')$. In this case, both algebras are graded by the integers and the elements of lowest degree in $\mathfrak{B}(X, M)$ and $\mathfrak{B}(X, M')$ have degree $-2 + 2\Delta$ and $-2 + 2\Delta'$, respectively. Since ϕ has to preserve degree, this implies $\Delta = \Delta'$ and hence

$$d(K) = \gcd(\Delta, 0) = \Delta = \Delta' = \gcd(\Delta', 0) = d(K').$$

Now assume that $d \geq 4$ and at least one of $d(K), d(K')$ is ≥ 4. Suppose that $d(K) \neq d(K')$ and there exists an isomorphism $\phi\colon \mathfrak{B}(X, M) \to \mathfrak{B}(X, M')$. Both algebras are freely generated by the images of the elements $\{q_{k,i}\}$ and $\{q'_{l,j}\}$, which we still denote by the same symbols.

By Lemma 10.13, there exists an integer $0 \leq b < d$ such that Q_b and Q'_b have different cardinality. Without loss of generality, we may assume that Q_b is empty and Q'_b infinite (otherwise we consider ϕ^{-1}). Let $q'_{r,s}$ be a generator in Q'_b. Then $q'_{r,s}$ is a polynomial in the images

$$\{\phi(q_{k,i})\}_{k \in \mathbb{N}, 0 \leq i \leq a},$$

with coefficients in $\mathbb{C}$ and we can write

$$q'_{r,s} = f(\phi(q_{k_1,i_1}), \dots, \phi(q_{k_v,i_v})) \in \mathbb{C}[\phi(q_{k_1,i_1}), \dots, \phi(q_{k_v,i_v})].$$

The images $\phi(q_{k,i})$ are themselves polynomials in the variables $\{q'_{l,j}\}$ with coefficients in $\mathbb{C}$. Expressed as a polynomial in the variables $\{q'_{l,j}\}$, at least one of the images $\phi(q_{k_w,i_w})$, $1 \leq w \leq v$, must contain a summand of the form $\alpha q'_{r,s}$ with $\alpha \in \mathbb{C}$ non-zero. Since ϕ preserves degrees, $\phi(q_{k_w,i_w})$ is homogeneous of degree

$$deg(\phi(q_{k_w,i_w})) = deg(\alpha q'_{r,s}) = deg(q'_{r,s}) \equiv 2b \bmod 2d.$$

This implies $deg(q_{k_w,i_w}) \equiv 2b \bmod 2d$, hence $q_{k_w,i_w} \in Q_b$. This is impossible, since $Q_b = \emptyset$. □

Lemma 10.15. *Suppose that either (i) $d(K) = d(K')$ or (ii) both numbers $d(K), d(K')$ are ≤ 3 and $d \neq 0$. Then the algebras $\mathfrak{A}(X, M)$ and $\mathfrak{A}(X, M')$ are isomorphic over $\mathfrak{L}(X)$.*

Proof. We can choose a basis $B_1, \dots, B_N$ of $H_2(X;\mathbb{Z})$ such that

$$c_1(B_1) = d(\xi) = d$$
$$c_1(B_n) = 0, \quad \text{for all } 2 \le n \le N.$$

Choose elements $A_0 \in H_2(M;\mathbb{Z})$ and $A_0' \in H_2(M';\mathbb{Z})$ such that

$$c_1(A_0) = \Delta, \; c_1(A_0') = \Delta'.$$

This is possible by Lemma 10.4. We will use these bases to define the algebras $\mathfrak{A}(X, M)$ and $\mathfrak{A}(X, M')$. Suppose that $d(K) = d(K')$. If $d = 0$, then

$$\Delta = \gcd(\Delta, 0) = d(K)$$
$$\Delta' = \gcd(\Delta', 0) = d(K').$$

This implies $deg(q_{k,i}) = deg(q'_{k,i})$ for all $k \in \mathbb{N}, 0 \le i \le a$. Hence the map

$$q_{k,i} \mapsto q'_{k,i}, \quad k \in \mathbb{N}, \, 0 \le i \le a,$$

induces a degree preserving isomorphism $\phi\colon \mathfrak{A}(X, M) \to \mathfrak{A}(X, M')$.

Suppose $d \ge 1$. Under our assumptions, the sets Q_b and Q_b' have the same cardinality for each $0 \le b < d$, cf. Lemma 10.13 and Example 10.12. Hence there exists a bijection

$$\psi\colon \mathbb{N} \times \{0, \dots, a\} \to \mathbb{N} \times \{0, \dots, a\}, (k, i) \mapsto \psi(k, i),$$

such that

$$deg(q_{k,i}) \equiv deg(q'_{\psi(k,i)}) \bmod 2d.$$

Since z_1' has degree $-2d$, there exists for each $(k, i) \in \mathbb{N} \times \{0, \dots, a\}$ an integer $\alpha(k, i) \in \mathbb{Z}$, such that

$$deg(q_{k,i}) = deg({z_1'}^{\alpha(k,i)} q'_{\psi(k,i)}).$$

The map

$$q_{k,i} \mapsto {z_1'}^{\alpha(k,i)} q'_{\psi(k,i)}, \quad k \in \mathbb{N}, \, 0 \le i \le a,$$

therefore induces a well-defined, degree preserving isomorphism $\phi\colon \mathfrak{A}(X, M) \to \mathfrak{A}(X, M')$ over $\mathfrak{L}(X)$. □

Combining Lemmas 10.14 and 10.15 we arrive at the following theorem.

Theorem 10.16. *The algebras $\mathfrak{A}(X, M)$ and $\mathfrak{A}(X, M')$ are isomorphic over $\mathfrak{L}(X)$ if and only if one of the following three conditions is satisfied:*

- $d \ge 1$ *and both* $d(K), d(K') \le 3$
- $d = 0$ *and* $d(K) = d(K')$
- $d \ge 4$ *and* $d(K) = d(K') \ge 4$.

Here we have used that an isomorphism of $\mathfrak{A}$-algebras induces an isomorphism of $\mathfrak{B}$-algebras.

The following result is described in [33], Proposition 2.9.1.

Theorem 10.17. *For a Boothby-Wang fibration $X \to M$ as above, the Morse-Bott contact homology $H_*^{cont}(X, \xi)$ specialized*[1] *at $t = 0$ is isomorphic to $\mathfrak{A}(X, M)$.*

Using Theorem 10.16 and Proposition 10.2 we get the following corollary. The part concerning equivalent contact structures follows, because equivalent contact structures have isomorphic contact homologies.

Corollary 10.18. *Let X be a closed, simply-connected 5-manifold which can be realized in two different ways as a Boothby-Wang fibration over closed, simply-connected symplectic 4-manifolds (M_1, ω_1) and (M_2, ω_2), whose symplectic forms represent integral and indivisible classes:*

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (M_1, \omega_1) & & (M_2, \omega_2) \end{tikzcd}$$

Denote the associated Boothby-Wang contact structures on X by ξ_1 and ξ_2 and the canonical classes of the symplectic structures by K_1 and K_2. Then:

- *The almost contact structures underlying ξ_1 and ξ_2 are equivalent if and only if $d(\xi_1) = d(\xi_2)$.*

Suppose that ξ_1 and ξ_2 are equivalent as contact structures.

- *If $d(\xi_1) = d(\xi_2) = 0$, then $d(K_1) = d(K_2)$.*
- *If $d(\xi_1) = d(\xi_2) \neq 0$, then either both $d(K_1), d(K_2) \leq 3$ or $d(K_1) = d(K_2)$.*

X.4 Applications

In order to apply Corollary 10.18 it is useful to perturb the symplectic form on a given symplectic manifold (M, ω), because in this way one can construct Boothby-Wang contact structures on different levels on the same total space over M.

Lemma 10.19. *Let (M, ω) be a minimal closed symplectic 4-manifold with $b_2^+(M) > 1$ and canonical class K. Then every class in $H^2(M; \mathbb{R})$ of the form $[\omega] + tK$ for a real number $t \geq 0$ can be represented by a symplectic form.*

Proof. Note that the canonical class K is a Seiberg-Witten basic class. Since M is assumed minimal, Proposition 3.10 and the argument in Corollary 3.11 show that K is represented by a disjoint collection of embedded symplectic surfaces in M all of which have non-negative self-intersection. The inflation procedure [83], which can be done on each of the surfaces separately and with the same parameter $t \geq 0$, shows that $[\omega] + tK$ is represented by a symplectic form on M. $\square$

We can now prove:

Theorem 10.20. *Let M be a closed, minimal simply-connected 4-manifold with $b_2^+(M) > 1$ and ω a symplectic form on M. Denote the canonical class of ω by K and let $m \geq 1$ be an arbitrary integer. Then there exists a symplectic form ω' on M, deformation equivalent to ω and representing an integral and indivisible class, such that the first Chern class of the associated Boothby-Wang contact structure ξ' has divisibility $d(\xi') = md(K)$.*

[1] Note that contact homology is actually a family of algebras which can be specialized at any $t \in H^*(X; \mathbb{R})$.

Proof. Let $k = d(K)$. We can assume that ω is integral and choose a basis for $H^2(M;\mathbb{Z})$ such that

$$K = k(1,0,\dots,0)$$
$$\omega = (\omega_1,\omega_2,0,\dots,0).$$

By a deformation we can assume that ω is not parallel to K, hence $\omega_2 \neq 0$. We can also assume that ω_1 is negative while ω_2 is positive: Consider the change of basis vectors

$$(1,0,0,\dots,0) \mapsto (1,0,0,\dots,0)$$
$$(0,1,0,\dots,0) \mapsto (q,\pm 1,0,\dots,0),$$

where q is some integer. Then the expression of ω changes to

$$(\omega_1 + q\omega_2, \pm\omega_2, 0,\dots,0).$$

Hence if q is large enough, has the correct sign and the $\pm$ sign is chosen correctly, the claim follows.

Suppose that $\sigma \in H^2(M;\mathbb{Z})$ is an indivisible class of the form

$$\sigma = (\sigma_1,\sigma_2,0,\dots,0)$$

which can be represented by a symplectic form, also denoted by σ. Let ζ denote the contact structure induced on the Boothby-Wang total space by σ. We claim that the divisibility $d(\zeta)$ is given by

$$d(\zeta) = k|\sigma_2|.$$

To prove this we write $K = -c_1(M) = rR + \gamma\sigma$, where $R = (R_1, R_2, 0..., 0)$. Then $k - \gamma\sigma_1$ and $\gamma\sigma_2$ are divisible by r. This implies that r divides $k\sigma_2$. Conversely note that by assumption σ_1,σ_2 are coprime. Let R_1, R_2 be integers with

$$1 = \sigma_2 R_1 - \sigma_1 R_2$$

and define

$$\gamma = -kR_2.$$

Then we can write

$$K = k\sigma_2 R - kR_2\sigma.$$

This proves the claim about $d(\zeta)$.

Suppose that $m \geq 1$. By multiplying the expression for ω with the positive number $\frac{m}{\omega_2}$ we see that the (rational) class

$$(\alpha, m, 0,\dots,0), \quad \alpha = \omega_1\frac{m}{\omega_2},$$

is represented by a symplectic form. Note that $\alpha < 0$. By the inflation trick in Lemma 10.19 with parameter $t = \frac{1}{k}(1-\alpha)$ it follows that

$$\omega' = (\alpha, m, 0,\dots,0) + (1-\alpha, 0,\dots,0)$$
$$= (1, m, 0,\dots,0)$$

is represented by a symplectic form ω'. The class ω' is indivisible. Let ξ' denote the induced Boothby-Wang contact structure. By our calculation above, $d(\xi') = mk$. □

Definition 10.21. For integers $d \geq 4$ and $r \geq 2$ denote by $Q(r, d)$ the number of elements of the following set:

$$Q(r,d) = \# \left\{ k \in \mathbb{N} \;\middle|\; \begin{array}{l} k \geq 4,\ k \text{ divides } d \text{ and there exists a simply-connected} \\ \text{symplectic 4-manifold } (M,\omega) \text{ with } b_2(M) = r \text{ and } b_2^+(M) > 1 \\ \text{whose canonical class } K \text{ has divisibility } d(K) = k. \end{array} \right\}$$

Lemma 10.22. *Let $d \geq 4$ and $r \geq 2$ be integers. Suppose that either*

- *d is odd and X the simply-connected 5-manifold $X = (r-2)S^2 \times S^3 \# S^2 \tilde{\times} S^3$, or*
- *d is even and X the simply-connected 5-manifold $X = (r-1)S^2 \times S^3$.*

In both cases, there exist at least $Q(r, d)$ many inequivalent contact structures on the level d on X.

Proof. Recall that a spin (non-spin) simply-connected 5-manifold has only even (odd) levels. Suppose that $d \geq 4$ is an integer and (M, ω) a simply-connected symplectic 4-manifold with $b_2(M) = r$ and $b_2^+(M) \geq 2$ whose canonical class has divisibility $k = d(K) \geq 4$ dividing d. We can write $d = mk$. By Lemma 6.2, the manifold M is minimal and by Theorem 10.20 there exists a symplectic structure ω' on M that induces on the Boothby-Wang total space X with $b_2(X) = r - 1$ a contact structure with $d(\xi) = d$. Since the symplectic form ω' is deformation equivalent to ω the canonical class K remains unchanged. By Corollary 10.18 the contact structures on the same non-zero level d on X coming from symplectic 4-manifolds with different divisibilities $k \geq 4$ of their canonical classes are pairwise inequivalent. $\square$

Definition 10.23. For an integer $d \geq 4$ let $N(d)$ denote the number of positive integers ≥ 4 dividing d. If d is even, let $N'(d)$ denote the number of odd divisors ≥ 4 of d.

Lemma 10.24. *Let $d \geq 4$ and $r \geq 2$ be integers.*

(1.) For any r we have $Q(r, d) \leq N(d)$.

(2.) If d is even and r is not congruent to 2 mod 4, then $Q(r, d) \leq N(d')$.

Proof. The first statement is clear by the definitions. For the second statement, suppose that M is a simply-connected symplectic spin 4-manifold. Then the intersection form Q_M is even and $b_2^+(M)$ odd. Note that $b_2^- = b_2^+ - \sigma$, hence $b_2(M) = 2b_2^+(M) - \sigma(M)$. Since Q_M is even, the signature $\sigma(M)$ is divisible by 8. This implies that $b_2(M)$ is congruent to 2 mod 4 because $b_2^+(M)$ is odd. Hence if r is *not* congruent to 2 mod 4 then there does not exist a simply-connected symplectic spin 4-manifold M with second Betti number r. Hence all elements of $Q(r, d)$ are in this case odd. $\square$

We can now use our geography results from Chapter VI to estimate the number $Q(r, d)$ for different values of r and d. For example, from symplectic structures on homotopy elliptic surfaces we get:

Lemma 10.25. *Let $n \geq 1$ and $d \geq 4$ be arbitrary integers.*

(1.) If d is odd, then $Q(12n - 2, d) = N(d)$.

(2.) If d is even, then $Q(24n - 2, d) = N(d)$ and $Q(24n - 15, d) = N'(d)$.

Proof. By Theorem 6.11 we have the following:

(1.) Suppose that $n > 1$ and $d \geq 4$ is odd. Then for every (odd) divisor $k \geq 4$ of d there exists a symplectic homotopy elliptic surface M with $b_2(M) = 12n - 2$, $b_2^+(M) \geq 3$ and $d(K) = k$. If $n = 1$ one can choose a Dolgachev surface M with $b_2(M) = 10$, $b_2^+(M) = 1$ and $d(K) = k$. Since the canonical class of a Dolgachev surface is represented by two disjoint tori of self-intersection zero, given by the multiple fibres, the proofs of Lemma 10.19 and Theorem 10.20 also work in this case.

(2.) Suppose that $d \geq 4$ is even and $m \geq 1$. Then for every *odd* divisor $k \geq 4$ of d there exists a symplectic homotopy elliptic surface M with $b_2(M) = 12m - 2$ and $d(K) = k$. For an *even* divisor $k \geq 4$ of d there exists a symplectic homotopy elliptic surface M with $b_2(M) = 12m - 2$ and $d(K) = k$ if and only if $m \geq 2$ is even.

Hence $Q(12m - 2, d) \geq N(d')$ if m is odd and $Q(12m - 2, d) = N(d)$ if m is even. Setting $m = 2n - 1$ in the first case and $m = 2n$ in the second case the claims follow.

□

With Lemma 10.22 we get:

Proposition 10.26. *Let $n \geq 1$ be an arbitrary integer.*

(1.) On every odd level $d \geq 5$ the 5-manifold $(12n - 4)S^2 \times S^3 \# S^2 \tilde{\times} S^3$ admits at least $N(d)$ inequivalent contact structures.

(2.) On every even level $d \geq 4$ the 5-manifold $(24n - 3)S^2 \times S^3$ admits at least $N(d)$ inequivalent contact structures.

(3.) On every even level $d \geq 4$ the 5-manifold $(24n - 15)S^2 \times S^3$ admits at least $N'(d)$ inequivalent contact structures.

In a similar way we can use other geography results from Chapter VI to find inequivalent contact structures on simply-connected 5-manifolds X with torsion free $H_2(X; \mathbb{Z})$.

Appendix A

Complements of codimension 2 submanifolds

Contents

In this appendix we derive some formulas for the first homology and the fundamental group of the complement of a closed, oriented codimension 2 submanifold F in a closed, oriented manifold M. We are in particular interested in the case of surfaces in 4-manifolds. However, the general case of codimension 2 submanifolds is not more difficult, hence we consider this case. Some of the results are well-known and used in many places in the literature, for instance Proposition A.3 on the fundamental group and Proposition A.2 on the first homology if $H_1(M) = 0$. We use the results in this chapter in particular in Sections V.1.4, V.3.2 and VI.3.2.

A.1 Definitions

Let M^n be a closed, oriented manifold and $F^{n-2} \subset M$ a closed, oriented submanifold of codimension 2 which represents a non-torsion class $[F] \in H_{n-2}(M;\mathbb{Z})$. We denote a closed tubular neighbourhood of F by $\nu(F)$ or νF and let

$$M' = M \setminus \operatorname{int} \nu F.$$

Then M' is an oriented manifold with boundary $\partial\nu F$. On the closed manifold M, the Poincaré dual of $[F]$ acts as a homomorphism on $H_2(M;\mathbb{Z})$,

$$\langle PD([F]), - \rangle \colon H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

By the assumption on $[F]$ the image of this homomorphism is non-zero and hence a subgroup of $\mathbb{Z}$ of the form $k\mathbb{Z}$ with $k > 0$. We assume that $[F]$ is divisible by k, i.e. there exists a class $A \in H_{n-2}(M;\mathbb{Z})$ such that $[F] = kA$. This is always true if $H_{n-2}(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$ is torsion free.

We fix the following notation for some of the inclusions. For simplicity, we denote the maps induced on homology and homotopy groups by the same symbol:

$i\colon F \to M$

$\rho\colon M' \to M$

$\mu\colon \partial\nu F \to M'$

$j\colon \sigma \to M'$, where σ denotes a *meridian* of F in M', i.e. a fibre of the circle bundle $\partial\nu F \to F$.

For any topological space X, we use as an abbreviation the symbols $H_*(X)$ and $H^*(X)$ to denote the homology and cohomology groups of X with $\mathbb{Z}$-coefficients. Other coefficients are denoted explicitly.

Let A' be the image of A under the homomorphism

$$f\colon H_{n-2}(M) \to H_{n-2}(M,F) \cong H_{n-2}(M', \partial M'), \tag{A.1}$$

where the first map comes from the long exact homology sequence for the pair (M,F) and the second map is by excision.

A.2 Calculation of the first integral homology

We begin with the calculation of the first homology of the complement of F in M. A similar calculation has been done in [67] and [120] for the case of a 4-manifold M and under the assumption $H_1(M) = 0$.

Lemma A.1. *There exists a short exact sequence*

$$0 \longrightarrow \frac{H^2(M)}{\mathbb{Z}PD[F]} \xrightarrow{\rho^*} H^2(M') \longrightarrow \textit{ker}(i\colon H_{n-3}(F) \to H_{n-3}(M)) \longrightarrow 0.$$

This sequence splits, because $H_{n-3}(F) \cong H^1(F)$ is torsion free.

Proof. We consider the long exact sequence in cohomology associated to the pair (M, M'):

$$\ldots \to H^2(M,M') \to H^2(M) \xrightarrow{\rho^*} H^2(M') \xrightarrow{\delta} H^3(M,M') \to H^3(M) \to \ldots.$$

By excision, Poincaré duality and the deformation retraction $\nu F \to F$ we have:

$$H^k(M,M') \cong H^k(\nu F, \partial\nu F) \cong H_{n-k}(\nu F) \cong H_{n-k}(F).$$

The map $H^k(M,M') \to H^k(M)$ is then under Poincaré duality equivalent to the map $i\colon H_{n-k}(F) \to H_{n-k}(M)$. With $k = 1, 2$, this proves the claim. □

We have the following proposition.

Proposition A.2. *For M and F as above, $H_1(M';\mathbb{Z}) \cong H_1(M;\mathbb{Z}) \oplus \mathbb{Z}_k$.*

Proof. We first show that $H^1(M') \cong H^1(M)$. This follows from the long exact sequence in homology for the pair (M,F):

$$0 \to H_{n-1}(M) \to H_{n-1}(M,F) \to H_{n-2}(F) \xrightarrow{i} H_{n-2}(M) \to H_{n-2}(M,F) \to H_{n-3}(F) \to H_{n-3}(M).$$

The map i is given by

$$i\colon H_{n-2}(F) \cong \mathbb{Z} \longrightarrow H_{n-2}(M), m \mapsto m \cdot [F].$$

Since $[F]$ is non-torsion, the map i is injective. This implies that $H_{n-1}(M,F) \cong H_{n-1}(M)$. Hence by excision and Poincaré duality

$$H^1(M') \cong H_{n-1}(M', \partial M') \cong H_{n-1}(M,F) \cong H_{n-1}(M) \cong H^1(M).$$

By the Lemma A.1 we see that

$$\begin{aligned} \mathrm{Tor}H^2(M') &\cong \mathrm{Tor}(H^2(M)/\mathbb{Z}PD[F]) \\ &\cong \mathrm{Tor}(H^2(M)/\mathbb{Z}kPD(A)) \\ &\cong \mathrm{Tor}H^2(M) \oplus \mathbb{Z}PD(A)/k\mathbb{Z}PD(A) \\ &\cong \mathrm{Tor}H^2(M) \oplus \mathbb{Z}_k PD(A). \end{aligned}$$

The third step follows because A is indivisible and of infinite order. The Universal Coefficient Theorem implies that

$$\mathrm{Tor}H^2(M) = \mathrm{Ext}(H_1(M), \mathbb{Z}) \cong \mathrm{Tor}H_1(M),$$

and similarly for M' (the second isomorphism is not canonical). This implies

$$\mathrm{Tor}H_1(M') \cong \mathrm{Tor}H_1(M) \oplus \mathbb{Z}_k.$$

Using again the Universal Coefficient Theorem we get

$$\begin{aligned} H_1(M') &\cong H^1(M') \oplus \mathrm{Tor}H_1(M') \\ &\cong H^1(M) \oplus \mathrm{Tor}H_1(M) \oplus \mathbb{Z}_k \\ &\cong H_1(M) \oplus \mathbb{Z}_k. \end{aligned}$$

□

A.3 Calculation of the fundamental group

In this section, we determine the relation between $\pi_1(M')$ and $\pi_1(M)$ which can be expressed as follows:

Proposition A.3. *The fundamental groups of M and M' are related by*

$$\pi_1(M) \cong \pi_1(M')/N(\sigma),$$

where $N(\sigma)$ denotes the normal subgroup in $\pi_1(M')$ generated by the meridian σ of F in M'.

Proof. We choose a base point in $\partial M'$, which we do not write down in the following. We want to apply the Seifert-van Kampen Theorem to the decomposition

$$M = M' \cup_\partial \nu(F).$$

We fix presentations

$$\begin{aligned} \pi_1(F) &= \langle \alpha_1, \ldots, \alpha_m \mid r_1, \ldots, r_n \rangle \\ \pi_1(M') &= \langle \beta_1, \ldots, \beta_k \mid q_1, \ldots, q_l \rangle, \end{aligned}$$

where the α_i and β_j are closed loops in F and M' starting and ending at the base point in $\partial M'$. Note that $\pi_1(\nu(F)) \cong \pi_1(F)$ since F is a strong deformation retract of $\nu(F)$. Let

$$\begin{aligned} \psi &\colon \pi_1(\partial\nu(F)) \to \pi_1(F) \\ \mu &\colon \pi_1(\partial\nu(F)) \to \pi_1(M') \end{aligned}$$

denote the canonical homomorphisms induced by the inclusions (and projection in the first case). The long exact homotopy sequence for the fibre bundle $S^1 \to \partial\nu(F) \to F$ gives an exact sequence

$$\ldots \to \pi_1(S^1) \to \pi_1(\partial\nu(F)) \xrightarrow{\psi} \pi_1(F) \to 1. \tag{A.2}$$

Hence we can choose generators $\gamma_1, \ldots, \gamma_{m+1}$ for $\pi_1(\partial\nu(F))$ such that $\gamma_{m+1} = \sigma$ and

$$\begin{aligned} \psi(\gamma_i) &= \alpha_i, \quad \text{for } i = 1, \ldots, m \\ \psi(\gamma_{m+1}) &= 1. \end{aligned}$$

We set

$$w_j = \mu(\gamma_j) \in \pi_1(M'), \quad 1 \leq j \leq m+1.$$

Under the natural inclusions i and ρ we can view all α_i and β_j as elements in $\pi_1(M)$. By Seifert-van Kampen

$$\pi_1(M) = \langle \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k \mid r_1, \ldots, r_n, q_1, \ldots, q_l, \alpha_1 w_1^{-1}, \ldots, \alpha_m w_m^{-1}, w_{m+1}^{-1} \rangle.$$

We want to simplify the presentation for $\pi_1(M)$. The relations $\alpha_i w_i^{-1} = 1$ imply that in $\pi_1(M)$ we get $\alpha_i = w_i$ for $1 \leq i \leq m$. Since the w_i are relations in the variables β_j we can write

$$\pi_1(M) \cong \langle \beta_1, \ldots, \beta_k \mid r_1, \ldots, r_n, q_1, \ldots, q_l, w_{m+1}^{-1} \rangle,$$

where the relations

$$r_i(\alpha_1, \ldots, \alpha_m)$$

become

$$r_i(w_1, \ldots, w_m).$$

The curve $r_i(\alpha_1, \ldots, \alpha_m)$ is null-homotopic on F. Since

$$\psi(r_i(\gamma_1, \ldots, \gamma_m)) = r_i(\alpha_1, \ldots, \alpha_m),$$

we see by the exact sequence (A.2) that $r_i(\gamma_1, \ldots, \gamma_m)$ is homotopic to a multiple $\gamma_{m+1}^{k_i}$ of the fibre. This implies that

$$r_i(w_1, \ldots, w_m) w_{m+1}^{-k_i}$$

is null-homotopic in M' and hence a product of the $q_1, \ldots, q_l$. Therefore

$$\begin{aligned} \pi_1(M) &\cong \langle \beta_1, \ldots, \beta_k \mid q_1, \ldots, q_l, w_{m+1} \rangle \\ &\cong \pi_1(M')/N(\sigma), \end{aligned}$$

since w_{m+1} is the class of the fibre in $\pi_1(M')$. □

Corollary A.4. *If M is simply-connected, then $\pi_1(M') = N(\sigma)$. Hence the fundamental group of the complement M' is normally generated by σ.*

Proposition A.3 implies that the sequence

$$1 \to N(\sigma) \xrightarrow{j} \pi_1(M') \xrightarrow{\rho} \pi_1(M) \to 1 \tag{A.3}$$

is exact. By the following lemma this sequence induces an exact sequence in homology.

Lemma A.5. *Let $A \xrightarrow{j} B \xrightarrow{\rho} C \longrightarrow 1$ be an exact sequence of groups. Then this induces an exact sequence on abelianizations $H(A) \xrightarrow{H(j)} H(B) \xrightarrow{H(\rho)} H(C) \longrightarrow 0$.*

Proof. By the universal property of abelianizations, we get a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{j} & B & \xrightarrow{\rho} & C & \longrightarrow & 1 \\ {\scriptstyle H}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle H}\downarrow & & \\ H(A) & \xrightarrow{H(j)} & H(B) & \xrightarrow{H(\rho)} & H(C) & \longrightarrow & 0 \end{array}$$

The map $H(\rho)$ is surjective, since $H \circ \rho$ is surjective. Moreover, $H(\rho) \circ H(j) = H(\rho \circ j) = 0$. Let $\beta \in H(B)$ with $H(\rho)(\beta) = 0$. Choose $b \in B$ with $H(b) = \beta$. Then $\rho(b)$ is a product of commutators in C,

$$\rho(b) = \Pi_i [c_i, c_i'].$$

Since ρ is surjective, we can choose preimages b_i, b_i' of c_i, c_i'. Let

$$b' = b(\Pi_i [b_i, b_i'])^{-1}.$$

Then $\rho(b') = 1$ and $H(b') = \beta$. Let $a \in A$ with $j(a) = b'$ and $\alpha = H(a)$. Then $H(j)(\alpha) = H(j(a)) = H(b') = \beta$. This proves exactness at $H(B)$. □

Corollary A.6. *The first integral homology groups of M' and M are related by the exact sequence*

$$0 \to \mathbb{Z}_k \xrightarrow{j} H_1(M';\mathbb{Z}) \xrightarrow{\rho} H_1(M;\mathbb{Z}) \to 0. \tag{A.4}$$

which splits. The image of j is generated by the class σ of the meridian of F in M'.

Proof. By equation (A.3) and Lemma A.5 we get a short exact sequence

$$0 \to H(N[\sigma]) \xrightarrow{j} H_1(M') \xrightarrow{\rho} H_1(M) \to 0.$$

The subgroup $H(N[\sigma])$ is equal to the cyclic subgroup $\langle\sigma\rangle$ generated by the class of a fibre, which is finite by Proposition A.2. Hence the map

$$\mathrm{Tor}H^2(M) \cong \mathrm{Ext}(H_1(M);\mathbb{Z}) \xrightarrow{(\rho_*)^*} \mathrm{Ext}(H_1(M');\mathbb{Z}) \cong \mathrm{Tor}H^2(M')$$

has cokernel isomorphic to $\langle\sigma\rangle$. Since the map $(\rho_*)^*$ is the same as the naturally induced map ρ^* on cohomology, which by the proof of Proposition A.2 has cokernel $\mathbb{Z}_k PD(A)'$, we see that $\langle\sigma\rangle \cong \mathbb{Z}_k$. □

A.4 Splittings for the first homology

By Theorem A.2 we know that the short exact sequence in Corollary A.6 splits. We want to determine an explicit splitting. We dualize the short exact sequence

$$0 \to \mathbb{Z}_k \xrightarrow{j} H_1(M') \xrightarrow{\rho} H_1(M) \to 0$$

to get the sequence

$$0 \to \mathrm{Hom}(H_1(M), \mathbb{Z}_k) \xrightarrow{\rho^*} \mathrm{Hom}(H_1(M'), \mathbb{Z}_k) \xrightarrow{j^*} \mathrm{Hom}(\mathbb{Z}_k, \mathbb{Z}_k).$$

Note with our convention from the beginning homology and cohomology groups without explicit coefficients have integral coefficients. A splitting of sequence (A.4) is determined by a homomorphism

$$s\colon H_1(M') \to \mathbb{Z}_k, \quad \text{with } s \circ j = \mathrm{Id}_{\mathbb{Z}_k},$$

or, equivalently, an element

$$s \in \mathrm{Hom}(H_1(M'), \mathbb{Z}_k), \quad \text{with } j^* s = \mathrm{Id}_{\mathbb{Z}_k}.$$

Suppose there exists such an element s. Then every other $s' \in \mathrm{Hom}(H_1(M'), \mathbb{Z}_k)$ with $j^* s = \mathrm{Id}_{\mathbb{Z}_k}$ is given by $s' = s + \rho^* t$ for some $t \in \mathrm{Hom}(H_1(M), \mathbb{Z}_k)$. This follows because $j^*(s' - s) = 0$ and by exactness of sequence (A.4). All such t can be chosen to define a splitting s'.

By the Universal Coefficient Theorem,

$$\mathrm{Hom}(H_1(M'), \mathbb{Z}_k) \cong H^1(M'; \mathbb{Z}_k)$$

and similarly for M. Hence a splitting is determined by an element $\alpha \in H^1(M'; \mathbb{Z}_k)$ with $\langle \alpha, j_* \sigma \rangle = 1$ and every other splitting α' is of the form $\alpha' = \alpha + \rho^* \beta$ where β is an (arbitrary) element in $H^1(M; \mathbb{Z}_k)$. We now want to construct a class in $H^1(M'; \mathbb{Z}_k)$ which defines a splitting.

Let $A \in H_{n-2}(M)$ be a class with $[F] = kA$, as above, and $A' \in H_{n-2}(M', \partial M')$ the associated class in M'. We consider the long exact sequence in cohomology related to the sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_k \to 0$:

$$\ldots \to H^1(M') \xrightarrow{p} H^1(M'; \mathbb{Z}_k) \xrightarrow{\beta} H^2(M') \xrightarrow{\cdot k} H^2(M') \to \ldots$$

Here β denotes the associated Bockstein homomorphism. Since A' is a k-torsion class on M' we have $k \cdot PD(A') = 0$. Hence there exists a class $\mathcal{A} \in H^1(M'; \mathbb{Z}_k)$ with $\beta(\mathcal{A}) = PD(A')$.

We can realize the Bockstein homomorphism as a connecting homomorphism explicitly in the following way (see e.g. [16]): We denote the singular chain complex of M' by C_*. Let $a \in \mathrm{Hom}(C_1, \mathbb{Z}_k)$ be a representative of $\mathcal{A}$. Then there exists an element $\hat{a} \in \mathrm{Hom}(C_1, \mathbb{Z})$ such that a is the mod k reduction of $\hat{a}$. Since $\delta a = 0$ we see that $\delta \hat{a} \in \mathrm{Hom}(C_2, \mathbb{Z})$ takes values in $k\mathbb{Z}$ and hence is divisible by k. Then the cochain $\frac{1}{k} \delta \hat{a}$ is coclosed and represents $PD(A)' \in H^2(M'; \mathbb{Z})$.

We consider the homomorphism

$$\begin{aligned} s_{\mathcal{A}}\colon H_1(M'; \mathbb{Z}) &\longrightarrow H_1(M; \mathbb{Z}) \oplus \mathbb{Z}_k \\ \alpha &\mapsto (\rho_* \alpha, \langle \mathcal{A}, \alpha \rangle). \end{aligned}$$

Proposition A.7. *The homomorphism $s_{\mathcal{A}}$ determines a splitting of the short exact sequence in Corollary A.6.*

Proof. Let $[\gamma] \in H_1(M';\mathbb{Z})$ denote the class of a fibre of the circle bundle $\partial\nu(F) \to F$ in M'. To prove that $s_{\mathcal{A}}$ determines a splitting of the short exact sequence (A.4) we have to show that

$$\langle \mathcal{A}, [\gamma] \rangle \equiv 1 \text{ mod } k.$$

Since $[\gamma]$ has order k in $H_1(M';\mathbb{Z})$ there exists a chain $\sigma \in C_2$ such that $k\gamma = \partial\sigma$. We get

$$\begin{aligned}\langle \mathcal{A}, [\gamma] \rangle &= \langle a, \gamma \rangle \equiv \langle \hat{a}, \gamma \rangle \mod k \\ &= \langle \hat{a}, \tfrac{1}{k}\partial\sigma \rangle = \langle \tfrac{1}{k}\delta\hat{a}, \sigma \rangle \mod k.\end{aligned}$$

We can cap off $\rho_*\sigma$ in M with k many 2-disks D^2, which are fibres of the normal bundle $\nu(F)$, to get a closed chain τ representing a class $[\tau] \in H_2(M;\mathbb{Z})$. Let c be a cocycle representing $PD(A) \in H^2(M;\mathbb{Z})$. Then we can write

$$\rho^* c = \tfrac{1}{k}\delta\hat{a} + \delta\mu,$$

for some $\mu \in \mathrm{Hom}(C_1, \mathbb{Z})$, since $\frac{1}{k}\delta\hat{a}$ represents $PD(A') = \rho^* PD(A)$. Then we have modulo k:

$$\begin{aligned}\langle \mathcal{A}, [\gamma] \rangle &\equiv \langle \rho^* c, \sigma \rangle - \langle \delta\mu, \sigma \rangle \\ &\equiv \langle c, \rho_*\sigma \rangle - k\langle \mu, \gamma \rangle \\ &\equiv \langle c, \tau \rangle - k\langle c, D^2 \rangle - k\langle \mu, \gamma \rangle \\ &\equiv \langle PD(A), [\tau] \rangle \\ &\equiv \tfrac{1}{k}\langle PD[F], [\tau] \rangle \mod k.\end{aligned}$$

We know that $\langle PD[F], [\tau] \rangle = k$ since the zero section of $\nu(F)$ intersects each fibre once. This implies the claim. $\square$

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