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Topological properties of asymptotic invariants and universal volume bounds

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Abstract

In this thesis, we prove that many asymptotic invariants of closed manifolds depend only on the image of the fundamental class under the classifying map of the universal covering. Examples include numerical invariants that reflect the asymptotic behaviour of the universal covering, like the minimal volume entropy and the spherical volume, as well as properties that are qualitative measures for the largeness of a manifold and its coverings, like enlargeability and hypersphericity.

Another important class of invariants that share the above invariance property originates from universal volume bounds. The main example is the systolic constant, which encodes the relation between short noncontractible loops and the volume of a manifold. Further interesting examples are provided by the optimal constants in Gromov's filling inequalities, for which we show that they depend only on the dimension and orientability.

Considering higher-dimensional generalizations of the systolic constant, a complete answer to the question about the existence of stable systolic inequalities is given. In the spirit of the results mentioned already, we also prove that the stable systolic constant depends only on the image of the fundamental class in a suitable Eilenberg-Mac Lane space.

Zusammenfassung

In dieser Arbeit wird gezeigt, dass viele asymptotische Invarianten geschlossener Mannigfaltigkeiten nur vom Bild der Fundamentalklasse unter der klassifizierenden Abbildung der universellen Überlagerung abhängen. Hierzu zählen sowohl numerische Invarianten, die das asymptotische Verhalten der universellen Überlagerung widerspiegeln, wie die minimale Volumenentropie und das sphärische Volumen, als auch Eigenschaften, die qualitative Maße für die Größe einer Mannigfaltigkeit und ihrer Überlagerungen darstellen, wie Vergrößerbarkeit und Hypersphärizität.

Eine weitere wichtige Klasse von Invarianten, die die obige Invarianzeigenschaft teilen, erhält man aus universellen Volumenschranken. Das wichtigste Beispiel hierfür ist die systolische Konstante, die das Verhältnis zwischen kurzen nichtzusammenziehbaren Schleifen und dem Volumen einer Mannigfaltigkeit wiedergibt. Weitere interessante Beispiele werden durch die optimalen Konstanten in Gromovs Filling-Ungleichungen gegeben, von denen gezeigt wird, dass sie nur von der Dimension und der Orientierbarkeit abhängen.

Bei der Betrachtung höher-dimensionaler Verallgemeinerungen der systolischen Konstante wird eine vollständige Antwort auf die Frage nach der Existenz stabiler systolischer Ungleichungen gefunden. In Analogie zu den oben erwähnten Ergebnissen wird bewiesen, dass die stabile systolische Konstante nur vom Bild der Fundamentalklasse in einem passenden Eilenberg-Mac Lane-Raum abhängt.

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Chapter 1

Introduction

This thesis is concerned with various invariants and properties of closed manifolds that describe asymptotic aspects of the universal covering or that arise as optimal constants in curvature-free bounds on the volume. We will prove that many of these invariants depend only on the image of the fundamental class under the classifying map of the universal covering. This behaviour will be called *homological invariance*.

Examples for asymptotic invariants include the minimal volume entropy, which describes the exponential volume growth of the universal covering, and the spherical volume, which is defined via immersions of the universal covering into the space of square-integrable funtions. Moreover, enlargeability and many other largeness properties of the universal covering like hypersphericity or macroscopic largeness are also homologically invariant in the above sense.

Universal volume bounds like Gromov's celebrated systolic and filling inequalities define, via the optimal constants in these inequalities, numerical invariants of manifolds. The most prominent example is the systolic constant, which determines the relation between the length of short noncontractible loops and the volume of the manifold. As an application of our results on homological invariance, an inequality between the minimal volume entropy and the systolic constant is derived.

The invariants mentioned so far actually depend on the fundamental group, that is, their values change if the fundamental group changes. The optimal constants in Gromov's filling inequalities do not. More precisely, we will show that they depend only on the dimension and orientability.

A slightly different kind of curvature-free volume bound is provided by the stable systolic inequalities. Here, the lower bound on the volume is not given by a one-dimensional quantity like the length of noncontractible loops but by the stabilized volume of higher-dimensional submanifolds that are not nullhomologous. Therefore, it is not surprising that homological invariance holds only after replacing the classifying map of the universal covering by a suitable map to an Eilenberg-Mac Lane space of higher degree. Moreover, we will give a complete answer to the freedom problem of stable systoles, that is, we will find a topological characterization for the existence and nonexistence of stable systolic inequalities.

In the following sections, we will give more details on the definitions of the invariants mentioned above, try to motivate our interest in them, and present the main results of the thesis. Some of these results are not stated in full generality to avoid too many technicalities.

1.1 Asymptotic invariants

The simplicial volume of manifolds was introduced by Gromov in his seminal paper on bounded cohomology [Gro82]. It is the value on the fundamental class of the seminorm that is dual to the seminorm of bounded cohomology. The simplicial volume plays a crucial role in the proof of Mostow's rigidity theorem by Gromov and Thurston. Moreover, it provides a lower bound for the minimal volume as we shall see below.

To be more precise, for a connected closed orientable manifold M of dimension n the simplicial volume is defined in the following way: consider a singular chain $c = \sum_{i=1}^{k} r_i\sigma_i$ with real coefficients r_i and singular simplices $\sigma_i : \Delta^n \to M$. Its ℓ^1-*norm* is given by $\|c\|_1 := \sum_{i=1}^{k} |r_i|$. The *simplicial volume* $\|M\|$ of the manifold is the infimum of the ℓ^1-norms of all real cycles that represent the fundamental class $[M] \in H_n(M;\mathbb{Z})$.

Gromov showed that the simplicial volume is entirely determined by the fundamental group $\pi_1(M)$ of the manifold and the classifying map $\Phi : M \to K(\pi_1(M), 1)$ of the universal covering. Recall that the Eilenberg-Mac Lane space $K(\pi_1(M), 1)$ is a connected CW complex with fundamental group $\pi_1(M)$ whose universal covering is contractible, and the classifying map Φ is a map that induces the identity on fundamental groups. Note that the homotopy type of the Eilenberg-Mac Lane space and the homotopy class of the classifying map are uniquely determined. The following theorem specifies the dependence of the simplicial volume on the classifying map. It may be seen as an archetypal statement on homological invariance.

Theorem 1.1 (Gromov). *Let M and N be two connected closed orientable manifolds having the same dimension n and the same fundamental group π. Let $\Phi : M \to K(\pi, 1)$ and $\Psi : N \to K(\pi, 1)$ denote the respective classifying maps of the universal coverings of M and N. If $\Phi_*[M] = \Psi_*[N] \in H_n(\pi;\mathbb{R})$, then $\|M\| = \|N\|$.*

With this now classical result in mind, we turn to related invariants, for which we will prove similar statements.

The simplicial volume provides a lower bound for another important invariant introduced in [Gro82]: the *minimal volume* $\operatorname{MinVol}(M)$. This is the infimal volume of all Riemannian metrics on M for which the absolute value of the sectional curvature is bounded by 1. By the scaling properties of the sectional curvature, this may also be written as

$$\operatorname{MinVol}(M) = \inf_g \|K_g\|^{n/2} \operatorname{Vol}(M, g),$$

where $\|K_g\|$ denotes the supremum of the absolute value of the sectional curvature, and the infimum is taken over all Riemannian metrics g on M.

In fact, there is a whole chain of inequalities that relates the minimal volume to the simplicial volume:

$$\frac{n^{n/2}}{n!}\|M\| \leq 2^n n^{n/2} T(M) \leq \lambda(M)^n \leq h(M)^n \leq (n-1)^n \operatorname{MinVol}(M),$$

where $T(M)$ is the spherical volume, $\lambda(M)$ is the minimal volume entropy, and $h(M)$ is the minimal topological entropy. The first two inequalities are proved in [BessCG91], the third one in [Man79], and the last one can be deduced from [Man81]. This chain appears for instance in [PatP03] (without the spherical volume), and in [Kot04] and [KędKM06]. We will continue with a short definition of these intermediate invariants.

The *volume entropy* of a Riemannian manifold is the exponential growth rate of the volume of a ball in the universal covering, that is to say, it is defined as

$$\lambda(M, g) := \lim_{r\to\infty} \tfrac{1}{r} \log \operatorname{Vol}(B(x, r))$$

where $B(x, r)$ is the ball of radius r around a point x in the universal covering of the manifold with respect to the lifted metric. Manning showed that this limit exists and is independent of the point $x \in \tilde{M}$ (see [Man79]). Using this asymptotic invariant, the *minimal volume entropy* of M is given by

$$\lambda(M) := \inf_g \lambda(M, g) \operatorname{Vol}(M, g)^{1/n}.$$

Note that the volume factor is necessary because of the scaling properties of the volume entropy. Just as well, one could restrict to metrics with unit volume.

The *minimal topological entropy* is defined in exactly the same way by replacing the volume entropy by the topological entropy of the geodesic flow on the unit tangent bundle of M. Recall that topological entropy measures the exponential complexity of the orbit structure of a dynamical system. (More details can be found for example in Paternain's book [Pat99].)

The spherical volume was introduced by Besson, Courtois, and Gallot and plays an important role in their solution to the minimal entropy problem for locally symmetric spaces of negative curvature (see [BessCG95]). To define the spherical volume, consider all $\pi_1(M)$-equivariant immersions of the universal covering $\tilde{M}$ into the unit sphere of the Hilbert space $L^2(\tilde{M})$. The pullback of the inner product defines a $\pi_1(M)$-invariant metric on the universal covering and thus a metric on M. The *spherical volume* is the infimum of the volumes of all metrics obtained in this way.

It is known that the minimal volume is sensitive to the differentiable structure of the manifold (see [Bes98] and also [Kot04]), and that the minimal volume entropy is homotopy and bordism invariant (see [Bab92], [Bab94], and [Bab95]). In chapter 2, we will prove that both the minimal volume entropy and the spherical volume are homologically invariant.

Theorem 1.2. *Let M and N be two connected closed orientable manifolds having the same dimension n and the same fundamental group π. Let $\Phi : M \to K(\pi, 1)$ and $\Psi : N \to K(\pi, 1)$ denote the respective classifying maps of the universal coverings of M and N. If $\Phi_*[M] = \Psi_*[N] \in H_n(\pi; \mathbb{Z})$, then*

$$\lambda(M) = \lambda(N) \quad \textit{and} \quad T(M) = T(N).$$

Note that in contrast to Theorem 1.1 integral coefficients are used here. It is an open question whether this theorem also holds with real coefficients instead. Another open question is whether the minimal topological entropy fulfills any topological invariance properties.

The techniques used in the proof of Theorem 1.2 stem from systolic geometry. A short introduction to the ideas and questions in this area is provided by [Ber08]. For more detailed informations see chapter 7.2 of Berger's book [Ber03], the survey article [CroK03], and Katz's book [Kat07].

1.2 Systolic geometry

The *systole* $\mathrm{sys}(M, g)$ of a Riemannian manifold is the length of the shortest loop that is not contractible. One of the most important questions in systolic geometry is whether an inequality

$$\mathrm{sys}(M, g)^n \leq C(M) \cdot \mathrm{Vol}(M, g)$$

holds for all Riemannian metrics g, and if it holds, what is the best constant $C(M)$. This is encoded in the *systolic constant* $\sigma(M)$ which is the infimum of the quotient $\operatorname{Vol}(M, g)/\operatorname{sys}(M, g)^n$ over all Riemannian metrics on M. In fact, the systolic constant is nonzero if and only if a systolic inequality as above holds. Moreover, the best constant is the reciprocal of $\sigma(M)$.

In the paper [Bab06], Babenko investigated the question whether the systolic constant is homologically invariant. He gave a positive answer in the case that the order of the class $\Phi_*[M] \in H_n(\pi_1(M); \mathbb{Z})$ equals the order of the fundamental group. This includes in particular the case where $\Phi_*[M]$ is a nontorsion element. Building on his ideas, we are able to remove this restriction (and moreover to prove Theorem 1.2).

Theorem 1.3. *Again, let M and N be two connected closed orientable n-dimensional manifolds having the same fundamental group π, and denote by Φ and Ψ the respective classifying maps. If $\Phi_*[M] = \Psi_*[N] \in H_n(\pi; \mathbb{Z})$, then $\sigma(M) = \sigma(N)$.*

In this case, integral coefficients are in fact necessary. This follows from Gromov's famous universal systolic inequality and a theorem by Babenko (see [Bab92], Theorem 8.2), which together imply that the systolic constant is nonzero if and only if $\Phi_*[M] \neq 0 \in H_n(\pi_1(M); \mathbb{Z})$. Note also that this statement of Gromov and Babenko is a complete answer to the above question whether a systolic inequality holds or not.

The systolic constant is an upper bound for the simplicial volume. This was shown by Gromov in his Filling paper [Gro83] where he proved that

$$\sigma(M) \geq c'_n \frac{\|M\|}{\log^n(1 + \|M\|)}$$

for some constant c'_n depending only on the dimension. In some special cases, including the case of orientable aspherical manifolds, Sabourau was able to replace the simplicial volume in this inequality by the minimal volume entropy (see [Sab06]). Using some ideas from geometric group theory, we will show that the simplicial volume may always be substituted by the minimal volume entropy.

Theorem 1.4. *Let M be a connected closed n-dimensional manifold. There exists a positive constant c_n depending only on n such that*

$$\sigma(M) \geq c_n \frac{\lambda(M)^n}{\log^n(1 + \lambda(M))}.$$

Note that the manifold is not assumed to be orientable in this theorem. Therefore, we will need stronger versions of Theorem 1.2 and Theorem 1.3 that include the nonorientable case to prove this inequality. These stronger statements may be found in chapter 2.

There are higher-dimensional analogs of the systole where one is not interested in loops but in submanifolds. Roughly speaking, the *k-systole* $\operatorname{sys}_k(M, g)$ is the volume of the smallest k-dimensional submanifold that is not nullhomologous. The *k-systolic constant* $\sigma_k(M)$ is defined as the infimum of the quotient $\operatorname{Vol}(M, g)/\operatorname{sys}_k(M, g)^{n/k}$ over all Riemannian metrics g on M.

It seems that this invariant is always zero for $k \geq 2$. This phenomenon is called *systolic freedom* since the k-systole does not bound the volume from below in this case. For orientable four-manifolds, systolic freedom was proved by Katz and Suciu in [KatSu99]. Moreover, they showed in [KatSu01] that every manifold is systolically free modulo torsion (see Theorem 4.12).

This suggests that the "right" invariants to study when it comes to higher-dimensional systoles are the so-called stable systoles. The *stable k-systole* $\operatorname{stabsys}_k(M, g)$ is defined as the minimum of the stabilized volume $\lim_{i\to\infty} \operatorname{Vol}_k(i\alpha)/i$ of all nontorsion homology classes $\alpha \in H_k(M; \mathbb{Z})$. (The volume of a homology class is the infimal volume of all submanifolds representing this class.) Here, the *stable k-systolic constant*

$$\sigma_k^{st}(M) := \inf_g \frac{\operatorname{Vol}(M, g)}{\operatorname{stabsys}_k(M, g)^{n/k}}$$

is known to be nonzero in many cases.

In chapter 4, we will give a complete answer to the freedom question for stable systoles.

Theorem 1.5. *Let M be a connected closed orientable manifold of dimension n. The stable k-systolic constant $\sigma_k^{st}(M)$ does not vanish if and only if n is a multiple of k, say $n = kp$, and there exist cohomology classes $\beta_1, \ldots, \beta_p \in H^k(M; \mathbb{R})$ such that $\beta_1 \smile \cdots \smile \beta_p \neq 0$ in $H^n(M; \mathbb{R})$.*

Moreover, if M is nonorientable and $k \geq 2$, then the stable k-systolic constant of M is always zero.

The existence of a stable systolic inequality under the conditions stated in the first part the theorem was shown by Gromov in [Gro83], 7.4.C. We will prove only the converse statement and the second part of the theorem, which both are results on the nonexistence of stable systolic inequalities.

As we have seen, the one-dimensional systolic constant from Theorem 1.3 is strongly related to the Eilenberg-Mac Lane space $K(\pi_1(M), 1)$. Similarly, the stable k-systolic constant is connected to the Eilenberg-Mac Lane space $K(\mathbb{Z}^b, k)$ where $b := b_k(M)$ is the k-th Betti number of M. There exist maps $\Phi : M \to K(\mathbb{Z}^b, k)$ that induce isomorphisms on k-dimensional homology modulo torsion. Using such maps, the following homological invariance result holds:

Theorem 1.6. *Let M and N be connected closed orientable manifolds of dimension n, and let $1 \leq k \leq n-1$. Suppose that $b_k(M) = b_k(N) =: b$ and that there are maps $\Phi : M \to K(\mathbb{Z}^b, k)$ and $\Psi : N \to K(\mathbb{Z}^b, k)$ such that the induced homomorphisms on k-dimensional homology modulo torsion are bijective and such that*

$$\Phi_*[M] = \Psi_*[N] \in H_n(K(\mathbb{Z}^b, k); \mathbb{R}).$$

Then the stable k-systolic constants coincide: $\sigma_k^{st}(M) = \sigma_k^{st}(N)$.

Note that real coefficients are used here. This contrasts the situation for the one-dimensional systolic constant where it is necessary to take integral coefficients.

A direct consequence of this theorem is that the stable k-systolic constant depends only on the *multilinear intersection form* on k-dimensional cohomology

$$\begin{aligned} Q_M^k : (H^k(M; \mathbb{Z}))^p &\to \mathbb{Z}, \\ (\beta_1, \ldots, \beta_p) &\mapsto \langle \beta_1 \smile \cdots \smile \beta_p, [M] \rangle, \end{aligned}$$

where $n = kp$ is the dimension of the manifold. In fact, $\sigma_k^{st}(M) = 0$ if and only if this intersection form vanishes (by Theorem 1.5), and moreover if the intersection forms of two manifolds are equivalent over $\mathbb{Z}$, then their stable k-systolic constants agree.

1.3 Further universal volume bounds

Let us return to the one-dimensional systole, i. e. the length of the shortest noncontractible loop. The best (smallest) constant $C(M)$ such that the systolic inequality

$$\mathrm{sys}(M, g)^n \leq C(M) \cdot \mathrm{Vol}(M, g)$$

holds for all Riemannian metrics g is given by

$$\mathrm{SR}(M) := \sup_g \frac{\mathrm{sys}(M, g)^n}{\mathrm{Vol}(M, g)}.$$

Note that this *systolic ratio* is just the reciprocal of the systolic constant $\sigma(M)$ considered before.

Gromov's universal systolic inequality states that there is a constant C_n depending only on the dimension n such that $\mathrm{SR}(M) \leq C_n$ for every connected closed n-dimensional manifold M for which $\Phi_*[M] \neq 0 \in$

$H_n(\pi_1(M);\mathbb{Z})$ where Φ denotes again the classifying map of the universal covering.

To prove this universal systolic inequality, Gromov introduced the filling radius and the filling volume of Riemannian manifolds. Roughly speaking, a filling of (M,g) is a complete Riemannian manifold (W,g') such that $\partial W = M$ and such that the induced path metrics satisfy $d_{g'}|M \equiv d_g$. The *filling volume* $\mathrm{FillVol}(M,g)$ is the infimal volume of all such fillings, and the *filling radius* is the infimal r such that (M,g) may be filled by some (W,g') satisfying $d_{g'}(M,w) \le r$ for all $w \in W$. (To bypass the bordism problem one actually considers fillings by pseudomanifolds W.)

Both filling invariants provide further universal volume bounds:

$$\mathrm{FillRad}(M,g)^n \le A_n \cdot \mathrm{Vol}(M,g) \quad \text{and}$$
$$\mathrm{FillVol}(M,g)^{n/(n+1)} \le B_n \cdot \mathrm{Vol}(M,g).$$

This is proved in Gromov's Filling paper [Gro83]. As before in the systolic context, the best constants in these inequalities are given by the following *filling ratios*

$$\mathrm{FR}(M) := \sup_g \frac{\mathrm{FillRad}(M,g)^n}{\mathrm{Vol}(M,g)} \quad \text{and}$$
$$\mathrm{FV}(M) := \sup_g \frac{\mathrm{FillVol}(M,g)^{n/(n+1)}}{\mathrm{Vol}(M,g)}.$$

In Theorem 1.3, we proved that the systolic ratio $\mathrm{SR}(M)$ is homologically invariant. Using similar methods, we will show the following surprising result concerning the filling ratios.

Theorem 1.7. *If M and N are connected closed manifolds of the same dimension $n \ge 3$ and if they are either both orientable or both nonorientable, then*

$$\mathrm{FR}(M) = \mathrm{FR}(N) \quad \textit{and} \quad \mathrm{FV}(M) = \mathrm{FV}(N).$$

Determining the exact values and the existence or nonexistence of maximizing Riemannian metrics remain open questions.

1.4 Largeness properties

To study obstructions to positive scalar curvature metrics, Gromov and others introduced several notions of largeness for Riemannian manifolds. For instance, (M,g) may be called large if it is enlargeable, or if its universal covering is hypereuclidean or hyperspherical, or if the filling radius of the universal covering is infinite. We will also consider coarse analogs of these properties. The term *large* will always serve as a placeholder for one of these properties.

If M is closed, then it can easily be seen that these largeness properties do not depend on the chosen Riemannian metric. Thus, they are topological invariants of the manifold. The definitions are given in the last chapter of the thesis. Here, we only recall the notion of enlargeability: let M be a connected closed orientable manifold of dimension n, and let g be any Riemannian metric on it. Then M is called *enlargeable* if for every $\varepsilon > 0$ there is a Riemannian covering $\bar{M}_\varepsilon$ and an ε-contracting map $\bar{M}_\varepsilon \to S^n$ to the unit sphere which is constant outside a compact set and of nonzero degree.

We will prove that enlargeability and each of the other largeness properties mentioned before is homologically invariant, and moreover that each of them determines a subspace in group homology consisting of classes that are not represented by large manifolds.

Theorem 1.8. *Let π be a finitely presented group. There is a subspace V_0 of the vector space $H_n(\pi;\mathbb{R})$ with the following property: if M is a connected closed orientable n-dimensional manifold with fundamental group π and if $\Phi : M \to K(\pi, 1)$ denotes the classifying map, then the class $\Phi_*[M]$ lies in V_0 if and only if the manifold is not large.*

It is unclear whether there are examples of groups for which one of these subspaces is not trivial. Note that there is an analogous subspace $V_0' \subset H_n(\pi;\mathbb{R})$ for the simplicial volume: the simplicial volume $\|M\|$ vanishes if and only if the class $\Phi_*[M]$ is contained in this subspace. (This is the nullspace of the seminorm induced by the ℓ^1-norm of singular chains.) One does not know whether there exists a connection between this subspace and the subspaces V_0 of the largeness properties. Since there are known examples of groups π for which V_0' is nonzero, one may conjecture that manifolds with nonvanishing simplicial volume are large in some sense, that is to say that $V_0 \subset V_0'$.

Enlargeable spin manifolds do not carry a metric of positive scalar curvature. This was shown by Gromov and Lawson in [GroL80] and [GroL83]. In fact, they also proved this result for area-enlargeable spin manifolds. Here, a closed orientable manifold is called *k-enlargeable* if for every $\varepsilon > 0$ there is a covering $\bar{M}_\varepsilon$ and a map $\bar{M}_\varepsilon \to S^n$ to the unit sphere which is constant outside a compact set and of nonzero degree, and which contracts the volume of any k-dimensional submanifold by a factor of ε. In the case $k = 2$ one speaks of *area-enlargeable* manifolds.

Although area-enlargeability does not seem to be homologically invariant, it shares one important property with the above notions of largeness:

every area-enlargeable closed manifold is *(rationally) essential*, that is, the homology class $\Phi_*[M] \in H_n(\pi_1(M); \mathbb{R})$ does not vanish. This was proved by Hanke and Schick using index theory (see [HanS06] and [HanS07]). We will extend this result to higher enlargeability.

Theorem 1.9. *Let M be a connected closed orientable manifold. If M is k-enlargeable and satisfies*

$$\pi_i(M) = 0 \quad \textit{for} \quad 2 \leq i \leq k-1,$$

then M is essential. In particular, area-enlargeable manifolds are essential.

It can be seen easily that the assumption on the homotopy groups is necessary. Note that for $k \geq n+1$, the assumptions are equivalent to M being aspherical. Thus, the conditions of this theorem interpolate between enlargeability and area-enlargeability on the one side and asphericity on the other side. For enlargeable and area-enlargeable spin manifolds, it is known that they do not admit a positive scalar curvature metric. For aspherical ones, this is only conjectured. Theorem 1.9 suggests that the conjecture on aspherical spin manifolds may be extended to k-enlargeable spin manifolds with trivial homotopy groups $\pi_i(M)$ for $2 \leq i \leq k-1$.

Remarks on the following chapters. Apart from the introductory chapter, the thesis consists of four other chapters. In the beginning of chapter 2, technical tools are developed that will be of great importance throughout the thesis. Later on in this chapter, more general versions of Theorems 1.2 and 1.3 are proved, and as an application of these results Theorem 1.4 is shown. Theorem 1.7 on the constancy of the filling ratios is derived in chapter 3. In chapter 4, we are concerned with higher-dimensional systoles and we prove Theorems 1.5 and 1.6. The final chapter 5 contains various definitions of largeness and the proofs of Theorems 1.8 and 1.9.

Parts of this thesis correspond to papers submitted for publication. A slightly shorter version of chapter 2 will appear as [Bru07a] in the journal *Geometric and Functional Analysis (GAFA)*. Chapter 3 corresponds to [Bru07b] and is to appear in *Journal für die reine und angewandte Mathematik (Crelle's Journal)*. Furthermore, chapter 4 is an extended version of [Bru07c], which will be published in *Mathematische Annalen*.

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Chapter 2

Homological invariance for asymptotic invariants and systolic inequalities

In this chapter, we will prove that certain asymptotic and systolic invariants of a connected closed manifold M depend only on the image of the fundamental class under the classifying map of the universal covering. This behaviour will be called *homological invariance*.

These invariants include the *minimal volume entropy* $\lambda(M)$, which describes the asymptotic volume growth of the universal covering, the *spherical volume* $T(M)$, which is an invariant intermediate between the minimal volume entropy and the simplicial volume, and the *(one-dimensional) systolic constant* $\sigma(M)$, which determines the relation between the lengths of short noncontractible loops and the volume of the manifold.

We will show more general versions of Theorems 1.2 and 1.3 that include the nonorientable case. To this end, we have to consider homology with local coefficients, more precisely with coefficients in the orientation bundle $\mathcal{O}$ of the manifold. Recall that a nonorientable manifold M does not possess a fundamental class with integral coefficients because $H_n(M;\mathbb{Z}) = 0$, where n denotes the dimension of M. But since $H_n(M;\mathcal{O}) \cong \mathbb{Z}$, the manifold admits a fundamental class with coefficients in the orientation bundle. In the orientable case the orientation bundle is trivial and homology with coefficients in the orientation bundle is just ordinary homology with integral coefficients.

Using this, we are able to prove the following theorem, which is the main result of this chapter.

Theorem 2.1. *Let M and N be two connected closed manifolds having the same fundamental group π. Let $\Phi : M \to K(\pi, 1)$ and $\Psi : N \to K(\pi, 1)$*

denote the respective classifying maps of the universal coverings of M and N. If the subgroups of orientation preserving loops of M and N coincide and if $\Phi_*[M] = \Psi_*[N]$, then

$$I(M) = I(N),$$

where I denotes either the systolic constant σ, the minimal volume entropy λ, or the spherical volume T.

Here, the fundamental classes $[M]$ and $[N]$ are to be understood with respect to coefficients in the orientation bundles of M and N. Note that in the orientable case the condition on the subgroups of orientation preserving loops is always fulfilled.

Many cases of this theorem for the systolic constant were known by work of Babenko (see [Bab06]), whose ideas we follow in parts of the proof. Sabourau applied these ideas to the minimal volume entropy (see [Sab06]). For the spherical volume this question has not been considered before.

Remark. Using a fundamental class with coefficients in the orientation bundle and cycles with coefficients in $\mathcal{O} \otimes \mathbb{R}$, one defines the *simplicial volume* for nonorientable manifolds in the same way as for orientable ones. Then Theorem 2.1 also holds for I the simplicial volume. In fact, the assumption $\Phi_*[M] = \Psi_*[N] \in H_n(K(\pi, 1); \mathcal{O})$ may be weakened to coefficients in $\mathcal{O} \otimes \mathbb{R}$. This extension of Theorem 1.1 can easily be deduced from the equality $\|M\| = \frac{1}{2}\|\bar{M}_{or}\|$ where $\bar{M}_{or}$ denotes the two-fold orientation covering of M.

To unify the treatment of the different invariants, we will introduce certain axioms that are satisfied by the systolic constant, the minimal volume entropy, and the spherical volume. In the proof of Theorem 2.1 we will use only these axioms and no other properties of the invariants. Thus, the theorem holds for all invariants I fulfilling the axioms. One more example for such an invariant is the *(one-dimensional) stable systolic constant* σ_1^{st}, a variation of the systolic constant.

Moreover, it will be convenient to consider relative versions of the invariants (relative to some homomorphism $\phi : \pi_1(M) \to \pi$ from the fundamental group to an arbitrary group) and to extend their definitions to simplicial complexes. The respective definitions and the axioms can be found in sections 2.2 and 2.3. The first paragraph of section 2.4 contains the proof of Theorem 2.1.

As an application of Theorem 2.1, we will look at manifolds whose fundamental groups have only two elements. The fact that $K(\mathbb{Z}_2, 1) = \mathbb{R}\mathrm{P}^\infty$ will allow us to derive a complete list of possible values for the systolic constant in this case. (For the minimal volume entropy, the spherical volume, and the stable systolic constant this is of no interest since these invariants vanish for finite fundamental groups.)

Corollary 2.2. *Let M be a connected closed n-dimensional manifold with fundamental group $\pi_1(M) = \mathbb{Z}_2$. Let $\alpha \in H^1(M;\mathbb{Z}_2)$ be the generator. Then*

$$\sigma(M) = \begin{cases} \sigma(\mathbb{R}\mathrm{P}^n) & \alpha^n \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$

where $\alpha^n \in H^n(M;\mathbb{Z}_2)$ denotes the n-fold cup product of the class α.

This was previously known only for orientable manifolds by another paper of Babenko (see [Bab04]). Note also that the exact value of $\sigma(\mathbb{R}\mathrm{P}^n)$ is unknown except in dimension two, where it is $2/\pi$ (see [Pu52]). This corollary will be proved in paragraph 2.4.4.

In section 2.5, we will investigate what happens to the minimal volume entropy when one enlarges the fundamental group by attaching 1-cells to the manifold. Using these observations, homological invariance, and computations from [Sab06], we will finally prove Theorem 1.4 in the last paragraph of this chapter:

Theorem 2.3. *Let M be a connected closed n-dimensional manifold. There exists a positive constant c_n depending only on n such that*

$$\sigma(M) \geq c_n \frac{\lambda(M)^n}{\log^n(1 + \lambda(M))}.$$

The proof will cover the dimensions $n \geq 3$. For surfaces, the theorem was shown in [KatSa05] by Katz and Sabourau. Moreover, Sabourau proved this inequality in special cases including the case of aspherical orientable manifolds (see [Sab06]). Note that the proof of Theorem 2.3 requires relative versions of both invariants involved in the formulation.

Theorem 2.3 sharpens a theorem of Gromov ([Gro83], Theorem 6.4.D') that stated the inequality

$$\sigma(M) \geq c'_n \frac{\|M\|}{\log^n(1 + \|M\|)}$$

where $\|M\|$ denotes the simplicial volume of M. Recall from the introduction that there is another inequality by Gromov ([Gro82], pages 35-37), improved by [BessCG91], Théorèmes 3.8 and 3.16, that says

$$\tfrac{n^{n/2}}{n!}\|M\| \leq \lambda(M)^n.$$

So in fact, Theorem 2.3 implies Gromov's inequality (up to constants).

Note also that these lower bounds for the systolic constant are optimal at least in dimensions two and three. A discussion of this optimality result may be found in [KatScV07].

The next section contains the technical core of this chapter, which will be used in chapters 3 and 4, too. It is concerned with maps from manifolds to CW complexes that will be deformed by elementary methods to gain useful normal forms for such maps.

2.1 Topological preliminaries

In this section, we show that every map from a manifold to a CW complex can be brought into a form convenient for many purposes. In the last paragraph, we define the notion of absolute degree and point out its geometrical meaning. The results of this section will be used throughout most chapters of the thesis.

2.1.1 The Hopf trick

Consider a proper map $f : (M, \partial M) \to (X, A)$ from a manifold M (with or without boundary) to a relative manifold (X, A), both having dimension n. Recall that a pair (X, A) is called a *relative manifold* if X is a Hausdorff space and $A \subset X$ is a closed subspace such that $X \setminus A$ is a manifold (see [Spa66], page 297). For example, every n-dimensional CW complex is a manifold relative to its $(n-1)$-skeleton.

Let $p \in X \setminus A$ be a point. Replacing f by a properly homotopic map if necessary, we may assume that f is smooth on the preimage of a small neighborhood of p, and moreover that f is transverse to p. The preimage of p then consists of finitely many points $p_1, \ldots, p_\ell$ in $M \setminus \partial M$. Choosing a local orientation of X at p, the map f induces local orientations of M at these points.

In this situation the following 'trick' due to Hopf applies, see [Hop30]. A modern presentation can be found in [Eps66], pages 378-380. (There, X is supposed to be a manifold. But in fact, it is enough that X is a manifold in a neighborhood of the point $p \in X$.)

Lemma 2.4 (Hopf trick). *Let $n \geq 3$. Assume now, that there is a path γ in M between two preimage points, say from p_1 to p_2, that reverses the induced orientations and that is mapped to a contractible loop in X. Then we may deform f on a compact subset of $M \setminus \partial M$ such that the number ℓ of preimage points of p is reduced by 2, and such that the resulting map is still transverse to p.*

Proof. There is a closed ball $D \subset X \setminus A$ with center p whose preimage $f^{-1}(D)$ consists of pairwise disjoint closed balls $D_1, \ldots, D_\ell$ in $M \setminus \partial M$ with centers $p_1, \ldots, p_\ell$, each of which is mapped diffeomorphically onto D.

Without loss of generality, we may assume that γ meets neither ∂M nor $D_3, \ldots, D_\ell$ and that inside D_1 and D_2 it is a straight line from the center p_i to a boundary point z_i such that $f(z_1) = f(z_2) =: z$. Moreover, we may assume that γ is a smooth embedding since M has at least dimension three. Choose a closed ball C that meets neither ∂M nor $D_3, \ldots, D_\ell$ and contains $D_1 \cup D_2 \cup \gamma$ as strong deformation retract in its interior. (One can easily construct such a ball using a tubular neighborhood of γ.)

Since the loop $f \circ \gamma$ is contractible by assumption, the part from z to z that lies outside $\mathring{D}$ is also contractible and this even in $X \setminus \mathring{D}$ since $n \geq 3$. This implies that there exists a contracting homotopy $h : S^1 \times [0,1] \to X$ that fixes the basepoint z of the loop such that $h(S^1 \times [0,1]) \cap D = z$.

Now, a first homotopy of f is defined as follows: let $B \subset \mathring{C}$ be a closed ball that still contains $D_1 \cup D_2 \cup \gamma$ as strong deformation retract in its interior. The homotopy is constant outside $\mathring{C}$ and on $D_1 \cup D_2$. On $\gamma \setminus (D_1 \cup D_2)$ it is constant until $t = \frac{1}{2}$ and then it contracts this loop to z by the homotopy h from above. Before $t = \frac{1}{2}$ it contracts B to $D_1 \cup D_2 \cup \gamma$. By the homotopy extension property applied to $f : C \setminus \mathring{B} \to X \setminus \mathring{D}$, this gives a homotopy on M that is constant outside C and we end up with a map $f' : M \to X$ that is still transverse to p such that $f'^{-1}(p)$ consists of the points $p_1, \ldots, p_\ell$. Moreover, it restricts to a map

$$f' : (B, \partial B) \to (D, \partial D).$$

Since γ is orientation reversing, this map has degree zero (whatever orientations we choose on B and D). Hence, by the Hurewicz theorem it represents zero in $\pi_n(D, \partial D)$, and thus we may deform it relative to ∂B to have image in ∂D. This deformation extends trivially to a homotopy on M. Therefore, we finally get a map homotopic to f which is transverse to p such that p has only $\ell - 2$ preimage points. □

2.1.2 Orientation issues

If M is a connected compact manifold of dimension n, then $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ in the orientable case and $H_n(M, \partial M; \mathbb{Z}) = 0$ in the nonorientable case. Moreover, $H_n(M, \partial M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ in any case. But since $\mathbb{Z}_2$ coefficients ignore much information, it is useful to consider local integer coefficient systems.

Recall that each local integer coefficient system on a connected locally path-connected topological space X that has a universal covering (for instance on a CW complex) is determined by a unique homomorphism

$$\rho : \pi_1(X) \to \mathbb{Z}_2 = \mathrm{Aut}(\mathbb{Z}).$$

We will denote this coefficient system by $\mathcal{O}_\rho$.

For a manifold M, there exists exactly one homomorphism $\rho : \pi_1(M) \to \mathbb{Z}_2$ such that $H_n(M, \partial M; \mathcal{O}_\rho) \cong \mathbb{Z}$. (For all other homomorphisms ρ this homology group is zero.) The kernel of this homomorphism ρ is the subgroup of all orientation preserving loops in M, and the local coefficient system $\mathcal{O}_\rho$ is called the *orientation bundle* of M. If not said otherwise, we will always use this bundle as local coefficient system. A generator $[M]_{\mathcal{O}_\rho}$ of $H_n(M, \partial M; \mathcal{O}_\rho) \cong \mathbb{Z}$ is called *fundamental class* of M.

If M is orientable, then the orientation bundle $\mathcal{O}_\rho$ is trivial and homology with coefficients in $\mathcal{O}_\rho$ is just homology with integer coefficients. In this case, a fundamental class is also denoted by $[M]_\mathbb{Z}$. In any case, the fundamental class with $\mathbb{Z}_2$ coefficients will be denoted by $[M]_{\mathbb{Z}_2} \in H_n(M, \partial M; \mathbb{Z}_2)$.

Remark. With respect to coefficients in the orientation bundle $\mathcal{O}_\rho$ all paths in M are orientation preserving.

2.1.3 Maps to n-dimensional CW complexes

Consider a map $f : (M, \partial M) \to (X, A)$ from a connected compact n-dimensional manifold to a pair of CW complexes whose induced homomorphism f_* on fundamental groups is surjective. If $\ker f_* \subset \ker \rho$, then the homomorphism $\rho : \pi_1(M) \to \mathbb{Z}_2$ induces a homomorphism $\rho : \pi_1(X) \to \mathbb{Z}_2$ and the induced homomorphisms on homology

$$f_* : H_*(M, \partial M; \mathcal{O}_\rho) \to H_*(X, A; \mathcal{O}_\rho)$$

are well-defined. If it is possible, we will always use coefficients in $\mathbb{K} = \mathcal{O}_\rho$ the orientation bundle of M. But if $\ker f_* \not\subset \ker \rho$, we have to take $\mathbb{K} = \mathbb{Z}_2$ coefficients.

Remark. Note that $\ker f_* \subset \ker \rho$ if and only if the covering $\tilde{M}_{f_*}$ associated to the subgroup $\ker f_* \subset \pi_1(M)$ is orientable.

Assume for the rest of this paragraph that X is n-dimensional and that $A \subset X$ is an $(n-1)$-dimensional subcomplex. Then

$$H_n(X, A; \mathbb{K}) \cong \ker(H_n(X, X^{(n-1)}; \mathbb{K}) \xrightarrow{\partial} H_{n-1}(X^{(n-1)}, A; \mathbb{K}))$$

by the long exact homology sequence of the triple $(X, X^{(n-1)}, A)$. Moreover, by excision $H_n(X, X^{(n-1)}; \mathbb{K})$ is isomorphic to

$$\bigoplus_{e\ n\text{-cell}} \mathbb{Z} \cdot e, \quad \text{respectively} \quad \bigoplus_{e\ n\text{-cell}} \mathbb{Z}_2 \cdot e.$$

Let $a \in H_n(X, A; \mathbb{K})$ be given by $\sum_{e\ n\text{-cell}} r_e \cdot e$ with $r_e = 0$ for all but finitely many n-cells e.

Lemma 2.5. *Let $n \geq 3$. If $f : (M, \partial M) \to (X, A)$ fulfills $f_*[M]_{\mathbb{K}} = a$ and is surjective on fundamental groups, then it is homotopic relative to ∂M to a map $f' : (M, \partial M) \to (X, A)$ for which there are $\sum_{e\ n\text{-cell}} |r_e|$ many pairwise disjoint closed balls $D_{ei_e}, i_e = 1, \ldots, |r_e|$ in $M \setminus \partial M$ such that*

$$f'^{-1}(\mathring{e}) = \mathring{D}_{e1} \cup \ldots \cup \mathring{D}_{e|r_e|}$$

and

$$f' : \mathring{D}_{e1} \cup \ldots \cup \mathring{D}_{e|r_e|} \to \mathring{e}$$

is a covering of mapping degree r_e. That is, it is an $|r_e|$-sheeted covering such that the orientations on $\mathring{D}_{ei}$ agree with respect to f'.

Notation. The absolute value on $\mathbb{Z}_2$ is defined as 0 for the trivial and as 1 for the nontrivial element.

Proof. First, remove the interiors of all n-cells e of X with $f(M) \cap \mathring{e} = \emptyset$ and with $r_e = 0$. This affects neither the surjectivity of $f_* : \pi_1(M) \twoheadrightarrow \pi_1(X)$ nor the equality $f_*[M]_{\mathbb{K}} = a$. (If X' denotes the complex obtained from X by removing those open n-cells, then $H_n(X', A; \mathbb{K}) \hookrightarrow H_n(X, A; \mathbb{K})$ is injective by the long exact sequence of the tripel (X, X', A) and $f_*[M]_{\mathbb{K}} = a$ is valid in $H_n(X', A; \mathbb{K})$, too.) Thus, there remain only finitely many n-cells because M is compact and $r_e = 0$ for almost all n-cells e.

We proceed by induction over the number of remaining n-cells of X. If it is zero, there is nothing to prove.

Now, let e be one of the n-cells of X. Choose a point $p \in \mathring{e}$ and assume without loss of generality that f is transverse to it. Denote its preimages by $p_1, \ldots, p_\ell$. The assumption implies that f has local mapping degree r_e at p. Hence $\ell \geq |r_e|$.

In case $\ker f_* \subset \ker \rho$, we may choose $d := \ell - |r_e|$ points from the points $p_1, \ldots, p_\ell$ such that one half of them is mapped orientation preservingly to p, the other half orientation reversingly (with respect to some choice of orientation of X at p). Take a path α from a point of the first kind to one of the second kind. Since $f_* : \pi_1(M) \twoheadrightarrow \pi_1(X)$ is surjective the loop $f \circ \alpha$ lies in its image. Let β be a loop based at the first point that is mapped to $f \circ \alpha$. Then $\gamma := \beta^{-1}\alpha$ has contractible image under f and is orientation reversing with respect to any choice of local orientations of M at $p_1, \ldots, p_\ell$ coming from p. Hence, we may apply the Hopf trick and reduce d by two. By induction, d finally becomes zero and $\ell = |r_e|$.

Next, consider the case that $\ker f_* \not\subset \ker \rho$. Choose a local orientation for X at p and thus also for M at the points p_i. Let α be a path between two such points. Proceeding as above, we may assume that its image is contractible in X. If α does not reverse the induced orientations, choose a loop β based at the starting point of α that is orientation reversing and in the kernel of f_*. Then $\gamma := \beta\alpha$ reverses the induced orientations and is mapped to a contractible loop. The Hopf trick reduces ℓ by two and induction shows that we may deform f until $\ell = |r_e|$.

Finally, choose a ball $D \subset \mathring{e}$ with center p such that $f^{-1}(D)$ consists of $|r_e|$ pairwise disjoint closed balls $D_{e1}, \ldots, D_{e|r_e|}$ in $M \setminus \partial M$, each mapped diffeomorphically and with the same orientation behaviour onto D. Compose f with a strong deformation retraction from $(X, X \setminus \mathring{D})$ to $(X, X \setminus \mathring{e})$.

The induction hypothesis applied to $M' := M \setminus (\mathring{D}_{e1} \cup \ldots \cup \mathring{D}_{e|r_e|})$, $X' := X \setminus \mathring{e}$ and $a' := \sum_{e' \neq e} r_{e'} \cdot e'$ finishes the proof. □

2.1.4 Maps to arbitrary CW complexes

Let M be a connected closed manifold of dimension $n \geq 2$. Consider a map $f : M \to X$ to a CW complex X that is surjective on fundamental group level. Let $a \in H_n(X^{(n)}; \mathbb{K})$ be a homology class in the n-skeleton of X (where we use $\mathbb{K} = \mathcal{O}_\rho$ if $\ker f_* \subset \ker \rho$ and $\mathbb{K} = \mathbb{Z}_2$ otherwise as always), and let $i : X^{(n)} \hookrightarrow X$ be the inclusion.

Lemma 2.6. *If $f_*[M]_{\mathbb{K}} = i_* a$, then we may deform f such that its image lies in the n-skeleton of X and such that $f_*[M]_{\mathbb{K}} = a \in H_n(X^{(n)}; \mathbb{K})$.*

In the case $\mathbb{K} = \mathcal{O}_\rho$ this is due to Babenko, see [Bab06], Lemme 3.10.

Proof. First consider the case $\mathbb{K} = \mathcal{O}_\rho$. The Hurewicz theorem gives an epimorphism

$$h : \pi_{n+1}(X^{(n+1)}, X^{(n)}) \twoheadrightarrow H_{n+1}(X^{(n+1)}, X^{(n)}; \mathcal{O}_\rho)$$

since $\pi_k(X^{(n+1)}, X^{(n)}) = 0$ for $k \leq n$. Hence, the commutative diagram with exact rows and vertical Hurewicz homomorphisms

$$\begin{array}{ccccc}
\pi_{n+1}(X^{(n+1)}, X^{(n)}) & \xrightarrow{\partial} & \pi_n(X^{(n)}) & \longrightarrow & \pi_n(X^{(n+1)}) = \pi_n(X) \\
\downarrow h & & \downarrow h & & \\
H_{n+1}(X^{(n+1)}, X^{(n)}; \mathcal{O}_\rho) & \xrightarrow{\partial} & H_n(X^{(n)}; \mathcal{O}_\rho) & \xrightarrow{j_*} & H_n(X^{(n+1)}; \mathcal{O}_\rho) = H_n(X; \mathcal{O}_\rho)
\end{array}$$

shows that the kernel of j_* equals the image of $h\partial : \pi_n(X^{(n+1)}, X^{(n)}) \to H_n(X^{(n)}; \mathcal{O}_\rho)$.

For the case $\mathbb{K} = \mathbb{Z}_2$ note that we may add a further row to the above diagram (with $\mathbb{Z}$ instead of $\mathcal{O}_\rho$) by applying the reduction map from $\mathbb{Z}$ to $\mathbb{Z}_2$ coefficients. The induced map

$$H_{n+1}(X^{(n+1)}, X^{(n)}; \mathbb{Z}) \twoheadrightarrow H_{n+1}(X^{(n+1)}, X^{(n)}; \mathbb{Z}_2)$$

is obviously surjective. Hence, we get the commutative diagram

$$\begin{array}{ccccc} \pi_{n+1}(X^{(n+1)}, X^{(n)}) & \xrightarrow{\partial} & \pi_n(X^{(n)}) & \longrightarrow & \pi_n(X) \\ \downarrow h & & \downarrow h & & \\ H_{n+1}(X^{(n+1)}, X^{(n)}; \mathbb{Z}_2) & \xrightarrow{\partial} & H_n(X^{(n)}; \mathbb{Z}_2) & \xrightarrow{j_*} & H_n(X; \mathbb{Z}_2) \end{array}$$

and again we have $\ker j_* = \operatorname{im} h\partial$.

By cellular approximation, we may assume that f maps to the n-skeleton of X. Since $f_*[M]_\mathbb{K} = i_* a \in H_n(X; \mathbb{K})$, we see that $j_* f_*[M]_\mathbb{K} = j_* a$, hence

$$a - f_*[M]_\mathbb{K} \in \ker j_*.$$

Let $s : S^n \to X^{(n)}$ be a preimage under h, i. e. we have $s_*[S^n]_\mathbb{K} = a - f_*[M]_\mathbb{K}$. We may assume that s is contractible in X. In fact, we can choose s in the image of the boundary homomorphism $\partial : \pi_{n+1}(X^{(n+1)}, X^{(n)}) \to \pi_n(X^{(n)})$.

Define

$$f' : M \to M \vee S^n \xrightarrow{f \vee s} X^{(n)},$$

where the first map contracts the boundary of a small ball in M. Then $f'_*[M]_\mathbb{K} = f_*[M]_\mathbb{K} + s_*[S^n]_\mathbb{K} = a \in H_n(X^{(n)}; \mathbb{K})$ and the maps f and f' are homotopic as maps to X by the choice of s. □

Combining Lemma 2.6 and Lemma 2.5 one immediately derives the following corollary.

Corollary 2.7. *Let M be a connected closed manifold of dimension $n \geq 3$, and let $f : M \to X$ be a map to a CW complex that is surjective on fundamental groups. Then f is homotopic to a map from M to the $(n-1)$-skeleton of X if and only if one of the following statements holds:*

(i) M is orientable, and $f_*[M]_\mathbb{Z} = 0$ in $H_n(X; \mathbb{Z})$.

(ii) M is nonorientable, f maps every orientation reversing loop to a noncontractible one, and $f_*[M]_{\mathcal{O}_\rho} = 0$ in $H_n(X; \mathcal{O}_\rho)$.

(iii) M is nonorientable, f maps some orientation reversing loop to a contractible one, and $f_*[M]_{\mathbb{Z}_2} = 0$ in $H_n(X; \mathbb{Z}_2)$.

The two lemmata imply sufficiency of the conditions (i) to (iii) in this corollary. But necessity is obvious by cellular homology.

2.1.5 Absolute and geometric degree

Let $f : (M, \partial M) \to (N, \partial N)$ be a map between two connected compact manifolds of dimension n. It factors as

$$(M, \partial M) \xrightarrow{\bar{f}} (\bar{N}, \partial \bar{N}) \xrightarrow{p} (N, \partial N)$$

where $p : \bar{N} \to N$ is the covering map that corresponds to the subgroup $f_*(\pi_1(M)) \subset \pi_1(N)$. Let j be the number of sheets of p. If $\ker \bar{f}_* = \ker f_* \subset \ker \rho$, then we may define the *degree* of f as zero for $j = \infty$ and as $j \cdot \deg(\bar{f})$ for $j < \infty$ where $\deg(\bar{f})$ is determined by

$$\bar{f}_*[M]_{\mathcal{O}_\rho} = \deg(\bar{f}) \cdot [\bar{N}]_{\mathcal{O}_\rho}.$$

This is to be understood as $\deg(\bar{f}) = 0$ if $H_n(\bar{N}, \partial \bar{N}; \mathcal{O}_\rho) = 0$. (This degree is defined only up to sign. We have to choose orientations to get a well-defined integer.)

Moreover, we define the *absolute degree* of f by

$$\deg_a(f) := \begin{cases} 0 & j = \infty \\ j \cdot |\deg(\bar{f})| & j < \infty, \ker f_* \subset \ker \rho \\ j \cdot |\deg_2(\bar{f})| & j < \infty, \ker f_* \not\subset \ker \rho \end{cases}$$

where $\deg_2(\bar{f})$ denotes the $\mathbb{Z}_2$ degree of $\bar{f}$. (This degree is well-defined without any choices.)

Remark. This definition coincides with the usual definition of absolute degree (see for example [Eps66] or [Sko87]). Note also that for maps of absolute degree one, the number j has to be one by definition. But $j = 1$ if and only if the induced map on fundamental groups is surjective. Thus, maps of absolute degree one are surjective on fundamental group level.

The *geometric degree* $\deg_g(f)$ of f is the smallest integer d for which there is a map $f' : (M, \partial M) \to (N, \partial N)$ homotopic to f relative to the boundary that is transverse to some point $p \in N \setminus \partial N$ such that $f'^{-1}(p)$ consists of d points. Note that always $\deg_a(f) \leq \deg_g(f)$.

Theorem 2.8 (Hopf, Kneser). *If $n \geq 3$, then $\deg_a(f) = \deg_g(f)$. In the two-dimensional case the same equality holds if one assumes that M and N are closed.*

Proof for $n \geq 3$. Choose a CW decomposition of $(N, \partial N)$ and lift it to the covering $(\bar{N}, \partial \bar{N})$. If the number j of sheets of p is infinite, then $\bar{N}$ is not compact and therefore $H_n(\bar{N}, \partial \bar{N}; \mathbb{K}) = 0$. Hence, $\bar{f}_*[M]_{\mathbb{K}} = 0$ and Corollary

2.7 shows that $\bar{f}$ contracts to the $(n-1)$-skeleton of $\bar{N}$. Thus, f also contracts to the $(n-1)$-skeleton of N by composition with p. In particular, $\deg_g(f) = 0$.

Now assume $j < \infty$. Applying Lemma 2.5 to $\bar{f}$ with $a = \deg_a(\bar{f}) \cdot [\bar{N}]_{\mathbb{K}}$, we get a homotopic map $\bar{f}'$ such that each open n-cell of $\bar{N}$ is covered by exactly $\deg_a(\bar{f})$ open n-cells in M. Hence, $f' := p \circ \bar{f}'$ is homotopic to f and has geometric degree equal to $j \cdot \deg_a(\bar{f}) = \deg_a(f)$. □

The Hopf part ($n \geq 3$) of this theorem was proved in [Hop30] by using the Hopf trick, see also [Eps66] for a modern presentation. In [Eps66], it is stated incorrectly that the equivalence of absolute and geometric degree also holds without further assumptions for $n = 2$. See [Sko87] for a discussion of this and a modern proof of Kneser's result from [Kne30] (the case $n = 2$ of the above theorem).

Our proof in fact shows slightly more, namely that each top-dimensional open cell of N is covered by exactly $\deg_a(f)$ open cells in M. Using the fact that smooth manifolds are triangulable, we get maps having the following nice property:

Definition 2.9. A simplicial map $f : X \to Y$ between two n-dimensional simplicial complexes is said to be (n, d)-*monotone* if the preimage of every open n-simplex of Y consists of at most d open n-simplices in X. It is called *strictly* (n, d)-*monotone* if the preimage of every open n-simplex of Y consists of exactly d open n-simplices in X.

Remark. Usually a map is called *monotone* if the preimage of any point is connected. In this sense, $(n, 1)$-monotone means that the map $f : X \to Y$ is monotone outside the $(n-1)$-skeleton, and (n, d)-monotone means that we may divide $X \setminus X^{(n-1)}$ into d sets such that f is monotone on each of these sets.

Corollary 2.10. *Let $f : (M, \partial M) \to (N, \partial N)$ be a map between connected compact manifolds of dimension $n \geq 3$, and let $d := \deg_a(f)$. Then f is homotopic to a strictly (n, d)-monotone map. In the two-dimensional case one has to assume that M and N are closed to get the same conclusion.*

In the closed case this corollary was proved in [Bab92], proof of Proposition 2.2, part (a). The two-dimensional case of this corollary is proved by using Kneser's theorem. In fact, Kneser constructed in his original proof a strictly $(2, d)$-monotone map homotopic to f (see [Kne30]).

2.2 Systolic constants and minimal volume entropy

In this section, we introduce (relative versions of) the one-dimensional systolic constant and the minimal volume entropy. Moreover, we define a stable version of the systolic constant.

To unify the treatment of these topological invariants, we investigate real-valued invariants that fulfill certain axioms. The systolic constants and the minimal volume entropy are shown to satisfy these axioms. Similar axioms will play an important role in chapters 3 and 4 and in the investigation of the spherical volume. The axioms proved in this section may be seen as archetypes for those later axioms.

The section concludes with a classification of *systolic* manifolds, i. e. manifolds with nonvanishing systolic constant. The analogous statement for the stable systolic constant is given in chapter 4.

2.2.1 Systolic constants

In systolic geometry, it is often convenient to work with Riemannian simplicial complexes (see for instance [Bab02] and [Bab06]).

Definition 2.11. By a Riemannian metric on a k-simplex Δ^k we understand the pullback of an arbitrary Riemannian metric on $\mathbb{R}^k$ via an affine linear embedding $\Delta^k \hookrightarrow \mathbb{R}^k$. A *(piecewise smooth) Riemannian metric* g on a simplicial complex X is given by a Riemannian metric g_τ on every simplex τ of X such that $g_{\tau'} \equiv g_\tau|\tau'$ whenever $\tau' \subset \tau$.

It is clear that any Riemannian manifold (M, g) becomes a Riemannian simplicial complex by choosing a smooth triangulation of M.

A Riemannian metric g enables us to measure the lengths of piecewise smooth curves in a simplicial complex X. As for Riemannian manifolds, one obtains in the connected case a path metric d_g on X. Moreover, if X is of dimension n, there is an obvious notion of n-dimensional Riemannian volume that coincides with the n-dimensional Hausdorff measure.

Let X be a connected finite simplicial complex of dimension n, and let $\phi : \pi_1(X) \to \pi$ be a group homomorphism. There is a corresponding map $\Phi : X \to K(\pi, 1)$ that induces this homomorphism ϕ on fundamental groups and that is determined uniquely up to homotopy by this property.

Definition 2.12. For a Riemannian metric g on X the *(one-dimensional) ϕ-systole* $\mathrm{sys}_\phi(X, g)$ is defined as the infimum of all lengths of closed piecewise smooth curves in X whose images under the corresponding map $\Phi : X \to$

$K(\pi, 1)$ are noncontractible. The *(one-dimensional) systolic constant relative to* ϕ is given by

$$\sigma_\phi(X) := \inf_g \frac{\mathrm{Vol}(X, g)}{\mathrm{sys}_\phi(X, g)^n},$$

where the infimum is taken over all Riemannian metrics g on X.

Remark. The systolic constant $\sigma_\phi(X)$ depends only on the kernel of ϕ, not on ϕ itself. But for practical reasons we will keep ϕ in the notation. Nevertheless, note that it is no actual restriction to assume that ϕ is surjective. The same remark applies to the definitions of the one-dimensional stable systole (Definition 2.14), the minimal volume entropy (Definition 2.15), and the spherical volume (Definition 2.25).

Consider the free Abelian group

$$H_1(X; \mathbb{Z})_\mathbb{R} := H_1(X; \mathbb{Z})/\mathrm{torsion},$$

whose rank is the first Betti number of X. Note that $K(H_1(X; \mathbb{Z})_\mathbb{R}, 1) = T^{b_1(X)}$ the $b_1(X)$-dimensional torus. The map corresponding to the canonical epimorphism $\pi_1(X) \twoheadrightarrow H_1(X; \mathbb{Z})_\mathbb{R}$ is called the *Jacobi map* of X.

Definition 2.13. For a class $a \in H_1(X; \mathbb{Z})_\mathbb{R}$ denote by $\ell_g(a)$ the length of the shortest loop γ in X that represents it. The *stable norm* of $a \in H_1(X; \mathbb{Z})_\mathbb{R}$ is defined as

$$\|a\|_g := \lim_{i \to \infty} \frac{\ell_g(ia)}{i}.$$

Note that $H_1(X; \mathbb{Z})_\mathbb{R} \subset H_1(X; \mathbb{R})$ is a lattice. Federer showed in [Fed74] that the stable norm extends to an actual norm on $H_1(X; \mathbb{R})$ (see also chapter 4).

Denote the map corresponding to the homomorphism $\phi : \pi_1(X) \to \pi$ by $\Phi : X \to K(\pi, 1)$. The stable norm on $H_1(X; \mathbb{R})$ induces via the homomorphism $\Phi_* : H_1(X; \mathbb{R}) \to H_1(\pi; \mathbb{R})$ a quotient norm on $H_1(\pi; \mathbb{R})$. (If Φ_* is not surjective, then this quotient norm takes finite values only on the image of Φ_*.)

Definition 2.14. The *stable* $(\phi, 1)$*-systole* $\mathrm{stabsys}_{(\phi,1)}(X, g)$ for a Riemannian metric g on X is defined as the minimum of the quotient norm on the nonzero elements of the lattice $H_1(\pi; \mathbb{Z})_\mathbb{R} \subset H_1(\pi; \mathbb{R})$. The *stable* $(\phi, 1)$*-systolic constant* is given by

$$\sigma^{st}_{\phi,1}(X) := \inf_g \frac{\mathrm{Vol}(X, g)}{\mathrm{stabsys}_{(\phi,1)}(X, g)^n}.$$

In the 'absolute' case, i. e. when $\phi : \pi_1(X) \to \pi$ is injective, we will speak of *systole* and *stable* 1-*systole* without any reference to ϕ. The respective constants are then called *systolic constant* and *stable* 1-*systolic constant*.

Remark. If $\phi : \pi_1(X) \twoheadrightarrow H_1(X;\mathbb{Z})$ is the Hurewicz epimorphism, then $\mathrm{sys}_\phi(X,g) =: \mathrm{sys}_1(X,g)$ will be called the 1-*systole* of (X,g) and $\sigma_\phi(X) =: \sigma_1(X)$ the 1-*systolic constant*. For $\phi : \pi_1(X) \twoheadrightarrow H_1(X;\mathbb{Z})_\mathbb{R}$ the canonical epimorphism $\mathrm{sys}_\phi(X,g) =: \mathrm{sys}_1^\infty(X,g)$ is called the 1-*systole modulo torsion* and $\sigma_\phi(X) =: \sigma_1^\infty(X)$ the 1-*systolic constant modulo torsion*. These notions and the notion of stable systole will be generalized to higher dimensions in chapter 4.

2.2.2 Minimal volume entropy

We continue to consider a connected finite n-dimensional simplicial complex X together with a homomorphism $\phi : \pi_1(X) \to \pi$. Let $\Phi : X \to K(\pi, 1)$ again denote the corresponding map.

Definition 2.15. Let $\tilde{X}_\phi$ be the Galois covering of X associated to the normal subgroup $\ker \phi \lhd \pi_1(X)$. For any Riemannian metric g on X define the *volume entropy relative to* ϕ as

$$\lambda_\phi(X,g) := \lim_{r\to\infty} \tfrac{1}{r} \log \mathrm{Vol}(B(x,r)),$$

where $B(x,r)$ is the ball of radius r around a point $x \in \tilde{X}_\phi$ with respect to the lifted metric. This limit exists and is independent of the center $x \in \tilde{X}_\phi$ (see [Man79]). One defines the *minimal volume entropy of* X *relative to* ϕ as

$$\lambda_\phi(X) := \inf_g \lambda_\phi(X,g)\,\mathrm{Vol}(X,g)^{1/n}.$$

If ϕ is injective, one just speaks of the *volume entropy* and the *minimal volume entropy* without mentioning the homomorphism ϕ.

To work with this definition quickly gets complicated. But there is an equivalent definition that is easier to handle.

Definition 2.16. Let G be a finitely generated group. A *norm* on G is a nonnegative function $L : G \to [0,\infty)$ such that

(i) $L(g) = 0 \;\Leftrightarrow\; g = 1$,

(ii) $L(g^{-1}) = L(g)$,

(iii) $L(gg') \le L(g) + L(g')$ (triangle inequality).

The *growth function* $\beta_L : [0,\infty) \to [1,\infty]$ of a norm L is defined by

$$\beta_L(r) := \#\{g \in G | L(g) \leq r\}.$$

If the limit

$$\lambda(G,L) := \lim_{r\to\infty} \tfrac{1}{r} \log \beta_L(r)$$

exists, it is called the *entropy* of G with respect to L.

Remark. We may use the inclusion $\iota_x : \Gamma \hookrightarrow \tilde{X}_\phi, \gamma \mapsto \gamma \cdot x$ of the Galois group $\Gamma := \pi_1(X)/\ker\phi$ into the Galois covering to induce a norm $L_{g,x}$ on Γ. Such norms will be called *Riemannian norms*. Then it can easily be seen that

$$\lambda_\phi(X,g) = \lambda(\Gamma, L_{g,x})$$

by using translates of a fundamental domain of the Galois action. (See for example [KaH95], Proposition 9.6.6 or [Sab06], Lemma 2.3.)

By work of Besson, Courtois, and Gallot (see [BessCG95]), it is known that locally symmetric metrics of negative curvature minimize the volume entropy. For instance, the minimal volume entropy of hyperbolic manifolds can be computed easily: since the universal covering is the hyperbolic space of constant curvature one the volume entropy equals $n-1$, and it only remains to compute the volume of the manifold.

In dimension $n = 2$, Katok proved that $\lambda(\Sigma) = \sqrt{2\pi|\chi(\Sigma)|}$ for closed surfaces of nonpositive Euler characteristic $\chi(\Sigma) \leq 0$ (see [Ka82]). The minimal volume entropy of the sphere and the real projective plane vanish.

2.2.3 Comparison axiom and homotopy invariance

For manifolds, invariants fulfilling the following comparison axiom behave reasonably well with respect to the absolute degree and are in particular invariant under homotopy equivalence.

Comparison axiom. Let X and Y be two connected finite simplicial complexes of dimension n, and let $\phi : \pi_1(X) \to \pi$ and $\psi : \pi_1(Y) \to \pi$ be group homomorphisms. If there exists an (n,d)-monotone map $f : X \to Y$ such that $\phi = \psi \circ f_*$, then

$$I_\phi(X) \leq d \cdot I_\psi(Y).$$

Lemma 2.17 (Babenko, Sabourau)**.** *The comparison axiom is fulfilled by $I = \sigma$, $I = \sigma_1^{st}$, and $I = \lambda^n$ (i. e. the minimal volume entropy to the power of n).*

Proofs of this lemma may be found in [Bab06], Proposition 3.2 and [Sab06], Lemma 3.5 (both for $d = 1$) and also in [Bab92], Propositions 2.2 and 8.7 (where $f_* : \pi_1(X) \to \pi_1(Y)$ is assumed to be surjective). Since this property of the systolic constants and the minimal volume entropy is of fundamental importance in this chapter, we include the proof.

Proof. Let $\varepsilon > 0$, and let g_2 be a Riemannian metric on Y. Choose a Riemannian metric g_1 on X, and define $g_1^t := f^* g_2 + t^2 g_1$. This is again a Riemannian metric on X. By choosing $t > 0$ small enough we can arrange that

$$\mathrm{Vol}(X, g_1^t) \leq d \cdot \mathrm{Vol}(Y, g_2) + \varepsilon.$$

Furthermore $f : (X, g_1^t) \to (Y, g_2)$ is nonexpanding. Therefore,

$$\mathrm{sys}_\phi(X, g_1^t) \geq \mathrm{sys}_\psi(Y, g_2)$$

and $\sigma_\phi(X) \leq d\sigma_\psi(Y)$ follows.

Since f is nonexpanding, $\ell_{g_1^t}(a) \geq \ell_{g_2}(f_* a)$ for every $a \in H_1(X;\mathbb{Z})_\mathbb{R}$. Thus, $\|a\|_{g_1^t} \geq \|f_* a\|_{g_2}$ and consequently

$$\mathrm{stabsys}_\phi(X, g_1^t) \geq \mathrm{stabsys}_\psi(Y, g_2).$$

As above $\sigma^{st}_{\phi,1}(X) \leq d\sigma^{st}_{\psi,1}(Y)$ follows.

Since $\phi = \psi \circ f_*$, the induced homomorphism

$$f_* : \pi_1(X)/\ker\phi \hookrightarrow \pi_1(Y)/\ker\psi$$

is injective. From the fact that f is nonexpanding follows that

$$L_{g_1^t, x}(\gamma) \geq L_{g_2, f(x)}(f_*(\gamma))$$

for every $\gamma \in \pi_1(X)/\ker\phi$, and together with the injectivity of f_* this implies

$$\beta_{g_1^t, x} \leq \beta_{g_2, f(x)}.$$

Therefore, $\lambda_\phi(X, g_1^t) \leq \lambda_\psi(Y, g_2)$ and finally $\lambda_\phi(X)^n \leq d\lambda_\psi(Y)^n$. □

From Corollary 2.10 we deduce:

Corollary 2.18. *Let M and N be two connected closed manifolds, and let $\psi : \pi_1(N) \to \pi$ be a group homomorphism. Let $f : M \to N$ be a map with $d := \deg_a(f)$. If I fulfills the comparison axiom, then*

$$I_{\psi\circ f_*}(M) \leq d \cdot I_\psi(N).$$

In particular, if f is a homotopy equivalence, then

$$I_{\psi\circ f_*}(M) = I_\psi(N).$$

If M is a smoothly triangulated connected closed manifold, then we have a priori two definitions for the systolic constants and the minimal volume entropy: one that uses only smooth Riemannian metrics and one that allows piecewise smooth Riemannian metrics in the sense above. But in fact, both definitions lead to the same values. In particular, both invariants are independent of which smooth triangulation of M we choose. Note that this independence of the chosen triangulation also follows from Corollary 2.18.

The following reasoning stems from [Bab92], Lemma 2.3 and the proof of Theorem 8.1.

Lemma 2.19. *Let M be a connected closed manifold of dimension n that is smoothly triangulated. If g is a piecewise smooth Riemannian metric on M, then there exists for every $\varepsilon > 0$ a smooth Riemannian metric g_ε on M such that*

(i) $g_\varepsilon \geq g$ *and*

(ii) $\mathrm{Vol}(M, g_\varepsilon) \leq \mathrm{Vol}(M, g) + \varepsilon$.

Proof. Choose a smooth Riemannian metric g_1 on M such that $g_1 \geq g$ (i. e. $g_1(v, v) \geq g(v, v)$ for all $v \in TM$ for which $g(v, v)$ is defined). Note that g is a smooth Riemannian metric on $M \setminus M^{(n-1)}$ and that $M^{(n-1)} \subset M$ is a zero set with respect to the n-dimensional volume.

Choose an open neighboorhood U of $M^{(n-1)}$ with $\mathrm{Vol}(U, g_1) < \varepsilon$. Define

$$g_\varepsilon := \lambda g + (1 - \lambda) g_1$$

with $\lambda : M \to [0, 1]$ a smooth function that is identically 1 outside of U and vanishes on a neighborhood of $M^{(n-1)}$. Then g_ε is a smooth Riemannian metric which obviously satisfies (i) and (ii). □

From (i) follows that $\mathrm{sys}_\phi(M, g_\varepsilon) \geq \mathrm{sys}_\phi(M, g)$. Together with (ii) this implies that the smooth systolic constant is bounded from above by the piecewise smooth systolic constant. But the converse inequality is obvious from the definition. Thus, both definitions of the systolic constant $\sigma_\phi(M)$ coincide. In an analogous manner, the same holds for the stable systolic constant and the minimal volume entropy.

The same technique can also be used to prove that these invariants coincide for a simplicial complex and its subdivisions. Note also that the analogous results hold for the spherical volume (see Definition 2.25) and similarly defined invariants in chapters 3 and 4.

2.2.4 Extension axiom

Since, given two manifolds, there may be no nontrivial map (say of absolute degree one) between them, we need a procedure to enlarge one of them such that we get a nontrivial map from the other manifold to the enlarged one. The next axiom shows how to enlarge a manifold and what happens to the invariants during the process.

Let $h : S^{k-1} \to X$ be a simplicial map with $1 \leq k < n$ such that $\Phi \circ h$ is contractible if $k = 2$. Define $X' := X \cup_h D^k$. This can be considered as a simplicial complex such that X is a subcomplex. Define moreover $\phi' : \pi_1(X) \to \pi$ as ϕ for $k \geq 3$ (the fundamental group has not changed), as the quotient map for $k = 2$, and as an arbitrary extension of ϕ for $k = 1$. Then we have $\phi' \circ i_* = \phi$ where $i : X \hookrightarrow X'$ is the inclusion.

Definition 2.20. An *extension* (X', ϕ') of (X, ϕ) is a simplicial complex that is obtained by a finite number of attachments in the way described above.

Extension axiom. Let (X', ϕ') be an extension of (X, ϕ) where $\phi : \pi_1(X) \twoheadrightarrow \pi$ is an epimorphism. Then

$$I_{\phi'}(X') = I_\phi(X).$$

Remark. The surjectivity assumption on ϕ guarantees that the corresponding Galois group $\pi \cong \pi_1(X)/\ker\phi$ remains the same for every extension of (X, ϕ). (Otherwise it could become bigger by attaching 1-cells, see paragraph 2.5.1.) In other words, the Galois covering $\tilde{X}'_{\phi'}$ is obtained from $\tilde{X}_\phi$ by π-equivariantly attaching cells of dimension $1 \leq k < n$.

Lemma 2.21 (Babenko, Sabourau)**.** *The extension axiom is fulfilled in the cases $I = \sigma$, $I = \sigma_1^{st}$, and $I = \lambda$.*

This is proved in [Bab06], Proposition 3.6 and [Sab06], Lemma 3.6. As above, the proof is included because of the importance of the extension axiom.

Proof. The proof is by induction. Thus, it is enough to look at the case of a single attachment.

The inclusion $i : X \hookrightarrow X'$ is an $(n, 1)$-monotone map, hence the comparison axiom implies $I_\phi(X) \leq I_{\phi'}(X')$ since $\phi = \phi' \circ i_*$ by definition.

To prove the other inequality, choose a Riemannian metric g on X and extend it to g' over X' in the following way: if $k = 1$, then $\tilde{X}'_{\phi'}$ is obtained from $\tilde{X}_\phi$ by π-equivariantly attaching 1-cells. Choose their (common) length in such a way that it does not decrease the distance between their starting and ending points. Now, let $k \geq 2$. Choose $R > 0$ such that $h : (S^{k-1}, g_R) \to$

(X, g) is nonexpanding where g_R is the round metric on S^{k-1} with radius R. Think of X' as

$$X \cup_h (S^{k-1} \times [0,1]) \cup_{S^{k-1}} S^k_+$$

and define g' as g on X, as

$$g' := ((1-t)h^*g + tg_R) \oplus dt^2$$

on $S^{k-1} \times [0,1]$, and as the round metric of radius R on the k-dimensional hemisphere S^k_+.

Let γ be a piecewise smooth curve in X' that joins two points in X. If γ meets the hemisphere S^k_+, we may substitute the arcs of γ in S^k_+ by geodesics in $S^{k-1} \times 1$ such that the new curve γ' is not longer than γ and homotopic to γ. Using the deformation retraction of $S^{k-1} \times [0,1]$ to $S^{k-1} \times 0$ we get a curve γ'' homotopic to γ' that is again not longer than γ' by the choice of g_R. (See also [Bab06], Lemme 3.5.)

The induced path metric $d_{g'}$ on X' is thus an extension of the path metric d_g on X induced by g. The same is true on the Galois coverings $\tilde{X}'_{\phi'}$ and $\tilde{X}_\phi$. Therefore, the induced metric $L_{g',x}$ on π equals the metric $L_{g,x}$ (choose $x \in X$), hence $\lambda_\phi(X) \geq \lambda_{\phi'}(X')$ since the volume has not changed.

Furthermore, the shortest loop in X' whose image in $K(\pi,1)$ is not contractible must meet X, hence may be chosen to lie entirely in X. Therefore $\sigma_\phi(X) \geq \sigma_{\phi'}(X')$.

The same is true for every loop in X. Hence, $\ell_g(a) = \ell'_g(i_*a)$ for every element $a \in H_1(X;\mathbb{Z})_\mathbb{R}$. Therefore, the norms $\|a\|_g$ and $\|i_*a\|_{g'}$ coincide for all $a \in H_1(X;\mathbb{Z})_\mathbb{R}$, and consequently $\sigma^{st}_{\phi,1}(X) \geq \sigma^{st}_{\phi',1}(X')$. □

2.2.5 Systolic manifolds

Using Corollary 2.7 and a famous theorem by Gromov, we are now able to give a homological classification of *ϕ-systolic* manifolds, i. e. of those manifolds M with $\sigma_\phi(M) > 0$.

Definition 2.22. A connected finite n-dimensional simplicial complex X is called *ϕ-essential* for a group homomorphism $\phi : \pi_1(X) \to \pi$ if the associated map $\Phi : X \to K(\pi,1)$ does not contract to the $(n-1)$-skeleton of $K(\pi,1)$.

In his Filling paper, Gromov proved the following universal systolic inequality ([Gro83], Appendix 2, (B'$_1$)):

Theorem 2.23 (Gromov). *If X is ϕ-essential, then*

$$\mathrm{sys}_\phi(X,g) \leq C_n \cdot \mathrm{Vol}(X,g)^{1/n}$$

for some universal constant $C_n > 0$.

So ϕ-essentialness implies ϕ-systolicity. Using the comparison axiom we see immediately that this is an equivalence:

$$\sigma_\phi(X) > 0 \quad \Leftrightarrow \quad X \ \phi\text{-essential.}$$

Together with Corollary 2.7, this equivalence implies the following classification of ϕ-systolic manifolds.

Corollary 2.24. *Let M be a connected closed manifold of dimension $n \geq 3$, and let $\phi : \pi_1(M) \twoheadrightarrow \pi$ be an epimorphism. Then*

$$\sigma_\phi(M) > 0 \quad \Leftrightarrow \quad \begin{cases} \Phi_*[M]_{\mathcal{O}_\rho} \neq 0 & \tilde{M}_\phi \text{ orientable,} \\ \Phi_*[M]_{\mathbb{Z}_2} \neq 0 & \tilde{M}_\phi \text{ nonorientable.} \end{cases}$$

In the case where $\tilde{M}_\phi$ is orientable, this corollary is due to Babenko (see [Bab92], Theorem 8.2). In chapter 4 of the thesis, we will investigate the analogous question for stable systolic constants (of arbitrary dimension).

Gromov's systolic inequality shows that there are many manifolds whose systolic constant is positive. But to compute the exact value in this case is very hard. In fact, it is known for only three essential manifolds (apart from the circle): the two-torus, $\sigma(T^2) = \sqrt{3}/2$ (Loewner, unpublished), the real projective plane, $\sigma(\mathbb{R}\mathrm{P}^2) = 2/\pi$ (Pu, [Pu52]), and the Klein bottle, $\sigma(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2) = 2\sqrt{2}/\pi$ (Bavard, [Bav86]).

For the stable systolic constant, the situation is similar: only $\sigma_1^{st}(T^2) = \sqrt{3}/2$ is known. (This is a direct consequence of Loewner's result.) The stable systolic constant of the real projective plane and the Klein bottle are zero since their first Betti numbers are zero respectively one, see also Corollary 4.13 in chapter 4.

2.3 Spherical volume

In this section, we want to investigate another invariant: the spherical volume T. Its definition is a bit more involved than the definitions of the minimal volume entropy and the systolic constants. Therefore, it is not easy to prove that T fulfills the comparison axiom of paragraph 2.2.3. But we are able to find weaker axioms that lead to the same conclusions. In particular, Corollary 2.18 is also valid in the case $I = T$.

The original definition of the spherical volume is due to Besson, Courtois, and Gallot ([BessCG91] and [BessCG95], see also [Sto02]). Inspired by the definition of the minimal volume entropy and the systolic constant, we introduce a relative version.

Again, let X be a connected finite simplicial complex of dimension n, and let $\phi : \pi_1(X) \to \pi$ be a group homomorphism. Denote by $\tilde{X}_\phi$ the Galois covering associated to the normal subgroup $\ker\phi \lhd \pi_1(X)$ and by $\Gamma := \pi_1(X)/\ker\phi$ the Galois group.

Definition 2.25. Let g be a Riemannian metric on X. Then $L^2(\tilde{X}_\phi)$ denotes the Hilbert space of square-integrable functions on $\tilde{X}_\phi$ with respect to the Riemannian volume of the lifted metric, and $S^\infty(\tilde{X}_\phi) \subset L^2(\tilde{X}_\phi)$ denotes its unit sphere. Note that Γ acts isometrically on both spaces by $\gamma \cdot \varphi(x) := \varphi(\gamma^{-1}x)$. Let $\mathcal{N}$ consist of those maps $F : \tilde{X}_\phi \to S^\infty(\tilde{X}_\phi)$ that are Γ-equivariant, Lipschitz continuous, and nonnegative, i. e. whose values are nonnegative functions. If $F \in \mathcal{N}$, then its restriction to the interior of the n-cells is differentiable almost everywhere by Rademacher's theorem and we can define

$$g_x^F(v_1, v_2) := \langle D_xF(v_1), D_xF(v_2)\rangle_{L^2(\tilde{X}_\phi)}$$

for almost all $x \in \tilde{X}_\phi, v_1, v_2 \in T_x\tilde{X}_\phi$. (Note that tangent spaces are well-defined for points inside top-dimensional simplices.) One finds that g^F is an almost everywhere defined positive semi-definite Γ-invariant metric on $\tilde{X}_\phi$.

This metric descends to X where it is also called g^F. We may define its volume form as 0 at points where g^F is degenerate or not defined and as the usual volume form at points where it is nondegenerate. Then dV_{g^F} is an integrable n-form on X. Hence, we can define

$$\begin{aligned} \operatorname{Vol}(X, g^F) &:= \int_X dV_{g^F} \quad \text{and} \\ T_\phi(X) &:= \inf_{F\in\mathcal{N}} \operatorname{Vol}(X, g^F). \end{aligned}$$

This second number is called the *spherical volume of X relative to ϕ.*

In the 'absolute' case (i. e. when ϕ is injective), one speaks of the *spherical volume* $T(X)$ and does not mention the homomorphism ϕ.

Remark. This definition is independent of the choice of the Riemannian metric g on X since the Hilbert spaces $L^2(\tilde{X}_\phi)$ for different Riemannian metrics are Γ-equivariantly isometric. (The change from the metric g' to the metric g is realized by multiplication with $\sqrt[4]{\det_g g'}$. This is a special case of the Radon-Nikodym theorem.) Moreover, the notion of Lipschitz continuity of $F : \tilde{X}_\phi \to S^\infty(\tilde{X}_\phi)$ does also not depend on which metric g we choose because X is compact.

If M admits a locally symmetric metric g_0 of negative curvature, then

$$T(M) = (4n)^{-n/2}\lambda(M)^n = (4n)^{-n/2}\lambda(M, g_0)^n \operatorname{Vol}(M, g_0)$$

by [BessCG95]. In particular, the spherical volume of hyperbolic manifolds is known. (See also the end of paragraph 2.2.2.)

Weak comparison axiom. Let X and Y be two connected finite simplicial complexes of dimension n, and let $\phi : \pi_1(X) \to \pi$ and $\psi : \pi_1(Y) \to \pi$ be group homomorphisms. If there exists a strictly (n,d)-monotone map $f : X \to Y$ such that $\phi = \psi \circ f_*$ and such that the induced homomorphism $f_* : \pi_1(X)/\ker\phi \xrightarrow{\cong} \pi_1(Y)/\ker\psi$ between the Galois groups is an isomorphism, then

$$I_\phi(X) \leq d \cdot I_\psi(Y).$$

Lemma 2.26. *The weak comparison axiom is fulfilled for $I = T$.*

Proof. Let g_2 be a Riemannian metric on Y. Define a Riemannian metric g_1 on X by using $f^* g_2$ on the nondegenerate simplices and extending it over all of X.

Since $f_* : \pi_1(X)/\ker\phi \xrightarrow{\cong} \pi_1(Y)/\ker\psi$ is an isomorphism, we get an equivariant lift $\tilde{f} : \tilde{X}_\phi \to \tilde{Y}_\psi$ of f that is again strictly (n,d)-monotone. The map

$$\begin{aligned} \mathcal{I} : L^2(\tilde{Y}_\psi) &\to L^2(\tilde{X}_\phi), \\ \varphi &\mapsto \tilde{\chi}/\sqrt{d} \cdot (\varphi \circ \tilde{f}), \end{aligned}$$

where $\tilde{\chi} : \tilde{X}_\phi \to \mathbb{R}$ is the characteristic map of the nondegenerate n-simplices, is an equivariant isometric homomorphism that preserves nonnegativity.

If $F : \tilde{Y}_\psi \to S^\infty(\tilde{Y}_\psi)$ is nonnegative equivariant Lipschitz, then consider the nonnegative equivariant Lipschitz map

$$\mathcal{I} \circ F \circ \tilde{f} : \tilde{X}_\phi \to S^\infty(\tilde{X}_\phi).$$

We have $g^{\mathcal{I} \circ F \circ \tilde{f}} = g^{F \circ \tilde{f}}$ since $\mathcal{I}$ is isometric and $g^{F \circ \tilde{f}} = f^* g^F$. Hence

$$\operatorname{Vol}(X, g^{\mathcal{I} \circ F \circ \tilde{f}}) = d \cdot \operatorname{Vol}(Y, g^F)$$

holds, which can be seen by looking at each open n-simplex of Y together with its preimage separately. (Note also that $f^* g^F$ is degenerate on the degenerate simplices in X.) Therefore,

$$T_\phi(X) \leq d \cdot T_\psi(Y). \qquad \square$$

Covering axiom. Let $f : X \to Y$ be a d-sheeted covering map of connected finite simplicial complexes, and let $\psi : \pi_1(Y) \to \pi$ be a homomorphism. Then

$$I_{\psi \circ f_*}(X) \leq d \cdot I_\psi(Y).$$

Lemma 2.27. *The covering axiom is true for $I = T$.*

Proof. This proof is essentially the same as the proof of Lemma 2.26 with one exception: the lifted map $\tilde{f} : \tilde{X}_{\psi\circ f_*} \to \tilde{Y}_\psi$ is a covering map with

$$D := [\ker\psi : f_*(\ker(\psi \circ f_*))]$$

sheets. Therefore, we have to replace the factor $\tilde{\chi}/\sqrt{d}$ in the definition of the isometry $\mathcal{I}$ by $1/\sqrt{D}$. (Note that in this case there are no degenerate simplices.) Then everything works out well. □

Zero axiom. Let X and Y be two connected finite simplicial complexes of dimension n, and let $\psi : \pi_1(Y) \to \pi$ be a group homomorphism. If $f : X \to Y$ is $(n, 0)$-monotone, then

$$I_{\psi\circ f_*}(X) = 0.$$

Lemma 2.28. *The zero axiom is valid for $I = T$.*

Proof. We have

$$2^n n^{n/2} T_\phi(X) \leq \lambda_\phi(X)^n$$

for all simplicial complexes (X, ϕ). (See [BessCG91], Théorème 3.8 or [Sto02], Proposition 4.1. There, the inequality is neither stated for simplicial complexes nor in the relative case, but the proof remains exactly the same.)

Since in the setting of the zero axiom $\lambda_{\psi\circ f_*}(X)^n = 0$ by the comparison axiom, we get $T_{\psi\circ f_*}(X) = 0$ from the cited inequality. □

Now, we can prove that Corollary 2.18 also holds for $I = T$.

Proposition 2.29. *Let M and N be two connected closed manifolds, and let $\psi : \pi_1(N) \to \pi$ be a group homomorphism. Let $f : M \to N$ be a map with $d := \deg_a(f)$. Then*

$$I_{\psi\circ f_*}(M) \leq d \cdot I_\psi(N)$$

for any invariant I that fulfills the weak comparison axiom, the covering axiom, and the zero axiom.

Proof. Denote by $p : \bar{N} \to N$ the connected covering of N associated to the subgroup $f_*(\pi_1(M)) \subset \pi_1(N)$. If $\bar{N}$ is not compact, then $\deg_a(f) = 0$ and f is homotopic to an $(n, 0)$-monotone map by Corollary 2.10. By the zero axiom $I_{\psi\circ f_*}(M) = 0$.

Assume now that $\bar{N}$ is compact. Note that f factorizes over $\bar{N}$

$$\begin{array}{ccc} & & \bar{N} \\ & \overset{\bar{f}}{\nearrow} & \downarrow p \\ M & \xrightarrow[f]{} & N \end{array}$$

such that $\bar{f}_* : \pi_1(M) \twoheadrightarrow \pi_1(\bar{N})$ is surjective, and that the absolute degree factors as $\deg_a(f) = \deg_a(\bar{f}) \deg_a(p)$. By Corollary 2.10 we may homotope $\bar{f}$ to be strictly $(n, \deg_a(\bar{f}))$-monotone. By the weak comparison axiom applied to this map and the covering axiom applied to p the proposition follows. □

For homotopy invariance even less assumptions are needed.

Corollary 2.30. *If $f : M \xrightarrow{\simeq} N$ is a homotopy eqivalence, then*

$$I_{\psi \circ f_*}(M) = I_\psi(N)$$

for every invariant I satisfying the weak comparison axiom.

The extension axiom does not need to be adjusted:

Lemma 2.31. *The invariant $I = T$ fulfills the extension axiom.*

Proof. We have $\pi_1(X)/\ker\phi \cong \pi \cong \pi_1(X')/\ker\phi'$, hence $\tilde{X}'_{\phi'}$ is obtained from $\tilde{X}_\phi$ by equivariant attachments of cells of dimension less than n. These cells are zero sets, thus canonically

$$L^2(\tilde{X}'_{\phi'}) = L^2(\tilde{X}_\phi).$$

Restriction to X defines a map $\mathcal{N}' \to \mathcal{N}$. Since the nonnegative part of $S^\infty(\tilde{X}_\phi)$ is contractible we may extend any map $F \in \mathcal{N}$ equivariantly over $\tilde{X}'_{\phi'}$ to get a map $F' \in \mathcal{N}'$. This gives a section $\mathcal{N} \to \mathcal{N}'$ of the above restriction map, which is therefore in particular surjective. Furthermore, $\mathrm{Vol}(X', g^{F'}) = \mathrm{Vol}(X, g^F)$ because the attached cells are of lower dimension, hence zero sets. Thus, $T_{\phi'}(X') = T_\phi(X)$. □

2.4 Homological invariance and first applications

Throughout this section, let I be an invariant that fulfills both the weak comparison axiom and the extension axiom. The main examples are of course the systolic constants, the minimal volume entropy, and the spherical volume.

In the first paragraph, homological invariance is proved in the form of a relative version of Theorem 2.1. Afterwards, we will apply this result in different situations. First, we will demonstrate that orientation-true degree one maps preserve the values of those invariants. This simplifies a rather long proof in [KędKM06]. As a second application, we will show that adding a simply-connected summand does not change the invariants under consideration. Furthermore, we will prove Corollary 2.2 about manifolds whose fundamental group consists of only two elements.

2.4.1 Homological invariance

The main result of this chapter is the following theorem, which includes Theorem 2.1 as a special case.

Theorem 2.32. *Let M and N be two connected closed manifolds of dimension $n \geq 3$, and let $\phi : \pi_1(M) \twoheadrightarrow \pi$ and $\psi : \pi_1(N) \twoheadrightarrow \pi$ be two epimorphisms. Denote by $\Phi : M \to K(\pi, 1)$ and $\Psi : N \to K(\pi, 1)$ the associated maps.*

(i) If there exists a homomorphism $\rho : \pi \to \mathbb{Z}_2$ such that both $\ker \rho\phi \lhd \pi_1(M)$ and $\ker \rho\psi \lhd \pi_1(N)$ are the respective subgroups of orientation preserving loops and if moreover

$$\Phi_*[M]_{\mathcal{O}_{\rho\phi}} = \Psi_*[N]_{\mathcal{O}_{\rho\psi}} \in H_n(\pi; \mathcal{O}_\rho)$$

holds, then $I_\phi(M) = I_\psi(N)$.

(ii) If $\tilde{N}_\psi$ is nonorientable and

$$\Phi_*[M]_{\mathbb{Z}_2} = \Psi_*[N]_{\mathbb{Z}_2} \in H_n(\pi; \mathbb{Z}_2),$$

then $I_\phi(M) \geq I_\psi(N)$.

Recall from section 2.1 that there is a homomorphism $\rho : \pi \to \mathbb{Z}_2$ such that $\ker \rho\psi \lhd \pi_1(N)$ is the subgroup of orientation preserving loops if and only if $\tilde{N}_\psi$ is orientable.

Note that part (i) of this theorem in the absolute case is exactly Theorem 2.1 for $n \geq 3$ (and for $I \in \{\sigma, \lambda, T\}$). But the two-dimensional case of Theorem 2.1 is trivial since two closed surfaces with the same fundamental group are diffeomorphic. For the systolic constants σ and σ_1^{st} part (i) is known in many cases by work of Babenko (see [Bab06]).

From part (ii) follows immediately:

Corollary 2.33. *If both $\tilde{M}_\phi$ and $\tilde{N}_\psi$ are nonorientable and*

$$\Phi_*[M]_{\mathbb{Z}_2} = \Psi_*[N]_{\mathbb{Z}_2} \in H_n(\pi; \mathbb{Z}_2),$$

then $I_\phi(M) = I_\psi(N)$.

For future use (see section 2.5) we will consider pseudomanifolds.

Definition 2.34 (see [Spa66], page 148)**.** A *connected closed n-dimensional pseudomanifold X* is a finite simplicial complex such that every simplex is the face of an n-simplex, every $(n-1)$-simplex is the face of exactly two n-simplices, and for every two n-simplices s and s' there exists a finite sequence $s = s_1, \ldots, s_m = s'$ of n-simplices such that s_i and s_{i+1} have an $(n-1)$-face in common.

Remark. Since every pseudomanifold X admits a CW decomposition with exactly one n-cell (see [Sab06], Lemma 2.2), we find that $H_n(X;\mathcal{O}_\rho)$ is either 0 or isomorphic to $\mathbb{Z}$ depending on the homomorphism $\rho : \pi_1(X) \to \mathbb{Z}_2$. Since there is no notion of orientation preserving paths in X, there may be more than one homomorphism $\rho : \pi_1(X) \to \mathbb{Z}_2$ (or indeed none) with $H_n(X;\mathcal{O}_\rho) \cong \mathbb{Z}$. Nevertheless, $H_n(X;\mathbb{Z}_2) = \mathbb{Z}_2$ in any case.

To prove Theorem 2.32 we need the following topological theorem, whose proof uses almost everything of section 2.1.

Theorem 2.35. *Let X be a connected closed pseudomanifold of dimension $n \geq 3$, and let N be a connected closed manifold of the same dimension. Let $\phi : \pi_1(X) \twoheadrightarrow \pi$ and $\psi : \pi_1(N) \twoheadrightarrow \pi$ be two epimorphisms, and let $\Phi : X \to K(\pi,1)$ and $\Psi : N \to K(\pi,1)$ be the associated maps.*

(i) If either there is a homomorphism $\rho : \pi \to \mathbb{Z}_2$ such that both homology groups $H_n(X;\mathcal{O}_{\rho\phi}) \neq 0$ and $H_n(N;\mathcal{O}_{\rho\psi}) \neq 0$ and

$$\Phi_*[X]_{\mathcal{O}_{\rho\phi}} = \Psi_*[N]_{\mathcal{O}_{\rho\psi}} \in H_n(\pi;\mathcal{O}_\rho),$$

(ii) or if $\tilde{N}_\psi$ is nonorientable and

$$\Phi_*[X]_{\mathbb{Z}_2} = \Psi_*[N]_{\mathbb{Z}_2} \in H_n(\pi;\mathbb{Z}_2),$$

then there exists an extension (X',ϕ') of (X,ϕ) and a strictly $(n,1)$-monotone map $h : N \to X'$ such that

$$\psi = \phi' \circ h_* \text{ and}$$
$$h_*[N]_{\mathbb{K}} = i_*[X]_{\mathbb{K}} \in H_n(X';\mathbb{K}),$$

where $i : X \hookrightarrow X'$ is the inclusion.

Proof. Use ϕ to identify $\pi_1(X)/\ker\phi = \pi$. Choose a CW decomposition of X. Now, attach (possibly infinitely many) 2-cells to X whose attaching loops generate $\ker\phi$. Thus, we get a CW complex $X(2)$ that has fundamental group $\pi_1(X(2)) = \pi$. Next attach 3-cells to $X(2)$ and kill $\pi_2(X(2))$ and then 4-cells to kill the third homotopoy group and so on. We obtain a sequence $X \subset X(2) \subset X(3) \subset \ldots$ of CW complexes that fulfill

$$\pi_1(X(k)) = \pi \text{ and } \pi_s(X(k)) = 0 \text{ for } 2 \leq s < k.$$

This gives a CW decomposition of $K(\pi,1)$, and we have

$$X(n-1) = K(\pi,1)^{(n-1)} \cup \left(\bigcup_{e\ n\text{-cell of } X} e \right) \quad \text{and}$$
$$X(k) = K(\pi,1)^{(k)} \qquad \text{for } k \geq n.$$

By Lemma 2.6, the map Ψ gives a map

$$g : N \to X(n)$$

such that $g_*[N]_{\mathbb{K}} = i_*[X]_{\mathbb{K}}$.

Lemma 2.5 shows that we may deform g to

$$\hat{g} : N \to X(n-1)$$

with $\hat{g}_*[N]_{\mathbb{K}} = i_*[X]_{\mathbb{K}}$ in $H_n(X(n-1); \mathbb{K})$. Moreover, $\hat{g}$ is strictly $(n,1)$-monotone.

By compactness, we may choose a finite subcomplex $X \subset X' \subset X(n-1)$ such that $\hat{g}(N) \subset X'$ and $\hat{g}_*[N]_{\mathbb{K}} = i_*[X]_{\mathbb{K}}$ in $H_n(X'; \mathbb{K})$. Together with the epimorphism $\phi' : \pi_1(X') \twoheadrightarrow \pi$ that is induced by the inclusion $X' \hookrightarrow X(n-1)$ this defines an extension of (X, ϕ). We finally obtain $h := \hat{g} : N \to X'$ having the asserted properties. □

Remark. Theorem 2.35 does not hold in dimension two. For example consider a closed oriented surface Σ of genus $g \geq 2$ and the torus T^2. Let $\Phi : \Sigma \to T^2$ be a degree one map. Since the torus is aspherical we are in the situation of case (i) of the theorem. Let X be an extension of Σ, i. e. it is obtained by attaching finitely many 1-cells to Σ. It is easy to see that there is no map $h : T^2 \to X$ that induces a nontrivial homomorphism in 2-dimensional homology.

Proof of Theorem 2.32. This is a direct consequence of Theorem 2.35 and the axioms: by the weak comparison axiom and the extension axiom we find

$$I_\psi(N) \leq I_{\phi'}(X') = I_\phi(X).$$

In case where $X = M$ is a manifold, we get the equality of (i) by changing the roles of M and N. □

2.4.2 Degree one maps

In the remainder of this section, we will suppose that I depends only on $\ker \phi$, not on ϕ itself. This allows us in many situations to assume without loss of generality that the group homomorphisms $\phi : \pi_1(X) \to \pi$ are surjective.

Theorem 2.32 has the following immediate consequence which improves the homotopy invariance from Corollary 2.30.

Definition 2.36. A map $f : M \to N$ between manifolds is called *orientation-true* if it maps orientation preserving loops to orientation preserving ones and orientation reversing loops to orientation reversing ones.

Corollary 2.37. *If $f : M \to N$ is an orientation-true map of absolute degree one between two connected closed manifolds of dimension $n \geq 3$, then*

$$I_{\psi\circ f_*}(M) = I_\psi(N)$$

for any homomorphism $\psi : \pi_1(N) \to \pi$.

Recall that maps of absolute degree one are always surjective on fundamental groups (see paragraph 2.1.5). Using this, Corollary 2.37 is a direct application of Theorem 2.32 respectively of Corollary 2.33.

In [KędKM06], Kędra, Kotschick, and Morita proved the following theorem (Theorem 4):

Theorem 2.38 (Kędra, Kotschick, Morita)**.** *Let M be a closed oriented manifold with nonvanishing volume flux group Γ_μ. Then M has a finite covering $\bar{M}$ whose minimal volume entropy $\lambda(\bar{M})$ vanishes.*

Their proof on pages 1260–1264 may be shortened and simplified in the following way: it starts with the construction of a map $\Phi : S^1 \times F \to \bar{M}$ from a closed oriented product manifold $S^1 \times F$ to a finite covering $\bar{M}$ of M. Lemma 24 on page 1261 states that Φ has degree one and induces an isomorphism on fundamental groups. Therefore, $\lambda(\bar{M}) = \lambda(S^1 \times F)$ by Corollary 2.37. Since the minimal volume entropy of $S^1 \times F$ vanishes by the vanishing of the minimal volume $\mathrm{MinVol}(S^1 \times F)$ (see [KędKM06] for details), this proves the cited theorem.

2.4.3 Adding simply-connected summands

It is rather difficult to investigate the behaviour of the invariants under connected sums. The easiest case is when one of the summands is simply-connected. The next corollary was already known for $I = \lambda$ and $I = \sigma$ by [Bab95].

Corollary 2.39. *Let M and N be two connected closed manifolds, and let $\phi : \pi_1(M) \to \pi$ be a homomorphism. If N is simply-connected, then*

$$I_\phi(M \# N) = I_\phi(M).$$

Proof. Let $n := \dim M = \dim N$. Note that for $n = 2$, we necessarily have $N \cong S^2$ and thus $M \# N \cong M$. So we may assume $n \geq 3$. Since the map $M \# N \to M$ is orientation-true, has absolute degree one, and induces an isomorphism on the fundamental group the respective values of I coincide by Corollary 2.37. □

This result is generalized to nonessential orientable summands in Corollary 2.46.

2.4.4 $\mathbb{Z}_2$-systoles

Next, we want to look at the case $\phi : \pi_1(M) \twoheadrightarrow \mathbb{Z}_2$ and in particular at manifolds with fundamental group $\mathbb{Z}_2$. Since for finite Galois groups the stable 1-systolic constant, the minimal volume entropy, and the spherical volume vanish, we concentrate on the systolic constant in this paragraph. Thus, we may use the classification from Corollary 2.24. Denote $\sigma_n := \sigma(\mathbb{R}\mathrm{P}^n) > 0$.

Corollary 2.40. *Let M be a connected closed manifold of dimension $n \geq 3$, and let $\phi : \pi_1(M) \twoheadrightarrow \mathbb{Z}_2$ be an epimorphism. Then $\sigma_\phi(M) \leq \sigma_n$, and $\sigma_\phi(M) = 0$ if and only if $\Phi_*[M]_{\mathbb{Z}_2} = 0$. Moreover, if $\tilde{M}_\phi$ is orientable, then*

$$\sigma_\phi(M) = \begin{cases} \sigma_n & \Phi_*[M]_{\mathbb{Z}_2} \neq 0 \\ 0 & \Phi_*[M]_{\mathbb{Z}_2} = 0 \end{cases}$$

In particular, $\sigma_\phi(M) = 0$ for M orientable and n even, and also for M nonorientable, $\tilde{M}_\phi$ orientable and n odd.

Proof. Note that $K(\mathbb{Z}_2, 1) = \mathbb{R}\mathrm{P}^\infty$. Hence

$$H_n(\mathbb{Z}_2; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & n \text{ odd} \\ 0 & n > 0 \text{ even} \end{cases}$$

and

$$H_n(\mathbb{Z}_2; \mathcal{O}_{\mathrm{Id}}) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z}_2 & n > 0 \text{ even} \end{cases}$$

Furthermore, $H_n(\mathbb{Z}_2; \mathbb{Z}_2) = \mathbb{Z}_2$ for all $n \geq 0$.

Note that $\Phi_*[M]_{\mathbb{K}} = 0$ if and only if $\Phi_*[M]_{\mathbb{Z}_2} = 0$. Hence, $\sigma_\phi(M) = 0$ if and only if $\Phi_*[M]_{\mathbb{Z}_2} = 0$ by the classification of ϕ-systolic manifolds. Moreover, $\Phi_*[M]_{\mathbb{K}} = 0$ in the two particular cases mentioned at the end of the corollary.

If $\Phi_*[M]_{\mathbb{Z}_2} \neq 0$, then $\Phi_*[M]_{\mathbb{K}} = i_*[\mathbb{R}\mathrm{P}^n]_{\mathbb{K}}$. Thus, Theorem 2.32 finishes the proof. □

Apart from the statement that $\sigma_\phi(M) \in \{0, \sigma_n\}$ for M nonorientable and $\tilde{M}_\phi$ orientable, this was already proved by Babenko in [Bab04].

In the special case $\pi_1(M) = \mathbb{Z}_2$, this corollary gives

$$\sigma(M) = \begin{cases} \sigma_n & \Phi_*[M]_{\mathbb{Z}_2} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\Phi : M \to \mathbb{R}\mathrm{P}^\infty$ denotes the classifying map of the universal covering. Note that this statement also holds for $n = 2$ since here $M \cong \mathbb{R}\mathrm{P}^2$.

Furthermore, the homomorphism $\Phi_* : H_n(M;\mathbb{Z}_2) \to H_n(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ is an isomorphism if and only if the induced homormophism Φ^* on n-dimensional cohomology (with $\mathbb{Z}_2$ coefficients) is an isomorphism. Since $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha']$ with $\alpha' \in H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ the generator, Corollary 2.2 follows immediately.

2.5 An inequality between the systolic constant and the minimal volume entropy

This section is devoted to the proof of Theorem 2.3. In the first paragraph, the extension axiom for the systolic constant and the minimal volume entropy is improved. This strong version of the extension axiom, Gromov's geometric cycles, and Sabourau's computations from [Sab06] are used in the second paragraph to prove Theorem 2.3. Note that in this proof, it is really necessary to consider relative versions of the invariants, even if one is only interested in the absolute case.

2.5.1 Strong extension axiom

In the extension axiom of paragraph 2.2.4, we assumed that the simplicial complex X is given together with a surjective homomorphism $\phi : \pi_1(X) \twoheadrightarrow \pi$. This guarantees that the Galois groups of the coverings $\tilde{X}'_{\phi'}$ and $\tilde{X}_\phi$ coincide. Without this assumption the Galois group of the extended complex X' can become extremely large compared to the original one. Think for instance of $X = T^n$ with $\phi : \mathbb{Z}^n \hookrightarrow \mathbb{Z}^n * \mathbb{Z}$ the inclusion in the first factor and $X' = T^n \vee S^1$ with ϕ' the identity. Here, the Galois group of $\tilde{X}_\phi$ has polynomial growth whereas the one of $\tilde{X}'_{\phi'}$ grows exponentially. Nevertheless, we can show that the systolic constant and the minimal volume entropy behave well in this situation.

Strong extension axiom. Let (X', ϕ') be an extension of (X, ϕ). Then

$$I_{\phi'}(X') = I_\phi(X).$$

We will show that the systolic constant and the minimal volume entropy satisfy this axiom. This is quite easy for the systolic constant, but in case of the minimal volume entropy there is some effort necessary. We will approximate Riemannian norms $L_{g,x}$ (see the remark after Definition 2.16) by simpler and more regular norms. This idea is due to Manning (see [Man05]).

Definition 2.41. Let G be a finitely generated group, and $S \subset G$ be finite generating set that is symmetric, that is $S^{-1} = S$. Throughout this paragraph, all generating sets will be assumed symmetric. For a norm $L : G \to [0, \infty)$ one defines another norm

$$N_{L,S}(g) := \inf\{\textstyle\sum_{i=1}^{n} L(s_i) | g = s_1 \cdots s_n, s_i \in S\}.$$

These norms will be called *generator norms.*

Remark. If one takes $L = 1$ the norm that assigns 1 to each nontrivial element of G, then $N_{1,S}$ is the well-known *word norm* or *word length* on G with respect to S.

Lemma 2.42. *The entropy (see Definition* 2.16*) of generator norms is well-defined, and one has*

$$\lambda(G, N_{L,S}) = \inf_{t>0}(\tfrac{1}{t} \log \beta_{N_{L,S}}(t) + \tfrac{1}{t} \log \#S).$$

Proof. Write $\beta := \beta_{N_{L,S}}$. We have

$$\beta(r + t) \leq \beta(r)\beta(t)\#S. \qquad (*)$$

This may be seen in the following way: if $N_{L,S}(g) \leq r + t$, then choose a minimal representation $g = s_1 \cdots s_n$, i. e. one fulfilling $N_{L,S}(g) = \sum_{i=1}^{n} L(s_i)$. Choose $k \in \mathbb{N}_0$ such that

$$\sum_{i=1}^{k} L(s_i) \leq r \quad \text{and} \quad \sum_{i=1}^{k+1} L(s_i) > r.$$

Define $g_1 := s_1 \cdots s_k$ and $g_2 := s_{k+2} \cdots s_n$. Then $g = g_1 s_{k+1} g_2$, and we have

$$N_{L,S}(g_1) = \sum_{i=1}^{k} L(s_i) \leq r \quad \text{and}$$
$$N_{L,S}(g_2) = \sum_{i=1}^{n} L(s_i) - \sum_{i=1}^{k+1} L(s_i) \leq r + t - r = t.$$

This proves equation $(*)$.

Now, let r and t be arbitrary positive real numbers, and choose $k \in \mathbb{N}_0$ such that $kt < r \leq (k + 1)t$. Then

$$\beta(r) \leq \beta((k + 1)t) \leq \beta(kt)\beta(t)\#S \leq \ldots \leq \beta(t)\beta(t)^k \#S^k$$

by $(*)$, and consequently

$$\tfrac{1}{r}\log\beta(r) \le \tfrac{1}{r}\log\beta(t) + \tfrac{k}{r}\log(\beta(t)\#S) \le \tfrac{1}{r}\log\beta(t) + \tfrac{1}{t}\log(\beta(t)\#S).$$

Therefore,

$$\limsup_{r\to\infty} \tfrac{1}{r}\log\beta(r) \le \tfrac{1}{t}\log(\beta(t)\#S)$$

for all $t > 0$. Hence

$$\limsup_{r\to\infty} \tfrac{1}{r}\log\beta(r) \le \inf_{t>0}(\tfrac{1}{t}\log\beta(t) + \tfrac{1}{t}\log\#S) \le \liminf_{t\to\infty} \tfrac{1}{t}\log\beta(t).$$

Thus, $\lambda(G, N_{L,S}) = \lim_{r\to\infty} \frac{1}{r}\log\beta(r)$ exists and fulfills the claimed equality. □

To prove the strong extension axiom for the minimal volume entropy we have to consider the case of one attached circle. The idea is to let its length grow to infinity. The following proposition investigates the analogous situation for generator norms.

Proposition 2.43. *Let G and H be finitely generated groups, and let L_G and L_H be generator norms with respect to the finite generating sets S and T, i. e. $L_G = N_{L_G,S}$ respectively $L_H = N_{L_H,T}$. Then $L_G * \varrho L_H$ is a generator norm on $G * H$ with respect to $S \cup T$ for every $\varrho > 0$. In this situation, the following equation holds:*

$$\lim_{\varrho\to\infty} \lambda(G * H, L_G * \varrho L_H) = \lambda(G, L_G).$$

Proof. Let $\varepsilon > 0$. Choose $R > 0$ such that

$$\tfrac{1}{R}\log\beta_{L_G}(R) \le \lambda(G, L_G) + \varepsilon \quad \text{and}$$
$$\tfrac{1}{R}\log(\#S + \#T) \le \varepsilon.$$

Whenever $\varrho \cdot \min_{t\in T} L_H(t) > R$, we have

$$\beta_{L_G*\varrho L_H}(R) = \beta_{L_G}(R)$$

since no elements of $H \setminus 1$ are involved, yet. Hence by Lemma 2.42

$$\begin{aligned}\lambda(G * H, L_G * \varrho L_H) &\le \tfrac{1}{R}\log\beta_{L_G*\varrho L_H}(R) + \tfrac{1}{R}\log(\#S + \#T)\\ &\le \lambda(G, L_G) + 2\varepsilon.\end{aligned}$$

Therefore $\limsup_{\varrho\to\infty} \lambda(G * H, L_G * \varrho L_H) \le \lambda(G, L_G)$. But

$$\lambda(G * H, L_G * \varrho L_H) \ge \lambda(G, L_G)$$

is obvious. Thus, the limit exists and equals $\lambda(G, L_G)$. □

In the next proposition, we want to use Manning's approximation result from [Man05]. Since we have to swap the limit $\varrho \to \infty$ with the approximation, we need to control the quality of the approximation.

Proposition 2.44. *Let (X,g) be a connected finite Riemannian simplicial complex, and let $\phi : \pi_1(X) \to \pi$ be a homomorphism. As usual denote by $L_{g,x}$ the induced norm on the Galois group $\Gamma := \pi_1(X)/\ker\phi$. Furthermore let L_H be a generator norm on a finitely generated group H with respect to the finite generating set $T \subset H$. Then*

$$\lim_{\varrho\to\infty} \lambda(\Gamma * H, L_{g,x} * \varrho L_H) = \lambda(\Gamma, L_{g,x}).$$

Proof. Write $L_g := L_{g,x}$. Choose a fundamental domain $F \subset \tilde{X}_\phi$ with diameter D and $x \in F$. Let R be an arbitrary positive real number. Write

$$h = \gamma_0 h_1 \gamma_1 \cdots h_n \gamma_n h_{n+1} \in \Gamma * H$$

with $\gamma_0 \in \Gamma, \gamma_1, \ldots, \gamma_n \in \Gamma \setminus 1, h_1, \ldots, h_n \in H \setminus 1, h_{n+1} \in H$, and choose $k \in \mathbb{N}_0$ such that

$$(k-1)R < L_g * \varrho L_H(h) = \sum_{i=0}^{n} L_g(\gamma_i) + \varrho \cdot \sum_{i=1}^{n+1} L_H(h_i) \leq kR.$$

Furthermore, let $k_0, \ldots, k_n \in \mathbb{N}_0$ such that

$$(k_i - 1)R < L_g(\gamma_i) \leq k_i R.$$

Think of the γ_i as shortest geodesics in $\tilde{X}_\phi$ starting at x and ending at $\gamma_i x$. Pick points $\alpha_{ij} \in \Gamma, j = 1, \ldots, k_i - 1$ such that

$$d(\gamma_i(jR), \alpha_{ij}x) \leq D$$

and set $\alpha_{i0} = 1, \alpha_{ik_i} = \gamma_i$. Then

$$L_g(\alpha_{ij}^{-1}\alpha_{i,j+1}) = d(\alpha_{ij}x, \alpha_{i,j+1}x) \leq R + 2D.$$

Put $S := \{\alpha \in \Gamma | L_g(\alpha) \leq R + 2D\}$. (This is a finite generating system of Γ as we just have shown.) Then

$$N_{L_g,S}(\gamma_i) \leq k_i \cdot (R + 2D)$$

and

$$\begin{aligned}
N_{L_g,S} * \varrho L_H(h) &\leq \sum_{i=0}^{n} k_i(R+2D) + \varrho \cdot \sum_{i=1}^{n+1} L_H(h_i) \\
&\leq \sum_{i=0}^{n} (k_i(R+2D) - (k_i - 1)R) + \sum_{i=0}^{n} L_g(\gamma_i) + \varrho \cdot \sum_{i=1}^{n+1} L_H(h_i) \\
&\leq \sum_{i=0}^{n} (k_i 2D + R) + kR \\
&\leq (k+n+1)(R+2D)
\end{aligned}$$

since $\sum_{i=0}^{n} k_i \leq k+n+1$. Hence

$$\beta_{L_g * \varrho L_H}(kR) \leq \beta_{N_{L_g,S} * \varrho L_H}((k+n+1)(R+2D))$$

and thus

$$\begin{aligned}
\limsup_{r\to\infty} \tfrac{1}{r} \log \beta_{L_g*\varrho L_H}(r) &= \limsup_{k\to\infty} \tfrac{1}{kR} \log \beta_{L_g*\varrho L_H}(kR) \\
&\leq \lim_{k\to\infty} \tfrac{1}{kR} \log \beta_{N_{L_g,S}*\varrho L_H}((k+n+1)(R+2D)) \\
&= \tfrac{R+2D}{R} \lambda(\Gamma * H, N_{L_g,S} * \varrho L_H).
\end{aligned}$$

Since every norm L satisfies $L \leq N_{L,S}$ by the triangle inequality, we see that

$$\liminf_{r\to\infty} \tfrac{1}{r} \log \beta_{L_g*\varrho L_H}(r) \geq \lambda(\Gamma * H, N_{L_g,S} * \varrho L_H).$$

Thus, the volume entropy of $L_g * \varrho L_H$ exists and equals the supremum

$$\lambda(\Gamma * H, L_g * \varrho L_H) = \sup_S \lambda(\Gamma * H, N_{L_g,S} * \varrho L_H)$$

over all finite generating systems S of Γ.

Moreover,

$$\lambda(\Gamma * H, L_g * \varrho L_H) \leq \tfrac{R+2D}{R} \lambda(\Gamma * H, N_{L_g,S} * \varrho L_H).$$

Now, if $\varrho \to \infty$, then by Proposition 2.43 the right-hand side goes to

$$\tfrac{R+2D}{R} \lambda(\Gamma, N_{L_g,S}) \leq \tfrac{R+2D}{R} \lambda(\Gamma, L_g).$$

Since $R > 0$ was arbitrary, we get

$$\limsup_{\varrho\to\infty} \lambda(\Gamma * H, L_g * \varrho L_H) \leq \lambda(\Gamma, L_g).$$

But $\lambda(\Gamma * H, L_g * \varrho L_H) \geq \lambda(\Gamma, L_g)$ is again obvious. Hence, the limit exists and equals $\lambda(\Gamma, L_g)$. □

Theorem 2.45. *The strong extension axiom is fulfilled by the systolic constant $I = \sigma$ and the minimal volume entropy $I = \lambda$.*

Proof. By the comparison axiom we have $I_\phi(X) \leq I_{\phi'}(X')$ since the inclusion $X \hookrightarrow X'$ is $(n,1)$-monotone. We may proceed by induction, attaching one cell at a time. The case where the Galois group $\Gamma := \pi_1(X)/\ker\phi$ does not change is already covered by the extension axiom since here ϕ and ϕ' factor into epimorphisms onto this quotient and the induced inclusion $\Gamma \hookrightarrow \pi$.

The case remaining to be investigated is therefore the following: let $X' = X \cup_h D^1$ where $h : S^0 \to X$ is simplicial, and let $\phi' : \pi_1(X') \cong \pi_1(X) * \mathbb{Z} \to \pi$ be an extension of the given homomorphism ϕ.

First, consider the simplicial complex $Y := X \vee S^1$ where the circle is attached at $h(1) \in X$. Using a path from $h(-1)$ to $h(1)$ in X, we get a homotopy equivalence $f : X' \to Y$ and thus may define $\psi := \phi' \circ f_*^{-1}$. Note that (Y, ψ) is again an extension of (X, ϕ) and that f is $(n,1)$-monotone and has an $(n,1)$-monotone homotopy inverse. Hence, $I_{\phi'}(X') = I_\psi(Y)$ and it remains to show that $I_\psi(Y) \leq I_\phi(X)$.

Case 1: $I = \sigma$. Let g be a Riemannian metric on X. Extend it over Y by assigning the length $\operatorname{sys}_\phi(X, g)$ to the attached circle. Then both $\operatorname{sys}_\psi(Y, g) = \operatorname{sys}_\phi(X, g)$ and $\operatorname{Vol}(Y, g) = \operatorname{Vol}(X, g)$, and consequently $\sigma_\psi(Y) \leq \sigma_\phi(X)$. (If the ϕ-systole of (X, g) is not finite, i. e. if $\ker\phi = \pi_1(X)$, then use a sequence of metrics where the length of the attached circle tends to infinity.)

Case 2: $I = \lambda$. Again, let a Riemannian metric g on X be given. Define g_ϱ to be the extension over Y that assigns the length $\varrho > 0$ to the circle S^1. Then $\operatorname{Vol}(Y, g_\varrho) = \operatorname{Vol}(X, g)$.

Take the attaching point $x = h(1)$ as base point. Then $\pi_1(Y) = \pi_1(X) * \mathbb{Z}$. Consider the homomorphism

$$\pi_1(Y) \xrightarrow{\phi * \mathrm{Id}} \pi * \mathbb{Z}.$$

We see that $\ker(\phi * \mathrm{Id}) \subset \ker\psi$, and thus

$$\lambda_{\phi*\mathrm{Id}}(Y) \geq \lambda_\psi(Y).$$

With $L : \mathbb{Z} \to [0, \infty)$ denoting the standard word norm we have

$$\lambda_{\phi*\mathrm{Id}}(Y, g_\varrho) = \lambda(\Gamma * \mathbb{Z}, L_{g,x} * \varrho L) \xrightarrow{\varrho\to\infty} \lambda(\Gamma, L_{g,x}) = \lambda_\phi(X, g)$$

by Proposition 2.44. Thus, $\lambda_\phi(X) \geq \lambda_{\phi*\mathrm{Id}}(Y) \geq \lambda_\psi(Y)$. □

Remark. The stable 1-systolic constant does not satisfy the strong extension axiom in this form. See paragraph 4.1.2 for a better suited version of the strong extension axiom in the case of stable systoles.

The strong extension axiom has the following consequence, which sharpens Corollary 2.39.

Corollary 2.46. *Let M and N be two connected closed manifolds of dimension $n \geq 3$, and let $\phi : \pi_1(M) \to \pi$ and $\psi : \pi_1(N) \to \pi'$ be two homomorphisms. If N is orientable and not ψ-essential, then*

$$I_{\phi*\psi}(M\#N) = I_\phi(M)$$

for I the systolic constant or the minimal volume entropy.

In the case of the systolic constant, this corollary was proved in [Bab03], Proposition 4.2.

Proof. First, consider the homomorphism $p : \pi * \pi' \twoheadrightarrow \pi$. By definition, $I_{\phi*\psi}(M\#N) \geq I_{p\circ(\phi*\psi)}(M\#N)$. But this second value equals $I_\phi(M)$ by Corollary 2.37. In this step, the assumption on the orientability of N is used.

On the other hand, since N is not ψ-essential the map $\Psi : N \to K(\pi', 1)$ corresponding to ψ may be chosen to have image in the $(n-1)$-skeleton. Thus, the map $\Phi \vee \Psi : M\#N \to K(\pi, 1) \vee K(\pi', 1)$ factorizes over $\mathrm{Id} \vee \Psi : M\#N \to M \vee K$ where $K \subset K(\pi', 1)^{(n-1)}$ is a finite subcomplex that contains the image of Ψ.

Therefore, $I_{\phi*\psi}(M\#N) \leq I_{\phi*\iota_*}(M \vee K)$ by the weak comparison axiom where $\iota : K \hookrightarrow K(\pi', 1)$ is the inclusion. But $M \vee K$ is an extension of M. Hence $I_{\phi*\iota_*}(M \vee K) = I_\phi(M)$ by the strong extension axiom, and consequently $I_{\phi*\psi}(M\#N) \leq I_\phi(M)$. □

The manifold N is not ψ-essential if and only if $\sigma_\psi(N) = 0$ (see paragraph 2.2.5). It is an open question whether the above corollary in the case of the minimal volume entropy also holds if one replaces the assumption that N is not ψ-essential by $\lambda_\psi(N) = 0$.

2.5.2 Geometric cycles

In this paragraph, we will investigate the relation between the systolic constant and the minimal volume entropy. In doing so, we will prove a relative analogue of Theorem 2.3.

Definition 2.47. Let π be a group, $\rho : \pi \to \mathbb{Z}_2$ a homomorphism, and $a \in H_n(\pi; \mathcal{O}_\rho)$ a homology class. Define

$$I(a) := \inf_{(X,\Psi)} I_{\Psi_*}(X),$$

where the infimum is taken over all *geometric cycles* (X, Ψ) representing the homology class a, i. e. over all maps $\Psi : X \to K(\pi, 1)$ from a connected closed n-dimensional pseudomanifold X to $K(\pi, 1)$ with $H_n(X; \mathcal{O}_{\rho\Psi_*}) \neq 0$ and $\Psi_*[X]_{\mathcal{O}_{\rho\Psi_*}} = a$. For coefficients in $\mathbb{Z}_2$ we use the analogous definition.

The next theorem shows that if there is a geometric cycle that is defined on a manifold and that is surjective on fundamental groups, then it is minimal.

Theorem 2.48. *If I fulfills the strong extension axiom and the weak comparison axiom, then*

$$I(\Phi_*[M]_{\mathbb{K}}) = I_{\Phi_*}(M)$$

for any connected closed manifold M of dimension $n \geq 3$ with $\Phi : M \to K(\pi, 1)$ such that $\Phi_* : \pi_1(M) \twoheadrightarrow \pi$ is surjective. (We use $\mathbb{K}$ the orientation bundle of M if the covering $\tilde{M}_{\Phi_*}$ is orientable and $\mathbb{K} = \mathbb{Z}_2$ otherwise.)

Proof. Let (X, Ψ) be a geometric cycle representing $\Phi_*[M]_{\mathbb{K}}$. Then there exists another geometric cycle (X', Ψ') representing the same homology class such that Ψ' maps the fundamental group of X' surjectively onto π and such that $I_{\Psi'_*}(X') = I_{\Psi_*}(X)$. This can be seen as follows: let $\gamma_1, \ldots, \gamma_m$ be generators of π. Think of the γ_i as closed curves in $K(\pi, 1)$ and define

$$X'' := X \vee \left(\bigvee_{i=1}^{m} S^1 \right) \quad \text{and}$$

$$\Psi'' := \Psi \vee \left(\bigvee_{i=1}^{m} \gamma_i \right) : X'' \to K(\pi, 1).$$

Then $\Psi''_*[X'']_{\mathbb{K}} = \Phi_*[M]_{\mathbb{K}}$ (here $[X'']_{\mathbb{K}}$ is the image of $[X]_{\mathbb{K}}$ under the inclusion $X \hookrightarrow X''$) and the induced homomorphism Ψ''_* on the fundamental group is an epimorphism. By the strong extension axiom $I_{\Psi''_*}(X'') = I_{\Psi_*}(X)$.

Now, consider the pseudomanifold

$$X' := X \# \left(\mathop{\#}_{i=1}^{m} (S^n/\{\pm \mathrm{pt}\}) \right),$$

where $S^n/\{\pm \mathrm{pt}\}$ is the n-sphere with two points identified. The projection of S^n to a closed interval such that $\pm \mathrm{pt}$ are mapped to the boundary points induces a map $S^n/\{\pm \mathrm{pt}\} \to S^1$. Let $p : X' \to X''$ be the composition of the projection

$$X' \to X \vee \left(\bigvee_{i=1}^{m} S^n/\{\pm \mathrm{pt}\} \right)$$

with this map on each $S^n/\{\pm\,\mathrm{pt}\}$. Define

$$\Psi' := \Psi'' \circ p : X' \to K(\pi, 1).$$

Note that p is a homotopy equivalence. Thus, Ψ' induces a surjection on fundamental groups, $H_n(X'; \mathbb{K}) \neq 0$, and (X', Ψ') represents $\Phi_*[M]_{\mathbb{K}}$.

Since p can be chosen strictly $(n, 1)$-monotone and has a strictly $(n, 1)$-monotone homotopy inverse, we get

$$I_{\Psi'_*}(X') = I_{\Psi''_*}(X'') = I_{\Psi_*}(X)$$

by the weak comparison axiom. From Theorem 2.35, the weak comparison axiom, and the extension axiom, it follows that $I_{\Phi_*}(M) \leq I_{\Psi'_*}(X') = I_{\Psi_*}(X)$. □

Now, we follow [Sab06], sections 4 and 5. First, we need the following theorem of Gromov about the existence of regular geometric cycles.

Theorem 2.49 ([Gro83], pages 70, 71)**.** *There exists a constant $A_n > 0$ such that for all $\varepsilon > 0$ there is a geometric cycle (X, Ψ) representing $a \in H_n(\pi; \mathbb{K})$ and a Riemannian metric g on X such that*

$$\sigma_{\Psi_*}(X, g) \leq (1 + \varepsilon)\sigma(a)$$

and

$$\mathrm{Vol}(B(x, R)) \geq A_n R^n$$

for all $x \in X$ and all $\varepsilon \leq R/\operatorname{sys}_{\Psi_*}(X, g) \leq \frac{1}{2}$. (Here, $B(x, R)$ denotes the ball around $x \in X$ of radius R with respect to g.) Such cycles are called ε-regular.

In fact, Gromov proved this only for $\mathbb{K} = \mathbb{Z}$ and $\mathbb{K} = \mathbb{Z}_2$ but it remains true for local integer coefficients with the same arguments. On ε-regular geometric cycles one can compare the systolic constant and the minimal volume entropy.

Proposition 2.50 ([Sab06], Proposition 4.1)**.** *Let (X, g, Ψ) be an ε-regular geometric cycle. Then*

$$\lambda_{\Psi_*}(X, g)\,\mathrm{Vol}(X, g)^{1/n} \leq \frac{\sigma_{\Psi_*}(X, g)^{1/n}}{\beta} \log \frac{\sigma_{\Psi_*}(X, g)}{A_n \alpha^n}$$

for all $\alpha \geq \varepsilon, \beta > 0$ with $4\alpha + \beta < \frac{1}{2}$.

From this proposition and Theorem 2.49 it follows directly that

$$\lambda(a) \leq \frac{\sigma(a)^{1/n}}{\beta} \log \frac{\sigma(a)}{A_n \alpha^n}$$

for all $\alpha, \beta > 0$ with $4\alpha + \beta < \frac{1}{2}$. The calculation from the proof of [Sab06], Theorem 5.1 shows:

Corollary 2.51. *There exists a constant $c_n > 0$ such that*

$$\sigma(a) \geq c_n \frac{\lambda(a)^n}{\log^n(1 + \lambda(a))}.$$

Combined with Theorem 2.48 this proves:

Theorem 2.52. *Let M be a connected closed manifold of dimension $n \geq 3$, and let $\phi : \pi_1(M) \twoheadrightarrow \pi$ be an epimorphism. There exists a positive constant c_n depending only on n such that*

$$\sigma_\phi(M) \geq c_n \frac{\lambda_\phi(M)^n}{\log^n(1 + \lambda_\phi(M))}.$$

Special cases of this statement were shown by Sabourau (see [Sab06]). Note that the absolute version of this theorem is in fact Theorem 2.3 for $n \geq 3$. The two-dimensional case was proved in [KatSa05]. Thus, Theorem 2.3 is shown.

Again, let M be a connected closed manifold of dimension n, and let g be a Riemannian metric on M. Denote by $\mathrm{Inj}(M, g)$ the *injectivity radius* of the Riemannian manifold (M, g). The *embolic constant* is defined by

$$\mathrm{Emb}(M) := \inf_g \frac{\mathrm{Vol}(M, g)}{\mathrm{Inj}(M, g)^n},$$

where the infimum is taken over all Riemannian metrics g on M. Berger proved in [Ber80] that

$$\mathrm{Emb}(M) \geq \frac{\mathrm{Vol}(S^n, g_1)}{\mathrm{Inj}(S^n, g_1)^n} = \mathrm{Emb}(S^n)$$

where g_1 denotes the round metric of radius 1 on the sphere. Note that $\mathrm{Inj}(S^n, g_1) = \pi$.

Obviously, $\mathrm{sys}(M, g) \geq 2\,\mathrm{Inj}(M, g)$ since every loop of shorter length lies in a ball of radius less than the injectivity radius and is contractible therefore. Hence $\mathrm{Emb}(M) \geq 2^n \sigma(M)$, and the following corollary is a direct consequence of Theorem 2.3.

Corollary 2.53. *Let M be a connected closed manifold of dimension n. There exists a positive constant c'_n depending only on n such that*

$$\operatorname{Emb}(M) \geq c'_n \frac{\lambda(M)^n}{\log^n(1+\lambda(M))}.$$

A direct proof of this inequality is given in [KatSa05], Theorem 2.3. Therein, Gromov's bound on the volume of balls in regular geometric cycles (Theorem 2.49) is replaced by the following bound by Croke which is valid in every Riemannian manifold. Namely, it is shown in [Cro80], Proposition 14 that

$$\operatorname{Vol}(B(x,R)) \geq A'_n R^n$$

for every $R \leq \operatorname{Inj}(M,g)/2$. Thus, an inequality analogous to the one in Proposition 2.50 with the systolic constant replaced by the embolic constant is valid for all manifolds, not only for regular geometric cycles. From this, Corollary 2.53 follows easily.

Chapter 3

Filling inequalities do not depend on topology

One of the most important curvature-free bounds on the volume of a Riemannian manifold is provided by Gromov's universal systolic inequality

$$\operatorname{sys}(M,g)^n \leq C_n \cdot \operatorname{Vol}(M,g), \tag{$*$}$$

which holds for all connected closed n-dimensional Riemannian manifolds (M,g) that are essential (see paragraph 2.2.5). Recall that M is called *essential* if there exists an aspherical complex K and a map $M \to K$ that does not contract to the $(n-1)$-skeleton of K. For instance, all aspherical manifolds are essential. The *(one-dimensional) systole* $\operatorname{sys}(M,g)$ is the length of the shortest noncontractible loop in M. A proof of $(*)$ can be found in [Gro83], Appendix 2, (B'_1). (Note also the paper [Gut06a], which contains a more detailed version of Gromov's proof.)

If one takes into account the topology of M, then the optimal (smallest) value of the constant $C(M)$ such that an inequality $(*)$ holds for all Riemannian metrics g on M can be improved. This best value is given by the *optimal systolic ratio*

$$\operatorname{SR}(M) := \sup_g \frac{\operatorname{sys}(M,g)^n}{\operatorname{Vol}(M,g)}.$$

(This is just the reciprocal of the one-dimensional systolic constant $\sigma(M)$ from chapter 2.)

Its exact value is known only for three essential manifolds apart from the trivial case of the circle: the two-torus ($\operatorname{SR}(T^2) = 2/\sqrt{3}$, Loewner, unpublished), the real projective plane ($\operatorname{SR}(\mathbb{R}\mathrm{P}^2) = \pi/2$, Pu, [Pu52]), and the Klein bottle ($\operatorname{SR}(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2) = \pi/2\sqrt{2}$, Bavard, [Bav86]).

Nevertheless, it is known that $\mathrm{SR}(M)$ varies with M. For example $\mathrm{SR}(T^n) \geq 1$ by a trivial computation for the flat torus that is obtained from the standard lattice in $\mathbb{R}^n$. But there exist hyperbolic manifolds M_{hyp} with arbitrarily small optimal systolic ratio: Gromov proved an upper bound for the optimal systolic ratio by the simplicial volume

$$\mathrm{SR}(M) \leq c'_n \frac{\log^n(1 + \|M\|)}{\|M\|},$$

which holds for all connected closed manifolds ([Gro83], Theorem 6.4.D', compare also Theorem 2.3 of the present text). For hyperbolic manifolds the simplicial volume is known to be proportional to the volume:

$$\|M_{hyp}\| = R_n^{-1} \cdot \mathrm{Vol}(M_{hyp}),$$

where R_n denotes the volume of the ideal regular n-simplex in hyperbolic space. (This is due to Gromov and Thurston, see [Gro82], page 11.) Since there are hyperbolic manifolds of arbitrarily large volume (just take a sequence of finite coverings with the number of sheets tending towards infinity), the above inequality shows that $\mathrm{SR}(M_{hyp})$ can become arbitrarily small.

To prove the systolic inequality $(*)$, Gromov introduced the *filling radius* and the *filling volume* of a Riemannian manifold. In [Gro83], Theorem 1.2.A and Theorem 2.3 he derived the universal filling inequalities

$$\begin{aligned} \mathrm{FillRad}(M, g)^n &\leq A_n \cdot \mathrm{Vol}(M, g) \quad \text{and} \\ \mathrm{FillVol}(M, g)^{n/(n+1)} &\leq B_n \cdot \mathrm{Vol}(M, g), \end{aligned}$$

that hold for all connected closed Riemannian manifolds. (Recently, Wenger found a shorter proof for the second inequality, see [Wen07].) Again fixing the manifold M, one defines in analogy to the systolic ratio two *optimal filling ratios*:

$$\begin{aligned} \mathrm{FR}(M) &:= \sup_g \frac{\mathrm{FillRad}(M, g)^n}{\mathrm{Vol}(M, g)} \quad \text{and} \\ \mathrm{FV}(M) &:= \sup_g \frac{\mathrm{FillVol}(M, g)^{n/(n+1)}}{\mathrm{Vol}(M, g)}. \end{aligned}$$

These topological invariants are the best values for the constants $A(M)$ and $B(M)$ such that the filling inequalities hold for all Riemannian metrics g on M.

In contrast to the behaviour of the optimal systolic ratio, the main result of this chapter is the following:

Theorem 3.1. *If M and N are two connected closed manifolds of the same dimension $n \geq 3$ and if they are either both orientable or both nonorientable, then*

$$\mathrm{FR}(M) = \mathrm{FR}(N) \quad \textit{and} \quad \mathrm{FV}(M) = \mathrm{FV}(N).$$

This is Theorem 1.7 from the introduction. It will be proved as a part of the more detailed Theorem 3.13. The proof uses an axiomatic approach, that was first introduced by Babenko in [Bab06] to investigate the optimal systolic ratio, and that was applied to further invariants in [Sab06] and in the second chapter of this thesis. Similar notions can already be found in works of Babenko, Katz, and Suciu on systolic freedom, see for example [BabK98], [BabKS98], [KatSu99], and [KatSu01]. The keywords are 'meromorphic map' and '(n, k)-morphism' (see also [CroK03], section 4.3).

The main idea is more or less the same as in the proof of Theorem 2.1 in chapter 2. Both optimal filling ratios fulfill a comparison axiom that roughly says that if there is a degree one map $M \to N$, then $\mathrm{F}(M) \geq \mathrm{F}(N)$. (Here and later on, F will always serve as a placeholder for FR or FV.) Moreover, they also satisfy an extension axiom which says that if one attaches finitely many cells of dimension strictly less than the dimension of the manifold, then the value of the filling ratios does not change. The CW complex obtained in this way is called an *extension* of the original manifold.

In contrast to chapter 2, the invariants under consideration do not depend on the fundamental group. Therefore, homological invariance reduces to constancy: if M and N have the same orientation behaviour, then there exists an extension X of N and a 'degree one' map $M \to X$. (This is a special case of the more general Theorem 2.35 from the preceeding chapter. In the orientable case, this can be seen more directly by using the Hurewicz theorem.) Applying both axioms one sees that

$$\mathrm{F}(M) \geq \mathrm{F}(X) = \mathrm{F}(N).$$

Inverting the roles of M and N gives equality and proves the theorem.

This chapter is organized as follows: first, the definitions of the filling invariants and some useful lemmata are recalled. The main part of the proof can be found in paragraph 3.1.2, where the optimal filling ratios are shown to satisfy the comparison and extension axioms. In section 3.2, the proof of Theorem 3.1 will be concluded. Finally, the results and some open questions are discussed.

Before starting to define the filling invariants, we want to direct the reader's attention to another curvature-free bound on the volume of a Riemannian manifold: the *embolic inequality*. (An extensive overview of curvature-free inequalities is provided by Chapter 7.2 of Berger's book [Ber03] and by the survey article [CroK03].)

Berger and Kazdan proved in [BerK80] that every connected closed Riemannian manifold (M, g) of dimension n fulfills

$$\operatorname{Inj}(M, g)^n \leq E_n \cdot \operatorname{Vol}(M, g),$$

where $\operatorname{Inj}(M, g)$ denotes the *injectivity radius*. Moreover, Berger showed that the best universal constant E_n is given by

$$\frac{\operatorname{Inj}(S^n, g_1)^n}{\operatorname{Vol}(S^n, g_1)} = \frac{\pi^n}{\sigma_n}$$

where σ_n denotes the volume of the n-dimensional round sphere of radius 1 (see [Ber80]).

As before, we may define an *optimal embolic ratio* $\operatorname{ER}(M)$ by taking the supremum of the ratio $\operatorname{Inj}(M, g)^n / \operatorname{Vol}(M, g)$ over all Riemannian metrics on M. (This is the reciprocal of the embolic constant $\operatorname{Emb}(M)$ defined in paragraph 2.5.2.) By Berger's theorem $\operatorname{ER}(M) \leq \operatorname{ER}(S^n)$.

In contrast to the filling ratios, the embolic ratio is not constant. This was shown by Croke in [Cro88] where he proved that $\operatorname{ER}(M) = \operatorname{ER}(S^n)$ if and only if M is homeomorphic to the sphere.

3.1 Optimal filling ratios

In the first paragraph of this section, the filling invariants and ratios are defined. Moreover, important techniques are recalled from Gromov's Filling paper [Gro83]. The second paragraph contains the proofs that both optimal filling ratios satisfy suitable comparison and extension axioms.

3.1.1 Filling radius and filling volume

The filling radius and the filling volume were introduced by Gromov in his Filling paper [Gro83]. Using piecewise smooth Riemannian metrics (see Definition 2.11), their definitions extend to simplicial complexes. Before recalling these definitions, we will focus on the so-called 'universal property' of the Banach space of all bounded functions on some set.

Definition 3.2. Let $f : Y \to X$ be a continuous map between metric spaces. The *dilation* of f is given by

$$\operatorname{dil}(f) := \sup_{y, y' \in Y, y \neq y'} \frac{d(f(y), f(y'))}{d(y, y')},$$

i. e. it is the smallest Lipschitz constant for f. We define the *dilation of f with respect to y* as

$$\operatorname{dil}(f,y) := \sup_{y' \in Y, y \neq y'} \frac{d(f(y), f(y'))}{d(y,y')}.$$

For a set V, let $L^\infty(V)$ denote the Banach space of all bounded functions $f : V \to \mathbb{R}$ with the uniform norm $\|f\|_\infty := \sup_{v\in V} |f(v)|$. It has the following *universal property*:

Lemma 3.3. *Let $Y \subset X$ be a nonempty subspace of a metric space, and let $f : Y \to L^\infty(V)$ be a Lipschitz map. Then the map $F : X \to L^\infty(V)$ defined by*

$$F_x(v) := \inf_{y\in Y}(f_y(v) + \operatorname{dil}(f,y) \cdot d(x,y))$$

is a Lipschitz continuous extension of f with $\operatorname{dil}(F) = \operatorname{dil}(f)$ *and* $\operatorname{dil}(F,y) = \operatorname{dil}(f,y)$ *for all $y \in Y$.*

The existence of an extension with the same dilation as f was shown by Gromov (see [Gro83], page 8). However, he used the extension

$$F'_x(v) := \inf_{y\in Y}(f_y(v) + \operatorname{dil}(f) \cdot d(x,y))$$

which in general does not have the property $\operatorname{dil}(F',y) = \operatorname{dil}(f,y)$ for all $y \in Y$. Since in paragraph 3.1.2 we will need that the dilation with respect to points in Y remains the same, we give a complete proof of Lemma 3.3.

Proof. A priori, F is a map to the space of functions from V to $[-\infty, \infty)$.

First note that F extends f, which can be seen as follows: for any $y \in Y$ one finds

$$\begin{aligned} 0 \le f_y(v) - F_y(v) &= \sup_{y'\in Y}(f_y(v) - f_{y'}(v) - \operatorname{dil}(f,y')d(y,y')) \\ &\le \sup_{y'\in Y}(\operatorname{dil}(f,y')d(y,y') - \operatorname{dil}(f,y')d(y,y')) = 0 \end{aligned}$$

because $\sup_{v\in V} |f_y(v) - f_{y'}(v)| = d(f_y, f_{y'}) \le \operatorname{dil}(f,y')d(y,y')$. Hence $F_y \equiv f_y$.

Furthermore, one has

$$\begin{aligned} F_x(v) &= \inf_{y\in Y}(f_y(v) + \operatorname{dil}(f,y)d(x,y)) \\ &\le \inf_{y\in Y}(f_y(v) + \operatorname{dil}(f,y)d(x',y) + \operatorname{dil}(f,y)d(x,x')) \\ &\le \inf_{y\in Y}(f_y(v) + \operatorname{dil}(f,y)d(x',y) + \operatorname{dil}(f)d(x,x')) \\ &= F_{x'}(v) + \operatorname{dil}(f)d(x,x') \end{aligned}$$

by the triangle inequality. This shows that $d(F_x, F_{x'}) = \|F_x - F_{x'}\|_\infty \leq \operatorname{dil}(f)d(x,x')$. In particular, F_x is bounded for every $x \in X$, and F is Lipschitz continuous with $\operatorname{dil}(F) \leq \operatorname{dil}(f)$. The converse inequality between the dilations is obvious since F extends f.

Note that one always has $d(f_y, f_{y'}) \leq \operatorname{dil}(f,y)d(x,y) + \operatorname{dil}(f,y')d(x,y')$ for any $x \in X$. Hence

$$\begin{aligned} d(F_y, F_x) &= \sup_{v\in V}\left|\sup_{y'\in Y}(f_y(v) - f_{y'}(v) - \operatorname{dil}(f,y')d(x,y'))\right| \\ &\leq \operatorname{dil}(f,y)d(x,y), \end{aligned}$$

which proves the claim $\operatorname{dil}(F,y) = \operatorname{dil}(f,y)$. □

Now, assume V to be a connected finite simplicial complex of dimension n. Let $\mathbb{K}$ denote $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_2$.

Definition 3.4. Let $\iota : V \hookrightarrow X$ be a topological embedding into a metric space X. The $\mathbb{K}$-*filling radius* of ι is defined as

$$\operatorname{FillRad}_{\mathbb{K}}(\iota : V \hookrightarrow X) := \inf\{r > 0 | \iota_* : H_n(V;\mathbb{K}) \to H_n(U_r(\iota V);\mathbb{K}) \text{ is zero}\},$$

where $U_r(\iota V)$ denotes the r-neighborhood of the image ιV in X.

To define the filling volume, we need to specify a choice of volume for singular Lipschitz chains, i. e. singular chains whose simplices are Lipschitz continuous. Following Gromov ([Gro83], page 11 and section 4.1), we define the volume of a singular Lipschitz simplex $\sigma : \Delta^n \to X$ in a metric space X by

$$\operatorname{Vol}(\sigma) := \inf\{\operatorname{Vol}(\Delta^n, g) | \sigma : (\Delta^n, g) \to X \text{ nonexpanding}\},$$

where g runs over all Riemannian metrics on Δ^n. If $c = \sum_i r_i\sigma_i \in C_n(X;\mathbb{K})$ is a singular Lipschitz chain, its volume is defined as

$$\operatorname{Vol}(c) := \textstyle\sum_i |r_i| \operatorname{Vol}(\sigma_i).$$

(In the case of $\mathbb{K} = \mathbb{Z}_2$ the ‘absolute value’ $|r|$ is understood as zero for the trivial element $r = 0$ and as one for $r \neq 0$.) For a Lipschitz cycle $z \in C_n(X;\mathbb{K})$ the $\mathbb{K}$-*filling volume* is given by

$$\operatorname{FillVol}_{\mathbb{K}}(z) := \inf_{\partial c = z} \operatorname{Vol}(c),$$

where the infimum is taken over all Lipschitz chains $c \in C_{n+1}(X;\mathbb{K})$ with boundary $\partial c = z$.

In case that X is a Banach space, this definition corresponds to the *hypereuclidean volume* (see [Gro83], page 33). This choice of volume has the following nice consequence:

Proposition 3.5 ([Gro83], Proposition 2.2.A)**.** *Let (M, g) be a connected closed orientable Riemannian manifold of dimension $n \geq 2$, and let W be an orientable $(n+1)$-dimensional manifold with boundary $\partial W = M$, for example $W = M \times [0, \infty)$. Then $\mathrm{FillVol}_{\mathbb{Z}}(M, g)$ equals the infimum of the volumes $\mathrm{Vol}(W, g')$ over all complete Riemannian metrics g' on W such that $d_{g'}|M \geq d_g$ holds for the induced path metrics.*

From now on, all singular simplices are assumed to be Lipschitz continuous.

The filling volume of an embedding $\iota : V \hookrightarrow X$ will be the filling volume of a canonical top-dimensional homology class. If V is a manifold, then the fundamental class provides such a class. In the case of simplicial complexes, we have to require the existence of a 'fundamental class'.

Definition 3.6. A connected finite n-dimensional simplicial complex V will be called $\mathbb{K}$-*orientable* if $H_n(V; \mathbb{K}) \cong \mathbb{K}$. As in the case of manifolds, a generator $[V]_{\mathbb{K}} \in H_n(V; \mathbb{K})$ is called *fundamental class*. In the case $\mathbb{K} = \mathbb{Q}$, we will always assume that $[V]_{\mathbb{Q}}$ lies in the integral lattice $H_n(V; \mathbb{Z}) \subset H_n(V; \mathbb{Q})$.

The last part of the definition simply says that for $\mathbb{K} = \mathbb{Q}$ one takes the integral fundamental class and allows filling cycles with rational coefficients.

Definition 3.7. Let V be a connected finite $\mathbb{K}$-orientable simplicial complex. If $\iota : V \hookrightarrow X$ is a Lipschitz embedding into a metric space X, then one defines the $\mathbb{K}$-*filling volume* of ι as

$$\mathrm{FillVol}_{\mathbb{K}}(\iota : V \hookrightarrow X) := \mathrm{FillVol}_{\mathbb{K}}(\iota_* z)$$

where z is a Lipschitz representative of $[V]_{\mathbb{K}}$.

This is independent of the representing fundamental cycle. Namely, let z and z' be two Lipschitz representatives of $[V]_{\mathbb{K}}$. Then there exists a Lipschitz chain $b \in C_{n+1}(V; \mathbb{K})$ such that $z' = z + \partial b$ (with $n = \dim V$). Therefore, it suffices to see that $\mathrm{Vol}(\iota_* b) = 0$. But $\mathrm{Vol}(\iota_* b) \leq L^{n+1} \cdot \mathrm{Vol}(b)$ where $L > 0$ is the Lipschitz constant of ι, and $\mathrm{Vol}(b) = 0$ since b is an $(n+1)$-cycle in an n-dimensional complex.

For a compact metric space (V, d) the *Kuratowski embedding*

$$\begin{aligned} \iota : V &\hookrightarrow L^\infty(V), \\ v &\mapsto d(v, \_) \end{aligned}$$

is an isometric embedding by the triangle inequality.

Let g be a piecewise smooth Riemannian metric on a connected finite ($\mathbb{K}$-orientable) simplicial complex V. With the induced path metric d_g, the complex V becomes a metric space. The associated Kuratowski embedding will be denoted by $\iota_g : V \hookrightarrow L^\infty(V)$. For this embedding the filling invariants are denoted by

$$\begin{aligned} \mathrm{FillRad}_{\mathbb{K}}(V,g) &:= \mathrm{FillRad}_{\mathbb{K}}(\iota_g) \quad \text{and} \\ \mathrm{FillVol}_{\mathbb{K}}(V,g) &:= \mathrm{FillVol}_{\mathbb{K}}(\iota_g). \end{aligned}$$

Definition 3.8. We define the *optimal filling ratios*

$$\begin{aligned} \mathrm{FR}_{\mathbb{K}}(V) &:= \sup_g \frac{\mathrm{FillRad}_{\mathbb{K}}(V,g)^n}{\mathrm{Vol}(V,g)} \quad \text{and} \\ \mathrm{FV}_{\mathbb{K}}(V) &:= \sup_g \frac{\mathrm{FillVol}_{\mathbb{K}}(V,g)^{n/(n+1)}}{\mathrm{Vol}(V,g)}. \end{aligned}$$

These numbers are the smallest constants such that the filling inequalities

$$\begin{aligned} \mathrm{FillRad}_{\mathbb{K}}(V,g)^n &\le A(V) \cdot \mathrm{Vol}(V,g) \quad \text{and} \\ \mathrm{FillVol}_{\mathbb{K}}(V,g)^{n/(n+1)} &\le B(V) \cdot \mathrm{Vol}(V,g) \end{aligned}$$

are satisfied for all Riemannian metrics g on V. By Gromov's universal filling inequalities, one knows that for manifolds M there are upper bounds for $\mathrm{FR}_{\mathbb{K}}(M)$ and $\mathrm{FV}_{\mathbb{K}}(M)$ depending only on the dimension. In particular, both filling ratios are finite for manifolds.

3.1.2 Axioms for filling ratios

The content of this paragraph is the proof that the filling ratios satisfy a comparison axiom and an extension axiom. A numerical invariant for simplicial complexes is said to fulfill a comparison axiom if the existence of a 'degree one' map $V \to W$ implies an inequality between the respective values of this invariant. It has to be specified which maps are of 'degree one' (for instance $(n,1)$-monotone maps as in Definition 2.9), and often there are further assumptions on the maps, like surjectivity on fundamental groups or on some homology groups. (Other examples can be found in chapters 2 and 4.)

An extension axiom is satisfied if attaching cells with dimension less than the topdimension does not change the value of the considered invariant. Again, further examples are provided by chapters 2 and 4.

The proof of Theorems 3.1 and 3.13 actually uses only these two axioms and no other properties of the filling ratios. It will be given in section 3.2.

Lemma 3.9 (Comparison axiom for FR and FV)**.** *If $f : V \to W$ is an $(n,1)$-monotone map between connected finite ($\mathbb{K}$-orientable) simplicial complexes of dimension n such that $f_* : H_n(V;\mathbb{K}) \twoheadrightarrow H_n(W;\mathbb{K})$ is surjective, then*

$$\mathrm{F}_{\mathbb{K}}(V) \geq \mathrm{F}_{\mathbb{K}}(W)$$

for both $\mathrm{F} = \mathrm{FR}$ *and* $\mathrm{F} = \mathrm{FV}$.

Proof. Let g_2 be a Riemannian metric on W. Choose a Riemannian metric g_1 on V and set $g_1^t := f^* g_2 + t^2 g_1$. This is again a Riemannian metric on V. One may choose $t > 0$ so small that

$$\mathrm{Vol}(V, g_1^t) \leq \mathrm{Vol}(W, g_2) + \varepsilon$$

for a given $\varepsilon > 0$. Denote the corresponding Kuratowski embeddings by ι_1^t and ι_2.

Since $f : (V, g_1^t) \to (W, g_2)$ is nonexpanding, there is a nonexpanding map $F : L^\infty(V) \to L^\infty(W)$ that extends $\iota_2 \circ f$ by the universal property of $L^\infty(W)$. (Think of $V \subset L^\infty(V)$ via ι_1^t.) Thus, there is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \iota_1^t}\big\downarrow & & \big\downarrow{\scriptstyle \iota_2} \\ L^\infty(V) & \xrightarrow{\;F\;} & L^\infty(W) \end{array}$$

which gives for every $r > 0$

$$\begin{array}{ccc} H_n(V;\mathbb{K}) & \xrightarrow{\;f_*\;} & H_n(W,\mathbb{K}) \\ {\scriptstyle (\iota_1^t)_*}\big\downarrow & & \big\downarrow{\scriptstyle (\iota_2)_*} \\ H_n(U_r(\iota_1^t V);\mathbb{K}) & \xrightarrow{\;F_*\;} & H_n(U_r(\iota_2 W);\mathbb{K}) \end{array}$$

Therefore $\mathrm{FillRad}_{\mathbb{K}}(V, g_1^t) \geq \mathrm{FillRad}_{\mathbb{K}}(W, g_2)$, and this gives the desired inequality $\mathrm{FR}_{\mathbb{K}}(V) \geq \mathrm{FR}_{\mathbb{K}}(W)$.

Let $z \in C_n(V;\mathbb{K})$ represent $[V]_{\mathbb{K}}$. Then $f_* z$ represents $f_*[V]_{\mathbb{K}} = \pm[W]_{\mathbb{K}}$ (in the case $\mathbb{K} = \mathbb{Q}$ look at the local degree and use that f is $(n,1)$-monotone), and one finds

$$\mathrm{FillVol}_{\mathbb{K}}((\iota_2)_*(f_* z)) = \mathrm{FillVol}_{\mathbb{K}}(F_*((\iota_1^t)_* z)) \leq \mathrm{FillVol}_{\mathbb{K}}((\iota_1^t)_* z).$$

Hence, $\mathrm{FillVol}_{\mathbb{K}}(W, g_2) \leq \mathrm{FillVol}_{\mathbb{K}}(V, g_1^t)$ and $\mathrm{FV}_{\mathbb{K}}(W) \leq \mathrm{FV}_{\mathbb{K}}(V)$. □

The proof of the extension axiom is more complicated. We will frequently use the following fact, which is a direct consequence of the universal property (Lemma 3.3).

Corollary 3.10. *If $i : (V, d_g) \hookrightarrow L^\infty(S)$ is an isometric embedding with S any set, then*

$$\mathrm{FillRad}_{\mathbb{K}}(i) = \mathrm{FillRad}_{\mathbb{K}}(V, g) \quad \textit{and}$$
$$\mathrm{FillVol}_{\mathbb{K}}(i) = \mathrm{FillVol}_{\mathbb{K}}(V, g).$$

We first investigate the extension axiom for the filling radius.

Proposition 3.11 (Extension axiom for FR)**.** *Let V' be an* extension *of V, that means V' is obtained from V by attaching finitely many cells of dimension less than $n := \dim V$. Then*

$$\mathrm{FR}_{\mathbb{K}}(V') = \mathrm{FR}_{\mathbb{K}}(V).$$

Proof. Since the inclusion $i : V \hookrightarrow V'$ is $(n, 1)$-monotone and induces an isomorphism

$$i_* : H_n(V; \mathbb{K}) \xrightarrow{\cong} H_n(V'; \mathbb{K}),$$

the comparison axiom implies $\mathrm{FR}_{\mathbb{K}}(V) \geq \mathrm{FR}_{\mathbb{K}}(V')$.

To prove the converse inequality, it suffices by induction to attach one k-cell at a time (with $k < n$). Let $h : S^{k-1} \to V$ be the (simplicial) attaching map, and let g be a Riemannian metric on V.

Consider all $R > 0$ such that $h : (S^{k-1}, g_R) \to (V, g)$ is nonexpanding, where g_R denotes the round metric with radius R on S^{k-1}. (In the case $k = 1$ choose $R > 0$ such that $23\pi R \geq d_g(h(-1), h(1))$.) Define Riemannian metrics g'_R on V' by thinking of V' as

$$V \quad \cup_h \quad (S^{k-1} \times [-1, 0]) \quad \cup \quad (S^{k-1} \times [0, 6]) \quad \cup \quad S^k_+$$

with S^k_+ a k-dimensional hemisphere and taking

$$g, \quad (-sh^*g + (1+s)g_R) \oplus \pi R ds^2, \quad g_R \oplus \pi R ds^2, \quad g_R$$

on the respective parts. Here, the last g_R denotes the round metric of radius R on the k-dimensional hemisphere. (For $k = 1$ use $5\pi R ds^2$ on $S^0 \times [-1, 0]$.)

Then the induced distance functions of g and g'_R coincide on V:

$$d_{g'_R}|V \equiv d_g,$$

i. e. the inclusion $i : (V, d_g) \hookrightarrow (V', d_{g'_R})$ is isometric as a map of metric spaces. Hence by Corollary 3.10, we have

$$\mathrm{FillRad}_{\mathbb{K}}(V, g) = \mathrm{FillRad}_{\mathbb{K}}(\iota_{g'_R} \circ i)$$

and therefore

$$\mathrm{FillRad}_{\mathbb{K}}(V', g'_R) \leq \mathrm{FillRad}_{\mathbb{K}}(V, g)$$

since for any $r > 0$ the r-neighborhood of $\iota_{g'_R} V'$ is larger than the one of $\iota_{g'_R} V$.

Furthermore, note that the inclusion $((S^{k-1} \times [1,6]) \cup S^k_+, d_{g'_R}) \subset (V', d_{g'_R})$ is isometric with respect to the induced path metrics. We will write $V'_R := \iota_{g'_R} V'$, $V_R := \iota_{g'_R} V$ and so on.

Next, we restrict our attention to radii $R > \mathrm{FillRad}_{\mathbb{K}}(V, g)$. Note that the r-neighborhood of $((S^{k-1} \times [1,6]) \cup S^k_+)_R$ does not meet the r-neighborhood of V_R for any $r < R$. Thus, we may think of $U_r(((S^{k-1} \times [1,6]) \cup S^k_+)_R)$ as some kind of tubular neighborhood and try to retract it to its core. This core is k-dimensional and plays therefore no role for n-dimensional homology. Hence, if $H_n(V; \mathbb{K})$ vanishes in $U_r(V'_R)$, then also in $U_r(V_R)$. We now concretize this idea.

Choose one of the radii R with $R > \mathrm{FillRad}_{\mathbb{K}}(V, g)$ and with $R > 1$ as reference radius and call it R_0. Since the Kuratowski embedding $\iota_0 := \iota_{g'_{R_0}}$ is not differentiable on the attached k-cell, we need to choose a smooth approximation to get an actual tubular neighborhood. Therefore, let

$$\iota : (S^{k-1} \times [3 + \delta/\pi R_0, 6]) \cup S^k_+ \hookrightarrow L^\infty(V')$$

be a smooth embedding such that

$$d(\iota(x), \iota_0(x)) < \delta$$

for all $x \in (S^{k-1} \times [3 + \delta/\pi R_0, 6]) \cup S^k_+$. Using the Kuratowski embedding ι_0 on $V' \setminus ((S^{k-1} \times [3,6]) \cup S^k_+)$ and linear interpolation on $S^{k-1} \times [3, 3 + \delta/\pi R_0]$, this defines a Lipschitz map $\iota : (V', g'_{R_0}) \to L^\infty(V')$ which is 3δ-close to ι_0. Denote by $K := \mathrm{dil}(\iota)$ its Lipschitz constant, and think of ι as a map $V'_{R_0} \to L^\infty(V')$.

With respect to $y \in V'_{R_0} \setminus ((S^{k-1} \times [2,6]) \cup S^k_+)_{R_0}$ the dilation $\mathrm{dil}(\iota, y)$ of ι is at most $1 + \delta$. This holds because

$$\begin{aligned} \frac{d(\iota(y), \iota(y'))}{d(y, y')} &\leq \frac{d(y, y') + d(y', \iota(y'))}{d(y, y')} \\ &\leq 1 + \frac{3\delta}{d(y, y')} \leq 1 + \delta \end{aligned}$$

for every $y' \in ((S^{k-1} \times [3,6]) \cup S^k_+)_{R_0}$ and because ι is isometric on $V'_{R_0} \setminus ((S^{k-1} \times [3,6]) \cup S^k_+)_{R_0}$.

Let $F : L^\infty(V') \to L^\infty(V')$ be an extension of $\iota : V'_{R_0} \to L^\infty(V')$ as in Lemma 3.3. Then

$$F(U_r(V'_{R_0} \setminus ((S^{k-1} \times [2,6]) \cup S^k_+)_{R_0})) \subset U_{r(1+\delta)}(V'_{R_0} \setminus ((S^{k-1} \times [2,6]) \cup S^k_+)_{R_0})$$

by the fact that $\operatorname{dil}(F,y)=\operatorname{dil}(\iota,y)\le 1+\delta$ for all $y\in V'_{R_0}\setminus((S^{k-1}\times[2,6])\cup S^k_+)_{R_0}$.

Denote by $E:=\iota((S^{k-1}\times[3+\delta/\pi R_0,6])\cup S^k_+)$ the smooth part of $\iota(V')$. Let $\nu(E)\to E$ be its normal bundle, and let $\tau: N\hookrightarrow L^\infty(V')$ be a tubular neighborhood for the trivial spray (i. e. the exponential map is given by vector addition) where $N\subset\nu(E)$ is open, fiberwise starshaped with respect to the zero section (thus it allows a deformation retraction to the zero section), and fiberwise bounded by $\operatorname{FillRad}_{\mathbb{K}}(V,g)/2$. Furthermore, assume that

$$\tau(N|\iota((S^{k-1}\times[3\tfrac{1}{2},6])\cup S^k_+))\subset U_{r_0}(((S^{k-1}\times[2,6])\cup S^k_+)_{R_0})$$

with $r_0:=\operatorname{FillRad}_{\mathbb{K}}(V',g'_{R_0})$. By compactness there is an $\varepsilon>0$ such that

$$F(U_\varepsilon(((S^{k-1}\times[4,6])\cup S^k_+)_{R_0}))\subset\tau(N|\iota((S^{k-1}\times[3\tfrac{1}{2},6])\cup S^k_+)).$$

Moreover, choosing $\varepsilon<(r_0-3\delta)/K$ guarantees that

$$F(U_\varepsilon((S^{k-1}\times[2,4])_{R_0}))\subset U_{r_0}((S^{k-1}\times[2,4])_{R_0}).$$

Claim. *There is an $R\ge R_0$ such that $H_n(V;\mathbb{K})$ vanishes for all real numbers $r>\operatorname{FillRad}_{\mathbb{K}}(V',g'_R)$ inside*

$$U_r(V'_{R_0}\setminus((S^{k-1}\times[2,6])\cup S^k_+)_{R_0})\ \cup\ U_\varepsilon(((S^{k-1}\times[2,6])\cup S^k_+)_{R_0}).$$

Proof of the claim. Choose $C\le 1$ such that $C\pi\operatorname{FillRad}_{\mathbb{K}}(V,g)<\varepsilon$, and choose

$$R\ge\max(R_0/C,\operatorname{diam}(V'_{R_0})/C\pi).$$

The identity on V' gives a nonexpanding homeomorphism $f: V'_R\to V'_{R_0}$. Let $\tilde F: L^\infty(V')\to L^\infty(V')$ be an extension of f as in Lemma 3.3. Then, for any $\operatorname{FillRad}_{\mathbb{K}}(V',g'_R)<r\le\pi\operatorname{FillRad}_{\mathbb{K}}(V,g)$ one finds

$$\tilde F(U_r(((S^{k-1}\times[2,6])\cup S^k_+)_R))\subset U_\varepsilon(((S^{k-1}\times[2,6])\cup S^k_+)_{R_0})$$

since $\operatorname{dil}(\tilde F,y)=\operatorname{dil}(f,y)\le C$ for any $y\in((S^{k-1}\times[2,6])\cup S^k_+)_R$. This follows from

$$\frac{d(f(y),f(y'))}{d(y,y')}\le C$$

which holds for all $y'\in V'_R$ since on $((S^{k-1}\times[1,6])\cup S^k_+)_R$ the map f is the contraction by the factor $R_0/R\le C$, and for the other y' note that the numerator is bounded from above by $\operatorname{diam}(V'_{R_0})$ and the denominator is bounded from below by πR.

Hence $\tilde F$ maps $U_r(V'_R)$ to

$$U_r(V'_{R_0}\setminus((S^{k-1}\times[2,6])\cup S^k_+)_{R_0})\cup U_\varepsilon(((S^{k-1}\times[2,6])\cup S^k_+)_{R_0}).$$

Therefore $H_n(V;\mathbb{K})$ vanishes therein, and the claim is proved. □

Note that $\mathrm{FillRad}_{\mathbb{K}}(V', g'_R) \geq \mathrm{FillRad}_{\mathbb{K}}(V', g'_{R_0}) = r_0$ by the universal property. Therefore, applying F shows that $H_n(V;\mathbb{K})$ also vanishes in

$$U_{r(1+\delta)}(V'_{R_0} \setminus ((S^{k-1} \times [4,6]) \cup S^k_+)_{R_0}) \cup \tau(N|\iota((S^{k-1} \times [3\tfrac{1}{2}, 6]) \cup S^k_+))$$

for all $\delta > 0$ and $r > \mathrm{FillRad}_{\mathbb{K}}(V', g'_R)$. Using the tubular neighborhood retraction (this is where the fiberwise bound on N comes in) and a Mayer-Vietoris argument one sees that $H_n(V;\mathbb{K})$ maps to zero in

$$U_{r(1+\delta)}((V' \setminus \mathring{S}^k_+)_{R_0}).$$

Since this holds for all $\delta > 0$ and $r > \mathrm{FillRad}_{\mathbb{K}}(V', g'_R)$ it follows that

$$\mathrm{FillRad}_{\mathbb{K}}(V' \setminus \mathring{S}^k_+, g'_{R_0}) \leq \mathrm{FillRad}_{\mathbb{K}}(V', g'_R).$$

The retraction $(V' \setminus \mathring{S}^k_+, g'_{R_0}) \to (V, g)$ is nonexpanding by the choice of g'_{R_0}, therefore

$$\mathrm{FillRad}_{\mathbb{K}}(V, g) \leq \mathrm{FillRad}_{\mathbb{K}}(V' \setminus \mathring{S}^k_+, g'_{R_0}) \leq \mathrm{FillRad}_{\mathbb{K}}(V', g'_R).$$

Note that we have actually proved that $\mathrm{FillRad}_{\mathbb{K}}(V, g) = \mathrm{FillRad}_{\mathbb{K}}(V', g'_R)$.

Since $\mathrm{Vol}(V, g) = \mathrm{Vol}(V', g'_R)$, one gets $\mathrm{FR}_{\mathbb{K}}(V) \leq \mathrm{FR}_{\mathbb{K}}(V')$. This finishes the proof of Proposition 3.11. □

To finish this paragraph, we will proof the extension axiom for FV. Note that an extension of a $\mathbb{K}$-orientable simplicial complex is again $\mathbb{K}$-orientable and has the same fundamental class.

Lemma 3.12 (Extension axiom for FV)**.** *Let V' be an extension of V. Then*

$$\mathrm{FV}_{\mathbb{K}}(V') = \mathrm{FV}_{\mathbb{K}}(V).$$

Proof. Since the inclusion $i : V \hookrightarrow V'$ is $(n, 1)$-monotone and induces an isomorphism in n-dimensional homology, the inequality $\mathrm{FV}_{\mathbb{K}}(V) \geq \mathrm{FV}_{\mathbb{K}}(V')$ holds by the comparison axiom.

Let g be a Riemannian metric on V, and extend it over V' such that the inclusion $i : V \hookrightarrow V'$ is isometric as a map of metric spaces (see the proof of Proposition 3.11). Call this Riemannian metric g'. Then $\iota_{g'} \circ i : V \hookrightarrow L^\infty(V')$ is an isometric embedding and $\mathrm{FillVol}_{\mathbb{K}}(V, g) = \mathrm{FillVol}_{\mathbb{K}}(\iota_{g'} \circ i)$ by Corollary 3.10. Choose a Lipschitz chain $z \in C_n(V;\mathbb{K})$ that represents $[V]_{\mathbb{K}}$. Then $i_* z$ is a Lipschitz chain that represents $[V']_{\mathbb{K}}$ and

$$\mathrm{FillVol}_{\mathbb{K}}(V, g) = \mathrm{FillVol}_{\mathbb{K}}(\iota_{g'*}(i_* z)) = \mathrm{FillVol}_{\mathbb{K}}(V', g').$$

Since $\mathrm{Vol}(V, g) = \mathrm{Vol}(V', g')$, the inequality $\mathrm{FV}_{\mathbb{K}}(V) \leq \mathrm{FV}_{\mathbb{K}}(V')$ follows. □

3.2 Constancy of the optimal filling ratios

In this section, we will prove the main theorem of this chapter, which states that both filling ratios FR and FV depend only on the dimension and orientability.

Theorem 3.13. *Let M and N be two connected closed manifolds of dimension $n \geq 3$. If either both are orientable or both are nonorientable, then*

$$\mathrm{F}_{\mathbb{K}}(M) = \mathrm{F}_{\mathbb{K}}(N)$$

for both $\mathrm{F} = \mathrm{FR}$ *and* $\mathrm{F} = \mathrm{FV}$. *If N is nonorientable, then*

$$\mathrm{F}_{\mathbb{Z}_2}(M) \leq \mathrm{F}_{\mathbb{Z}_2}(N).$$

This theorem obviously includes Theorem 3.1. The proof relies on the following observation which is a special case of Theorem 2.35 for the trivial group $\pi = 1$.

Corollary 3.14. *Let M and N be two connected closed manifolds of dimension $n \geq 3$. If either both M and N are orientable or if N is nonorientable, then there exists an extension V of M and an $(n, 1)$-monotone map $h : N \to V$ with $h_*[N]_{\mathbb{K}} = i_*[M]_{\mathbb{K}} = [V]_{\mathbb{K}}$, where $i : M \hookrightarrow V$ is the inclusion.*

Note that the classifying space of the trivial group is (homotopy equivalent to) a point. Hence, the conditions from Theorem 2.35 are reduced to this simple requirement on the orientations of M and N.

Although this corollary has already been proved, we will illuminate the case where M is orientable from a slightly different angle and give a short and concise proof of the corollary under this assumption.

Proof for orientable M. In a first step, we choose an $(n, 1)$-monotone map $N \to S^n$. For example, we could choose a ball inside N and take the map that contracts everything outside this ball to a point.

We may assume that M is a subcomplex of a contractible complex K (the classifying space of the trivial group). In fact, choose a CW decomposition of M with a single zero-cell and define K as the cone over M where the one-cell that connects the zero-cell of M with the apex is collapsed. Then K is finite and obtained from M by attachment of positive-dimensional cells only.

Denote by $M(k)$ the union of M with the k-skeleton of K. Then the complex $M(n)$ and the pair $(M(n), M(n-1))$ are both $(n-1)$-connected. Since $M(n-1)$ is simply-connected (because $n \geq 3$), the two vertical Hurewicz homomorphisms on the right-hand side of the following diagram are isomorphisms:

$$\begin{array}{ccccccc} & & \pi_n(M(n-1)) & \longrightarrow & \pi_n(M(n)) & \longrightarrow & \pi_n(M(n), M(n-1)) \\ & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0 & \longrightarrow & H_n(M(n-1);\mathbb{Z}) & \longrightarrow & H_n(M(n);\mathbb{Z}) & \longrightarrow & H_n(M(n), M(n-1);\mathbb{Z}) \end{array}$$

A diagram chase shows that the first vertical Hurewicz homomorphism $\pi_n(M(n-1)) \to H_n(M(n-1);\mathbb{Z})$ is surjective. Therefore, there is a map $s : S^n \to M(n-1)$ such that $s_*[S^n]_{\mathbb{K}} = i_*[M]_{\mathbb{K}}$. (Here, the orientability of M is of crucial importance. If M was nonorientable, then $H_n(M(n-1);\mathbb{Z})$ would be zero and there would be no map $S^n \to M(n-1)$ of degree one.)

Note that $V := M(n-1)$ is an extension of M. By Lemma 2.5, the map s is homotopic to an $(n,1)$-monotone map. The composition with the chosen map $N \to S^n$ defines the map $h : N \to V$. □

The argument for the case of nonorientable M is similar but more involved. The reader is referred to Theorem 2.35.

Proof of Theorem 3.13. By Corollary 3.14 there is an extension V of M and an $(n,1)$-monotone map $h : N \to V$ with $h_*[N]_{\mathbb{K}} = [V]_{\mathbb{K}}$. Therefore,

$$\mathrm{F}_{\mathbb{K}}(N) \geq \mathrm{F}_{\mathbb{K}}(V) = \mathrm{F}_{\mathbb{K}}(M)$$

by the comparison axiom and the extension axiom from paragraph 3.1.2. Changing the roles of M and N gives equality in the cases where M and N are both orientable or both nonorientable. □

Note that the proof used only the axioms and no other properties of the filling ratios. Thus, it works for all numerical invariants of simplicial complexes that satisfy the comparison and extension axiom. Since we do not know another invariant satisfying these axioms, we refrained from stating Theorem 3.13 in this generality.

As a consequence of Theorem 3.13, we have eight distinguished positive numbers in each dimension $n \geq 3$ that satisfy the following inequalities:

$$\mathrm{FR}^{or}_{\mathbb{Q}}(n) \leq \mathrm{FR}^{or}_{\mathbb{Z}}(n) \geq \mathrm{FR}^{or}_{\mathbb{Z}_2}(n) \leq \mathrm{FR}^{non\text{-}or}_{\mathbb{Z}_2}(n) \text{ and}$$
$$\mathrm{FV}^{or}_{\mathbb{Q}}(n) \leq \mathrm{FV}^{or}_{\mathbb{Z}}(n) \geq \mathrm{FV}^{or}_{\mathbb{Z}_2}(n) \leq \mathrm{FV}^{non\text{-}or}_{\mathbb{Z}_2}(n).$$

The first two inequalities of each line are direct consequences of the definitions, the last inequalities stem from Theorem 3.13. We do not know about strict inequalities or equalities.

The exact values of these constants are not known. Actually, the filling radius is only known in the following cases:

$$\begin{aligned}
\operatorname{FillRad}_{\mathbb{K}}(\mathbb{RP}^n, g_1) &= \pi/6, \\
\operatorname{FillRad}_{\mathbb{K}}(S^n, g_1) &= \tfrac{1}{2}\arccos(-\tfrac{1}{n+1}), \\
\operatorname{FillRad}_{\mathbb{Z}}(\mathbb{CP}^2, g_{FS}) &= \tfrac{1}{2}\arccos(-\tfrac{1}{3}), \\
\operatorname{FillRad}_{\mathbb{Q}}(\mathbb{CP}^k, g_{FS}) &= \tfrac{1}{2}\arccos(-\tfrac{1}{3}),
\end{aligned}$$

where g_1 denotes the round metrics of constant curvature one and g_{FS} the Fubini-Study metric, and $\mathbb{K}$ stands for all possible coefficient rings chosen from $\mathbb{Z}$, $\mathbb{Z}_2$, and $\mathbb{Q}$. (See [Kat83] and [Kat91].) Note also that

$$\operatorname{FillRad}_{\mathbb{Z}}(\mathbb{CP}^3, g_{FS}) > \operatorname{FillRad}_{\mathbb{Q}}(\mathbb{CP}^3, g_{FS})$$

by [Kat91], Theorem 0.3.

By computing the ratio $\operatorname{FillRad}_{\mathbb{K}}(M, g)^n / \operatorname{Vol}(M, g)$ for these examples, it follows that the round projective space is not maximizing FR in dimensions $n \neq 1$ and that the standard complex space is not maximizing in even dimensions $n \geq 4$. By this calculations, one is tempted to conjecture that the supremum that defines FR is a maximum and that the round metric on the sphere maximizes this ratio.

For the filling volume the situation is far more vague: one does not know its exact value for a single Riemannian manifold, not even for the circle. (See [Kat07], chapter 8.)

In the case of surfaces the comparison axiom has the following consequence.

Corollary 3.15. *Denote the connected closed surface of genus g by Σ_g in the orientable case and by N_g in the nonorientable case. Then*

$$\mathrm{F}_{\mathbb{K}}(\Sigma_g) \leq \mathrm{F}_{\mathbb{K}}(\Sigma_{g+1})$$

and

$$\mathrm{F}_{\mathbb{Z}_2}(N_g) \leq \mathrm{F}_{\mathbb{Z}_2}(N_{g+1}).$$

Moreover, since $N_{2g+1} \cong \Sigma_g \# \mathbb{RP}^2$ the inequality

$$\mathrm{F}_{\mathbb{Z}_2}(\Sigma_g) \leq \mathrm{F}_{\mathbb{Z}_2}(N_{2g+1})$$

holds.

It would be interesting to know whether equality always holds or whether strict inequality can indeed occur in the first two inequalities of the corollary.

The following example may give an idea what can go wrong in dimension $n = 2$.

In paragraph 3.1.1, we already mentioned the following fact (Proposition 3.5): let M be a connected closed orientable manifold of dimension at least two, and let d be a metric on M (not necessarily steming from a Riemannian metric). If W is another orientable manifold with boundary $\partial W = M$, then

$$\mathrm{FillVol}_{\mathbb{Z}}(M, d) = \inf\{\mathrm{Vol}(W, g') \ \mid \ d_{g'}|M \geq d\}$$

where g' runs over all complete Riemannian metrics on W.

This was proved in [Gro83], Proposition 2.2.A. Note that this infimum does not depend on the topology of W. (Indeed, one can always take $W = M \times [0, \infty)$.) So this theorem resembles Theorem 3.13 and can be (and actually is) proved by similar methods. But in [Gro83], 2.2.B (2) it is shown that this proposition does not hold if the dimension of W is two. Thus, by analogy, it may well be that strict inequalities occur in Corollary 3.15.

However, note that Gromov's counterexample is not Riemannian in the sense that the investigated metric d on the circle $\partial W = S^1$ is not geodesic.

Chapter 4

On manifolds satisfying stable systolic inequalities

Let M be a connected closed manifold of dimension n, and let g be a Riemannian metric on it. In chapter 2, we investigated the *stable* 1-*systole* $\mathrm{stabsys}_1(M,g)$. It is defined as the minimum of the stable norm on the nonzero elements of the integral lattice $H_1(M;\mathbb{Z})_{\mathbb{R}}$ in $H_1(M;\mathbb{R})$. The *stable norm* is the stabilization of the functional on $H_1(M;\mathbb{Z})_{\mathbb{R}}$ that is induced by the length of loops.

In an analogous way, one can define the *stable k-systole* $\mathrm{stabsys}_k(M,g)$ as the minimum of the stable norm on the nonzero classes in the lattice $H_k(M;\mathbb{Z})_{\mathbb{R}}$. In this case, the functional that is stabilized to obtain the *stable norm* is induced by the k-dimensional volume of Lipschitz cycles.

The main focus of this chapter is on the existence and nonexistence of stable systolic inequalities of the form

$$\mathrm{stabsys}_k(M,g)^{n/k} \leq C(M) \cdot \mathrm{Vol}(M,g), \qquad (*)$$

in which the constant $C(M)$ does not depend on the metric g. Therefore, it is natural to look at the *stable k-systolic constant*

$$\sigma_k^{st}(M) := \inf_g \frac{\mathrm{Vol}(M,g)}{\mathrm{stabsys}_k(M,g)^{n/k}} ,$$

where the infimum is taken over all Riemannian metrics g on M. (If the k-th Betti number $b_k(M)$ is zero, then the stable k-systolic constant is understood as zero.) Obviously, $\sigma_k^{st}(M) > 0$ if and only if M satisfies a stable systolic inequality $(*)$. Moreover, the reciprocal of $\sigma_k^{st}(M)$ is the best constant $C(M)$ such that $(*)$ holds.

For example, it is known by work of Gromov that

$$\sigma_2^{st}(\mathbb{C}\mathrm{P}^n) = 1/n! ,$$

see [Gro99], Theorem 4.36. (A more detailed proof may be found in [Kat07], section 13.2.) In fact, this is the only example of a higher-dimensional stable systolic constant whose value is known and not zero.

In his Filling paper, Gromov gave a sufficient condition for an orientable manifold to satisfy a more general stable systolic inequality.

Theorem 4.1 ([Gro83], 7.4.C)**.** *Let M be a connected closed orientable manifold of dimension n, and let $(k_1, \dots, k_p)$ be a partition of n, i. e. an unordered sequence of positive integers such that $\sum_{i=1}^p k_i = n$. If there are cohomology classes $\beta_i \in H^{k_i}(M; \mathbb{R})$ such that their cup product $\beta_1 \smile \cdots \smile \beta_p \in H^n(M; \mathbb{R})$ does not vanish, then*

$$\prod_{i=1}^{p} \operatorname{stabsys}_{k_i}(M, g) \leq C \cdot \operatorname{Vol}(M, g)$$

for a constant $C > 0$ depending only on the dimension n, the partition $(k_1, \dots, k_p)$, and the Betti numbers $b_{k_i}(M)$ of M.

For the dependence on the Betti numbers and the partition the reader is referred to [BanK03], Theorem 2.1. Applied to stable systolic inequalities $(*)$, that is to the case of partitions $(k, \dots, k)$, this theorem implies that the stable k-systolic constant $\sigma_k^{st}(M)$ of a connected closed orientable manifold of dimension $n = kp$ is positive if there are cohomology classes $\beta_1, \dots, \beta_p \in H^k(M; \mathbb{R})$ such that $\beta_1 \smile \cdots \smile \beta_p \neq 0$. (See also [BanK03], Theorem 2.7.)

For example, cohomologically symplectic manifolds (i. e. even-dimensional manifolds M^{2n} possessing a cohomology class $\omega \in H^2(M; \mathbb{R})$ such that $\omega^n \neq 0$) have nonvanishing stable systolic constants in all even dimensions that divide the dimension of M.

In [Bab92], Theorem 8.2 (c), Babenko showed that in the case $k = 1$ the condition stated above is also necessary for the existence of a stable 1-systolic inequality. More precisely, he proved that if the Jacobi mapping $\Phi : M \to T^b$, with $b := b_1(M)$ the first Betti number, maps the fundamental class of M to zero, then $\sigma_1^{st}(M) = 0$. (This also follows from the comparison axiom in chapter 2 together with Corollary 2.7.) But if $\Phi_*[M]_{\mathbb{Z}} \neq 0$, then there are cohomology classes $\beta_1', \dots, \beta_n' \in H^1(T^b; \mathbb{R})$ such that

$$\langle \beta_1' \smile \cdots \smile \beta_n', \Phi_*[M]_{\mathbb{Z}} \rangle \neq 0$$

since the cohomology of the torus is generated by classes of degree one. Therefore, the cohomology classes $\beta_i := \Phi^* \beta_i' \in H^1(M; \mathbb{R})$ have nonvanishing cup product, and the stable 1-systolic constant of M is nonzero by Gromov's theorem.

Extending this, we will show the following equivalence, which is valid for all integers $1 \leq k \leq n-1$. (This is the first part of Theorem 1.5 from the introduction. The second part is a special case of Theorem 4.3 below.)

Theorem 4.2. *Let M be a connected closed orientable manifold of dimension n. The stable k-systolic constant $\sigma_k^{st}(M)$ does not vanish if and only if n is a multiple of k, say $n = kp$, and there exist cohomology classes $\beta_1, \ldots, \beta_p \in H^k(M;\mathbb{R})$ such that $\beta_1 \smile \cdots \smile \beta_p \neq 0$ in $H^n(M;\mathbb{R})$.*

For instance, by Poincaré duality, the stable middle-dimensional systolic constant of even-dimensional manifolds vanishes if and only if the middle-dimensional Betti number is zero.

For nonorientable manifolds, we will prove a kind of 'general stable systolic freedom' whenever no one-dimensional systoles are involved. (The term *(stable) systolic freedom* refers to the absence of a (stable) systolic inequality, see for instance the articles [KatSu99], [KatSu01], and [Bab02] for other kinds of systolic freedom.)

Theorem 4.3. *Let M be a connected closed nonorientable manifold of dimension n. Let $p_k \geq 0$ be nonnegative real numbers for $k = 2, \ldots, n-1$. Then there is no constant $C > 0$ such that*

$$\prod_{k=2}^{n-1} \mathrm{stabsys}_k(M,g)^{p_k} \leq C \cdot \mathrm{Vol}(M,g)$$

holds for all Riemannian metrics g on M. In particular, $\sigma_k^{st}(M) = 0$ for $k = 2, \ldots, n-1$.

Denote by $b := b_k(M)$ the k-th Betti number of M. There exists a map $\Phi : M \to K(\mathbb{Z}^b, k)$ that induces an isomorphism

$$H_k(M;\mathbb{Z})_{\mathbb{R}} \xrightarrow{\cong} H_k(K(\mathbb{Z}^b,k);\mathbb{Z}).$$

To see this, note that by the canonical isomorphism

$$[M, K(\mathbb{Z}^b,k)] \cong H^k(M;\mathbb{Z})^b$$

it suffices to choose classes $\beta_1, \ldots, \beta_b \in H^k(M;\mathbb{Z})$ that represent a basis of $H^k(M;\mathbb{Z})_{\mathbb{R}}$. This choice corresponds to a map

$$\Phi : M \to K(\mathbb{Z}^b, k)$$

such that the canonical basis $\delta_1, \ldots, \delta_b$ of $H^k(K(\mathbb{Z}^b,k);\mathbb{Z})$ is pulled back to $\beta_i = \Phi^*\delta_i$. Thus, Φ induces an isomorphism on k-dimensional cohomology modulo torsion and consequently also on the integral lattices of k-dimensional homology. Note however that the homotopy class of this map is not uniquely determined by the isomorphism $H_k(M;\mathbb{Z})_\mathbb{R} \cong H_k(K(\mathbb{Z}^b,k);\mathbb{Z})$ except when $H^k(M;\mathbb{Z})$ is torsion-free, which happens for example if $k=1$.

As in chapter 2 for asymptotic invariants and the (stable) 1-systolic constant, comparison and extension techniques will allow us to prove homological invariance for stable systolic constants as announced in Theorem 1.6.

Theorem 4.4. *Let M and N be two connected closed orientable manifolds of dimension n, and let $1 \le k \le n-1$. Suppose that $b_k(M) = b_k(N) =: b$ and that there are maps $\Phi : M \to K(\mathbb{Z}^b,k)$ and $\Psi : N \to K(\mathbb{Z}^b,k)$ such that the induced homomorphisms on the integral lattices of k-dimensional homology are bijective and such that*

$$\Phi_*[M]_\mathbb{Z} = \Psi_*[N]_\mathbb{Z} \in H_n(K(\mathbb{Z}^b,k);\mathbb{R}).$$

Then the stable k-systolic constants coincide: $\sigma_k^{st}(M) = \sigma_k^{st}(N)$.

Note that in contrast to the one-dimensional systolic constant of chapter 2, the stable systolic constant is homologically invariant with respect to real coefficients.

As a direct consequence, two manifolds have the same stable k-systolic constants whenever there exists a degree one mapping between them that induces an isomorphism of the integral lattices of k-dimensional homology.

Let M be a connected closed orientable manifold of dimension $n = kp$. Consider the *multilinear intersection form*

$$\begin{aligned} Q_M^k : (H^k(M;\mathbb{Z})_\mathbb{R})^p &\to \mathbb{Z} \\ (\beta_1,\dots,\beta_p) &\mapsto \langle \beta_1 \smile \cdots \smile \beta_p, [M]_\mathbb{Z}\rangle. \end{aligned}$$

By Theorem 4.4, this form vanishes identically if and only if $\sigma_k^{st}(M) = 0$.

Using the computation of the real cohomology ring of the Eilenberg-Mac Lane space $K(\mathbb{Z},k)$ by Cartan and Serre, we are able to derive the following corollary of Theorem 4.4.

Corollary 4.5. *Let M and N be two connected closed orientable manifolds of dimension $n = kp$. If the multilinear intersection forms Q_M^k and Q_N^k are equivalent over $\mathbb{Z}$, then*

$$\sigma_k^{st}(M) = \sigma_k^{st}(N).$$

In the next section, we will define higher-dimensional systoles and give an axiomatic approach to stable systolic constants. This will be used to prove Theorem 4.4 in paragraph 4.2.3. Theorem 4.2 and Theorem 4.3 will be proved in paragraph 4.2.2 using some topological facts on spheres and their loop spaces that are presented in paragraph 4.2.1.

4.1 On higher-dimensional systoles

In chapter 2, we considered one-dimensional systoles. By replacing loops by closed submanifolds or more generally by Lipschitz cycles, higher-dimensional systoles can be defined in an analogous manner. These definitions are given in the first half of this section. In the second half, we show that higher-dimensional stable systolic constants fulfill suitable comparison and extension axioms.

4.1.1 Higher-dimensional systolic constants

Originally, systoles were defined for Riemannian manifolds. But using piecewise smooth Riemannian metrics (see Definition 2.11), the definitions extend to simplicial complexes, see also [Bab02], section 2.

Let X be a connected finite simplicial complex of dimension n, and let g be a Riemannian metric on it. Let $1 \leq k \leq n-1$ be an integer. The volume of an integral or real k-dimensional Lipschitz cycle $c = \sum_i r_i \sigma_i$ is given by

$$\mathrm{Vol}_k(c) := \textstyle\sum_i |r_i| \, \mathrm{Vol}_k(\Delta^k, \sigma_i^* g).$$

To see that this is well-defined, note that the pullback 'metric' $\sigma_i^* g$ is almost everywhere defined by Rademacher's theorem and is positive semidefinite. Thus, it has an almost everywhere defined 'volume form', and $\mathrm{Vol}_k(\Delta^k, \sigma_i^* g)$ is the integral of this k-form over Δ^k. (See Definition 2.25 for a similar discussion.)

For an integral homology class $\alpha \in H_k(X;\mathbb{Z})$ the *volume* $\mathrm{Vol}_k(\alpha)$ is defined as the infimum of the volumes of all integral Lipschitz cycles representing α. The *stable norm* $\|\alpha\|$ of a real homology class $\alpha \in H_k(X;\mathbb{R})$ is defined in the same way but using all real Lipschitz cycles representing α. Federer showed that the stable norm is in fact a norm, and moreover that

$$\|\alpha\| = \lim_{i\to\infty} \tfrac{1}{i} \, \mathrm{Vol}_k(i\alpha)$$

holds for all integral homology classes α. (This is proved in [Fed74], sections 4 and 5. See also [Gro99], section 4.C.).

Definition 4.6. The *k-systole*, the *k-systole modulo torsion*, and the *stable k-systole* of the Riemannian simplicial complex (X, g) are defined as

$$\begin{aligned}
\mathrm{sys}_k(X,g) &:= \inf_{\alpha \in H_k(X;\mathbb{Z})\setminus 0} \mathrm{Vol}_k(\alpha), \\
\mathrm{sys}_k^{\infty}(X,g) &:= \inf_{\alpha \in H_k(X;\mathbb{Z})_{\mathbb{R}}\setminus 0} \mathrm{Vol}_k(\alpha), \quad \text{and} \\
\mathrm{stabsys}_k(X,g) &:= \min_{\alpha \in H_k(X;\mathbb{Z})_{\mathbb{R}}\setminus 0} \|\alpha\|,
\end{aligned}$$

where $H_k(X;\mathbb{Z})_{\mathbb{R}}$ denotes the integral lattice in $H_k(X;\mathbb{R})$.

Remark. In the case $k = 1$, the 1-systole $\mathrm{sys}_1(X,g)$ does not coincide with the one-dimensional systole $\mathrm{sys}(X,g)$ from chapter 2: the 1-systole is the length of the shortest loop in X which is not nullhomologous, whereas the systole is the length of the shortest loop which is not nullhomotopic. However, if $\phi : \pi_1(X) \twoheadrightarrow H_1(X;\mathbb{Z})$ denotes the Hurewicz homomorphism, then $\mathrm{sys}_1(X,g) = \mathrm{sys}_\phi(X,g) \geq \mathrm{sys}(X,g)$ in the notation of chapter 2. Analogously, $\mathrm{sys}_1^\infty(X,g) = \mathrm{sys}_{\phi'}(X,g)$ where ϕ' is the composition of the Hurewicz homomorphism ϕ and the canonical epimorphism $H_1(X;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})_{\mathbb{R}}$. Moreover, note that the stable 1-systole $\mathrm{stabsys}_1(X,g)$ coincides with the one-dimensional stable systole $\mathrm{stabsys}(X,g)$ from chapter 2.

The main focus of this chapter is on the existence and nonexistence of stable systolic inequalities of the form

$$\mathrm{stabsys}_k(X,g)^{n/k} \leq C(X) \cdot \mathrm{Vol}(X,g), \qquad (\dagger)$$

in which the constant $C(X)$ does not depend on the metric g. Therefore, it is natural to look at the following systolic constants.

Definition 4.7. Let X be a connected finite simplicial complex of dimension n. The *k-systolic constant*, the *k-systolic constant modulo torsion*, and the *stable k-systolic constant* are given by

$$\begin{aligned}
\sigma_k(X) &:= \inf_g \frac{\mathrm{Vol}(X,g)}{\mathrm{sys}_k(X,g)^{n/k}}\,, \\
\sigma_k^\infty(X) &:= \inf_g \frac{\mathrm{Vol}(X,g)}{\mathrm{sys}_k^\infty(X,g)^{n/k}}, \quad \text{and} \\
\sigma_k^{st}(X) &:= \inf_g \frac{\mathrm{Vol}(X,g)}{\mathrm{stabsys}_k(X,g)^{n/k}}\,,
\end{aligned}$$

where the infima are taken over all Riemannian metrics on X.

Obviously, $\sigma_k^{st}(X) > 0$ if and only if X satisfies a stable systolic inequality ($\dagger$). Positivity of $\sigma_k(X)$ or $\sigma_k^\infty(X)$ is equivalent to the existence of an analogous systolic inequality.

Remark. For manifolds, one can also define systolic constants by allowing smooth Riemannian metrics only. But these constants coincide with the systolic constants defined above, compare the discussion at the end of paragraph 2.2.3.

4.1.2 Axioms for stable systolic constants

As in section 2.2, we will consider comparison and extension axioms for real-valued invariants I of connected finite simplicial complexes. Subsequently, we will show that the stable systolic constant fulfills those axioms. In contrast to the approach in chapter 2, we do not consider relative systolic constants. But look at the proof of Lemma 4.9 and in particular at Definition 4.10 for 'relative' notions in this context.

Comparison axiom. Let X and Y be two connected finite simplicial complexes of dimension n. If there exists an (n, d)-monotone map $f : X \to Y$ such that the induced homomorphism $f_* : H_k(X; \mathbb{Z})_{\mathbb{R}} \hookrightarrow H_k(Y; \mathbb{Z})_{\mathbb{R}}$ is injective and the image $f_*(H_k(X; \mathbb{Z})_{\mathbb{R}})$ is contained in $r \cdot H_k(Y; \mathbb{Z})_{\mathbb{R}}$ for a positive integer r, then

$$I(X) \leq d/r^{n/k} \cdot I(Y).$$

Extension axiom. Let X be a connected finite n-dimensional simplicial complex, and let X' be an *extension* of X, i. e. X' is obtained from X by attachment of finitely many cells of dimension $1 \leq \ell \leq n-1$ such that the inclusion $X \hookrightarrow X'$ induces the composition of a split monomorphism $H_k(X; \mathbb{Z})_{\mathbb{R}} \hookrightarrow H_k(X'; \mathbb{Z})_{\mathbb{R}}$ with multiplication by some positive integer r. Then

$$I(X') = r^{n/k} \cdot I(X).$$

We will prove that both axioms are satisfied for the stable k-systolic constant. Similar ideas may be found in various papers on systolic invariants under the keywords 'meromorphic map' and '(n, k)-morphism'. (See for example [BabK98], [BabKS98], [KatSu99], and [KatSu01].)

The k-systolic constant modulo torsion and the k-systolic constant fulfill similar axioms. Since such axioms are not needed later on, we do not investigate them.

Lemma 4.8. *The comparison axiom holds for $I = \sigma_k^{st}$ (with $1 \leq k \leq n-1$).*

See for instance [Bab02], Proposition 2.2.7 for a similar argument. There, a kind of systolic freedom (i. e. the vanishing of a suitably defined systolic constant) is pulled back. Here, the systolic constants of Y may also be nonzero. The used pullback technique goes back to [Bab92], Proposition 2.2.

Proof. Choose Riemannian metrics g_1 and g_2 on X and Y respectively. Then

$$g_1^t := f^* g_2 + t^2 g_1$$

with $t > 0$ is again a Riemannian metric on X. Choosing $t > 0$ small enough it can be arranged that

$$\mathrm{Vol}(X, g_1^t) \leq d \cdot \mathrm{Vol}(Y, g_2) + \varepsilon$$

for any given $\varepsilon > 0$. Moreover,

$$f : (X, g_1^t) \to (Y, g_2)$$

is nonexpanding and thus decreases the volume of Lipschitz cycles and the stable norm of homology classes. Since $f_* : H_k(X;\mathbb{Z})_\mathbb{R} \hookrightarrow H_k(Y;\mathbb{Z})_\mathbb{R}$ is injective and since $f_*(H_k(X;\mathbb{Z})_\mathbb{R}) \subset r \cdot H_k(Y;\mathbb{Z})_\mathbb{R}$, it follows that

$$\mathrm{stabsys}_k(X, g_1^t) \geq r \cdot \mathrm{stabsys}_k(Y, g_2)$$

by the fact that the stable norm is a norm. Therefore, $\sigma_k^{st}(X) \leq d/r^{n/k} \cdot \sigma_k^{st}(Y)$. □

Lemma 4.9. *The stable k-systolic constant satisfies the extension axiom.*

In [BanKSW06], Proposition 7.3, this is proved for a special case. Note also section 10 of the cited paper. With some adjustments the proof carries over to the general case. A similar argument was first used in [BabK98], Lemma 6.1.

Proof. The inclusion $i : X \hookrightarrow X'$ is $(n,1)$-monotone, induces a monomorphism on the integral lattices of k-dimensional homology, and the image $i_*(H_k(X;\mathbb{Z})_\mathbb{R}) \subset r \cdot H_k(X';\mathbb{Z})_\mathbb{R}$, hence $\sigma_k^{st}(X) \leq 1/r^{n/k} \cdot \sigma_k^{st}(X')$ by the comparison axiom.

To prove the converse inequality, we want to use induction over the number of attached cells. But it may happen that the attachment of some k-cells increases the k-th Betti number and that later on the attachment of some $(k+1)$-cells decreases it again. Then the induced homomorphism on real k-dimensional homology may be injective on the whole but it is not injective at every step of the induction. Therefore, it is useful to introduce the following 'relative' version of the stable k-systolic constant. (Compare the definition of the relative one-dimensional systoles in section 2.2.)

Definition 4.10. Let b be a positive integer, and let $\phi : H_k(X;\mathbb{Z})_\mathbb{R} \to \mathbb{Z}^b$ be a homomorphism. The induced homomorphism $H_k(X;\mathbb{R}) \to \mathbb{R}^b$ will also be denoted by ϕ. For a metric g on X, the *stable (ϕ,k)-systole* $\mathrm{stabsys}_{\phi,k}(X,g)$ is defined as the minimum of the quotient norm of the stable norm on the nonzero elements of the lattice $\mathbb{Z}^b$ in $\mathbb{R}^b$. The *stable (ϕ,k)-systolic constant* is given by

$$\sigma_{\phi,k}^{st}(X) := \inf_g \frac{\mathrm{Vol}(X,g)}{\mathrm{stabsys}_{\phi,k}(X,g)^{n/k}}.$$

For ϕ a split monomorphism, this definition coincides with the original 'absolute' one. Note also that for $k = 1$ this relative definition of the stable systole coincides with the definition from paragraph 2.2.1.

Now, an *extension* (X', ϕ') of (X, ϕ) consists of a simplicial complex X' that is obtained from X by attaching finitely many cells of dimension $1 \leq \ell \leq n-1$ and of a homomorphism $\phi' : H_k(X'; \mathbb{Z})_{\mathbb{R}} \to \mathbb{Z}^b$ such that $\phi = \phi' \circ i_*$ with $i : X \hookrightarrow X'$ the inclusion. We will prove the following relative version of the extension axiom.

Claim. *If (X', ϕ') is an extension of (X, ϕ), then $\sigma^{st}_{\phi',k}(X') \leq \sigma^{st}_{\phi,k}(X)$.*

In fact, equality holds because a relative version of the comparison axiom is also fulfilled. Moreover, this claim implies that σ^{st}_k satisfies the original extension axiom: taking ϕ' as an isomorphism and ϕ as the composition $\phi' \circ i_*$ one gets

$$\sigma^{st}_k(X') = \sigma^{st}_{\phi',k}(X') \leq \sigma^{st}_{\phi,k}(X).$$

Furthermore, $\mathrm{stabsys}_k(X, g) = r \cdot \mathrm{stabsys}_{\phi,k}(X, g)$ since the stable norm is a norm. Therefore, $\sigma^{st}_{\phi,k}(X) = r^{n/k} \cdot \sigma^{st}_k(X)$ and the extension axiom follows.

With this relative version of extension it is possible to proceed by induction over the number of attached cells. To prove the claim it suffices therefore to consider the case where a single ℓ-cell is attached to X.

Note that the volume of X' equals the volume of X since the attached cell is of lower dimension, hence it is a set of measure zero with respect to any n-dimensional volume.

Let g be a Riemannian metric on X, and let $h : S^{\ell-1} \to X$ be the simplicial attaching map. Choose $R > 0$ such that $h : (S^{\ell-1}, g_R) \to (X, g)$ is nonexpanding where g_R denotes the round metric of radius R. Define a Riemannian metric on $X' = X \cup_h D^\ell$ in the following way: think of X' as divided into four pieces

$$X \quad \cup_h \quad (S^{\ell-1} \times [-1, 0]) \quad \cup \quad (S^{\ell-1} \times [0, L]) \quad \cup \quad S^\ell_+$$

and take the following Riemannian metrics on the respective pieces

$$g, \quad ((1+t)g_R - th^*g) \oplus dt^2, \quad g_R \oplus dt^2, \quad g_R,$$

where (S^ℓ_+, g_R) is an ℓ-dimensional round hemisphere of radius R and $L > 0$ is some (large) number. This gives a Riemannian metric g_L on X'.

If $\mathrm{stabsys}_{\phi',k}(X', g_L) \geq \mathrm{stabsys}_{\phi,k}(X, g)$ for some $L > 0$, we are done. Hence, we may assume that $\mathrm{stabsys}_{\phi',k}(X', g_L) < \mathrm{stabsys}_{\phi,k}(X, g)$ for every $L > 0$.

Let $\alpha' \in H_k(X';\mathbb{R})$ represent via ϕ a nonzero class in $\mathbb{Z}^b$ such that $\|\alpha'\| = \mathrm{stabsys}_{\phi',k}(X', g_L)$, and let $c \in C_k(X';\mathbb{R})$ be a real cycle representing α' such that $\mathrm{Vol}_k(c) \leq \|\alpha'\| + \varepsilon$.

Next, we apply the coarea formula to the projection p of $S^{\ell-1} \times [0, L]$ to the second factor. Denote $c_t := c \cap p^{-1}(t)$. Then

$$\int_0^L \mathrm{Vol}_{k-1}(c_t) dt \leq \mathrm{Vol}_k(c),$$

and therefore there is a t_0 such that

$$\mathrm{Vol}_{k-1}(c_{t_0}) \leq \mathrm{Vol}_k(c)/L \leq (\|\alpha'\| + \varepsilon)/L.$$

Since the right hand side is bounded by $(\mathrm{stabsys}_{\phi,k}(X, g) + \varepsilon)/L$, we can force the volume of c_{t_0} to be arbitrarily small by choosing L very large. By the isomperimetric inequality for small cycles (see [Gro83], Sublemma 3.4.B') applied to $S^{\ell-1} \times t_0$ there is a filling d of c_{t_0} of volume

$$\mathrm{Vol}_k(d) \leq C_{R,\ell} \cdot \mathrm{Vol}_{k-1}(c_{t_0})^{k/(k-1)},$$

with a constant $C_{R,\ell} > 0$ depending only on the radius R and the dimension ℓ. Assuming $\mathrm{Vol}_{k-1}(c_{t_0}) \leq 1$, we get a 'linear isoperimetric inequality':

$$\mathrm{Vol}_k(d) \leq C_{R,\ell} \cdot \mathrm{Vol}_{k-1}(c_{t_0}).$$

The cycle c decomposes into two pieces along c_{t_0}, that is to say $c = c_+ \cup_{c_{t_0}} c_-$. Define another cycle

$$c' := c_+ \cup_{c_{t_0}} d = c - (c_- \cup_{c_{t_0}} (-d)).$$

Since the cycle $c_- \cup_{c_{t_0}} (-d)$ is contained in the attached ℓ-cell, it is null-homologous. Thus, c' also represents α'. Moreover,

$$\begin{aligned}\mathrm{Vol}_k(c') &\leq \|\alpha'\| + \varepsilon + C_{R,\ell}(\|\alpha'\| + \varepsilon)/L \\ &= (\|\alpha'\| + \varepsilon)(1 + C_{R,\ell}/L).\end{aligned}$$

The map that contracts the cylinder $S^{\ell-1} \times [-1, L]$ to X is nonexpanding. Hence, the image c'' of c' under this retraction satisfies the same volume bound and still represents α'.

The homology class $\alpha \in H_k(X;\mathbb{R})$ represented by c'' in X is a preimage of α'. Therefore, it represents a nonzero element of the lattice $\mathbb{Z}^b \subset \mathbb{R}^b$. Moreover,

$$\begin{aligned}\|\alpha\| &\leq \mathrm{Vol}_k(c'') \\ &\leq (\|\alpha'\| + \varepsilon)(1 + C_{R,\ell}/L) \\ &= (\mathrm{stabsys}_{\phi',k}(X', g_L) + \varepsilon)(1 + C_{R,\ell}/L).\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, we see that

$$\mathrm{stabsys}_{\phi,k}(X,g) \leq \mathrm{stabsys}_{\phi',k}(X',g_L)(1 + C_{R,\ell}/L).$$

For L tending to infinity, this implies $\sigma^{st}_{\phi,k}(X) \geq \sigma^{st}_{\phi',k}(X')$. Thus, the claim is proved, and the extension axiom is valid for $I = \sigma^{st}_k$. □

4.2 Stable systolic constants

This section starts with a short review of some topological properties of spheres and their loop spaces. Those properties will be used to show that every finite-dimensional CW complex admits a map to a product of spheres or loop spaces that induces an isomorphism on real homology in a given dimension. In the beginning of paragraph 4.2.2, Katz's and Suciu's result on systolic freedom modulo torsion is recalled. Similar ideas are then used to prove Theorem 4.2 and Theorem 4.3. In the remainder of this chapter, we show that stable systolic constants are homologically invariant and that they depend only on the multilinear intersection form.

4.2.1 Spheres and their loop spaces

In this paragraph, we will briefly recall some topological properties of spheres and their loop spaces that will be used in the proofs of Theorem 4.2 and Theorem 4.3.

If k is odd, let L_k denote the k-dimensional sphere S^k. The cohomology ring is the exterior algebra

$$H^*(L_k;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha]$$

with $\alpha \in H^k(L_k;\mathbb{Z})$ a generator. Therefore, by the Künneth formula

$$H^*(L_k^b;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1,\ldots,\alpha_b]$$

with α_i of degree k. Furthermore, we will always use the CW decomposition of S^k consisting of one 0-cell and one k-cell.

It is known by work of Serre (see [Ser51]) that the homotopy groups $\pi_m(S^k)$ for $m > k$ are finite (recall that k is odd). Sullivan showed in [Sul74] that the selfmap $S^k \to S^k$ of degree d induces a homomorphism $\pi_m(S^k) \to \pi_m(S^k)$ which is nilpotent on d-torsion. Therefore, there exists a nonzero degree selfmap of S^k that induces the trivial homomorphism on $\pi_m(S^k)$.

Finally, note that the map $S^k \to S^k$ of degree two multiplies each homology class in L_k^b of positive dimension by some power of two. In particular, the induced homomorphism on homology with $\mathbb{Z}_2$ coefficients is trivial.

If k is even, let L_k be a CW complex that is homotopy equivalent to the based loop space ΩS^{k+1} of the $(k+1)$-dimensional sphere. More precisely, let L_k be the James reduced product $J(S^k)$ (see for example [Hat02], pages 224–225 and section 4.J). The CW structure on L_k consists of one cell in each dimension divisible by k, and the cohomology ring is the divided polynomial algebra

$$H^*(L_k;\mathbb{Z}) \cong \Gamma_{\mathbb{Z}}[\alpha]$$

with $\alpha \in H^k(L_k;\mathbb{Z})$. (Recall that this is almost a polynomial algebra. In fact, the generator in degree kp equals $\alpha^p/p!$. Using real coefficients, the cohomology ring is a polynomial algebra.) By the Künneth formula

$$H^*(L_k^b;\mathbb{Z}) \cong \Gamma_{\mathbb{Z}}[\alpha_1,\dots,\alpha_b],$$

where the classes α_i are of degree k.

Since $\pi_m(\Omega S^{k+1}) = \pi_{m+1}(S^{k+1})$, the homotopy groups $\pi_m(L_k)$ are finite for $m > k$ (recall that k is even now). The group structure in $\pi_m(L_k)$ coincides with the one coming from loop multiplication, hence one can easily construct selfmaps $L_k \to L_k$ that induce multiplication by some positive integer on $H^k(L_k;\mathbb{Z})$ and trivial homomorphisms on $\pi_m(L_k)$.

The map $\Omega S^{k+1} \to \Omega S^{k+1}$ that assigns to each loop its double induces multiplication by two in $\pi_k(L_k^b)$ and therefore also in k-dimensional homology $H_k(L_k^b;\mathbb{Z})$ and cohomology $H^k(L_k^b;\mathbb{Z})$. Hence, it induces multiplication by some power of two in every $H^\ell(L_k^b;\mathbb{Z})$ with $\ell > 0$ and thus in every $H_\ell(L_k^b;\mathbb{Z})$, as well. Consequently, the induced homomorphism on homology with $\mathbb{Z}_2$ coefficients is trivial.

Summary. The CW complexes L_k^b have cells only in dimensions divisible by k, their real cohomology rings are generated by elements of degree k, and there exist selfmaps $h_m : L_k^b \to L_k^b$ for all $m > k$ that induce multiplication by some positive integer on $H^k(L_k^b;\mathbb{Z})$ (and therefore also on $H_k(L_k^b;\mathbb{Z})$) and vanish on $\pi_m(L_k^b)$. Moreover, there exists a selfmap $L_k^b \to L_k^b$ that induces the zero homomorphism in homology of nonzero dimension with coefficients in $\mathbb{Z}_2$ but that is bijective on $H_k(L_k^b;\mathbb{R})$.

Using these properties of L_k, we are able to prove the following lemma. It stems from [KatSu99], sections 4 and 10.

Lemma 4.11. *Let X be a connected CW complex of dimension n. Let $1 \le k \le n-1$, and assume that the k-th Betti number $b := b_k(X)$ is finite. Then* *there is a map $f : X \to L^b_k$ that induces an isomorphism*

$$f_* : H_k(X;\mathbb{R}) \xrightarrow{\cong} H_k(L^b_k;\mathbb{R}).$$

In fact, f can be chosen such that the induced map on the integral lattices corresponds to multiplication by some positive integer.

Proof. The case $k = 1$ is easy because L^b_1 is just the b-dimensional torus T^b, which is $K(\mathbb{Z}^b, 1)$. The canonical epimorphism

$$\pi_1(X) \twoheadrightarrow H_1(X;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})_{\mathbb{R}} \cong \mathbb{Z}^b$$

is induced by the so-called *Jacobi map* $f : X \to T^b$. The induced homomorphism $f_* : H_1(X;\mathbb{R}) \to H_1(T^b;\mathbb{R})$ is consequently an isomorphism, which is moreover an isomorphism of the integral lattices.

Now, let $2 \le k \le n-1$. Choose a CW decomposition of $K(\mathbb{Z}^b, k)$ such that the $(k+1)$-skeleton is $\bigvee^b S^k$, the wedge sum of b spheres of dimension k. As shown in the introductory paragraph of this chapter, there is a map $X \to K(\mathbb{Z}^b, k)$ that induces an isomorphism on the integral lattices of homology in dimension k. By cellular approximation, this gives a map $X^{(k+1)} \to \bigvee^b S^k$. Note that the $(k+1)$-skeleton of L^b_k is also $\bigvee^b S^k$. Thus, we have a map

$$f^{(k+1)} : X^{(k+1)} \to L^b_k$$

that induces an isomorphism on the integral lattices of homology in degree k.

Let $h_{k+1} : L^b_k \to L^b_k$ be as in the summary above, i. e. it induces the trivial homomorphism on the $(k+1)$-dimensional homotopy group and multiplication by some positive integer on real homology of degree k. Then the composition $h_{k+1} \circ f^{(k+1)} : X^{(k+1)} \to L^b_k$ extends over $X^{(k+2)}$ since it is zero on the $(k+1)$-dimensional homotopy groups. Call this extension $f^{(k+2)} : X^{(k+2)} \to L^b_k$. Repeating this process finally gives a map

$$f : X \to L^b_k$$

for which the induced monomorphism $H_k(X;\mathbb{Z})_{\mathbb{R}} \hookrightarrow H_k(L^b_k;\mathbb{Z})_{\mathbb{R}}$ corresponds to multiplication by some positive integer. $\square$

4.2.2 Existence of stable systolic inequalities

In this paragraph, Theorem 4.2 and Theorem 4.3 are proved. Both proofs rely on the properties of spheres and their loop spaces that are listed in paragraph 4.2.1.

Before we start to prove those two theorems, we want to recall some facts about the systolic constant and the systolic constant modulo torsion. In the articles [KatSu99] and [KatSu01], Katz and Suciu showed that every manifold is systolically free modulo torsion for systoles of dimension at least two.

Theorem 4.12 (Katz, Suciu). *Let M be a connected closed manifold of dimension n, and let $2 \leq k \leq n-1$. Then*

$$\sigma_k^{\infty}(M) = 0.$$

This kills any interest in the systolic constant modulo torsion. By definition, $\sigma_k(M) \geq \sigma_k^{\infty}(M)$. But it is widely believed that the k-systolic constant also vanishes for every manifold. For orientable manifolds of dimension four this was shown in [KatSu99].

Using ideas similar to those of [KatSu99], in particular the map from Lemma 4.11, we now prove Theorem 4.2 and Theorem 4.3.

Proof of Theorem 4.2. Denote $b := b_k(M)$. If k does not divide n, then the n-skeleton and the $(n-1)$-skeleton of L_k^b coincide, and $\sigma_k^{st}(M) = 0$ by the comparison axiom applied to the map $f : M \to (L_k^b)^{(n-1)}$ of Lemma 4.11 (which is $(n, 0)$-monotone due to the dimension of the range). Assume now that $n = kp$.

If $n = 2$, then $k = 1$ and it is known that, apart from the sphere, all closed orientable surfaces satisfy a stable 1-systolic inequality. Since their cohomology rings are generated in degree one, Theorem 4.2 is true in this case, and we may restrict our attention to $n \geq 3$.

Let $f : M \to L_k^b$ induce an isomorphism on real homology of degree k. If $f_*[M]_{\mathbb{Z}} = 0$ in $H_n(L_k^b; \mathbb{R})$, then $f_*[M]_{\mathbb{Z}} = 0$ also in $H_n(L_k^b; \mathbb{Z})$ since the homology of L_k^b is torsion-free. By Corollary 2.7 one can homotope f so that its image lies in the $(n-1)$-skeleton of L_k^b (i. e. f is $(n, 0)$-monotone), and by the comparison axiom $\sigma_k^{st}(M)$ vanishes. The theorem of Gromov stated in the introduction of this chapter (Theorem 4.1) shows that there cannot be cohomology classes $\beta_1, \ldots, \beta_p \in H^k(M; \mathbb{R})$ having nonvanishing product.

On the other hand, if $f_*[M]_{\mathbb{Z}} \neq 0$, then there are cohomology classes $\beta_1', \ldots, \beta_p' \in H^k(L_k^b; \mathbb{R})$ such that

$$\langle \beta_1' \smile \ldots \smile \beta_p', f_*[M]_{\mathbb{Z}} \rangle \neq 0$$

since the cohomology of L_k^b is generated by classes of degree k. Therefore, the cohomology classes $\beta_i := f^* \beta_i' \in H^k(M; \mathbb{R})$ have nonvanishing product, and the stable k-systolic constant of M is nonzero by Theorem 4.1. This finishes the proof of Theorem 4.2. □

Proof of Theorem 4.3. Let $f_k : M \to L_k^{b_k(M)}$ be as in Lemma 4.11. Define

$$F := (f_2, \dots, f_{n-1}) : M \longrightarrow L := \prod_{k=2}^{n-1} L_k^{b_k(M)}.$$

Then by the Künneth formula, F induces monomorphisms on k-dimensional real homology for all $k = 2, \dots, n-1$.

There is a selfmap $L \to L$ that maps all homology classes (of nonzero dimension) to an even multiple. Composing F with this map, one gets a map (still called F) that is injective on real homology of dimensions $k = 2, \dots, n-1$ and that is zero on $H_n(M; \mathbb{Z}_2)$, i. e. $F_*[M]_{\mathbb{Z}_2} = 0$ in $H_n(L; \mathbb{Z}_2)$.

By Corollary 2.7 it is possible to deform F so that its range lies in the $(n-1)$-skeleton of L. (Note that L is simply-connected and that we may assume $n \geq 3$ since the theorem is empty for $n = 2$.)

Choose Riemannian metrics g_1 and g_2 on M and on the $(n-1)$-skeleton $L^{(n-1)}$ of L, respectively. Then

$$g_1^t := F^* g_2 + t^2 g_1$$

with $t > 0$ is again a (piecewise smooth) Riemannian metric on M. Choosing $t > 0$ small enough, it can be arranged that

$$\mathrm{Vol}(M, g_1^t) \leq \varepsilon$$

for any given $\varepsilon > 0$. Moreover,

$$F : (M, g_1^t) \to (L^{(n-1)}, g_2)$$

is nonexpanding. Since $F_* : H_k(M; \mathbb{Z})_{\mathbb{R}} \hookrightarrow H_k(L^{(n-1)}; \mathbb{Z})_{\mathbb{R}}$ is injective for all $k = 2, \dots, n-1$, it follows that

$$\mathrm{stabsys}_k(M, g_1^t) \geq \mathrm{stabsys}_k(L^{(n-1)}, g_2).$$

Therefore, the left-hand side of

$$\prod_{k=2}^{n-1} \mathrm{stabsys}_k(M, g_1^t)^{p_k} \leq C \cdot \mathrm{Vol}(M, g_1^t)$$

is bounded from below by the constant $\prod_k \mathrm{stabsys}_k(L^{(n-1)}, g_2)^{p_k}$, whereas the right-hand side can be made arbitrarily small. Thus, there is no constant $C > 0$ such that this inequality is satisfied for all metrics on M. $\square$

Finally, we will investigate the stable 1-systolic constant and the 1-systolic constant modulo torsion. Let $b := b_1(M)$ denote the first Betti number, and let $\Phi : M \to T^b$ be the *Jacobi map*, i. e. a map that induces the canonical epimorphism $\phi = \Phi_* : \pi_1(M) \twoheadrightarrow H_1(M;\mathbb{Z})_\mathbb{R}$. (Note that this map is uniquely determined up to homotopy by this epimorphism.)

Recall that $\sigma_1^\infty(M) = \sigma_\phi(M)$ in the notation of chapter 2. In paragraph 2.2.5, we saw that $\sigma_1^\infty(M)$ is zero if and only if the Jacobi map $\Phi : M \to T^b$ is homotopic to a map with range in the $(n-1)$-skeleton of the torus. (This is due to Gromov and Babenko, see chapter 2 for references.)

By the comparison axiom, the stable 1-systolic constant vanishes if Φ maps to the $(n-1)$-skeleton. Since $\sigma_1^{st}(M) \geq \sigma_1^\infty(M)$ by definition, it follows therefore that the same characterization is true for the vanishing of $\sigma_1^{st}(M)$. Using Corollary 2.7, we get the following corollary.

Corollary 4.13. *The stable* 1*-systolic constant* $\sigma_1^{st}(M)$ *and the* 1*-systolic constant modulo torsion* $\sigma_1^\infty(M)$ *vanish if and only if the fundamental class of* M *with coefficients in* $\mathbb{Z}$, $\mathcal{O}$, *or* $\mathbb{Z}_2$ *(according to the orientation behaviour of the Jacobi map) is mapped to zero by the Jacobi map.*

Recall that $\mathcal{O}$ denotes the orientation bundle of M. For surfaces this can easily be seen directly. Note in particular that the only surfaces for which the stable 1-systolic constant and the 1-systolic constant modulo torsion vanish are the sphere, the real projective plane, and the Klein bottle.

4.2.3 Homological invariance for stable systolic constants

The first aim of this paragraph is to prove Theorem 4.4. Then, we apply the theorem to the case of projective spaces over division algebras.

We will need the following corollary from section 2.1.

Corollary 4.14. *Let* M *and* N *be two connected closed orientable manifolds of dimension* $n \geq 3$*. Let* $i : M \hookrightarrow X$ *be an embedding into a simplicial complex, and let* $f : N \to X$ *be a map such that the induced homomorphism on fundamental groups is surjective and such that* $f_*[N]_\mathbb{Z} = i_*[M]_\mathbb{Z}$ *in* $H_n(X;\mathbb{Z})$*. Identify* M *and its image in* X*. Then* f *is homotopic to an* $(n,1)$*-monotone map* $N \to M \cup X^{(n-1)}$*.*

Proof. By Lemma 2.6 we may assume that X is n-dimensional, and by Lemma 2.5 the map f can be deformed to an $(n,1)$-monotone map with image in $M \cup X^{(n-1)}$. □

Proof of Theorem 4.4. Note that the theorem is trivial for $n = 2$ because the orientable surfaces are classified by their first Betti number. Hence, we may restrict to $n \geq 3$.

The images of the fundamental classes of M and N in $H_n(K(\mathbb{Z}^b, k); \mathbb{Z})$ may differ by a torsion class. Thus, Corollary 4.14 cannot be applied directly. But we can bypass this problem in the following way.

Let $f : K(\mathbb{Z}^b, k)^{(n+1)} \to L_k^b$ be a map as in Lemma 4.11. (Since $K(\mathbb{Z}^b, k)$ is not finite-dimensional in general, we have to consider some skeleton to be able to apply Lemma 4.11.) In particular, the induced map on the integral lattices of k-dimensional homology corresponds to multiplication by some positive integer r. Then $f_*\Phi_*[M]_\mathbb{Z} = f_*\Psi_*[N]_\mathbb{Z}$ in $H_n(L_k^b; \mathbb{Z})$ since the homology of L_k^b is torsion-free.

Using the mapping cylinder of $f \circ \Phi$, we may assume without loss of generality that $f \circ \Phi$ is the inclusion of a subcomplex into L_k^b. If we start with a CW decomposition of M with only one 0-cell, then there is a 1-cell in the mapping cylinder connecting this 0-cell with the 0-cell of L_k^b. Collapsing this 1-cell, we may assume that L_k^b is obtained from M by attaching only cells of positive dimension.

By Corollary 4.14, $f \circ \Psi$ is homotopic to an $(n, 1)$-monotone map

$$\Psi' : N \longrightarrow X := M \cup (L_k^b)^{(n-1)}.$$

Note that X is a finite complex. In particular, it is an extension of M. By the comparison and extension axiom, we get

$$\sigma_k^{st}(N) \leq 1/r^{n/k} \cdot \sigma_k^{st}(X) = \sigma_k^{st}(M).$$

Changing the roles of M and N gives equality. □

Remark. In the cases $k = 1$ and $k = 2$, it is not necessary to use a map $f : K(\mathbb{Z}^b, k) \to L_k^b$ because the homology of $K(\mathbb{Z}, 1) = S^1$ and $K(\mathbb{Z}, 2) = \mathbb{C}\mathrm{P}^\infty$ is already torsion-free.

A direct consequence of Theorem 4.4 is the following corollary.

Corollary 4.15. *Let $f : M \to N$ be a degree one map between connected closed orientable manifolds such that the induced homomorphism*

$$f_* : H_k(M; \mathbb{Z})_\mathbb{R} \xrightarrow{\cong} H_k(N; \mathbb{Z})_\mathbb{R}$$

is bijective. Then $\sigma_k^{st}(M) = \sigma_k^{st}(N)$.

For example, the inclusion $\mathbb{C}^{2n+1} \times 0 \hookrightarrow \mathbb{C}^{2n+2} = \mathbb{H}^{n+1}$ gives a degree one map $\mathbb{C}\mathrm{P}^{2n} \to \mathbb{H}\mathrm{P}^n$ from complex to quaternionic projective space that induces isomorphisms on homology in all dimensions divisible by 4. Therefore, the corollary implies

$$\sigma_{4k}^{st}(\mathbb{H}\mathrm{P}^n) = \sigma_{4k}^{st}(\mathbb{C}\mathrm{P}^{2n}),$$

which is nonzero if and only if k divides n by Theorem 4.2. In particular,

$$\sigma_{4n}^{st}(\mathbb{H}\mathrm{P}^{2n}) = \sigma_{4n}^{st}(\mathbb{C}\mathrm{P}^{4n}).$$

This last equation is Theorem 1.2 from [BanKSW06]. However, note that our proof is not essentially different from the proof of Bangert, Katz, Shnider, and Weinberger. In fact, their reasoning is along the same lines but only for the special case of complex and quaternionic projective spaces.

Note also that in dimension eight, they show that

$$\sigma_4^{st}(\mathbb{H}\mathrm{P}^2) = \sigma_4^{st}(\mathbb{C}\mathrm{P}^4) \in [\tfrac{1}{14}, \tfrac{1}{6}].$$

In fact, the calculation of this stable systolic constant is possibly within reach, see [BanKSW06].

4.2.4 The multilinear intersection form

Consider two simply-connected closed four-manifolds M and N. If their intersection forms are equivalent over $\mathbb{Z}$, then the manifolds are homotopy equivalent by the theorem of Milnor and Whitehead. Therefore, their stable 2-systolic constants coincide. (For this only the comparison axiom is needed since every homotopy equivalence of manifolds is homotopic to an $(n, 1)$-monotone map by Corollary 2.10.) We will see that one can drop the assumption on the fundamental group using Theorem 4.4.

More generally, let M be a connected closed orientable manifold of dimension kp. Consider the *multilinear intersection form*

$$\begin{aligned} Q_M^k : (H^k(M;\mathbb{Z})_{\mathbb{R}})^p &\to \mathbb{Z}, \\ (\beta_1, \ldots, \beta_p) &\mapsto \langle \beta_1 \smile \cdots \smile \beta_p, [M]_{\mathbb{Z}} \rangle. \end{aligned}$$

In the case $p = 2$, this is the usual intersection form.

Before we start with the proof of Corollary 4.5, note that a similar result was derived by Hamilton (see [Ham06], Theorem 1.2): if two closed orientable four-manifolds with $b_2^+ = 1$ have equivalent intersection forms, then their conformal systolic constants agree. Here, the *conformal systolic constant* $\mathrm{CS}(M)$ is the supremum of the *conformal systole* over all Riemannian metrics on M. (See [Kat07] for more details on conformal systoles.)

Proof of Corollary 4.5. Write $b := b_k(M) = b_k(N)$. Choose maps $\Phi : M \to K(\mathbb{Z}^b, k)$ and $\Psi : N \to K(\mathbb{Z}^b, k)$ inducing isomorphisms on the integral lattices of k-dimensional cohomology such that the isomorphism

$$\Psi^* \circ (\Phi^*)^{-1} : H^k(M;\mathbb{Z})_{\mathbb{R}} \xrightarrow{\cong} H^k(N;\mathbb{Z})_{\mathbb{R}}$$

is an equivalence of the multilinear intersection forms Q^k_M and Q^k_N. Then

$$\begin{aligned}\langle \beta_1 \smile \cdots \smile \beta_p, \Phi_*[M]_{\mathbb{Z}} \rangle &= \langle \Phi^*\beta_1 \smile \cdots \smile \Phi^*\beta_p, [M]_{\mathbb{Z}} \rangle \\ &= \langle \Psi^*\beta_1 \smile \cdots \smile \Psi^*\beta_p, [N]_{\mathbb{Z}} \rangle \\ &= \langle \beta_1 \smile \cdots \smile \beta_p, \Psi_*[N]_{\mathbb{Z}} \rangle\end{aligned}$$

for all cohomology classes $\beta_1, \ldots, \beta_p$ in $H^k(K(\mathbb{Z}^b, k); \mathbb{R})$.

Since the cohomology ring of the Eilenberg-Mac Lane space $K(\mathbb{Z}^b, k)$ equals the exterior algebra $\Lambda_{\mathbb{R}}[\alpha_1, \ldots, \alpha_b]$ if k is odd and the polynomial algebra $\mathbb{R}[\alpha_1, \ldots, \alpha_b]$ if k is even by work of Cartan and Serre (see for example [Whi78], page 670), it follows that the classes $\Phi_*[M]_{\mathbb{Z}}$ and $\Psi_*[N]_{\mathbb{Z}}$ coincide in $H_n(K(\mathbb{Z}^b, k); \mathbb{R})$. By Theorem 4.4, the stable k-systolic constants of M and N are equal. □

Consider the octonionic projective plane $\mathbb{O}\mathrm{P}^2$. Its cohomology ring is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^3)$ with α of degree eight. Thus, the intersection form $Q^8_{\mathbb{O}\mathrm{P}^2}$ on eight-dimensional cohomology is given by

$$\begin{aligned}Q^8_{\mathbb{O}\mathrm{P}^2} : H^8(\mathbb{O}\mathrm{P}^2;\mathbb{Z}) \times H^8(\mathbb{O}\mathrm{P}^2;\mathbb{Z}) &\to \mathbb{Z}, \\ (\alpha, \alpha) &\longmapsto 1.\end{aligned}$$

The intersection forms of $\mathbb{C}\mathrm{P}^8$ and $\mathbb{H}\mathrm{P}^4$ on the respective eight-dimensional cohomology groups are obviously equivalent over $\mathbb{Z}$ to the intersection form of the octonionic projective plane. Therefore,

$$\sigma_8^{st}(\mathbb{O}\mathrm{P}^2) = \sigma_8^{st}(\mathbb{H}\mathrm{P}^4) = \sigma_8^{st}(\mathbb{C}\mathrm{P}^8).$$

Note that this shows that neither the canonical metric of the octonionic projective plane nor the symmetric metric of the quaternionic projective four-space are systolically optimal. In fact, Berger showed in [Ber72] that

$$\begin{aligned}\mathrm{Vol}(\mathbb{C}\mathrm{P}^8, g_0)/\operatorname{stabsys}_8(\mathbb{C}\mathrm{P}^8, g_0)^2 &= 1/70, \\ \mathrm{Vol}(\mathbb{H}\mathrm{P}^4, g_0)/\operatorname{stabsys}_8(\mathbb{H}\mathrm{P}^4, g_0)^2 &= 5/126, \\ \mathrm{Vol}(\mathbb{O}\mathrm{P}^2, g_0)/\operatorname{stabsys}_8(\mathbb{O}\mathrm{P}^2, g_0)^2 &= 7/66,\end{aligned}$$

where g_0 denotes the respective canonical Riemannian metrics.

Chapter 5

Enlargeability is homologically invariant

In this last chapter of the thesis, we are concerned with enlargeability of closed manifolds and with largeness properties of their universal coverings.

Recall that a connected closed orientable manifold M of dimension n is called *enlargeable* if for every $\varepsilon > 0$ there exists a covering $\bar{M}_\varepsilon$ and an ε-contracting map $\bar{M}_\varepsilon \to S^n$ to the unit sphere that is constant outside a compact set and of nonzero degree. Gromov and Lawson proved in [GroL80] and [GroL83] that an enlargeable spin manifold does not carry a metric of positive scalar curvature.

An important question in differential geometry is which topological properties follow from the existence of a positive scalar curvature metric. Related to this is the question about topological consequences of enlargeability. Hanke and Schick showed that enlargeable manifolds have the following property:

Definition 5.1. Let $\Phi : M \to B\pi_1(M)$ be the classifying map of the universal covering. The manifold M will be called *rationally essential* if $\Phi_*[M]_\mathbb{Z} \neq 0 \in H_n(B\pi_1(M); \mathbb{Q})$.

Remark. Note that $B\pi_1(M) = K(\pi_1(M), 1)$. Since the former notation seems to be more common in this context, it will be used throughout this chapter. Note also that for orientable manifolds the definition of essentialness from paragraph 2.2.5 is equivalent to $\Phi_*[M]_\mathbb{Z} \neq 0 \in H_n(B\pi_1(M); \mathbb{Z})$ by Corollary 2.7.

In [HanS06] and [HanS07], enlargeable manifolds are shown to be rationally essential using index theory. Relying on ideas from coarse geometry, it is proved in [HanKRS07] that a manifold whose universal covering is hyperspherical is also rationally essential. (The universal covering of a closed orientable manifold M is called *hyperspherical* if M is enlargeable and the covering $\bar{M}_\varepsilon$ in the definition of enlargeability may always be chosen as the universal covering.)

More generally, the following notions of largeness will be considered: M is enlargeable, the universal covering $\tilde{M}$ is hypereuclidean or hyperspherical, and the filling radius of $\tilde{M}$ is infinite. Moreover, we will investigate coarse analogs of the last three properties: $\tilde{M}$ is coarsely hypereuclidean, coarsely hyperspherical, or macroscopically large.

The definitions of these largeness properties are given in the next section, where we will also prove that infinte filling radius is in fact equivalent to its coarse analog macroscopic largeness (see Proposition 5.12). The word *large* will always be used as a placeholder for one these properties.

Extending the results of [HanS06], [HanS07], and [HanKRS07], large manifolds are shown to be rationally essential. The proof does not use index theory or coarse geometry (apart from the case of the three coarse largeness properties, of course). In fact, largeness is homologically invariant, and moreover each largeness property determines a subspace in group homology consisting of all classes represented by nonlarge manifolds. (This corresponds to Theorem 1.8 from the introductory chapter 1.)

Theorem 5.2. *Let π be a finitely presented group. There is a subspace V_0 of the vector space $H_n(B\pi;\mathbb{Q})$ with the following property: if M is a connected closed orientable n-dimensional manifold with fundamental group π and classifying map $\Phi : M \to B\pi$, then the class $\Phi_*[M]_{\mathbb{Z}}$ lies in V_0 if and only if the manifold is not large.*

Although some of these subspaces are contained in others, it is not known if they may differ or if they always coincide. It is unclear whether there are examples for which one of these subspaces is nontrivial. (Note that for $n \leq 3$ the subspace V_0 is in general not determined by the property in the theorem. For instance, there are finitely presented groups π which are not fundamental groups of manifolds of dimension $n \leq 3$. Thus, every subspace of $H_n(B\pi;\mathbb{Q})$ satisfies the property in the theorem, which is empty in this case. But look at paragraph 5.2.1 and particularly at Theorem 5.18 for the “right” definition of V_0 in any dimension.)

In the definition of enlargeability, the maps $\bar{M}_\varepsilon \to S^n$ are required to contract distances. If one replaces this condition by the requirement that they contract the volume of k-dimensional submanifolds, then M is called *k-enlargeable*. In the case $k = 2$, this property is also called *area-enlargeability*.

Relying on index theory, Hanke and Schick showed in [HanS07] that area-enlargeable manifolds are rationally essential. We are able to extend this result to higher k and obtain Theorem 1.9 from the introduction.

Theorem 5.3. *Let M be a connected closed orientable manifold. If M is k-enlargeable and satisfies*

$$\pi_i(M) = 0 \quad \textit{for} \quad 2 \le i \le k-1,$$

then M is rationally essential. In particular, area-enlargeable manifolds are rationally essential.

For $k \le 2$ the condition on the homotopy groups is to be understood as empty. Note that for $k > 2$ the condition is in fact necessary: for M any enlargeable manifold, the product $M \times S^2$ is 3-enlargeable but the classifying map $M \times S^2 \to B\pi_1(M)$ sends the fundamental class to zero, i. e. $M \times S^2$ is not rationally essential. In fact, the second homotopy group $\pi_2(M \times S^2) = \pi_2(M) \times \pi_2(S^2)$ is not trivial.

For $k \ge n+1$ the k-dimensional volume of any subset of S^n is zero, of course. Hence, the assumptions of Theorem 5.3 boil down to $\pi_i(M) = 0$ for $2 \le i \le k-1$ with $k-1 \ge n$. By the Hurewicz theorem, this implies that the universal covering of M has trivial homotopy groups in all degrees. Thus, it is contractible. Otherwise said, the manifold M is aspherical in this case.

Area-enlargeable spin manifolds do not carry a metric of positive scalar curvature. This was shown by Gromov and Lawson in [GroL83]. It is conjectured that no aspherical manifold carries a metric of positive scalar curvature. The conditions from Theorem 5.3 interpolate between these two cases: enlargeable and area-enlargeable manifolds on the one side and aspherical ones on the other side. Thus, it seems natural to conjecture that those conditions are also an obstruction to positive scalar curvature (in the spin case). In fact, the strong Novikov conjecture implies that rationally essential spin manifolds do not admit a positive scalar curvature metric (see [Ros83]).

In the next section, largeness properties for complete Riemannian manifolds are defined and compared to each other. The proofs of Theorem 5.2 and Theorem 5.3 are given in the last section.

5.1 Large Riemannian manifolds

Let $f : (M, g) \to (N, g')$ be a smooth map between two Riemannian manifolds, and let k be a positive integer.

Definition 5.4. The *k-dilation* of f is defined as

$$\mathrm{dil}_k(f) := \sup_{p \in M} \|\Lambda^k df_p\|,$$

the supremum of the norm of the k-fold exterior product of the differential df. Note that for $k = 1$ the *dilation* $\mathrm{dil}(f) := \mathrm{dil}_1(f)$ is the smallest Lipschitz constant for f.

Said differently, the k-dilation is the smallest number ε such that for any k-dimensional submanifold $D \subset M$ the k-dimensional volume of the image $f(D) \subset N$ is bounded by $\varepsilon \cdot \mathrm{Vol}_k(D)$.

Let $p \in M$ be a point, and let n be the dimension of M. Denote by $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ the eigenvalues of the Gram matrix of the pullback $df_p^* g'_{f(p)}$ with respect to g_p. Then $\|\Lambda^k df_p\| = \lambda_1 \cdot \ldots \cdot \lambda_k$. Therefore, the inequality

$$\mathrm{dil}_\ell(f)^{1/\ell} \leq \mathrm{dil}_k(f)^{1/k} \qquad (*)$$

holds for all $\ell \geq k$.

Let (V, g) be a connected complete orientable Riemannian manifold of dimension n. A choice of orientation for V defines a fundamental class $[V]_{\mathbb{Z}} \in H_n^{lf}(V; \mathbb{Z})$ in locally finite homology. Then the mapping degree is well-defined for proper maps to oriented manifolds and for maps to closed oriented manifolds that are constant outside a compact set.

We will recall various notions of largeness for (V, g), most of which where first formulated by Gromov (see for example [GroL83], [Gro86], and [Gro96]). Compare also [Cai94] and [Gut06b].

Definition 5.5. The Riemannian manifold (V, g) is called *k-hypereuclidean* if there is a proper map

$$f : (V, g) \to (\mathbb{R}^n, g_0)$$

to the Euclidean space of nonzero degree with finite k-dilation. It is called *k-hyperspherical* if for every $\varepsilon > 0$ there is a map

$$f_\varepsilon : (V, g) \to (S^n, g_1)$$

to the unit sphere that is constant outside a compact set and of nonzero degree such that $\mathrm{dil}_k(f_\varepsilon) \leq \varepsilon$. For $k = 1$ we will omit the number, and for $k = 2$ we will speak of *area-hypereuclidean* and *area-hyperspherical* manifolds.

By the inequality $(*)$, every k-hypereuclidean or k-hyperspherical manifold is also ℓ-hypereuclidean respectively ℓ-hyperspherical for any $\ell \geq k$. Since $\mathbb{R}^n$ is obviously hyperspherical, any k-hypereuclidean manifold is also k-hyperspherical. Note also that both notions depend only on the bi-Lipschitz type of the metric g.

Closely related to this is the notion of enlargeability. It was introduced by Gromov and Lawson in [GroL80] and [GroL83].

Definition 5.6. A connected orientable n-dimensional manifold V is called *k-enlargeable* if for every complete Riemannian metric g on V and every $\varepsilon > 0$ there is a Riemannian covering $(\bar{V}_\varepsilon, g)$ of V and a map

$$f_\varepsilon : (\bar{V}_\varepsilon, g) \to (S^n, g_1)$$

that is constant outside a compact set and of nonzero degree such that $\mathrm{dil}_k(f_\varepsilon) \leq \varepsilon$. If all coverings $\bar{V}_\varepsilon$ may be chosen spin, then V will be called *spin k-enlargeable*. As before, we will omit the number in the case $k = 1$ and speak of *area-enlargeable* manifolds in the case $k = 2$.

If V is closed, then all Riemannian metrics on V are bi-Lipschitz to each other and it is enough that V satisfies the above conditions with respect to one Riemannian metric.

The significance of this notion is demonstrated by the following theorem, which is proved in [GroL83], Theorem 6.12.

Theorem 5.7 (Gromov, Lawson). *If V is spin area-enlargeable, then it does not carry a complete Riemannian metric of positive scalar curvature.*

Next, we will investigate the notion of infinite filling radius. Recall that every Riemannian metric g induces a path metric d_g on V. Denote by $L^\infty(V)$ the vector space of all functions on V with the uniform 'norm' $\|\_\|_\infty$. Note that this is not a norm since it may take infinite values. Therefore, the induced 'metric' is not an actual metric. Nevertheless, the *Kuratowski embedding*

$$\begin{aligned} \iota_g : (V, d_g) &\hookrightarrow L^\infty(V), \\ v &\mapsto d_g(v, \_) \end{aligned}$$

is an isometric embedding by the triangle inequality.

One could replace $L^\infty(V)$ by its affine subspace $L^\infty(V)_b$ that is parallel to the Banach space of all bounded functions on V and contains the distance function $d_g(v, \_)$ for some $v \in V$. Then the image of the Kuratowski embedding is contained in $L^\infty(V)_b$, and the 'norm' $\|\_\|_\infty$ induces an actual metric on $L^\infty(V)_b$. Since all points of $L^\infty(V)$ outside of this affine subspace are already infinitely far away from it, this would not change the following definition.

Definition 5.8. The *filling radius* is defined as

$$\mathrm{FillRad}(V, g) := \inf\{r > 0 | \iota_{g*}[V]_\mathbb{Z} = 0 \in H_n^{lf}(U_r(\iota_g V); \mathbb{Q})\}$$

where $U_r(\iota_g V) \subset L^\infty(V)$ denotes the r-neighborhood of the image $\iota_g V$.

Remark. Note that for closed manifolds $L^\infty(V)_b$ is the vector space of all bounded functions on V. Therefore, the above definition of the filling radius coincides with the definition from chapter 3. For noncompact manifolds the filling radius does not need to be finite. For instance, the filling radius of the Euclidean space is infinite.

The space $L^\infty(S)$ of all functions on an arbitrary set S fulfills the following *universal property*, which is proved in the same way as Lemma 3.3.

Lemma 5.9 ([Gro83], page 8). *If $Y \subset X$ is a subspace of a metric space and if $f : Y \to L^\infty(S)$ is an L-Lipschitz map, then there exists an extension $F : X \to L^\infty(S)$ which is also L-Lipschitz.*

Note that if f is proper (i. e. preimages of bounded sets are bounded) and $d(\_, Y)$ is uniformly bounded in X, then F is proper, too. This shows in particular that the property $\mathrm{FillRad}(V, g) = \infty$ depends only on the bi-Lipschitz type of the metric g.

Lemma 5.10 (Gromov, see [Gro86]). *If (V, g) is hyperspherical, then its filling radius is infinite.*

Proof. Assume that (V, g) is hyperspherical and that $\mathrm{FillRad}(V, g) < r$ for some finite r. Choose $\varepsilon > 0$ such that $\varepsilon r < \mathrm{FillRad}(S^n, g_1)$. Let $f_\varepsilon : (V, g) \to (S^n, g_1)$ be an ε-contracting map that sends the complement of a compact set $K \subset V$ to a point $p \in S^n$ and that has nonzero degree.

Identify V and S^n with their images under the respective Kuratowski embeddings. By the universal property, there is an ε-contracting map $F : L^\infty(V)_b \to L^\infty(S^n)$ that extends f_ε. Then $U_r(V)$ is mapped to $U_{\varepsilon r}(S^n)$ and $U_r(V \setminus K)$ to $U_{\varepsilon r}(p) \subset U_{\varepsilon r}(S^n)$. Therefore,

$$\deg(f_\varepsilon)[S^n]_{\mathbb{Z}} = f_{\varepsilon *}[V]_{\mathbb{Z}} = 0 \in H_n(U_{\varepsilon r}(S^n), U_{\varepsilon r}(p); \mathbb{Q}).$$

But $\deg(f_\varepsilon) \neq 0$, and since $\mathrm{FillRad}(S^n, g_1) > \varepsilon r$ it follows that

$$[S^n]_{\mathbb{Z}} \neq 0 \in H_n(U_{\varepsilon r}(S^n); \mathbb{Q}) \cong H_n(U_{\varepsilon r}(S^n), U_{\varepsilon r}(p); \mathbb{Q}).$$

This contradiction shows that the filling radius of (V, g) has to be infinite. □

Remark. We have seen that hypereuclidean implies hyperspherical implies infinite filling radius. It is not known whether these implications are equivalences or not.

In [GonY00], Gong and Yu used coarse algebraic topology to define another notion of largeness, which is said to be closely related to Gromov's definitions. In fact, we will show that it is equivalent to the property of having infinite filling radius.

First, we will recall the definition of coarse homology. For more details on coarse geometry we refer to Roe's book [Roe03] and particularly to chapter 5 on coarse algebraic topology.

Let X be a metric space. A cover $\mathcal{U}$ of X is called *uniform* if the diameters of its sets are uniformly bounded and if every bounded set in X meets only finitely many sets of $\mathcal{U}$. A collection $\{\mathcal{U}_i\}$ of uniform covers is called *anti-Čech system* if for every $r > 0$ there exists a cover $\mathcal{U}_i$ with Lebesgue number at least r.

The *nerve* of a cover $\mathcal{U}$ will be denoted by $|\mathcal{U}|$. It is the simplicial complex whose simplices are finite subsets of $\mathcal{U}$ with nonempty intersection in X. In particular, the set of vertices is $\mathcal{U}$. The nerve of a uniform cover is locally finite.

If $\mathcal{U}$ and $\mathcal{V}$ are two uniform covers such that the Lebesgue number of $\mathcal{V}$ is bigger than the uniform bound on the diameters of the sets of $\mathcal{U}$, then there is a proper simplicial map $|\mathcal{U}| \to |\mathcal{V}|$ mapping each vertex $U \in \mathcal{U}$ to some vertex $V \in \mathcal{V}$ that contains U. The proper homotopy class of this map is uniquely determined.

Given an anti-Čech system $\{\mathcal{U}_i\}$ one defines the *coarse homology* of X as

$$HX_k(X;\mathbb{Q}) := \varinjlim H_k^{lf}(|\mathcal{U}_i|;\mathbb{Q}).$$

This is independent of the choice of the anti-Čech system.

If X is assumed to be proper (i. e. bounded closed sets are compact), then for any uniform cover $\mathcal{U}$ there is a proper map $X \to |\mathcal{U}|$ that sends each point $x \in X$ to a point in the simplex spanned by those $U \in \mathcal{U}$ that contain x. Moreover, the proper homotopy class of such a map is uniquely determined. Therefore, one gets an induced homomorphism

$$c : H_k^{lf}(X;\mathbb{Q}) \to HX_k(X;\mathbb{Q}),$$

which will be called the *character homomorphism* of X.

Definition 5.11. A connected complete orientable n-dimensional Riemannian manifold (V, g) is called *macroscopically large* if

$$c[V]_{\mathbb{Z}} \neq 0 \in HX_n(V;\mathbb{Q}).$$

Note that this property depends only on the quasi-isometry class of the metric. We will show that macroscopic largeness is equivalent to infinite filling radius. This proves in particular that the property of infinite filling radius depends also only on the quasi-isometry class of the Riemannian metric.

Proposition 5.12. *Let (V, g) be a connected complete orientable Riemannian manifold. Then (V, g) is macroscopically large if and only if its filling radius is infinite.*

We will need the notion of coarse map. The general definition is a bit involved. But recall from [Roe03], chapter 1.3 that a map $f : X \to Y$ from a path matric space to a metric space is *coarse* if and only if it is large scale Lipschitz and proper (i. e. preimages of bounded sets are bounded).

Proof. Identify V with its image under the Kuratowski embedding, and denote the dimension of V by n.

First assume that $\mathrm{FillRad}(V, g) < r$ for some finite r. Then there is a locally finite complex $X \subset U_r(V)$ containing V such that $[V]_{\mathbb{Z}} = 0 \in H_n^{lf}(X; \mathbb{Q})$. Moreover, the inclusion $V \hookrightarrow X$ is a coarse equivalence since the coarse map that assigns to a point $x \in X$ a point $v \in V$ such that $d(x, v) \le r$ is an inverse. The commutative diagram

$$\begin{array}{ccc} H_n^{lf}(V;\mathbb{Q}) & \longrightarrow & H_n^{lf}(X;\mathbb{Q}) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ HX_n(V;\mathbb{Q}) & \xrightarrow{\cong} & HX_n(X;\mathbb{Q}) \end{array}$$

shows that $c[V]_{\mathbb{Z}} = 0 \in HX_n(V; \mathbb{Q})$, i. e. (V, g) is not macroscopically large. (Note that $U_r(V)$ is also coarsely equivalent to V but that it is not proper. Therefore, it is not clear whether it admits a character homomorphism.)

To prove the converse implication, assume that (V, g) is not macroscopically large. By the definition of the direct limit there is a uniform cover $\mathcal{U}$ of V such that $\phi_*[V]_{\mathbb{Z}} = 0 \in H_n^{lf}(|\mathcal{U}|; \mathbb{Q})$ where $\phi : V \to |\mathcal{U}|$ is a proper map that sends each point $v \in V$ to a point in the simplex spanned by those $U \in \mathcal{U}$ that contain v. Let $r > 0$ be an upper bound on the diameters of the sets of $\mathcal{U}$.

Define a map $\psi : |\mathcal{U}| \to L^\infty(V)$ by sending each vertex $U \in \mathcal{U}$ to some point $\psi(U) \in U \subset V$ and by extending this linearly over each simplex of the nerve.

Let p be a point in $|\mathcal{U}|$. It may be written as $p = \sum \lambda_i U_i$ with $\sum \lambda_i = 1$, $\lambda_i > 0$, and $U_i \in \mathcal{U}$ such that $\bigcap U_i \neq \emptyset$. Then $\psi(p) = \sum \lambda_i \psi(U_i)$ and

$$\begin{aligned} d(\psi(p), \psi(U_1)) &= \Big\| \sum \lambda_i \psi(U_i) - \psi(U_1) \Big\|_\infty \\ &\le \sum \lambda_i \|\psi(U_i) - \psi(U_1)\|_\infty \\ &\le 2r \end{aligned}$$

since $U_i \cap U_1 \neq \emptyset$. This shows that the image of ψ lies in the $2r$-neighborhood of V in $L^\infty(V)$. Hence

$$(\psi \circ \phi)_*[V]_{\mathbb{Z}} = 0 \in H_n^{lf}(U_{2r}(V); \mathbb{Q}).$$

Let $v \in V$ be a point. Say v lies in the sets $U_1, \dots, U_m \in \mathcal{U}$ and in no other set of $\mathcal{U}$. Then $\phi(v) = \sum \lambda_i U_i$ for some $\lambda_i \geq 0$ with $\sum \lambda_i = 1$. Therefore,

$$\begin{aligned} d(\psi(\phi(v)), v) &= \left\| \sum \lambda_i \psi(U_i) - v \right\|_\infty \\ &\leq \sum \lambda_i \|\psi(U_i) - v\|_\infty \\ &\leq r \end{aligned}$$

since $v \in U_i$. Thus, the linear homotopy from the inclusion $V \hookrightarrow L^\infty(V)$ to $\psi \circ \phi$ is proper and lies entirely in $U_r(V)$. Therefore

$$[V]_{\mathbb{Z}} = (\psi \circ \phi)_*[V]_{\mathbb{Z}} \in H_n^{lf}(U_r(V); \mathbb{Q}),$$

and consequently $[V]_{\mathbb{Z}} = 0 \in H_n^{lf}(U_{2r}(V); \mathbb{Q})$ and $\mathrm{FillRad}(V, g) \leq 2r < \infty$. □

Lemma 5.10 and Proposition 5.12 show that hyperspherical manifolds are macroscopically large. This is proved directly in [HanKRS07], Theorem 1.3 (1) using the *balloon space* B^n. This path metric space is defined as a real half-line $[0, \infty)$ with an n-dimensional round sphere S_i^n of radius i attached at each positive integer $i \in [0, \infty)$.

Proposition 5.13 ([HanKRS07], Proposition 2.2)**.** *The coarse homology in dimension n of the balloon space is given by*

$$HX_n(B^n; \mathbb{Q}) \cong \left(\prod_{i=1}^{\infty} \mathbb{Q} \right) / \left(\bigoplus_{i=1}^{\infty} \mathbb{Q} \right).$$

Moreover, its locally finite homology is given by $H_n^{lf}(B^n; \mathbb{Q}) \cong \prod_{i=1}^\infty \mathbb{Q}$, and the character homomorphism $c : H_n^{lf}(B^n; \mathbb{Q}) \to HX_n(B^n; \mathbb{Q})$ is the canonical projection.

Using this computation, the following characterization of hyperspherical manifolds follows (see [HanKRS07], Proposition 3.1).

Lemma 5.14. *A connected complete orientable Riemannian manifold (V, g) of dimension n is hyperspherical if and only if there exists a proper Lipschitz map $f : (V, g) \to B^n$ such that $f_*[V]_{\mathbb{Z}} \neq 0 \in HX_n(B^n; \mathbb{Q})$.*

Proof. First, assume that (V, g) is hyperspherical. We will construct a sequence of closed balls

$$\emptyset = B_0 \subset B_1 \subset B_2 \subset \ldots \subset V$$

that exhausts V and a sequence of 1-Lipschitz maps $f_i : B_i \setminus \mathring{B}_{i-1} \to S^n_i \vee [i, i+1] \subset B^n$ such that $f_i(\partial B_{i-1}) = i$, $f_i(\partial B_i) = i + 1$, and such that f_i is of nonzero degree as a map to S^n_i.

Assume that the balls and maps have been constructed up to index $i - 1$. Let S^n_R be the round sphere of radius R with R large. Choose a 1-Lipschitz map $f'_i : (V, g) \to S^n_R$ that is constant outside a compact set K_i and that is of nonzero degree. Without loss of generality, we may assume that $B_{i-1} \subset K_i$ and that $f'_i(B_{i-1})$ and the point $f'_i(V \setminus K_i)$ avoid a ball of radius πi inside S^n_R (choose for instance $R \geq 2i + r/\pi$ with r the radius of B_{i-1}). Let $g_i : S^n_R \to S^n_i$ be a nonexpanding map that contracts everything outside this ball of radius πi to a point.

Choose a ball $B_i \subset V$ such that $K_i \subset B_i$ and such that $d(\partial B_i, K_i) \geq 1$. Define f_i as follows:

$$f_i(v) := \begin{cases} g_i \circ f'_i(v) & \text{for } v \in B_i \setminus \mathring{B}_{i-1}, d(v, \partial B_i) \geq 1 \\ i + 1 - d(v, \partial B_i) & \text{for } v \in B_i, d(v, \partial B_i) \leq 1 \end{cases}$$

Then f_i has the asserted properties.

All the maps f_i together define a proper 1-Lipschitz map $f : (V, g) \to B^n$ such that every entry of $f_*[V]_\mathbb{Z} \in H^{lf}_n(B^n; \mathbb{Q}) \cong \prod_{i=1}^\infty \mathbb{Q}$ is nonzero, in particular $f_*[V]_\mathbb{Z} \neq 0 \in HX_n(B^n; \mathbb{Q})$.

Now, let $f : (V, g) \to B^n$ be a proper Lipschitz map such that $f_*[V]_\mathbb{Z} \neq 0 \in HX_n(B^n; \mathbb{Q})$. Let $\varepsilon > 0$, and choose an integer $i \geq \mathrm{dil}(f)/\varepsilon$ such that the i-th entry of $f_*[V]_\mathbb{Z} \in H^{lf}_n(B^n; \mathbb{Q}) \cong \prod_{i=1}^\infty \mathbb{Q}$ is not zero. This is possible since by assumption there are infinitely many nonvanishing entries.

Let f_ε be the composition of f with the canonical quotient map from B^n to the i-th sphere S^n_i and the dilation from this sphere of radius i to the unit sphere. Then f_ε is constant outside a compact set, has nonzero degree, and its dilation is given by $\mathrm{dil}(f)/i \leq \varepsilon$. This proves that (V, g) is hyperspherical. □

Definition 5.15. The Riemannian manifold (V, g) is called *coarsely hypereuclidean* if there is a coarse map

$$f : (V, g) \to (\mathbb{R}^n, g_0)$$

to the Euclidean space such that $f_*[V]_\mathbb{Z} \neq 0 \in HX_n(\mathbb{R}^n; \mathbb{Q}) \cong \mathbb{Q}$. It is called *coarsely hyperspherical* if there is a coarse map

$$f : (V, g) \to B^n$$

to the balloon space such that $f_*[V]_{\mathbb{Z}} \neq 0 \in HX_n(B^n;\mathbb{Q})$.

These two notions depend only on the quasi-isometry class of the metric. Obviously, hypereuclidean manifolds are coarsely hypereuclidean, and hyperspherical manifolds are coarsely hyperspherical. Moreover, coarse hypereuclideaness implies coarse hypersphericity, which again implies macroscopic largeness. It is not known if any of these implications are equivalences.

5.2 Essentialness and homological invariance

This section consists of two parts. In the first one, we introduce versions of the largeness properties defined above for maps from manifolds to CW complexes. Moreover, we state and prove a theorem on homological invariance for these properties that implies Theorem 5.2. The second half of this section is devoted to the proof of Theorem 5.3.

5.2.1 Large homology classes

Let X be a connected CW complex. A homology class $a \in H_n(X;\mathbb{Q})$ is said to be *represented* by a map $\Phi : M \to X$ where M is a connected closed orientable n-dimensional manifold if Φ maps the fundamental class of M to a. By Thom's work [Tho54], a nonzero multiple of each homology class can be represented in such a way.

If $\pi_1(X)$ is finitely generated, then by connected sum with enough copies of $S^{n-1} \times S^1$ one can easily achieve that the induced homomorphism $\Phi_* : \pi_1(M) \to \pi_1(X)$ is surjective whenever the dimension n is at least two. Moreover, if $\pi_1(X)$ is finitely presented, then it is possible in dimensions $n \geq 4$ to alter M by surgery such that its fundamental group is isomorphic to $\pi_1(X)$. (This fails for $n \leq 3$ because not every finitely presented group is the fundamental group of a surface or a three-manifold.)

To prove Theorem 5.2, the following notions of largeness for maps are handy because they allow us to avoid this surgery process.

Definition 5.16. Let X be a connected CW complex with countable fundamental group, and let $\Phi : M \to X$ be a map from a connected closed orientable n-dimensional manifold. Let g be a Riemannian metric on M.

(i) The map Φ is called *enlargeable* if for every $\varepsilon > 0$ there is a connected covering $\bar{X}_\varepsilon$ such that there exists an ε-contracting map $f_\varepsilon : (\bar{M}_\varepsilon, g) \to (S^n, g_1)$ from the pullback $\bar{M}_\varepsilon := \Phi^*\bar{X}_\varepsilon$ that is constant outside a compact set and of nonzero degree.

(ii) The map is called *(coarsely) hypereuclidean*, *(coarsely) hyperspherical*, respectively *macroscopically large* if the pullback $\tilde{M}_\Phi := \Phi^*\tilde{X}$ has the according property where $\tilde{X}$ denotes the universal covering of the CW complex X.

Note that these definitions do not depend on the metric g since M is compact. The assumption that the fundamental group of X is countable guarantees that every connected covering of X has at most countably many sheets. Thus, the pullback to M is always a manifold.

The term *large* will serve as a placeholder for one of the six properties of maps that are defined above. First, we will prove that largeness depends only on the represented class, not on the representation itself.

Theorem 5.17. *Let X be a connected CW complex with countable fundamental group, let M and N be two connected closed orientable manifolds of dimension n, and let $\Phi : M \to X$ and $\Psi : N \to X$ be two maps. If $\Phi_*[M]_\mathbb{Z} = q \cdot \Psi_*[N]_\mathbb{Z}$ in $H_n(X;\mathbb{Q})$ for some rational number $q \neq 0$, then Φ is large if and only if Ψ is large.*

Proof. By symmetry it suffices to consider the case where Φ is large. Assume without loss of generality that $M \subset X$ is a subcomplex and that $\Phi : M \hookrightarrow X$ is the inclusion (replace X by the mapping cylinder of Φ, which is homotopy equivalent to X). Then there exists a finite subcomplex $S \subset X$ containing M and $\Psi(N)$ such that $\Phi_*[M]_\mathbb{Z} = q \cdot \Psi_*[N]_\mathbb{Z}$ in $H_n(S;\mathbb{Q})$.

Case 1: Let Φ be enlargeable. Since S is obtained from M by successive attachments of finitely many cells, it has the following property: there is a piecewise smooth Riemannian metric (see Definition 2.11) on S such that for every $\varepsilon > 0$ there is a covering $\bar{X}_\varepsilon$ and an ε-contracting map $f_\varepsilon : (\bar{S}_\varepsilon, g) \to (S^n, g_1)$ from the pullback $\bar{S}_\varepsilon := i^*\bar{X}_\varepsilon$ that is constant outside a compact set such that $f_{\varepsilon *}[\bar{M}_\varepsilon]_\mathbb{Z} \neq 0$ where $\bar{M}_\varepsilon \subset \bar{S}_\varepsilon$ is the corresponding covering of M. This property of the inclusion $i : S \hookrightarrow X$ will be called *enlargeability with respect to $M \subset S$.*

To check that the inclusion of S has this property, we proceed by induction over the number of attached cells. Say we have attached all cells but one and obtained (S', g) such that the inclusion $i : S' \hookrightarrow X$ is enlargeable with respect to M. Let $h : S^{k-1} \to S'$ be the attaching map of the last remaining cell, i. e. $S = S' \cup_h D^k$. Let d denote the diameter of the image of a lift of h to the universal covering of S'.

Extend the metric g over S in the following way: think of S as

$$S' \quad \cup_h \quad (S^{k-1} \times [-1,0]) \quad \cup \quad (S^{k-1} \times [0,d]) \quad \cup \quad S^k_+$$

and define the metric by taking

$$g, \quad (-th^*g + (1+t)g_r) \oplus dt^2, \quad g_r \oplus dt^2, \quad g_r$$

on the respective parts. Here, g_r is the round metric of radius r on S^{k-1}, respectively on the hemisphere S^k_+, with r chosen such that $h : (S^{k-1}, g_r) \to (S', g)$ is 1-contracting.

Let $\varepsilon > 0$, and let $\bar{X}_\varepsilon$, $\bar{S}'_\varepsilon$, and $f'_\varepsilon : \bar{S}'_\varepsilon \to S^n$ be as in the claim. Let $\bar{S}_\varepsilon$ be the pullback of $\bar{X}_\varepsilon$ over S. Then $\bar{S}_\varepsilon$ is obtained by attachment of one k-cell for each lift h_i of h to $\bar{S}'_\varepsilon$. The diameters of the h_i are at most d, and the ε-contracting map f'_ε maps $h_i(S^{k-1})$ to a ball of radius εd.

Therefore, the map f'_ε can be extended in the following way: project $S^{k-1} \times [-1, 0]$ to the first factor and apply $f'_\varepsilon \circ h_i$, the cylinder lines $x \times [0, d]$ are mapped to the shortest geodesic connecting $f'_\varepsilon \circ h_i(x)$ and the center of the ball containing $f'_\varepsilon \circ h_i(S^{k-1})$, and the cap S^k_+ is mapped to this center. If $\varepsilon > 0$ is small enough, this map is welldefined. (If $h_i(S^{k-1})$ is mapped to a point, then we extend f'_ε by mapping the attached cell to this point, of course.) The map thus defined is ε-contracting and does not map $[\bar{M}_\varepsilon]_{\mathbb{Z}}$ to zero since it extends f'_ε. Therefore, the inclusion of S into X is enlargeable with respect to M as claimed.

Choose a Riemannian metric g' on N such that $\Psi : (N, g') \to (S, g)$ is 1-Lipschitz. Let $\bar{N}_\varepsilon$ be the pullback of $\bar{S}_\varepsilon$ via Ψ. Then Ψ lifts to a proper 1-Lipschitz map $\bar{\Psi}_\varepsilon : \bar{N}_\varepsilon \to \bar{S}_\varepsilon$ such that $\bar{\Psi}_{\varepsilon *}[\bar{N}_\varepsilon]_{\mathbb{Z}} = 1/q \cdot [\bar{M}_\varepsilon]_{\mathbb{Z}}$. Hence,

$$f_\varepsilon \circ \bar{\Psi}_\varepsilon : (\bar{N}_\varepsilon, g') \to (S^n, g_0)$$

is an ε-contracting map which is constant outside a compact set and of nonzero degree. This shows that Ψ is enlargeable.

Case 2: If Φ is hypereuclidean or hyperspherical, then the same argument proves Theorem 5.17. Note that in the hypereuclidean case the map $\tilde{S}_i \to \mathbb{R}^n$ constructed as above is indeed proper (and Lipschitz). Here, $\tilde{S}_i$ denotes the pullback of the universal covering $\tilde{X}$.

Case 3: Let Φ be macroscopically large. Then $\tilde{M}_\Phi$, $\tilde{N}_\Psi$ and $\tilde{S}_i$ are coarsely equivalent to each other with the coarse equivalences given by the lifts $\tilde{\Phi} : \tilde{M}_\Phi \to \tilde{S}_i$ and $\tilde{\Psi} : \tilde{N}_\Psi \to \tilde{S}_i$ of Φ and Ψ respectively. Hence, the following commutative diagram shows that $\tilde{N}_\Psi$ is macroscopically large:

$$\begin{array}{ccc}
H^{lf}_n(\tilde{N}_\Psi;\mathbb{Q}) & \xrightarrow{c} & HX_n(\tilde{N}_\Psi;\mathbb{Q}) \\
\downarrow{\scriptstyle \tilde{\Psi}_*} & & \downarrow{\scriptstyle \tilde{\Psi}_*}\ \cong \\
H^{lf}_n(\tilde{S}_i;\mathbb{Q}) & \xrightarrow{c} & HX_n(\tilde{S}_i;\mathbb{Q}) \\
\uparrow{\scriptstyle \tilde{\Phi}_*} & & \uparrow{\scriptstyle \tilde{\Phi}_*}\ \cong \\
H^{lf}_n(\tilde{M}_\Phi;\mathbb{Q}) & \xrightarrow{c} & HX_n(\tilde{M}_\Phi;\mathbb{Q})
\end{array}$$

From $c([\tilde{M}_\Phi]_\mathbb{Z}) \neq 0$ and $\tilde{\Phi}_*[\tilde{M}_\Phi]_\mathbb{Z} = q \cdot \tilde{\Psi}_*[\tilde{N}_\Psi]_\mathbb{Z} \in H_n^{lf}(\tilde{S}_i;\mathbb{Q})$ follows that the class $\tilde{\Psi}_* c([\tilde{N}_\Psi]_\mathbb{Z}) = 1/q \cdot \tilde{\Phi}_* c([\tilde{M}_\Phi]_\mathbb{Z})$ is not zero. Consequently, $c([\tilde{N}_\Psi]_\mathbb{Z}) \neq 0$, i. e. Ψ is macroscopically large.

Case 4: If Φ is coarsely hypereuclidean or coarsely hyperspherical, then compose the coarse equivalence $(\tilde{\Phi})^{-1} \circ \tilde{\Psi} : \tilde{N}_\Psi \to \tilde{M}_\Phi$ with the coarse map $\tilde{M}_\Phi \to \mathbb{R}^n$ respectively $\tilde{M}_\Phi \to B^n$. This composition maps the fundamental class of $\tilde{N}_\Psi$ to a nonzero class in the coarse homology of the Euclidean space respectively of the balloon space. Thus, Ψ is coarsely hypereuclidean respectively coarsely hyperspherical.

With this last case Theorem 5.17 is finally proved. □

A homology class $a \in H_n(X;\mathbb{Q})$ will be called *large* if there exists a large representation $\Phi : M \to X$ of a nonzero multiple of a. By Theorem 5.17, every representation of a nonzero multiple of a has then this property. Recall that by Thom's work for every homology class there is a nonzero multiple that is representable by a manifold (see [Tho54]).

Theorem 5.18. *Let X be a connected CW complex with countable fundamental group. The nonlarge homology classes in $H_n(X;\mathbb{Q})$ form a subspace.*

Obviously, this theorem generalizes Theorem 5.2.

Proof. Let $V_0 \subset H_n(X;\mathbb{Q})$ be the subset of all nonlarge homology classes.

First, note that the zero homology class is not large: $0 \in V_0$. (This may be restated by saying that largeness implies rational essentialness.) To see this, let $\Phi : M \to X$ be a representation of $0 \in H_n(X;\mathbb{Q})$. As in the preceeding proof assume without loss of generality that M is a subcomplex of X and that Φ is the inclusion. Then there is a subcomplex $M \subset S \subset X$ such that $[M]_\mathbb{Z} = 0 \in H_n(S;\mathbb{Q})$. Hence $[\bar{M}_\varepsilon]_\mathbb{Z} = 0 \in H_n^{lf}(\bar{S}_\varepsilon;\mathbb{Q})$ and $[\tilde{M}_\Phi]_\mathbb{Z} = 0 \in H_n^{lf}(\tilde{S}_i;\mathbb{Q})$, and the reasoning of the proof of Theorem 5.17 shows that Φ cannot be large.

There is also an easier argument: the zero class is represented by a constant map from the n-sphere. Since S^n is its own universal covering, no map with domain S^n can be large. This shows that the subspace V_0 contains the image of the Hurewicz homomorphism $\pi_n(X) \to H_n(X;\mathbb{Q})$, in particular $0 \in V_0$.

Since Theorem 5.17 shows that V_0 is closed under multiplication by scalars, it remains to consider addition. Let a and b be two nonlarge homology classes. There is a nonzero rational number q such that qa and qb are representable by manifolds, say by $\Phi : M \to X$ and by $\Psi : N \to X$ respectively. Then Φ and Ψ are both not large.

The class $q(a+b)$ is represented by

$$\Theta : M\#N \to M \vee N \xrightarrow{\Phi\vee\Psi} B\pi$$

where the first map contracts the 'belt sphere' of the connected sum.

Case 1: Assume Θ is enlargeable. Attach an n-cell to $M\#N$ along the belt sphere. Denote the thus obtained complex by S. The obvious extension $\Theta' : S \to X$ of Θ is enlargeable with respect to $M\#N$ (see case 1 in the proof of Theorem 5.17). The coverings $\bar{S}_\varepsilon$ are obtained from the respective coverings $\bar{M}_\varepsilon$ and $\bar{N}_\varepsilon$ by identifying the lifts of balls. Therefore,

$$H_n^{lf}(\bar{S}_\varepsilon;\mathbb{Q}) \cong H_n^{lf}(\bar{M}_\varepsilon;\mathbb{Q}) \oplus H_n^{lf}(\bar{N}_\varepsilon;\mathbb{Q})$$

and $[(\overline{M\#N})_\varepsilon]_\mathbb{Z}$ corresponds to $[\bar{M}_\varepsilon]_\mathbb{Z} + [\bar{N}_\varepsilon]_\mathbb{Z}$. Consequently, either Φ or Ψ is enlargeable.

Case 2: Assume that Θ is (coarsely) hypereuclidean or (coarsely) hyperspherical. Consider the pullback coverings

$$(\widetilde{M\#N})_\Theta, \tilde{S}_{\Theta'}, \tilde{M}_\Phi \text{ and } \tilde{N}_\Psi$$

of the universal covering $\tilde{X}$. As in the preceeding proof $\tilde{S}_{\Theta'}$ is (coarsely) hypereuclidean respectively (coarsely) hyperspherical. Since

$$H_n^{lf}(\tilde{S}_{\Theta'};\mathbb{Q}) \cong H_n^{lf}(\tilde{M}_\Phi;\mathbb{Q}) \oplus H_n^{lf}(\tilde{N}_\Psi;\mathbb{Q})$$

it follows that either $\tilde{M}_\Phi$ or $\tilde{N}_\Psi$ has the same property.

Case 3: Assume that Θ is macroscopically large. The pullback coverings

$$(\widetilde{M\#N})_\Theta, \tilde{S}_{\Theta'}, \tilde{M}_\Phi \text{ and } \tilde{N}_\Psi$$

are coarsely equivalent to each other. Therefore, all four coarse homology groups agree and since

$$H_n^{lf}(\tilde{S}_{\Theta'};\mathbb{Q}) \cong H_n^{lf}(\tilde{M}_\Phi;\mathbb{Q}) \oplus H_n^{lf}(\tilde{N}_\Psi;\mathbb{Q})$$

the following commutative diagram shows that either $\tilde{M}_\Phi$ or $\tilde{N}_\Psi$ is macroscopically large:

$$\begin{array}{ccc} H_n^{lf}((\widetilde{M\#N})_\Theta;\mathbb{Q}) & \xrightarrow{c} & HX_n((\widetilde{M\#N})_\Theta;\mathbb{Q}) \\ \downarrow & & \downarrow \cong \\ H_n^{lf}(\tilde{S}_{\Theta'};\mathbb{Q}) & \xrightarrow{c} & HX_n(\tilde{S}_{\Theta'};\mathbb{Q}) \end{array}$$

This finishes the proof of Theorem 5.18. □

The subspace $V_0 \subset H_n(X;\mathbb{Q})$ contains all spherical homology classes, i. e. all classes representable by maps from the n-sphere. If $X = B\pi$ for some countable group π, then the only spherical class is the zero class. We do not know any example of a group π for which V_0 is not trivial.

5.2.2 Higher enlargeability implies essentialness

The notion of k-dilation extends in an obvious way to piecewise smooth maps of simplicial complexes with piecewise smooth Riemannian metrics (see Definition 2.11). For Riemannian manifolds this new definition coincides with Definition 5.4. Moreover, the inequality

$$\mathrm{dil}_\ell(f)^{1/\ell} \leq \mathrm{dil}_k(f)^{1/k}$$

for $\ell \geq k$ remains valid.

Let X be a connected finite simplicial complex, and let $a \in H_n(X;\mathbb{Q})$ be a cohomology class. If $\bar{X} \to X$ is a (possibly infinite) covering, then a defines a locally finite homology class $\bar{a} \in H_n^{lf}(\bar{X};\mathbb{Q})$ as follows: if $\sum r_\sigma \sigma$ is a cycle in X that represents a, then $\bar{a}$ is represented by the locally finite cycle $\sum r_\sigma \bar{\sigma}$ where $\bar{\sigma}$ runs over all lifts of σ to $\bar{X}$.

Choose a Riemannian metric g on X. Then X is called *k-enlargeable with respect to a* if for every $\varepsilon > 0$ there exists a Riemannian covering $(\bar{X}_\varepsilon, g)$ of (X, g) and a (piecewise smooth) map $f_\varepsilon : (\bar{X}_\varepsilon, g) \to (S^n, g_1)$ with k-dilation at most ε which is constant outside a compact set and fulfills $f_{\varepsilon *}\bar{a} \neq 0$ where $\bar{a} \in H_n^{lf}(\bar{X}_\varepsilon;\mathbb{Q})$ is defined as above.

By the compactness of X this definition is independent of the choice of the Riemannian metric g.

Proposition 5.19. *Let X be a connected finite simplicial complex, and let $\ell \leq n$ be a positive integer. Assume that X is ℓ-enlargeable with respect to a homology class $a \in H_n(X;\mathbb{Q})$. Let X' be obtained from X by attachment of finitely many $(\ell+1)$-cells to the ℓ-skeleton $X^{(\ell)}$. Then X' is $(\ell+1)$-enlargeable with respect to the image of a in $H_n(X';\mathbb{Q})$.*

In the proof we will need the following lemma.

Lemma 5.20. *There exists a constant $C_n > 0$ depending only on n such that for any map $f : N \to S^n$ from an ℓ-dimensional manifold N with $\ell < n$ to the unit n-sphere there is a map $f' : N \to S^n$ to an $(\ell-1)$-dimensional subcomplex of S^n such that $d(f(x), f'(x)) \leq C_n \cdot \mathrm{Vol}_\ell(f(N))^{1/\ell}$ for all $x \in N$.*

Gromov showed a similar statement for the Euclidean space instead of the unit sphere ([Gro83], Proposition 3.1.A). The above lemma may be proved in an analogous manner or can easily be deduced from Gromov's result.

Proof of Proposition 5.19. Let g be a Riemannian metric on X. Since the attached cells do not interfere with each other we may assume to simplify matters that there is only one $(\ell+1)$-cell to attach. Let $h : S^\ell \to X^{(\ell)} \subset X$

be the (simplicial) attaching map. Choose a lift $\tilde{h} : S^\ell \to \tilde{X}$ to the universal covering, and denote the ℓ-dimensional volume of the image $\tilde{h}(S^\ell)$ by v.

First assume $\ell < n$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta \cdot C_n(\delta v)^{1/\ell} \leq \varepsilon$ and such that $\delta^{\frac{\ell+1}{\ell}} \leq \varepsilon$. Let $f_\delta : \bar{X}_\delta \to S^n$ be constant outside a compact set such that $f_{\delta *}\bar{a} \neq 0$ and $\operatorname{dil}_\ell(f_\delta) \leq \delta$. Note that

$$\operatorname{dil}_{\ell+1}(f_\delta) \leq \operatorname{dil}_\ell(f_\delta)^{\frac{\ell+1}{\ell}} \leq \delta^{\frac{\ell+1}{\ell}} \leq \varepsilon.$$

We extend the given metric g on X over the attached $(\ell+1)$-cell in the following way: think of X' as

$$X \quad \cup_h \quad (S^\ell \times [-1,0]) \quad \cup \quad (S^\ell \times [0,1]) \quad \cup \quad S^{\ell+1}_+$$

and define the metric by taking

$$g, \quad (-th^*g + (1+t)g_r) \oplus dt^2, \quad g_r \oplus dt^2, \quad g_r$$

on the respective parts. Here, g_r is the round metric of radius r on S^ℓ, respectively on the hemisphere $S^{\ell+1}_+$, with r chosen such that $h : (S^\ell, g_r) \to (X, g)$ is 1-contracting.

Moreover, we extend f_δ over the attached cells as follows: on $S^\ell \times [-1,0]$ we use the projection to S^ℓ and apply the attaching map h_i (which is some lift of h) and f_δ. The ℓ-dimensional volume of $f_\delta \circ h_i(S^\ell)$ is at most δv. By Lemma 5.20 there is a map $f' : S^\ell \to S^n$ to an $(\ell-1)$-dimensional subcomplex of the n-sphere such that $d(f', f_\delta \circ h_i) \leq C_n(\delta v)^{1/\ell}$. The cylinder lines $x \times [0,1]$ are mapped to minimizing geodesics from $f_\delta \circ h_i(x)$ to $f'(x)$. (For δ small enough this gives a well-defined map.) The remaining cap $S^{\ell+1}_+$ may be seen as the cone over $S^\ell \times 1$ and is mapped to some cone over the $(\ell-1)$-dimensional subcomplex to which $S^\ell \times 1$ is mapped. (If the attaching map is contracted to one point by f_δ, then the whole attached cell shall be mapped to this point, of course.)

By the choice of δ, this new map has $(\ell+1)$-dilation at most ε: on $S^\ell \times [-1,0]$ and the cap $S^{\ell+1}_+$ because they are mapped to ℓ-dimensional subcomplexes, which are zero sets for the $(\ell+1)$-dimensional volume, and on $S^\ell \times [0,1]$ because the ℓ-dimensional volume of the first factor is decreased by a factor of δ and the second factor is $(C_n(\delta v)^{1/\ell})$-contracted. Thus, X' is $(\ell+1)$-enlargeable with respect to the image of a in $H_n(X';\mathbb{Q})$.

Finally assume $\ell = n$. For $\varepsilon > 0$ such that $\varepsilon v < \operatorname{Vol}_n(S^n)$, the composition with f_ε of any lift h_i of the attaching map cannot be surjective. Hence, $f_\varepsilon \circ h_i$ is nullhomotopic and we may extend f_ε over $\bar{X}'_\varepsilon$. Since S^n is n-dimensional, every map to it has zero $(n+1)$-dilation. □

Next, we will show Theorem 5.3 by an inductive argument. In this proof, Proposition 5.19 will serve as the induction step.

Proof of Theorem 5.3. Let M be k-enlargeable, and let $\pi_i(M)$ be trivial for $2 \leq i \leq k-1$. Then it is possible to construct $B\pi_1(M)$ from M by attaching only cells of dimension at least $k+1$. We may assume that the image of the attaching map of every ℓ-cell lies in the $(\ell-1)$-skeleton.

If M is not rationally essential, then there is a finite subcomplex $X \subset B\pi_1(M)$ containing M such that $[M]_{\mathbb{Z}} = 0$ in $H_n(X;\mathbb{Q})$. We may assume that X is of dimension $n+1$. Then by an induction using Proposition 5.19, the simplicial complex X is $(n+1)$-enlargeable with respect to $[M]_{\mathbb{Z}} \in H_n(X;\mathbb{Q})$. But this contradicts the fact that $[M]_{\mathbb{Z}}$ vanishes in X. Therefore, M has to be rationally essential. □

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