Black hole attractors and the entropy function in four- and five-dimensional $N = 2$ supergravity

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Zusammenfassung

Extremale schwarze Löcher in Theorien, bei denen die Gravitation an abelsche Eichfelder und neutrale Skalare koppelt, wie sie bei der Niederenergie-Beschreibung der Kompaktifizierung der Stringtheorie auf Calabi–Yau-Mannigfaltigkeiten auftreten, zeigen das Attraktor-Phänomen: Am Ereignishorizont nehmen die Skalare Werte an, die durch die Ladungen, welche das Schwarze Loch trägt, festgelegt werden sowie unabhängig von den Werten im Unendlichen sind. Das ist so, weil die in Vektorfeldern enthaltene Energie am Ereignishorizont als effektives Potenzial wirkt (als black-hole-Potenzial), und die Skalare in seine Minima führt.

Im Falle von symmetrischen schwarzen Löchern in Theorien bei denen die Eichpotenziale in der Lagrangefunktion nur über Feldstärken erscheinen, kann das Attraktor-Phänomen alternativ mittels eines Variationsprinzips basierend auf der sogenannten Entropiefunktion beschrieben werden. Diese ist definiert als Legendre-Transformierte der Lagrangedichte in Bezug auf die elektrischen Felder, wobei über den Horizont integriert wird. Stationäritätsbedingungen für die Entropiefunktion nehmen dann die Form von Attraktorgleichungen an, die die Werte der Skalare am Horizont mit den Ladungen des schwarzen Loches in Beziehung setzen; der stationäre Wert selbst liefert die Entropie des schwarzen Loches.

In der vorliegenden Arbeit untersuchen wir den Zusammenhang zwischen der Entropiefunktion und dem black-hole-Potenzial im Fall von vierdimensionaler $N = 2$ Supergravitation und zeigen, dass bei Abwesenheit von Korrekturen höherer Ordnung der Lagrangefunktion beide Begriffe äquivalent sind. Wir veranschaulichen deren praktische Anwendung, indem wir eine supersymmetrische und eine nicht-supersymmetrische Lösung für die Attraktorgleichungen eines Konifold-Präpotenzials angeben.

Über die Untersuchung eines Zusammenhangs zwischen vier- und fünf-dimensionalen schwarzen Löchern erweitern wir die Definition der Entropiefunktion auf eine Klasse rotierender schwarzer Löcher in $N = 2$ Supergravitation mit kubischen Präpotenzialen. Auf diese Klasse war die ursprüngliche Definition nicht anwendbar aufgrund der Brechung der Rotationssymmetrie sowie des expliziten Auftretens der Eichpotenziale im Chern–Simons Term. Wieder geben wir zwei Typen von Lösungen für die die jeweiligen Attraktor-Gleichungen an.

Weiterhin erlaubt es uns die Verknüpfung zwischen vier- und fünf-dimensionalen schwarzen Löchern fünf-dimensionale Fluss-Differentialgleichungen erster Ordnung abzuleiten, welche die Form der Felder vom Unendlichen bis zum Horizont festlegen, als auch mittels dimensionaler Reduktion nicht-supersymmetrische Lösungen in vier Dimensionen zu konstruieren.

Schlussendlich können vier-dimensionale extremale schwarze Löcher in $N = 2$ Supergravitation als gewisse zwei-dimensionale String-Kompaktifizierungen mit Flüssen aufgefasst werden. Durch diese Tatsache motiviert, postuliert das jüngst vorgeschlagene entropische Prinzip als Wahrscheinlichkeitsmass auf dem Raum dieser String-Kompaktifizierungen die ins Exponential erhobene Entropie der zugehörigen schwarzen Löcher. Mittels des Konifold-Beispiels finden wir, dass das entropische Prinzip Kompaktifizierungen begünstigt, die in Infrarot-freien Eichtheorien resultieren.

Abstract

Extremal black holes in theories of gravity coupled to abelian gauge fields and neutral scalars, such as those arising in the low-energy description of compactifications of string theory on Calabi–Yau manifolds, exhibit the attractor phenomenon: on the event horizon the scalars settle to values determined by the charges carried by the black hole and independent of the values at infinity. It is so, because on the horizon the energy contained in vector fields acts as an effective potential (the black hole potential), driving the scalars towards its minima.

For spherically symmetric black holes in theories where gauge potentials appear in the Lagrangian solely through field strengths, the attractor phenomenon can be alternatively described by a variational principle based on the so-called entropy function, defined as the Legendre transform with respect to electric fields of the Lagrangian density integrated over the horizon. Stationarity conditions for the entropy function then take the form of attractor equations relating the horizon values of the scalars to the black hole charges, while the stationary value itself yields the entropy of the black hole.

In this study we examine the relationship between the entropy function and the black hole potential in four-dimensional $N = 2$ supergravity and demonstrate that in the absence of higher-order corrections to the Lagrangian these two notions are equivalent. We also exemplify their practical application by finding a supersymmetric and a non-supersymmetric solution to the attractor equations for a conifold prepotential.

Exploiting a connection between four- and five-dimensional black holes we then extend the definition of the entropy function to a class of rotating black holes in five-dimensional $N = 2$ supergravity with cubic prepotentials, to which the original formulation did not apply because of broken spherical symmetry and explicit dependence of the Lagrangian on the gauge potentials in the Chern–Simons term. We also display two types of solutions to the respective attractor equations.

The link between four- and five-dimensional black holes allows us further to derive five-dimensional first-order differential flow equations governing the profile of the fields from infinity to the horizon and construct non-supersymmetric solutions in four dimensions by dimensional reduction.

Finally, four-dimensional extremal black holes in $N = 2$ supergravity can be also viewed as certain two-dimensional string compactifications with fluxes. Motivated by this fact the recently proposed entropic principle postulates as a probability measure on the space of these string compactifications the exponentiated entropy of the corresponding black holes. Invoking the conifold example we find that the entropic principle would favor compactifications that result in infrared-free gauge theories.

Contents

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From the bottom of my heart I thank my family. For everything.

Understandest thou what thou readest?
And he said,
How can I, except some man should guide me?

Acts 8:30–31

Chapter 1

Prolegomena

On the 27th of November, 1783, a letter by the Rev. John Michell to Henry Cavendish was read before the Royal Society [118]:

> *[...] If there should really exist in nature any bodies whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us, or of there should exist any other bodies of a somewhat smaller size which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from sight; yet, if any luminous bodies infer their existence of the central ones with some degree of probability, as this might afford a clue to some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis; but as the consequences of such a supposition are very obvious, I shall not prosecute them any further. [...]*

Michell's predictions, even though deeply rooted in 18th century concepts about gravity and light and fallen into long oblivion, were well ahead of his time. Not only did he envisage objects whose escape velocity exceeds the speed of light, rendering them completely dark (for which J. A. Wheeler nearly 200 years later coined the name 'black holes'), but also proposed an indirect method of detecting them, which is essentially one of those currently employed. At present there is empirical evidence that black holes are in fact ubiquitous in the universe, occupying centers of most galaxies (cf. [125]).

Astrophysical significance of black holes would be a sufficiently good reason to study them in detail, but their unusual properties make them interesting in their own right. In modern theoretical physics, as Juan Maldacena aptly put it, they have acquired the status of 'the hydrogen atom of quantum gravity' [117], for it is in black holes that the need to reconcile general relativity with quantum mechanics becomes most apparent: Black holes do not conform to the laws of thermodynamics unless a quantum effect—the Hawking radiation—is taken into account, but if the quantization is restricted to the electromagnetic radiation and does not include gravity itself, the purely thermal Hawking radiation violates unitary evolution of states in quantum mechanics (irrecoverably destroying all information that has ever entered the black hole). The resolution of this 'information paradox' can be hoped for only in a quantum theory of gravity.

The crucial step in any attempted quantum description of black holes consists in identifying their microstates. A proposed model can be then tested by verifying whether statistical Boltzmann's entropy agrees with the entropy inferred from the macroscopic properties of the black hole. In theories involving scalar fields the latter will in general depend on the horizon values of the scalars. For charged extremal black holes this poses a potential problem regardless of the detail of the model, because the microscopic entropy is fully determined by quantized charges and therefore should not depend on any continuously varying parameters. It turns out, however, that a phenomenon known as the attractor mechanism ensures that the horizon values of the scalars are not arbitrary, but also determined by the charges.

The attractor mechanism was first established for supersymmetric black holes [71, 150, 69, 70], and later extended to non-supersymmetric extremal black holes [66, 81] in four dimensions. In the absence of higher-curvature corrections to the action the attractor equations constraining the scalars at the horizon to be functions of the moduli arise as extremization conditions for the effective potential, known as the black hole potential [69, 66, 81, 85], and intuitively understood as the electromagnetic energy of vector fields in a scalar medium.

A different way to describe the attractor mechanism is the entropy function formalism [146, 147]. In this approach one defines an entropy function, whose extremization determines the values of the scalar fields at the horizon. The entropy of the black hole is then given by the value of the entropy function at the extremum. The original enunciation defines the entropy function as a partial Legendre transform with respect to electric fields of the Lagrangian density integrated over the event horizon and applies to spherically symmetric black holes in a broader class of theories than the black hole potential, namely arbitrary theories of gravity (including possible higher-curvature corrections) coupled to abelian gauge fields and neutral scalars, provided that the gauge potentials appear in the Lagrangian solely through field strengths (or are immaterial for a given solution).

The attractor mechanism reduces the problem of finding the horizon values of the fields to solving a set of equations, but to obtain full solutions interpolating between the asymptotic values of fields at infinity and at the horizon, one still needs to solve the (second-order) differential equations of motion. A subset of solutions can however be derived by rewriting the action as a sum of squares of first-order flow equations [66, 123, 60]. The interpolating solutions are then given in terms of harmonic functions [71, 69, 11, 140, 141, 15]. It is always the case for supersymmetric solutions, but non-supersymmetric examples expressed in harmonic functions have been also found [107] and [37, 4] demonstrated a class of non-supersymmetric solutions described by first-order equations.

In this thesis we shall concern ourselves with black hole attractor mechanism in four- and five-dimensional $N = 2$ supergravity. The amount of supersymmetry in this theory (8 supercharges in four and five space-time dimensions) already permits non-trivial dynamics, but is simultaneously restrictive enough to substantially simplify the analysis, as the theory is completely specified by a single function called the prepotential. In a broader context, since $N = 2$ supergravities provide low energy field-theoretical description of Calabi–Yau compactifications in string- and M-theory, the results obtained in the supergravity regime might be directly employed to test the string-theoretical microscopic models of these black holes.

Independently of that, the near-horizon solutions of four-dimensional extremal $N = 2$ black holes are equivalent to compactifications with fluxes of type IIB string theory on $\mathcal{X} \times S^2$ (where $\mathcal{X}$ is a Calabi–Yau three-fold) when the non-compact 2-dimensional space-time is of the anti-de Sitter type [133]. This led to the entropic principle of [133, 92], which posits the exponentiated black hole entropy as a probability density for cosmological selection of flux compactifications, enabling an additional interpretation of specific black hole attractors.

This thesis may be divided into two parts: The first is an exposition of the preliminaries, the second collects the research papers [28, 30, 31, 32] to which I contributed in the course of my doctoral studies, adapted to form a coherent entity. Extensive literature already exists on black hole attractors. Recently also both thorough and very brief reviews of various aspects of the subject have appeared, e.g. [3, 27, 26, 45, 53, 54, 91, 116, 121, 122, 134] (what a pity that they were not yet available when I started my studies!) and it would be conceited of me to think that I could impart the entire content matter presented there better then the respective distinguished authors. I therefore strove to provide instead a concise survey of the selected aspects that build the immediate foundation and context for the new results. To avoid repeating what can easily be accessed elsewhere and add a touch of freshness to the textbook knowledge I chose, where it could be done without detriment to the explanation, to depart from the typical line of presentation by including comments less commonly found in other sources (like the construction of embedding diagrams in place of the prevalent Carter–Penrose diagrams) or by emphasizing certain aspects to a greater extent than some less specialized texts (e.g. the distinction between coordinates on a Calabi–Yau manifold and four different types of coordinates used in the literature for its complex structure moduli space). Aiming at the reader's benefit has affected also the style of referencing in the introductory portion, which gives priority to the potential contemporary usefulness of quoted texts rather than to the influence of individual original works on the historical development of this domain of physics, particularly in the early period.

The outline of the dissertation is as follows:

- Basic properties of extremal black holes are recalled in chapter 2.

- Chapter 3 introduces the reader to the attractor phenomenon, the black hole potential, in particular in four-dimensional $N = 2$ supergravity, and the entropy function.

- Chapter 4, corresponding to the paper [30], demonstrates the equivalence of the two approaches when the higher-order corrections to the gravitational part of the supergravity Lagrangian are absent and illustrates the practical advantages and disadvantages of both by finding new attractor solutions in the case of the one-modulus prepotential associated to a conifold.

- Chapter 5, based on [32], generalizes the original definition of the entropy function to extremal five-dimensional black holes with one rotation parameter by exploiting the relation between extremal black hole solutions in five- and in four-dimensional $N = 2$ supergravity theories with cubic prepotentials. Two types of solutions to the associated attractor equations are displayed.

- The same connection serves in chapter 6 to construct and solve four-dimensional flow equations by dimensional reduction from five dimensions [28]. This provides a new perspective on the non-uniqueness of the rewriting of the action as perfect squares.

- The conifold example of chapter 3 is revoked again in chapter 7 in the context of the entropic principle to argue that in the supersymmetric case flux compactifications leading to infrared-free theories would be favored, also when higher-curvature corrections to the Lagrangian are taken into account [31].

- Conclusions and outlook close the main text.

Chapter 2

Black holes

2.1 Black holes in the Einstein–Maxwell theory

The essence of Einstein's general relativity, the interplay between energy and space-time geometry, entails a prediction that a large enough concentration of mass (or better said, energy) curves the surrounding space-time so strongly that nothing, not even light, can escape from inside an invisible, semi-permeable border—the event horizon—separating the interior—the black hole—from the rest of the Universe. More precisely (see e.g. [97, 120]), a black hole region of an asymptotically flat space-time is the part not contained in the causal past of future null infinity (future timelike infinity for asymptotically anti-de Sitter space-times). The boundary of such a region is a null hypersurface, called the (future) event horizon. For a rigorous definition, which turns out to be a challenge in itself and requires considerable mathematical sophistication, we refer the reader to [156].

Einstein's theory is probably most succinctly expressed in the Einstein–Hilbert action

$$S = \frac{1}{16\pi G_d}\int d^d x\sqrt{-g}\mathcal{R}\,, \tag{2.1.1}$$

(with Newton's constant G_d in d spacetime dimensions, the metric determinant g and the Ricci scalar $\mathcal{R}$) from which the vacuum Einstein's equations follow:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 0\,. \tag{2.1.2}$$

Taking the trace (in $d > 2$) we infer their equivalent form, the vanishing of the Ricci tensor $R_{\mu\nu}$:

$$R_{\mu\nu} = 0\,, \tag{2.1.3}$$

meaning that the space-time must be Ricci flat.

Finding exact solutions to Einstein's equations, especially with matter added, can be very involved, but exploitation of symmetries by inserting to the equations appropriately crafted trial solutions (ansätze) often significantly simplifies the task. Symmetries of space-time are encoded in Killing vectors, generating isometries of the metric. A Killing vector

$\xi = \xi^\mu \partial_\mu$ satisfies the Killing equation (see eg. [128])

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,. \tag{2.1.4}$$

In asymptotically flat space-times the normalization is typically chosen in such a way that $\xi^2 \to -1$ at infinity.

Birkhoff's theorem (first discovered by Jebsen, cf. [103]) asserts that [97] any spherically symmetric solution to vacuum Einstein's equations in 4 dimensions must be stationary (that is, possess a timelike Killing vector field) and asymptotically flat. This means that the simplest example of a black hole provided by the Schwarzschild solution (in coordinates (t, r, θ, ϕ))

$$ds^2 = -\left(1 - \frac{2G_4 M}{r}\right) dt^2 + \left(1 - \frac{2G_4 M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2\,, \tag{2.1.5}$$

with the two-sphere metric

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{2.1.6}$$

which was the first nontrivial exact solution to Einstein's equations ever found,[1] must be also unique.

In the parametrization (2.1.5) the horizon is located at the Schwarzschild radius $r_{\mathrm{S}} = 2G_4 M$ and even though the metric exhibits a singularity there, it is only an artifact of the coordinate system used and the solution remains regular. The singularity at $r = 0$ however is real, as the curvature invariant $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 48(G_4 M)^2/r^6$ tends to infinity. The radial coordinate becomes timelike inside the Schwarzschild radius and this fact exhibits the no-return property of the event horizon: once an object crosses the horizon, it must inevitably continue its motion in the direction of decreasing r until it reaches the central singularity. To aid imagination, we can visualize the Schwarzschild space-time using conformal Carter–Penrose diagrams; as they can be commonly found in the literature, we have chosen in Fig. 2.1 to present instead two types of embedding diagrams [1, 83, 111].

Far away from the hole, where the curvature becomes weak and we may expand $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ around the flat Minkowski background $\eta_{\mu\nu}$, the Newtonian approximation reveals the meaning of the parameter M. The geodesic equation, describing the motion of a particle freely falling (zero proper acceleration) along the world line $x^\mu(\tau)$

$$\frac{d}{d\tau}\frac{dx^\mu}{d\tau} = 0 \tag{2.1.7}$$

reduces (cf. eg. [33]) to Newton's second law

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \tag{2.1.8}$$

and by comparison with the acceleration produced by the central gravitational potential $-G_4 M/r$ we may interpret M in $g_{00} \approx -(1 - 2G_4 M/r)$ as the mass of the black hole.

[1] Schwarzschild found his eponymous solution just one year after Einstein's publication of general theory of relativity. The causal structure of this spacetime was understood only much later.

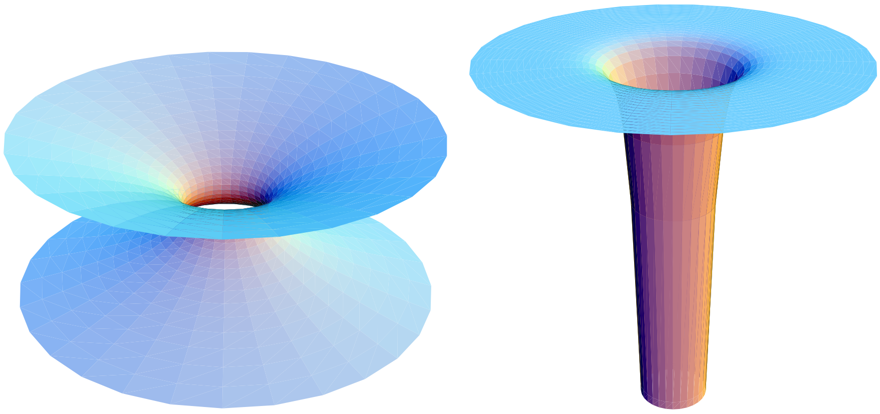

Figure 2.1: Embedding diagrams of the Schwarzschild space-time. The left graph depicts a 2D slice of the t = const hypersurface, embedded in a 3D Euclidean space $ds^2 = dr^2 + dz^2 + r^2 d\varphi^2$ (and projected onto the page). The rotational paraboloid $z(r) = \pm 2\sqrt{2G_4 M}\sqrt{r - 2G_4 M}$ reproduces (2.1.5) for $\theta = \pi/2$. Mathematical admissibility of both signs indicates a wormhole. The right graph displays the curvature (here: the negative of the invariant $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$; the Ricci scalar vanishes) in the equatorial plane.

The theorem of Birkhoff can be generalized to the Einstein–Maxwell theory in four dimensions, given by the action

$$S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\,(\mathcal{R} - F_{\mu\nu}F^{\mu\nu})\,, \tag{2.1.9}$$

where it implies the uniqueness of the Reissner–Nordström (RN) black hole

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2G_4 M}{r} + \frac{Q^2 + P^2}{r^2}\right) dt^2 \\ &+ \left(1 - \frac{2G_4 M}{r} + \frac{Q^2 + P^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2\,, \end{aligned} \tag{2.1.10}$$

$$F_{tr} = \frac{Q}{r^2}\,, \qquad F_{\theta\phi} = P\sin\theta \tag{2.1.11}$$

as a spherically symmetric, stationary, asymptotically flat solution outside a charge distribution. The parameters Q and P indeed correspond to the electric and magnetic charge, as defined by the volume integrals of the respective charge densities. These, by Stokes's theorem and Maxwell's equations, can be written as surface integrals of the field strength F and its Hodge dual $\star F$

$$Q = \frac{1}{4\pi}\oint_{S^2_\infty} \star F\,, \qquad P = \frac{1}{4\pi}\oint_{S^2_\infty} F\,, \tag{2.1.12}$$

where the integrals are evaluated at spatial infinity.

Depending on the relationship between the charges and the mass the Reissner-Nordström solutions represent three qualitatively different space-times:

- $G_4^2M^2 > Q^2 + P^2$: two horizons at

$$r_\pm = G_4M \pm \sqrt{(G_4M)^2 - (Q^2 + P^2)}\,, \tag{2.1.13}$$

of which the outer is the event horizon and the inner is the so-called Cauchy horizon. This space-time has an intriguing causal structure, but since it will not play any role in what follows, we shall not expand on it here, referring the reader to the plentiful literature (eg. [33, 111, 151]).

- $G_4^2M^2 = Q^2 + P^2$ (extremal case, relevant for this thesis): horizons coalesce. The line element (2.1.10) takes the form

$$ds^2 = -\left(1 - \frac{G_4M}{r}\right)^2 dt^2 + \left(1 - \frac{G_4M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2\,, \tag{2.1.14}$$

or, after the change of coordinates $r \to r - G_4M$

$$ds^2 = -\left(1 + \frac{G_4M}{r}\right)^{-2} dt^2 + \left(1 + \frac{G_4M}{r}\right)^2 (dr^2 + r^2 d\Omega_2^2)\,. \tag{2.1.15}$$

In the near-horizon limit

$$ds^2 = -\left(\frac{G_4M}{r}\right)^{-2} dt^2 + \left(\frac{G_4M}{r}\right)^2 dr^2 + (G_4M)^2 d\Omega_2^2\,, \tag{2.1.16}$$

in which, after yet another coordinate transformation, $r \to (G_4M)^2/r$,

$$ds^2 = \left(\frac{G_4M}{r}\right)^2 (-dt^2 + dr^2) + (G_4M)^2 d\Omega_2^2\,, \tag{2.1.17}$$

we recognize the Bertotti–Robinson metric: the product of a two-dimensional space of constant curvature, the anti-de Sitter space, and a two-sphere.

- $G_4^2M^2 < Q^2 + P^2$: no horizon extant, naked singularity. This situation is believed to be unphysical (the total energy would be smaller than the electromagnetic energy alone) and disallowed by the (as yet unproven) cosmic censorship conjecture, which forbids formation of singularities from a gravitational collapse in an asymptotically flat space-time, initially non-singular on some space-like hypersurface.[2]

2.2 Thermodynamics of black holes

From the uniqueness of the Schwarzschild and Reissner–Nordström solutions it follows that regardless of the details of the gravitational collapse or whatever process brought the

[2] The electron violates the extremality bound, but it is not a black hole, because its Compton wavelength is larger than its Schwarzschild radius.

black hole into existence, the final object is entirely characterized by just a few parameters measured at infinity; in fact, adding the information about the angular momentum J fully exhausts the specification in four dimensions. As J. A. Wheeler put it: 'Black holes have no hair' (except for the above four: M, Q, P and J). Albeit this would no longer be true for the Einstein–Yang–Mills system or other kinds of fields added [97], the black hole might not be stable either. Even though uniqueness is generally lost in higher dimensions, under more restrictive assumptions certain results persist [138, 64].

A thermodynamical system in equilibrium has a tellingly similar property: its state can be described by several macroscopic variables (state parameters), even though the description of microscopic dynamics might be very complicated. What is more, black hole mechanics obeys laws [8] bearing a striking resemblance to those of thermodynamics [16] (for a modern perspective see [158, 42, 139]).

The zeroth law states that surface gravity, defined below, remains constant across the horizon, in analogy to the constancy of temperature throughout a system in thermal equilibrium (zeroth law of thermodynamics). In Einstein's gravity horizons of all stationary black holes are Killing horizons, in other words, the vector field normal to the horizon is Killing; this is also true in higher derivative gravity for stationary, axisymmetric black holes with the so-called t-φ orthogonality property (see [158, 120] and references therein). As the horizon is null, the norm of the corresponding Killing vector ξ is constant on the horizon (namely, zero: $\xi^\mu \xi_\mu = 0$), so its gradient must be perpendicular to the horizon and hence parallel to ξ:

$$\nabla^\mu(\xi^\nu \xi_\nu) = -2\kappa \xi^\mu \,. \tag{2.2.1}$$

The proportionality coefficient κ coincides with the surface gravity, which is the acceleration of a static particle on the horizon of a stationary black hole, as measured at spatial infinity [156, 151]. For the Reissner–Nordström black hole we have

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2} \,. \tag{2.2.2}$$

In the extremal limit the surface gravity vanishes.

The first law connects variations in the black hole parameters analogously to the first law of thermodynamics:

$$dE = Td\mathcal{S} + \text{work terms.} \tag{2.2.3}$$

For the RN black hole, from the variation of the event horizon area

$$A = \int_{r=r_+} \sqrt{g_{\varphi\varphi} g_{\theta\theta}}\, d\varphi\, d\theta = 4\pi r_+^2 \,, \tag{2.2.4}$$

by virtue of the uniqueness theorem regarded as function of the mass and the charges, one immediately finds

$$\delta M = \frac{\kappa_+}{8\pi G_4} \delta A + \Phi_\mathrm{e} \delta Q + \Phi_\mathrm{m} \delta P \,, \tag{2.2.5}$$

where $\Phi_\mathrm{e} = Q/r_+$ and $\Phi_\mathrm{m} = P/r_+$ have the interpretation of electric and (scalar) magnetic potential. (It is not without a reason that the relation (2.2.5) is usually displayed for the purely electric case; for the intricacies of the inclusion of dyonic charges see [9]). Given the connection between the surface gravity and the temperature, the horizon area plays in eq. (2.2.5) the role of entropy in (2.2.3).

The connection between the entropy and the area of the event horizon is strengthened by the second law: the horizon area cannot decrease with time, provided that the matter energy-momentum tensor $T_{\mu\nu}$ satisfies the null energy condition ($T_{\mu\nu}k^\mu k^\nu \geq 0$ for all null k^μ) and the space-time (of dimension $d \geq 3$) is 'strongly asymptotically predictable' [158] (which, in practical terms, means that the censorship hypothesis is valid).

There exists also the third law of black hole mechanics, even though it is probably fair to say that its status seems to be less firm than the remaining ones [136]. This very fact is also paralleled by the third law of thermodynamics, which is arguably more a property of ordinary matter than a fundamental law of nature. The stronger Planck's statement that the entropy must tend to a universal constant value (which can be taken equal to zero) when the temperature approaches absolute zero would be violated in systems with degenerate ground states, as is its analogue for black holes: for instance the extremal RN solution has vanishing surface gravity, but nonzero horizon area. The weaker Nernst formulation (the absolute temperature cannot be reduced to the absolute zero in a finite number of operations) does have an analogue, though, as worded and proven by Israel [100]: "no continuous process, in which the energy tensor of accreted matter remains bounded and satisfies the weak energy condition in a neighborhood of an apparent horizon can reduce the surface gravity of a black hole to zero within a finite advanced time." The weak energy condition similarly to the null energy condition reads $T_{\mu\nu}k^\mu k^\nu \geq 0$, but now for all future-directed timelike (rather than null) k^μ, and 'advanced time' refers to the combination $t + r$.

The above similitude strongly speaks in favor of attributing to the surface gravity and the horizon area physical significance as the temperature and the entropy of the black hole. Indeed, since owing to the analogy between the Boltzmann factor $\mathrm{e}^{-\beta H}$ and the time evolution operator of quantum mechanics $\mathrm{e}^{-\mathrm{i}Ht/\hbar}$ the partition function of a thermodynamical system can be written as a Euclidean path integral (see [96])

$$\mathscr{Z} = \mathrm{tr}\, \mathrm{e}^{-\beta H} = \int \mathscr{D}[\phi] \mathrm{e}^{-\int_0^\beta d\tau L} \tag{2.2.6}$$

with periodic boundary conditions in the imaginary time $\mathrm{i}t = \tau \sim \tau + \hbar\beta$, we may quickly calculate the temperature corresponding to the Wick-rotated RN line element

$$ds^2 = \frac{(r - r_-)(r - r_+)}{r^2} d\tau^2 + \left(\frac{(r - r_-)(r - r_+)}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_2^2 \,. \tag{2.2.7}$$

Near the event horizon we obtain approximately:

$$ds^2 = \frac{r_+ - r_-}{r_+^2} u \, d\tau^2 + \left(\frac{r_+ - r_-}{r_+^2} u \right)^{-1} dr^2 + r_+^2 d\Omega_2^2 \,, \tag{2.2.8}$$

where $u = r - r_+$. Introducing a new variable $\xi = 2\sqrt{r_+^2 u/(r_+ - r_-)}$ transforms the line element into

$$ds^2 = \frac{1}{4}\left(\frac{r_+ - r_-}{r_+^2}\right)^2 \xi^2 d\tau^2 + d\xi^2 + r_+^2 d\Omega_2^2\,, \tag{2.2.9}$$

which has the form describing a product of flat Euclidean space (in polar coordinates) and an S^2

$$ds^2 = \tilde{r}^2 d\tilde{\theta} + d\tilde{r}^2 + dr_+^2 d\Omega_2^2\,, \tag{2.2.10}$$

provided that the angular variable $\tilde{\theta} = \frac{r_+ - r_-}{2r_+^2}\tau$ has period 2π (otherwise the flat part of the metric would describe the surface of a cone with a singularity at the tip $\tilde{r} = 0$). In terms of the Euclidean time

$$\tau \sim \tau + 4\pi\frac{r_+^2}{r_+ - r_-} =: \tau + \hbar\beta\,. \tag{2.2.11}$$

From the definition $\beta = 1/(k_{\mathrm{B}}T)$ we finally have

$$T = \frac{\hbar}{k_{\mathrm{B}}}\frac{r_+ - r_-}{4\pi r_+^2} = \frac{\hbar}{2\pi k_{\mathrm{B}}}\kappa_+\,. \tag{2.2.12}$$

As a further confirmation one can proceed with the calculation of the entropy using the Gibbons and Hawking's Euclidean action method [82] (see also [96, 159]). Applying (2.2.6) to the metric itself we take the partition function of the space-time

$$\mathcal{Z} = \int \mathscr{D}[g] e^{-\tilde{I}[g]}, \tag{2.2.13}$$

and expect the path integral to be well approximated by the stationary point contribution $e^{-\tilde{I}[g_{\mathrm{RN}}]}$ yielded by a classical solution to the equations of motion, namely the RN black hole. $\tilde{I}$ stands for the Euclidean Einstein–Hilbert action with the boundary term (needed to remove second derivatives of the metric through integration by parts, as required by the path integral approach)

$$I = -\frac{1}{16\pi G_4}\int \mathcal{R}\sqrt{g}\, d^4x - \frac{1}{8\pi G_4}\int K\sqrt{h}\, d^3x, \tag{2.2.14}$$

after the (infinite) flat-space contribution has been subtracted:

$$\tilde{I} = I_{\mathrm{RN}} - I_{\mathrm{flat}}\,. \tag{2.2.15}$$

In the above K denotes the extrinsic curvature and h is the determinant of the 3-metric induced on the boundary at a constant r, ultimately taken to infinity. Rather than calculating the boundary term directly, we will employ the relation

$$\int K\sqrt{h}\, d^3x = \frac{\partial}{\partial n}\int \sqrt{h}\, d^3x, \tag{2.2.16}$$

where n is the unit outward vector normal to the boundary.

For the case at hand $n = n^r \partial_r$ and thus, using the Euclidean normalization $n^r n_r = g_{rr} n^r n^r = +1$ and (2.2.7), we find

$$\frac{\partial}{\partial n} = \sqrt{\frac{(r-r_-)(r-r_+)}{r^2}}\,\partial_r. \tag{2.2.17}$$

From the induced metric on the slice $S^1 \times S^2$ (Euclidean time $\times$ the boundary of $\mathbb{R}^3$)

$$h_{ij} = \operatorname{diag}\left(\frac{(r-r_-)(r-r_+)}{r^2}, r^2, r^2\sin^2\theta\right) \tag{2.2.18}$$

we obtain the action (only the boundary term contributes)

$$I_{\mathrm{RN}} = -\frac{1}{8\pi G_4}\frac{\partial}{\partial n}\int \sqrt{h}\, d^3x = -\frac{4r^2 - 3(r_- + r_+)r + 2r_- r_+}{4r}\beta. \tag{2.2.19}$$

Analogously, for the flat background

$$h_{ij} = \operatorname{diag}\left(1, r^2, r^2\sin^2\theta\right) \tag{2.2.20}$$

we have

$$G_4 I_{\mathrm{flat}} = -\beta\sqrt{(r-r_-)(r-r_+)}, \tag{2.2.21}$$

so that the difference becomes (in the last step we have recalled (2.2.11))

$$\begin{aligned} G_4\tilde{I} &= -\beta \lim_{r\to\infty}\left(\frac{4r^2 - 3(r_- + r_+)r + 2r_- r_+}{4r} - \sqrt{(r-r_-)(r-r_+)}\right) \\ &= \frac{1}{4}(r_+ + r_-)\beta = 4\pi r_+^2 \frac{r_+ + r_-}{r_+ - r_-}. \end{aligned} \tag{2.2.22}$$

By standard thermodynamics (F_{H} stands for the Helmholtz free energy)

$$\mathcal{S} = k_{\mathrm{B}}\beta(E - F_{\mathrm{H}})\,, \qquad E = -\frac{\partial \log \mathcal{Z}}{\partial \beta}\,, \qquad F_{\mathrm{H}} = -\frac{1}{\beta}\log\mathcal{Z}\,, \tag{2.2.23}$$

so that

$$\mathcal{S}/k_{\mathrm{B}} = \beta\frac{\partial \tilde{I}}{\partial\beta} - \tilde{I} = \beta\left(\frac{\partial\tilde{I}}{\partial r_+}\frac{\partial r_+}{\partial\beta} + \frac{\partial\tilde{I}}{\partial r_-}\frac{\partial r_-}{\partial\beta}\right) - \tilde{I}, \tag{2.2.24}$$

where, again from differentiating the expression (2.2.11) for β,

$$1 = 4\pi\left(\frac{r_+(r_+ - 2r_-)}{(r_+ - r_-)^2}\frac{\partial r_+}{\partial\beta} + \frac{r_+^2}{(r_+ - r_-)^2}\frac{\partial r_-}{\partial\beta}\right), \qquad \frac{\partial r_-}{\partial\beta} = \frac{4\pi r_+^2}{\beta^2}. \tag{2.2.25}$$

Substituting $\tilde{I}$ of (2.2.22) gives finally

$$\mathcal{S} = \frac{k_{\mathrm{B}}}{G_4}\pi r_+^2 = \frac{k_{\mathrm{B}}}{4G_4}A\,, \tag{2.2.26}$$

known as the Bekenstein area law.

A few comments are in order. The recognition of one fourth of the event horizon area (in Planck's units) as the black hole entropy must be regarded as surprising at the very least: one would expect an extensive thermodynamical variable to be proportional to the volume, not the area. One is tempted to see in this phenomenon evidence for the veracity of 't Hooft and Susskind's holographic principle (see [21] for a review and precise formulation), which conjectures that the fundamental degrees of freedom of a quantum gravity theory in a certain volume should be associated with its boundary, at most one per quarter of Planck's area.

What is more, classically the identification $T \sim \kappa$ would contradict the fact that black holes are perfectly absorbing bodies and therefore should have zero temperature, but Hawking's discovery [95] that black holes emit thermal radiation of quantum-mechanical origin with the characteristic temperature $T_{\mathrm{H}} = \frac{\hbar}{2\pi k_{\mathrm{B}}}\kappa$ completed the thermodynamical interpretation of black hole properties. The Hawking effect can be given an intuitive explanation: when a pair of virtual particles is spontaneously created by vacuum fluctuations in the vicinity of the black hole horizon, one of the particles can cross the event horizon. If the other particle remains outside, the pair cannot recombine and the particle becomes a real particle, perceived by an external observer as a quantum of radiation emitted by the black hole. As the outgoing particle carries positive energy, the energy of its absorbed counterpart must be negative, decreasing the mass of the black hole. Consequently the horizon area will also decrease (the energy condition in the second law of black hole mechanics is infringed by the infalling matter, so the second law of black hole mechanics must not be applied), but the second law of thermodynamics stays in force, because the entropy carried by the radiation at least compensates the reduction of black hole entropy.

Since the Hawking temperature of a Schwarzschild black hole is inversely proportional to the mass, the black hole will become hotter as it radiates (it therefore has negative specific heat). From Stefan–Boltzmann's law we know that the radiant emittance of a black body grows proportionally to T^4. The total radiated power will be thus proportional to $1/M^2$ (the horizon area grows as M^2) and so the black hole will evaporate in the time of the order M^3. Even for black holes of merely solar masses the Hawking temperature ($\sim 10^{-6}\,\mathrm{K}$) is far smaller than the temperature of the cosmic microwave background (precluding detection of the Hawking radiation) and their lifetime exceeds the present age of the Universe by 54 orders of magnitude. As noted earlier extremal black holes have vanishing surface gravity and temperature, thus they do not radiate and are stable.

2.3 Wald's entropy formula

A derivation of black hole entropy on the basis of the first law, but applicable to an arbitrary theory invariant under general coordinate transformations, possibly including in the Lagrangian terms of higher order in the Riemann tensor or its derivatives, has been developed by Wald [157] and identifies the entropy as the Noether charge corresponding to the Killing horizon isometry. For Einstein's gravity Wald's construction reduces to

Bekenstein's formula, but in general it contains corrections to the area law.

Following the reviews in [102, 126] (for a derivation with explicit gauge fields see [78]) consider Lagrangian density[3] $\sqrt{-g}\mathscr{L}$, depending on a certain number of dynamical fields, here collectively denoted by ψ. Its variation under general fields transformation $\psi \to \psi + \delta\psi$ amounts to

$$\delta(\sqrt{-g}\mathscr{L}) = \sqrt{-g}E \cdot \delta\psi + \sqrt{-g}\nabla_\mu \theta^\mu(\delta\psi)\,, \tag{2.3.1}$$

where the dot product substitutes the sum over fields and contraction of indices. The equations of motion read $E = 0$. If the field variation leaves the Lagrangian density invariant, $\delta(\sqrt{-g}\mathscr{L}) = 0$, then (2.3.1) immediately implies that θ^μ is the Noether current conserved on shell (that is, when the equations of motion are satisfied, $E = 0$): $\nabla_\mu \theta^\mu(\delta\psi) = 0$.

Under diffeomorphisms generated by a vector field ξ^μ the dynamical fields ψ transform by a Lie derivative, $\delta\psi = \mathcal{L}_\xi \psi$. This cannot change the action, which we assumed to be invariant, but the Lagrangian density transforms by a total derivative

$$(\sqrt{-g}\mathscr{L})\big|_{\psi + \mathcal{L}_\xi \psi} = \mathcal{L}_\xi(\sqrt{-g}\mathscr{L}) = \sqrt{-g}\nabla_\mu(\xi^\mu \mathscr{L})\,. \tag{2.3.2}$$

Consequently, the Noether current conserved on shell becomes

$$J^\mu = \theta^\mu(\mathcal{L}_\xi \psi) - \xi^\mu \mathscr{L}\,. \tag{2.3.3}$$

For any local symmetry the Noether current can be written as the divergence of a globally defined and antisymmetric in indices Noether potential $Q^{\mu\nu}$, being a local function of the fields and linear in the transformation parameter

$$J^\mu = \nabla_\nu Q^{\mu\nu}\,, \tag{2.3.4}$$

up to terms vanishing on shell. By Stokes's (Gauß–Ostrogradsky's) theorem the Noether charge contained in a space-like volume Σ can be evaluated (here for a d-dimensional space-time) as

$$\oint_{\partial\Sigma} d^{d-2}x \sqrt{h}\epsilon_{\mu\nu} Q^{\mu\nu}\,, \tag{2.3.5}$$

where h stands for the determinant of the induced metric and $\epsilon_{\mu\nu}$ is the binormal form on $\partial\Sigma$.

Wald points out that if there exists a Hamiltonian H generating the evolution along ξ^μ (assumed constant), then

$$\delta H = \delta \int_\Sigma dV_\mu J^\mu - \int_\Sigma dV_\mu \nabla_\nu(\xi^\mu \theta^\nu - \xi^\nu \theta^\mu)\,, \tag{2.3.6}$$

where dV denotes the volume element of Σ. Moreover, for the special case when the variation transforms one solution to another solution, we can use (2.3.4) to recast δH into

[3] In the remaining text we will frequently refer to the Lagrangian density (which is also a scalar density with respect to general coordinate transformations) and to the Lorentz scalar $\mathscr{L}$ simply as 'Lagrangian'.

the form of surface integrals over the boundary $\partial\Sigma$. If in addition ξ^μ is Killing, then $\delta H = 0$, which relates the surface integrals to one another.

Choosing Σ to extend from infinity to the space-like cross-section of the Killing horizon, on which the Killing vector vanishes (known as the bifurcation surface), we obtain from the above an equation between quantities evaluated at infinity and on the horizon. In the general spinning but stationary case we can write the Killing vector as $\xi = \partial_t + \Omega^{(a)}\partial_{\varphi^{(a)}}$, where $\Omega^{(a)}$, $a = 1, \ldots, \lfloor (d-1)/2 \rfloor$, are called angular velocities of the horizon (in d dimensions we expect $\lfloor (d-1)/2 \rfloor$ angular momentum invariants [127]). The integrals calculated at infinity turn out to be Komar expressions (c.f. [156]) for the variations of the mass and angular momenta: $\delta M - \Omega^{(a)}\delta J_{(a)}$, provided that the space-time admits the proper notion of asymptotic flatness, so that the integrals are well defined. The expression on the bifurcation surface B, completing the first law, is

$$\delta \oint_B d^{d-2}x\sqrt{h} = \frac{\kappa}{2\pi}\delta\mathcal{S}\,, \tag{2.3.7}$$

with $\mathcal{S} = 2\pi \oint_B d^{d-2}x\sqrt{h}\epsilon_{\mu\nu}Q^{\mu\nu}(\tilde{\chi})$. The dependence of the Noether potential on the Killing field and its derivatives can be removed by the exploitation of Killing vector identities and the fact that at the bifurcation surface $\nabla_\mu\tilde{\chi}_\nu = \epsilon_{\mu\nu}$, to give a purely geometric functional of the metric and the matter fields, $\tilde{Q}^{\mu\nu}$. The resulting definition of the entropy

$$\mathcal{S} = 2\pi \oint d^{d-2}x\sqrt{h}\epsilon_{\mu\nu}\tilde{Q}^{\mu\nu}\,, \tag{2.3.8}$$

has been proved to be valid not only on B, but on an arbitrary cross-section of the Killing horizon. Formula (2.3.7) reqiures that the surface gravity be nonzero. To define the entropy for extremal black holes one should consider a non-extremal solution and take the appropriate limit of the result (but see [136] for a critical appraisal of this procedure).

Assuming a specific form of the Lagrangian one can derive from equation (2.3.8) more explicit formulae. For instance, when the Lagrangian

$$\mathscr{L} = \mathscr{L}(\psi, \nabla_\mu\psi, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}, \nabla_{(\lambda_1}\nabla_{\lambda_2)}R_{\mu\nu\rho\sigma}, \ldots)\,, \tag{2.3.9}$$

apart from matter fields ψ and their first derivatives, the metric and the Riemann tensor (regarded formally as independent of the metric), contains an arbitrary but finite number n of symmetrized derivatives of the Riemann tensor, the entropy becomes

$$\mathcal{S} = -2\pi \oint d^{d-2}x\sqrt{h}\sum_{m=0}^{n}(-1)^m\nabla_{(\lambda_1}\cdots\nabla_{\lambda_m)}Z^{\lambda_1\cdots\lambda_m:\mu\nu\rho\sigma}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{2.3.10}$$

where

$$Z^{\lambda_1\cdots\lambda_m:\mu\nu\rho\sigma} = \frac{\partial\mathscr{L}}{\partial(\nabla_{(\lambda_1}\cdots\nabla_{\lambda_m)}R_{\mu\nu\rho\sigma})}\,. \tag{2.3.11}$$

When we disallow derivatives of the Riemann tensor, but still permit arbitrary powers thereof, we obtain for static black holes

$$\mathcal{S} = 2\pi \oint \frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,. \tag{2.3.12}$$

In particular, if we add to Einstein's gravity a term quadratic in the Ricci scalar $\mathscr{L} = \frac{1}{16\pi G}(\mathcal{R} + \alpha\mathcal{R}^2)$, this formula yields, using $\partial\mathcal{R}/\partial R_{\mu\nu\rho\sigma} = g^{\mu\rho}g^{\nu\sigma}$ and the normalization $\epsilon^{\mu\nu}\epsilon_{\mu\nu} = -2$,

$$\mathcal{S} = \frac{1}{4G_d}\oint d^{d-2}x\sqrt{h}(1 + 2\alpha\mathcal{R})\,. \tag{2.3.13}$$

We see that already in this very simple example Bekenstein's formula (the first term) is corrected, so that the entropy is no longer exactly proportional to the area (or—in higher dimensions—content) of the event horizon.

Chapter 3

Attractor mechanism

3.1 Electromagnetic duality

In this section (based on [5, 62, 72, 82, 109, 154]) we investigate Einstein–Maxwell's theory extended in a different manner: rather than a single electromagnetic field, let us consider a number of abelian vector fields, labeled by capital Latin indices, with gauge kinetic couplings dependent on neutral scalar fields, labeled by lowercase Latin letters (note that g_{ij} is the metric on the scalar manifold and not the spacetime metric $g_{\mu\nu}$):

$$8\pi G_4 \mathscr{L} = \frac{1}{2}\mathcal{R} - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{4}\mathcal{I}_{IJ}(\phi)F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{4}\mathcal{R}_{IJ}(\phi)F^I_{\mu\nu}\star F^{J\mu\nu}\,. \tag{3.1.1}$$

For the positivity of the kinetic energy g_{ij} and $\mathcal{I}_{IJ}$ ought to be positive and negative definite, respectively; we also take the matrices g_{ij}, $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ to be symmetric.

Let us recast the Lagrangian density of the vector fields, $\mathscr{L}_1$, in another way, decomposing the gauge field strengths into self-dual and anti-self-dual parts:

$$F^{+I} = \frac{1}{2}(F^I + \tilde{F}^I)\,, \qquad F^{-I} = \frac{1}{2}(F^I - \tilde{F}^I)\,, \tag{3.1.2}$$

where the dual field strengths $\tilde{F}^I$

$$\tilde{F}^I = \mathrm{i}\star F^I \tag{3.1.3}$$

include an extra imaginary unit to compensate for $\star^2 F^I = -F^I$ (for any p-form and d-dimensional metric with signature s, $\star^2 = (-1)^{p(d-p)+s}$). In these conventions $\bar{F}^{+I} = F^{-I}$. Introducing the shorthands

$$\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + \mathrm{i}\mathcal{I}_{IJ}\,, \qquad \hat{\tau} = 4\pi(\mathcal{R} + \mathcal{I}\star) \tag{3.1.4}$$

and the canonically conjugate tensors

$$G_I^{+\mu\nu} = 2\mathrm{i}\frac{\partial\mathscr{L}}{\partial F^{+I}_{\mu\nu}} = -\bar{\mathcal{N}}_{IJ}F^{+J\mu\nu}\,, \qquad G_I^{-\mu\nu} = 2\mathrm{i}\frac{\partial\mathscr{L}}{\partial F^{-I}_{\mu\nu}} = \mathcal{N}_{IJ}F^{-J\mu\nu}\,, \tag{3.1.5}$$

brings $\mathscr{L}_1$ into

$$\begin{aligned}\mathscr{L}_1 &= -\frac{1}{2}\operatorname{Im}(\bar{\mathcal{N}}_{IJ}F^{+I}_{\mu\nu}F^{+J\mu\nu}) = \frac{1}{2}\operatorname{Im}(F^{+I}G^{+}_{I})\\ &= \frac{1}{2}\operatorname{Im}(\mathcal{N}_{IJ}F^{-I}_{\mu\nu}F^{-J\mu\nu}) = \frac{1}{2}\operatorname{Im}(F^{-I}G^{-}_{I})\end{aligned} \tag{3.1.6}$$

or equally compactly in the form language

$$\sqrt{-g}\mathscr{L}_1 d^4x = \frac{1}{2}\mathcal{I}_{IJ}F^I\wedge\star F^J + \frac{1}{2}\mathcal{R}_{IJ}F^I\wedge F^J = -\frac{1}{8\pi}F^I\wedge\hat{\tau}F^J\,. \tag{3.1.7}$$

The operator $\hat{\tau}$ corresponds to the matrix

$$\tau_{IJ} = \frac{\vartheta_{IJ}}{2\pi} + 4\pi\mathrm{i}\left(\frac{1}{g^2}\right)_{IJ} \tag{3.1.8}$$

of theta angles and inverse square couplings of the theory. Theta angles are coefficients of the topological terms (proportional to the instanton number). Consequently, as the name ‘angle’ suggests, values differing by an integral multiple of 2π are equivalent: $\vartheta \sim \vartheta + 2\pi$ and so are the couplings: $\tau \sim \tau + 1$.

To exhibit another interpretation of $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$, we rewrite the spin-1 part of the Lagrangian, $\mathscr{L}_1$, in a non-covariant way. In an orthonormal frame (boldface symbols refer to three-vectors in space)

$$\mathscr{L}_1 = -\frac{1}{2}\mathcal{I}_{IJ}(\mathbf{E}^I\cdot\mathbf{E}^J - \mathbf{B}^I\cdot\mathbf{B}^J) - \mathcal{R}_{IJ}\mathbf{E}^I\cdot\mathbf{B}^J\,. \tag{3.1.9}$$

Due to the scalar dependence we might think of the situation as of electromagnetic fields in a scalar medium with constitutive relations

$$\mathbf{D}_I = \mathcal{I}_{IJ}\mathbf{E}^J + \mathcal{R}_{IJ}\mathbf{B}^J\,, \tag{3.1.10}$$

$$\mathbf{H}_I = \mathcal{I}_{IJ}\mathbf{B}^J - \mathcal{R}_{IJ}\mathbf{E}^J\,. \tag{3.1.11}$$

or, conversely,

$$\begin{pmatrix}\mathbf{H}\\ \mathbf{E}\end{pmatrix} = M\begin{pmatrix}\mathbf{B}\\ \mathbf{D}\end{pmatrix}, \qquad M = \begin{pmatrix}\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1}\\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1}\end{pmatrix}. \tag{3.1.12}$$

The scalar-dependent matrix $M = M(\phi)$ is real and unimodular: $\det M = 1$; the coefficients play the role of permeabilities and permittivities of the medium. The electromagnetic energy density (the 00 component of the energy-momentum tensor) can be written as

$$-\frac{1}{2}(\mathbf{H}_I\cdot\mathbf{B}^I + \mathbf{D}_I\cdot\mathbf{E}^I) = -\frac{1}{2}\begin{pmatrix}\mathbf{B} & \mathbf{D}\end{pmatrix}M(\phi)\begin{pmatrix}\mathbf{B}\\ \mathbf{D}\end{pmatrix}. \tag{3.1.13}$$

The energy density (3.1.13) together with the Bianchi identities and the equations of motion

$$\partial^\mu \operatorname{Im} F^{+I}_{\mu\nu} = 0\,, \qquad \partial_\mu \operatorname{Im} G^{+\mu\nu}_I = 0 \tag{3.1.14}$$

remains invariant under these transformations $S \in SL(2n, \mathbb{R})$, where n is the number of vector fields

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix} \to S \begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix}, \qquad M \to (S^{\mathrm{T}})^{-1} M S^{-1}, \tag{3.1.15}$$

which do not affect the metric g_{ij} on the scalar manifold. Canonically conjugate quantities form a symplectic vector, which transforms as

$$\begin{pmatrix} F^I \\ G_I \end{pmatrix} \to S \begin{pmatrix} F^I \\ G_I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^I \\ G_I \end{pmatrix}. \tag{3.1.16}$$

To preserve this structure with (3.1.5) S must be a symplectic matrix, that is one satisfying

$$S^{\mathrm{T}} \Omega S = \Omega, \qquad \Omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \tag{3.1.17}$$

or

$$A^{\mathrm{T}} C - C^{\mathrm{T}} A = 0, \qquad B^{\mathrm{T}} D - D^{\mathrm{T}} B = 0, \qquad A^{\mathrm{T}} D - C^{\mathrm{T}} B = \mathbb{I}. \tag{3.1.18}$$

Since the magnetic and electric charges, defined as

$$p^I = \frac{1}{4\pi} \oint_{S^2_\infty} F^I, \qquad q_I = \frac{1}{4\pi} \oint_{S^2_\infty} \star G_I, \tag{3.1.19}$$

where $G = (G^+ + G^-)/\mathrm{i}$, should be ultimately quantized by the Dirac condition (that is: take values in some integral lattice and its reciprocal) and yet transform according to

$$\begin{pmatrix} p \\ q \end{pmatrix} \to S \begin{pmatrix} p \\ q \end{pmatrix}, \tag{3.1.20}$$

we see that the duality group needs to be restricted to $Sp(2n, \mathbb{Z})$.

Note that the Lagrangian density itself is not invariant under electromagnetic duality transformations and that Lagrangian density and the energy density are Legendre transforms of one another. Moreover the duality, because it relates different couplings, is not a symmetry of the theory, but an equivalence of different theories under redefinitions of the charges.

3.2 Black hole potential

The above theory has in four dimensions two classical vacua (ground states with constant scalars, $\partial_\mu \phi^i = 0$, and covariantly constant Maxwell fields, $\nabla_\mu F^I_{\nu\rho} = 0$): the Minkowski space-time with $F^I_{\mu\nu} = 0$ and arbitrary values of the scalars, and the Bertotti–Robinson space-time $AdS_2 \times S^2$ with

$$F^I = \frac{4\pi p^I}{A_{S^2}} \eta_{S^2}, \qquad \star G_I = \frac{4\pi q_I}{A_{S^2}} \eta_{S^2}, \tag{3.2.1}$$

where A_{S^2} stands for the surface of the 2-sphere and η_{S^2} for its surface element.

In contrast to the Minkowski vacuum the values of the scalars in the Bertotti–Robinson vacuum are constrained, even though they have no explicit potential and are therefore, by definition, moduli. The (scalar-dependent) electromagnetic energy (3.1.13), after solving for the gauge fields in terms of the black hole charges, plays the role of the effective potential, which the scalars have to extremize [69, 66, 81]:

$$V_{\mathrm{BH}}(\phi, p, q) = -\frac{1}{2}\begin{pmatrix} p & q \end{pmatrix} M(\phi) \begin{pmatrix} p \\ q \end{pmatrix}. \tag{3.2.2}$$

If the extremum

$$\frac{\partial V_{\mathrm{BH}}(\phi, p, q)}{\partial \phi} = 0 \tag{3.2.3}$$

is unique, it fully determines the values of the scalars as functions of the charges. What is more, since the extremal value of the potential corresponds to the square radius of the S^2, the solution—in particular the entropy—is completely characterized by the charges. This phenomenon is known as the attractor mechanism.

The supersymmetric extrema of the black hole potential are always minima [66] and are thus proper attractors, but no similar generally valid assertions can be made in the non-supersymmetric case. When the extremum is not a minimum (but a saddle point or a maximum), for specifically fine-tuned asymptotic values the moduli will still evolve to that particular extremum as one nears the horizon, but arbitrarily small deviations of the asymptotic values will lead to a different solution on the horizon. Such solutions are oxymoronically called unstable attractors or un-attractors. Another possibility is the existence of flat directions of the black hole potential, resulting in neutral stability of the solution: as the entropy is specified by the extremal value, it remains determined by the charges, but the values of the moduli can freely change along the flat directions (which for two degrees of freedom might be pictured as valleys in the potential surface). Finally, it might happen that the minimum is not unique: then each is surrounded by a basin of attraction and horizon solutions are labeled apart from the charges by an additional discrete parameter, usually referred to as the area code.

To see the workings of the attractor mechanism in detail let us consider, taking the example from [85, 108, 152] and [3, 154], the general static line element

$$ds^2 = -\mathrm{e}^{2U(\mathbf{x})}dt^2 + \mathrm{e}^{-2U(\mathbf{x})}\gamma_{mn}(\mathbf{x})dx^m dx^n \,. \tag{3.2.4}$$

Einstein's equations become

$$-\frac{1}{2}R_{mn}(\gamma) + \partial_m U \partial_m U + \frac{1}{2}g_{ij}\partial_m\phi^i\partial_n\phi^j - \mathrm{e}^{2U}V_{mn} = 0\,, \tag{3.2.5}$$

$$\nabla_m\partial^m U - \mathrm{e}^{2U}\gamma^{mn}V_{mn} = 0\,, \tag{3.2.6}$$

where

$$V_{mn} = -\frac{1}{2}\begin{pmatrix} F^I_m & G_{Im} \end{pmatrix} M \begin{pmatrix} F^J_n \\ G_{Jn} \end{pmatrix} \tag{3.2.7}$$

and

$$F^I_m = \frac{1}{2}\gamma_{mn}\gamma^{-1/2}\underline{\varepsilon}^{npq}F^I_{pq}\,, \qquad G_{Im} = \frac{1}{2}\gamma_{mn}\gamma^{-1/2}\underline{\varepsilon}^{npq}G_{Ipq}\,. \tag{3.2.8}$$

Under the assumption that all functions depend only on one (spatial) coordinate τ, conveniently chosen as in [108]

$$\gamma_{mn}dx^m dx^n = \frac{c^4}{\sinh^4 c\tau}d\tau^2 + \frac{c^2}{\sinh^2 c\tau}(d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{3.2.9}$$

the equations of motion and Bianchi identities can be solved in terms of harmonic functions:

$$F^I_m = \partial_m H^I\,, \qquad G_{Im} = \partial_m H_I \tag{3.2.10}$$

$$H = \begin{pmatrix} H^I \\ H_I \end{pmatrix} = \begin{pmatrix} p^I \\ q_I \end{pmatrix}\tau + \begin{pmatrix} h^I \\ h_I \end{pmatrix} = \Gamma\tau + h\,. \tag{3.2.11}$$

Einstein's equations reduce to

$$c^2 - U'^2 - \frac{1}{2}g_{ij}\phi'^i\phi'^j + \mathrm{e}^{2U}V_{\mathrm{BH}} = 0\,, \tag{3.2.12}$$

$$-U'' + \mathrm{e}^{2U}V_{\mathrm{BH}} = 0\,. \tag{3.2.13}$$

Note that the second equation also follows from the effective Lagrangian resulting from the substitutions

$$\mathscr{L} = U'^2 + \mathrm{e}^{2U}V_{\mathrm{BH}} + \frac{1}{2}g_{ij}\phi'^i\phi'^j\,, \tag{3.2.14}$$

but the first is a constraint, which ensures self-consistency of the ansatz.

The case $c \to 0$ corresponds to extremality. Writing the extremal metric as

$$ds^2 = -a(r)^2dt^2 + a(r)^{-2}dr^2 + b(r)^2d\Omega^2\,, \tag{3.2.15}$$

[85, 152] cast the radial equations of motion for the scalars in the suggestive manner:

$$\partial_r(a^2b^2g_{ij}\partial_r\phi^j) = \frac{1}{2b^2}\frac{\partial V_{\mathrm{BH}}}{\partial\phi^i}\,. \tag{3.2.16}$$

For the (stable) attractor two conditions are sufficient: the existence of a critical point of the potential at some ϕ^i_0

$$\left.\frac{\partial V_{\mathrm{BH}}}{\partial\phi^i}\right|_{\phi^i_0} = 0 \tag{3.2.17}$$

and the positive definiteness of the Hessian $\partial_i\partial_j V_{\mathrm{BH}}|_{\phi^i_0}$. Then there exists an extremal solution with constant scalars $\phi^i(r) = \phi^i_0$ (called double extremal), which satisfies (3.2.16). The value of the potential at the extremum turns out to be the squared radius of the horizon

$$V_{\mathrm{BH}}(\phi^i_0) = b^2_{\mathrm{h}}\,, \tag{3.2.18}$$

so the Bekenstein–Hawking entropy is simply

$$\mathcal{S} = \frac{1}{4}A = \pi V_{\mathrm{BH}}(\phi^i_0)\,. \tag{3.2.19}$$

3.3 Special geometry of $N = 2$ supergravity

A concrete realization of the type of theories described in the previous section is $N = 2$ supergravity coupled to abelian multiplets [154] (see [56, 58, 57] for the original articles), which arises as an effective low-energy description in type II string theories compactified on a Calabi–Yau three-fold. (Alternatively one can take the 11-dimensional M-theory as the starting point and compactify on a product of a Calabi–Yau manifold and a circle.) Compactification partly breaks supersymmetry, leaving 8 supercharges, combined into 2 independent Lorentz spinors.

Theories with 8 supercharges are theories with the largest amount of supersymmetry permitting arbitrary functions in their definitions. 16 supersymmetries would be more restrictive: specification of the number of fields and dimension determines the geometry of the scalar manifold. And finally, the maximal 32 supersymmetries do not allow matter couplings and the dimensionality fixes the geometry.

The massless field content of $N = 2$ supergravity in $3 + 1$ space-time dimensions can be organized into three types of multiplets labeled by the highest spin. The gravity multiplet consists of the spin-2 graviton, two spin-3/2 gravitini and the spin-1 graviphoton. Each of the n_{v} vector multiplets contains one spin-1 photon, two spin-1/2 fermions and two real scalars (spin 0), which can be combined into one complex field. There are also n_{h} hypermultiplets with two spin-1/2 hyperfermions and four hyperscalars each. In type IIB the scalars in vector multiplets parametrize the moduli space of deformations of the Kähler form ('volume') on the Calabi–Yau, while the hypermultiplet scalars span the moduli space of complex structure ('shape') deformations of the Calabi–Yau manifold. In type IIA the situation is reversed. Note that the number of gauge fields, due to the graviphoton, exceeds by one the number n_{v} of complex scalars in vector multiplets. The hypermultiplets are immaterial for the black hole solutions.

To understand the notion of a Calabi–Yau manifold and describe the geometry of scalar manifold in 4-dimensional $N = 2$ supergravity, known as special (or special Kähler) geometry (reviews can be found in [43, 74, 153]), let us summarize some necessary facts about complex manifolds [22, 89, 128] (see also [10], Chapter 9). A $2n$-dimensional real manifold $\mathcal{X}$ can be viewed as an almost complex manifold of complex dimension n if it admits an almost complex structure, that is a globally defined linear map J on the tangent space, satisfying $J^2 = -\mathbb{I}$. With the aid of the almost complex structure one can distinguish between holomorphic and anti-holomorphic vectors (with eigenvalues $\pm\mathrm{i}$), so in an adapted coordinate basis ζ^i the action of the complex structure corresponds to the multiplication by the imaginary unit: $J(\partial_{\zeta^i}) = \mathrm{i}\partial_{\zeta^i}$, $J(\partial_{\bar{\zeta}^i}) = -\mathrm{i}\partial_{\bar{\zeta}^i}$. If the canonical form of the almost complex structure can be extended to a neighborhood of any point, the almost complex manifold is a complex manifold.

The metric on a manifold $\mathcal{X}$ is called Hermitian with respect to J if it satisfies $g(J(u), J(v)) = g(u, v)$ for all vectors u and v of the tangent space $T_p\mathcal{X}$, at each point $p \in \mathcal{X}$. With $g_{ij} = g(\partial_{\zeta^i}, \partial_{\zeta^j})$, $g_{i\bar{\jmath}} = g(\partial_{\zeta^i}, \partial_{\bar{\zeta}^j})$ etc. the hermiticity condition implies that the metric is block off-diagonal: $g_{ij} = g_{\bar{\imath}\bar{\jmath}} = 0$. From the reality of the metric we also find $\overline{g_{ij}} = g_{\bar{\imath}\bar{\jmath}}$, $\overline{g_{\bar{\imath}j}} = g_{i\bar{\jmath}}$ and from the symmetry: $g_{ij} = g_{ji}$, $g_{\bar{\imath}j} = g_{j\bar{\imath}}$. A complex manifold always admits a Hermitian metric and a complex manifold endowed with a Hermitian metric is said to be a Hermitian manifold.

Further let us introduce the associated Kähler form

$$\mathcal{K}(u, v) = \frac{1}{2\pi} g(J(u), v) \,, \tag{3.3.1}$$

which in adapted coordinates can be explicitly represented by

$$\mathcal{K} = \frac{\mathrm{i}}{2\pi} g_{i\bar{\jmath}} \, d\zeta^i \wedge d\bar{\zeta}^j \,. \tag{3.3.2}$$

A Hermitian manifold is called a Kähler manifold if the Kähler form is closed, $d\mathcal{K} = 0$ (which is equivalent to the integrability condition $\nabla J = 0$: J must be covariantly constant). Closed Kähler form implies that $\partial_k g_{i\bar{\jmath}} = \partial_{\bar{k}} g_{i\bar{\jmath}} = 0$, so the metric can be expressed as the second derivative of a (real) function known as the Kähler potential:

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K(\zeta, \bar{\zeta}) \,. \tag{3.3.3}$$

The Kähler potential is not unique, because the Kähler transformation

$$K(\zeta, \bar{\zeta}) \to K(\zeta, \bar{\zeta}) + f(\zeta) + \overline{f(\zeta)} \tag{3.3.4}$$

by some holomorphic function $f(\zeta)$ leaves the metric invariant.

Calabi–Yau manifolds are compact and Ricci-flat Kähler manifolds (other, not always equivalent definitions exist). As conjectured by Calabi and proved by Yau a necessary and sufficient condition for Ricci flatness is the vanishing of the first Chern class, implying in turn the existence of the unique (up to rescaling) globally defined and nowhere vanishing holomorphic $(n, 0)$-form (usually denoted Ω; we use the letter Θ to avoid confusion with the skew-symmetric matrix Ω in (3.1.17)). For the CY three-fold in suitably chosen coordinates:

$$\Theta = \Theta(\zeta) \, d\zeta^1 \wedge d\zeta^2 \wedge d\zeta^3 \,. \tag{3.3.5}$$

The complex structure moduli (denoted here collectively by z) of a Calabi–Yau $\mathcal{X}$ form a special Kähler manifold in its own right, $\mathcal{M}$, with the Kähler potential given in terms of the holomorphic $(3, 0)$-form [23]

$$K(z, \bar{z}) = -\log \left(-\mathrm{i} \int_{\mathcal{X}} \Theta \wedge \bar{\Theta} \right) . \tag{3.3.6}$$

Let us introduce for the third integral homology of $\mathcal{X}$, $H_3(\mathcal{X}, \mathbb{Z})$, the canonical basis of $b_3 = h_{3,0} + h_{2,1} + h_{1,2} + h_{0,3} = 2(h_{2,1} + 1)$ three-cycles (A^I, B_I), $I = 0, \ldots, h_{2,1}$. The dual cohomology basis (α_I, β^I) is by definition:

$$\int_{A^I} \alpha_J := \delta^I_J \,, \qquad \int_{B_J} \beta^I := \delta^I_J \,. \tag{3.3.7}$$

The 'canonical' bases are such that $\int_{A^I} \beta^J = 0$, $\int_{B_I} \alpha_J = 0$ and (β^I, α_J) are minus Poincaré duals of (A^I, B_I) respectively:

$$\int_{\mathcal{X}} \alpha_J \wedge \beta^I = \int_{A^I} \alpha_J \,, \qquad \int_{B_J} \beta^I = \int_{\mathcal{X}} \beta^I \wedge \alpha_J \,, \tag{3.3.8}$$

which means that the oriented intersection numbers [22] of the cycles obey

$$A^I \cap B_J := \int_{\mathcal{X}} \beta^I \wedge \alpha_J = -\delta^I_J = -B_J \cap A^I \,, \tag{3.3.9}$$

$$A^I \cap A^J = 0 \,, \qquad B_I \cap B_J = 0 \,. \tag{3.3.10}$$

Any closed 3-form ω can be now expanded in the dual basis:

$$\omega = \big(\textstyle\int_{A^I} \omega\big)\alpha_I + \big(\textstyle\int_{B_I} \omega\big)\beta^I \quad \text{(sum over indices)} \tag{3.3.11}$$

and for a wedge product of two such forms the Riemann bilinear relation (see e.g. [130, 155]) can be established:

$$\begin{aligned}\int_{\mathcal{X}} \omega \wedge \eta &= \int_{\mathcal{X}} \Big(\alpha_I \textstyle\int_{A^I} \omega + \beta^I \int_{B_I} \omega\Big) \wedge \Big(\alpha_J \int_{A^J} \eta + \beta^J \int_{B_J} \eta\Big) \\ &= \delta^J_I \int_{A^I} \omega \int_{B_J} \eta - \delta^I_J \int_{B_I} \omega \int_{A^J} \eta = \int_{A^I} \omega \int_{B_I} \eta - \int_{B_I} \omega \int_{A^I} \eta \,.\end{aligned} \tag{3.3.12}$$

Defining the holomorphic periods of Θ over the cycles (the $F_I(z)$ should not be confused with the gauge fields $F^I_{\mu\nu}$ from the preceding section):

$$X^I(z) := \int_{A^I} \Theta \,, \qquad F_I(z) := \int_{B_I} \Theta \,, \tag{3.3.13}$$

we can write

$$\int_{A^I} \bar{\Theta} = \overline{\int_{A^I} \Theta} = \bar{X}^I(\bar{z}) \,, \quad \int_{B_I} \bar{\Theta} = \overline{\int_{B_I} \Theta} = \bar{F}_I(\bar{z}) \,, \tag{3.3.14}$$

and by (3.3.12):

$$\int_{\mathcal{X}} \Theta \wedge \bar{\Theta} = X^I(z)\bar{F}_I(\bar{z}) - F_I(z)\bar{X}^I(\bar{z}) \,. \tag{3.3.15}$$

Locally $X^I(z)$ completely specify the complex structure of $\mathcal{X}$ (see [23]), therefore $F_I(z)$ must be expressible as functions of $X^I(z)$, which play the role of projective (homogeneous) coordinates on the moduli space $\mathcal{M}$ of complex structures. Furthermore, it turns out that

$$F_I(X(z)) = \frac{\partial F(X(z))}{\partial X^I(z)} \,, \tag{3.3.16}$$

with $F(X(z))$ called the prepotential, being a holomorphic, homogeneous function of degree 2, i.e. $F(\lambda X(z)) = \lambda^2 F(X(z))$ for any $\lambda \in \mathbb{C} \setminus \{0\}$ (provided that the prepotential exists, see below). The complex structure moduli themselves can be regarded as affine (inhomogeneous) coordinates on the moduli space, e.g. $z^A = X^A(z)/X^0(z)$, $A = 1, \ldots, h_{2,1}$, in terms of which by homogeneity of the prepotential $F(X(z)) =: -\mathrm{i}X^0(z)^2\mathcal{F}(z)$, $F_A = -\mathrm{i}X^0\mathcal{F}_A = -\mathrm{i}X^0\partial\mathcal{F}/\partial z^A$, and $F_0 = -\mathrm{i}X^0(2\mathcal{F} - z^A\mathcal{F}_A)$, the last equation expressing Euler's homogeneous function theorem (the imaginary unit appears in the foregoing formulae by convention). In type IIB compactifications $h_{2,1} = n_\mathrm{v}$.

The Kähler potential (3.3.6) becomes finally

$$\begin{aligned} K(z, \bar{z}) &= -\log\Big[\mathrm{i}\Big(\bar{X}^I(\bar{z})F_I(X(z)) - X^I(z)\bar{F}_I(\bar{X}(\bar{z}))\Big)\Big] \\ &= -\log\Big[|X^0(z)|^2\Big(2(\mathcal{F} + \bar{\mathcal{F}}) - (z^A - \bar{z}^A)(\mathcal{F}_A - \bar{\mathcal{F}}_A)\Big)\Big]. \end{aligned} \tag{3.3.17}$$

Note that the rescaling

$$X^I(z) \to \mathrm{e}^{-f(z)}X^I(z) \qquad \text{implies} \qquad F_I(X(z)) \to \mathrm{e}^{-f(z)}F_I(X(z)) \tag{3.3.18}$$

(for the F_I are homogeneous of degree 1) and corresponds to the Kähler transformation

$$K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + \overline{f(z)}. \tag{3.3.19}$$

The prepotential specifies also the vector couplings. The matrix $\mathcal{N}$ of (3.1.4) takes the form [55]

$$\mathcal{N}_{IJ} = \bar{F}_{IJ}(X(z)) + 2\mathrm{i}\frac{\operatorname{Im} F_{IK}(X(z))\operatorname{Im} F_{JL}(X(z))\, X^K(z)\, X^L(z)}{\operatorname{Im} F_{MN}(X(z))\, X^M(z)\, X^N(z)}\,, \tag{3.3.20}$$

where $F_{IJ}(X(z)) = \partial^2 F(X(z))/\partial X^I(z)\partial X^J(z)$.

As we have just witnessed, special geometry is entirely determined by the prepotential. A different choice of the homology basis would lead to a different symplectic vector $(X'^I, F'_I)^\mathrm{T}$, for which no prepotential might exist, but a suitable symplectic transformation [74, 43] can be used to rotate the vector back to $(X^I, F_I)^\mathrm{T} = S\,(X'^I, F'_I)^\mathrm{T}$ with $S \in Sp(2(h_{2,1} + 1), \mathbb{Z})$.

For a Calabi–Yau conifold $\mathcal{X}_\mathrm{c}$, F_I can be determined near the singular points [149, 90]: each singular point (node) determines a "vanishing cycle" (a cycle, whose period vanishes as we approach the singularity). As we encircle the locus of a vanishing cycle, say A^1, (which is a complex codimension one submanifold in the moduli space $\mathcal{M}_\mathrm{c}$, if A^1 is the only vanishing cycle) the homology basis undergoes a monodromy transformation

$$B_1 \to B_1 + (B_I \cap A^1)A^1, \quad F_1(X(z)) \to F_1(X(z)) + (B_I \cap A^1)X^1(z)\,. \tag{3.3.21}$$

For the canonical basis ($B_I \cap A^J = \delta_I^J$) this transformation property implies that near $X^1(z) = 0$, which has the meaning of the conifold deformation parameter:

$$F_1(X^1(z)) = \frac{1}{2\pi\mathrm{i}}X^1(z)\log X^1(z) + \text{single-valued}(X^1(z))\,. \tag{3.3.22}$$

Using these results we can evaluate the Kähler potential in the case of a single complex structure modulus and a single node (with more moduli the monodromy considerations need to be appropriately modified [90], but the procedure remains the same). Taking $X^0(z) = 1$ and $z = X^1(z)$ (so-called 'special' coordinates [11]) we have $\mathcal{F}_1 = \frac{1}{2\pi \mathrm{i}} z \log z + \mathrm{sv}(z)$, $\mathcal{F} = \frac{1}{4\pi \mathrm{i}}(z^2 \log z - \frac{1}{2} z^2) + \mathrm{sv}(z)$ and from (3.3.17):

$$K(z, \bar{z}) = -\log \left[\frac{1}{2\pi} \left(|z|^2 \log |z|^2 + \mathrm{sv}(z, \bar{z}) \right) \right] . \tag{3.3.23}$$

An important class of prepotentials, which will play a major role also in this thesis, arises from dimensional reduction of five-dimensional $N = 2$ supergravity [94]. This theory is specified by a constant, fully symmetric third-rank tensor C_{ABC} appearing in the Chern–Simons term. Upon dimensional reduction this tensor (identified with the triple intersection numbers of the compactification Calabi–Yau) determines the prepotential commonly referred to as 'cubic'

$$F(X(z)) = -\frac{1}{3!} C_{ABC} \frac{X^A(z) X^B(z) X^C(z)}{X^0(z)} . \tag{3.3.24}$$

The corresponding geometry (in both five and four dimensions) bears the name 'very special' and is elaborated on in appendix B.

3.4 Attractor equations in special geometry

In four-dimensional $N = 2$ supergravity without higher-order corrections the attractor equations (3.2.3) can be given a much more explicit form. To derive it, let us introduce—with a slight abuse of notation—new variables X^I related to the holomorphic coordinates $X^I(z)$ of the previous section by [57, 148, 35, 34, 23, 49]

$$X^I = \mathrm{e}^{K(z,\bar{z})/2} X^I(z) . \tag{3.4.1}$$

It still holds that $F_I(X) = \partial F / \partial X^I$ and the physical scalars are given by $z^A = X^A / X^0$.

With the Kähler potential normalized as in the previous section the symplectic vector

$$V = \begin{pmatrix} X^I \\ F_I(X) \end{pmatrix} \tag{3.4.2}$$

satisfies the dilatational gauge-fixing constraint

$$\mathrm{i} \bar{V}^{\mathrm{T}} \Omega V = \mathrm{i} \left[\bar{X}^I F_I(X) - X^I \bar{F}_I(X) \right] = \mathrm{e}^K \mathrm{e}^{-K} = 1 , \tag{3.4.3}$$

which guarantees that the Einstein term in the action has standard normalization (cf. [154] for a pedagogic treatment of this point).

From the symplectic vectors

$$Q = \begin{pmatrix} p^I & q_I \end{pmatrix}^{\mathrm{T}} \tag{3.4.4}$$

and V we can build the function (the the graviphoton charge)

$$Z(X) = Q^{\mathrm{T}} \Omega V = p^I F_I(X) - q_I X^I , \tag{3.4.5}$$

which by definition of symplectic transformations (3.1.17) is a symplectic invariant and which in an asymptotically flat background agrees with the central charge, when evaluated at infinity. Under Kähler transformations (3.3.18), (3.3.19) $Z(X)$ transforms as

$$Z(X) \to \mathrm{e}^{-\frac{1}{2}\left[f(z)-\bar{f}(\bar{z})\right]} Z(X)\,, \tag{3.4.6}$$

and hence has the Kähler weight 1/2. Consequently, the Kähler covariant derivative of Z reads

$$D_A Z(X) = \partial_A Z(X) + \frac{1}{2}(\partial_A K) Z(X)\,, \qquad \partial_A = \frac{\partial}{\partial z^A}\,. \tag{3.4.7}$$

Observe that the X^I are covariantly holomorphic, i.e. $\bar{D}_{\bar{A}} X^I = 0$.

Recall that the black hole potential was then given by (3.2.2)

$$V_{\mathrm{BH}} = -\frac{1}{2} Q^{\mathrm{T}} M(\mathcal{N}) Q\,, \tag{3.4.8}$$

where the notation now stresses that the scalar-field dependence in the matrix M of (3.1.12) occurs through the formula (3.3.20) for the matrix $\mathcal{N}$ of vector couplings, which remains true with $X^I(z)$ replaced by X^I, because the rescaling affects equally the numerator and the denominator. The black hole potential can be expressed in terms of the central charge $Z(X)$ and derivatives thereof as follows [38, 69, 66]. Using the special geometry identities (see [38])

$$F_I = \mathcal{N}_{IJ} X^J\,, \tag{3.4.9}$$

$$D_A F_I = \bar{\mathcal{N}}_{IJ} D_A X^J\,, \tag{3.4.10}$$

$$-\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1\,IJ} = \bar{X}^I X^J + g^{A\bar{B}} D_A X^I \bar{D}_{\bar{B}} \bar{X}^J\,, \tag{3.4.11}$$

we compute $-\mathrm{i}Q + \Omega\, M(\mathcal{N})\, Q$ and obtain [18]

$$-\,\mathrm{i}Q + \Omega\, M(\mathcal{N})\, Q = 2\left(Z(X)\bar{V} + g^{A\bar{B}} D_A V \bar{D}_{\bar{B}} \bar{Z}(\bar{X})\right). \tag{3.4.12}$$

Decomposing (3.4.12) into imaginary and real part yields

$$-\,\mathrm{i}Q = Z(X)\bar{V} - \bar{Z}(\bar{X})V + g^{A\bar{B}}\left(D_A V \bar{D}_{\bar{B}}\bar{Z}(\bar{X}) - D_A Z(X) \bar{D}_{\bar{B}}\bar{V}\right), \tag{3.4.13}$$

$$\Omega\, M(\mathcal{N})\, Q = Z(X)\bar{V} + \bar{Z}(\bar{X})V + g^{A\bar{B}}\left(D_A V \bar{D}_{\bar{B}}\bar{Z}(\bar{X}) + D_A Z(X) \bar{D}_{\bar{B}}\bar{V}\right). \tag{3.4.14}$$

Contracting (3.4.14) with $Q^{\mathrm{T}}\,\Omega$ results in

$$V_{\mathrm{BH}} = -\frac{1}{2} Q^{\mathrm{T}} M(\mathcal{N}) Q = |Z(X)|^2 + g^{A\bar{B}} D_A Z(X) \bar{D}_{\bar{B}} \bar{Z}(\bar{X})\,, \tag{3.4.15}$$

where we used (3.4.5). This expresses the black hole potential V_{BH} in terms of $Z(X)$ and derivatives thereof.

Extrema of the black hole potential with respect to z^A satisfy [66]

$$\partial_A V_{\rm BH} = 0 \;\Leftrightarrow\; 2\,\bar{Z}(\bar{X})\,D_A Z(X) + g^{B\bar{C}}\,(\mathcal{D}_A D_B Z(X))\,\bar{D}_{\bar{C}}\bar{Z}(\bar{X}) = 0\,, \qquad (3.4.16)$$

where $\mathcal{D}$ is a fully covariant Kähler derivative, including the Levi–Civita connection of the Kähler metric, so $\mathcal{D}_A D_B Z = \left(D_A\,\delta^C_B - \Gamma^C_{AB}\right) D_C Z$. By virtue of the special geometry relation

$$\mathcal{D}_A D_B V = \mathrm{i}\,C_{ABC}\,\bar{D}^C\bar{V} \quad , \quad \bar{D}^C = g^{C\bar{C}}\,\bar{D}_{\bar{C}}\,, \qquad (3.4.17)$$

the double derivative in (3.4.16) can be replaced by

$$\mathcal{D}_A D_B Z(X) = \mathrm{i}\,C_{ABC}\,\bar{D}^C\bar{Z}(\bar{X})\,, \qquad (3.4.18)$$

where

$$C_{ABC} = \mathrm{e}^K F_{IJK}(X(z))\,\frac{\partial X^I(z)}{\partial z^A}\,\frac{\partial X^J(z)}{\partial z^B}\,\frac{\partial X^K(z)}{\partial z^C}\,. \qquad (3.4.19)$$

For later use we also note that [38]

$$\frac{1}{2}Q^{\rm T}\,M(F)\,Q = g^{A\bar{B}} D_A Z(X) \bar{D}_{\bar{B}}\bar{Z}(\bar{X}) - |Z(X)|^2\,, \qquad (3.4.20)$$

where, again abusing the notation a little, $M(F)$ stands for the matrix analogous to $M(\mathcal{N})$ of (3.1.12), but with $\mathcal{N}_{IJ}$ replaced by F_{IJ}. This can be checked by using the identity [52]

$$\left(N^{-1}\right)^{IJ} = g^{A\bar{B}} D_A X^I \bar{D}_{\bar{B}}\bar{X}^J - X^I\bar{X}^J\,, \qquad N_{IJ} = \mathrm{i}\left(\bar{F}_{IJ} - F_{IJ}\right)\,. \qquad (3.4.21)$$

The black hole potential (3.4.15) is expressed in terms of X^I and z^A. It will be useful to express it in yet another rescaled set of projective variables Y^I [11] that are Kähler-invariant:

$$\Pi(Y) = \bar{Z}(\bar{X}) V = \begin{pmatrix} Y^I \\ F_I(Y) \end{pmatrix}\,. \qquad (3.4.22)$$

In terms of the Y-variables, equations (3.4.13) and (3.4.15) become

$$\begin{aligned} -\,\mathrm{i}Q &= \bar{\Pi} - \Pi + Z(Y)^{-1} g^{A\bar{B}}\left(\partial_A\Pi\,\bar{\partial}_{\bar{B}}\bar{Z}(\bar{Y}) - \partial_A Z(Y)\bar{\partial}_{\bar{B}}\bar{\Pi}\right)\,, & (3.4.23)\\ V_{\rm BH} &= Z(Y) + Z(Y)^{-1} g^{A\bar{B}}\partial_A Z(Y)\bar{\partial}_{\bar{B}}\bar{Z}(\bar{Y})\,, & (3.4.24) \end{aligned}$$

respectively, where

$$Z(Y) = Q^{\rm T}\,\Omega\,\Pi = p^I F_I(Y) - q_I Y^I\,. \qquad (3.4.25)$$

Observe that $Z(Y)$ is real, i.e. $Z(Y) = |Z(X)|^2 = \bar{Z}(\bar{Y})$, and that it may be written as [11]

$$Z(Y) = |Z(X)|^2\,\mathrm{i}\left(\bar{X}^I F_I - X^I\bar{F}_I\right) = \mathrm{i}\left(\bar{Y}^I F_I(Y) - Y^I\bar{F}_I(\bar{Y})\right) = |Y^0|^2\,\mathrm{e}^{-G(z,\bar{z})}\,, \quad (3.4.26)$$

where we used the constraint (3.4.3) in the first step, and $Y^0 = \bar{Z}(\bar{X})\,\mathrm{e}^{K/2}\,X^0(z)$ in the last step, and where

$$G(z,\bar{z}) = K(z,\bar{z}) + \log|X^0(z)|^2\,. \qquad (3.4.27)$$

In the Kähler gauge $X^0(z) = 1$, $G = K$.

In the following, we will assume that $Z(Y) \neq 0$. Inserting the extremization condition (3.4.16),

$$\partial_A Z(Y) = -\frac{1}{2Z(Y)} g^{B\bar{C}} \mathcal{D}_A D_B Z(Y)\,\bar{\partial}_{\bar{C}}\bar{Z}(\bar{Y})\,, \qquad (3.4.28)$$

into (3.4.23) yields the desired attractor equations [106]

$$Q = 2\,\mathrm{Im}\left(\Pi(Y) + \frac{1}{2}(Z(Y))^{-2}\,g^{A\bar{B}} g^{\bar{D}E} \bar{\mathcal{D}}_{\bar{B}} \bar{D}_{\bar{D}} \bar{Z}(\bar{Y})\,\partial_A \Pi\,\partial_E Z(Y)\right)\,. \qquad (3.4.29)$$

In the supersymmetric case only the first term on the right-hand side remains and the attractor equations reduce to [11]

$$p^I = 2\,\mathrm{Im}\,Y^I\,, \qquad q_I = 2\,\mathrm{Im}\,F_I\,. \qquad (3.4.30)$$

Using (3.4.18), the double derivative in (3.4.29) can, in the Kähler gauge $X^0(z) = 1$, be written as

$$\mathcal{D}_A D_B Z(Y) = \mathrm{i}\,\frac{\bar{Z}(\bar{X})}{Z(X)}\,C_{ABC}\,g^{C\bar{C}}\,\partial_{\bar{C}}\bar{Z}(\bar{Y}) = \mathrm{i}\,\frac{Y^0}{\bar{Y}^0}\,C_{ABC}\,g^{C\bar{C}}\,\partial_{\bar{C}}\bar{Z}(\bar{Y})\,, \qquad (3.4.31)$$

where

$$Y^0 = \bar{Z}(\bar{X})X^0 = \mathrm{e}^{K/2}\bar{Z}(\bar{X}) = \mathrm{e}^{K}\left[p^I \bar{F}_I(\bar{X}(\bar{z})) - q_0 - q_A \bar{z}^A\right]\,. \qquad (3.4.32)$$

The entropy of an extremal black hole is determined by the value of the black hole potential (3.4.24)

$$V_{\mathrm{BH}} = \Big(Z(Y) + Z(Y)^{-1} g^{A\bar{B}} \partial_A Z(Y) \bar{\partial}_{\bar{B}} \bar{Z}(\bar{Y})\Big) \qquad (3.4.33)$$

at the extremum of the potential [69, 66, 81],

$$\mathcal{S}/\pi = V_{\mathrm{BH}}|_{\mathrm{extr}}\,. \qquad (3.4.34)$$

In the supersymmetric case, where $\partial_A Z = 0$, the entropy reduces to

$$\mathcal{S}_{\mathrm{BPS}}/\pi = Z(Y)|_{\mathrm{extr}}\,. \qquad (3.4.35)$$

On the other hand, in the non-supersymmetric case, and restricting to prepotentials $\mathcal{F}(z)$ that only depend on one single modulus $z^1 = z$, it was shown in [18] that (3.4.34) can be written as

$$\mathcal{S}/\pi = Z(Y)\left.\left(1 + 4\,\frac{g_{z\bar{z}}^3}{|C_{111}|^2}\right)\right|_{\mathrm{extr}}\,. \qquad (3.4.36)$$

Using (3.4.26), the entropy can be expressed as a function of the modulus z,

$$\mathcal{S}/\pi = |Y^0|^2\,\mathrm{e}^{-G(z,\bar{z})}\left(1 + 4\,\epsilon\,\frac{g_{z\bar{z}}^3}{|C_{111}|^2}\right)\,, \qquad (3.4.37)$$

where $\epsilon = 0, 1$ for supersymmetric and non-supersymmetric black holes, respectively.

3.5 Sen's entropy function

As we have seen in the previous section the attractor mechanism in gravity coupled to abelian gauge fields and neutral scalars does not rely on supersymmetry, but rather on the extremality of the black hole. One can however ask if the attractor behavior persists also in the presence of higher-derivative corrections to the Lagrangian. Since the extremum of the black hole potential corresponds to the area of the event horizon, and Wald's entropy generally deviates from the area law, one should not expect the approach presented so far to be directly applicable. There exists an alternative description, conceived by Sen [146, 147], based on a rewriting of Wald's formula and thus valid for a very general class of theories in arbitrary dimensions.

Without higher-curvature corrections, extremal black holes distinguish themselves as accommodating the maximum allowed amount of charge or angular momentum for a given mass. In arbitrary higher-derivative gravity Sen defines extremal black holes as those with the near-horizon geometry of $AdS_2 \times S^{d-2}$, that however means also a restriction to static black holes (spherical symmetry). Further, Sen's original construction, which we summarize below, requires that the Lagrangian density depend on the gauge fields solely through the gauge field strengths, immediately excluding theories with Chern–Simons terms, such as the five-dimensional $N = 2$ supergravity (unless the CS terms do not play a role in the solution).

Consistently with the assumed symmetries we write the near-horizon metric (cf. (2.1.16)) and gauge field strengths as

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) , \tag{3.5.1}$$

$$\phi^i = \phi^i_0 \,, \qquad F^I_{rt} = e^I \,, \qquad F^I_{\theta\varphi} = p^I \sin\theta \,, \tag{3.5.2}$$

with constant v_1, v_2, ϕ^i_0, e^I and p_I. (Since in the subsequent part we shall concern ourselves with the case of four space-time dimensions, we give here the four-dimensional formulae.) The spacetime is a product of two 2-dimensional spaces, and in a 2 dimensions the Riemann tensor has only one independent component, proportional to the determinant of the metric. Consequently, the non-vanishing components of the 4-dimensional Riemann tensor are

$$R_{\alpha\beta\gamma\delta} = -v_1^{-1} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \,, \qquad \alpha, \beta, \gamma, \delta = r, t \,, \tag{3.5.3}$$

$$R_{mnpq} = v_2^{-1} (g_{mp} g_{nq} - g_{mq} g_{np}) \,, \qquad m, n, p, q = \theta, \varphi \,. \tag{3.5.4}$$

Integrating equations of motion following from Maupertuis's stationary action principle over the angular coordinates amounts to multiplication by a constant due to spherical symmetry. The equations of motion themselves, with the ansatz made above, reduce to equations for the unknown parameters. As a consequence one can obtain the same equations from what we shall call the reduced Lagrangian

$$\mathcal{F} = \int d\theta d\varphi \sqrt{-g} \mathscr{L} \bigg|_{\text{horizon}} \tag{3.5.5}$$

by extremizing with respect to the parameters:

$$\frac{\partial \mathcal{F}}{\partial \phi_0^i} = 0\,, \qquad \frac{\partial \mathcal{F}}{\partial v_{1,2}} = 0\,. \tag{3.5.6}$$

Furthermore, the electric charges q_I are canonically conjugate to the electric fields e^I:

$$q_I = -2\frac{\partial \mathcal{F}}{\partial e^I}\,. \tag{3.5.7}$$

(in the conventions of [144] adopted in the next chapter)

Wald's formula simplifies significantly to read

$$\mathcal{S} = 8\pi \frac{\partial \mathscr{L}}{\partial R_{rtrt}} g_{rr} g_{tt} A\,, \tag{3.5.8}$$

where A is the area of the horizon. Denoting by $\mathcal{F}_\lambda$ the analog of the reduced Lagrangian (3.5.5) where all occurrences of R_{rtrt} have been replaced by λR_{rtrt} with a constant λ, Sen shows that

$$\left.\frac{\partial \mathcal{F}_\lambda}{\partial \lambda}\right|_{\lambda=1} = \int d\theta d\varphi \sqrt{-g} R_{\alpha\beta\gamma\delta} \frac{\partial \mathscr{L}}{\partial R_{\alpha\beta\gamma\delta}} = -\frac{1}{2\pi}\mathcal{S}\,. \tag{3.5.9}$$

and argues that $\mathcal{F}_\lambda(\phi_0, v, e, p)$ must be of the form $v_1 f(\phi_0, v_2, p, \lambda v_1^{-1}, e v_1^{-1})$ for some function f. Then it follows that

$$\lambda \frac{\partial \mathcal{F}_\lambda}{\partial \lambda} + v_1 \frac{\partial \mathcal{F}_\lambda}{\partial v_1} + e^I \frac{\partial \mathcal{F}_\lambda}{\partial e^I} - \mathcal{F}_\lambda = 0 \tag{3.5.10}$$

and, setting $\lambda = 1$ and using previous results,

$$\mathcal{S} = 2\pi \left.\left(-\frac{1}{2} e^I q_I - \mathcal{F} \right)\right|_{\text{eom}}\,. \tag{3.5.11}$$

But equations of motion are just relations between the charges and the values of the fields on the horizon, in other words the attractor equations. The conclusion is therefore that taking the Legendre transform of the reduced Lagrangian yields a quantity

$$\mathcal{E} = 2\pi \left(-\frac{1}{2} e^I q_I - \mathcal{F} \right) \tag{3.5.12}$$

that possesses two properties:

1. Its extremization produces the attractor equations.
2. Its stationary value yields the entropy of the black hole.

An object with these characteristics Sen denominates the entropy function. Note that Sen's construction reproduces Wald's formula, so that the prescription applies to gravity theories with arbitrary higher-order corrections.

Chapter 4

Black hole potential and the entropy function

4.1 Attractors with the conifold prepotential

As we have seen in the previous chapter, in the absence of higher curvature interactions the attractor phenomenon can be described both in the black hole potential and in the entropy function approach and the defining properties of the entropy function conspicuously remind one of the characteristics of the black hole potential. (But the coincidence of extrema of two functions does not imply automatically that the functions are equal.) Here we compare both methods in the setting of $N = 2$ supergravity in four dimensions. We show that the entropy function is equivalent to the black hole potential (as one might expect from the fact that the Legendre transform of the electromagnetic Lagrangian is the electromagnetic Hamiltonian—the energy in vector fields) and we give the attractor equations in a new form derived directly from the entropy function.

We solve the attractor equations for the one-modulus prepotential associated with the conifold of the mirror quintic in type IIB [25] for extremal black holes carrying two non-vanishing charges. The advantage of the entropy function approach is relative simplicity, allowing us to obtain exact solutions to the attractor equations: a supersymmetric and a non-supersymmetric one. These two solutions and their entropies are not related in a simple way to one another, unlike in the class of extremal type IIA (large volume) black hole solutions carrying D0 and D4 charge. There, the two entropies are mapped into each other by reversing the sign of the D0 charge [152, 108].

Usually one writes the black hole entropy as a function of the charges, but in the context of the entropic principle [133, 92], to which we devote chapter 7, one regards the entropy, through the attractor equations, as a function on the moduli space. In the one-modulus case, an explicit expression exists for the entropy of an extremal black hole as a function of the scalar fields [11, 18]. For the conifold prepotential we find that, whereas the entropy attains a local maximum at the conifold point for the supersymmetric solution [31] (see also [73, 18]), it possesses a local minimum there for the non-supersymmetric solution.

Nonetheless, the entropy of the non-supersymmetric solution has a local maximum in the vicinity of the conifold point, and the point corresponding to this local maximum represents a stable solution to the attractor equations. At this local maximum, the entropy is larger than the entropy of the supersymmetric solution at the conifold point.

4.2 Solutions in the black hole potential approach

In this section we consider a specific one-modulus prepotential and solve the attractor equations (3.4.29) following from the black hole potential (3.4.24), for two non-vanishing charges. Then, using (3.4.34), we compute the entropy of the resulting black hole. We refer to [152, 84, 108] for other examples.

The prepotential we assume is the conifold prepotential [25]

$$F(Y) = -\mathrm{i}\left(Y^0\right)^2 \mathcal{F}(T) = -\mathrm{i}\left(Y^0\right)^2 \left(\frac{\beta}{2\pi} T^2 \log T + a\right) , \tag{4.2.1}$$

where $T = -\mathrm{i}z = -\mathrm{i}Y^1/Y^0$, β is a real negative constant and a is a complex constant with $\mathrm{Re}\, a > 0$.

For simplicity we consider extremal black holes with two non-vanishing charges q_0 and p^1, so that the charge vector Q is given by

$$Q = \begin{pmatrix} 0 \\ p^1 \\ q_0 \\ 0 \end{pmatrix} . \tag{4.2.2}$$

In the following, we calculate all the quantities that appear in (3.4.29) for the prepotential (4.2.1). We work in the Kähler gauge $X^0(z) = 1$. The resulting exact expressions are displayed in subsection 4.2.1. Since these expressions are complicated, we approximate them in subsection 4.2.2 so as to be able to solve (3.4.29).

4.2.1 Intermediate results

Computing the derivative $\mathcal{F}_T = \partial\mathcal{F}/\partial T = \beta T\left(2\log T + 1\right)/(2\pi)$ and inserting it into (3.3.17) yields in the Kähler gauge $X^0(z) = 1$ the Kähler potential

$$K(T,\bar{T}) = -\log\left(4\,\mathrm{Re}\, a - \frac{\beta}{2\pi}\left(T+\bar{T}\right)^2 - \frac{2\beta}{\pi}|T|^2 \log|T|\right) . \tag{4.2.3}$$

Computing $Y^1 = \mathrm{i}TY^0$, $F_0 = \partial F/\partial Y^0 = -2\mathrm{i}aY^0 + \mathrm{i}\beta T^2 Y^0/(2\pi)$ and $F_1 = \partial F/\partial Y^1 = -\beta T Y^0 \log T/\pi - \beta T Y^0/(2\pi)$, we obtain for the vector $\Pi(Y)$,

$$\Pi(Y) = \begin{pmatrix} Y^0 \\ Y^1 \\ F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} Y^0 \\ \mathrm{i}TY^0 \\ -2\mathrm{i}aY^0 + \mathrm{i}\frac{\beta T^2}{2\pi}Y^0 \\ -\frac{\beta T}{\pi}Y^0 \log T - \frac{\beta T}{2\pi}Y^0 \end{pmatrix} . \tag{4.2.4}$$

The central charge (3.4.25) takes the form

$$Z(Y) = -\frac{\beta}{\pi} p^1 Y^0 T \log T - \frac{\beta}{2\pi} p^1 Y^0 T - q_0 Y^0 \,. \tag{4.2.5}$$

Using (3.4.32), we obtain

$$\begin{aligned} \partial_z Z(Y) &= -\mathrm{i}\partial_T Z = \frac{\mathrm{i}\beta}{\pi} p^1 Y^0 \left(\log T + \frac{3}{2}\right) - \mathrm{i}\left(\partial_T K\right) Z \,, & (4.2.6)\\ \partial_z \Pi(Y) &= -\mathrm{i}\partial_T \Pi = \begin{pmatrix} -\mathrm{i}\left(\partial_T K\right) Y^0 \\ Y^0\left(1 + T\left(\partial_T K\right)\right) \\ Y^0\left(-2a\left(\partial_T K\right) + \frac{\beta T}{\pi} + \frac{\beta T^2}{2\pi}\left(\partial_T K\right)\right) \\ \frac{\mathrm{i}\beta}{\pi} Y^0\left(1 + T\left(\partial_T K\right)\right)\log T + \frac{\mathrm{i}\beta}{2\pi} Y^0\left(3 + T\left(\partial_T K\right)\right) \end{pmatrix}, & (4.2.7) \end{aligned}$$

where

$$\partial_T Y^0 = \left(\partial_T K\right) Y^0 = \frac{\frac{\beta Y^0}{\pi}\left(2\bar{T} + T + 2\bar{T}\log|T|\right)}{4\,\mathrm{Re}\,a - \frac{\beta}{2\pi}\left(T + \bar{T}\right)^2 - \frac{2\beta}{\pi}|T|^2\log|T|} \,. \tag{4.2.8}$$

The metric $g_{T\bar{T}} = \partial_T \bar{\partial}_{\bar{T}} K$ is computed to be

$$g_{T\bar{T}} = \frac{\frac{4\beta}{\pi}\,\mathrm{Re}\,a\left(3 + 2\log|T|\right) + \frac{\beta^2}{\pi^2}\left(\frac{1}{2}T^2 + 2|T|^2 + \frac{1}{2}\bar{T}^2\right) + \frac{2\beta^2}{\pi^2}\left(T^2 + \bar{T}^2\right)\log|T|}{\left(4\,\mathrm{Re}\,a - \frac{\beta}{2\pi}\left(T + \bar{T}\right)^2 - \frac{2\beta}{\pi}|T|^2\log|T|\right)^2} \,. \tag{4.2.9}$$

Using $\bar{F}(\bar{X}(\bar{z})) = \mathrm{i}\,\bar{\mathcal{F}}(\bar{z})$, we have

$$\bar{C}_{\bar{1}\bar{1}\bar{1}} = \mathrm{e}^K \frac{\partial^3 \bar{F}(\bar{X}(\bar{z}))}{\partial \bar{z}^3} = \mathrm{e}^K \frac{\beta}{\pi \bar{T}} \,. \tag{4.2.10}$$

Inserting this into (3.4.31) gives

$$\bar{\mathcal{D}}_{\bar{z}}\bar{\partial}_{\bar{z}}\bar{Z}(\bar{Y}) = \frac{-\mathrm{i}\bar{Y}^0\left(4\,\mathrm{Re}\,a - \frac{\beta}{2\pi}\left(T+\bar{T}\right)^2 - \frac{2\beta}{\pi}|T|^2\log|T|\right)\left(\frac{\mathrm{i}\beta}{\pi}p^1 Y^0\left(\log T + \frac{3}{2}\right) - \mathrm{i}\left(\partial_T K\right) Z\right)}{Y^0\bar{T}\left(4\,\mathrm{Re}\,a\left(3 + 2\log|T|\right) + \frac{\beta}{\pi}\left(\frac{1}{2}T^2 + 2|T|^2 + \frac{1}{2}\bar{T}^2\right) + \frac{2\beta}{\pi}\left(T^2 + \bar{T}^2\right)\log|T|\right)} \,. \tag{4.2.11}$$

For the two-charge case (4.2.2), the black hole potential (3.4.24) is invariant under the exchange

$$Y^0 \leftrightarrow \bar{Y}^0 \,, \qquad T \leftrightarrow \bar{T} \,. \tag{4.2.12}$$

This can be seen as follows. Under the exchange (4.2.12), $Z(Y)$ given in (4.2.5) transforms into

$$Z(Y) \to Z(\bar{Y}) = -\frac{\beta\bar{T}}{\pi} p^1 \bar{Y}^0 \log\bar{T} - \frac{\beta\bar{T}}{2\pi} p^1 \bar{Y}^0 - q_0 \bar{Y}^0 = \bar{Z}(\bar{Y}) \,. \tag{4.2.13}$$

On the other hand, since $Z(Y)$ is, by construction, a real quantity, i.e. $Z(Y) = |Z(X)|^2 = \bar{Z}(\bar{Y})$, it follows that $Z(Y) = Z(\bar{Y})$ under (4.2.12). Similarly, $g^{T\bar{T}}\partial_T Z(Y)\partial_{\bar{T}}\bar{Z}(\bar{Y})$ is invariant under the exchange (4.2.12). Thus, analogously to [144], we will look for the class of solutions to the attractor equations (3.4.29) that are invariant under (4.2.12), namely for solutions with real Y^0 and real T. To further ease the computations we also assume that $T \geq 0$ when solving the attractor equations.

4.2.2 Approximate solutions

Now we approximate the expressions calculated in the last section by only keeping the leading terms in the limit $T \to 0$, i.e. we consider T in the vicinity of the conifold point. We obtain

$$\begin{aligned}
g_{z\bar{z}} &\approx \frac{\beta}{2\pi \operatorname{Re} a} \log|T| \,, && (4.2.14)\\
\partial_T Y^0 &\approx \frac{\beta Y^0 \bar{T} \log|T|}{2 \operatorname{Re} a} \,, && (4.2.15)\\
Z(Y) &\approx -\frac{\beta p^1 Y^0}{\pi} T \log T - q_0 Y^0 \,, && (4.2.16)\\
\partial_z Z(Y) &\approx \frac{\mathrm{i}\beta p^1 Y^0}{\pi} \log T + \frac{\mathrm{i}\beta q_0 Y^0}{2 \operatorname{Re} a} \bar{T} \log|T| \,, && (4.2.17)\\
\bar{\mathcal{D}}_{\bar{z}} \bar{\partial}_{\bar{z}} \bar{Z}(\bar{Y}) &\approx \frac{\bar{Y}^0 \beta \left(\frac{p^1}{\pi} \log T + \frac{q_0}{2 \operatorname{Re} a} \bar{T} \log|T| \right)}{2\bar{T} \log|T|} \,, && (4.2.18)\\
\partial_z \Pi(Y) &\approx \begin{pmatrix} -\frac{\mathrm{i}\beta Y^0 \bar{T} \log|T|}{2 \operatorname{Re} a} \\ Y^0 \\ -\frac{a\beta Y^0}{\pi \operatorname{Re} a} \bar{T} \log|T| \\ \frac{\mathrm{i}\beta Y^0}{\pi} \log T \end{pmatrix} . && (4.2.19)
\end{aligned}$$

Setting $Y^0 = \bar{Y}^0$ and $T = \bar{T}$ as mentioned above, we then obtain for the attractor equations (3.4.29),

$$\begin{aligned}
\begin{pmatrix} 0 \\ p^1 \\ q_0 \\ 0 \end{pmatrix} &= 2 \operatorname{Im} \left[\begin{pmatrix} Y^0 \\ \mathrm{i} T Y^0 \\ -2\mathrm{i} a Y^0 + \mathrm{i} \frac{\beta T^2}{2\pi} Y^0 \\ -\frac{\beta T}{\pi} Y^0 \log T - \frac{\beta T}{2\pi} Y^0 \end{pmatrix} \right] \\
&\quad + 2 \operatorname{Im} \left[\frac{\mathrm{i}\pi^2 \left(2 \operatorname{Re} a\, p^1 + q_0 T\right)^2}{4T \log T \left(\beta p^1 T \log T + \pi q_0\right)^2} \begin{pmatrix} -\frac{\mathrm{i}\beta Y^0 T \log T}{2 \operatorname{Re} a} \\ Y^0 \\ -\frac{a\beta Y^0}{\pi \operatorname{Re} a} T \log T \\ \frac{\mathrm{i}\beta Y^0}{\pi} \log T \end{pmatrix} \right] . \qquad (4.2.20)
\end{aligned}$$

Taking the imaginary part results in the following two attractor equations involving p^1 and q_0,

$$p^1 = 2TY^0 + Y^0 \frac{\pi^2 \left(2\,\mathrm{Re}\, a\, p^1 + q_0 T\right)^2}{2T \log T \left(\beta p^1 T \log T + \pi q_0\right)^2}\,, \tag{4.2.21a}$$

$$q_0 = -4\,\mathrm{Re}\, a Y^0 + \frac{\beta T^2}{\pi} Y^0 - \frac{a\beta\pi Y^0 \left(2\,\mathrm{Re}\, a\, p^1 + q_0 T\right)^2}{2\,\mathrm{Re}\, a \left(\beta p^1 T \log T + \pi q_0\right)^2}\,. \tag{4.2.21b}$$

These equations can be approximately solved to leading order in T in the limit $T \to 0$. We find the following two approximate solutions, one of which is supersymmetric. The supersymmetric one ($\partial_z Z(Y) = 0$) is given by [31]

$$p^1 \approx 2Y^0 T\,, \tag{4.2.22a}$$

$$q_0 \approx -4\,\mathrm{Re}\, a\, Y^0\,, \tag{4.2.22b}$$

whereas the non-supersymmetric solution ($\partial_z Z(Y) \neq 0$) reads

$$p^1 \approx 8Y^0 T \log T\,, \tag{4.2.23a}$$

$$q_0 \approx -4\,\mathrm{Re}\, a\, Y^0\,. \tag{4.2.23b}$$

Solving for the modulus T yields

$$Y^0 \approx -\frac{q_0}{4\,\mathrm{Re}\, a}\,, \tag{4.2.24a}$$

$$T \approx -2\,\mathrm{Re}\, a\, \frac{p^1}{q_0}\,, \tag{4.2.24b}$$

in the supersymmetric case, and

$$Y^0 \approx -\frac{q_0}{4\,\mathrm{Re}\, a}\,, \tag{4.2.25a}$$

$$T \log T \approx -\frac{1}{2}\,\mathrm{Re}\, a\, \frac{p^1}{q_0}\,. \tag{4.2.25b}$$

in the non-supersymmetric case.

The conditions $T \ll 1$ and $T \geq 0$ constrain the choice of the charges. In the supersymmetric case, we have to choose p^1 and q_0 such that $p^1/q_0 < 0$ and $|p^1| \ll |q_0|$, whereas in the non-supersymmetric case the charges have to satisfy $p^1/q_0 > 0$ and $|p^1| \ll |q_0|$.

Next, we calculate the entropy in the limit $T \to 0$. Inserting (4.2.22) into (3.4.35) yields

$$\mathcal{S}/\pi \approx \left(Y^0\right)^2 \left(4\,\mathrm{Re}\, a\, - \frac{2\beta}{\pi} T^2 \log T\right) \tag{4.2.26}$$

in the supersymmetric case [31], whereas inserting (4.2.23) into (3.4.34) yields

$$\mathcal{S}/\pi \approx \left(Y^0\right)^2 \left(4\,\mathrm{Re}\, a\, + \frac{32\beta}{\pi} T^2 \left(\log T\right)^3\right) \tag{4.2.27}$$

in the non-supersymmetric case. Both are in accordance with the expression obtained from (3.4.37).

We observe that the solutions (4.2.24) and (4.2.25) (and their associated entropies (4.2.26) and (4.2.27)) are not related in a simple way to one another, in contrast to what happens in the case of cubic prepotentials [152, 108]. For the cubic prepotential $F(Y) = -(Y^1)^3/Y^0$, the supersymmetric solution to the attractor equations (3.4.29) is given by $Y^0 = p^1/(2T)\,,\ T = \sqrt{|q_0/p^1|}$ with $q_0/p^1 < 0$, whereas the non-supersymmetric solution is given by $Y^0 = p^1/(4T)\,,\ T = \sqrt{q_0/p^1}$ with $q_0/p^1 > 0$. The values of the T-modulus are mapped into one another under $q_0 \to -q_0$. The associated entropies, $\mathcal{S} = 2\pi\sqrt{|q_0\,(p^1)^3|}$ and $\mathcal{S} = 2\pi\sqrt{q_0\,(p^1)^3}$, respectively, are also mapped into one another under this transformation. For the case of the conifold prepotential, there is no such simple transformation relating the two solutions given above.

4.3 Entropy function approach

In the following, we show that the entropy function (3.5.12) evaluated for an $N = 2$ supergravity Lagrangian without R^2-interactions is equivalent to the black hole potential, and we give the associated attractor equations (see also [2]). These results were obtained in collaboration extended to Bernard de Wit and Swapna Mahapatra. Further results and extensions, including the generalization to R^2-interactions, appeared in [29].

The advantage of the entropy function approach over the black hole potential method is relative simplicity, as the formulation does not involve covariant derivatives nor mixing of Y^I (or X^I) and z^A variables (the natural variables for the central charge are X^I, while the differentiation in the black hole potential approach is with respect to z^A). What is more, the entropy function readily lends itself to the inclusion of higher-order corrections.

4.3.1 Equivalence to the black hole potential

The entropy function (3.5.12) evaluated for an $N = 2$ supergravity Lagrangian, here considered without R^2 corrections, can be rewritten, after the electric fields e^I have been eliminated through their equations of motion, entirely in terms of the Y^I variables and the charges (see eq. (3.11) in [144]),

$$\mathcal{E}(Y,\bar{Y},p,q)/\pi = Q^{\mathrm{T}} m(\tau)\, Q - 2\mathrm{i}\, Q^{\mathrm{T}} m(\tau)\,\bar{\Pi} + 2\mathrm{i}\,\Pi^{\mathrm{T}} m(\tau)\, Q - 2\mathrm{i}\,\bar{\Pi}^{\mathrm{T}}\Omega\,\Pi\,, \tag{4.3.1}$$

where Q, Π and Ω have been defined in (3.4.4), (3.4.22) and (3.1.17), respectively,

$$m(\tau) = \begin{pmatrix} \bar{\tau}N^{-1}\tau & -\bar{\tau}N^{-1} \\ -N^{-1}\tau & N^{-1} \end{pmatrix}, \qquad \tau_{IJ} \equiv F_{IJ}\,, \qquad N = \mathrm{i}(\bar{\tau}-\tau)\,, \tag{4.3.2}$$

and the relationship between the Y-variables used here and those of [144] is

$$\begin{pmatrix} Y^I \\ F_I \end{pmatrix} \leftrightarrow \frac{v\,\bar{w}}{4} \begin{pmatrix} x^I \\ F_I \end{pmatrix}_{[144]} \qquad (v_1 = v_2 = v)\,. \tag{4.3.3}$$

Due to the homogeneity of the prepotential, τ is homogeneous of degree 0, so it is not subject to rescaling. Observe that $m(\tau)$ is Hermitian ($m^{\mathrm{T}} = \bar{m}$), because τ is symmetric and N is real (so that N^{-1} is both real and symmetric).

Thanks to these properties, as well as $m - \bar{m} = \mathrm{i}\Omega$, and the fact that a transpose of a scalar is equal to itself, $\mathcal{E}$ can be recast into the form

$$\begin{aligned}\mathcal{E}(Y,\bar{Y},p,q)/\pi &= \frac{1}{2}Q^{\mathrm{T}}M(\tau)\,Q + \mathrm{i}Q^{\mathrm{T}}M(\tau)\,(\Pi-\bar{\Pi}) + Q^{\mathrm{T}}\Omega\,(\Pi+\bar{\Pi}) - 2\mathrm{i}\,\bar{\Pi}^{\mathrm{T}}\Omega\,\Pi \\ &= \frac{1}{2}\left(Q+\mathrm{i}(\Pi-\bar{\Pi})\right)^{\mathrm{T}}M(\tau)\left(Q+\mathrm{i}(\Pi-\bar{\Pi})\right) + \frac{1}{2}(\Pi-\bar{\Pi})^{\mathrm{T}}M(\tau)\,(\Pi-\bar{\Pi}) \\ &\quad + Q^{\mathrm{T}}\Omega\,(\Pi+\bar{\Pi}) - 2\mathrm{i}\,\bar{\Pi}^{\mathrm{T}}\Omega\,\Pi\,,\end{aligned} \tag{4.3.4}$$

where $M(\tau) = m(\tau) + \bar{m}(\tau)$ is the same symmetric matrix as in (3.1.12), but evaluated for F_{IJ} instead of $\mathcal{N}_{IJ}$,

$$M(\tau) = \begin{pmatrix} \mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1}\end{pmatrix} = \begin{pmatrix} \bar{\tau}N^{-1}\tau + \tau N^{-1}\bar{\tau} & -(\tau+\bar{\tau})N^{-1} \\ -N^{-1}(\tau+\bar{\tau}) & 2N^{-1}\end{pmatrix}\,, \tag{4.3.5}$$

and now

$$\mathcal{R} = \mathrm{Re}\,\tau\,, \qquad \mathcal{I} = \mathrm{Im}\,\tau\,. \tag{4.3.6}$$

By direct expansion, exploiting the homogeneity relation

$$Y^I F_{IJ} = F_J\,, \tag{4.3.7}$$

the symmetry in indices and the definition of N we have

$$\frac{1}{2}(\Pi-\bar{\Pi})^{\mathrm{T}}M(\tau)\,(\Pi-\bar{\Pi}) = \mathrm{i}\bar{\Pi}^{\mathrm{T}}\Omega\,\Pi\,. \tag{4.3.8}$$

As a result the entropy function can be represented as a sum of two entities

$$\mathcal{E}(Y,\bar{Y},p,q)/\pi = \Sigma(Y,\bar{Y},p,q) + \frac{1}{2}\left(Q+\mathrm{i}(\Pi-\bar{\Pi})\right)^{\mathrm{T}}M(\tau)\left(Q+\mathrm{i}(\Pi-\bar{\Pi})\right)\,, \tag{4.3.9}$$

where [11]

$$\begin{aligned}\Sigma(Y,\bar{Y},p,q) &= -\mathrm{i}\bar{\Pi}^{\mathrm{T}}\Omega\,\Pi + Q^{\mathrm{T}}\Omega\,(\Pi+\bar{\Pi}) \\ &= -\mathrm{i}(\bar{Y}^I F_I - Y^I\bar{F}_I) + p^I(F_I+\bar{F}_I) - q_I(Y^I+\bar{Y}^I)\,.\end{aligned} \tag{4.3.10}$$

We will now identify (4.3.9) with the two parts of the black hole potential (3.4.24).

Substitution of (3.4.26) into Σ implies that

$$\Sigma(Y,\bar{Y},p,q) = \mathrm{i}(\bar{Y}^I F_I - Y^I\bar{F}_I) = Z(Y)\,, \tag{4.3.11}$$

in which we recognize the first term in V_{BH}.

The product in (4.3.9) decomposes into (cf. (4.3.4))

$$\frac{1}{2}\left[Q^{\mathrm{T}}M(\tau)\,Q + \mathrm{i}Q^{\mathrm{T}}M(\tau)\,(\Pi-\bar{\Pi}) + \mathrm{i}(\Pi-\bar{\Pi})^{\mathrm{T}}M(\tau)\,Q - (\Pi-\bar{\Pi})^{\mathrm{T}}M(\tau)\,(\Pi-\bar{\Pi})\right]. \tag{4.3.12}$$

The first term above becomes, by virtue of (3.4.20),

$$\frac{1}{2}Q^{\mathrm{T}}M(\tau)\,Q = Z(Y)^{-1}g^{A\bar{B}}\partial_A Z(Y)\bar{\partial}_{\bar{B}}\bar{Z}(Y) - Z(Y)\,. \tag{4.3.13}$$

To be precise, the quoted identity concerns $M(\tau(X))$ and not $M(\tau(Y))$ as here, but as we have indicated, τ is homogeneous of degree 0 and $\tau(X)=\tau(Y)$.

Recall from (4.3.4) that as a consequence of $M=M^{\mathrm{T}}$, the second and third term in (4.3.12) are equal to one another. Applying the same techniques as in the derivation of (4.3.8) we obtain

$$\frac{\mathrm{i}}{2}(\Pi-\bar{\Pi})^{\mathrm{T}}M(\tau)\,Q = \frac{1}{2}\left(p^I(F_I+\bar{F}_I) - q_I(Y^I+\bar{Y}^I)\right) = \frac{1}{2}\left(Z(Y)+\bar{Z}(Y)\right) = Z(Y)\,, \tag{4.3.14}$$

where we used that $Z(Y)$ is real.

Finally, collecting the partial results confirms that (cf. (3.4.33))

$$\mathcal{E}/\pi = Z(Y) + Z(Y)^{-1}g^{A\bar{B}}\partial_A Z(Y)\bar{\partial}_{\bar{B}}\bar{Z}(\bar{Y}) = V_{\mathrm{BH}}\,. \tag{4.3.15}$$

4.3.2 Attractor equations

The attractor equations can be derived by demanding that $\mathcal{E}$, given in (4.3.9) and (4.3.10), be stationary with respect to independent variations in Y^I and $\bar{Y}^I$. By invoking homogeneity of the prepotential (4.3.7) we obtain after [113]

$$\delta\Sigma = -(\tilde{q}_J-\tilde{p}^I F_{IJ})\delta Y^J - (\tilde{q}_J-\tilde{p}^I\bar{F}_{IJ})\delta\bar{Y}^J = \tilde{p}^I(\delta F_I+\delta\bar{F}_I) - \tilde{q}_I(\delta Y^I+\delta\bar{Y}^I) = 0\,, \tag{4.3.16}$$

from which the equations for supersymmetric attractors immediately follow. Analogously, if more tediously, from $\delta\mathcal{E}=0$ we arrive at the general, non-supersymmetric equations

$$2v_I + \bar{v}_J N^{JK}F_{KIL}\tilde{p}^L - \mathrm{i}\bar{v}_J N^{JK}F_{KIL}N^{LM}v_M = 0\,, \qquad v_I = \tilde{q}_I - F_{IJ}\tilde{p}^J\,. \tag{4.3.17}$$

(and their complex conjugates), where

$$\tilde{q}_I = q_I - 2\,\mathrm{Im}\,F_I\,, \qquad \tilde{p}^I = p^I - 2\,\mathrm{Im}\,Y^I\,, \qquad N^{IJ}\equiv (N^{-1})^{IJ}. \tag{4.3.18}$$

Note that the quantities (4.3.18) are real.

The structure of (4.3.17) makes it evident that $v_I=0$ is still a solution. Indeed, from $v_I=0$ and $\bar{v}_I=0$ one infers that if $\mathrm{Im}\,F_{IJ}$ is nonsingular, then

$$\tilde{p}^I = 0\,, \qquad \tilde{q}_I = 0\,, \tag{4.3.19}$$

which are the supersymmetric attractor equations (3.4.30).

4.3.3 Exact solutions

In the entropy function formalism the attractor equations for the conifold prepotential (4.2.1) and two non-zero charges q_0 and p^1 take a sufficiently simple form to allow a manageable exact solution. The system of equations (4.3.17) reduces (under the same simplifying assumptions as in subsection 4.2.2, namely $Y^0 = \bar{Y}^0, T = \bar{T}$ and $T \geq 0$) to two independent simultaneous equations,

$$8\beta Y^0 T^2 - 8\beta p^1 T + 8\pi q_0 + 32\pi \operatorname{Re} a\, Y^0 + \frac{\beta(T(\beta p^1 T - 2\pi q_0) - 4\pi p^1 \operatorname{Re} a)^2}{Y^0 \left(\beta T^2 + 4\pi \operatorname{Re} a\right)^2} = 0\,, \tag{4.3.20a}$$

$$4p^1(2\log T + 3) - 8Y^0 T(2\log T + 3) - \frac{(T(\beta p^1 T - 2\pi q_0) - 4\pi p^1 \operatorname{Re} a)^2}{Y^0 T \left(\beta T^2 + 4\pi \operatorname{Re} a\right)^2} = 0\,, \tag{4.3.20b}$$

which, as we have already seen, possess two (pairs of) solutions: one preserving half of supersymmetries and one supersymmetry-breaking.

The supersymmetric solution

$$p^1 = 2Y^0 T\,, \tag{4.3.21a}$$

$$q_0 = -4\operatorname{Re} a\, Y^0 + \frac{\beta}{\pi} Y^0 T^2 \tag{4.3.21b}$$

can be directly compared with the approximate solution (4.2.22).

For comparison with (4.2.23) the exact non-supersymmetric solution

$$p^1 = \frac{Y^0 T \left(\beta^2 T^4 (2\log T(\log T + 3) + 5) + 8\pi \operatorname{Re} a \left(\beta T^2 (\log T + 2) + \pi \operatorname{Re} a (4\log T + 7)\right)\right)}{\left(\beta T^2 (\log T + 1) - 2\pi \operatorname{Re} a\right)^2}\,, \tag{4.3.22a}$$

$$\begin{aligned} q_0 = &-\frac{Y^0}{2\pi \left(\beta T^2(\log T + 1) - 2\pi \operatorname{Re} a\right)^2} \\ &\Big[\beta^3 T^6 \left(2\log T(\log T + 2) + 1\right) + 8\pi \operatorname{Re} a \Big(\beta^2 T^4 \left(\log T(5\log T + 11) + 5\right) \\ &+ \pi \operatorname{Re} a \left(\beta T^2 \left(4\log T(2\log T + 3) + 1\right) + 4\pi \operatorname{Re} a\right)\Big)\Big]\,, \end{aligned} \tag{4.3.22b}$$

needs to be expanded for small T,

$$p^1 = 8Y^0 T\log T + 14 Y^0 T + \mathcal{O}(T^3)\,, \tag{4.3.23a}$$

$$q_0 = -4\operatorname{Re} a\, Y^0 - \frac{\beta}{\pi} Y^0 T^2 \left(8(\log T)^2 + 16\log T + 5\right) + \mathcal{O}(T^3) \tag{4.3.23b}$$

and turns out not to be related in a simple way to the supersymmetric solution, as we already mentioned when discussing the approximate result (4.2.23) (see the end of subsection 4.2.2).

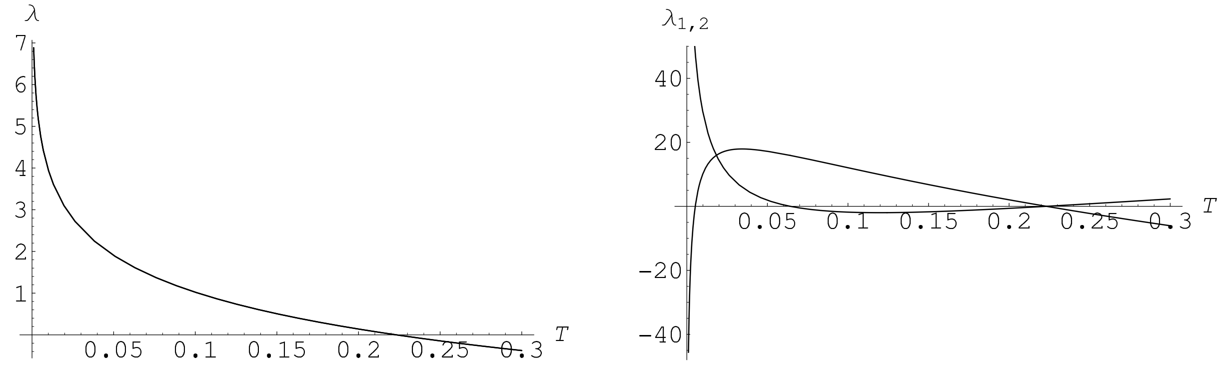

Figure 4.1: Eigenvalues $\lambda_{1,2}$ of the Hessian of the black hole potential as functions of T for $Y^0 = 1$, $\beta = -1/2$, $a = 1$ in the supersymmetric (left) and non-supersymmetric case (right).

4.4 Stability of solutions

To verify whether a solution to the attractor equations is stable, viz. indeed represents an attractor, one needs to check if it furnishes the black hole potential, regarded as a function of the moduli for a given set of charges, with a minimum. In practice it might be again more feasible to avail oneself of the expressions derived from the entropy function (4.3.9), where now Y^1 has been replaced by $\mathrm{i}TY^0$, and Y^0 is (in the Kähler gauge $X^0(z) = 1$) expressed in terms of T as $Y^0 = \mathrm{e}^{K(T,\bar{T})}\left(-\frac{\beta}{\pi}p^1\bar{T}(\frac{1}{2} + \log\bar{T}) - q_0\right)$.

The quality of critical points of a real-valued function can be determined with the aid of its Hessian matrix (unless the second derivatives vanish). If, as here, the function has complex arguments z^A, by 'Hessian matrix' we mean the Hessian computed with respect to real variables ($x^A = \frac{1}{2}(z^A + \bar{z}^A)$ and $y^A = \frac{1}{2\mathrm{i}}(z^A - \bar{z}^A)$), which may be expressed by the matrix of complex derivatives (using $\partial_x = \partial + \bar{\partial}$ and $\partial_y = \mathrm{i}(\partial - \bar{\partial})$) through the following block-matrix equation

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \mathrm{i} \\ \mathbb{I} & -\mathrm{i} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \frac{\partial^2 f}{\partial z^2} & \frac{\partial^2 f}{\partial z \partial \bar{z}} \\ \frac{\partial^2 f}{\partial z \partial \bar{z}} & \frac{\partial^2 f}{\partial \bar{z}^2} \end{pmatrix} \begin{pmatrix} \mathbb{I} & \mathrm{i} \\ \mathbb{I} & -\mathrm{i} \end{pmatrix}. \tag{4.4.1}$$

(See also [18]). Whenever the Hessian is positive definite at a stationary point, this point must be a minimum.

The Hessian matrix of $V_{\mathrm{BH}}(z, \bar{z}, p, q)$ with the supersymmetric solution (4.3.21) substituted after differentiation has one double eigenvalue, positive for sufficiently small $T = -\mathrm{i}z$ (Fig. 4.1, left). This means that the supersymmetric solution is stable, in accordance with the universal statement [66] that for supersymmetric solutions (the relevant part of) the Hessian is proportional to the Kähler metric, rendering all supersymmetric solutions attractors, as long as the metric remains positive definite. (Notice that the fact that at some point the eigenvalue becomes negative signals that the assumed prepotential is no longer a good approximation there.)

For the non-supersymmetric solution (4.3.22) the Hessian of the black hole potential has two distinct eigenvalues, which exhibit complicated behavior as T varies (Fig. 4.1, right). For very small T the eigenvalues have opposite signs, indicating a saddle point of the potential (the solution is unstable), but in the range approximately $T \in [0.005, 0.06]$ the eigenvalues are both positive, so the solution becomes an attractor (provided that the prepotential in this region can be still reliably described by (4.2.1)).

4.5 Extrema of the entropy in the moduli space

In anticipation of chapter 7, where we will discuss the entropic principle [133, 92], let us display the black hole entropy as a function on the moduli space, rather than, as otherwise common, a function of the charges. (We ignore at this point the question whether a solution to the attractor equations with integral charges can be found for an arbitrary point in the moduli space.)

Inserting the supersymmetric solution (4.3.21) into (4.3.9) yields the entropy

$$\mathcal{S} = 2(Y^0)^2 \left(2\pi \operatorname{Re} a - \beta T^2(\log T + 1)\right) , \tag{4.5.1}$$

in agreement with (3.4.37),

$$\mathcal{S}_{\mathrm{BPS}} = \pi |Y^0|^2 \mathrm{e}^{-G(z,\bar{z})} = 2|Y^0|^2 \left(2\pi \operatorname{Re} a - \beta |T|^2 \log|T| - \beta (\operatorname{Re} T)^2\right) . \tag{4.5.2}$$

The entropy $\mathcal{S}$, regarded as a function of z (or T) for constant Y^0, has a local maximum at the conifold point $T = 0$ [31], as shown in Fig. 4.2. The left graph corresponds to our explicit solution (4.5.1) for two charges (constrained to the positive T semi-axis), while the right represents the general formula (4.5.2), without restrictions on the charges.

Inserting the non-supersymmetric solution (4.3.22) into (4.3.9) yields the entropy

$$\begin{aligned}
\frac{\mathcal{S}}{\pi (Y^0)^2} = {} & \frac{1}{4\pi \left(\beta T^2(\log T + 1) - 2\pi \operatorname{Re} a\right)^4} \\
& \left[\frac{1}{\beta T^2 + 4\pi \operatorname{Re} a} \Bigg(\beta T^2 (2\log T + 3)^3 \left(\beta T^2 + 4\pi \operatorname{Re} a\right)^5 \right. \\
& - \Big(\beta^2 T^4 \left(-9\beta T^2 + 4\left(a\pi - \beta T^2\right)(\log T)^2 + 4\left(2a\pi - 3\beta T^2\right)\log T + 4a\pi\right) \\
& - 8\pi\beta \operatorname{Re} a\, T^2 \left(5\beta T^2 (\log T)^2 + 2\left(7\beta T^2 + a\pi\right)\log T + 2\left(5\beta T^2 + a\pi\right)\right) \\
& - 32\pi^3 (\operatorname{Re} a)^3 + 16\pi^2 (\operatorname{Re} a)^2 \left(-4\beta T^2 (\log T)^2 - 7\beta T^2 - 10\beta T^2 \log T + a\pi\right) \\
& \left. + 4\pi\bar{a} \left(\beta T^2(\log T + 1) - 2\pi \operatorname{Re} a\right)^2 \Big)^2 \Bigg) - 8\left(\beta T^2(\log T + 1) - 2\pi \operatorname{Re} a\right)^5 \right] \\
= {} & 4 \operatorname{Re} a + \frac{\beta}{\pi} T^2 \left(32(\log T)^3 + 144(\log T)^2 + 214\log T + 106\right) + \mathcal{O}(T^3) ,
\end{aligned} \tag{4.5.3}$$

again in agreement with (3.4.37),

$$\begin{aligned}\mathcal{S}_{\text{non-BPS}} &= \pi|Y^0|^2\mathrm{e}^{-G(z,\bar{z})}\left(1+4\frac{g_{z\bar{z}}^3}{|C_{111}|^2}\right)\\ &= 2|Y^0|^2\left(2\pi\,\mathrm{Re}\,a-\beta|T|^2\log|T|-\beta(\mathrm{Re}\,T)^2\right)\\ &\left(1+\beta|T|^2\frac{\left(\beta|T|^2(1-2\log|T|)+2\beta(\mathrm{Re}\,T)^2\,(1+2\log|T|)+4\pi\,\mathrm{Re}\,a\,(3+2\log|T|)\right)^3}{4\left(2\pi\,\mathrm{Re}\,a-\beta|T|^2\log|T|-\beta(\mathrm{Re}\,T)^2\right)^4}\right).\end{aligned} \tag{4.5.4}$$

In contrast to the supersymmetric case, $\mathcal{S}$ attains for constant Y^0 a local minimum at the conifold point. There exists, however, also a local maximum around $T\approx 0.05$ (Fig. 4.3). As mentioned at the end of the previous section, this point is an attractor, provided that one can still trust the prepotential (4.2.1) there. If we keep Y^0 constant as in [92], the maximal value of the non-supersymmetric entropy is higher than that of the supersymmetric entropy.

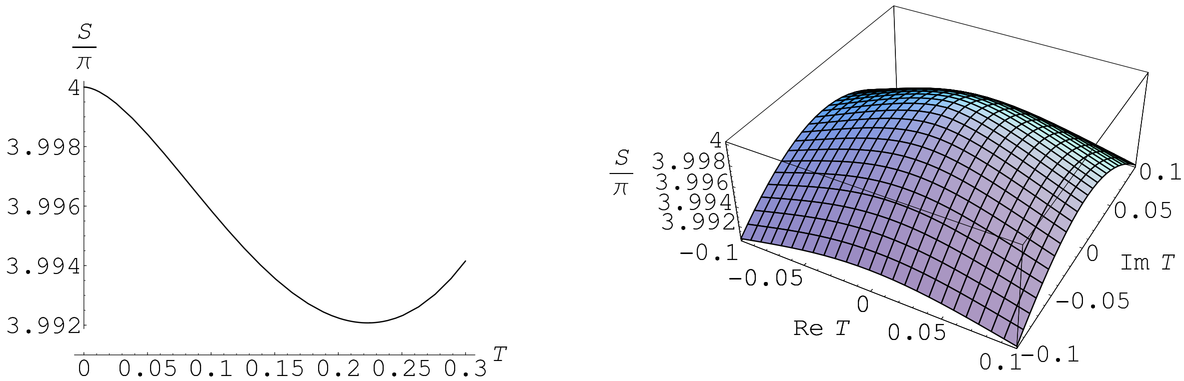

Figure 4.2: $\mathcal{S}/\pi$ as a function of T for $Y^0 = 1$, $\beta = -1/2$, $a = 1$ in the supersymmetric case. The left graph is a cross section along the positive T semi-axis through the surface in the right graph.

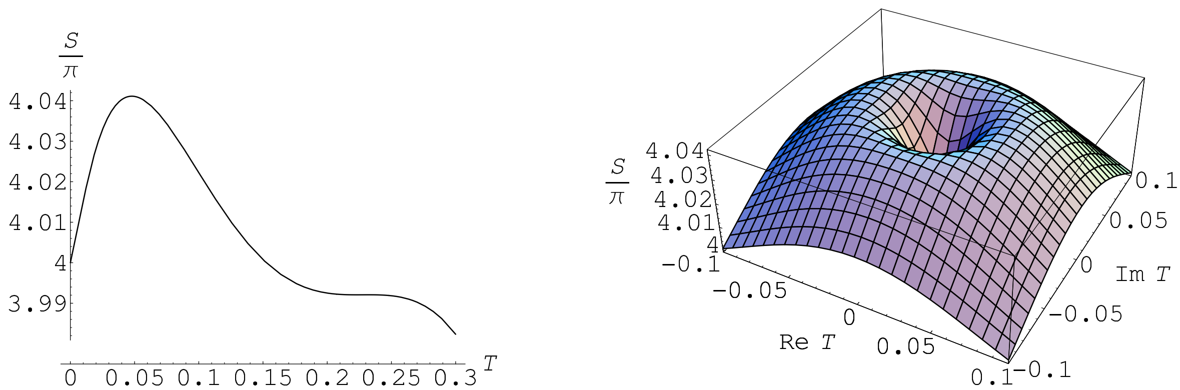

Figure 4.3: $\mathcal{S}/\pi$ as a function of T for $Y^0 = 1$, $\beta = -1/2$, $a = 1$ in the non-supersymmetric case, similarly to Fig. 4.2.

Chapter 5

Entropy function for five-dimensional rotating black holes

5.1 Extremal black holes in five and four dimensions

Extremal black holes in five dimensions can be related to extremal black holes in four dimensions. This connection is implemented by placing the five-dimensional black hole in a Taub-NUT geometry, and by using the modulus of the Taub-NUT space to interpolate between the five and the four-dimensional description. In the vicinity of the NUT charge, spacetime looks five-dimensional, whereas far away from the NUT the spacetime looks four-dimensional. This connection was first established in [77, 76] for supersymmetric black holes in the context of $N = 2$ supergravity theories that in four dimensions are based on cubic prepotentials, and was further discussed in [14].

In the following, we focus on rotating extremal black holes in five dimensions which are connected to static extremal black holes in four dimensions in the way described above. We use this link to define the entropy function for these rotating black hole solutions in the context of $N = 2$ supergravity theories with cubic prepotentials. In four dimensions, the static extremal black holes we consider carry charges (P^I, Q_I), where $P^0 \neq 0$ corresponds to the NUT charge in five dimensions (in this chapter we distinguish the charges as measured in four and five dimensions by using capital and small letters, respectively). These four-dimensional black holes are connected to rotating five-dimensional black holes with one independent angular momentum parameter. The five-dimensional $N = 2$ supergravity theories contain Chern–Simons terms for the abelian gauge fields, so that the definition of the entropy function given in [146, 6] cannot be directly applied whenever these terms play a role for the given background. Therefore, we define the entropy function for these rotating five-dimensional black holes to equal the entropy function of the associated static black holes in four dimensions. The latter was computed for $N = 2$ supergravity theories in [144, 29]. Then, we specialize to the case of black holes with non-vanishing charges (P^0, Q_I), which in five dimensions correspond to rotating electrically charged extremal black holes in a Taub-NUT geometry. Extremization of the entropy function yields a set of attractor equations for the various parameters characterizing the near-horizon solution. We check that these attractor equations are equivalent to the equations of motion in five dimensions evaluated in the black hole background. We construct two types of solutions to the attractor equations and we compute their entropy.

Our approach for defining the entropy function in the presence of Chern–Simons terms is based on dimensional reduction, and is therefore similar to the approach used in [143] for defining the entropy function of the three-dimensional BTZ black hole. Related results for rotating AdS_5 black holes have appeared in [124].

5.2 Dimensional reduction

Extremal black holes in five dimensions can be connected to extremal black holes in four dimensions, as described above. In the following, we focus on rotating black holes in five dimensions which are connected to static black holes in four dimensions. The associated near-horizon geometries are related by dimensional reduction over a compact direction of radius R. In the context of five-dimensional theories based on n abelian gauge fields A_5^A and real scalar fields X^A $(A = 1, \dots, n)$ coupled to gravity, the reduction is based on the following formulae (see for instance [94, 135]),

$$\begin{aligned} ds_5^2 &= \mathrm{e}^{2\phi}\, ds_4^2 + \mathrm{e}^{-4\phi}\,(dx^5 - A_4^0)^2 \quad , \quad dx^5 = R\, d\psi \ , \\ A_5^A &= A_4^A + C^A\,(dx^5 - A_4^0) \ , \\ \hat{X}^A &= \mathrm{e}^{-2\phi}\, X^A \ , \end{aligned} \tag{5.2.1}$$

where the A_4^I denote the four-dimensional abelian gauge fields (with $I = 0, A$).

We will focus on $N = 2$ supergravity theories that are based on cubic prepotentials in four dimensions. As we review in appendix B, the rescaled scalar fields $\hat{X}^A$ and the Kaluza-Klein scalars C^A are combined into the four-dimensional complex scalar fields z^A [94],

$$z^A = C^A + \mathrm{i}\hat{X}^A \ . \tag{5.2.2}$$

We take the fields C^A and $\hat{X}^A$, and hence also z^A, to be dimensionless.

The near-horizon geometry of the rotating five-dimensional black hole is taken to be a squashed $AdS_2 \times S^3$ given by [124]

$$ds_5^2 = v_1(-r^2dt^2 + \frac{dr^2}{r^2}) + \frac{v_2}{4}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) + \frac{v_2 v_3}{4}\left(d\psi + \cos\theta\, d\varphi - \alpha\, r dt\right)^2 \ , \tag{5.2.3}$$

where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$. The parameters v_1, v_2, v_3 and α are constant. The near-horizon geometry of the associated static four-dimensional black hole is of the $AdS_2 \times S^2$ type,

$$ds_4^2 = \tilde{v}_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \tilde{v}_2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{5.2.4}$$

with constant parameters $\tilde{v}_1$ and $\tilde{v}_2$. Using (5.2.1), we find the following relations,

$$\begin{aligned} e^{-4\phi} &= \frac{v_2 v_3}{4R^2} \,, \\ A_4^0 &= R\,(-\cos\theta d\varphi + \alpha r dt) \,, \end{aligned} \tag{5.2.5}$$

as well as

$$\tilde{v}_1 = v_1 \sqrt{\frac{v_2 v_3}{4R^2}} \quad , \quad \tilde{v}_2 = \frac{v_2}{4} \sqrt{\frac{v_2 v_3}{4R^2}} \,, \tag{5.2.6}$$

and hence

$$\tilde{v}_1\, \tilde{v}_2 = \frac{v_1\, v_2^2\, v_3}{16R^2} \quad , \quad \frac{\tilde{v}_1}{\tilde{v}_2} = 4\, \frac{v_1}{v_2} \,. \tag{5.2.7}$$

We denote the electric fields in four and five dimensions by $F_{rt} = e$. Hence, we rewrite A_4^0 as

$$A_4^0 = e_4^0\, r\, dt - p^0 R \cos\theta d\varphi \,, \tag{5.2.8}$$

with $e_4^0 = \alpha R$ and the NUT charge $p^0 = 1$.

5.3 Entropy function in five dimensions

The entropy function (3.5.12) is derived from the reduced Lagrangian. As we reviewed, the reduced Lagrangian $\mathcal{F}$ is obtained by evaluating the Lagrangian in the near-horizon black hole background and integrating over the horizon. In five and four dimensions,

$$\begin{aligned} \mathcal{F}_5 &= \int d\psi\, d\theta\, d\phi\, \sqrt{-G}\, \mathscr{L}_5 \,, \\ \mathcal{F}_4 &= \int d\theta\, d\phi\, \sqrt{-g}\, \mathscr{L}_4 \,. \end{aligned} \tag{5.3.1}$$

In the presence of Chern–Simons terms, however, the definition of the entropy function given in [146, 6] is not directly applicable whenever these terms play a role for the given background. This is the situation encountered in $N = 2$ supergravity theories in five dimensions, but not in four dimensions. Therefore, we use dimensional reduction to define the entropy function $\mathcal{E}_5$ for rotating black holes in five dimensions in terms of the entropy function $\mathcal{E}_4$ for the associated four-dimensional black holes,

$$\mathcal{E}_5 = \mathcal{E}_4 \,. \tag{5.3.2}$$

5.3.1 Rotating electrically charged black holes

Here we consider rotating electrically charged extremal black hole solutions in $N = 2$ supergravity theories in five dimensions. The bosonic part of the five-dimensional Lagrangian is given by (B.14). The black hole solutions carry NUT charge $p^0 = 1$ as well as electric charges q_A. The near-horizon solution is specified in terms of constant scalars X^A, the line element (5.2.3) and the five-dimensional gauge potentials A_5^A,

$$A_5^A = e_5^A\, r\, dt + C^A\, R\, (d\psi + \cos\theta d\varphi)\ , \tag{5.3.3}$$

where $F_{rt}^A = e_5^A$ denotes the electric field in five dimensions. Both e_5^A and C^A are constant.

These five-dimensional rotating extremal black holes are connected to static electrically charged extremal black holes in four dimensions with constant scalars z^A, line element (5.2.4) and four-dimensional gauge potentials A_4^I given by (5.2.8) and

$$A_4^A = e_4^A\, r\, dt\ . \tag{5.3.4}$$

The five- and four-dimensional electric fields are related by

$$e_5^A = e_4^A - C^A\, e_4^0 = e_4^A - \alpha\, R\, C^A \tag{5.3.5}$$

according to (5.2.1). In our conventions, the electric fields in five and four dimensions have length dimension one.

As reviewed in appendix B, the five- and four-dimensional actions (B.14) and (B.24) are identical upon dimensional reduction over x^5, up to boundary terms which are usually discarded and which arise when integrating the Chern–Simons term in (B.14) by parts. However, when evaluating these actions in a background with constant C^A, as is the case for the near-horizon solutions under consideration, they are not any longer equal to one another. Namely, evaluating the Chern–Simons term in (B.14) for constant C^A, and using $F_5^A = F_4^A - C^A\, F_4^0$ (see (5.2.1)), we obtain with the help of (B.32)

$$\begin{aligned} C_{ABC}\, F_5^A \wedge F_5^B \wedge A_5^C \ &= \ \frac{1}{2}\, R\, d\psi\, d^4x\, \sqrt{-g}\, \mathrm{Re}\mathcal{N}_{IJ}\, F^I_{4\mu\nu}\, \star F_4^{J\mu\nu} \\ &\quad - R\, C_{ABC}\, \left(C^A\, C^B\, F_4^C - \frac{2}{3}\, C^A\, C^B\, C^C\, F_4^0\right) \wedge F_4^0 \wedge d\psi\ . \end{aligned} \tag{5.3.6}$$

Thus, the actions differ by

$$\begin{aligned} 8\pi\, (S_5 - S_4) \ &= \ \frac{1}{6G_4} \int d^4x\, \sqrt{-g}\, \mathrm{Re}\mathcal{N}_{IJ}\, F^I_{4\mu\nu}\, \star F_4^{J\mu\nu} \\ &\quad + \frac{1}{6G_4} \int C_{ABC}\, \left(C^A\, C^B\, F_4^C - \frac{2}{3}\, C^A\, C^B\, C^C\, F_4^0\right) \wedge F_4^0\ . \end{aligned} \tag{5.3.7}$$

Similarly, in the background specified by (5.3.3), the reduced Lagrangians (5.3.1) differ by

$$\mathcal{F}_5 - \mathcal{F}_4 = \frac{1}{12G_4}\, R\, C_{ABC}\, C^A\, C^B\, e_4^C\ . \tag{5.3.8}$$

This has to be taken into account when using (5.3.2) to define the entropy function in five dimensions in terms of $\mathcal{E}_4$. The entropy function of static extremal black holes in four dimensions is the Legendre transform of the reduced Lagrangian $\mathcal{F}_4$ with respect to the electric fields and reads [146]

$$\mathcal{E}_4 = 2\pi \left(-\frac{1}{2}\, e_4^I \, Q_I \, G_4^{-1/2} - \mathcal{F}_4 \right) \,, \tag{5.3.9}$$

where we denote the four-dimensional electric charges by Q_I. The normalizations are as in [144, 29], with the additional $G_4^{-1/2}$ to ensure that $\mathcal{E}_4$ is dimensionless. Using (5.3.5), (5.3.8) and (5.3.2), we now express (5.3.9) as

$$\begin{aligned} \mathcal{E}_5 \;=\; & 2\pi \Bigg[-\frac{1}{2}\, \alpha \left(J + R\, C^A \left(q_A \, G_5^{-1/3} - \frac{2\pi\, R^2}{3G_5}\, C_{ABC}\, C^B\, C^C \right) \right) \\ & -\frac{1}{2}\, e_5^A \left(q_A \, G_5^{-1/3} - \frac{2\pi\, R^2}{3G_5}\, C_{ABC}\, C^B\, C^C \right) - \mathcal{F}_5 \Bigg] \,, \end{aligned} \tag{5.3.10}$$

where the five-dimensional quantities (J, q_A) are given in terms of the four-dimensional electric charges (Q_0, Q_A) by

$$\begin{aligned} J &= Q_0 \, R \, G_4^{-1/2} \,, \\ q_A \, G_5^{-1/3} &= Q_A \, G_4^{-1/2} \,. \end{aligned} \tag{5.3.11}$$

In (5.3.18) below, J will be related to the angular momentum in five-dimensions. Observe that in the presence of the C^A, the electric charges q_A are shifted by a term proportional to $C_{ABC}\, C^B\, C^C$. This shift, which is due to (5.3.8) and thus has its origin in the presence of the Chern–Simons term in the five-dimensional action (B.14), has also been observed in [124]. In addition, we note that J also gets shifted by terms involving C^A. This shift ensures that extrema of $\mathcal{E}_5$ satisfy all the five-dimensional equations of motion. This we now demonstrate by explicitly checking the equation of motion for $A^A_{5\psi}$, as follows. Using (5.3.3), we compute

$$\begin{aligned} \mathcal{F}_5 \;=\; & \pi \, \frac{v_1 \, (v_2^3 \, v_3)^{1/2}}{4G_5} \left[-\frac{1}{v_1} + \frac{4 - v_3}{v_2} + \frac{v_2 \, v_3 \, \alpha^2}{16 v_1^2} + \frac{G_{AB}\, e_5^A \, e_5^B}{2 v_1^2} - 8R^2 \, \frac{G_{AB}\, C^A \, C^B}{v_2^2} \right] \\ & -\frac{2\pi}{3G_5}\, R^2 \, C_{ABC}\, C^A \, C^B \, e_5^C \,. \end{aligned} \tag{5.3.12}$$

Then, varying the entropy function $\mathcal{E}_5$ with respect to the electric fields e_5^A and setting $\partial_e \mathcal{E}_5 = 0$ yields

$$\frac{\pi}{4G_5} \, \frac{(v_2^3 \, v_3)^{1/2}}{v_1} \, G_{AB}\, e_5^B = -\frac{1}{2}\, \hat{q}_A \,, \tag{5.3.13}$$

while varying with respect to C^A and setting $\partial_C \mathcal{E}_5 = 0$ gives

$$-\frac{\alpha}{2}\, \hat{q}_A + \frac{2\pi R}{G_5}\, C_{ABC}\, C^B \, e_5^C + \frac{4\pi R}{G_5} \, \frac{v_1 (v_2^3 \, v_3)^{1/2}}{v_2^2} \, G_{AB} C^B = 0 \,, \tag{5.3.14}$$

where we introduced

$$\hat{q}_A = q_A\, G_5^{-1/3} - \frac{2\pi R^2}{G_5} C_{ABC}\, C^B\, C^C \;, \tag{5.3.15}$$

for convenience. Combining (5.3.13) and (5.3.14) results in

$$\alpha\, \frac{(v_2^3\, v_3)^{1/2}}{v_1}\, G_{AB}\, e_5^B + 8R\, C_{ABC}\, C^B\, e_5^C + 16R\, \frac{v_1 (v_2^3\, v_3)^{1/2}}{v_2^2}\, G_{AB} C^B = 0 \;, \tag{5.3.16}$$

which is precisely the equation of motion for $A^A_{5\psi}$ evaluated in the black hole background. Observe that when $\alpha\, q_A \neq 0$, then generically also $C^A \neq 0$. We also note that when expressed in terms of four-dimensional quantities, $\hat{q}_A$ equals $\hat{q}_A = G_4^{-1/2}\,(Q_A - \mathrm{Re}\mathcal{N}_{A0}\, P^0)$, where P^0 is given by (5.3.30).

The entropy function (5.3.10) depends on a set of constant parameters, namely e_5^A, X^A, C^A, v_1, v_2, v_3 and α, whose horizon values are determined by extremizing $\mathcal{E}_5$. To this end, we compute the (remaining) extremization equations. Inserting (5.3.13) into (5.3.10) gives

$$\begin{aligned}\mathcal{E}_5 \;=\;& 2\pi\,\alpha\left[-\frac{1}{2}J - \frac{1}{2}R\, C^A\left(q_A\, G_5^{-1/3} - \frac{2\pi R^2}{3G_5}\, C_{ABC}\, C^B\, C^C\right)\right] + G_5\, \frac{v_1}{(v_2^3\, v_3)^{1/2}}\, \hat{q}_A\, G^{AB}\, \hat{q}_B \\ & - \frac{\pi^2}{2G_5} v_1 (v_2^3\, v_3)^{1/2}\left[-\frac{1}{v_1} + \frac{4 - v_3}{v_2} + \frac{v_2\, v_3 \alpha^2}{16 v_1^2} - 8R^2 \frac{G_{AB} C^A C^B}{v_2^2}\right] \;.\end{aligned} \tag{5.3.17}$$

Demanding $\partial_\alpha \mathcal{E}_5 = 0$ results in the expression for the angular momentum,

$$\frac{\pi}{32 G_5}\frac{v_2^{5/2} v_3^{3/2}}{v_1}\,\alpha = -\frac{1}{2}J - \frac{1}{2}R\, C^A\left(q_A\, G_5^{-1/3} - \frac{2\pi R^2}{3G_5}\, C_{ABC}\, C^B\, C^C\right) \;. \tag{5.3.18}$$

Computing $\partial_{v_i} \mathcal{E}_5 = 0$ (with $i = 1, 2, 3$), we obtain

$$\begin{aligned} v_1 &= \frac{v_2}{4} \;, \\ v_2\, v_3\left[2 v_2 + v_2\, v_3 (1 - 2\alpha^2)\right] &= \frac{2 G_5^2}{\pi^2}\, \hat{q}_A\, G^{AB}\, \hat{q}_B \;, \\ 2 v_2 - v_2 v_3 (2 - \alpha^2) &= 8R^2\, G_{AB}\, C^A\, C^B \;. \end{aligned} \tag{5.3.19}$$

Observe that the first of these conditions yields $\tilde{v}_1 = \tilde{v}_2$, as can be seen from (5.2.7). This implies the vanishing of the Ricci scalar for the associated four-dimensional geometry.

Inserting the relations (5.3.18) and (5.3.19) into (5.3.17) results in

$$\mathcal{E}_5 = \frac{\pi^2}{2G_5}\left(v_2^3\, v_3\right)^{1/2} \;, \tag{5.3.20}$$

which exactly equals the macroscopic entropy $\mathcal{S}_{\text{macro}} = A_5/(4G_5)$ of the rotating black hole, where A_5 denotes the horizon area.

Introducing the abbreviations

$$\begin{aligned}\Omega &= \frac{2G_5^2}{\pi^2}\,\frac{1}{\sqrt{v_2\,v_3}}\,\hat{q}_A\,G^{AB}\,\hat{q}_B\;,\\ \Delta &= 8R^2\,\sqrt{v_2\,v_3}\,G_{AB}\,C^A\,C^B\;,\\ \Gamma &= \frac{8G_5}{\pi}\left[-\frac{1}{2}J-\frac{1}{2}R\,C^A\left(q_A\,G_5^{-1/3}-\frac{2\pi R^2}{3G_5}\,C_{ABC}\,C^B\,C^C\right)\right]\;,\end{aligned}\tag{5.3.21}$$

we obtain from (5.3.18) and (5.3.19) the following two equations,

$$\begin{aligned}3(v_2\,v_3)^{3/2}-3\frac{\Gamma^2}{(v_2v_3)^{3/2}} &= \Omega-\Delta\;,\\ \sqrt{v_2\,v_3}\,(6v_2-3v_2v_3) &= \Omega+2\Delta\;.\end{aligned}\tag{5.3.22}$$

Solving the first of these equations yields (with v_2v_3 positive)

$$(v_2\,v_3)^{3/2}=\frac{1}{6}\,(\Omega-\Delta)+\sqrt{\Gamma^2+\frac{1}{36}(\Omega-\Delta)^2}\;.\tag{5.3.23}$$

Inserting this into the second equation of (5.3.22) gives

$$\left(v_2^3\,v_3\right)^{1/2}=\frac{1}{4}\,(\Omega+\Delta)+\frac{1}{2}\sqrt{\Gamma^2+\frac{1}{36}(\Omega-\Delta)^2}\;.\tag{5.3.24}$$

Thus, by taking suitable ratios of (5.3.23) and (5.3.24), we obtain v_2 and v_3 expressed in terms of Ω, Δ and Γ. Now, recalling the definition of $\hat{X}^A$ in (5.2.1) and using (5.2.5), we have $\sqrt{v_2\,v_3}\,G_{AB}=2R\,\hat{G}_{AB}$, where

$$\hat{G}_{AB}=-C_{ABC}\,\hat{X}^C+9\,\frac{\hat{X}_A\,\hat{X}_B}{\hat{\mathcal{V}}}\;,\tag{5.3.25}$$

with $\hat{X}_A$ and $\hat{\mathcal{V}}$ defined in (B.33). Therefore Ω, Δ and Γ, and hence also the horizon area (5.3.24), are entirely determined in terms of the scalar fields $\hat{X}^A$ and C^A and the charges. The horizon values of $\hat{X}^A$ and C^A are in turn determined in terms of the charges by solving the respective extremization equations. The extremization equations for the C^A are given by (5.3.14), while the extremization equations for the $\hat{X}^A$ are obtained by setting $\partial_{\hat{X}^A}\mathcal{E}_5=0$. Rather than computing the horizon values in this way, we will determine them by solving the associated attractor equations in four dimensions. This will be done in the next subsection.

Finally, let us consider static black holes. When the rotation parameter α is set to zero, we have $\Gamma=0$ and (5.3.16) can be abbreviated as $D_{AB}\,C^B=0$. In the following we will assume that D_{AB} is invertible so that $C^A=0$. We then infer from (5.3.19) and (5.3.21) that $v_3=1$, $\Delta=0$ and

$$\Omega=\frac{G_5^{4/3}}{\pi^2 R}\,q_A\,\hat{G}^{AB}\,q_B\;,\tag{5.3.26}$$

which is the black hole potential in five dimensions for static electrically charged black holes [68]. Using (5.3.11), we obtain for (5.3.20),

$$\mathcal{E}_5 = \frac{2\pi}{3} \, Q_A \, \hat{G}^{AB} \, Q_B \; . \tag{5.3.27}$$

From (B.32) and (5.3.25) we infer that $\hat{G}_{AB} = -\mathrm{Im}\mathcal{N}_{AB}$. With the help of (B.32), (B.33) and (5.3.24) we compute

$$\mathrm{Im}\mathcal{N}_{00} = -\hat{\mathcal{V}} = \frac{1}{12\pi} \frac{G_5}{R^3} \, Q_A \, [(\mathrm{Im}\mathcal{N})^{-1}]^{AB} \, Q_B \; , \tag{5.3.28}$$

where we used $\mathrm{Im}\mathcal{N}_{A0} = 0$. It follows that we can rewrite (5.3.27) as

$$\mathcal{E}_5 = -\frac{2\pi}{4} \left[(P^0)^2 \, \mathrm{Im}\mathcal{N}_{00} + Q_A \, [(\mathrm{Im}\mathcal{N})^{-1}]^{AB} \, Q_B \right] \; , \tag{5.3.29}$$

where

$$P^0 = p^0 \, \frac{R}{G_4^{1/2}} \quad , \quad p^0 = 1 \; . \tag{5.3.30}$$

Thus, (5.3.27) precisely equals the four-dimensional black hole potential,

$$\mathcal{E}_4 = -\frac{2\pi}{4} (Q_I - \mathcal{N}_{IK} P^K) \, [(\mathrm{Im}\mathcal{N})^{-1}]^{IJ} \, (Q_J - \bar{\mathcal{N}}_{JL} P^L) \; , \tag{5.3.31}$$

for the case at hand with $C^A = 0$ and non-vanishing charges (P^0, Q_A), as it should. In (5.3.31) (P^I, Q_J) denote the magnetic and electric charges in four dimensions, respectively.

5.3.2 Attractor equations and examples

The four-dimensional entropy function (5.3.31) can be rewritten [29] into (cf. (4.3.9))

$$\mathcal{E}_4 = \pi \left[\Sigma + \left(\tilde{Q}_I - F_{IJ} \tilde{P}^J \right) N^{IK} \left(\tilde{Q}_K - \bar{F}_{KL} \tilde{P}^L \right) \right] \; , \tag{5.3.32}$$

where in the notation of this chapter

$$\Sigma = -\mathrm{i} \left(\bar{Y}^I F_I - Y^I \bar{F}_I \right) - Q_I (Y^I + \bar{Y}^I) + P^I (F_I + \bar{F}_I) \; , \tag{5.3.33}$$

$$N_{IJ} = \mathrm{i} \left(\bar{F}_{IJ} - F_{IJ} \right) \; , \qquad \tilde{Q}_I = Q_I + \mathrm{i}(F_I - \bar{F}_I) \; , \qquad \tilde{P}^I = P^I + \mathrm{i}(Y^I - \bar{Y}^I) \; . \tag{5.3.34}$$

The scalar fields (5.2.2) are again expressed in terms of the Y^I by $z^A = Y^A/Y^0$. The horizon values of the scalar fields $\hat{X}^A$ and C^A can be conveniently determined by solving the attractor equations for the Y^I in four dimensions, which read [30, 29] (cf. (4.3.17))

$$-2(\tilde{Q}_J - F_{JK} \, \tilde{P}^K) + \mathrm{i}(\tilde{Q}_I - \bar{F}_{IM} \, \tilde{P}^M) N^{IR} \, F_{RSJ} \, N^{SK} (\tilde{Q}_K - \bar{F}_{KL} \, \tilde{P}^L) = 0 \; . \tag{5.3.35}$$

Contracting with Y^I results in

$$\mathrm{i} \left(\bar{Y}^I \, F_I - Y^I \, \bar{F}_I \right) = P^I \, F_I - Q_I \, Y^I \; . \tag{5.3.36}$$

Supersymmetric black holes satisfy $\tilde{Q}_I = \tilde{P}^J = 0$.

In the following, we will discuss two classes of four-dimensional non-supersymmetric extremal black holes which are connected to five-dimensional black holes. These have a non-vanishing P^0 given by (5.3.30). The first class consists of black holes with non-vanishing charges (P^0, Q_A) in heterotic-like theories with prepotential $F(Y) = -Y^1 Y^a \eta_{ab} Y^b / Y^0$, where η_{ab} denotes a symmetric matrix with the inverse η^{ab} ($\eta^{ab}\eta_{bc} = \delta^a_c$) and $a, b = 2, \ldots, n$. These black holes are static in five dimensions. Taking $P^0 > 0$ and $Q_1\, Q_a\, \eta^{ab}\, Q_b < 0$, we find that the attractor equations (5.3.35) are solved by

$$\begin{aligned} Y^0 &= -\frac{\mathrm{i}}{4}\, P^0 \,, \\ Y^1 &= \frac{1}{8}\sqrt{-\frac{P^0\, Q_a\, \eta^{ab}\, Q_b}{Q_1}} \,, \\ Y^a &= -\frac{1}{4}\sqrt{-\frac{P^0\, Q_1}{Q_c\, \eta^{cd}\, Q_d}}\; \eta^{ab}\, Q_b \,. \end{aligned} \tag{5.3.37}$$

The z^A read,

$$\begin{aligned} z^1 &= \mathrm{i}\, \hat{X}^1 = \frac{\mathrm{i}}{2}\sqrt{-\frac{Q_a\, \eta^{ab}\, Q_b}{P^0\, Q_1}} \,, \\ z^a &= \mathrm{i}\, \hat{X}^a = -\mathrm{i}\sqrt{-\frac{Q_1}{P^0\, Q_c\, \eta^{cd}\, Q_d}}\; \eta^{ab}\, Q_b \,. \end{aligned} \tag{5.3.38}$$

Requiring $\hat{\mathcal{V}} > 0$ for consistency (see (B.33)) restricts the charges to $Q_a\, \eta^{ab}\, Q_b > 0$ and $Q_1 < 0$. Using (5.3.37), (5.3.32), (5.3.30) and (5.3.11), the entropy is computed to be

$$\mathcal{S}_5 = \pi\sqrt{-P^0\, Q_1\, Q_a\, \eta^{ab}\, Q_b} = \frac{\sqrt{\pi}}{2}\sqrt{-q_1\, q_a\, \eta^{ab}\, q_b} \,. \tag{5.3.39}$$

Upon performing the rescaling $q_A \to (4\pi)^{1/3}\, q_A$, the entropy (5.3.39) attains its standard form. For the case $n = 3$ with non-vanishing $\eta_{23} = \eta_{32} = \frac{1}{2}$, the so-called STU model, the above solution has been given in [107] and found to be stable. Requiring the moduli S, T and U to lie in the Kähler cone imposes the additional restriction $Q_2 < 0$ and $Q_3 < 0$.

The solution (5.3.37) is non-supersymmetric in four dimensions, since $\tilde{Q}_A \neq 0, \tilde{P}^0 \neq 0$. We now check the supersymmetry of the associated five-dimensional solution. An electrically charged supersymmetric solution in five dimensions satisfies the condition $\mathcal{A}_A = 0$ [39, 41, 142, 40], where in our conventions (see appendix B)

$$\mathcal{A}_A = q_A - 2\, \mathrm{e}^{6\phi}\, Z(\hat{X})\, \hat{X}_A \quad , \quad Z(\hat{X}) = q_A\, \hat{X}^A \,, \tag{5.3.40}$$

with $\hat{X}_A$ given in (B.33). Computing $\mathcal{A}_A$ for the solution (5.3.37) using (5.3.11), we find that $\mathcal{A}_A = G_5^{1/3}\, G_4^{-1/2}\, (Q_A + 3P^0\, \hat{X}_A) = 0$. The entropy (5.3.39) takes the supersymmetric form $\mathcal{S}_5 = (2\pi)^{1/2}\, 3^{-3/2}\, |q_A\, X^A|^{3/2}$. Solutions which are supersymmetric from a higher-dimensional point of view, but non-supersymmetric from a lower-dimensional point of view, have been discussed in [131, 63] and occur when dimensionally reducing geometries that are $U(1)$-fibrations, as in our case. In string theory one can generate new solutions from a given one by using duality transformations. Two configurations which are related in this manner must be both supersymmetric or both non-supersymmetric in four dimensions. Hence, the configurations obtained in this way from the (P^0, Q_A) solution (5.3.37) will be non-supersymmetric in four dimensions. Those with a positive P^0 can be lifted to five-dimensional solutions which, depending on the specific duality transformation, may or may not be supersymmetric.

The second class of solutions we consider consists of black holes with non-vanishing charges (P^0, Q_0). They correspond to rotating black holes in five dimensions, of the type discussed in [137, 101, 115, 6, 65, 7, 48], which are not supersymmetric. We use the prepotential $F(Y) = -Y^1\, Y^2\, Y^3 / Y^0$. Taking $P^0\, Q_0 > 0$ and $\mathrm{Re} z^A = 0$, we find that the attractor equations (5.3.35) are solved by

$$\begin{aligned} Y^0 &= -\frac{(1-\mathrm{i})}{8}\, P^0 \,, \\ Y^1\, Y^2\, Y^3 &= \mathrm{i}(1-\mathrm{i})^3\, \frac{(P^0)^2\, Q_0}{512} \,. \end{aligned} \tag{5.3.41}$$

Observe that the attractor equations do not determine the individual values Y^1, Y^2 and Y^3, because the entropy function has two flat directions. The coupling constant $\mathrm{Im}\mathcal{N}_{00}$ and the entropy are, however, determined in terms of the charges. The former takes the value $-\mathrm{Im}\mathcal{N}_{00} = \mathrm{i} z^1\, z^2\, z^3 = Q_0/P^0 > 0$. We also find that $\hat{\mathcal{V}} > 0$, as required by consistency. From (5.3.23) and (5.3.24) we obtain $v_2 = \frac{1}{2}\, |\Gamma|^{2/3}$ and $v_3 = 2$, and from (5.3.18) we have $\alpha = \mathrm{sgn}\, \Gamma$. Using (5.3.41), (5.3.32), (5.3.30) and (5.3.11), the entropy is computed to be

$$\mathcal{S}_5 = \pi\, P^0\, Q_0 = \pi\, J \,. \tag{5.3.42}$$

We close this section by displaying the relation between the five-dimensional quantity $Z(\hat{X})\, \mathrm{e}^{6\phi}$ appearing in (5.3.40) and the four-dimensional Y^0 for the case of static black holes with $C^A = 0$. From (5.3.36) we obtain (with $Q_0 = P^A = 0$, and with P^0 given by (5.3.30))

$$Z(\hat{X})\, \mathrm{e}^{6\phi} = \frac{\mathrm{i}}{2}\, \frac{G_5^{1/3}}{G_4^{1/2}} \left(-8\, Y^0 + \mathrm{i}\frac{R}{\sqrt{G_4}} \right) \,, \tag{5.3.43}$$

where we used $Y^0 = -\bar{Y}^0$, which follows from the reality of (5.3.43). For a supersymmetric solution in four dimensions, $\tilde{P}^0 = 0$ and hence $Y^0 = \mathrm{i}P^0/2$, so that

$$Z(\hat{X})\, \mathrm{e}^{6\phi} = 6\pi\, R^2\, G_5^{-2/3} \,. \tag{5.3.44}$$

5.4 More general black holes

In the above we concentrated our attention on rotating electrically charged black holes with one independent angular momentum parameter, for simplicity. General charged static black holes in four dimensions also carry magnetic charges P^A, and these charges can be easily incorporated into our formulae by adding a term $-P^A\, R\, \cos\theta\, d\varphi$ to both (5.3.3) and (5.3.4). Their entropy function is given by (5.3.32), and the entropy function of the associated five-dimensional rotating black holes is then defined by (5.3.2).

In five dimensions, rotating extremal black holes may carry two independent angular momentum parameters [127]. These black holes will be connected to rotating extremal black holes in four dimensions. The entropy function of these five-dimensional black holes can then again be defined in terms of the entropy function of the associated rotating four-dimensional black holes. The entropy function for rotating attractors in four dimensions has recently been discussed in [6].

Indeed, shortly after [32] a generalization to arbitrary extremal rotating black holes as well as black rings appeared in [86].

Chapter 6

Flow equations in very special geometry

6.1 First-order equations for interpolating solutions

Interpolating solutions describing single-center static supersymmetric black holes [71, 150, 69, 70] in four-dimensional $N = 2$ supergravity theories at the two-derivative level can be obtained by solving a set of first-order differential (flow) equations [66, 123, 60]. Such solutions are given in terms of harmonic functions [71, 69, 11, 140, 141, 15].

First-order flow equations exist when the effective black hole potential can be expressed in terms of a "superpotential" W. Then, the effective two-derivative Lagrangian can be written as a sum of squares of first-order flow equations involving W. The rewriting of the black hole potential in terms of W is, however, not unique [37]. A given black hole potential may thus give rise to different first-order flow equations, and the resulting black hole solutions may or may not be supersymmetric.

In this chapter, we use very special geometry to construct first-order flow equations for five-dimensional rotating electrically charged extremal black holes in a Taub-NUT geometry, generalizing the results obtained in [116] for static extremal black holes in asymptotically flat spacetime in five dimensions. The "superpotentials" W_5 we employ are constructed out of the five-dimensional central charge by rotating the electric charges with non-trivial elements belonging to the invariance group of the inverse matrix G^{AB} associated with the kinetic terms for the Maxwell fields. We solve the flow equations and obtain interpolating solutions describing extremal black holes in a Taub-NUT geometry in five dimensions. Then we use the 5d/4d-connection presented in the previous chapter to obtain four-dimensional first-order flow equations, based on four-dimensional "superpotentials" W_4, from the five-dimensional flow equations. In this way, we give a new interpretation of the results of [37]. The solutions to the four-dimensional flow equations that we present describe single-center static dyonic extremal black holes in four dimensions, whose magnetic charge is the NUT charge. Some of the non-supersymmetric solutions in four dimensions we find are connected to supersymmetric solutions in five dimensions, as already observed in [32] (see the previous chapter). This feature is related to the $U(1)$-fibration of the Taub-NUT geometry [131, 63]. For this set of solutions, the associated first-order flow equations in four dimensions may hence be explained in terms of hidden supersymmetry.

6.2 Extremal black holes in five and four dimensions

As before, we concentrate on single-center rotating extremal black holes in five dimensions which are connected to single-center static extremal black holes in four dimensions. We do this in the context of $N = 2$ supergravity theories with cubic prepotentials (cf. appendix B for the conventions used). In five dimensions, the black holes we consider are again electrically charged and may carry one independent angular momentum parameter. The NUT charge is denoted by p^0 and is positive. In four dimensions, the associated static black holes have charges (p^0, q_0, q_A), where q_0 is related to the rotation in five dimensions [77, 76].

The dimensional reduction procedure in equations (5.2.1) and (5.2.2) remains unchanged. The five-dimensional scalar fields X^A satisfy the constraint (B.1). The quantity $\mathrm{e}^{-6\phi}$ is related to the four-dimensional Kähler potential $K(z, \bar{z})$ given in (B.18) by

$$\mathrm{e}^{-6\phi} = \frac{1}{6\mathcal{V}}\, C_{ABC}\, \hat{X}^A \hat{X}^B \hat{X}^C = \frac{1}{8\mathcal{V}}\, \mathrm{e}^{-K} \; . \tag{6.2.1}$$

We take the five-dimensional line element and the five-dimensional gauge fields A_5^A to be given by

$$\begin{aligned} ds_5^2 &= G_{MN}\, dx^M\, dx^N = -f^2(r)\,(dt + w)^2 + f^{-1}(r)\, ds^2_{\mathrm{HK}} \; , && (6.2.2)\\ A_5^A &= \chi^A(r)\,(dt + w) \; , && (6.2.3) \end{aligned}$$

where ds^2_{HK} describes the line element of a four-dimensional hyper-Kähler manifold. We set

$$\begin{aligned} ds^2_{\mathrm{HK}} &= N\, \left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right) + R^2 N^{-1} (d\psi + p^0\, \cos\theta\, d\varphi)^2 \; , \\ w &= w_5(r)\,(d\psi + p^0\, \cos\theta\, d\varphi) + w_4(r)\, \cos\theta\, d\varphi \; . \end{aligned} \tag{6.2.4}$$

Here $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$ and N denotes a harmonic function in three spatial dimensions,

$$N = h^0 + \frac{p^0\, R}{r} \quad , \quad p^0 > 0 \; . \tag{6.2.5}$$

When $h^0 = 0$ and $p^0 = 1$, the line element ds^2_{HK} describes a four-dimensional flat space, whereas, when $h^0 > 0$, it describes a Taub-NUT space. In the following, we will take $h^0 > 0\, ,\, p^0 > 0$.

Using (5.2.1) and reducing over ψ results in

$$\mathrm{e}^{-4\phi} = \frac{1}{fN} - \left(\frac{f w_5}{R}\right)^2 \tag{6.2.6}$$

as well as

$$A_4^0 = \mathrm{e}^{4\phi} \frac{f^2 w_5}{R} (dt + w_4 \cos\theta\, d\varphi) - R p^0 \cos\theta\, d\varphi \, . \tag{6.2.7}$$

The associated four-dimensional line element and the four-dimensional gauge fields A_4^A are

$$
\begin{aligned}
ds_4^2 &= -\mathrm{e}^{2U}(dt + w_4 \cos\theta \, d\varphi)^2 + \mathrm{e}^{-2U}\left(dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)\right) , && (6.2.8) \\
A_4^A &= \frac{\mathrm{e}^{4\phi}}{fN} \, \chi^A \, (dt + w_4 \cos\theta \, d\varphi) , && (6.2.9)
\end{aligned}
$$

with

$$\mathrm{e}^{2U} = \mathrm{e}^{2\phi} \frac{f}{N} \, . \tag{6.2.10}$$

The Kaluza-Klein scalars C^A are given by

$$C^A = \frac{w_5}{R} \, \chi^A \, . \tag{6.2.11}$$

For a static black hole in four dimensions $w_4 = 0$. The resulting black hole carries charges (p^0, q_0, q_A), but no magnetic charges p^A.

In the following, we will set the value of the Taub-NUT modulus to $R = 1$, for convenience.

6.3 Flow equations in five dimensions

To derive first-order flow equations for five-dimensional rotating electrically charged black holes in the geometry (6.2.4), we start by recalling the bosonic part of the five-dimensional $N = 2$ supergravity action S

$$
\begin{aligned}
\frac{8\pi G_5}{\mathcal{V}} \, S &= \int d^5x \sqrt{-G} \left(\mathcal{R} - g_{ij} \, \partial_M \phi^i \, \partial^M \phi^j - \frac{1}{2} \, G_{AB} F^A_{MN} \, F^{BMN} \right) \\
&\quad - \frac{1}{6\mathcal{V}} \int C_{ABC} \, F^A \wedge F^B \wedge A^C \, . && (6.3.1)
\end{aligned}
$$

We use the conventions of [32], which we summarize in appendix B. In particular, we set $2\mathcal{V} = 1$.

We take $\phi^i = \phi^i(r)$ and insert the ansatz (6.2.2) and (6.2.3) into the action (6.3.1). We write the result as

$$\frac{8\pi G_5}{\mathcal{V}} \, S = S_1 + S_2 \, , \tag{6.3.2}$$

where S_2 contains only terms that are proportional to w and derivatives thereof. We obtain

(we refer to appendix C for some of the details)

$$
\begin{aligned}
S_1 &= \frac{1}{2}\int dt\, dr\, d\theta\, d\varphi\, d\psi\ \sin\theta \\
&\qquad\qquad \left[-3r^2 f^{-2}(f')^2 - 2r^2 g_{ij}\phi'^i\phi'^j + 2r^2 f^{-2} G_{AB}\chi'^A\,\chi'^B + 2\partial_r\left(r^2 f^{-1} f'\right)\right]\ , \\
S_2 &= \frac{1}{2}\int dt\, dr\, d\theta\, d\varphi\, d\psi\ \sin\theta \\
&\quad \left[\frac{f}{r^2 N}\left[(p^0 w_5 + w_4)^2 + r^4 N^2 w_5'^2 + r^2\cot^2\theta\, w_4'^2\right]\left(f^2 - 2G_{AB}\chi^A\chi^B\right)\right. \\
&\quad \left. +\frac{2}{3\mathcal{V}} C_{ABC}\left(p^0 w_5 + w_4\right)\chi^A\chi^B\chi^C\, w_5'\right]\ ,
\end{aligned}
\tag{6.3.3}
$$

where $' = \partial_r$.

The terms in S_1 can be written as

$$
\begin{aligned}
S_1 &= \frac{1}{2}\int dt\, dr\, d\theta\, d\varphi\, d\psi\ \sin\theta\left[-3r^2 f^{-2}(f')^2 - 2r^2 g_{ij}\phi'^i\phi'^j\right. \\
&\qquad +2r^{-2} f^{-2} G_{AB}\left(r^2\chi'^A + f^2 G^{AC} q_C\right)\left(r^2\chi'^B + f^2 G^{BD} q_D\right) - 2r^{-2} f^2\, q_A G^{AB} q_B \\
&\qquad \left. +2\partial_r\left(r^2 f^{-1} f' - 2q_A\,\chi^A\right)\right]\ .
\end{aligned}
\tag{6.3.4}
$$

The term proportional to $q_A G^{AB} q_B$ is the black hole potential which, with the help of (B.13), can be written as

$$
q_A G^{AB} q_B = \frac{2}{3}|Z_5|^2 + g^{ij}\,\partial_i|Z_5|\,\partial_j|Z_5|\ , \tag{6.3.5}
$$

where

$$
Z_5 = q_A\, X^A \tag{6.3.6}
$$

denotes the (real) central charge in five dimensions. The rewriting (6.3.5) is, however, not unique, as discussed in [37] in the four-dimensional context. Whenever the inverse vector fields kinetic matrix G^{AB} possesses an invariance group with elements $R^A{}_B$, i.e.

$$
R^A{}_C\, G^{CD}\, R^B{}_D = G^{AB}, \tag{6.3.7}
$$

and if $R^A{}_B$ is a constant real matrix, then $q_A G^{AB} q_B$ can more generally be written as[1]

$$
q_A G^{AB} q_B = \frac{2}{3}|W_5|^2 + g^{ij}\,\partial_i|W_5|\,\partial_j|W_5|\ , \tag{6.3.8}
$$

where

$$
W_5 = Q_A\, X^A \quad , \quad Q_A = q_B\, R^B{}_A\ . \tag{6.3.9}
$$

The case (6.3.5) is contained in (6.3.8) with $R^A{}_B = \delta^A{}_B$. Specific examples based on (6.3.9) will be discussed in section 6.5.

[1] There may exist other rewritings of the black hole potential which are not captured by (6.3.9).

The rewriting of the black hole potential in terms of (6.3.9) results in a rewriting of S_1, as follows. First, we observe that S_1 can be cast in the form (6.3.4), with q_A replaced by Q_A everywhere. Then, using (6.3.8) we obtain

$$\begin{aligned} S_1 \;=\;& \frac{1}{2}\int dt\, dr\, d\theta\, d\varphi\, d\psi\, \sin\theta \Big[-3\tau^2 f^2 \left(\partial_\tau f^{-1} - \frac{2}{3}|W_5|\right)^2 \\ & -2\tau^2 g_{ij}\left(\partial_\tau \phi^i + f g^{il}\partial_l |W_5|\right)\left(\partial_\tau \phi^j + f g^{jk}\partial_k |W_5|\right) \\ & +2\tau^2 f^{-2} G_{AB}\left(\partial_\tau \chi^A - f^2 G^{AC} Q_C\right)\left(\partial_\tau \chi^B - f^2 G^{BD} Q_D\right) \\ & +2\partial_r\left(r^2 f^{-1} f' - 2Q_A\,\chi^A - 2f\,|W_5|\right)\Big] \,, \end{aligned} \tag{6.3.10}$$

where

$$\tau = \frac{1}{r}\,. \tag{6.3.11}$$

The last term in (6.3.10) denotes a total derivative, and we will comment on its interpretation below. Thus, up to a total derivative term, S_1 is expressed in terms of squares of first-order flow equations which, when requiring stationarity of S_1 with respect to variations of the fields, result in

$$\partial_\tau f^{-1} = \frac{2}{3}|W_5|\,, \tag{6.3.12a}$$

$$\partial_\tau \chi^A = f^2 G^{AB} Q_B\,, \tag{6.3.12b}$$

$$\partial_\tau \phi^i = -f g^{ij}\partial_j |W_5|\,. \tag{6.3.12c}$$

They are analogous to the flow equations for static supersymmetric black holes in asymptotically flat spacetime in five dimensions derived in [116].

Next, we rewrite S_2 as a sum of squares, as follows. Defining

$$\tilde{\chi}^A = \chi^A + s f X^A\,, \tag{6.3.13}$$

with a certain proportionality constant s, to be determined in (6.3.21), we rewrite

$$G_{AB}\chi^A\chi^B = G_{AB}\tilde{\chi}^A\tilde{\chi}^B + \frac{3}{2}s^2 f^2 - \frac{3sf}{\mathcal{V}}\tilde{\chi}^A X_A \tag{6.3.14}$$

and

$$C_{ABC}\chi^A\chi^B\chi^C = C_{ABC}\tilde{\chi}^A\tilde{\chi}^B(\tilde{\chi}^C - 3sfX^C) + 6s^2 f^2(3\tilde{\chi}^A X_A - s\mathcal{V}f)\,. \tag{6.3.15}$$

Then, we obtain for S_2,

$$
\begin{aligned}
S_2 \;=\; & \frac{1}{2}\int dt\, dr\, d\theta\, d\varphi\, d\psi\, \sin\theta \\
& \Big\{ -\frac{2f}{r^2 N}\left[(p^0 w_5 + w_4)^2 + r^4 N^2 w_5'^2 + r^2 \cot^2\theta\, w_4'^2\right] G_{AB}\tilde{\chi}^A\tilde{\chi}^B \\
& +\frac{2}{3\mathcal{V}}(p^0 w_5 + w_4) w_5'\, C_{ABC}\,\tilde{\chi}^A\tilde{\chi}^B(\tilde{\chi}^C - 3sfX^C) \\
& +\frac{2sf^2}{\mathcal{V}r^2 N}\left[(p^0 w_5 + w_4)^2 + r^4 N^2 w_5'^2 + r^2\cot^2\theta\, w_4'^2\right]\left[3\tilde{\chi}^A X_A + \frac{(1-3s^2)\mathcal{V}}{2s} f\right] \\
& +\frac{4s^2 f^2}{\mathcal{V}}(p^0 w_5 + w_4)\, w_5'\,(3\tilde{\chi}^A X_A - s\mathcal{V}f)\Big\}\;.
\end{aligned}
\tag{6.3.16}
$$

The first two lines already form a sum of squares (of $\tilde{\chi}^A$). The last two can also be combined into perfect squares provided that $s = \pm 1$, yielding

$$
\frac{2sf^2}{\mathcal{V}r^2 N}\left[\left(p^0 w_5 + w_4 + sr^2 N w_5'\right)^2 + r^2\cot^2\theta\, w_4'^2\right](3\tilde{\chi}^A X_A - s\mathcal{V}f)\,. \tag{6.3.17}
$$

Thus, the additional first-order flow equations following from the stationarity of S_2 are

$$
\tilde{\chi}^A = 0\;, \tag{6.3.18a}
$$

$$
\partial_\tau w_4 = 0\;, \tag{6.3.18b}
$$

$$
sN\partial_\tau w_5 = p^0 w_5 + w_4\;. \tag{6.3.18c}
$$

The coefficient s is related to the sign of W_5, as follows. On the solution, we have from (6.3.13) and (6.3.18a) that $\chi^A = -sfX^A$, and both sides of this equation are a function of τ. Differentiating χ^A with respect to τ and using the chain rule as well as (6.3.12c), we obtain

$$
\begin{aligned}
\partial_\tau\chi^A &= -s\left(-f^2\partial_\tau f^{-1}X^A + f\partial_\tau\phi^i\partial_i X^A\right) \\
&= s\,\mathrm{sgn}(W_5) f^2\left(\frac{2}{3}X^A X^B + g^{ij}\partial_i X^A\partial_j X^B\right) Q_B\,.
\end{aligned}
\tag{6.3.19}
$$

Invoking the identity (B.13) we have

$$
\partial_\tau\chi^A = s\,\mathrm{sgn}(W_5) f^2 G^{AB} Q_B\,, \tag{6.3.20}
$$

and comparison with (6.3.12b) gives

$$
s = \mathrm{sgn}(W_5)\,. \tag{6.3.21}
$$

Summarizing, we find that there are two classes of first-order flow equations specified by the sign of W_5. They are given by (6.3.12) and (6.3.18). Observe that (6.3.18a) is the solution to (6.3.12b), and that on a solution to (6.3.12), the black hole potential (6.3.8) can also be written as

$$2f^2\left(\frac{2}{3}|W_5|^2 + g^{ij}\,\partial_i|W_5|\,\partial_j|W_5|\right) = 3f^2(\partial_\tau f^{-1})^2 + 2g_{ij}\partial_\tau\phi^i\partial_\tau\phi^j\ . \tag{6.3.22}$$

It can be checked that the five-dimensional Einstein-, Maxwell- and scalar field equations of motion derived from (6.3.1) are satisfied for these two classes of flow equations.

The flow equations (6.3.12) are solved by [142, 40, 79] (recall that $2\mathcal{V} = 1$)

$$f^{-1} = \frac{2}{3}\,H_A\,X^A\ , \tag{6.3.23a}$$

$$\chi^A = -sfX^A\ , \tag{6.3.23b}$$

$$f^{-1}\,X_A = \frac{1}{3}\,H_A\ , \tag{6.3.23c}$$

where H_A denotes a harmonic function in three space dimensions,

$$H_A = h_A + |Q_A|\tau \quad , \quad h_A > 0\ . \tag{6.3.24}$$

The H_A are taken to be positive since this, together with the requirement for the X^A to lie inside the Kähler cone (i.e. $X^A > 0$), ensures that $f^{-1} > 0$ along the flow.

The remaining flow equations of (6.3.18) are solved by

$$w_5 = H_0 = h_0 + q_0\,\tau\,, \qquad w_4 = h^0 q_0 - h_0 p^0 \tag{6.3.25}$$

for $s = 1$, and by

$$w_5 = c\,N^{-1} - \frac{w_4}{p^0}\,, \qquad c = \text{const}\,, \qquad w_4 = \text{const} \tag{6.3.26}$$

for $s = -1$. The $s = 1$ solution is standard [79], and it describes a rotating supersymmetric solution in five dimensions provided that $Q_A = q_A$ (i.e. $R^A{}_B = \delta^A{}_B$). The $s = -1$ solution, on the other hand, is non-standard. This solution is non-supersymmetric, since one of the conditions for supersymmetry derived in [79], namely that the self-dual part of dw vanishes, is violated.

In the absence of rotation (i.e. $w = 0$), the five-dimensional solutions (6.3.23) are supersymmetric provided that $Q_A = q_A$.

Finally, let us comment on the boundary term in (6.3.10) which, on the solution (6.3.23), equals $\frac{2}{3}f|W_5|$ evaluated at spatial infinity. This value is independent of both p^0 and the five-dimensional rotation parameter w_5. Therefore, it does not equal the ADM mass of the associated four-dimensional black hole which, in general, depends on both p^0 and w_5 (cf. (6.4.18)). The ADM mass of the $s = 1$ solution (6.3.25) is lower than the one of the $s = -1$ solution (6.3.26).

6.4 Flow equations in four dimensions

In the following, we relate the five-dimensional flow equations (6.3.12) and (6.3.18) to four-dimensional ones by using the dictionary given in section 6.2. We set $w_4 = 0$ in order to obtain static solutions in four dimensions. The four-dimensional flow equations then take the form

$$\begin{aligned} \partial_\tau U &= \mathrm{e}^U W_4 \,, \\ \partial_\tau z^A &= 2\mathrm{e}^U g^{A\bar{B}} \partial_{\bar{B}} W_4 \,, \end{aligned} \tag{6.4.1}$$

with a suitably identified W_4 as in [37, 4]. Observe that for a supersymmetric flow in four dimensions [66, 123, 60],

$$-W_4 = |Z_4| \,, \tag{6.4.2}$$

with Z_4 given in (B.21).

6.4.1 Black holes with $w_5 = 0$

In the absence of rotation in five dimensions we infer from (6.2.6) and (6.2.10) that the four-dimensional quantity e^U is expressed as

$$\mathrm{e}^{-4U} = \mathrm{e}^{4\phi} f^{-4} = N f^{-3} \tag{6.4.3}$$

in terms of five-dimensional quantities. Differentiating with respect to $\tau = 1/r$ and using (6.2.5) gives

$$\partial_\tau \mathrm{e}^{-U} = \frac{1}{4}\left(\mathrm{e}^{3U} f^{-3} p^0 + 3\,\mathrm{e}^{\phi}\,\partial_\tau f^{-1}\right) \,. \tag{6.4.4}$$

Using (6.2.1) we obtain

$$\partial_\tau \mathrm{e}^{-U} = \frac{1}{8}\left(\mathrm{e}^{-K/2} p^0 + 12\,\mathrm{e}^{K/2}\,\mathrm{e}^{-2\phi}\,\partial_\tau f^{-1}\right) \,. \tag{6.4.5}$$

Inserting the five-dimensional flow equation for f^{-1} given in (6.3.12a) into the above expression yields

$$\partial_\tau U = \mathrm{e}^U W_4 \,, \tag{6.4.6}$$

with

$$W_4 = -\frac{1}{8}\,\mathrm{e}^{K/2}\,\left(\mathrm{e}^{-K} p^0 + 4\,\left|Q_A(z^A - \bar{z}^A)\right|\right) \,. \tag{6.4.7}$$

Here we used that $C^A = 0$ according to (6.2.11), so that $z^A = \mathrm{i}\hat{X}^A$.

Similarly, it can be checked that the flow equation for the scalar fields z^A is given by (6.4.1) with W_4 given by (6.4.7). This is done in appendix D.

Setting $T^A = -\mathrm{i}z^A$ and taking the real part of T^A to lie inside the Kähler cone, i.e. $T^A + \bar{T}^A > 0$, yields (with $p^0 > 0$)

$$-W_4 = \frac{1}{8}\,\mathrm{e}^{K/2}\,\left(\mathrm{e}^{-K} p^0 + 4\,\left|Q_A\,(T^A + \bar{T}^A)\right|\right) > 0 \,, \tag{6.4.8}$$

which is non-vanishing along the flow. This we now compare with the absolute value of the central charge Z_4 given in (B.23),

$$|Z_4| = \frac{1}{8}\, \mathrm{e}^{K/2}\, \left|\mathrm{e}^{-K}\, p^0 + 4\, q_A\, (T^A + \bar{T}^A)\right| \,. \tag{6.4.9}$$

Both expressions only agree provided that $Q_A(T^A + \bar{T}^A) > 0$ and $Q_A = q_A$, in which case the flow is supersymmetric in four dimensions, since it is derived from $|Z_4|$. Otherwise the flow is non-supersymmetric. First-order flow equations based on (6.4.8) were first obtained in [37] in the context of the STU-model (which corresponds to $C_{123} = 1$) by using a different approach.

The solution to the first-order flow equations (6.4.1) based on (6.4.8) reads

$$\begin{aligned} \mathrm{e}^{-2U} &= \frac{2}{3} N^{1/2}\, H_A\, f^{-1/2}\, X^A \,, \\ z^A &= \mathrm{i}\hat{X}^A = \mathrm{i}N^{-1/2}\, f^{-1/2}\, X^A \,, \end{aligned} \tag{6.4.10}$$

where $f^{-1/2}\, X^A$ is the solution to (6.3.23c), and where we used (6.4.3). This solution was first derived in [107] for the STU-model by solving the four-dimensional equations of motion. The horizon is at $\tau = \infty$, and the scalar fields z^A are attracted to constant values there. Computing e^{-2U} at the horizon yields the entropy of the extremal black hole

$$\mathcal{S} = \pi\, |W_4|^2_{\mathrm{hor}} \tag{6.4.11}$$

in accordance with [37, 4]. An equivalent expression for the entropy can be found in [144, 29].

We may apply duality transformations to obtain first-order flow equations for extremal non-supersymmetric solutions in four dimensions carrying other types of charges. This was also discussed in [107]. We consider the prepotential (we refer to appendix B and to [11] for some of the conventions used)

$$F(Y) = -\frac{1}{6}\, \frac{C_{ABC}\, Y^A Y^B Y^C}{Y^0} \,, \tag{6.4.12}$$

and apply, for instance, the non-perturbative duality transformation $(Y^I, F_I) = (-\tilde{F}_I, \tilde{Y}^I)$. This gives rise to extremal black hole solutions of the type recently discussed in [152, 108, 129], as follows. The $z^A = Y^A/Y^0$ can then be expressed as $z^A = \tilde{F}_A/\tilde{F}_0$ and the charges (p^0, Q_A) become equal to $(-\tilde{q}_0, \tilde{P}^A)$ (and similarly for the respective harmonic functions). Observe that since $Q_A = q_B\, R^B{}_A$, the charge $\tilde{P}^A$ does not equal $\tilde{p}^A = q_A$ in general. Since the combination $\mathrm{i}\left(\bar{Y}^I\, F_I - Y^I\, \bar{F}_I\right)$ is invariant under symplectic transformations, and because it is equal to $|Y^0|^2\, \mathrm{e}^{-K}$, it follows that

$$|\tilde{Y}^0|^2\, \mathrm{e}^{-\tilde{K}} = |Y^0|^2\, \mathrm{e}^{-K} \,. \tag{6.4.13}$$

Hence we can write

$$\frac{\mathrm{e}^{\tilde{K}/2}}{|\tilde{Y}^0|} = \frac{\mathrm{e}^{K/2}}{|Y^0|} = \frac{\mathrm{e}^{K/2}}{|\tilde{F}_0|} \,. \tag{6.4.14}$$

Also, using $z^A = \mathrm{i}\hat{X}^A$ as well as (B.15) and (B.18), we have $|F_0| = \frac{1}{8}|Y^0|\,\mathrm{e}^{-K}$. Then, we find that (6.4.8) can be expressed in terms of the transformed quantities as

$$\begin{aligned} -W_4 &= \frac{1}{8}\frac{\mathrm{e}^{K/2}}{|\tilde{F}_0|}\left(-\mathrm{e}^{-K}\,|\tilde{F}_0|\,\tilde{q}_0 + 8\,|\tilde{P}^A\,\tilde{F}_A|\right) \\ &= \mathrm{e}^{\tilde{K}/2}\left(-\tilde{q}_0 + \left|\tilde{P}^A\,\frac{\tilde{F}_A}{\tilde{Y}^0}\right|\right) . \end{aligned} \tag{6.4.15}$$

Observe that $z^A = -\bar{z}^A$ implies that also $\tilde{z}^A = -\bar{\tilde{z}}^A$. Thus, extremal four-dimensional black hole solutions with charges $(\tilde{q}_0, \tilde{P}^A)$ and with scalar fields satisfying $\tilde{z}^A = -\bar{\tilde{z}}^A$ can be obtained by solving the first-order flow equations (6.4.1) based on (6.4.15). Observe that $-\tilde{q}_0 = p^0 > 0$ and $\tilde{F}_A/\tilde{Y}^0 = -Y^A/F_0 > 0$. Hence, only when $\tilde{P}^A = \tilde{p}^A$ and $\tilde{P}^A\tilde{F}_A/\tilde{Y}^0 > 0$ is $-W_4 = |Z_4|$, as can be seen from (B.21), and the resulting flow is supersymmetric. Otherwise, the flow is non-supersymmetric.

6.4.2 Black holes with $w_5 \neq 0$

Now we derive the four-dimensional flow equations associated with rotating black holes with $w_5 \neq 0$ in five dimensions. From (6.2.6) and (6.2.10) we obtain

$$\mathrm{e}^{2U} = \mathrm{e}^{2\phi}\, f\, N^{-1} = \mathrm{e}^{-2\phi}\, f^2\, \Delta , \tag{6.4.16}$$

where

$$\Delta = 1 + \mathrm{e}^{4\phi}\,(f\,w_5)^2 . \tag{6.4.17}$$

Using (6.4.16) we have

$$\mathrm{e}^{-4U} = N\,f^{-3} - (N\,w_5)^2 \tag{6.4.18}$$

as well as

$$\begin{aligned} \mathrm{e}^{3U}\,f^{-3} &= \mathrm{e}^{-3\phi}\,\Delta^{3/2} , \\ \mathrm{e}^{3U}\,N\,f^{-2} &= \mathrm{e}^{\phi}\,\Delta^{1/2} . \end{aligned} \tag{6.4.19}$$

Combining (6.2.11) with (6.3.23b) results in

$$C^A = -s\,w_5\,f\,\mathrm{e}^{2\phi}\,\hat{X}^A , \tag{6.4.20}$$

and hence

$$z^A = \alpha\,\hat{X}^A \quad , \quad \alpha = -s\,w_5\,f\,\mathrm{e}^{2\phi} + \mathrm{i} . \tag{6.4.21}$$

Comparing (6.4.17) with (6.4.21) we obtain

$$\Delta = |\alpha|^2 \quad , \quad 4 - 3\Delta = 1 - 3\,(\mathrm{Re}\alpha)^2 . \tag{6.4.22}$$

We begin by considering the $s = 1$ solution given by (6.3.25), with $w_4 = 0$, which implies

$$w_5 = H_0 = q_0 N/p^0 . \tag{6.4.23}$$

From (6.4.19) and (6.4.17) we have

$$e^{3U}\,N\,w_5^2 = e^{-3\phi}\,\Delta^{1/2}\,(\Delta-1)\ . \tag{6.4.24}$$

Using (6.4.18), (6.4.23) and the flow equation (6.3.12a), we obtain

$$\partial_\tau\, e^{-U} = \frac{1}{4}e^{3U}\left(p^0 f^{-3} + 2N f^{-2} W_5 - 4p^0\,N\,H_0^2\right)\ . \tag{6.4.25}$$

Inserting (6.4.19) and (6.4.24) into (6.4.25) results in

$$\partial_\tau\, e^{-U} = \frac{1}{4}\,e^{3\phi}\,\Delta^{1/2}\left(p^0\,e^{-6\phi}\,(4-3\Delta) + 2Q_A\,\hat{X}^A\right)\ . \tag{6.4.26}$$

The right hand side of (6.4.26) is identified with the "superpotential"

$$-W_4 = \frac{1}{4}\,e^{3\phi}\,\Delta^{1/2}\left[p^0\,e^{-6\phi}\,\left(1-3\,(\mathrm{Re}\alpha)^2\right) + 2Q_A\,\hat{X}^A\right]\ . \tag{6.4.27}$$

This we compare with Z_4 given in (B.23), which for the case at hand reads

$$Z_4 = \frac{1}{2}\,e^{K/2}\,\alpha\,\left[-p^0\,e^{-6\phi}\,\left(1-3\,(\mathrm{Re}\alpha)^2\right) - 2q_A\,\hat{X}^A\right]\ . \tag{6.4.28}$$

Using (6.2.1) we find that (6.4.2) holds when $Q_A = q_A$. Then, the $s=1$ flow is a supersymmetric flow in four dimensions. It is non-supersymmetric in all other cases.

Likewise, it can be checked that the flow equation for the scalar fields z^A is given by (6.4.1) with W_4 of (6.4.27).

The solution to the first-order flow equation (6.4.1) based on (6.4.27) reads

$$\begin{aligned} e^{-4U} &= \frac{4}{9}N\,\left(H_A\,f^{-1/2}\,X^A\right)^2 - (N\,H_0)^2\ , \\ z^A &= \alpha\,\hat{X}^A = \frac{3}{2}\left(\frac{-N\,H_0 + \mathrm{i}e^{-2U}}{N\,H_B\,f^{-1/2}\,X^B}\right)\,f^{-1/2}\,X^A\ , \end{aligned} \tag{6.4.29}$$

where $f^{-1/2}\,X^A$ is the solution to (6.3.23c). The entropy is again given by $\mathcal{S} = \pi\,|W_4|^2_{\mathrm{hor}}$.

Next, we consider the $s=-1$ solution given in (6.3.26), with $w_4 = 0$. From (6.4.18) we obtain

$$e^{-4U} = N\,f^{-3} - c^2\ , \tag{6.4.30}$$

which we demand to be positive at spatial infinity (i.e. at $\tau = 0$) to ensure that e^{-4U} remains non-vanishing along the flow. Using the flow equation (6.3.12a) as well as (6.4.19) we get

$$\begin{aligned} \partial_\tau\, e^{-U} &= \frac{1}{4}e^{3U}\left(p^0 f^{-3} - 2N f^{-2}\,W_5\right) \\ &= \frac{1}{4}e^{3\phi}\,\Delta^{1/2}\left(e^{-6\phi}\,\Delta\,p^0 - 2Q_A\,\hat{X}^A\right)\ . \end{aligned} \tag{6.4.31}$$

With the help of (6.4.21) and (6.4.22) this can be rewritten as

$$\partial_\tau \, \mathrm{e}^{-U} = -W_4 \tag{6.4.32}$$

with

$$-\,W_4 = \mathrm{e}^{K/2} \left| \frac{p^0}{6} \, C_{ABC} z^A \, z^B \, \bar{z}^C - Q_A \, z^A \right| \, . \tag{6.4.33}$$

Similarly, one can verify that the flow equation for the scalar fields z^A is given by (6.4.1) with W_4 as in (6.4.33). This is done in appendix D. Thus, we find that the $s = -1$ solution is described by a first-order flow equation based on (6.4.33). Inspection of (6.4.31) shows that W_4 is non-vanishing along the flow. An example of a flow of this type was constructed recently in [37], where also the stability of the solution is discussed.

The solution to the first-order flow equation (6.4.1) based on (6.4.33) reads

$$\begin{aligned} \mathrm{e}^{-4U} &= \frac{4}{9} N \, \left(H_A \, f^{-1/2} \, X^A\right)^2 - c^2 \, , \\ z^A &= \alpha \, \hat{X}^A = \frac{3}{2} \left(\frac{c + \mathrm{i}\mathrm{e}^{-2U}}{N \, H_B \, f^{-1/2} \, X^B} \right) f^{-1/2} \, X^A \, , \end{aligned} \tag{6.4.34}$$

where $f^{-1/2} \, X^A$ is the solution to (6.3.23c). When setting $c = 0$, the solution (6.4.34) reduces to the static case (6.4.10). The solution (6.4.34) is such that the axions C^A vanish at the horizon (i.e. at $\tau = \infty$), so that $z^A_{\text{hor}} = \mathrm{i}\hat{X}^A_{\text{hor}}$, and the entropy is given by (6.4.11). Away from the horizon the C^A are non-vanishing and therefore the axions are subject to a non-trivial flow.

Finally, we may apply the non-perturbative duality transformation discussed below (6.4.11) to (6.4.33). Using (B.15) we have $F_0 = \frac{1}{8} \, \alpha^3 \, Y^0 \, \mathrm{e}^{-K}$, and hence

$$-\,W_4 = \mathrm{e}^{\tilde{K}/2} \left| \frac{\bar{\alpha}}{\alpha} \, \tilde{q}_0 - \tilde{P}^A \, \frac{\tilde{F}_A}{\tilde{Y}^0} \right| \, . \tag{6.4.35}$$

The phase $\bar{\alpha}/\alpha$ can be expressed in terms of the transformed quantities as follows. For the non-perturbative duality transformation under consideration, $\tilde{F}(\tilde{Y}) = F(Y)$ (see for instance [51]). Since $(\bar{\alpha}/\alpha)^3 = \bar{F}(\bar{Y})/F(Y)$, it follows

$$\left(\frac{\bar{\alpha}}{\alpha}\right)^3 = \frac{\bar{\tilde{F}}(\bar{\tilde{Y}})}{\tilde{F}(\tilde{Y})} \, . \tag{6.4.36}$$

6.5 Multiple W_5 for a given black hole potential

In this section we extend our discussion of section 6.3 on the black hole potential and its description in terms of W_5.

One of the key features that allow to identify classes of four-dimensional stable extremal black holes described by first-order differential equations in [37] is the degenerate description of the four-dimensional effective potential in terms of a "superpotential" W_4. The solutions are supersymmetric only when the latter coincides with the four-dimensional central charge Z_4.

A similar analysis can be carried out in five dimensions. As discussed in section 6.3, whenever the five-dimensional effective potential is expressed in terms of a "superpotential" W_5 as in (6.3.8), first-order flow equations for the various fields describing the extremal black hole can be obtained. The associated solutions may be supersymmetric in five dimensions when W_5 equals the five-dimensional central charge $Z_5 = q_A\, X^A$.

In section 6.3 we focused on W_5's which are obtained by studying the invariance group of the inverse matrix G^{AB} (see (6.3.7)). This is similar to the discussion in [37] of the invariances of the complex matrix $\mathcal{M}$ appearing in the four-dimensional black hole potential, only that it is simpler in five dimensions because G^{AB} is real. It may be useful to note that although the matrices $R^A{}_B$ are part of the invariance group of the norm defined by the inverse metric G^{AB}, they can also be interpreted as transformations on the moduli space of very special geometry by using relations such as (B.9).

Rather than attempting to characterize the general form of such matrices $R^A{}_B$, let us in the following discuss a few classes of very special geometries for which it is possible to find non-trivial solutions to (6.3.7). Generically, G^{AB} (with $A, B = 1, \dots, n$) has non-zero entries in all of its matrix elements and possesses n different eigenvalues. Then, the only allowed matrix $R^A{}_B$ is the identity matrix. Non-trivial solutions can be found when G^{AB} has m identical eigenvalues, in which case $R^A{}_B$ is an orthogonal matrix in $O(m)$. A further specialization arises when G^{AB} becomes (block) diagonal. Consider, for instance, the class of scalar manifolds

$$\mathcal{M} = \frac{SO(n-1,1)}{SO(n-1)} \times SO(1,1)\ , \qquad (6.5.1)$$

which is associated to the Jordan algebra $J = \Sigma_{n-1} \times \mathbb{R}$, where Σ_{n-1} is the Jordan algebra of degree two corresponding to a quadratic form of signature $(+-\dots-)$. The inverse G^{AB} therefore factorizes into a generic $(n-1)\times(n-1)$-block and a single entry for the extra $SO(1,1)$ factor [94]. This means that the associated "superpotential" can be chosen as

$$W_5 = \pm q_1 X^1 + q_a X^a \quad , \quad a = 2, \dots, n\ , \qquad (6.5.2)$$

where X^1 is the ambient vector space coordinate associated with the $SO(1,1)$ factor. The exceptional case $n = 2$ with moduli a and b has a metric which further degenerates to a diagonal form and therefore admits arbitrary sign changes in front of any of the three charges allowed by the model in the "superpotential". Let us discuss this in more detail. The scalar manifold is simply $SO(1,1) \times SO(1,1)$ and it can be obtained as a hypersurface in an ambient vector space, parameterized by the X^A coordinates. The matrix $G_{AB}(X^C)$ is the metric of this ambient space and has non-trivial entries in all of its elements as functions of X^C. After the restriction to the hypersurface characterized by $2\mathcal{V} = 1$, this metric reduces to the diagonal form

$$G_{AB} = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{a^2b^2}{2} \end{pmatrix} . \tag{6.5.3}$$

From (6.5.3) it is clear that any diagonal matrix $R^A{}_B$ with entries ± 1 solves (6.3.7) and therefore leaves the black hole potential invariant. The various “superpotentials” are then defined by $W_5 = q_A R^A{}_B X^B$, and an example thereof is (6.5.2). Only when the matrix $R^A{}_B$ is the identity is it possible to obtain supersymmetric solutions.

In order to obtain further insights into the degenerate description of the black hole potential in terms of “superpotentials”, additional guidance beyond the discussion of invariances of the inverse metric G^{AB} is needed. In this respect it may be useful to note that some of the static solutions (6.4.10) based on $W_5 = Z_5$ are supersymmetric in five dimensions, but non-supersymmetric in four dimensions, as already observed in [32]. This feature is related to the $U(1)$-fibration of the Taub-NUT geometry (6.2.4) [131, 63]. Other solutions, such as the rotating $s = -1$ solution given in (6.3.26), are neither supersymmetric in five nor in four dimensions, but are nevertheless derived from first-order flow equations. For the former solutions the existence of first-order flow equations may be explained by hidden supersymmetry. It would be interesting to study further how the appearance of non-supersymmetric first-order equations in four dimensions is related to supersymmetry in higher dimensions or to fake supergravity [75, 36].

It is known that group theoretical tools, such as the analysis of orbits of U-duality groups in $N \geq 2$ supergravity theories in four and five dimensions [67, 68, 17], can be used to classify both BPS and non-BPS states. U-duality can also be used to shed light on the degenerate description of the black hole potential in terms of “superpotentials” W. In the recent work [4] it was shown that in four-dimensional supergravity theories with $N > 2$, some of the W’s giving rise to the black hole potential can be written in terms of linear combinations of U-duality invariants. Both the supersymmetric and non-supersymmetric critical points of the black hole potential are then related to its different rewritings in terms of these invariants. It remains to be seen whether these results can be extended to the case of general $N = 2$ supergravity theories in four and five dimensions.

Chapter 7

Entropic principle

7.1 Black hole attractors and flux vacua

As we have seen in section 3.4, in four dimensions the near horizon geometry of supersymmetric black hole solutions is characterized by attractor equations which, at the two-derivative level, follow from the extremization condition of the black hole central charge Z, i.e. $DZ = 0$. The latter exhibits an interesting similarity to the condition $DW = 0$ for supersymmetric flux vacua, where W denotes the flux-generated superpotential in type II or F-theory compactifications. A connection [133] between black holes and flux compactifications is provided by type IIB supersymmetric black hole solutions, for which the near horizon condition $DZ = 0$ can be viewed as the extremization condition of a five-form flux superpotential W generated upon compactifying type IIB string theory on $S^2 \times \mathcal{X}$, where $\mathcal{X}$ denotes a Calabi–Yau three-fold (for related work see [44, 13, 61, 105]).

Concretely, one takes the five-form flux to be of the type

$$F_{[5]} = \omega \wedge F_{[3]} \,, \tag{7.1.1}$$

where the two-form ω is a unit form on the S^2 and the three-form $F_{[3]}$ threads the cycles of the Calabi–Yau. Denoting the resulting quantized fluxes by

$$p^I = \int_{A^I} F_{[3]} \,, \qquad q_I = \int_{B_I} F_{[3]} \tag{7.1.2}$$

(compare with section 3.3) the authors of [133] find that the supersymmetry conditions for the fluxes, expressed as extremization conditions for the superpotential (with the holomorphic $(3,0)$-form Θ on $\mathcal{X}$)

$$W = \int_{S^2 \times \mathcal{X}} F_{[5]} \wedge \Theta \,, \tag{7.1.3}$$

take the form of attractor equations for a supersymmetric four-dimensional black hole with charges (p^I, q_I) (3.4.30)

$$p^I = 2\,\mathrm{Im}\, Y^I \,, \qquad q_I = 2\,\mathrm{Im}\, F_I \,, \tag{7.1.4}$$

together with the requirement that the remaining $(1+1)$-dimensional space be the AdS_2, which matches the near-horizon geometry of the black hole. The black hole itself can be realized by wrapping D3 branes p^I times on A^I and q_I times on B_I. The (negative) cosmological constant of the AdS_2 is determined by the value of W at the extremum, which is also the area of the two-sphere, and hence the Bekenstein–Hawking entropy [133].

In view of this connection, it was suggested in [133, 92] to interpret the exponentiated entropy of large supersymmetric black holes in Calabi–Yau compactifications as an entropic function for supersymmetric flux compactifications on $AdS_2 \times S^2 \times \mathcal{X}$. At the two-derivative level, recalling (3.4.35) and (3.4.26), the entropy of supersymmetric black holes takes the form

$$\mathcal{S} = \pi \, |Y^0|^2 \, \mathrm{e}^{-G(z,\bar{z})} \,, \tag{7.1.5}$$

where, in a certain gauge, $G(z, \bar{z})$ reduces to the Kähler potential for the moduli fields $z^A = Y^A/Y^0$ belonging to vector multiplets labelled by $A = 1, \ldots, n$. The fields Y^0 and z^A are expressed in terms of the black hole charges (p^I, q_I) by the attractor equations. Once the charges are identified with fluxes, each choice (Y^0, z^A) translates into a particular flux compactification. By fixing Y^0 to a specific value Y^0_{f}, the entropy (7.1.5) can be viewed as a function over the moduli space of the Calabi–Yau three-fold, and to each point z^A in moduli space one assigns a (suitably normalized) probability density $\mathrm{e}^{\mathcal{S}}$ (entropic principle [133, 92]).

The entropy of supersymmetric black holes is corrected by higher-curvature interactions [114]. Therefore, the probability density for AdS_2 vacua with five-form fluxes is modified due to R^2-interactions. In this chapter, we will be interested in studying the maximization of the entropy, viewed as a function over the moduli space of the Calabi–Yau three-fold, in the presence of higher-curvature corrections.

As in chapter 4 we study the entropy extremization in the neighborhood of the conifold point. Our results differ from [92], because in contrast to [92], we do not consider the extremization of the Hartle–Hawking type wave function $|\psi_{0,0}|^2$,but instead the extremization of the wave function $|\psi_{p,q}|^2$, whose value at the attractor point is the exponential of the entropy [133]. We find that singularities where an excess of additional massless hypermultiplets appear, correspond to local maxima of the entropy. Therefore, following the entropic principle, the associated vacua would have a higher probability.

We demonstrate that the gravitational coupling $F^{(1)}$ leads to an enhancement of the maximization of the entropy at these singularities. For the case of the conifold, we also take into account the contribution from the higher coupling functions $F^{(g)}$ by resorting to the non-perturbative expression of the topological free energy computed in [87, 88] for the resolved conifold. We find that the entropy is maximized at the conifold point for the case of real topological string coupling constant, whereas it ceases to have a maximum at the conifold point for complex values of the coupling constant.

7.2 The entropic function

To use the exponentiated entropy of large supersymmetric black holes in Calabi–Yau compactifications (3.4.35), (3.4.26)

$$\mathcal{S} = \pi\,\mathrm{i}\left(\bar{Y}^I\,F_I^{(0)}(Y) - Y^I\,\bar{F}_I^{(0)}(\bar{Y})\right) = \pi\,|Y^0|^2\,\mathrm{e}^{-G(z,\bar{z})}\,, \tag{7.2.1}$$

where

$$F^{(0)}(Y) = -\mathrm{i}\,(Y^0)^2\,\mathcal{F}^{(0)}(z)\,, \qquad z^A = Y^A/Y^0\,, \tag{7.2.2}$$

denotes a holomorphic prepotential without higher-order corrections (the indices run over $I = 0,\ldots,n$ and $A = 1,\ldots,n$), as a probability density for supersymmetric flux compactifications, the entropy has to be expressed as a function on the moduli space of Calabi–Yau compactifications. The fields Y^I are determined in terms of the charges carried by the black hole by virtue of the attractor equations (see section 7.3), but since the number of physical moduli z^A is one less than the number of pairs of black hole charges (p^I, q_I), one has to fix one particular charge combination in order to discuss the maximization of the entropy with respect to the z^A.

One possibility would be to allow Y^0 to vary in a prescribed way as one moves around in moduli space, i.e. $Y^0(z,\bar{z})$. A more economical possibility consists in assigning the same value Y^0_{f} to all points in moduli space, i.e. $Y^0 = Y^0_{\mathrm{f}}$ [92]. The fields Y^0 and z^A are expressed in terms of the black hole charges by the attractor equations [71, 150, 69], as will be reviewed in section 7.3. These charges are in turn interpreted as flux data. Therefore, to each particular flux compactification we can assign a probability density proportional to $\mathrm{e}^{\mathcal{S}}|_{Y^0_{\mathrm{f}}}$. Fixing Y^0 to a particular Y^0_{f} means choosing a codimension one hypersurface in the complex space of charges (assuming that the charges are continuous), which provides a mapping between moduli z^A and charges (p^I, q_I).

Having fixed $Y^0 = Y^0_{\mathrm{f}}$, one may look for maxima of (7.2.1) in moduli space, i.e. for maxima of $\mathrm{e}^{-G(z,\bar{z})}$ in a certain domain. Local extrema in the interior of this domain satisfy $\partial_A\,\mathrm{e}^{-G(z,\bar{z})} = 0$. In order to determine the nature of these critical points one can analyze the definiteness of the matrix of second derivatives of $\mathrm{e}^{-G(z,\bar{z})}$. At a critical point, $\partial_A\partial_{\bar{B}}\,\mathrm{e}^{-G} = -g_{A\bar{B}}\,\mathrm{e}^{-G}$, where $g_{A\bar{B}} = \partial_A\partial_{\bar{B}}K$ denotes the Kähler metric. Recall

$$G(z,\bar{z}) = K(z,\bar{z}) + \log|X^0(z)|^2\,. \tag{7.2.3}$$

In the gauge $X^0(z) = 1$, we have $G = K$. On the other hand, in the gauge $X^0(z) = W(X(z))$, where $W(X(z))$ denotes the holomorphic central charge $W(X(z)) = \mathrm{e}^{-K/2}Z$, we have $G = K + \log|W|^2 = 2\log|Z|$ [18]. If $\mathcal{F}^{(0)}_{ABC}$ and $g_{A\bar{B}}$ are finite (where $\mathcal{F}^{(0)}_{ABC}$ is the triple derivative of the prepotential), then $\partial_A\partial_B\,\mathrm{e}^{-G}$ vanishes there. This can be best seen [18] in the gauge $X^0(z) = W(X(z))$, where the vanishing of $\partial_A\partial_B\,\mathrm{e}^{-G}$ translates into the vanishing of $\partial_A\partial_B\,|Z|$. The latter is guaranteed to hold by virtue of special geometry [66], provided that $\mathcal{F}^{(0)}_{ABC}\,g^{C\bar{C}}$ is finite. By direct calculation, the vanishing of $\partial_A\partial_B\,\mathrm{e}^{-G}$ implies that $(z^C - \bar{z}^C)\,\mathcal{F}^{(0)}_{ABC}$ vanish. Thus, it follows that if the metric $g_{A\bar{B}}$ is positive definite, the critical point is a maximum of e^{-G} [73, 18].

In Calabi–Yau compactifications, and for large values of the moduli z^A, $\mathcal{F}^{(0)}(z)$ is a cubic expression in the z^A and therefore, the entropy (7.2.1) grows to infinity as $z^A \to \infty$. Hence, in order to study the maximization of $\mathrm{e}^{-G(z,\bar{z})}$ in a well-posed way, we restrict ourselves to a finite region in moduli space and ask, whether the entropy has local maxima in this region. As we will discuss in section 7.4, a class of such points is provided by singularities of the Calabi–Yau three-fold at which an excess of (charged) hypermultiplets becomes massless. Examples thereof are the conifold of the mirror quintic [25] as well as singularities associated with the appearance of non-abelian gauge symmetries with a non-asymptotically free spectrum [112, 110].

Consider the case when the singularity is characterized by a vanishing modulus, which we now denote $V = -\mathrm{i}z^1$ with $\mathcal{F}^{(0)}(z) \sim p(T) - V^2 \log V$, where $p(T)$ denotes a function of the remaining moduli, which are held fixed. This results in $\mathcal{F}^{(0)}_{VV} \sim -\log V$ as well as $\mathcal{F}^{(0)}_{VVV} \sim -V^{-1}$, which diverges as $V \to 0$, while $(z^1 - \bar{z}^1)\,\mathcal{F}^{(0)}_{111} \sim (V + \bar{V})\mathcal{F}^{(0)}_{VVV} \sim (1 + \bar{V}/V)$ remains finite. This is thus an example where $\mathcal{F}^{(0)}_{ABC}$ tends to infinity in such a way that $\partial_A \partial_B\, \mathrm{e}^{-G}$ remains finite and non-vanishing at the singularity. The function e^{-G} has an extremum at $V = 0$ (see section 7.4). Since the metric $g_{A\bar{B}}$ diverges at the singularity, it follows that $\partial_A \partial_B\, \mathrm{e}^{-G}$ is smaller than $\partial_A \partial_{\bar{B}}\, \mathrm{e}^{-G}$. Since the metric $g_{A\bar{B}}$ is positive definite near the singularity, the extremum of e^{-G} at $V = 0$ is a local maximum.

The maximization of the entropy may be further enhanced when taking into account higher-curvature corrections. This will be discussed in section 7.5. In the presence of such corrections, the entropy ceases to be given by the area law (7.2.1). For the case of a certain class of terms quadratic in the Riemann tensor encoded in a holomorphic homogeneous function $F(Y, \Upsilon)$, the macroscopic entropy, computed from the associated effective $N = 2$ Wilsonian action using Wald's formula (2.3.12), is given by [114]

$$\mathcal{S} = \pi\Big(\mathrm{i}\left(\bar{Y}^I F_I(Y,\Upsilon) - Y^I \bar{F}_I(\bar{Y},\bar{\Upsilon})\right) + 4\,\mathrm{Im}\,(\Upsilon\, F_\Upsilon)\Big)\,, \tag{7.2.4}$$

where $F_I = \partial F/\partial Y^I$ and $F_\Upsilon = \partial F/\partial \Upsilon$. Here Υ denotes the square of the (rescaled) graviphoton 'field strength', which takes the value $\Upsilon = -64$ at the horizon of the black hole. The Y^I are determined in terms of the black hole charges by the attractor equations (7.3.1). The term $\pi\,\mathrm{i}\left[\bar{Y}^I F_I(Y,\Upsilon) - Y^I \bar{F}_I(\bar{Y},\bar{\Upsilon})\right]$ describes the R^2-corrected area of the black hole, while the term $4\pi\,\mathrm{Im}\,(\Upsilon\, F_\Upsilon)$ describes the deviation from the area law due to the presence of higher-curvature interactions (as paralleled by the second term in the simplest example (2.3.13)).

7.3 Choice of Y^0

In the presence of higher-curvature corrections encoded in $F(Y, \Upsilon)$, the attractor equations determining the near-horizon values of Y^I take the form [114]

$$Y^I - \bar{Y}^I = \mathrm{i}p^I\,, \qquad F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon}) = \mathrm{i}q_I\,, \tag{7.3.1}$$

where (p^I, q_I) denote the magnetic and electric charges of a supersymmetric black hole, respectively. Since $F(Y,\Upsilon)$ is weighted homogeneous, i.e. $F(\lambda Y, \lambda^2\Upsilon) = \lambda^2 F(Y,\Upsilon)$ for any $\lambda \in \mathbb{C}\setminus\{0\}$, then by Euler's theorem

$$2\,F - Y^I F_I = 2\,\Upsilon F_\Upsilon\,, \tag{7.3.2}$$

and consequently

$$F_0 = \frac{1}{Y^0}\left(2F - 2\Upsilon\, F_\Upsilon - z^A\,\frac{\partial F}{\partial z^A}\right), \tag{7.3.3}$$

where $z^A = Y^A/Y^0$. Using (7.3.1) we compute

$$z^A \pm \bar z^A = \frac{1}{2|Y^0|^2}\Big(\mp\mathrm{i}\,p^A(Y^0 \mp \bar Y^0) \pm (Y^A + \bar Y^A)(Y^0 \pm \bar Y^0)\Big)\,. \tag{7.3.4}$$

As discussed in the previous section, we would like to fix the value of Y^0 to a constant value Y^0_f throughout moduli space. Inspection of (7.3.4) suggests to take either $Y^0_\mathrm{f} = \bar Y^0_\mathrm{f}$ or $Y^0_\mathrm{f} = -\bar Y^0_\mathrm{f}$, since this leads to a simplification of the expression. Note, however, that in order to be able to connect four-dimensional supersymmetric black holes to spinning supersymmetric black holes in five dimensions [77, 14], both q_0 and p^0 have to be non-vanishing, which requires taking Y^0_f to be complex.

Setting $Y^0_\mathrm{f} = \bar Y^0_\mathrm{f}$ implies $p^0 = 0$. Then, (7.3.4) reduces to

$$z^A - \bar z^A = \frac{\mathrm{i}}{Y^0_\mathrm{f}}\,p^A\,, \tag{7.3.5}$$

and the second attractor equation in (7.3.1) gives

$$\frac{\partial F}{\partial z^A} - \frac{\partial \bar F}{\partial \bar z^A} = \mathrm{i}Y^0_\mathrm{f}\,q_A\,. \tag{7.3.6}$$

The value of Y^0_f is determined by the equation $F_0 - \bar F_0 = \mathrm{i}q_0$, and is expressed in terms of $z, \bar z$ and q_0. For instance, when neglecting R^2-interactions, and using (7.3.2), it follows that

$$F_0 = -\mathrm{i}Y^0\left(2\mathcal{F}^{(0)}(z) - z^A\mathcal{F}^{(0)}_A(z)\right), \tag{7.3.7}$$

where we set $F = F^{(0)} = -\mathrm{i}(Y^0)^2\,\mathcal{F}^{(0)}(z)$, and where $\mathcal{F}^{(0)}_A = \partial\mathcal{F}^{(0)}/\partial z^A$. Then

$$Y^0_\mathrm{f} = \bar Y^0_\mathrm{f} = -\frac{q_0}{2(\mathcal{F}^{(0)} + \bar{\mathcal{F}}^{(0)}) - (z^A\mathcal{F}^{(0)}_A + \bar z^A\bar{\mathcal{F}}^{(0)}_A)}\,. \tag{7.3.8}$$

For a fixed value Y^0_f, one moves around in moduli space by changing the charges (p^A, q_A), which results in a change of z^A according to (7.3.5) and (7.3.6). However, in order to keep Y^0_f constant as one varies z^A, it follows from (7.3.8) that one must change q_0 in a continuous fashion. Since q_0 is quantized, we need to take q_0 to be large in order to be able to treat it as a continuous variable. Observe that when $Y_{\rm f}^0$ is large, a unit change in the charges (p^A, q_A) corresponds to a quasi-continuous change of the z^A.

Similarly, choosing $Y_{\rm f}^0 = -\bar{Y}_{\rm f}^0$ yields $Y_{\rm f}^0 = {\rm i}p^0/2$ fixed to a particular value. Then, (7.3.4) and (7.3.1) yield

$$\begin{aligned} z^A + \bar{z}^A &= 2\,\frac{p^A}{p^0}\,, \\ \frac{\partial F}{\partial z^A} + \frac{\partial \bar{F}}{\partial \bar{z}^A} &= -\frac{q_A\,p^0}{2}\,. \end{aligned} \tag{7.3.9}$$

A choice of charges (p^A, q_A) determines a point z^A, and the remaining equation $F_0 - \bar{F}_0 = {\rm i}q_0$ determines the value of q_0. This value will, generically, not be an integer, and therefore consistency requires again taking q_0 to be large in order to be able to treat it as a continuous variable.

Observe that Y^0 is related to the topological string coupling $g_{\rm top}$ by (see (E.26))

$$(Y^0)^2\, g_{\rm top}^2 = 4\pi^2\,. \tag{7.3.10}$$

Therefore, we will be interested in taking Y^0 to be real (i.e. $g_{\rm top}$ real) or complex, but not purely imaginary.

7.4 Entropy maximization near singularities

Let us examine the case when one of the $T^A - {\rm i}z^A = -{\rm i}Y^A/Y^0$ is taken to be small. This time we will denote this modulus by $V = -{\rm i}z^1 = -{\rm i}Y^1/Y^0$. We consider the situation where ${\rm e}^{-G}$ is extremized as $V \to 0$, while Y^0 and the remaining moduli T^a are kept fixed. As our basic example, we take the $\mathcal{F}^{(0)}$ of chapter 4,

$$\mathcal{F}^{(0)}(V) = \frac{\beta}{2\pi}\,V^2 \log V + a\,, \tag{7.4.1}$$

where β denotes a real constant, and where the constant a is complex. We compute

$$\begin{aligned} {\rm e}^{-G(V,\bar{V})} &= 2\left(\mathcal{F}^{(0)} + \bar{\mathcal{F}}^{(0)}\right) - (V + \bar{V})\left(\mathcal{F}^{(0)}_V + \bar{\mathcal{F}}^{(0)}_{\bar{V}}\right) \\ &= 4\,{\rm Re}\,a - \frac{\beta}{2\pi}(V+\bar{V})^2 - \frac{2\beta}{\pi}|V|^2 \log|V|\,. \end{aligned} \tag{7.4.2}$$

(cf. (4.5.2)) and note that ${\rm e}^{-G(V,\bar{V})}$ has a local maximum at $V = 0$ for negative β. The value at this maximum is given by $4\,{\rm Re}\,a \equiv {\rm e}^{-G_0}$. This is displayed in Fig. 7.1 (cf. Fig. 4.2). We take ${\rm Re}\,a > 0$ to ensure that ${\rm e}^{-G(V,\bar{V})}$ is positive in the vicinity of $V = 0$.

Observe that adding a cubic polynomial (and in particular a linear term) in V to (7.4.1) does not affect the leading behavior of (7.4.2) near $V = 0$.

Next, we compute the metric on the moduli space near $V = 0$. Using

$$g_{V\bar{V}} = \partial_V \partial_{\bar{V}} G = -{\rm e}^G\,\partial_V\partial_{\bar{V}}{\rm e}^{-G} + G_V\,G_{\bar{V}}\,, \tag{7.4.3}$$

and taking $V \to 0$, we find $G_V = -\mathrm{e}^G \partial_V \mathrm{e}^{-G} \to 0$ and

$$g_{V\bar{V}} \approx \frac{\beta}{\pi}\,\mathrm{e}^{G_0}\,\log|V|^2\,. \tag{7.4.4}$$

Note that the result (7.4.4) depends crucially on having $\mathrm{Re}\,a > 0$. Furthermore, for the metric to be positive definite as $V \to 0$, the constant β has to be negative.

We also compute the gauge couplings associated with (7.4.1) near $V \approx 0$. Using (3.1.8) and (3.3.20) [55]

$$\left(\frac{1}{g^2}\right)_{IJ} = \frac{\mathrm{i}}{4}\left(\mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ}\right), \qquad \mathcal{N}_{IJ} = \bar{F}_{IJ} + 2\mathrm{i}\,\frac{\mathrm{Im}\,F_{IK}\,\mathrm{Im}\,F_{JL}Y^K Y^L}{\mathrm{Im}\,F_{MN}Y^M Y^N}\,, \tag{7.4.5}$$

we find, upon diagonalization, that one of the gauge couplings remains approximately constant, while the other coupling exhibits a logarithmic running,

$$g^{-2} \approx \mathrm{Re}\,a\,, \qquad \tilde{g}^{-2} \approx \frac{\beta}{4\pi}\log|V|^2\,. \tag{7.4.6}$$

Observe that for negative β, the coupling $\tilde{g}$ becomes small as $V \to 0$.

The basic example (7.4.1), with $\beta = -1/2$, describes the conifold singularity of the mirror quintic in type IIB with $V = \psi - 1$ (cf. (E.13)) [25, 80]. The metric (7.4.4) is precisely the metric at the conifold point $\psi = 1$ (see table 2 of [25]). Since the conifold singularity is associated with the appearance of one additional massless hypermultiplet [149], we see that we have entropy maximization when a hypermultiplet becomes massless at the singularity. Note that the character of the extremum of the entropy (7.2.1) at $V = 0$ is independent of the value of Y^0_{f}.

Next, consider the resolved conifold in type IIA. The associated $\mathcal{F}^{(0)}$ is described by (7.4.1) with $\beta = -1/2$ and $a = 0$ (cf. (E.13)) [88]. We compute the Kähler metric and the gauge coupling in the associated field theory. We decouple gravity by restoring Planck's mass in (7.5.3) with G/M_{Pl}^2 and z^A/M_{Pl}, and by expanding both sides of (7.5.3) in powers of M_{Pl}^{-2} [145],

$$G = \hat{K}(z,\bar{z}) + f(z) + \bar{f}(\bar{z}) + \mathcal{O}(M_{\mathrm{Pl}}^{-2})\,, \tag{7.4.7}$$

where

$$\hat{K} = -\left(\bar{z}^A\,\mathcal{F}_A + z^A\bar{\mathcal{F}}_A\right). \tag{7.4.8}$$

Using (7.4.1), we obtain near $V = -\mathrm{i}z^1 \to 0$,

$$\hat{K} \approx \frac{2\beta}{\pi}|V|^2\log|V|\,. \tag{7.4.9}$$

Computing the corresponding Kähler metric near $V \to 0$ yields

$$g_{V\bar{V}} = \partial_V\partial_{\bar{V}}\hat{K} \approx \frac{2\beta}{\pi}\log|V|\,, \tag{7.4.10}$$

which is positive definite for $\beta < 0$. The gauge coupling is computed from (7.4.5) with $\mathcal{N}_{IJ} = \bar{F}_{IJ}$ [50]. We obtain

$$\tilde{g}^{-2} = \frac{\mathrm{i}}{4}\left(\bar{F}_{11} - F_{11}\right) \approx \frac{\beta}{4\pi}\log|V|^2\,, \tag{7.4.11}$$

in agreement with (7.4.6).

More generally, whenever the singularity in moduli space is such that a sufficiently large number of (charged) hypermultiplets becomes massless there, so that the resulting β is negative,[1] the function e^{-G} exhibits a local maximum. Examples thereof are singularities associated with the appearance of non-abelian gauge symmetries with a non-asymptotically free spectrum [112, 110]. A concrete example is provided by the so-called heterotic ST model, which is a two-Kähler moduli model with a dual type IIA description in terms of a hypersurface of degree 12 in weighted projective space $P^4_{(1,1,2,2,6)}$ with Euler characteristic $\chi = -252$ [104]. The type IIA dual description is based on (E.2) with $V = S - T$ and $n_{0,1} = 2$. From (E.7) and (E.9) we infer that $\beta = -1$ and that a is positive. At $V = 0$, a gauge symmetry enhancement takes place, whereby a $U(1)$ group is enlarged to an $SU(2)$, with four additional (charged) hypermultiplets becoming massless there [112, 110]. The entropy of axion-free black holes in this model does indeed have a maximum at $S = T$ (cf. eq. (4.34) in [12]).

7.5 Entropy maximization in the presence of R^2- interactions

Next, let us discuss entropy maximization in the presence of higher-curvature interactions encoded in $F(Y, \Upsilon)$. Usually, the generalized prepotential $F(Y, \Upsilon)$ is assumed to have a perturbative expansion of the form

$$F(Y,\Upsilon) = \sum_{g=0}^{\infty} F^{(g)}(Y)\,\Upsilon^g\,. \tag{7.5.1}$$

Then, expanding the entropy (7.2.4) in terms of the coupling functions $F^{(g)}(Y)$ yields

$$\begin{aligned}\mathcal{S}/\pi = {}& |Y^0|^2\,\mathrm{e}^{-G(z,\bar z)} - 2\mathrm{i}\Upsilon\left(F^{(1)} - \bar F^{(1)}\right) - \mathrm{i}\Upsilon(z^A - \bar z^A)\left(\frac{\bar Y^0}{Y^0}\frac{\partial F^{(1)}}{\partial z^A} + \frac{Y^0}{\bar Y^0}\frac{\partial \bar F^{(1)}}{\partial \bar z^A}\right)\\ & + 2\mathrm{i}\sum_{g=2}^{\infty} F^{(g)}(Y)\,\Upsilon^g\left(-g + (1-g)\,\frac{\bar Y^0}{Y^0}\right) - 2\mathrm{i}\sum_{g=2}^{\infty}\bar F^{(g)}(\bar Y)\,\Upsilon^g\left(-g + (1-g)\,\frac{Y^0}{\bar Y^0}\right)\\ & - \mathrm{i}(z^A - \bar z^A)\sum_{g=2}^{\infty}\Upsilon^g\left(\frac{\bar Y^0}{Y^0}\frac{\partial F^{(g)}}{\partial z^A} + \frac{Y^0}{\bar Y^0}\frac{\partial \bar F^{(g)}}{\partial \bar z^A}\right),\end{aligned} \tag{7.5.2}$$

[1] Note that many texts use the opposite sign convention.

where $G(z,\bar{z})$ is

$$\mathrm{e}^{-G(z,\bar{z})} = 2\left(\mathcal{F}^{(0)} + \bar{\mathcal{F}}^{(0)}\right) - (z^A - \bar{z}^A)\left(\mathcal{F}^{(0)}_A - \bar{\mathcal{F}}^{(0)}_A\right). \tag{7.5.3}$$

Let us first discuss the effect of the gravitational coupling function $F^{(1)}$ on the maximization of the entropy. Let us again consider a singularity of the type (7.4.1) associated with a vanishing modulus $V = -\mathrm{i}z^1 = -\mathrm{i}Y^1/Y^0$, while the other moduli are non-vanishing and kept fixed. From (E.11) it follows that $F^{(1)}$ takes the form

$$F^{(1)} \approx -\frac{\mathrm{i}}{64 \cdot 12\pi}\beta \log V \tag{7.5.4}$$

near $V = 0$. We compute (using $\Upsilon = -64$)

$$-2\mathrm{i}\Upsilon\left(F^{(1)} - \bar{F}^{(1)}\right) = 4\,\mathrm{Im}\left(\Upsilon\, F^{(1)}\right) = \frac{\beta}{3\pi}\log|V|\,, \tag{7.5.5}$$

which for negative β reaches a maximum as $V \to 0$, i.e. $\mathrm{Im}\left(\Upsilon\, F^{(1)}\right) \to +\infty$.

On the other hand, the term proportional to $F^{(1)}_1$ in (7.5.2) only contributes with the phases of Y^0 and Y^1,

$$-\mathrm{i}\Upsilon(z^1 - \bar{z}^1)\left(\frac{\bar{Y}^0}{Y^0}\frac{\partial F^{(1)}}{\partial z^1} + \frac{Y^0}{\bar{Y}^0}\frac{\partial \bar{F}^{(1)}}{\partial \bar{z}^1}\right) = \frac{\beta}{6\pi}\big(\cos(2\theta_0) - \cos(2\theta_1)\big), \tag{7.5.6}$$

where $Y^0 = |Y^0|\,\exp(\mathrm{i}\theta_0)$ and $Y^1 = |Y^1|\,\exp(\mathrm{i}\theta_1)$.

We conclude that, for negative β, not only does e^{-G} have a maximum at $V = 0$, but also the R^2-corrected entropy (7.5.2) based on $F^{(0)}$ and $F^{(1)}$. Moreover, the contribution of the coupling function $F^{(1)}$ is such that it enhances the maximization of the entropy.

The gravitational coupling function $F^{(1)}$ (as well as the higher $F^{(g)}$) is known to receive non-holomorphic corrections [19, 20]. For instance, for the quintic three-fold [19] (and up to an overall constant),

$$\mathrm{Re}\, F^{(1)} = \log\left(g^{-\frac{1}{2}}_{\psi\bar{\psi}}\, \mathrm{e}^{\frac{62}{3}K}\, |\psi^{\frac{62}{3}}\,(1-\psi^5)^{-\frac{1}{6}}|^2\right). \tag{7.5.7}$$

Near $V = -\mathrm{i}z \equiv \psi - 1 \approx 0$, $K =$ constant and $g_{\psi\bar{\psi}} \sim -\log|V|$ [25], so that

$$\mathrm{Re}\, F^{(1)} \sim -\log(-\log|V|) - \tfrac{1}{6}\log|V|^2\,. \tag{7.5.8}$$

Therefore, as $V \to 0$, the behavior of the non-holomorphic term is less singular than the behavior of the holomorphic term, and it can be dropped from the maximization analysis.

Let us express $V = -\mathrm{i}Y^1/Y^0$ in terms of the charges q_0, q_1, p^0 and p^1 by solving the attractor equations (7.3.1). We take $\mathcal{F}^{(0)}$ to be given by (7.4.1) and $F^{(1)}$ to be given by (7.5.4). For simplicity, we take Y^1 to be imaginary and Y^0 to be real, so that V is real. Then $Y^1 = \mathrm{i}\,p^1/2$, and Y^0 is determined by

$$4(\mathrm{Re}\, a)(Y^0)^2 + q_0 Y^0 + \frac{\beta}{6\pi} - \frac{\beta}{4\pi}(p^1)^2 = 0\,. \tag{7.5.9}$$

A large value of Y^0 can be obtained by choosing $|q_0|$ to be large (assuming $\mathrm{Re}\, a \neq 0$), whereas a small value of V can be achieved by sending $p^1 \to 0$. Observe that taking Y^0 to be fixed at a large value is natural, since (7.5.1) is based on the perturbative expansion of the topological string free energy, and Y^0 is related to the inverse topological string coupling constant by $Y^0 = 2\pi\, g_{\rm top}^{-1}$ (cf. (E.26)).

Next, let us discuss the effect of the higher $F^{(g)}$ (with $g \geq 2$) on the maximization of the entropy. It is known that the higher $F^{(g)}$ also exhibit a singular behavior at $V = 0$. For concreteness, we consider the conifold singularity of the mirror quintic [25]. Near the conifold point $z = Y^1/Y^0 \to 0$ [20, 80],

$$F^{(g)}(Y) = \mathrm{i}\frac{A_g}{(Y^0)^{2g-2}\, z^{2g-2}}\,, \qquad g \geq 2\,, \tag{7.5.10}$$

where A_g denote real constants which are expressed [80] in terms of the Bernoulli numbers B_{2g} and are alternating in sign. (cf. (E.28)).

Inserting (7.5.10) into (7.5.2) yields

$$\begin{aligned}\mathcal{S}/\pi = {} & |Y^0|^2\, \mathrm{e}^{-G(z,\bar z)} - 2\mathrm{i}\Upsilon\left(F^{(1)} - \bar F^{(1)}\right) - \mathrm{i}\Upsilon(z-\bar z)\left(\frac{\bar Y^0}{Y^0}\frac{\partial F^{(1)}}{\partial z} + \frac{Y^0}{\bar Y^0}\frac{\partial \bar F^{(1)}}{\partial \bar z}\right) \\ & - 2\sum_{g=2}^{\infty} A_g\, \Upsilon^g\, (Y^1)^{2-2g}\left(-g + (1-g)\,\frac{\bar Y^1}{Y^1}\right) \\ & - 2\sum_{g=2}^{\infty} A_g\, \Upsilon^g\, (\bar Y^1)^{2-2g}\left(-g + (1-g)\,\frac{Y^1}{\bar Y^1}\right).\end{aligned} \tag{7.5.11}$$

We observe that the contribution from the higher $F^{(g)}$ ($g \geq 2$) to (7.5.11) does not cancel out. This is problematic, since the $F^{(g)}$ become increasingly singular as $z \to 0$. For instance, when taking Y^1 to be purely imaginary, the combination $\Upsilon^g (Y^1)^{2-2g}$ is negative for all $g \geq 2$, and the total contribution of the higher $F^{(g)}$ to the entropy does not have a definite sign due to the alternating sign of B_{2g}. On the other hand, if we take Y^1 to be real, then the total contribution of the higher $F^{(g)}$ to the entropy is positive, since the combination $A_g \Upsilon^g$ is positive for all $g \geq 2$. This contribution becomes infinitely large at $Y^1 = 0$. Thus, to get a better handle on the behavior of the entropy in the presence of higher-curvature corrections near the conifold point $V = 0$, it is best to use the non-perturbative expression for $F(Y, \Upsilon)$ for the conifold given in [87, 92], rather than to rely on the perturbative expansion (7.5.1). This will be done next.

For the resolved conifold in type IIA, $F(Y, \Upsilon)$ is given by (see appendix A) [87, 92],

$$F(Y, \Upsilon) = -C\sum_{n=1}^{\infty} n \log\left(1 - q^n Q\right), \tag{7.5.12}$$

where we neglected the Q-independent terms, since they do not affect the extremization with respect to Q. Here $q = \mathrm{e}^{-g_{\rm top}}$ and $Q = \mathrm{e}^{-2\pi V}$ (we hope that no confusion with the electric charges should arise). We impose the physical restrictions $\mathrm{Re}\, g_{\mathrm{top}} > 0$ and $\mathrm{Re}\, V \geq 0$. The topological string coupling constant g_{top} and the constant C are expressed in terms of Y^0 and of Υ via (E.26).

Under the assumption of uniform convergence, inserting (7.5.12) into (7.2.4) yields the entropy

$$\mathcal{S} = -\sum_{n=1}^{\infty} \mathrm{Re}\left(\frac{nq^n Q \log q \big[n \log q + 2 \log|Q| \big]}{\log \bar{q}\, (1 - q^n Q)} + \frac{n \big[nq^n Q \log q - 2(1 - q^n Q) \log(1 - q^n Q) \big]}{1 - q^n Q} \right). \tag{7.5.13}$$

In order to determine the nature of the extrema of (7.5.13) we numerically approximated this expression by a finite sum of a sufficiently large number of terms. To improve the accuracy of the approximation, the summation was performed in the order of decreasing n, so that subsequent summands were of comparable magnitude to the partial sums.

Let us first consider the case when g_{top} (and hence q) is taken to be real. We find that the entropy attains a maximum at $Q = 1$, i.e. at the conifold point $V = 0$, regardless of the strength of the coupling g_{top}, as displayed in Fig. 7.2. Observe that the maximum at $Q = 1$ occurs at the boundary of the allowed domain, where derivative tests do not directly apply. $\mathcal{S}(q, Q)$ is periodic in $\arg(Q) = -\,\mathrm{Im}(2\pi V)$.

As the string coupling becomes weaker, the convergence of the series slows down, and the number of terms needed to be taken into account grows roughly proportionally to the inverse coupling, independently of Q. This is shown in Fig. 7.3. We observe that apart from the magnitude, the value of g_{top} has little influence on the shape of $\mathcal{S}(q, Q)$.

Next, let us subtract the tree-level contribution to the entropy, $\mathcal{S}^{(0)}$, computed from

$$F^{(0)} = C\, g_{\mathrm{top}}^{-2}\ \mathrm{Li}_3(Q)\,, \tag{7.5.14}$$

so as to exhibit the contribution to the entropy from the higher-curvature corrections. The tree-level contribution can be written as

$$\mathcal{S}^{(0)} = \frac{2\,\mathrm{Re}\big[\mathrm{Li}_3(Q) - \log|Q|\ \mathrm{Li}_2(Q)\big]}{|\log q|^2}\,. \tag{7.5.15}$$

When treated as a function of Q for a fixed q (observe that $\mathcal{S}^{(0)}$ does not depend on $\arg(g_{\mathrm{top}})$), $\mathcal{S}^{(0)}$ has a shape similar to the shape of $\mathcal{S}$.

The difference $\mathcal{S} - \mathcal{S}^{(0)}$ amounts to the contribution to the entropy of the higher-curvature terms. It depends on Q and g_{top}, as can be seen in Fig. 7.4. At the conifold point $V = 0$, the difference $\mathcal{S} - \mathcal{S}^{(0)}$ is positive for small coupling g_{top}, whereas it becomes negative for large values of g_{top}. At weak coupling higher-order corrections become negligible. At strong coupling the corrections, albeit smaller, are comparable to $\mathcal{S}^{(0)}$ and negative, resulting in $\mathcal{S} \ll \mathcal{S}^{(0)}$ ($\mathcal{S}^{(0)}$ decreases as g_{top}^{-2}, while $\mathcal{S}$ decreases as $g_{\mathrm{top}} \mathrm{e}^{-g_{\mathrm{top}}}$ for large g_{top}). This is displayed in Fig. 7.5, where we plotted $(\mathcal{S} - \mathcal{S}^{(0)})/\mathcal{S}$.

Finally, if we allow g_{top} to be complex the behavior of $\mathcal{S}$ changes markedly. As $\mathrm{Re}(V)$ tends to zero, we notice increasingly pronounced oscillations, whose amplitude and period sensitively depend on $g_{\rm top}$ (see Fig. 7.6). In effect, the maximum formerly at $V = 0$ is displaced and new local extrema appear.

Note that the entropy (7.5.13) is not necessarily positive, because we have not included the contribution stemming from the Euler characteristic of the Calabi–Yau three-fold (see (E.9)) and of other moduli (which, if present, we have taken to be constant).

7.6 Relation to OSV free energy

According to [132], the entropy (7.2.4) can be rewritten as

$$\mathcal{S} = E - L\,, \tag{7.6.1}$$

where E denotes the OSV free energy which, in the conventions of [113], reads

$$E = 4\pi\,\mathrm{Im}\,F\,, \tag{7.6.2}$$

and where the Legendre transform piece L is given by

$$L = \pi\,q_I\,\phi^I = 4\pi\,\mathrm{Im}\,F_I\,\mathrm{Re}\,Y^I \tag{7.6.3}$$

by virtue of the attractor equations

$$q_I = 4\,\frac{\partial\,\mathrm{Im}\,F}{\partial\phi^I} \tag{7.6.4}$$

with $\phi^I = 2\,\mathrm{Re}\,Y^I$.

The function $F(Y,\Upsilon)$ is related to the topological partition function $F_{\rm top}$ by $F(Y,\Upsilon) = -\mathrm{i}F_{\rm top}/(2\pi)$ (see (E.25) and (E.26) with $\Upsilon = -64$). Using $E = -(F_{\rm top} + \bar{F}_{\rm top})$ and

$$Z_{\rm top} = \mathrm{e}^{-F_{\rm top}}\,, \tag{7.6.5}$$

we obtain

$$\mathrm{e}^{\mathcal{S}} = |Z_{\rm top}|^2\,\mathrm{e}^{-L}\,. \tag{7.6.6}$$

For the resolved conifold in type IIA, the free energy E and L, computed from (7.5.12), are given by

$$\begin{aligned} E &= 2\sum_{n=1}^{\infty} n\,\mathrm{Re}\left(\log(1-q^nQ)\right)\,, \\ L &= 2\sum_{n=1}^{\infty} n\left[\mathrm{Im}\left(\frac{q^nQ\log q}{1-q^nQ}\right)\mathrm{Im}\left(\frac{\log Q}{\log q}\right)\right. \\ &\qquad\left. + \,\mathrm{Re}\left(\frac{1}{\log q}\right)\mathrm{Re}\left(\frac{q^nQ\log q(n\log q+\log Q)}{1-q^nQ}\right)\right]. \end{aligned} \tag{7.6.7}$$

By numerically approximating these expressions as before, we find that for real coupling $g_{\rm top}$, the OSV free energy E is minimized at the conifold point $Q = 1$, see Fig. 7.7. The entropy $\mathcal{S}$ is maximized at the conifold, as discussed before.

At the conifold point, E and L as functions of real q have the behaviour displayed in Fig. 7.8. In the limit $g_{\rm top} \to 0$, we thus find that at the conifold point,

$$E = \frac{1}{2} L = -\mathcal{S} \tag{7.6.8}$$

(which holds for the sums, but not term by term). Hence, at the conifold point,

$$\mathrm{e}^{\mathcal{S}} = |Z_{\rm top}|^{-2} \,. \tag{7.6.9}$$

Observe that (7.6.9) is in agreement with (7.6.6) at the conifold point.

The relation (7.6.9) can also be derived from the prepotential $\mathcal{F}^{(0)}$ given in (7.4.1), as follows. At $V = 0$, by (7.2.1) and (7.4.2),

$$\mathcal{S} = 2\pi \, |Y^0|^2 \, \left(\mathcal{F}^{(0)} + \bar{\mathcal{F}}^{(0)}\right) . \tag{7.6.10}$$

Taking Y^0 to be real (which corresponds to real $g_{\rm top}$), and using $F = -\mathrm{i}(Y^0)^2 \, \mathcal{F}^{(0)}$, we compute the OSV free energy (7.6.2) and find precisely

$$E = -\mathcal{S} \,. \tag{7.6.11}$$

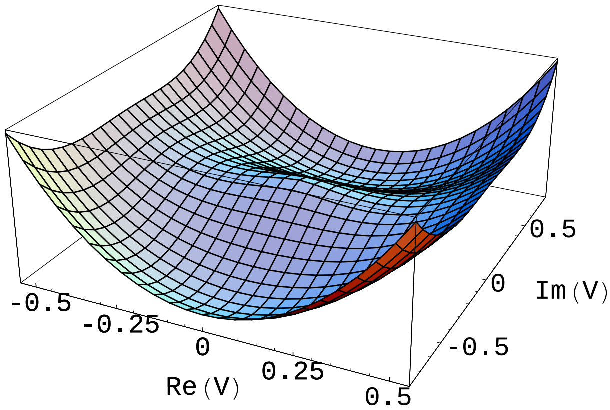

Figure 7.1: e^{-G} exhibits a local maximum at $V = 0$ for negative β.

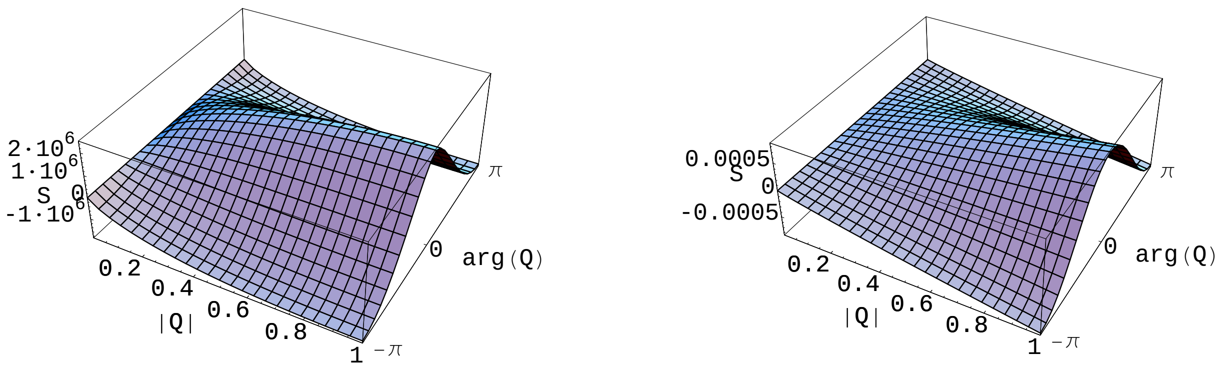

Figure 7.2: $\mathcal{S}$ as a function of Q for $g_{\text{top}} = 0.001$ (left) and $g_{\text{top}} = 10$ (right). The conifold point corresponds to $|Q| = e^{-\operatorname{Re}(2\pi V)} = 1$ and $\arg(Q) = -\operatorname{Im}(2\pi V) = 0$. The number of terms taken into account was 100001 and 1001, respectively.

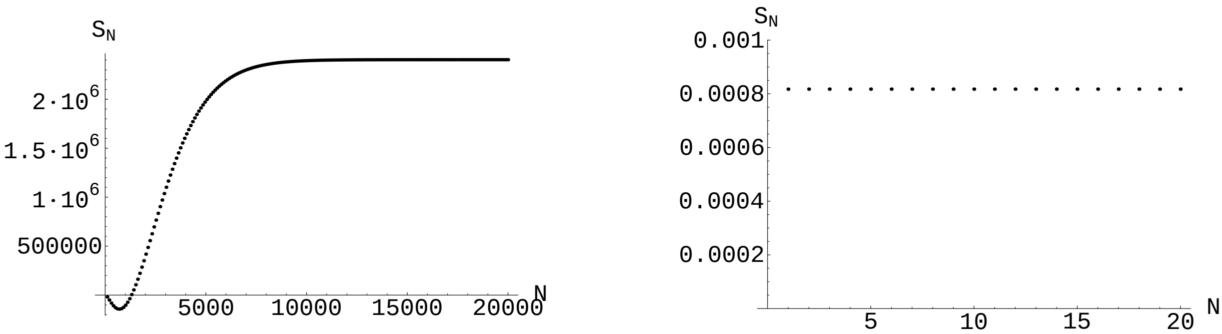

Figure 7.3: Sums of the first N terms (partial sums) of (7.5.13) for $g_{\text{top}} = 0.001$ (left) and $g_{\text{top}} = 10$ (right) at the conifold point $V = 0$ ($Q = 1$).

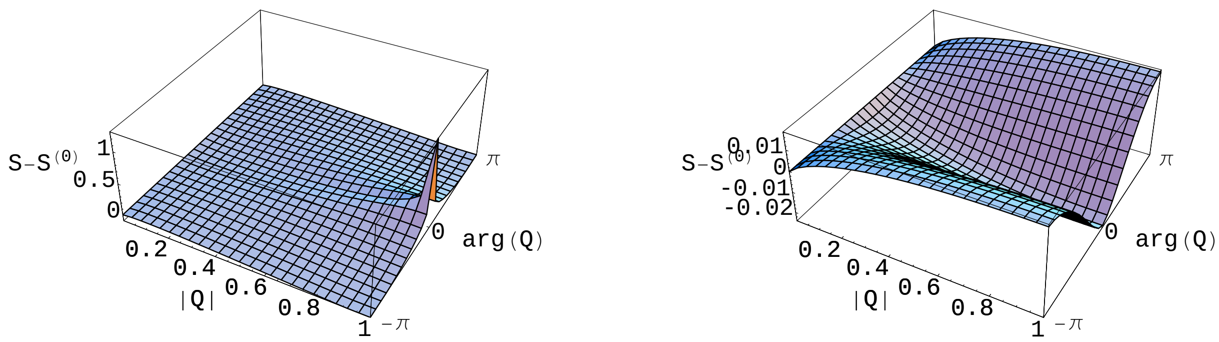

Figure 7.4: $\mathcal{S}-\mathcal{S}^{(0)}$ as a function of Q for $g_{\rm top}=0.001$ (left) and $g_{\rm top}=10$ (right).

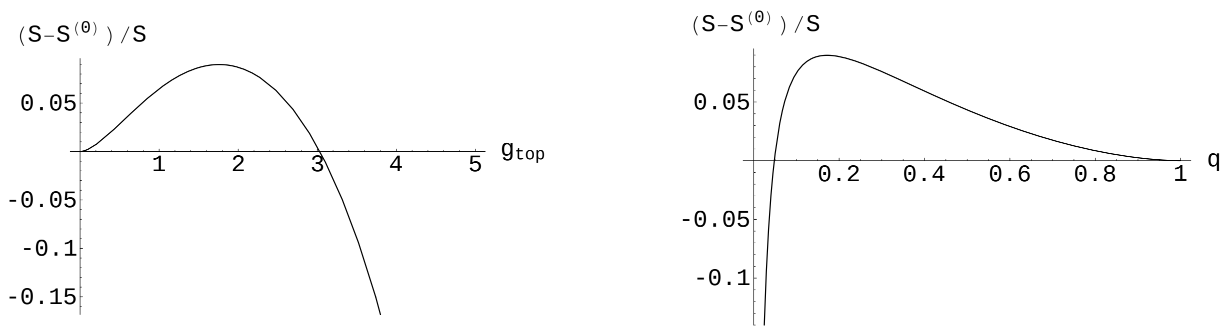

Figure 7.5: The ratio $(\mathcal{S}-\mathcal{S}^{(0)})/\mathcal{S}$ as a function of $g_{\rm top}$ or q at $Q=1$.

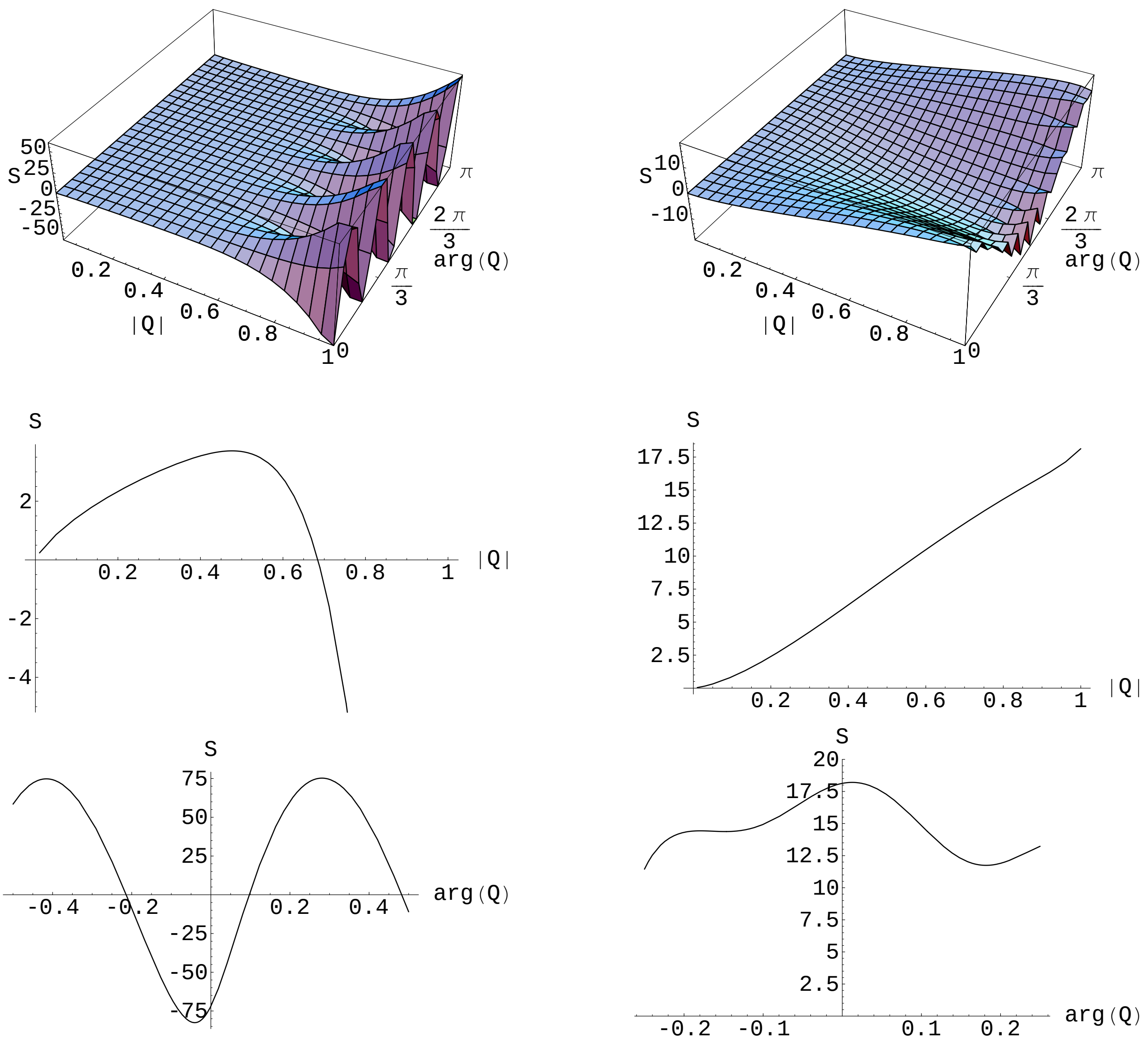

Figure 7.6: $\mathcal{S}$ as a function of Q for complex g_{top}. Left: $g_{\text{top}} = 0.01 - 0.7\text{i}$, right: $g_{\text{top}} = 0.01 + 3\text{i}$. Note that the range of $\arg(Q)$ in the 3-dimensional graphs has been cut by half to exhibit the point $Q = 1$ more clearly (but the periodicity remains 2π). The 2-dimensional graphs show in greater detail the edges of the surfaces closest to the viewer (cross-sections through the surfaces along $\arg(Q) = 0$ and $|Q| = 1$).

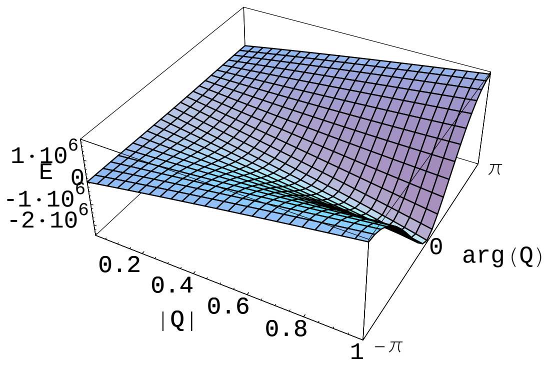

Figure 7.7: E and L as functions of Q for $g_{\text{top}} = 0.001$ (compare with the left graph in Fig. 7.2).

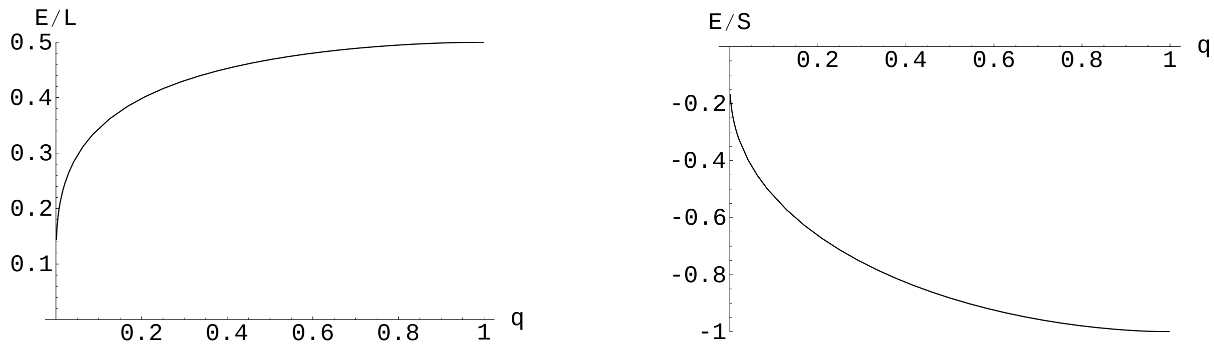

Figure 7.8: Ratios E/L and $E/\mathcal{S}$ at the conifold point, plotted as functions of real q.

Chapter 8

Conclusions and outlook

In this thesis we investigated the attractor phenomenon for extremal single-center black holes in four- and five-dimensional $N = 2$ supergravity with emphasis on the entropy function formalism. We have shown that in the absence of higher-order corrections the entropy function is equivalent to the more intuitive black hole potential, but has the important advantage of being applicable also when the corrections are present. In addition, the attractor equations when directly derived from the entropy function take a more manageable form, which allowed us to find exact solutions for the one-modulus prepotential associated with the conifold. Nevertheless, it is not clear at the moment how to use the entropy function in theories with higher-curvature corrections to verify the stability of the solutions and this question indicates a possible direction of further research. A related challenge is the generalization to multi-center black hole solutions, suggested [107] to be the stable counterparts of the unstable single-center (un-)attractors.

The relationship between four- and five-dimensional black holes, which we employed to generalize the original definition of the entropy function to a class of rotating five-dimensional black holes, shed some new light on the issue of supersymmetry of extremal black hole solutions in four space-time dimensions. The near-horizon analysis provided a concrete example of a solution supersymmetric in five dimensions, which upon dimensional reduction can become non-supersymmetric. This might induce one to suspect that extremal non-supersymmetric solutions can be derived from first-order flow equations (rather than second-order equations of motion) whenever they are related to a supersymmetric solution in one dimension higher, since supersymmetric solutions always admit a first-order description. The derivation of four-dimensional flow equations by dimensional reduction from five dimensions demonstrated however that first-order description in four dimensions is not necessarily contingent on supersymmetry in five dimensions. Complete understanding of this problem is still lacking, it would therefore be desirable to explore it in the future in more detail.

The extremal solutions we found for the one-modulus conifold prepotential in turn attest that, unlike in previously known cases, the non-supersymmetric solution does not have to be related to the corresponding supersymmetric solution by a mere sign reversal of the charges. Incorporating higher-order corrections revealed difficulties in applying the standard perturbative expansion of the generalized prepotential in the graviphoton field strength to black hole attractors. A non-perturbative expression should be used instead, but its closed form is not known, making the calculations much more complicated.

The conifold example also served to prove that the entropic principle would predict the cosmological emergence of supersymmetric flux vacua involving infrared-free theories as more likely than asymptotically-free theories, contrary to what was hoped for by the original authors of the entropic proposal. If we were to apply the entropic principle also to non-supersymmetric vacua, our example would evince that for the conifold a non-supersymmetric vacuum in the vicinity of the conifold point would be even more likely (in agreement with what one may expect: in our world supersymmetry, even if exists, evidently must be broken).

The results pertaining to probabilities should be interpreted with caution: at the moment the entropic principle remains a conjecture concerning only specific flux compactifications resulting in the two-dimensional anti-de Sitter space-time, and so the analysis cannot pretend to unconditionally hold for our universe. Nonetheless the entropic idea seems very compelling, as it is one of the very few proposals for natural vacuum- (or model-) selection in string theory and does not rely on anthropic arguments ('the world has to be as it is, because otherwise life could not develop and so there would not be anyone to ponder the question'). Finding a suitable selection rule would answer some of the most profound conundrums in physics.

Once a mathematical curiosity denied physical significance, black holes have firmly established their due place in science and popular culture alike. In theoretical physics they continue to be a particularly ample source of insight, especially in those areas which are currently, and will surely long remain, beyond the reach of experiments. This is certainly the case with unraveling the structure of quantum gravity. One can only speculate whether they could help us understand the nature of the universe as a whole.

Appendix A

Notation and conventions

We work in Planck's units $c = \hbar = k_{\mathrm{B}} = 1$, occasionally restoring these constants in the formulae. Especially when needed for dimensional reduction, Newton's constants G_4 and G_5 (related (B.25) by the length of the compact dimension) are explicitly written. The space-time metric is taken to have signature $(-,+,+,+)$.

We define the totally antisymmetric Levi-Civita permutation symbol in a pseudo-Riemannian space (and any coordinate system) with

$$\underline{\varepsilon}_{012\ldots} = 1 = -\underline{\varepsilon}^{012\ldots} \tag{A.1}$$

and the corresponding tensor

$$\varepsilon_{\mu_1\mu_2\mu_3\cdots} = \sqrt{|g|}\,\underline{\varepsilon}_{\mu_1\mu_2\mu_3\ldots}\,, \tag{A.2}$$

$$\varepsilon^{\mu_1\mu_2\mu_3\cdots} = \frac{1}{\sqrt{|g|}}\underline{\varepsilon}^{\mu_1\mu_2\mu_3\cdots}\,. \tag{A.3}$$

Here the ellipsis has been used to accommodate arbitrary but finite dimensionality. Raising and lowering of the indices proceeds, as for any other proper tensor, with the metric (this is the convention adopted by [119] and [135]). The minus sign (A.1) is then necessary due to the Lorentzian signature, and consequently negative determinant, of the metric—let us recall that for any square matrix A,

$$\underline{\varepsilon}_{a_1a_2a_3\ldots}A_1^{a_1}A_2^{a_2}A_3^{a_3}\cdots = \det A\,. \tag{A.4}$$

Note that some texts, for instance [33] adopt the definition without the minus in equation (A.1), then an extra minus sign appears in (A.3). Other, such as [154] define ε in tangent space in the same way as our Levi-Civita symbol, and use the notation $\varepsilon_{\mu_1\mu_2\mu_3\ldots}$ for the tensor *density* $e^{-1}e^{a_1}_{\mu_1}e^{a_2}_{\mu_2}e^{a_3}_{\mu_3}\cdots\varepsilon_{a_1a_2a_3\ldots}$ (e^a_μ is the vielbein), designed so that it remains invariant in any coordinate system, at the expense of the fact that the analogously specified density with upper indices is not obtained by raising the indices on ε with the metric. Since

$$e = \det e^a_\mu = \sqrt{-g} \tag{A.5}$$

that tensor density evaluates by (A.4) to what we have denoted $\underline{\varepsilon}_{\mu_1\mu_2\mu_3\dots}$. The disadvantage of our convention is that, as we have just seen and perhaps contrary to what the symbols might suggest, $\underline{\varepsilon}_{\mu_1\mu_2\mu_3\dots}$ does not correspond to $\underline{\varepsilon}_{a_1a_2a_3\dots}$ converted to world indices (instead, in our notation, $\varepsilon_{\mu_1\mu_2\mu_3\dots}$ does).

Hodge dual of a p-form $\omega = \frac{1}{p!}\omega_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ is defined as

$$(\star\omega)_{\mu_1\dots\mu_q} = \frac{1}{p!}\varepsilon_{\mu_1\dots\mu_q}{}^{\nu_1\dots\nu_p}\omega_{\nu_1\dots\nu_p}\,, \tag{A.6}$$

where $p + q = d$ (the dimension of space-time).

For the Levi–Civita (Christoffel) connection we use

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{A.7}$$

and for the Riemann tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu\Gamma^{\rho}_{\nu\sigma} - \partial_\nu\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,. \tag{A.8}$$

The Ricci tensor and scalar are

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}\,, \qquad \mathcal{R} = R^{\mu}{}_{\mu}\,. \tag{A.9}$$

Appendix B

Very special geometry and dimensional reduction

Here we collect various elements of $N = 2$ supergravity theories in four and in five dimensions. We also review the reduction of the five-dimensional action based on very special geometry to the four-dimensional action based on special geometry. This will explain our conventions, which differ slightly from the ones used in [94, 59, 93, 14]. For notational simplicity, we drop the subscripts on the five- and four-dimensional gauge fields.

The five-dimensional $N = 2$ supergravity action is based on the cubic polynomial [94]

$$\mathcal{V} = \frac{1}{6} C_{ABC} X^A X^B X^C \,, \tag{B.1}$$

where the $X^A(\phi)$ are real scalar fields satisfying the constraint $\mathcal{V} = \text{constant}$.[1] In our conventions $2\mathcal{V} = 1$ [32].

From the definitions

$$G_{AB}(X) = -\frac{1}{2} \partial_A \partial_B \log \mathcal{V}|_{\mathcal{V}=\text{constant}} \,, \tag{B.2}$$

and

$$X_A = \frac{1}{6} C_{ABC} X^B X^C \,, \tag{B.3}$$

it follows that

$$G_{AB} = \frac{1}{\mathcal{V}} \left(-\frac{1}{2} C_{ABC} X^C + \frac{9}{2} \frac{X_A X_B}{\mathcal{V}} \right) \tag{B.4}$$

$$X^A X_A = \mathcal{V} \tag{B.5}$$

and

$$X_A = \frac{2\mathcal{V}}{3} G_{AB} X^B \,, \qquad X^A = \frac{3}{2\mathcal{V}} G^{AB} X_B \,, \tag{B.6}$$

[1] In an effective field theory description of M-theory compactifications on Calabi–Yau three-folds $\mathcal{V}$ corresponds to the volume of the Calabi–Yau, which belongs to a hypermultiplet, and as hypermultiplets decouple, it sets a constraint on the vector multiplet scalars.

where $G_{AB}\, G^{BC} = \delta_A^C$.

As $\mathcal{V}$ is a constant, i.e. $\partial_i \mathcal{V} = 0$ and $\partial_i \mathcal{V} = \frac{3}{2}(\partial_i X_A) X^A$, we also have

$$(\partial_i X_A) X^A = 0 \tag{B.7}$$

and consequently, by the definitions (B.3) and (B.4),

$$G_{AB} \partial_i X^B = -\frac{3}{2\mathcal{V}} \partial_i X_A \,. \tag{B.8}$$

Here $\partial_i X^A = \frac{\partial}{\partial \phi^i} X^A(\phi)$, where ϕ^i denote the physical scalar fields.

The metric on the scalar manifold is defined by

$$g_{ij} = G_{AB} \partial_i X^A \partial_j X^B \,. \tag{B.9}$$

The index structure dictates that

$$g^{ij} \partial_i X^A \partial_j X^B = a(G^{AB} - b X^A X^B) \tag{B.10}$$

with constant coefficients a and b. Contraction with X_B must vanish because of eq. (B.7). From eq. (B.6) we then have

$$0 = g^{ij} \partial_i X^A \partial_j X^B X_B = a(\tfrac{2}{3} \mathcal{V} X^A - b X^A \mathcal{V}) \,, \tag{B.11}$$

which fixes $b = \frac{2}{3}$. To determine the coefficient a we contract eq. (B.10) with G_{AB}, invoke the definition (B.9) and observe that the number of physical scalars is one less than the number of vector fields, n,

$$n - 1 = g_{ij} g^{ij} = G_{AB} g^{ij} \partial_i X^A \partial_j X^B = a(G_{AB} G^{AB} - \tfrac{2}{3} G_{AB} X^A X^B) = a(n-1) \,. \tag{B.12}$$

This implies that $a = 1$, so that finally

$$g^{ij} \partial_i X^A \partial_j X^B = G^{AB} - \frac{2}{3} X^A X^B \,. \tag{B.13}$$

The bosonic part of the five-dimensional $N = 2$ supergravity action reads

$$\begin{aligned} S_5 \;=\; & \frac{1}{8\pi G_5} \left[\int d^5x \sqrt{-G} \left(\frac{1}{2} \mathcal{R}_G - \frac{1}{2}\, G_{AB}\, \partial_M X^A\, \partial^M X^B - \frac{1}{4} G_{AB}\, F^A_{MN} F^{BMN} \right) \right. \\ & \left. - \frac{1}{6} \int C_{ABC}\, F^A \wedge F^B \wedge A^C \right] \,, \end{aligned} \tag{B.14}$$

where G denotes the determinant of the spacetime metric in five dimensions.

The four-dimensional $N = 2$ supergravity action corresponding to (B.1) is based on the prepotential [58, 57]

$$F(Y) = -\frac{1}{6} \frac{C_{ABC}\, Y^A Y^B Y^C}{Y^0} \,, \tag{B.15}$$

where the Y^I are complex scalar fields ($I = 0, A$). The four-dimensional gauge couplings $\mathcal{N}_{IJ}$ are given by

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2\mathrm{i}\frac{\mathrm{Im}\, F_{IK}\ \mathrm{Im}\, F_{JL}\, Y^K Y^L}{\mathrm{Im}\, F_{MN}\, Y^M Y^N} \ , \tag{B.16}$$

where $F_I = \partial F/\partial Y^I$, $F_{IJ} = \partial^2 F/\partial Y^I \partial Y^J$.

The four-dimensional physical scalar fields z^A are

$$z^A = \frac{Y^A}{Y^0} \ . \tag{B.17}$$

The four-dimensional Kähler potential $K(z, \bar{z})$ derived from (B.15) is

$$\mathrm{e}^{-K} = \frac{\mathrm{i}}{6} C_{ABC} \, (z^A - \bar{z}^A) \, (z^B - \bar{z}^B) \, (z^C - \bar{z}^C) \ . \tag{B.18}$$

The Kähler metric $g_{A\bar{B}} = \frac{\partial}{\partial z^A}\frac{\partial}{\partial \bar{z}^B} K$ derived from (B.18) satisfies the relation

$$g_{A\bar{B}} = \frac{1}{2}\, \mathrm{e}^{4\phi}\, G_{AB} \ . \tag{B.19}$$

The four-dimensional quantity $Z(Y)$ is given by

$$Z(Y) = p^I\, F_I(Y) - q_I\, Y^I \ , \tag{B.20}$$

where $F_I = \partial F(Y)/\partial Y^I$. The associated four-dimensional complex central charge Z_4 reads

$$Z_4 = \mathrm{e}^{K/2}\, Z(Y)/Y^0 \ . \tag{B.21}$$

For a prepotential of the form (B.15) we obtain

$$Z(Y) = Y^0 \left(\frac{p^0}{6}\, C_{ABC} z^A z^B z^C - \frac{p^A}{2}\, C_{ABC} z^B z^C - q_0 - q_A z^A \right) \ , \tag{B.22}$$

and hence,

$$Z_4 = \mathrm{e}^{K/2} \left(\frac{p^0}{6}\, C_{ABC} z^A z^B z^C - \frac{p^A}{2}\, C_{ABC} z^B z^C - q_0 - q_A z^A \right) \ . \tag{B.23}$$

The bosonic part of the four-dimensional $N = 2$ supergravity action reads

$$\begin{aligned} S_4 \ = \ \frac{1}{8\pi G_4} \int d^4x \,\sqrt{-g}\, \Big(& \frac{1}{2}\mathcal{R}_g - g_{A\bar{B}}\, \partial_\mu z^A \partial^\mu \bar{z}^B + \frac{1}{4}\, \mathrm{Im}\mathcal{N}_{IJ}\, F^I_{\mu\nu}\, F^{J\mu\nu} \\ & -\frac{1}{4}\mathrm{Re}\mathcal{N}_{IJ}\, F^I_{\mu\nu} \star F^{J\mu\nu} \Big) \ , \end{aligned} \tag{B.24}$$

where g denotes the determinant of the spacetime metric in four dimensions.

Now we perform the reduction of (B.14) along x^5 down to four dimensions using (5.2.1). We take the various fields to be independent of the fifth coordinate x^5. Setting $x^5 = R\psi\,,\ 0 \leq \psi < 4\pi$, we use that the five- and four-dimensional Newton constants are related by

$$G_5 = 4\pi\, R\, G_4\ . \tag{B.25}$$

Reducing the gauge kinetic terms $G_{AB}F^AF^B$ gives rise to a scalar kinetic term of the form

$$-\frac{1}{4}\sqrt{-G}\, G_{AB}\, F^A F^B \rightarrow -\frac{1}{2}\,\sqrt{-g}\,\mathrm{e}^{4\phi}\, G_{AB}\,\partial_\mu C^A \partial^\mu C^B\ , \tag{B.26}$$

whereas reducing $\mathcal{R}_G - G_{AB}\partial_M X^A \partial^M X^B$ gives rise to scalar kinetic terms for $\hat{X}^A = \mathrm{e}^{-2\phi}\, X^A$,

$$\sqrt{-G}\left(\frac{1}{2}\mathcal{R}_G - \frac{1}{2}G_{AB}\partial_M X^A\partial^M X^B\right) \rightarrow \sqrt{-g}\left(\frac{1}{2}\mathcal{R}_g - \frac{1}{2}\mathrm{e}^{4\phi}\, G_{AB}\,\partial_\mu \hat{X}^A\,\partial^\mu \hat{X}^B\right)\ . \tag{B.27}$$

Eqs. (B.26) and (B.27) can be combined into

$$\sqrt{-g}\left(\frac{1}{2}\mathcal{R}_g - \frac{1}{2}\,\mathrm{e}^{4\phi}\, G_{AB}\,\partial_\mu z^A\,\partial^\mu \bar{z}^{\bar{B}}\right)\ , \tag{B.28}$$

where z^A is defined as in (5.2.2):

$$z^A = C^A + \mathrm{i}\hat{X}^A\ . \tag{B.29}$$

Using (B.18) we compute $g_{A\bar{B}} = \frac{1}{2}\mathrm{e}^{4\phi}\, G_{AB}$, and hence (B.28) can be written as

$$\sqrt{-g}\left(\frac{1}{2}\mathcal{R}_g - g_{A\bar{B}}\,\partial_\mu z^A\,\partial^\mu \bar{z}^{\bar{B}}\right)\ . \tag{B.30}$$

In addition, reducing $\mathcal{R}_G$ and $G_{AB}F^AF^B$ also gives rise to the four-dimensional gauge kinetic terms

$$\begin{aligned}
\sqrt{-G}\,\mathcal{R}_G \quad &\rightarrow \quad \sqrt{-g}\left(\mathcal{R}_g - \frac{1}{4}\mathrm{e}^{-6\phi}\, F^0\, F^0\right)\ , \qquad\qquad \text{(B.31)}\\
-\frac{1}{2}\,\sqrt{-G}\, G_{AB}\, F^A\, F^B \quad &\rightarrow \quad -\frac{1}{2}\,\sqrt{-g}\,\mathrm{e}^{-2\phi}\, G_{AB}\left[F^AF^B - 2C^B\, F^A\, F^0 + C^A\, C^B\, F^0 F^0\right]\ .
\end{aligned}$$

This we compare with $\mathrm{Im}\mathcal{N}_{IJ}\, F^I\, F^J$ in four dimensions. To this end, we compute the couplings $\mathcal{N}_{IJ}$ for the prepotential (B.15) and we express them in terms of the fields $\hat{X}^A$ and C^A using (5.2.2),

$$\begin{aligned}
\mathcal{N}_{00} &= -\frac{1}{3}\, C_{ABC}\, C^A C^B C^C - \mathrm{i}\left[2\,\mathrm{e}^{-2\phi}\,\mathcal{V}\, G_{AB}\, C^A C^B + \hat{\mathcal{V}}\right]\ ,\\
\mathcal{N}_{0A} &= \frac{1}{2}\, C_{ABC}\, C^B C^C + 2\mathrm{i}\,\mathrm{e}^{-2\phi}\,\mathcal{V}\, G_{AB}\, C^B\ ,\\
\mathcal{N}_{AB} &= -\, C_{ABC}\, C^C - 2\mathrm{i}\,\mathrm{e}^{-2\phi}\,\mathcal{V}\, G_{AB}\ ,
\end{aligned} \tag{B.32}$$

where

$$\hat{\mathcal{V}} = \hat{X}_A\,\hat{X}^A \quad , \quad \hat{X}_A = \frac{1}{6}C_{ABC}\hat{X}^B\hat{X}^C \quad , \quad \mathrm{e}^{-6\phi} = \mathcal{V}^{-1}\,\hat{\mathcal{V}}\ . \tag{B.33}$$

Hence we find that the sum of the field strength terms on the right hand side of (B.31) equals

$$\frac{1}{4\mathcal{V}}\,\mathrm{Im}\mathcal{N}_{IJ}\,F^I\,F^J\ . \tag{B.34}$$

Thus, requiring the matching of the five-dimensional gauge kinetic term $-\frac{1}{4}G_{AB}F^AF^B$ in (B.14) with the four-dimensional gauge kinetic term $\frac{1}{4}\mathrm{Im}\mathcal{N}_{IJ}\,F^I\,F^J$ in (B.24) yields the normalization condition

$$2\,\mathcal{V} = 1\ . \tag{B.35}$$

Next, we reduce the five-dimensional Chern–Simons term $C_{ABC}\,F^A \wedge F^B \wedge A^C$ in (B.14). Using (5.2.1), we first observe that $C_{ABC}\,F^A \wedge F^B \wedge A^C_\psi d\psi$ can be expressed in terms of four-dimensional gauge fields as,

$$\begin{aligned} C_{ABC}\,F^A \wedge F^B \wedge A^C_\psi\,d\psi \;&=\; R\,C_{ABC}\,[\,C^A\,F^B \wedge F^C - C^A\,C^B\,F^C \wedge F^0 \\ &\qquad + \frac{1}{3}C^A\,C^B\,C^C\,F^0 \wedge F^0\,] \wedge d\psi\ , \end{aligned} \tag{B.36}$$

up to a total derivative term. The field strengths on the right hand side are four-dimensional, and $A^C_\psi = R\,C^C$. Using

$$C_{ABC}\,C^A\,F^B \wedge F^C \wedge d\psi = -\frac{1}{2}d\psi\,d^4x\,\sqrt{-g}\,C_{ABC}\,C^A\,F^B\star F^C\ , \tag{B.37}$$

and similarly for the other terms in (B.36), we obtain (up to a total derivative)

$$C_{ABC}\,F^A \wedge F^B \wedge A^C_\psi\,d\psi = \frac{1}{2}\,R\,d\psi\,d^4x\,\sqrt{-g}\,\mathrm{Re}\mathcal{N}_{IJ}\,F^I\star F^J\ , \tag{B.38}$$

where we used (B.32). Then, using

$$C_{ABC}\,F^A \wedge F^B \wedge A^C = 3C_{ABC}\,F^A \wedge F^B \wedge A^C_\psi\,d\psi\ , \tag{B.39}$$

which holds up to a total derivative term, we obtain

$$\frac{1}{6G_5}\int C_{ABC}\,F^A \wedge F^B \wedge A^C = \frac{1}{4G_4}\int d^4x\,\sqrt{-g}\,\mathrm{Re}\mathcal{N}_{IJ}\,F^I\star F^J\ . \tag{B.40}$$

Thus, dimensional reduction of (B.14) yields (B.24), up to boundary terms.

Appendix C

Evaluation of the action in five dimensions

The square root of the determinant of the metric (6.2.2), (6.2.4) is

$$\sqrt{-G} = \frac{Rr^2 N \sin\theta}{f} \; . \tag{C.1}$$

The inverse metric reads

$$G^{MN} = \begin{pmatrix} -1/f^2 + f\left(\frac{Nw_5^2}{R^2} + \frac{w_4^2}{r^2N}\frac{\cos^2\theta}{\sin^2\theta}\right) & 0 & 0 & -\frac{fw_4}{r^2N}\frac{\cos\theta}{\sin^2\theta} & f\left(-\frac{Nw_5}{R^2} + \frac{p^0w_4}{r^2N}\frac{\cos^2\theta}{\sin^2\theta}\right) \\ 0 & f/N & 0 & 0 & 0 \\ 0 & 0 & f/(r^2N) & 0 & 0 \\ -\frac{fw_4}{r^2N}\frac{\cos\theta}{\sin^2\theta} & 0 & 0 & \frac{f}{r^2N}\frac{1}{\sin^2\theta} & -\frac{p^0f}{r^2N}\frac{\cos\theta}{\sin^2\theta} \\ f\left(-\frac{Nw_5}{R^2} + \frac{p^0w_4}{r^2N}\frac{\cos^2\theta}{\sin^2\theta}\right) & 0 & 0 & -\frac{p^0f}{r^2N}\frac{\cos\theta}{\sin^2\theta} & f\left(\frac{N}{R^2} + \frac{(p^0)^2}{r^2N}\frac{\cos^2\theta}{\sin^2\theta}\right) \end{pmatrix} \tag{C.2}$$

and the Ricci scalar is

$$\begin{aligned} \mathcal{R} = \frac{1}{2r^4fN^3}\Big[& f^5N\left((p^0w_5+w_4)^2 + \frac{r^4N^2}{R^2}w_5'^2 + r^2\cot^2\theta\, w_4'^2\right) \\ & - 5r^4N^2f'^2 + 2r^3fN^2(2f'+rf'') \\ & - f^2\left((p^0)^2R^2 - r^4N'^2 + 2r^3N(2N'+rN'')\right)\Big], \end{aligned} \tag{C.3}$$

where the prime denotes the derivative with respect to the radial coordinate r, i.e. $' = \partial/\partial r$. The last line in the expression (C.3) above vanishes on account of the definiton (6.2.5).

Inserting the ansatz

$$A_5^A = \chi^A(r)\,(dt+w) + p^A\,\cos\theta\, d\varphi \tag{C.4}$$

into the gauge kinetic term in (6.3.1) yields

$$
\begin{aligned}
-\frac{1}{2}\sqrt{-G}\,G_{AB}F^A_{MN}F^{BMN} = &-\frac{R\sin\theta}{r^2f^2N}G_{AB}\Big[f^3p^Ap^B + f^3(p^0w_5+w_4)(p^A\chi^B+\chi^Ap^B) \\
&+ f^3\Big((p^0w_5+w_4)^2 + \frac{r^4N^2}{R^2}w_5'^2 + r^2\cot^2\theta\, w_4'^2\Big)\chi^A\chi^B \\
&- r^4N\chi'^A\chi'^B\Big]\ .
\end{aligned} \tag{C.5}
$$

The Chern–Simons term in (6.3.1) evaluates to

$$
\begin{aligned}
-\frac{1}{6\mathcal{V}}C_{ABC}\,F^A\wedge F^B\wedge A^C \;=\; &\frac{1}{3\mathcal{V}}\sin\theta\, C_{ABC}\left[p^A+(p^0w_5+w_4)\chi^A\right]\chi^B\chi^Cw_5' \\
&dt\wedge dr\wedge d\theta\wedge d\varphi\wedge d\psi\ .
\end{aligned} \tag{C.6}
$$

Appendix D

Flow equations for the complex scalar fields

Here, we derive the flow equation for the complex scalars z^A. We set $2\mathcal{V} = 1$. We begin by first considering the case discussed in subsection 6.4.1, so that $z^A = \mathrm{i}\hat{X}^A$. Using (6.4.3) we obtain

$$\partial_\tau \hat{X}^A = \partial_\tau (f^{-2}\mathrm{e}^{2U}) X^A + \mathrm{e}^{-2\phi}\, \partial_\tau \phi^i\, \partial_i X^A\,, \tag{D.1}$$

where ∂_i stands for the derivatives with respect to the physical scalars ϕ^i in five dimensions. From (6.3.12a) and (6.4.5) we get

$$\begin{aligned}\partial_\tau (f^{-2}\mathrm{e}^{2U}) &= \frac{4}{3}\mathrm{e}^{2U} f^{-1} |Q_A X^A| - 2 f^{-2}\mathrm{e}^{3U}\partial_\tau \mathrm{e}^{-U} \\ &= \frac{4}{3} f |Q_A \hat{X}^A| - \frac{1}{4} f^{-2}\mathrm{e}^{3U}\left(\mathrm{e}^{-K/2}p^0 + 4\mathrm{e}^{K/2}|Q_A \hat{X}^A|\right)\,.\end{aligned} \tag{D.2}$$

Using the flow equation for ϕ^i given in (6.3.12c) we obtain

$$\partial_\tau \phi^i\, \partial_i X^A = -s\, f\, \mathrm{e}^{4\phi}\left(\frac{1}{2}g^{A\bar{B}} - \frac{2}{3}\hat{X}^A \hat{X}^B\right) Q_B\,, \tag{D.3}$$

where we also employed (B.13) and (B.19). Inserting (D.2) and (D.3) into (D.1) and using (6.4.3) and (6.2.1) yields

$$\partial_\tau \hat{X}^A = \mathrm{e}^U \left(2s\,\mathrm{e}^{K/2}\hat{X}^A\hat{X}^B Q_B - \frac{1}{4}\mathrm{e}^{-K/2}\hat{X}^A p^0 - s\,\mathrm{e}^{K/2} g^{A\bar{B}} Q_B\right)\,. \tag{D.4}$$

With the help of (B.6), (B.19) and (6.2.1) we can express $\hat{X}^A$ as

$$\hat{X}^A = \mathrm{e}^K g^{A\bar{B}} C_{BCD}\hat{X}^C\hat{X}^D\,, \tag{D.5}$$

so that equation (D.4) becomes

$$\partial_\tau \hat{X}^A = \mathrm{e}^U g^{A\bar{B}}\left(2s\,\mathrm{e}^{3K/2} C_{BCD}\hat{X}^C\hat{X}^D Q_E \hat{X}^E - \frac{1}{4}\mathrm{e}^{K/2} C_{BCD}\hat{X}^C\hat{X}^D p^0 - s\,\mathrm{e}^{K/2} Q_B\right)\,. \tag{D.6}$$

This expression precisely agrees with the flow equation for $\partial_\tau z^A$ given in (6.4.1) and based on (6.4.7). Namely, evaluating

$$\begin{aligned}
\partial_\tau z^A &= 2\mathrm{e}^U g^{A\bar{B}} \partial_{\bar{B}} W_4 \\
&= -\frac{1}{4}\mathrm{e}^U g^{A\bar{B}} \partial_{\bar{B}} \left(\mathrm{e}^{-K/2} p^0 + 8\mathrm{e}^{K/2} \left|Q_A \hat{X}^A\right|\right) \\
&= -\mathrm{e}^U g^{A\bar{B}} \left[\left(\frac{1}{8}\,\mathrm{e}^{K/2} p^0 - s\,\mathrm{e}^{3K/2} Q_C \hat{X}^C\right) \partial_{\bar{B}}(\mathrm{e}^{-K}) + \mathrm{i} s\,\mathrm{e}^{K/2} Q_B\right] \\
&= \mathrm{i}\mathrm{e}^U g^{A\bar{B}} \left[\left(-\frac{1}{4}\,\mathrm{e}^{K/2} p^0 + 2s\,\mathrm{e}^{3K/2} Q_E \hat{X}^E\right) C_{BCD} \hat{X}^C \hat{X}^D - s\,\mathrm{e}^{K/2} Q_B\right]
\end{aligned} \tag{D.7}$$

shows that (D.7) precisely equals (D.6). In deriving (D.7) we used the relations

$$\partial_{\bar{B}} \left|Q_A \hat{X}^A\right| = s\, Q_A \partial_{\bar{B}} \hat{X}^A = \frac{\mathrm{i}}{2} s\, Q_B \tag{D.8}$$

and (from (B.18))

$$\partial_{\bar{A}}(\mathrm{e}^{-K}) = -\frac{\mathrm{i}}{2} C_{ABC}(z^B - \bar{z}^B)(z^C - \bar{z}^C) = 2\mathrm{i} C_{ABC} \hat{X}^B \hat{X}^C\,. \tag{D.9}$$

Next, we consider the $s = -1$ solution described in subsection 6.4.2. Proceeding as above, we compute

$$\begin{aligned}
\partial_\tau z^A &= \partial_\tau \left(\alpha\, \mathrm{e}^{-2\phi}\, X^A\right) \\
&= \Big(-\frac{2}{3}\alpha f^2 N^{-1} \mathrm{e}^{-2U} \left|W_5\right| - p^0 \alpha f N^{-2} \mathrm{e}^{-2U} \\
&\quad + \frac{\mathrm{i}}{2}\Delta^{1/2} f N^{-1} \mathrm{e}^{-U+3\phi} \left(p^0\, \Delta\, \mathrm{e}^{-6\phi} - 2Q_B \hat{X}^B\right)\Big) X^A \\
&\quad + \alpha f \mathrm{e}^{2\phi} \left(\frac{1}{2} g^{A\bar{B}} - \frac{2}{3}\hat{X}^A \hat{X}^B\right) Q_B\,.
\end{aligned} \tag{D.10}$$

Comparing with $\partial_{\bar{B}} W_4$ based on (6.4.33),

$$\begin{aligned}
\partial_{\bar{B}} W_4 &= \mathrm{i}\mathrm{e}^{3K/2} \Delta^{1/2} C_{BEF} \hat{X}^E \hat{X}^F \left(\frac{1}{8} p^0\, \Delta\, \mathrm{e}^{-K} - Q_A \hat{X}^A\right) \\
&\quad - \frac{1}{2}\alpha\, \mathrm{e}^{K/2} \Delta^{-1/2} \left(\frac{1}{2} p^0\, \Delta\, C_{BEF} \hat{X}^E \hat{X}^F - Q_B\right)\,,
\end{aligned} \tag{D.11}$$

we find that (6.4.1) precisely holds.

Appendix E

Normalization of the generalized prepotential

In type IIA, $F^{(0)}(Y)$ has the following expansion [25, 24, 98, 99]

$$F^{(0)}(Y) = (Y^0)^2 \left(-\frac{C_{ABC} z^A z^B z^C}{6} + h(z) - \frac{1}{(2\pi \mathrm{i})^3} \sum_{d_A} n_d \, \mathrm{Li}_3(\mathrm{e}^{2\pi \mathrm{i} d_A z^A}) \right), \tag{E.1}$$

where C_{ABC} are the intersection numbers and n_d denote rational instanton numbers. The quadratic polynomial $h(z)$ contains a constant term given by $\mathrm{i}\,\chi\,\zeta(3)/(2(2\pi)^3)$, where χ denotes the Euler characteristic. Using (7.2.2) yields

$$\mathcal{F}^{(0)}(z) = -\frac{\mathrm{i}}{6} C_{ABC} z^A z^B z^C + \mathrm{i} h(z) + \frac{1}{(2\pi)^3} \sum_{d_A} n_d \, \mathrm{Li}_3(\mathrm{e}^{2\pi \mathrm{i}\, d_A z^A}) \,. \tag{E.2}$$

Observe that in the limit of large positive $\mathrm{Im}\, z^A$, $\mathrm{e}^{-G(z,\bar{z})}$ (computed from (7.5.3)) is positive, as it should.

The coupling function $F^{(1)}(Y)$ is given by[1] [19, 24, 99]

$$F^{(1)}(Y) = -\frac{\mathrm{i}}{256\pi} \left[-\frac{2\pi \mathrm{i}}{12} c_{2A}\, z^A - \sum_{d_A} \left(2 n_d^{(1)} \log(\eta(\mathrm{e}^{2\pi \mathrm{i} d_A z^A})) + \frac{n_d}{6} \log(1 - \mathrm{e}^{2\pi \mathrm{i} d_A z^A}) \right) \right]. \tag{E.3}$$

Consider a singularity associated with the vanishing of one of the moduli $T^A = -\mathrm{i} z^A$. We denote this modulus by V. The other moduli are taken to be large, so that we may approximate

$$\sum_{d_A} n_d \, \mathrm{Li}_3(\mathrm{e}^{-2\pi d_A T^A}) \approx \sum_{d_V} n_{0,0,\ldots,d_V} \, \mathrm{Li}_3(\mathrm{e}^{-2\pi d_V V}) \,. \tag{E.4}$$

Let us assume that that the only non-vanishing instanton number $n_{0,0,\ldots,d_V}$ is the one with $d_V = 1$. Using

$$\mathrm{Li}_3(\mathrm{e}^{-x}) = \zeta(3) - \frac{\pi^2}{6} x + \left(\frac{3}{4} - \frac{1}{2} \log x \right) x^2 + \mathcal{O}(x^3) \,, \tag{E.5}$$

[1] We use the normalization given in [114].

we find that for $V \approx 0$, the function $\mathcal{F}^{(0)}$ can be approximated by

$$\mathcal{F}^{(0)} = -\frac{C_{ABC}T^A T^B T^C}{6} + \mathrm{i}\tilde{h}(\mathrm{i}T) + \frac{\beta}{2\pi} V^2 \log V \,, \tag{E.6}$$

where

$$\beta = -\frac{n_{0,0,\ldots,1}}{2} \,. \tag{E.7}$$

The instanton number $n_{0,0,\ldots,1}$ counts the difference of charged hyper- and vector multiplets becoming massless at $V = 0$, i.e.

$$n_{0,0,\ldots,1} = n_{\mathrm{h}} - n_{\mathrm{v}} \,. \tag{E.8}$$

Note that the quadratic polynomial $\mathrm{i}\tilde{h}$ contains a constant term a given by

$$a = (2-\chi)\,\frac{\zeta(3)}{2(2\pi)^3} \,. \tag{E.9}$$

Similarly, we find that near $V = 0$,

$$F^{(1)}(Y) = -\frac{\mathrm{i}}{256\pi}\left[\frac{2\pi}{12} c_{2A}\, T^A - 2\sum_{d_A} n_d^{(1)} \log(\eta(\mathrm{e}^{-2\pi d_A T^A})) + \frac{\beta}{3}\log V\right] . \tag{E.10}$$

Therefore, near $V = 0$ we obtain

$$\begin{aligned} F(Y,\Upsilon) &= \sum_{g=0}^{\infty} F^{(g)}(Y)\Upsilon^g = F^{(0)}(Y) + F^{(1)}(Y)\Upsilon + \cdots \\ &= -\frac{\mathrm{i}(Y^0)^2}{2\pi}\beta\, V^2 \log V - \frac{\mathrm{i}\Upsilon}{64\cdot 12\pi}\beta \,\log V + \cdots \,, \end{aligned} \tag{E.11}$$

where we displayed only the terms proportional to $\log V$.

The function $F(Y,\Upsilon)$ is proportional to the topological free energy $F_{\mathrm{top}}(g_{\mathrm{top}}, z)$. In order to determine the precise relation between supergravity and topological string quantities, we consider the case of the resolved conifold in type IIA. First, observe that for this case the functions $\mathcal{F}^{(0)}$ and $F^{(1)}$ are given by [88]

$$\begin{aligned} \mathcal{F}^{(0)} &= -\frac{V^3}{12} + \mathrm{i}h(\mathrm{i}V) + \frac{1}{(2\pi)^3}\sum_n \frac{\mathrm{e}^{-2\pi n V}}{n^3} \,, \\ F^{(1)} &= -\frac{\mathrm{i}}{256\pi}\left[\frac{2\pi}{12}\, c_2\, V - \frac{1}{6}\log(1-\mathrm{e}^{-2\pi V})\right] , \end{aligned} \tag{E.12}$$

where $h(\mathrm{i}V)$ denotes a quadratic polynomial in V, and where $c_2 = -1$. Observe that $\chi = 2$, so that $a = 0$. Using (E.5) and (E.9), it follows that near $V = 0$,

$$\begin{aligned} \mathcal{F}^{(0)} &\approx -\frac{1}{4\pi} V^2 \log V \,, \\ F^{(1)} &\approx \frac{\mathrm{i}}{128\cdot 12\pi}\log V \,. \end{aligned} \tag{E.13}$$

Then, comparison with (E.6) and (E.10) yields $\beta = -1/2$.

The topological free energy for the resolved conifold reads [87, 92]

$$F_{\rm top} = -\sum_{n=1}^{\infty} n \log(1 - q^n Q)\,, \tag{E.14}$$

where $q = \mathrm{e}^{-g_{\rm top}}$ and $Q = \mathrm{e}^{-t}$, and where we neglected the Q-independent terms. We now review the standard argument leading to the expansion of $F_{\rm top}$ in powers of $g_{\rm top}$. Using the Laurent expansions

$$\log(1 - z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}\,, \qquad |z| < 1\,, \tag{E.15}$$

and

$$\sum_{n=1}^{\infty} n z^n = \frac{z}{(1 - z)^2}\,, \qquad |z| < 1\,, \tag{E.16}$$

we obtain for $|q^k Q| < 1$ and $|q^k| < 1$,

$$F_{\rm top} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{n q^{kn} Q^k}{k} = \sum_{k=1}^{\infty} \frac{q^k Q^k}{k(1 - q^k)^2} = \sum_{k=1}^{\infty} \frac{Q^k}{4k \sinh^2(k g_{\rm top}/2)}\,. \tag{E.17}$$

The conditions $|q^k| < 1$ and $|q^k Q| < 1$ imply that $\mathrm{Re}\, g_{\rm top} > 0$ and $\mathrm{Re}\, t > -\mathrm{Re}\, g_{\rm top}$, the former condition being automatically satisfied for physical coupling and the latter being fulfilled when $\mathrm{Re}\, t$ is interpreted as the volume of the two-cycle.

The expression (E.17) can be further rewritten with the help of Bernoulli numbers B_n, defined by

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}\,, \qquad |z| < 2\pi\,, \tag{E.18}$$

and satisfying

$$B_{2n+1} = 0 \quad (n > 0), \qquad B_{2n} = (-1)^{n-1} |B_{2n}|\,. \tag{E.19}$$

The first few values are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$ and $B_4 = -1/30$. Subtracting from (E.18) its derivative multiplied by z we obtain

$$\frac{(z/2)^2}{\sinh^2(z/2)} = B_0 + \sum_{n=1}^{\infty} B_n \frac{(1 - n) z^n}{n!}\,, \tag{E.20}$$

and so, by virtue of the properties of B_n,

$$F_{\rm top} = \sum_{k=1}^{\infty} \frac{Q^k}{k^3} \left(g_{\rm top}^{-2} + \sum_{g=1}^{\infty} (-1)^g \, \frac{(2g - 1)}{(2g)!} |B_{2g}| k^{2g} g_{\rm top}^{2g-2} \right). \tag{E.21}$$

This can be written in terms of the polylogarithms

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}\,, \qquad |z| < 1\,, \tag{E.22}$$

as

$$F_{\text{top}} = g_{\text{top}}^{-2}\,\text{Li}_3(\text{e}^{-t}) + \sum_{g=1}^{\infty}(-1)^g\,\frac{(2g-1)}{(2g)!}\,|B_{2g}|\,g_{\text{top}}^{2g-2}\,\text{Li}_{3-2g}(\text{e}^{-t})\,. \tag{E.23}$$

In the limit $t \to 0$, we obtain [80]

$$\begin{aligned} F_{\text{top}} = & -\frac{1}{2}\left(\frac{t}{g_{\text{top}}}\right)^2 \log t + \frac{1}{12}\log t - \sum_{g\geq 2}^{\infty}\frac{B_{2g}}{2g(2g-2)}\left(\frac{g_{\text{top}}}{t}\right)^{2g-2} \\ & + g_{\text{top}}^{-2}\,\zeta(3) + \sum_{g\geq 2}^{\infty}(-1)^g\,\frac{(2g-1)}{(2g)!}\,|B_{2g}|\,g_{\text{top}}^{2g-2}\,\zeta(3-2g)\,, \end{aligned} \tag{E.24}$$

where we made use of the identity $\text{Li}_s(1) = \zeta(s)$.

Observe that when deriving (E.23) the expansion (E.18) was used, which is valid under the condition $|z| < 2\pi$, or $|k g_{\text{top}}| < 2\pi$. In (E.17), however, this condition is satisfied only up to a certain integer k. The result (E.24) is therefore not rigorous. A careful analysis of the asymptotic expansion at weak topological coupling g_{top} has been given in [46, 47].

Finally, substituting $t = 2\pi V$ and comparing (E.24) with (E.11) yields

$$F(Y,\Upsilon) = C F_{\text{top}}(g_{\text{top}}, z) \tag{E.25}$$

with

$$\begin{aligned} C &= \frac{\text{i}\,\Upsilon}{128\pi}\,, \\ g_{\text{top}}^2 &= -\frac{\pi^2\Upsilon}{16(Y^0)^2}\,, \end{aligned} \tag{E.26}$$

where we used $\beta = -1/2$.

For the conifold, it follows from (E.24) and (E.26) that in the limit $V = -\text{i}Y^1/Y^0 \to 0$, the higher coupling functions $F^{(g)}(Y)$ are given by

$$F^{(g)}(Y) = \text{i}\frac{A_g}{(Y^1)^{2g-2}}\,, \qquad g \geq 2\,, \tag{E.27}$$

where

$$A_g = -\frac{4^{2-2g}}{256\pi}\,\frac{B_{2g}}{g(2g-2)}\,. \tag{E.28}$$

Observe that the coefficients A_g are alternating in sign.

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Curriculum Vitae

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Work

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Gabriel Lopes Cardoso, Johannes M. Oberreuter and Jan Perz,
"Entropy function for rotating extremal black holes in very special geometry,"
JHEP 05 (2007) 025, `hep-th/0701176`

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"Extremal non-BPS black holes and entropy extremization,"
JHEP 09 (2006) 078, `hep-th/0607202`

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Appointed so-called ‘e-man’ (student responsible for teaching evaluation) by the Director of the Julius Institute, Utrecht University

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Chosen (with O. Mac Conamhna) by the Head of the Department of Mathematical Physics at the NUI, Galway for an interview with the external tuition assessment committee