Effective Gluon Interactions from Superstring Disk Amplitudes

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Diese Arbeit widme ich meinen Eltern.

Wenn wir anfangen, etwas zu glauben,
so nicht einen einzelnen Satz, sondern
ein ganzes System von Sätzen.
Nicht einzelne Axiome leuchten mir ein,
sondern ein Sytem worin sich Folgen und
Prämisen gegenseitig stützen.

L. Wittgenstein (1950)

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Contents

Chapter 1

Introduction

Great has been the interest man has shown for his environment and for the data his senses can perceive, since we can remember. This fact, combined with a strong desire for predictability of events, has led to what is known today as science. Of course, it has to be stressed, that mainly the name "science" has remained the same, its meaning and techniques have changed during the history.

If people thought that "the earth rests on the back of an elephant which rests on the back of a giant tortoise which rests on the face of a limitless sea" as documented about 1800 B.C. in the Indian writings of *Upanishadic Apocryphia*, some time later, in the ancient Greece, the pre–Socratic philosophers began to analytically criticize such images: they continued asking the question of origins, but also added some others, as e.g. "What is reality", "Where does the variety of things we observe comes from" and finally what are the patterns, or laws if we want, which govern nature. At that time also the basic language, in which science shall be later formulated, was settled and given rigor[1], the mathematics. The immediate question arose wether there is any way in which we might describe nature by mathematical language. Mathematics was born from natural concepts, such as distinct, numerable objects and various sets with their intersections and unifications; those incipient ideas were first idealized and then generalized: the idealization came as the continuation of natural numbers to infinity or later, as the infinite exactness of rational and real numbers; the generalization was and still is today the definition of new objects inspired by the older: from natural to whole numbers and then to rationals, reals, complex and so on,[1]. It shouldn't though be surprising that similarities between nature and mathematics arose. Even at that time various natural observations could be mapped to mathematical concepts, which was rather looked at as a mystery. However, the "mathematical" rigor as known today, was still missing, such that great mathematical discoveries were rejected for their "incompatibility" with the natural way of thinking[2], although the notion of "proof"

[1]Although mathematics was "invented" earlier, at the end of the summerish period and during the babylonian time, its notion of exactness was first settled about 500 B.C. Then, the notion of truth was introduced as a consequence of the mathematical proof, thus making possible the separation between conjecture and truth.

[2]The most famous example, if also very embarrassing, took place in the Pythagorean school, where

was known at that time.

Throughout the years one feature though persisted, being maybe the thread of science: the notion of ”law” and its manner of definition. People have simply observed regularities in nature and after a number of repetitions (which had to be high enough), named these as ”laws” or even as ”nature laws”. Neither the number of repetitions not the boarder between a simple law and a natural law is defined sharply.

A best example for the establishing of such a law, is the oldest of the sciences, the science of celestial bodies. It's not easy to overlook the strong regularities governing the movement of extraterrestrial bodies, which naturally leads us to the concept of law. Also its mapping to mathematics is quite tempting and even the creation of ”new” mathematics was inspired by it, as known from the first theoretical physicist ever, Isaac Newton.

It was this period which marked the beginning of the new scientific era, as known today. The scheme of ”doing physics” was implicitly accepted and can be described as the following: experimental work leads to better and better understanding of the phenomena studied which finally can be described by a proposed theory, formulated in terms of mathematical equations. In newer times, this scheme has been reversed, one starts very often with one or more conjectured theories which are then ”proved” by the experiment.

It was maybe for the first time in the mid of the 20th century, when science was more rigorously analyzed and even casted into a definition. It was K.R. Popper to whom we owe these merits he presented in his major work ”The logic of scientific discovery”, [2]. For the first time a sharp limitation was drawn between science and metaphysics. His strongest discovery was the unprovability of a theory. Although very surprising, this idea is still today not very spread, or at least not as it should be. Popper gave some simple, basic criteria for a conjectured theory such that it can be named a scientific theory:

- self consistency
- consistency with valid existing theories
- power of predictability
- falsifiability

(1.1)

I will comment a little on those criteria and on their critics. First of all, those properties are definitions and not god given qualities. They have been stated in order to establish a system in which we can work together and understand each other. They cannot be proved nor can they be deduced from more basic assumptions: they are something like an ”ethics” for scientific behavior.

The first criteria given, states that a theory should be free of any inconsistencies which can be achieved by only applying the mathematical apparatus on it: thus no

for the first time a proof for the existence of nonrational numbers was given but not accepted and its proponent even killed.

comparison with any experiment or observational data is needed just applying the mathematical logic should single out theories which are inconsistent. For sure such an agreement is of value, since if any inconsistency is observed inside the theory, than the whole mathematical language is not any more usable for describing that theory. So nothing is said about the nature, just the mathematical language we are using for describing theories would have to be rejected, surely a very dramatic consequence!

The second property of a scientific theory, its compatibility with already established theories, is an argumentative issue. For sure, ideally, a new proposed theory should be in total agreement with other existing theories and also explain and predict new phenomena. However, quite often, especially in latter times, a new proposed theory is a radical change of point of view or even a change of paradigm, which is contradictory to older theories. In former centuries, when scientific theories where established for the first time, this criterion was for sure more simple to be applied: since one phenomenon was attacked for the first time, no other conjectures about it (or at least conjectures of mathematical nature) could be contradicted. Nowadays, various generations of conjectures have been established about all possible phenomena, such that an inconsistency with other theories is unavoidable. Examples are the special relativity or the quantum principle, which both couldn't be consistent with newtonian mechanics, if the latter was considered a fundamental theory. Many other examples can be given, but the situation stays the same: we don't have an applicable rule to decide which theory has to be kept and which has to be abandoned. For instance, all theories formulated today have to be "lorentz invariant", i.e. in accordance with the special theory of relativity. If they aren't, they will be automatically rejected. It is the sensibility of the scientist which guides one between experimental evidence and willingness of changing the paradigm.

The third criterion, is maybe beyond any controversy, since it states that a theory should make some predictions or at least explain more than the others did. It is almost a tautology, since the very (implicit) definition of the scientific theory is such that it explains deeper or predicts more phenomena than its ancestors did.

We turn now our attention to the last and maybe the most known and important criterion proposed by Popper for a scientific theory: its falsifiability. Simply stated, it just means, that a theory has to be build such that it can be proven as false: thus its predictions have to be measurable and the experimenter has to be in the position to prove those prediction as false. The accent is here clearly on the falsification and not on the provability. The reason for that is that such a complex topic as a theory cannot be proven. In order to understand this counterintuitive statement let's look at some explicit example: Newton's attraction of massive bodies. It states that bodies attract according to the law proportional to the inverse square of their distance. We further suppose we don't know that this law is false in order for our game to be played. If we try to *prove* this theory we have to show first that the $\frac{1}{r^2}$–law holds. To do this, we have to rule out any other law, in which for example other polynomials involving the distance can show up, even with tiny suppressing factors. The consequence is we have to measure the experimental data with infinite accuracy! This is a first impossibility of proving Newton's attraction law. But even supposed we could measure with infinite accuracy, we aren't done, since Newton's law makes a prediction about **all** bodies in the universe, we have thus to check this law for every single pair of bodies! We see, in order to prove a law we are faced with many impossibilities. A theory's failure however, can be shown with a small amount of energy. So the way science works is in principle the following: the theoretician either simply guesses new theories, guided by experimental evidence or the physicist is inspired by already stated, older theories and deforms those in order to fit them to experimental data or to his wishes. This way, we obtain a rich spectrum of different theories which are then consecutively exposed to tests and ruled out one after the other, until new theories are available, the latter going the very same way as the former. At this point, we cannot disregard a very obvious similarity with the biological evolution governed by genetic mutation and natural selection: given the incipient theory, it undergoes some processes of (external) mutation, since it doesn't change itself but it *is* changed by the physicist. Furthermore, those changes aren't random, since the physicist has either discovered some intrinsic inconsistency, or has a higher sense of beauty which the actual theory doesn't satisfy yet, or finally has strong hints from the experimental side. The latter reason clearly represents the natural selection, although the other two mutation mechanisms can also be regarded as natural selection. Thus, physics is as "cruel" to its theories as nature is to its life forms. With every new, accepted theory we get one step nearer to the absolute truth (assumed such a thing exists), but uncountable trials have been thrown away, and are often regarded as useless, although they represent inevitable rungs of an (infinite) ladder.

Much more can be said on upper criteria for a reasonable scientific theory, we shall however limit ourself to what we have already said.

Although the type of science defined above seems to be quite reasonable and accepted by all scientists there could be strong evidence for its failure to characterize a scientific behavior. Or the other way around, it seems that we are faced in the last years with a deep paradigm change concerning the way in which we attack physical topics.

We refer now to a possible revolution in physics, which has started latently in the late 1980ies, supposed string theory is the right and only way we can describe nature. Then, for the first time, Candelas, Horowitz, Strominger and Witten, [3], proposed compactifications for space–time emergent from the string theory in order to make contact with the observed four–dimensional space–time. This process has been carried on until today when we are faced with a single theory with a huge number of vacua, of which every describing a possible world, with its own natural constants and coupling strengths. We stress again: should string theory be truth, then the so called landscape involves a number of vacua of the order of 10^{500} which destroys every desired uniqueness of our world. Many physicists today regard this phenomena as the single one able to deliver an explanation of the fine tuning observed in our Standard Model.

Accepting this point of view means a very drastic change of paradigm, since we have to give up the predictability of a scientific theory and further give up the uniqueness of our world regarding it as just a possible realization of the huge number of allowed vacua. The whole nature of this paradigm is not even worked out, since there are still unclear effects accompanying it: once we reject the falsifiability then we can never prove string theory to be false. So we have to accept it without any reason. On the other hand side it is exactly string theory which induced this change of paradigm, since it predicts those various numbers of vacua. We can't neglect this circularity, which maybe might be cured in the future.

We stop our general introductory exposition about science and its nature here and dedicate ourself in the next section to the theory accepted by most physicists today. We will try to keep the explanations in that introductory section as simple as possible, for we would like to emphasize the physical concepts and reveal them as good as we can and not hide them behind technicalities.

Standard model and the theory of gravity

Now, before going on, time has come to reveal the main ideas of the "standard model", that is the model widely accepted today and used to predict nature at its smallest and also largest scales.

At this stage ones attention will immediately be attracted by the clear decomposition of the theories. Thus on the one hand we have a classical theory formulated in terms of geometrical means, being not described with quantum mechanical principles, the theory of gravity, and the other hand, we have a quantum mechanical field theory, composed of three distinct pieces, each concerned with the description of elementary particles. This might seem a little bit unsatisfactory, but on this just a little later. Let us first describe in more detail each of those theories, and then concern about the structure of the whole construction of the standard model.

Special and general relativity

Gravity is best described nowadays by the Einstein's *General Theory of Gravity*,[4, 5, 6], whose mathematical formulation was first given by A. Einstein, in 1916. This is the logical continuation and generalization of the special theory of relativity, first formulated in 1905 by the same person. The restricted version of the relativity has as its basic physical principles two axioms, the constance of the speed of light and the generality of physical laws in all *inertial systems*[3], but has a much more greater underlying change of paradigm, which revolutionized physics until that time. This paradigm states the existence of a new four–dimensional space–time, the unification of three–dimensional Newtonian space with time, which should be the real frame in which we live. Since descriptions of the same physical event in different inertial frames should lead to the same mathematical form, the transformations which arise naturally predict new effects such as time dilatation or space contractions. At that time those predictions were nearly incredible and hard to digest. This is also the reason why he never got a Nobel Prize for that work. However, the physical community was confident with his competency and also the power of his theories, reason for which he was finally

[3] An inertial system of reference is a coordinate system which is not accelerated with respect to some other coordinate system, thus moving with constant velocity or being in rest.

given after all the Nobel prize, however for his *photoelectrical effect*, to be discussed later.

The nature of the definition of inertial frames is at its very heart quite restrictive, since no acceleration is at all allowed. Thus, at least in principle, the way to generalizations was given. The latter were achieved as already stressed, in 1916, when Einstein related the curvature of space–time with the total energy contained in it, this being the general theory of relativity. Its description is given in terms of *differential geometry*, where the unified space–time is described by an real Riemannian manifold $\mathbb{R}^4, \eta$. Furthermore, the space–time is highly dynamic, i.e. it reacts on the matter filling it and also gets back reacted from it. This interplay is described by differential tensor equations which are extremely nonlinear. They relate essentially the Einstein tensor $G_{\mu\nu}$, being responsible for the curvature of the space–time manifold, with the momentum–energy tensor $T_{\mu\nu}$ coming from the presence of matter. Unfortunately the theory cares just about the "mechanical" properties of matter, such as mass distribution, momentum, angular momentum, never being concerned with microscopical attributes of matter. Usually matter is simply put into the theory and one observes the evolution of the space–time–matter complex. For sure this is in practice by far not easily to be solved exactly, but in principle it is solvable, and this is achieved by computer aid. The theory was created almost one century ago, and none of its parts have been changed during the decades. Today we still work with the same formulation left by Einstein.

The quantum world

Roughly at the same time as relativity was developed, accurate experimental results gave strong evidence for (at least partial) *quantization* of matter. First M. Planck, postulated 1900 the appearance of energy in discrete quantities, $h\nu$[4], and five years later Einstein again described *photons*, as discrete entities of light and also postulated that the other kinds of matter should be quantized. Thus, two deeply different descriptions of matter were given, on the one hand the classical wave representation, on the other the newer quantum representation. This fact was later on generalized by L. de Broglie, who postulated for all matter or radiation a dual description in terms of particles and waves, given by $\lambda = \frac{h}{p}$, where λ is the wavelength and p the momentum of the described entity. Already those thoughts have lead to simple atomic models which could roughly describe some of the properties encountered when dealing real atoms. More directed work and intuitional postulates which culminated with W. Heisenberg's uncertainty principle $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$, which states that the accuracy of the measurement of the momentum multiplied by the accuracy of the measured position of the one and same entity cannot be smaller than the quantity on the right. This is a constraint with respect to the accuracy of measurement involving one quantum entity, which eventually lead to reformulation of the momentum and space coordinates as non commuting operators. This paradigm completed somehow the principles of *quantum mechanics*.

[4] Here h denotes the basic unit in which energy can be measured, of magnitude $h = 6.6 \cdot 10^{-34} J\,s$, named after his postulator, and ν the frequency of radiation.

For more on that subject see e.g. [13, 14]. Thus simple state equations could be constructed allowing for an toy theory which could deliver simple and exact measurements and predictions. However, for a more accurate and appropriate microscopic description of matter this wasn't enough. One soon understood that the principles of special relativity, the old field theoretical descriptions dating from Newton and the new quantum mechanics should be unified, in the same manner Newtonian physics was unified with the quantum principles to the quantum mechanics. This was eventually achieved in the framework of *quantum field theory*, [7, 8, 10], were we essentially have an infinite, uncountable, set of quantum mechanical systems, for we take the continuum limit from (quantum) mechanics to (quantum) fields. So quite a straightforward quantization method arose: in the existing Lorentz-invariant field theory[5] the involved fields were expanded in modes, and those were imposed (anti-)commutation relations on, such that momentum and position became operator valued functions of the space–time. This procedure is nowadays known as "canonical quantization". Success soon crowned the work and people were able to calculate for example the radiation of quantum transitions. Decades later, more sophisticated and powerful techniques were put into the theory, from all of which maybe the *path integral* technique, [11], is most powerful and intuitive. This way, just two axioms about quantum mechanics and a strong mathematical apparatus allow for various, precise and elegant results. The first axioms states that the probability from evolving from some initial state $|i>$ to a final state $|f>$ is given by the square of the absolute value of a transition amplitude, call it $K(i, f)$. The second axioms states that this transition amplitude is the sum over all possible paths from $|i>$ to $|f>$ weighted with $e^{iS/\hbar}$, as best described in [12]:

$$P(i, f) = |K(i, f)|^2, \tag{1.2}$$

with

$$K(i, f) = \sum_{paths} e^{iS/\hbar} = \int \mathcal{D}x(t) e^{iS/\hbar}. \tag{1.3}$$

The rest of the story is concerned with the implementation of that formal sum, technique which rests on an old idea from P. Dirac. This way, we can think of the path integral approach as the continuum generalization of the two–slit experiment to the n–slit experiment with m screens,[15]. Once quantization is done we have to eliminate the encountered infinities by the techniques of renormalization. One is then able to successfully calculate scattering cross sections, decay times of bound states and other intrinsic quantum properties such as the anomalous magnetic momentum of the electron, which is one of the very best examples showing the accuracy of the quantum field theories. Moreover, intuitive scenarios are even able to recover the mechanisms of

[5] At that time, the only developed field theory was the theory of light, the electrodynamics. Gravitation was (and still is) described in a geometrical manner, the other two nuclear forces weren't known at that time.)

general like charge repulsion in Maxwell theory and mass attraction in gravity, see e.g. [16]! Moreover, quantum electrodynamics, the quantum version of Maxwell's theory, proved as one of the most accurate and best describing theories ever treated.

More experiments in the late fifties and beginning sixties lead to the discovery of a huge amount of particles, the so called *particle zoo*. Initiated by those discoveries two more forces of nuclear nature (the strong and the weak nuclear force) were postulated on order to better describe subatomic and nuclear structures and also to incorporate all the particles already encountered. The description proved to be a very straightforward generalization of the quantum electrodynamics if the latter was formulated as a Yang–Mills theory with gauge group $U(1)$, the former being a *gauge theory*, i.e. a theory describing interactions between fermions (which are half integer spin particles, making up the "matter") by minimally coupling them to bosons (integer spin particles responsible for interactions) – the force particles, also known as *gauge bosons.* The gauge boson arise almost naturally within the theory of fermions by just imposing invariance under gauge transformations, see e.g. [17]. Then, the weak nuclear fore is described as a $SU(2)$ Yang–Mills theory and the strong force as a $SU(3)$ Yang–Mills theory. Thus the standard model can be written $U(1) \times SU(2) \times SU(3)$ as an acronym of its governing gauge transformation properties.

After this grand unification, all particles discovered until then proved to be mainly resonances of the few underlying basic particles, which could be described by only three fundamental forces: the electro–magnetic, the weak and strong nuclear force! They were formulated in terms of Yang–Mills theories, This is what we call the *standard model*, the model accepted today as describing all particles that have been sofar observed[6].

Criticism of the Standard Model

For sure, this is a great step man has done but we cannot neglect its shortcomings. The greatest maybe, it's the incompatibility of gravitation with quantum effects. This testifies the incompleteness of either one of the two theories or even both. The general theory of relativity is a purely deterministic, "mechanic" theory, where in principle everything can be approximated to desired order. This highly violates the quantum principles, where "particles" are described by their wave functions and any observable quantity is a probabilistic value. (An explicit path followed by a "particle" doesn't even make sense here.) Great effort has been made in order to cure this problem but they remained mainly unfruitful. One promising solution might be *Loop Quantum Gravity* [18, 19], which applies the techniques of *loop quantization* to diffeomorphic invariant theories aiming a resulting quantum gravity. However, this project is still in work and no evidence for a breakthrough yet exists.

$ 1.000.000 has been offered to the person who might be able to solve the "hierarchy problem" of the Yang–Mills theories. The very prize signalizes the importance of the problem and its difficulty. There are big mass gaps in the spectrum of "elementary"

[6] An additional mechanism for conferring mass to certain force particles has been put in the theory, known as *Higgs mechanism* .

particles coming from Yang–Mills theories which nobody can explain and also don't seem to be deductible from the theory alone. This is a big shortcoming of the standard model on which intense work is spent. A possible outcome may be the existence of *supersymmetry* [7].

Another weakness is the required use of the technique called renormalization. When performing those mathematical operations it often looks as some "undesired", infinite quantities were simply hidden for convenience. The whole apparatus has often been criticized and the encountered infinities are a strong hint for the standard model just being an approximation, some effective theory, build on top of a more basic underlying theory. This is reason enough to search for that conjectured theory.

Mass emergence is another misunderstood topic. Yang–Mills theories predict the right number of particles with the right properties except their mass: after it, all gauge bosons should be massless. This is unfortunately not the case, thus a mechanism for mass "acquisition" was proposed, the *Higgs mechanism.* Also a massive Higgs–particle should exist, which has never been discovered.

The particle observation basis Super–Kamiokande has detected in 1998 neutrino oscillations. Those are clear signs for the non–zero mass of neutrinos, which are predicted as massless in the standard model. One more inconsistency which has to be solved.

As a last remark, the number of parameters inserted into the theory[8], are about 19 (!). It is expected from a scientific theory to have relatively few parameters, such that by minimal input one gets maximal output, i.e. experimental predictions. Furthermore under those 19 parameters a lot of them are masses of particles, and those should really be predicted, i.e. come from the output of the theory and not viceversa.

These problems, some of which are of aesthetical nature, others of computational nature and some are of deeply comprehensive nature cannot be neglected. For sure, the standard model together with gravity have excellently served the science and predicted many useful and unexpected events. This should however, hardly be a reason to keep those theories up, despite their shortcomings and inconsistencies. As long as no better theory is known, the standard model and gravity is the best we have, so we have to keep it. On the other hand, we have something new, then it's time to throw away the old habits and accommodate for the newer. This may seem very "egoistic" and "selfish", but this is how science works. Thus, after the short summary and outline of the work, the next chapter should be dedicated to a new and alternative theory, which might be a way out of the labyrinth of inconsistencies presented above.

Summary and outline

Main part of this work is the calculation of the six–gluon superstring scattering amplitude on the disk in order to extract the relevant part for the effective action, which

[8] Those parameters are simply real numbers, which allow for an numerical computation of desired quantities with the rules given by theory.

describes the behavior of the scattered states. There have been already indirect approaches in order to achieve that, see e.g. [20, 21], which aimed to match the spectrum coming from string theory with that from higher order Yang–Mills theories. Another approach has recently been made in [22], considering BPS-solutions to the equations of motion in gauge theories. All those methods being indirect emphasize the importance of the topics presented here, since they might be a direct check for the former. Nevertheless it will become clear that scattering amplitudes with more than five external gluons [23, 24, 25], are extremely complicated to compute. Moreover, this amplitude has only once been computed in [26], thus it is the one and only string scattering amplitude with six external states. The difficulties encountered in the case of the five–point amplitude become in our case insuperable, such that a completely new and powerful method will be presented, which not only solves the six–gluon problem but also exhibits marvelous relations between triple hypergeometric functions and generally speaking makes a strong connection between string theory and number theory. The method can shortly be described as equating all the permutations of the position of the six vertex operators (representing the external states), of which two are in the (-1)–ghost picture and the rest are in the 0–ghost picture, since those permutations although looking different they describe the same S–matrix and thus are same. This will generate a huge system of algebraic equations, thus translating the problem of the six–point amplitude to the solving of the system of equations. Variables of that system are various triple hypergeometric functions, the system establishing this way mathematical relations between those functions, which turn out to be mathematical identities, one being able to prove that with the necessary amount of time and mathematical skills. Since those identities are also extremely difficult to establish, they cannot be found in the literature, see e.g. [27]. Those relations are also written explicitly down for the first time in [26]. Furthermore, in order to expand the string expression in the momenta[9] of the six gluon deep mathematical work will be done involving (triple) hypergeometric functions. That way, using different representations for those functions, like integral or sum representations, remarkable relations between string theory (represented by hypergeometric functions) and number theory (multiple zeta functions) will be established. To give a taste of that, a simple and already many years known example is

$$\int_0^1 dx \int_0^1 dy \; (1-xy)^{\alpha' s-1} = \frac{H_{\alpha' s}}{\alpha' s} = \zeta(2) - \alpha' s\; \zeta(3) + (\alpha' s)^2\; \zeta(4) - (\alpha' s)^3\; \zeta(5) + \ldots .$$

On the left hand side we have the integral which appears in certain expressions during the computation of five–gluon scattering amplitudes, $\alpha' s$ being one Mandelstam variable multiplied by the string tension. On the right hand side we have the harmonic number $H_{\alpha' s}$, which is finally series expanded in an alternative infinite sum of consecutive values of the Riemann zeta–function, ζ. When now looking at similar simple expressions in the six–point case we derive

[9] Much more should one speak about the Mandelstam variables, since only those kinematic invariants will be used.

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\ (1-xyz)^{\alpha' s-2} = \zeta(2) + \alpha' s\ [\ \zeta(2) - 2\ \zeta(3)\ ]$$
$$+ (\alpha' s)^2\ [\ \zeta(2) - 2\ \zeta(3) + \frac{5}{4}\ \zeta(4)\ ] + \dots$$

where again s is some kinematic invariant and $\zeta(p)$ is the Riemann zeta–function, often encountered in number theory and defined by

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p} \quad , \quad s \geq 2.$$

Its generalization is straightforward and given by so called multiple zeta–function, of which e.g. the triple one is given in the following equation:

$$\zeta(s_1, s_2, s_3) = \sum_{\substack{m_i=1 \\ m_3<m_2<m_1}}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} m_3^{s_3}} \quad , \quad s_1 \geq 2\ ,\ s_2, s_3 \geq 1\ .$$

Such multiple zeta–sums are encountered in various integral representations showing up in the six–gluon scattering like is the case in

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \frac{(1-xy)^{\alpha' s}}{(1-xyz)^2} = \zeta(2) - \alpha' s\ \zeta(3) + (s\alpha')^2\ \zeta(2,1,1) + \dots.$$

Those were simple examples encountered in the computations and are given as a pedagogical foretaste for the mathematics to come. The next degree of complexity is an integral with two Mandelstam variables

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \frac{(1-xy)^{\alpha' s}\ (1-yz)^{\alpha' t}}{(1-xyz)^2} = \zeta(2) - \alpha'\ (s+t)\ \zeta(3)$$
$$+ \alpha'^2\ [\ s^2 + t^2 + \frac{7}{4}\ st\ ]\ \zeta(4) + \dots.$$

which requires already a much more thorough treatment. Although the right hand side looks quite innocent, hard work has been spent in order to establish the result, which is the consequence of evaluating following triple sum

$$\sum_{m_i=1}^{\infty} \frac{m_3}{m_1\ m_2\ (m_1+m_3)\ (m_2+m_3)\ (m_1+m_2+m_3)} = \frac{7}{4}\ \zeta(4),$$

which is of much more bigger degree of complexity than its nice "cousin" the triple zeta–sum. They are related to Witten zeta–functions[10] and Euler/Zagier sums. In general, they cannot be simply expressed in terms of basic zeta–numbers.

Thus, to sum up, the main results presented in this work are

- An efficient method to calculate supersymmetric $N-$point tree $-$ level string amplitudes is presented.
- The six gluon open superstring disk amplitude can be expressed through a basis of six triple hypergeometic functions, which encode the full $\alpha' -$ dependence.
- Material to obtain the $\alpha' -$ expansion of these functions is derived : We calculate many multiple Euler $-$ Zagier sums including multiple harmonic series.

Those results will be presented as follows: after this short introduction a more technical one will follow and throw some light on the topic of superstrings. There, the main building blocks of string theory (actions, supersymmetry, quantization, D–branes) are introduced and after that, superstring theory is considered as a whole, giving a close look to the main predictions coming out of it. This part will be concerned on establishing the contact between theory and experiment, thus compactification will be discussed and some realistic model shown. Also the topic of dualities in string theory and its landscape will be discussed, since this is of high interest from the theoretical point of view. In the second chapter, the notion of low energy effective action will be set in relation with string theory. First, the concept of effective theory will be defined as the low energy limit with respect to some parameter incorporated in the theory and then some very prominent examples are given. The latter will be widely spread over the broad spectrum of physics. Finally, the technique of taking this limit are presented and then a first more technical look is given to Born–Infeld action, an example of special interest under the effective actions. Further on, some light is shed on the techniques of computing superstring tree–amplitudes in chapter three. After introducing the basic concepts of amplitudes which naturally arises from summing over all paths of the string function, more general features of the S–matrix are revealed and finally, as a pedagogical step and preparation for what shall come, the four gluon superstring tree amplitude is exposed in great detail. Already here, we can grasp the main new technique used later to compute the six–point function.

The author has tried to keep those first three chapters as basic and comprehensive as possible. The number of equations is reduced to a minimum and formulated as straightforward as possible. Those chapters might serve as a basic of the respective subjects treated there. However, since also other such introductions exist, a variety of literature sources is given.

The six–gluon scattering amplitude, being the main topic of that work, will finally be introduced in chapter four. Since its complexity is rather high, the whole theme will be an intricate interrelation between physics and mathematics. This chapter deals

[10]Witten zeta–functions are defined by $W(a,b,c)=\sum_{m,n=1}^{\infty}\frac{1}{m^a\ n^b\ (m+n)^c}$

with the physics involved and thus the general expression from superstring theory will be derived. This way, the S–matrix for six gluons is written down and its kinematics analyzed: after showing how, in principle, the S–matrix can be obtained with the help of the new ghost–picture method, few remarks are made about the integration regions involved in the scattering, since those are in one to one correspondence to the different representations of the S–matrix. Further, the system of equations which is equivalent to solving the problem is presented and its six–dimensional solution displayed. The S–matrix will then expanded in its Mandelstam variables.

The next large chapter is devoted to the mathematics encountered and used to solve the problems presented in the antecedent chapter. There we just listed the various mathematical results which will be now derived in detail. Thus, a very first step is to exactly define the functions encountered in the S–matrix, the triple hypergeometric functions, and analyze their properties. This is done in the frame of special functions, where also generalized hypergeometric functions are treated and various higher generalizations thereof. Expanding those functions will automatically lead to an exposition of infinite sums. Thus, departing from the basic definition of harmonic numbers and the Riemann zeta function we will analyze euler sums, multiple sums and triple zeta functions. Also the dual representations thereof, the series and the integral representation, are treated and set in relation to special functions.

The last chapter will be dedicated to field theory, where a more thorough look at the Born–Infeld action will be given and also the difficulties shown, when extracting it from string theory. Thus, starting from the expanded S–matrix, we analyze it with respect to its momentum structure and also the encountered transcendental numbers. Next, we will set up the field theory ansatz, and deal with various Feynman diagrams categorized with respect to their topology. Finally, all used Feynman diagrams are collected and presented and the way is shown how to get to the effective action from the calculated S–matrix.

Part I

Superstring Theory, Consistency and Amplitudes

Chapter 2

Superstring theory – a survey

2.1 Ingredients

Rather innocent is the main assumption of string theory, i.e. it states the one–dimensionality of the "basic matter" (which was former known as particles), contrary to the traditional point of view, where particles are zero-dimensional points. This will have unimagined consequences, like the higher dimensionality of our space–time or the very large spectrum of models which can be extracted from string theory, also known as string landscape. We will try to treat those topics here and also completely introduce the superstring with all its features. A lot of good introductions on that topic exist, our being even one of the shortest. Thus, for further reading there are a lot of good introductions and overviews. Without claiming to be complete I shall refer to following works: [28, 29, 32, 34, 35], although there is plenty of literature on the net. In the following we will give a short introduction, the main emphasize being on the techniques used there and also on the underlying ideas and concepts.

2.1.1 Actions

In classical relativistic theories, i.e. those not being quantized, a particle is described by a Lagrangian which is set to be the trace marked by the particle when moving in space–time. In a more physical language this is nothing else then describing the motion of that particle by giving its position in space–time within a k–tuple of coordinates specifying its position and momentum, where k is the degree of freedom of the underlying particle. This will create a D–dimensional curve. Afterwards we will take the action be the length of that world line. As usual in mechanics, the variation of that Lagrangian will reveal the equations of motion, and concomitant the minimal length of that curve. This action shows automatically parametrization invariance and of course Poincaré invariance, for being constructed in such a way.

When moving now to the concept of extended "particles", the string, we will go a similar way in order to set up a Lagrangian. Since we are dealing now with strings, opposite to zero–dimensional particles, the former being two dimensional objects in space-time, their classical action, call it Nambu-Goto, is taken to be the area integral over the world–sheet[1] mimicking the relativistic action of a classical particle

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \, (-\det \partial_a X^\mu \partial_b X_\mu)^{1/2} \tag{2.1}$$

Here, α' is related to the string tension, which is defined as mass per unit length of string, and is given in terms of the tension as $T = \frac{1}{2\pi\alpha'}$. This "mechanical"[2] action has again two symmetries: the Poincaré–invariance which is somehow obvious for we started from the relativistic one–dimensional particle, and the diffeomorphic invariance allowing one to choose the preferred parametrization, which is again given by the very construction of the action. The latter is also intuitionally expected, since the action, and hence the physics, should be independent of the chosen way to parameterize the surface. On the one hand, the action above is quite intuitive and thus seems familiar, since we started with the zero–dimensional particle and generalized it to the one–dimensional string. However, for from the technical point of view, the Nambu–Goto action is very hard to treat because of its nonlinearity. Especially later on, when we will quantize the theory this will be an impossible thing to do with that action. This was soon recognized, and a classically equivalent form for the action was chosen. This was done by adding a new field γ_{ab}, an explicit metric on the world–sheet. This action, called Polyakov action, is given by

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu. \tag{2.2}$$

In the formula above, γ_{ab} is the newly introduced world–sheet metric, and its determinant is named γ. Haven got rid of the square root, this form has the enormous advantage of being linear. This is the only reason one has chosen the Polyakov action to work with instead of the older Nambu–Goto action. Additionally, it has on more invariance typical for two dimensional objects, the Weyl–invariance, which will play an important role when quantizing. It is worth to emphasize again that two action are at classical level fully equivalent. This can easily be shown when analyzing the equations of motion of the metric γ_{ab} which follow from the Polyakov–action (2.2) when varying it with respect to its metric:

$$\delta_\gamma S_P[X^\mu, \gamma_{ab}] \quad \rightarrow \quad \partial_a X^\mu \partial_b X_\mu = \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X_\mu \; . \tag{2.3}$$

Here we just have the equations of motion for the metric which can further be massaged to show that the induced metric from the Nambu–Goto action is proportional to

[1] By world–sheet we denote the trace which a string leaves in the space–time, similar to that a particle leaves in field theory, the only difference being its dimension.

[2] I am speaking about a *mechanical* action in order to emphasize we have here a classical, non-quantized object, which is really a (massless) string

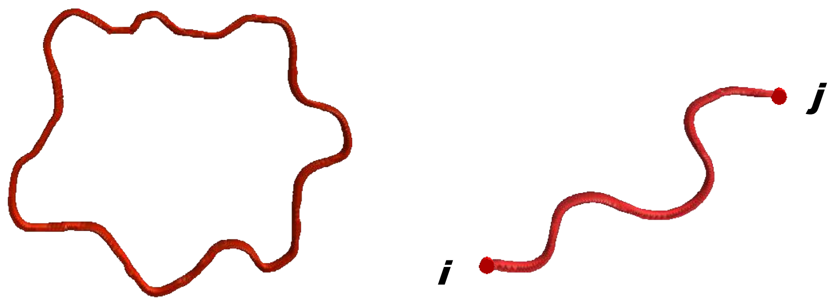


Figure 2.1: Closed string and open one with charges at its ends

the world–sheet metric. Maybe we should emphasize that the two are just *proportional*, since the metric from the Polyakov action can still be rescaled with one multiplicative factor such that the action stays the same, this just being the Weyl symmetry.

Since we are just analyzing a string which has the ability to vibrate, one might wish to see that mathematically, fact which can be establish in the same manner as the equality of the two actions: varying the Polyakov–action this time with respect with the string coordinates X^μ leads to their equations of motion

$$\delta_X S_P[X^\mu, \gamma_{ab}] \quad \rightarrow \quad (-\gamma)^{1/2} \nabla^2 X^\mu = 0, \tag{2.4}$$

which can clearly be recognized as the equation of a vibrating string. One more important fact is hidden in the derivation of the last equation: we had to use boundary conditions in order to eliminate the the surface term in the varied action. There are basically two different of boundary conditions: one that allows the string ends to freely move in the space, this allowing for an open string, and such conditions that close the string. We see, that, the two kinds of strings, open and closed, arise naturally out of the equations of motion. No additional supposition is made on the theory to become that. Those open and closed strings are depicted in figure 2.1. Furthermore the former boundary conditions can also be modified such that the string ends are fixed! Those conditions will break the Poincaré invariance in the D–dimensional space–time, but we will worry about that later.

We should give as a last remark on the classical string its mode expansion:

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'} \sum_{k \neq 0} \frac{\alpha_k^\mu}{k} e^{-ik\tau} \cos(k\,\sigma). \tag{2.5}$$

Upper equation displays the mode expansion of the open string, as the open and closed string are different, since they obey different boundary conditions. We recognize the string position x^μ, its momentum p^μ and the oscillator modes α_k^μ, which after quantization, will become different string states, thus they will represent the particles in the string spectrum, like e.g. the photon, graviton, and so on.

2.1.2 Supersymmetry

This section is dedicated to *supersymmetry*[3], one of the very important and beautiful assumptions and concomitant of highly theoretical interest. This basic symmetry is put by hand into the theory. However, strong theoretical reasons and advantages are in favor of that symmetry, like the hope for one possible solution of the hierarchy problem in Yang–Mills theories[4], symmetry which basically doubles the spectrum of particles, requiring a supersymmetric partner to each standard particle. For more literature on that see e.g. [37, 38, 39, 40]. Another more appropriate reason for requiring supersymmetric invariance in the string action[5] is the absence of any fermionic states in the bosonic string; supersymmetry will ensure us the presence of fermionic particles coming from the superstring.

The mathematical framework of supersymmetry is given by the minimal relaxation of the requirements on the Poincaré algebra, such that the *S–matrix*[6] still preserves its unitarity and other possible internal symmetries. It was shown by Coleman and Mandula (1967), [42], that those are the only possible reasonable properties a S–matrix could have. Now, the already mentioned relaxation consists in introducing anticommutators in the Poincaré algebra of the corresponding symmetry shown by the S–matrix, since those changes still preserve its attributes. It was shown later on, by Haag, Lopuszánski, Sohnius (1975) [43], that supersymmetry is the only possible symmetry compatible with the new requirements. So this step is quite easy to understand but again with very deep reaching consequences. The algebra describing Poincaré invariance of the S–matrix is enlarged by also considering some anticommuting quantities, which are to be specified and thus constrained just in a while. In mathematical language the anticommutators change the usual algebra to an $\mathbb{Z}_2$–graded algebra, meaning that additional, fermionic charges are put into the Poincaré algebra and have nontrivial (anti-)commutation relations with themselves and the Poincaré charges. We can give a closer look at that algebra in the Appendix A, where it is exactly listed for the case of one supersymmetric partner, i.e. the case $N = 1$ supersymmetry. Representations of the new algebra are bosonic *and* fermionic states, with the attribute of being pared with each other: as stated at the beginning of that section, every (fermionic) bosonic particle existing until supersymmetry gets a twin particle with the same quantum properties except spin which is in that case (integer) half integer. The number of those additional charges (the number of supersymmetric partners) can be varied, though it cannot exceed eight. For sure, in realistic models, the number of supersymmetries should be one, since not even those super partners have been yet discovered. This is also a good motivation for searching for mechanisms of breaking supersymmetry. Some

[3]Again, there are a lot of good overviews on that topic but still a very best one is the classical book of Wess and Bagger,[36]. See also [41]

[4]See previous discussion about that topic in the critics about the standard model.

[5]Since the *super*string inherits its name from *super*symmetry we will talk from now on about the superstring opposite to the bosonic string

[6]This notion designates in its explicit form a function depending on energy, spin, polarization, charge and other quantum numbers, giving the intensity distribution with respect to the angle of the outcome of a scattering experiment.

of them will be presented little later. With that symmetry given the Polyakov action looks like the following

$$S = \frac{1}{4\pi\alpha'} \int \sqrt{\gamma} \left[\gamma^{ab} \partial_a X^\mu \partial^b X_\mu + \frac{i}{2} \psi_\mu \partial\!\!\!/ \psi^\mu + \frac{i}{2} (\chi_a \gamma^4 \gamma^a \psi^\mu) \left(\partial_b X^\mu - \frac{i}{4} \chi_b \psi^\mu \right) \right] . \tag{2.6}$$

Remembering this is a classical action one would like to quantize it, in order to make the transition from mechanics to quantum systems which should deliver the microscopic description of the matter.

2.1.3 Quantization

This process is carried exactly as discussed in the introduction referring to quantum field theories, although we will encounter some difficulties, which are related with the additional degrees of freedom we have, i.e. the Weyl symmetry. Also, for the sake of brevity we will restrict ourself to the bosonic string, since conceptually the superstring is handled the same way. We will begin with the usual commutation relation imposed on the conjugated variables

$$[x^\mu(\sigma, \tau), p^\nu(\sigma, \tau)] = i\eta^{\mu\nu} \delta(\sigma - \sigma'), \tag{2.7}$$

which translates into commutation relation between their modes

$$\begin{aligned} [\alpha^\mu_m, \alpha_n \nu] &= m\delta_{m+n,0}\eta^{\mu\nu} \\ [\bar{\alpha}^\mu_m, \bar{\alpha}_n \nu] &= m\delta_{m+n,0}\eta^{\mu\nu} \end{aligned} \tag{2.8}$$

However, we are not finished yet. Since we started with a system with a highly degree of symmetry, we still have to preserve that symmetry after quantization, which will translate into physical constraints on an arbitrary given state $|\phi>$. What we exactly mean here is the conformal symmetry, also known as Weyl symmetry, which will be imposed in the following manner, see e.g. [30], [31], [33]: given the momentum–energy tensor calculated with the usual technique form the superstring Lagrangian, we define the Virasoro operators as its modes

$$L_m = \int_0^\pi d\sigma T_{--} e^{-im(\tau-\sigma)} \ , \ \bar{L}_m = \int_0^\pi d\sigma T_{++} e^{-im(\tau+\sigma)}, \tag{2.9}$$

where we have given the energy–momentum tensor in conformal coordinates

$$T_{++} = \frac{1}{2}\,\partial_+ X \cdot \partial_+ X \quad , \quad T_{--} = \frac{1}{2}\,\partial_- X \cdot \partial_- X \quad , \quad T_{+-} = T_{-+} = 0 \ , \tag{2.10}$$

and also have used the derivative with respect with conformal coordinates $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. Now we want to impose energy–momentum conservation and the equations of motion of $T_{\mu\nu}$ (derived again as current conservation from diffeomorphism invariance of the string Lagrangian)

$$\nabla^\alpha T_{\alpha\beta} = 0 \quad , \quad T_{\alpha\beta} = 0 \tag{2.11}$$

which finally translate as conditions on the physical string quantum states via the Virasoro operators:

$$\begin{aligned} L_n \left|\phi\right\rangle = 0 \quad , \quad n > 0 \ , \\ (L_0 - a)\left|\phi\right\rangle = 0 \ , \end{aligned} \tag{2.12}$$

where L_n are conformal operators generators of the conformal transformation and also of the group $SL(2,\mathbb{C})$ as mentioned before. However, when computing their algebra, also known as Virasoro algebra, which is then the algebra of the conformal symmetry we see that it doesn't close anymore as in the classical case, for we have picked up an anomaly from quantization

$$\{L_m, L_n\} = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \ , \tag{2.13}$$

Though the right hand side of (2.13) has a second term which is somehow unusual: the algebra isn't closed anymore with that term, but this is nothing else than the already presented conformal anomaly. It can be and it has to be removed, fact which will put constraints on the theory! Fixing up this anomaly will eventually determine the coefficient a and also restore the conformal symmetry. However, we shall mention that this anomaly is just the result of the violation of a classical symmetry when going over to the quantized version. This process should always save the classical symmetries and in our special case concerning the conformal symmetry tremendous consequences follow: additional constraints about the embedding Minkowsky space–time arise, coming exactly from the anomaly caused by the Weyl–invariance; it turns out for the space–time to be 10-dimensional in the case of supersymmetric strings! This was for sure and still is one of the predictions of string theory with greatest impact for our knowledge about the world, even if not verified yet. To read more about conformal field theory see [77]. The process of quantizing gauge theories, which certainly is the case for the superstring theory, is a very well understood issue today. We will argue more on that topic in section 4.1, where we will also present some other quantization procedure which proves more convenient for introducing interactions in the theory.

2.1.4 D–branes

Before going on and looking how one can bring together the predicted ten-dimensional Minkowsky with our observed space–time, it is worth to present one capital concept in

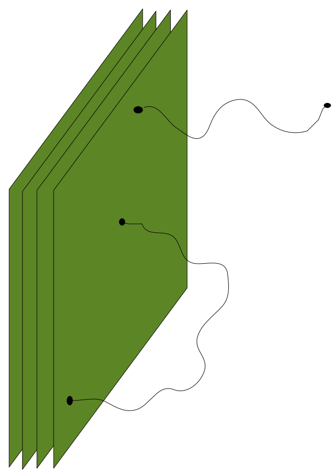


Figure 2.2: Open strings with different boundary conditions seen as ending on a D–brane

string theory, the D–brane. We have already mentioned in section 2.1.1, that various boundary conditions can be imposed on the open string, as depicted in figure 2.1 in order to derive its equation of motion: its ends, being free, can obey different boundary conditions, as imposed when solving mathematically the equation of a vibrating string. These boundaries can either let the ends free, such that momentum flowing along the string will be conserved, or they can confine the string ends to some subspace (which can be imagined as a hypersurface) of the 10–dimensional space they are living in. Further, it has been proven worth to let those hypersurfaces, call them *D–branes*, have a life of their own, i.e. one interprets the boundary conditions as emerging from objects already existing, on which open strings can end, see fig. 2.2. For more on that see [44, 45, 46, 47].

Beyond its ability to vibrate as a genuine string, the open string can also carry some charges attached to its ends, the Chan–Paton–charges, for those are compatible with Poincaré–invariance and don't affect the world sheet symmetries. Since each single string state will be enhanced to a matrix $|ij\rangle$, depending in which state the ends of that string are, the string amplitudes will be invariant under a $U(N)$ transformation. This way, a global symmetry of the world sheet emerges to a gauge symmetry in the space–time. So when taking into consideration the massless modes[7] of the open string we will be talking about some massless gauge particles in the space–time. So we are at the point we could start a discussion about possible models for the observed nature as emerging out of string theory. Before doing that, however, we still have to take care of the extra dimensions arising in a consistent string theory as explained in section 2.1.3.

[7]Since we started with a vibrating string the modes will be grouped into an infinite tower of excitations, of which the lowest is massless.

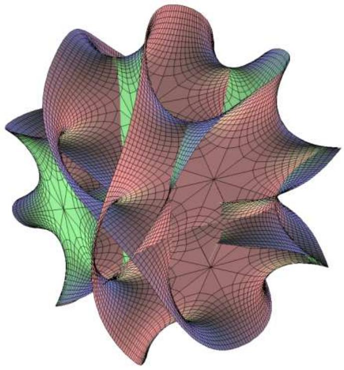


Figure 2.3: Three dimensional slice through a six dimensional particular Calabi–Yau space

2.1.5 Compactification

Given that richness of particles and also the promising gauge counterparts for the standard model particles, we're aiming now to make contact with the four dimensional Minkowsky space–time. Since string theory should be a description of our nature it lives in the same space–time as we live in. We remember, that the 10–dimensional space–time, in which strings are embedded, arose from canceling the conformal anomaly. Thus this space–time property is crucial for the strings if we want them to be anomaly free, i.e. to be well defined and free of inconsistencies. On the other hand side, our Minkowsky space–time is known (from a reach experimental palette) to be four dimensional. However those experiments have a finite accuracy, so from a rigorous point of view, all what we can say is, our world seems to be four dimensional until the energy scale it was probed. This way one possibility arises to cure the apparent contradiction: We may compactify the rest six dimensions present in string theory, such that they will not be ruled out by actual experiments. The radius, if we choose to have some circular compact dimensions, of the six coordinates should be just smaller than the distance which can be probed by actual experiments.

This for sure, seems a little bit arbitrary and also a kind of cheating, since a real experiment will never be able to exclude every arbitrary tiny distance, so it may seem we are just hiding our theory behind the non vanishing experimental limits. However, we will see in a while, this is not that hopeless. String theory is powerful enough to also give constraints on the size and even on the shape of the compactified space. Thus a first property of the six–dimensional compactified space should be its compactness, which is required by empirical reasons[8]. Furthermore, also its shape is constrained by e.g. preservation of supersymmetry and also simplicity arguments lead theoreticians to chose Calabi–Yau spaces as the prototype of the underling space. This way our Minkowsky space–time becomes a direct product $\mathbb{R}^{1,3} \times CY_3$ consisting

[8]for if there had been non–compact dimensions in our space–time, they would have been discovered long time ago, which is not the case

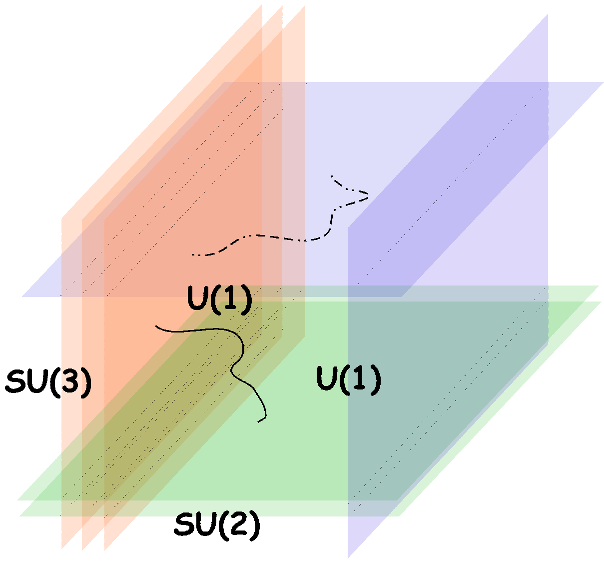


Figure 2.4: Possible brane scenario for the standard model: strings stretching between different brane stacks confer the "particles" desired quantum properties, like coupling to different gauge groups.

of the observed, infinitely extended four dimensional space–time and a compact six dimensional manifold CY_3. A Calabi–Yau space is an extremely mathematically rich manifold, i.e. there are lots of freedoms for designing its shape. A glimpse of such a object is given in picture 2.3, where a slice through this six–dimensional space is shown. For more on the mathematics of those spaces, see e.g. [48, 49, 50, 51]. We will inspect in more detail now the degrees of freedom of those spaces and thus of the new string theory, which is now quite similar to our world. As it will turn out, those degrees of freedom can be used to fine tune the model and eventually to get a description for a real world[9].

2.2 Results and predictions

2.2.1 Models

Additional to the internal degrees of freedom of a given Calabi–Yau, which are the size and the shape of it, additional nontrivial space "properties", like e.g. *orbifolds* and *orientifolds* (see for example [52]) can be added to the Calabi–Yau space leading to a very rich spectrum of possible compact manifolds. This originates from following very interesting effect: although the space–time is "just" a direct product of the four dimensional Minkowsky and Calabi-Yau manifold it turns out that the effective physics

[9]In order to make oneself a better image of those circumstances, we might imagine a radio, the buttons of which can be turned in order to reach the desired effect. However, our "string theory radio" has lots of buttons such that searching for a special effect might prove quite long–dated

in our experienced world will get influenced by the choice of the Calabi-Yau, the orbifolds, respective orientifolds imposed on it[10] and also on the manner the D–branes are arranged and curved in the Calabi–Yau space (e.g. they can be wrapped around cycles). We would like now to extract the particle content out of the string and see different mechanisms by which the standard model could be reproduced. In contrast to Quantum Field Theory we do not add up the different "matter" and "force" particles by putting together the corresponding Lagrangian. The string is supposed to be the essence of everything, for the "matter" as well as for the "force" particles. Thus the whole spectrum should arise just from the intrinsic degrees of freedom of the string and its excitations. It should thus be clear that D–branes play a crucial role, since they were the support of open strings with Chan–Paton factors, which on their part created the massless gauge particles. Furthermore the gauge group $SU(N)$ is given by the number of D–brans N on top of each other. Remembering that our standard model was given by the gauge group $U(1) \times SU(2) \times SU(3)$ we can imagine that forming an intersecting brane model, with different stack of three, two and one D–branes on top of each other, will simulate quite good the standard model. Such a scenario is depicted in figure 2.4. But even in this model, not every freedom has been chosen, for also the intersecting angles can for example preserve or not supersymmetry. For such models consult the work [53] and also the references therein. However, if we were at the beginning searching desperately for a theory to embrace the standard model and also cure its shortcomings, we have found now an apparatus with a lot of parameters and thus degrees of freedom, which are responsible for creating different *vacua*! So we have now plenty of models and have to search for the right one in the huge number of possibilities. General agreement hasn't yet been established, so the numbers still vary, but it is generally accepted that the possible number of different vacua should range from about 10^{500} until even infinity. This high sensitivity of the effective action, which should describe our world, is very dramatic insofar every choice of an CY_3 gives one "possible world" with its own matter content, its own interactions and its own coupling strength. So we still have just a few highly constricted theories, but we have a huge, maybe not even finite, number of vacua (for more on that see [54, 55, 56]). The problem is known as the "Landscape problem" and causes great worries to the string theory community. Different ways has been searched for, in order to be able to hold up string theory, even philosophical points of view haven't been spared in physics, e.g. the *anthropic principle*[11].To read more on that see e.g. [57, 58, 59, 60] So far we could try to categorize the different string theories, however the landscape problem remains.

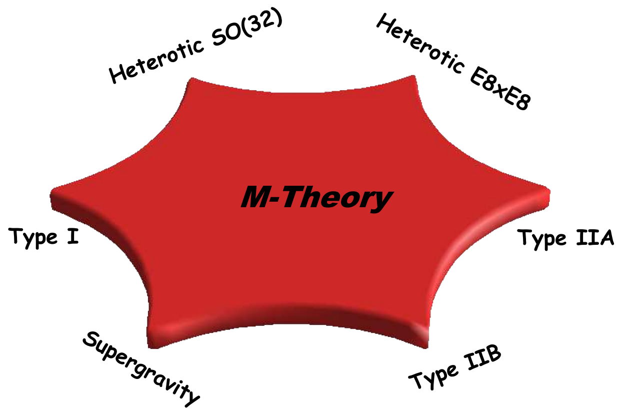


Figure 2.5: String theories and supergravity as different limits of an conjectured underlying basic theory

2.2.2 Dualities

When we analyzed the spectrum of string theory in order to attempt making a model for the physical world, we encountered closed and open strings, the latter with Chan–Paton charges attached to their ends. This former property of being closed or open will be a very important classification tool for our theories: open strings can make loops when interacting, and form closed strings, but closed strings remain closed without breaking ever. Furthermore, constraints are imposed on the gauge group $U(N)$ emerging from D–branes, by consistency requirements. These reduce the gauge group to $SO(32)$. As specified before closed and open strings are part of two different theories. Also the quality of being orientable or not is one criterium of categorizing string theories. Thus, the string world–sheet swept out by the evolving string in the space–time can have one orientation, such that one of its surfaces keeps always pointing outwards, or it may not be orientable, like the Moebius strip or Klein bottle in the case of open respectively closed strings. Already now, we have some distinct, clear delimited theories: the one containing open and thus also closed strings, we will call it ”Type I” theory, with gauge group $SO(32)$; two types of closed string theories, called ”Type IIA” and ”Type IIB”, depending if the world–sheet is orientable or not; finally there are two *heterotic* string theories with gauge groups $SO(32)$ respectively $E_8 \times E_8$. These types of string theories are a mixture between a bosonic string and a superstring, as already the adjective ”heterotic” suggests. Such a variety of different string theories (not vacua!) may not necessary be desired, for we started unifying the different interactions and wanted just *one* theory to describe the whole nature and ended up with five theories (despite the huge amount of the possible vacua). However, a rich web of dualities has

[10]Those have the property to project out some of the string states, fact which has very dramatic effects, like reducing the degree of supersymmetry or even breaking it completely

[11]The anthropic principle declares that under the many possibilities of universes our has been chosen just by chance and no wondering is necessary about that lucky choice, since hadn’t been that the case, we couldn’t have notice anything by the simple reason of not being created!

been discovered, relating all upper theories to each other. Dualities relate different regimes of coupling strengths of the five theories[12]. This is eventually a strong hint for an underlying, more basic, theory, whereas the five string theories are just effective theories of different limits of the underlying theory, call it *M–Theory*, see fig. 2.5. The whole network of dualities and its implications are thoroughly discussed in [65, 66, 67]. See also [64] for more technical examples of dualities in string theory. The theory named *Sugra*(abbreviation of Supergravity) depicted in the figure should also be one limit of M–Theory and is a supersymmetric formulation of Einstein's general relativity. Thus there are strong hints that string theory is just some corner of a much more richer M–theory. This M–theory contains the already developed five string theories and also the supergravity. Thus, there is still hope that somehow, this more powerful theory could concretely be formulated and be also restringing enough to cure maybe the landscape problem by just delivering some pick up mechanism for the right vacuum.

[12] The notion of dualities is very nicely introduced in field theory in [61, 62, 63]

Chapter 3

The concept of effective theories

As we saw in the previous chapter superstring theory may contain a lot of information about our environment. The number of particles, their quantum mechanical properties like charge, spin, mass, color and so on can be recovered from the right model of superstring theory. The number of gauge groups and their kind and dimension can also emerge from strings and even the properties of particles to couple to different forces, i.e. to be sensitive to different gauge particles is possible. All those different possibilities give rise to different (world)–scenarios which maybe could be extracted out of the rich string landscape. However, even if we find or we are able to predict the right landscape, i.e. the right vacuum, the full connection between theory and reality is not established, since the theory is separated from experiment by a huge energy barrier. We need thus a very last piece to connect the two regimes. We will qualitatively analyze this last piece in this chapter. Given one vacuum chosen out of the landscape it will surely contain more information about our world than we posses at this time, for string theory should predict new effects and also explain old ones which weren't understood. Since we describe today all phenomena by quantized field theories, which on their part are encoded in the language of Lagrangians we expect the new information from the string theory as new Lagrangians describing the dynamics of new particles and/or as corrections in some parameter to the Lagrangians we already have. Exactly those Lagrangians, respectively the corrections thereof we want to extract from string theory. This principle is known under the name of *effective theory*.

3.1 Definition and examples

Almost everywhere in physics the concept of "effective theory" finds its place. In almost all cases, the underlying (microscopic) theory is very complicated, nonlinear and hard or even impossible to solve. In order to still describe the system or at least say something about it, a new theory is build with new parameters which describe more "roughly" the system. This way, not all degrees of freedom are taken over and the new effective theory describes the phenomena at a new length scale (which is in one to one correspondence with the energy scale) being bigger than the microscopic one, whatever this one may be. It's important to notice that possible microscopic structures are fully neglected, such that the system under consideration is seen as "atomic", without any substructure. Of course, when doing such assumptions, the effective theory will show infinities when going too high with the energy: since the scale at which that theory is defined, is finite (for we neglected any substructure of constituent system, and this means per definitionem we look somehow diffuse at our system), we cannot go deeper since there we encounter effects given by the substructure of the systems. Also the new defined parameters of the effective theory may not even exist in the microscopic "exact" theory, but they work very well in the framework, i.e. at the scale at which they were introduced. This is also the reason for potential divergencies of those parameters when increasing the energy, since then we probe scales at which those parameters are not defined, this being signalized by infinities. Now we will analyze some examples such that all those abstract facts will become quite clear and even familiar.

Prominent examples there are many, but one of the very oldest and thus classical is thermodynamics. In order to describe one particle, classically, there aren't any problems encountered, two particles are described with the same easiness, but in the case of 10^{23} particles the situation changes drastically. One cannot keep anymore the individual degrees of freedom of every particle since not even a supercomputer could master that. Instead effective measurable quantities are introduced, such as temperature, pressure of entropy which don't even make sense in systems with a few particles[1]: they are genuinely effective quantities. This way a description of such a system is possible at all. On the other hand, that description is a very good and accurate one, despite the fact that all microscopic properties of the single particle are neglected. We thus easily understand that "pressure" is defined only in the case of many particles. Gradually reducing the number of the particles will at some point create problems since we reach scales where the quantities cease to be properly defined.

Somehow orthogonal to that example is the following one: in nuclear physics, when describing nuclei of heavy elements on makes use of quantum hadrodynamics. In that case, the number of particles is not necessary large but the underlying, basic theory, *the quantum chromodynamics*[2], is very complicated making it impossible to treat hadron compound systems. This is also the reason we called this example orthogonal to the last one: there we had very simple descriptive methods but a huge number of particles. Here we have relatively few particles but an enormous rich and complicated interaction. (From the point of view of thermodynamics this theory may even not work well, for we might be exactly at the boarder where the particle number is high enough such that the microscopic theory breaks down but the effective theory still doesn't work properly). One single hadron has a substructure being made up by at least two quarks. So an interaction between two ore more hadrons is actually an interaction between quarks which is best described by quantum chromodynamics. Effective potentials are

[1]There arises immediately the question which is the limit where the effective theory ceases to work properly. This should however not concern us, since we are dealing always with systems which can be doubtlessly described by effective quantities.

[2]Quantum chromodynamics is a $SU(3)$–gauge theory describing the strong nuclear interaction between the "colored quarks" by means of exchanged gluons.

introduced with the effect that even an effective field theory is created: thus the force between the observed particles is mediated by "mesons". Those particles are at their own also compound of quarks, but here they are regarded as being the "elementary" force carrying entities. Thus one neglects the quarks and treats the hadron as being "atomic" and described by an effective theory which gives them effective properties, without asking where they come from.

Another example is the BCS–theory which describes superconductivity in metals. The underlying theory in that case is the quantum mechanics of solid bodies, and especially of electrons and phonons[3] Those interact with the effect of binding two electrons in a Cooper–pair. This pair of electrons behaves as a bosonic particle, not obeying the rules for fermionic states.

3.2 Techniques

Despite the number of particles and that of gauge groups emergent from string theory, the latter also makes predictions which can be analyzed within the respective low energy effective action. The energy scale set in the framework of effective theories will also be the battle field where all the concepts introduced before, i.e. string theory, and effective actions will meet together supplied by some other mathematical ideas, to be presented in next chapters. As stated before, string theory has infinitely many vibrating modes, almost all of which are massive. So if would like to set up an effective theory and consider all those excitations, it would require an infinite number of differential equation coming from the effective action in order to describe them. The solution is just to neglect the majority of the string states. Since the string tension is in the region of the Planck mass (10^{19} GeV) the massive modes are extremely heavy such that we can be sure that this simplification will not affect the effective theory, at least not at the scale we are looking at. It would require enormous energies to come in the regime where also the massive states would show up in the action with new effects. Thus, the heavy modes can simply be integrated out, such that in the effective action just the massless modes contribute. It is worth to notice that the number of the massless modes is finite and even rather small.

Several ways exist in which we can come to that effective theory beginning with string theory. One way is to formulate string theory from the beginning on in an given background. A background denotes just the fields describing the space–time in which the string evolves, thus the well known metric. Generally, this background is usually given in terms of an traceless metric $G_{\mu\nu}$, its antisymmetric part $B_{\mu\nu}$ and the trace Φ. These quantities are respectively called the graviton, antisymmetric tensor field and the dilaton. The formulation of string theory is, as already said, very restrictive, since we have to preserve all the symmetries from the classical action. This way also the background fields are constrained. When imposing conformal invariance on

[3] A phonon denotes a vibrating mode of a lattice in a solid body. It proves that such vibrations can propagate and even stay localized showing particle properties, fact which animated physicists to describe it as a (quasi)particle.

string theory in that background, also the background defined above has to fulfill some requirements. That consistency is nothing else than differential equations which are exactly the equations of motion for the named fields, see e.g. [68]. We shall though not follow that method because its difficulty but will approach another method, somehow more simpler to calculate the effective action. We will make use of the fact that correlation functions calculated in field theory are equal to the corresponding amplitudes in string theory. By "corresponding amplitudes" it is meant that one takes the same topology of the interaction[4], same number and sort of interacting particles and of course the same background. This method we will use in the present work.

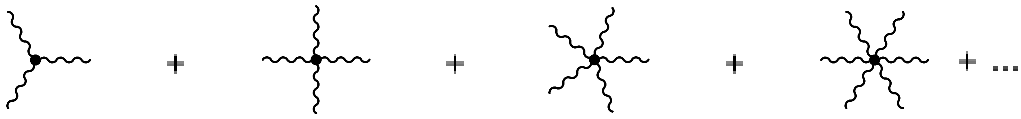
As a next step we shall have some thoughts on the field theory. Since we have to compare the field theory with string theory (by computing the same scattering process) we could try to guess a Lagrangian for the field theory and then compare it with string theory and finally just adjust it. Thus one could write down the most general, non–redundant ansatz. The non–redundancy should be emphasized here, for a field theory Lagrangian is not unique, since field shifts and redefinitions don't change the physics (S–matrix). (More specific field operations which let the physics unchanged will be presented in the Born–Infeld section). So when an ansatz is written down, great attention is to be paid in order not to count the same term twice or often, just because it is written different! Such an ansatz, which is also valid for D space–time dimensions, could be of the following form

$$S_{eff} = \alpha'^{\,-D/2} \sum_{n,m}{}' \alpha'^{\,1/2(n+m)}\, c_{nm}\, \partial^n \Phi^m, \tag{3.1}$$

where the coefficients c_{nm} are unknowns, to be determined from the corresponding string scattering amplitudes. Actually those are the quantities which are wishful to be determined, for they exactly encode the ratio between the consecutive terms in the action. As the notation already suggests, c_{nm} are ordered with respect to the number of derivatives acting on the number of fields. Each term is a combination of derivatives and the respective fields under consideration. Thus c_{nm} exactly determines how many terms contain just powers of Φ or how many contain a number of derivatives acting on the specific number of fields Φ. As stated above, the prime on the sum indicates we have built the expression such that each term present is unique, i.e. every term is counted once and no field transformation or Lagrangian symmetry is able to relate two different terms in the expansion (3.1). Furthermore, the expression is organized as a power series in the string tension α', which will necessary be encountered when computing the amplitudes. Possible other constants may enter the series, like the string coupling constant g_{String}. The latter organizes also the string loop expansion. Last but not least, each term is in one to one correspondence with a specific Feynman–diagram.The correspondence can be established when considering solely the number

[4]Feynman graphs can be classified in tree–, one loop–, two loop–, etc, diagrams with N external particles. Further, tree–diagrams can be reducible or not, depending wether some internal states are propagated in the process.

of interacting fields given in one specific term (thus neglecting possible derivatives) and also writing just the contact diagrams for that term. The series can thus also be "written" pictorially as



which is the same expression written as the corresponding sum of Feynman graphs. Thus the first term in the "Feynman representation" stands for the terms $\Phi^3, \partial^2\Phi^3, \partial^4\Phi^3, \ldots,$ and all other higher derivative terms with only three fields[5] Φ. Correspondingly, the next Feynman diagram represents all four–field interactions, regardless of the number of derivatives acting on them.

We are not yet familiar with string amplitudes, but when given they are a product of momenta k_i and fields Φ_i. In order to establish the equality between string and field theory the string momenta have to be replaced with derivatives ∂_i and in case of non abelian theories with the corresponding covariant derivative $D = \partial_i + [A_i, *]$. This method is usually used in field theory.

3.3 Born-Infeld–action: a first look

In this section we will look very carefully at a special example of an effective action, namely at the Born–Infeld action. This field theory describes approximately the dynamics of D–branes in string theory. Since the Lagrangian given by that effective theory is the only one which describes D–branes in the low energy regime it is worth to be studied. We will first list the known form of this Lagrangian and corrections to it will be discussed in chapter 7.

As we already know, open strings carry Chan–Paton charges attached at their ends. Those charges with the corresponding massless vector modes of the string are responsible for the gauge group. Ignoring the massive modes, we aim obtaining an effective theory for the massless gauge modes of the string which are confined to the surface of the D–brane. This will naturally describe the excitations and consequently the oscillations of the latter, as well as the modes dynamics. It is then naturally expected, since we are dealing with gauge modes coming from the superstring, when looking at the corresponding low energy effective action, to obtain some Yang–Mills theory. This is indeed the case, a highly nonlinear gauge theory is obtained, named Born–Infeld action. This action describes the low energy behavior of open strings or

[5] As a matter of fact we have just chosen such terms which are Lorentz invariant. This is the reason we have written just even numbers of derivatives connected with each other. In the case of vector fields Φ^μ there is also the possibility of contracting the derivative with the fields itself. This will be the case when treating the non abelian Born–Infeld action.

equivalently the action for the corresponding D–brane to which the open strings are attached. As the gauge group under which the Chan–Paton charges transforms can vary, one obtains different gauge theories. For the simplest case of an $U(1)$ gauge group the low energy theory will contain nonlinear corrections to the well known $U(1)$ gauge theory, which is just electrodynamics! Historically, Born and Infeld searched for a nonlinear extension of electrodynamics, arriving at the same non linear $U(1)$–theory which also describes a single D–Brane. This is the reason for the nomenclature in case of effective actions for Dp–branes. The action for the abelian string, was derived 1986 by Tseytlin, [69] and is of following form:

$$\alpha'^{-1/2-p/2} \int d^{p+1}x \sqrt{1 + (2\pi\alpha' F_{\mu\nu})^2}. \tag{3.2}$$

The space–time integral is performed in ten dimensions since this is the condition for an anomaly free superstring theory. Further, inside the square root, which causes the high degree of nonlinearity, we recognize the squared gauge field strength $F_{\mu\nu}$[6] multiplied by the string tension α'. Series expanding this action in α', the first term will describe "usual" electrodynamics, $F_{\mu\nu}F_{\mu\nu}$ being the action for the $U(1)$–gauge boson. Consecutive terms are corrections to that classical action. One very interesting and enormous useful property of (3.2), is its closed form. So one in principle knows the correction to arbitrary order in α', one just needs to series expand the formula 3.2 to desired order.

Since the Born–Infeld action is the only theory describing the behavior of D–branes, it is of burning interest to see how the action for more D–branes at top of each other looks like, this being the case for non abelian gauge group. As explained before, when dealing with more D–branes at top of each other, the gauge group of the massless vector particle generated by the open string gets enhanced to $U(N)$, where N is the number of D–branes. So in that case, the Born–Infeld–actin has to describe a non abelian, nonlinear gauge theory. This action is of high interest and until now, unfortunately, just a perturbative expression (in α') has been obtained. Thus, for each higher order term big effort has to be done, i.e. the corresponding string amplitudes is to be calculated, and then the effective action to that order can be extracted. Up to α'^2–order this action has been computed to (see [69])

$$\begin{aligned}\mathcal{L}^{Dp}_{\text{effective}} = \mathrm{Tr}\Big\{\, F^2_{mn} - \frac{1}{3}\,(2\pi\alpha')^2\,\Big(\, & F_{ab}F_{bd}F_{ca}F_{dc} + \frac{1}{2}\,F_{ab}F_{bc}F_{cd}F_{da} \\ & -\frac{1}{4}\,F_{ab}F_{ba}F_{cd}F_{dc} - \frac{1}{8}\,F_{ab}F_{cd}F_{ba}F_{dc}\,\Big) + \mathcal{O}(\alpha'^3)\,\Big\}\ . \end{aligned}\tag{3.3}$$

In this formula,Tr denotes the trace over the gauge indices(which for the sake of clarity have been suppressed) since we deal with non abelian fields. The explicit

[6] As usual in field theory the field strength is given by the exterior derivative of the one–form field dA, or in components by $\partial_\mu A_\nu - \partial_\nu A_\mu$. In the case of non–abelian fields there is an additional commutator $[A_\mu, A_\nu]$

indices in the formula are just space–time indices coming from both the field A_μ and the derivative ∂_μ, respectively the momentum k^μ. For the more exact definitions of the non–abelian objects see Appendix J. As a first remark, one should notice that the action 3.3 can be obtained by just series expanding equation 3.2. However, when going further with the expansion in α' the non abelian series departs from 3.3, already at α'^4–order. This is not fully unexpected since in the non–abelian action the field definition is different as the one in the abelian case. The action given in 3.3 is the naive generalization of the action 3.2 for the non–abelian case, however it proves not to be a valid one. When calculating properly the non–abelian action (as will be intensively presented in the present work), terms like D^2F^4 appear at the respective order in α', which cannot be anymore neglected as in the abelian case[7]. Furthermore those terms are multiplied by transcendental numbers, like $\zeta(3)$, $\zeta(4)$[8], etc, which could never arise from the expansion of a square root! Those terms can uniquely be computed with the help of string amplitudes. While D^2F^4–term comes just from a higher momentum expansion of the four–point amplitude [71], terms like F^5 have to be extracted from the five–gluon amplitude in string theory, [72, 73, 74]. To conclude, the non abelian Born–Infeld is today fully known until $\alpha'3$. In order to have the full result also from the α'^4–order the six–gluon amplitude from superstring is needed, which will be the main topic of that thesis. This amplitude will make possible to have full control about all terms of the form $D^{2n}F^6$, like shown in following table:

α'^0	1	$\mathbf{F^2}$			
α'^1	0	F^3	D^2F^2		
α'^2	$\zeta(2)$	$\mathbf{F^4}$	D^2F^3	D^4F^2	
α'^3	$\zeta(3)$	$\mathbf{F^5}$	$\mathbf{D^2F^4}$	D^6F^2	
α'^4	$\zeta(4)$	$\mathbf{F^6}$	$\mathbf{D^4F^4}$	$\mathbf{D^2F^5}$	
α'^5	$\zeta(5)$	$\mathbf{F^7}$	$\mathbf{D^6F^4}$	$\mathbf{D^4F^5}$	$\mathbf{D^2F^6}$
$\vdots$		$\dots$	$\dots$	$\dots$	$\dots\dots$

Table 1: *Higher order F–terms appearing at a given α'–order in the supersymmetric D–brane action .*

In the table, just the bold terms have to be considered, since the the other ones either can be derived from the bold terms or simply vanish by requirements coming from string theory, as is the case for the term F^3. This term is found to be nonexistent when computing the corresponding superstring three–point function. Furthermore, the dependence of the other terms may be proven by using Bianchi identities, equations of motion of the on–shell gauge fields, partial integration and finally field redefinitions. This is a tedious work to be done but very important in order to obtain the correct field theory. Further, the left column shows the α'–order and the zeta factor by which

[7] Due to $[D_\mu, D_\nu]\ F_{\rho\sigma} = -i\ [F_{\mu\nu},\ F_{\rho\sigma}]$ derivative terms can be converted into commutators of non–derivative terms and viceversa. Derivative terms could be fully neglected in the abelian case, see [69].

[8] $\zeta(3)$ is the Riemann ζ–function evaluated at the real number 3.

the term is multiplied. As a last remark it's worth to notice that in the case of odd powers of the field strength $F_{\mu\nu}$ the zeta factors are atomic numbers, i.e. there is not some more basic expression for them in terms of other mathematical constants, as is the case for the even zeta numbers which can be expressed in terms of powers of π. This fact is a good criterion for organizing the odd terms from string theory, since the zeta numbers do not mix as is the case for even zeta numbers: for that we can look at terms coming from the α'^4–order multiplied by $\zeta(4) = \frac{\pi^4}{96}$ and at terms from the second order in α' multiplied by $\zeta(2) = \frac{\pi^2}{6}$. A product of two terms of second order is also multiplied by π^4 and some rational numbers exactly as terms from the fourth order do. This makes it harder to disentangle them and exactly trace them back to their origins.

Chapter 4

Tree–level scattering of open superstrings on Dp–branes

4.1 Tree–level amplitudes: the basics

This section should serve as a very introduction to the topic of superstring scattering at the tree-level. For that we will study in some detail one more quantization method, namely the functional integral. Since the consequent full approach would require a huge amount of time and knowledge in mathematics, we will restrict ourself to the basic ideas and results and sometimes derive them rather heuristically, without being very precise mathematically. Also, for the sake of brevity we will dedicate the following considerations the bosonic string, its supersymmetric generalization being not that complicated and reviewed in the next section.

4.1.1 Path integral formulation

As mentioned already in the introduction, one very elegant although quite formal procedure to deal with in quantum field theories is the Feynman path integral. However, when treating interactions in string theory one is more or less "forced" to use that technique, since processes involving more strings are automatically included in the theory, they just being splitting and/or joining of strings, which on their own are just different topologies of the world–sheet, one being eventually able to express those topologies in terms of computable mathematical quantities.

In field theory, path integration means representing amplitudes as a sum over all possible paths occurring between initial and final states, weighted with the classical action.

It turns now out, that this same tool will be a quite natural way to introduce interactions in string theory. In the language of string theory, the path integration will run over all different world–sheets which connect the initial state with the final one. Thus there will be, as in the case of field theory, uncountable infinitely many world–sheets, forming all possible shapes, which connect the initial string states with the final configuration. But since we are summing over **all** world–sheets, we are summing over all possible topologies, and exactly those topologies can be interpreted as strings merging and diverging, fact which is exactly what we imagine under interaction in string theory. Of course the question arises, how is this sum to be implemented? the formal answer is

$$\int [dXdg] \ e^{(-S)}. \tag{4.1}$$

We see the weight $e^{(-S)}$ being the classical action of the string, which is just the Polyakov–action, and the path integration is performed, similarly to field theory, over all fields encountered in the action: the metric g [1], which is responsible for the world–sheet topology and the field X, this being the string itself.

Now we have to remember that our theory had a very strong local symmetry, namely the Weyl $\times$ diffeomorphism–invariance, which allowed for arbitrary local transformations of the metric. Thus different looking metrics are connected by that symmetry and identified. This way different world–sheets are related to each other, the consequence of that being that we perform the path integral over infinitely many metrics which are in fact the same. Thus the path integral over **all** metrics $g_{\mu\nu}$, will surely overcount the number of world–sheets and our path integral will be infinite! This is nothing else than the conformal anomaly encountered in the canonical quantization and which we have fixed by imposing the closure of the Virasoro algebra, (2.13). We will fix it here again, by imposing an explicit gauge choice. This procedure is already very well known in gauge theories where one faces the same problem, see more on that in [9]. There, the action is symmetric under some continuous space–dependent transformation of the fields which relates the latter to each other. It will thus be clear that a naive path integration over the fields will be divergent since overcounting. We recognize thus the problem described above in string theory. The way out is to simply consider solely independent metrics, so the ones not being related by Weyl and diffeomorphic transformations:

$$\int \frac{[\,dXdg\,]}{V_{\mathrm{diff}\times\mathrm{Weyl}}} \ e^{(-S)}. \tag{4.2}$$

Here we have denoted by $V_{\mathrm{diff}\times\mathrm{Weyl}}$ the volume of the conformal group, by which we divided the functional integral, signaling we have fixed that symmetry. For sure this is a highly formal expression but using the Faddeev–Popov determinant, one can eliminate the volume and eventually introduce a correlation function of three bosonic ghosts, which allows us to fix the position of three vertices of those involved in the string scattering.

[1] We have performed here a Wick rotation: the Minkowski world–sheet metric γ is replaced with a Euclidean metric g, changing signature from $(-,+)$ to $(+,+)$.

4.1.2 Gauge fixing

In order to brake the symmetry we have to choose an explicit gauge. Thus we will require the metric satisfy some condition $F(g_{\mu\nu}) = 0$ and after that multiply the functional by the respective delta functional

$$\int [\, dXdg\,]\ \delta_{\text{gauge}}\left(F(g_{\mu\nu})\right)\ e^{(-S)}. \tag{4.3}$$

We shall choose for F the expression $(g - \hat{g})$, where $\hat{g}$ is just the flat Minkowski metric. However, the naive insertion of the delta function will change the integration measure, thus we will also have to insert its corresponding determinant such that the expression in its whole equals one:

$$1 = \Delta_{FP}(g) \int [d\mu]\ \delta(g - \hat{g}^{\mu}) \tag{4.4}$$

In upper formula we integrate over all gauge transformations, i.e. Weyl and diffeomorphism transformation of the metric. Δ_{FP} denotes the corresponding determinant coming from the variable transform in the delta functional, and is named after Faddeev and Popov. This is the correct expression we want to insert into (4.2), such that the integral won't be changed, and we arrive at the final formal formula for the string functional:

$$Z[\hat{g}] = \int \frac{[\, d\mu\, dX\, dg\,]}{V_{\text{diff}\times\text{Weyl}}}\ \Delta_{FP}(g)\ \delta(g - \hat{g}^{\mu})\ e^{(-S)}. \tag{4.5}$$

This way we have succeeded in eliminating the superfluous world–sheets, for the delta functional will exactly filter out just the ones satisfying the desired condition F. We can thus carry out the path integral over the metric and rename the variables. Moreover, we have inserted before the identity into the string functional as an integral over gauge transformations with measure $d\mu$. The volume of the $PSL(2, \mathbb{C})$–symmetry which is the (diff×Weyl)–symmetry in the denominator of our formula will be exactly canceled by that integral, and we are left with

$$Z[\hat{g}] = \int [\, dX\,]\ \Delta_{FP}(\hat{g})\ e^{(-S)}. \tag{4.6}$$

The Faddeev–Popov determinant will be analyzed in more detail in appendix B.

4.1.3 Gauge anomaly and ghosts

Of course we want now to evaluate the expression (4.6) which is still a little bit difficult because of the formal determinant of Faddeev and Popov. Though it can be computed: starting with the general conformal transformation

$$g^{\mu}_{ab} = e^{[\,2\omega(\sigma)\,]} \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}\; g_{cd}, \tag{4.7}$$

which we have separated into pure diffeomorphisms and pure Weyl transformations, we can expand it near the identity and invert relation (4.4) for the determinant. Representing it as an integral over ghost fields, as explained in appendix B will eventually lead us to following form of the string functional

$$Z[\hat{g}] = \int [\,dX\, db\, dc\,]\; e^{-S_X - S_g}, \tag{4.8}$$

where we have integrated over ghost fields b and c as explained in the appendix and the action S_g denotes

$$S_g = \frac{1}{2\pi}\int d^2\sigma\;\hat{g}^{1/2} b_{ab}\hat{\nabla}^a c^b, \tag{4.9}$$

the action for the ghost fields. The hat on the Nabla–operator indicates we have used the gauged metric, i.e. the unit flat Minkowski metric. More about the ghost integration over the corresponding Lagrange density will be said just in the next subsection, where its consequences will be a fixing of positions of three vertices. However, before we shall treat the vertices one open question still remains: to what extent is the Weyl symmetry preserved after quantization? i.e. is the string functional $Z[g]$ independent of the gauge imposed on the metric g_{ab}?

The Weyl invariance means classically that the energy–momentum tensor T_{ab} is traceless. In the path integral case, we obtain T_{ab} by varying the Polyakov action with respect to the metric g_{ab}, which eventually gives for the operator trace

$$T^a_{\ a} = f\, R, \tag{4.10}$$

with f some proportionality constant and R the Ricci scalar. To check this equality in the quantum case and also to determine the constant f we can look at the transformation properties of both sides of equation (4.10). Comparing the two transformations and also setting them equal will set the Minkowski space–time dimension for the bosonic string to 26 and in the case of supersymmetric strings the space–time dimension proves to be ten. So again, this is the dimension in which superstring theory is free of any anomalies. String theory can also be formulated in different dimensions, though the price to be paid is the loss of consistency. This is again a confirmation of the results already achieved in the introduction, where we have quantized the string with another method. Hence, the equivalence of the two methods and also the correctness of the results is proved. Now we shall move to vertices and finally to string interactions.

4.1.4 Ghosts and moduli

Time has come to care about the ghost and the determinant coming from integration them out. We started with an diverging integral over redundant metrics

$$Z[\hat{g}] = \sum_{\text{Topologies}} \int [\, dX\, db\, dc \,]\; e^{-S_X - S_g} \;, \tag{4.11}$$

caused by the strong conformal symmetry which the theory shew. After fixing that symmetry, such that the integral became finite, (4.8) we expect now an integral over metrics, which are somehow parameterized by a variable; this parameter we will call "modulus". In our case it exactly describes different metrics, such that they cannot be related to each other by an conformal transformation,

$$Z[\hat{g}] = \sum_{\text{Topologies}} \int [\, dX\, db\, dc \,] \int d^f p\; e^{-S_X - S_g} \;, \tag{4.12}$$

Here we have signalized that, by introducing an integral over some f moduli p, which are to describe the metric of the associated world–sheet geometry. A simple case will deliver a good example of how one can pictorially imagine that.

Two dimensional spaces are simple enough such that they can be completely categorized in terms of Riemann surfaces, [75, 76]. Fortunately, the string sweeps out a compact two dimensional surfaces, which can thus be easily analyzed. In the case of closed strings, the situation is even simpler, since the surfaces ca be conformally mapped to the sphere or to tori with different numbers of handles, thus those surfaces being enumerable by just the numbers of handles, g, they have. When dealing with close strings, the sum over topologies in formula (4.12) will run over g, and thus in the language of Feynman diagrams will organize the interactions in tree– and loop–interactions.

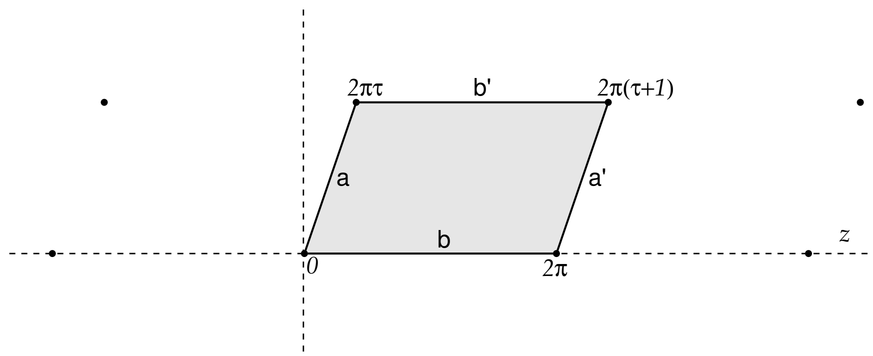


Figure 4.1: Generating basic vectors for the algebraic torus with complex modulus τ.

Lets pick up one simple interaction though non trivial, the simplest loop–diagram, thus lets fix $g = 1$. In that case we will have to do with a torus which will be the world–sheet swept out by the string. Above we have depicted the "algebraic" torus, which is defined as identifying the opposite sides of the upper parallelogram, $(\sigma^1, \sigma^2) \sim (\sigma^1, \sigma^2) + 2\pi(p, q)$. As usual, (σ^1, σ^2) denote the two coordinates of the string world–sheet, thus time and one–dimensional space. Further we have $(p, q) \in \mathbb{Z} \times \mathbb{Z}$. Thus its curvature is that of the flat space. Further, we can use our (diff $\times$ Weyl)–symmetry to bring the metric to the form

$$d^2 s = |d\sigma^1 + \tau d\sigma^2|^2 \ . \tag{4.13}$$

Here we have introduced a complex parameter τ, which is nothing else than the modulus described earlier, which should pick us exactly the inequivalent metrics g_{ab}. The parameter is defined as having its real part positive, consequently it lives in the complex upper half–plane. However, one immediately notices in formula (4.13) that the metric is still invariant under two transformation of the modulus, namely

$$T : \tau \to \tau + 1 \quad , \quad S : \tau \to \tau = -\frac{1}{\tau} \tag{4.14}$$

Those two transformations form the group $PSL(2, \mathbb{Z})$. This is found when noticing that the T and S transformations are equivalent to

$$\tau \to \frac{a\tau + b}{c\tau + d} \ , \tag{4.15}$$

where (a, b, c, d) are all integers and $(ad - bc) = 1$. Thus we can form a matrix U which acts on τ as $\tau \to U\tau$ with the requirement $U \in PSL(2, \mathbb{Z})$[2]. Since this symmetry produces redundant metrics we should also fix it. This is formally done by requiring the modulus τ to be constrained in $\mathbb{C}/PSL(2, \mathbb{Z})$, thus in complex plane modulo the symmetry group. This new space can also be computed and is shown in the figure 4.2.

We thus arrive at the following form for an string scattering function

$$\mathcal{A}_N = \int \frac{[\,dX\, dg\,]}{V_{\text{Diff}\times\text{Weyl}}} \int d\tau \ e^{(-S_X)} \prod_{i=1}^{N} \int d^2\sigma^i \ \mathcal{V}_i(k_i, \sigma_i) \ , \tag{4.16}$$

where the integration over τ is taken to be within the region $\mathbb{C}/PSL(2, \mathbb{Z})$. However when we have vertices inserted as above, thus not dealing with the "naked" partition function, there is another physical way in order to avoid that symmetry. We can fix the position of κ vertices, where κ is related to the Euler χ number and the genus g as

[2] This group is also known as the modular group for obvious reasons. A highly interesting and very nowadays very active branch of mathematics is concerned with modular groups, generalizations thereof and also modular functions and forms, [99]–[102].

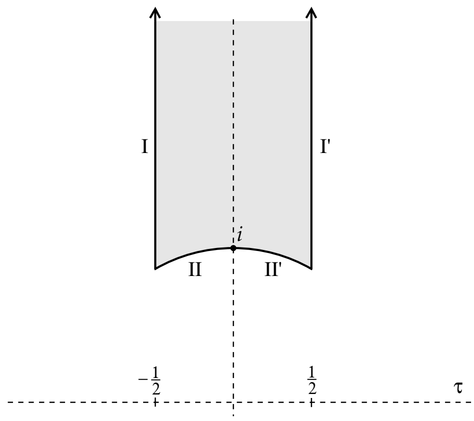


Figure 4.2: Fundamental region of the complex modulus τ of a torus.

$$2\kappa = 3\chi \quad , \quad -3\chi = 6g - 6 \; . \tag{4.17}$$

In the case of tree–amplitudes, g will be set to zero and κ will equal three, allowing us thus to fix the position of three arbitrary vertices. This terminates our discussion about moduli integration. Next section will serve as a definition of vertex operators and after that we will be in the position to calculate simple tree–amplitudes.

4.1.5 From vertices to amplitudes

In order to calculate scattering amplitudes between different states, we will follow the same procedure used in usual quantum field theory together with the path integral representation. There we insert vertex operators into the path integral, thus obtaining correlation functions. We were concerned in the last section with the anomaly free and thus consistent formulation of the path integral function of string theory with no vertices inserted. This functional, (4.8), is also known as a partition function. Mimicking the programm usual in quantum field theories we will insert now string vertices into the partition function, the former representing different string states. The corresponding relation for N external states is given by following formula

$$\begin{aligned} \mathcal{A}_N &= \int \frac{[\, dX\, dg\,]}{V_{\text{Diff}\times\text{Weyl}}}\; e^{(-S_X)} \prod_{i=1}^{N} \int d^2\sigma^i \;\mathcal{V}_i(k_i,\sigma_i) \qquad (4.18)\\ &= \int [\, dX\, db\, dc\,]\; e^{(-S_X - S_g)} \mathcal{V}_1(k_1,z_1)\; \mathcal{V}_2(k_2,z_2)\; \mathcal{V}_3(k_3,z_3) \prod_{i=1}^{N-3} \int d^2\sigma^i \;\mathcal{V}_i(k_i,\sigma_i) \; . \end{aligned}$$

In this expression we recognize the "empty" path integral 4.2, which is just the string partition function. Beyond that we have included additional vertices $\mathcal{V}_j(k_j,\sigma_j)$ integrated over their positions and which will be defined in a while. As argued in the

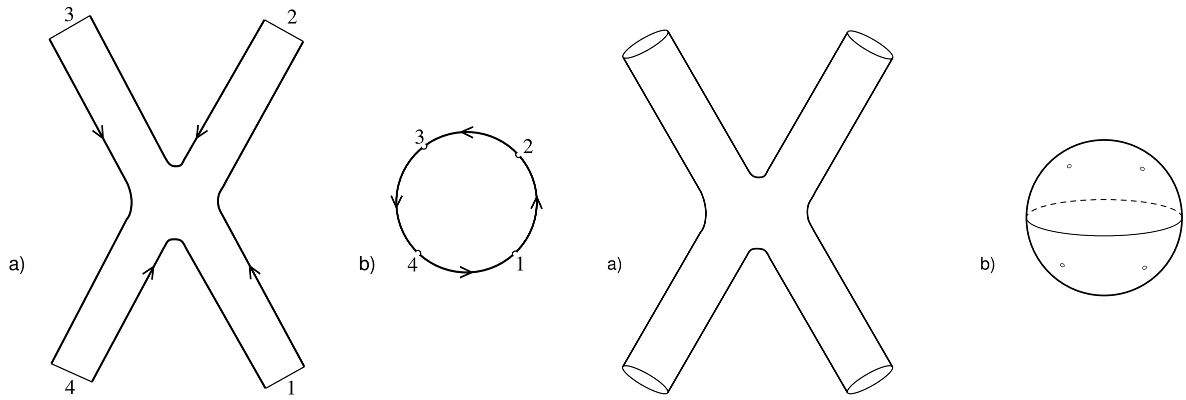


Figure 4.3: Four-point amplitude and its corresponding conformal mapping. Left – open states; right – closed states.

previous section, of all the vertices three are fixed at arbitrary positions, such that the additional encountered symmetry $PSL(2,\mathbb{Z})$ of the metric moduli space is also fixed.

A pictorial description of this scattering process can be seen in figure 4.3. For the sake of concreteness we have picked up the special example of four external states. Also the genus has been reduced from $g = 1$ to $g = 0$ such that we have a tree–amplitude. Moreover this simple amplitude will also serve us introducing the computational techniques. Back to the figure, the world–sheet is conformally mapped to the upper half of the complex plane, which further is isomorph to the disk. The same procedure is applied to the closed string, as seen in the same picture, where the world–sheet is mapped to the sphere. Furthermore, in the case of open strings, external states are mapped to regions of the disk boundary. Since the closed string world–sheet doesn't have any boundaries, its external states are mapped to so called punctures on the sphere. Thus we can think of vertices as being inserted either on the sphere or at the boundary of the disk.

We can now "convert" the path integral expression to Wick contractions of operators inserted into the functions, in a similar manner as done in conformal field theory, thus arriving at following expression for the N–point string interaction

$$\mathcal{A}_N = \left\langle \mathcal{V}_1(k_1, z_1)\ \mathcal{V}_2(k_2, z_2)\ \mathcal{V}_3(k_3, z_3)\ \prod_{i=1}^{N-3} \int d^2\sigma^i\ \mathcal{V}_i(k_i, \sigma_i) \right\rangle \langle\ c(z_1)\ c(z_2)\ c(z_3)\ \rangle\ . \tag{4.19}$$

New about this formula is the fact that we have converted the ghost–determinant into an expectation value of the three ghosts c_i, which are in one to one relation to the number of positions fixed. Since we have now an explicit expression for computing string amplitudes, the very last step towards their calculation is the specification of the string vertices. We should find a suitable definition for them and see how different external states, which we have already interpreted as different particles, i.e. graviton, gauge boson, etc., can be expressed by vertices.

First of all, since vertices are string states ϕ, they have to obey the consistency requirements imposed on string theory, this being the conformal invariance, as explicitly stated in the old covariant quantization method in (2.12):

$$\begin{aligned} L_n|\phi\rangle = \bar{L}_n|\phi\rangle &= 0, \\ (L_0-1)|\phi\rangle = (\bar{L}_0-1)|\phi\rangle &= 0, \qquad (4.20) \\ (L_0-\bar{L}_0)|\phi\rangle &= 0, \qquad (4.21) \end{aligned}$$

Those were conditions imposed for the sake of consistent quantization. In this chapter though, we have already quantized the string with the means of path integration, without imposing the conditions (4.21). However,a major fact of quantization was the (re)establishing of conformal invariance which we have done. So conditions (4.21) will hold also now. This technique will just reveal to us the form of the string states, and not fix any inconsistencies.

We will thus define the string states as conformal fields[3], i.e.

$$|\phi\rangle = \phi(0)|0\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0\rangle. \qquad (4.22)$$

Since we map string states to conformal fields, this operation is being also known as the state/field mapping. Further, since all conformal fields are generated by the conformal operators and since we have all those operators, we can reproduce the whole string spectrum by just acting on the vacuum with the conformal operators L_n. Field representations of the latter can be worked out, see e.g. [77]. The lowest mode of the string is a scalar, known as Tachyon, with momentum k^μ:

$$|k\rangle = \lim_{z,\bar{z}\to 0} : e^{ik_\mu X^\mu} : |0\rangle. \qquad (4.23)$$

By applying to it the requirements 4.21 it can be shown that its momentum obeys $k^2 = 2$ and thus not surprisingly $m^2 = -2$. Its negative mass is the reason this particle is called Tachyon.

Higher excitation states will be defined in complete analogy. We let the creation operators act on the vacuum, the lowest string state, and then apply to them the consistency conditions coming from conformal invariance.

Since the work of finding the vertices has in principle been done, we are in the position of calculating tree amplitudes, at least some of the simple ones. The three-tachyon amplitude on the disk, for example, is given by

$$\begin{aligned} S_{D_2}(k_1,k_2,k_3) = \langle : c^1 e^{ik_1\cdot X}(y_1) :: c^2 e^{ik_2\cdot X}(y_2) :: c^3 e^{ik_2\cdot X}(y_3) :\rangle \\ + (k_2 \leftrightarrow k_3) \qquad (4.24) \end{aligned}$$

[3] Conformal fields are fields which transform in a special well defined way when acted with conformal operators, see [78]

Formula 4.24 might appear a little bit unexpected and a lot of new notation has been introduced. First of all, the expression on the right hand side is an expectation value of operators, as already encountered in quantum field theory. The double point notation $:\phi:$ is just a signal for normal ordering, the pendant of the "time ordering" encountered in field theory. As explained earlier, the contraction of operators is to be viewed as in quantum field theory where exactly the same happens: one starts with a path integral over a number of fields and at the end this expression is equivalent to contracting the respective fields. Those contractions are calculated using the usual Wick–contraction techniques. Now to the fields involved: c^1, c^2, c^3 are the bosonic ghosts, already mentioned before. They are responsible for the gauging of the position of exactly **three** vertices but not for fixing their cyclic order, that's why we have the additional last term on the right hand side with the two momenta k_1, k_2 exchanged. Finally, $e^{ik_j\cdot X}(y)$ is the tachyon at position y.

Notice that all integrals over the positions have disappeared, since we deal with an **three**–point function, but from gauging the conformal symmetry we are free to fix the position of exactly **three** vertices, thus the three–point function is somehow trivial. Last ingredient is now the correlation function of the the fields present in 4.24, those being given by

$$\langle c(y_1)c(y_2)c(y_3)\rangle = (y_1-y_2)(y_1-y_3)(y_2-y_3)$$

and

$$\langle\prod_j e^{ik_j\cdot X}(y_j)\rangle = \prod_{i<j}|y_i-y_j|^{k_i\cdot k_j} \tag{4.25}$$

We can now evaluate the amplitude. Last thing to consider is conservation of momentum $\sum k_i = 0$ and $k_i^2 = 2$, where the latter relation was found before by conformal requirements. With those relations implemented, we obtain:

$$\begin{aligned} S_{D_2}(k_1,k_2,k_3) &= \delta^D(\Sigma_i k_i)\times|y_{12}|^{1+2\alpha' k_1\cdot k_2}\times|y_{13}|^{1+2\alpha' k_1\cdot k_3}\times|y_{23}|^{1+2\alpha' k_2\cdot k_3} \\ &= \delta^D(k_1+k_2+k_3), \end{aligned} \tag{4.26}$$

with $|y_{ij}| = (y_i - y_j)$. Generalizing to the four–point tachyon amplitude on the disk, is almost trivial, this one being given by

$$\begin{aligned} &S_{D_2}(k_1,k_2,k_3,k_4) \\ &\quad = \int_{-\infty}^{\infty} dy_4 \langle\prod_{j=1}^{3} :c^j(y_j)e^{ik_j\cdot X(y_j)}::e^{ik_4\cdot X(y_4)}(y_4):\rangle + (k_2\leftrightarrow k_3). \end{aligned} \tag{4.27}$$

We recognize all the ingredients already studied in the last amplitude, with the single difference we have now four vertices, thus the last one cannot anymore be fixed, such that an integral over its position has appeared. Introducing the Mandelstam variables

$$s = (k_1 + k_2)^2 \quad t = (k_1 + k_3)^2 \quad u = (k_1 + k_4)^2, \tag{4.28}$$

which can be seen to be just simple scalar products of different momenta. Their conservation relation holds

$$s + t + u = \sum m_i = -8. \tag{4.29}$$

Again performing the Wick contractions and after that fixing the values of the vertices positions at $y_1 = 0$, $y_2 = 1$, and $y_3 \to \infty$ gives

$$S_{D_2}(k_1, k_2, k_3, k_4) = \delta^D(\Sigma_i k_i) \left[\int_{\infty}^{\infty} dy |y|^{-\alpha' u - 2} |1 - y|^{-\alpha' t - 2} + (t \to s) \right]. \tag{4.30}$$

The integral is a standard one and found in every formulary, see e.g.[79]. For that, we have to divide the integration interval into three pieces

$$(-\infty, +\infty) = (-\infty, 0] \;\cup\; [0, 1] \;\cup\; [1, \infty),$$

and perform the variable transform $x' = \frac{1}{x}$. The result will be the famous Euler–Beta integral:

$$B(a, b) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} = \int_0^1 dy\; y^{a-1}(1 - y)^{b-1},$$

with $\Gamma(x)$ being the known Gamma function obeying the functional relation $\Gamma(x + 1) = x\Gamma(x)$. The field of such functions, their generalization and various representations thereof will thoroughly be covered in part II of the present work.

4.2 Tree-level amplitudes of open superstrings

4.2.1 Setting the stage

Now that we have introduced string amplitudes and successfully computed a simple one we should talk in more detail about the frame in which we will calculate to main amplitude this work is concerned with. This section thus will set up the frame in which future work will be done, namely the dynamics of Yang–Mills fields fixed on D–branes preserving maximal supersymmetry. This discussion holds in all types of superstring theories except the heterotic ones. As roughly explained in the introductional chapter about the superstring, Yang–Mills fields are the massless modes of superstrings attached to D–branes. As in quantum field theory, they are named A_μ and originate from the bosonic string sector, opposite to their superpartners, the gauginos ψ_μ, which come from the fermionic sector. Since D–branes have various dimensions, we will denote their dimension by p and include it in the name, i.e. call them Dp–branes. As already known, Dp–branes are boundary conditions of the open string interpreted as infinitely extended objects[4]. Those boundary conditions will make them observable also in the superstring modes, such as A_μ and ψ_μ. It soon becomes clear that when dealing with Dp–branes where $p < 9$ we also have to consider fields obeying Dirichlet boundary conditions, though being transversal to the brane. In this work though we will not consider those cases; this is not a simplification since the effects can easily be obtained from the D9–brane by T–duality. For more on T–duality, see [80, 81]. Thus, our setup will be given by gauge fields living on D9–branes with maximal supersymmetry and gauge group $SO(32)$.

We have now to treat scatterings of gauge states, which will be just a slightly generalization of the amplitude (4.27), the difference coming from the fact that we have now to insert the vertices responsible for gauge states. The amplitude will generally look as in quantum chromodynamics, namely

$$\mathcal{A}_N(k_1,\xi_1,a_1;\ldots;k_N,\xi_N,a_N) = \sum_{\pi\,\in\,\overline{(1,2,\ldots,N)}\,'} \mathrm{Tr}(\lambda^{\pi(1)}\lambda^{\pi(2)}\ldots\lambda^{\pi(N)})\; A^{\pi}. \tag{4.31}$$

Few words have to be said about that formal expression: it symbolizes an S–matrix of N gluons, characterized by their polarizations ξ_i and momenta k_i. The function A^π will fully capture the dynamics of the interaction, thus depending on momenta and polarization and being the straightforward generalization of the four point amplitude. The sum multiplying it is though new, but known from quantum chromodynamics where one also deals with gluons. It is the sum over all cyclically inequivalent (symbolized by the prime) permutations $(\pi(1), \pi(2), \ldots, \pi(N))$ over all color gluon indices. Thus the elements of the sum are taken from the permutation group S_N modulo the cyclically equivalent ones, $\mathbb{Z}_N$. Since the dimension of that group is $\dim(\mathrm{S_N}/\mathbb{Z}_\mathrm{N}) = (\mathrm{N}-1)!$ we have exactly $(N-1)!$ elements. Also the trace Tr goes over the color indices of the Chan–Paton factors λ^k.

Before setting the full frame one more piece is needed. We have to specify the topology of our world–sheet (which will be the disk, since we calculate tree–amplitudes) on which the vertices are to be inserted, this being done by fixing the ghost number to be (-2). We are free to choose the ghost–picture in which the vertices are given and this we use to take **two** vertices in ghost picture (-1) and the rest of them in ghost picture 0, this ensuring a total ghost number of (-2). This requirement is a

[4] Two types of boundary conditions are possible, one that fixes the position of the string called Dirichlet condition (this is also where the D–brane inherits its name) and one which allows the string to move freely, called Neumann condition

direct implication of the superdiffeomorphism invariance of the superstring action. On the disk the conformal Killing volume V_{CKG}^{-1} is accounted for by fixing three positions and introducing the respective c–ghost correlator, as we have already seen before. The (-1)–ghost picture gauge boson vertex in superstring theory is given by

$$V_{A^a}^{(-1)}(z,k) = \lambda^a \ \xi_\mu \ e^{-\phi(z)} \ \psi^\mu(z) \ e^{ik_\rho X^\rho(z)} \tag{4.32}$$

and the one in zero–ghost picture by

$$V_{A^a}^{(0)}(z,k) = \lambda^a \ \xi_\mu \ [\ \partial X^\mu(z) + i \ (k\psi) \ \psi^\mu(z) \ ] \ e^{ik_\rho X^\rho(z)}. \tag{4.33}$$

Since those vertices represent gauge particles, they carry a group charge called color, λ^i; in order to fully specify the kinematic of the particle we introduce their polarization ξ_μ and momentum k_μ. As usual the scalar product of a polarization vector with its momentum is zero, $\xi_\mu \cdot k^\mu = 0$ and momentum is conserved $\sum k_i = 0$. Thus, now we are ready to set up an S–matrix for open string gluons by just inserting in (4.19) two (-1)–ghost vertices and $(N-2)$ vertices in the zero-ghost picture arriving at

$$\begin{aligned}\mathcal{A}_N(k_1,\xi_1,a_1;\ldots;k_N,\xi_N,a_N) \\ = \prod_{r=1}^{3} \int d^2 z_r \ \langle V_{A^{a_1}}^{(-1)}(z_1) \ V_{A^{a_2}}^{(-1)}(z_2) \ V_{A^{a_3}}^{(0)}(z_3) \ldots \ V_{A^{a_N}}^{(0)}(z_N)\rangle.\end{aligned} \tag{4.34}$$

We notice that the first two vertices are put in the (-1)–ghost picture while the rest of them is in the zero–ghost picture. This is an arbitrary choice as will be seen in a while. We also have to specify the ordering of the Chan–Paton factors and much more important the integration region. Since we are calculating tree–diagrams in open string theory, our world–sheet is conformally equivalent to a disk and the vertices are inserted at the boundary as depicted in figure (4.3). This immediately implies we have real integrals along the boundary of the upper complex plane, thus along the real axis $\mathbb{R}$. But one more constraint is put on the integration order: we have to integrate the position of each vertex from the position of the antecedent one until the next vertex, while the first is integrated from minus infinity until the second and the last is integrated until plus infinity. Thus the region of integration is in one to one correspondence with the Chan–Paton order $\text{Tr}(\lambda^{\pi(1)}\lambda^{\pi(2)}\ldots\lambda^{\pi(N)})$ and the former may be specified as

$$\mathcal{I}_\pi = \{ \ \text{Im}(z_j) = 0 \mid z_{\pi(1)} < z_{\pi(2)} < \ldots < z_{\pi(N)} \ \}, \tag{4.35}$$

the constraint $\text{Im}(z_j) = 0$ being equivalent to a real integration.

4.2.2 A review of the four–point function

With the tools studied in the last section we are now in the position to fully understand and compute the four open string gluon S–matrix on the disk, which now simplifies from the formula (4.31) to

$$\mathcal{A}_4(k_1,\xi_1,a_1;k_2,\xi_2,a_2;k_3,\xi_3,a_3;k_4,\xi_4,a_4) = \sum_{\substack{\pi\ \in\ \overline{(1,2,3,4)} \\ 6\ \text{permutations}\ \pi}} \mathrm{Tr}(\lambda^{\pi(1)}\lambda^{\pi(2)}\lambda^{\pi(3)}\lambda^{\pi(4)})\ A^{\pi}\ . \tag{4.36}$$

In order to compute the kinematics of that amplitude we have to start with equation (4.34). After inserting the vertices we have simply to evaluate the Wick contractions between the conformal fields. Those can be found in every standard book on the subject and are collected in following expression

$$\begin{aligned}
\langle X^{\mu}(z_1)X^{\nu}(z_2)\rangle &= -g^{\mu\nu}\ \ln(z_1-z_2)\ ,\\
\langle \psi^{\mu}(z_1)\psi^{\nu}(z_2)\rangle &= -\frac{g^{\mu\nu}}{z_1-z_2}\ ,\\
\langle e^{ik_1X(z_1)}e^{ik_2X(z_2)}\rangle &= \mid z_1-z_2\mid^{k_1k_2}\ ,\\
\langle e^{-\phi(z_1)}e^{-\phi(z_2)}\rangle &= \frac{1}{z_1-z_2}\ ,\\
\langle c(y_1)c(y_2)c(y_3)\rangle &= (y_1-y_2)(y_1-y_3)(y_2-y_3)\ .
\end{aligned}\tag{4.37}$$

We notice first the new field ϕ which is just the superpartner of the bosonic ghost c[5]. It's important to notice that the correlators decouple, i.e. the bosonic fields X^{μ} have independent contraction rules from that of the fermionic modes ψ^{μ} or the super–ghost fields ϕ and the ghosts c. This fact causes a factorization of the correlation function which facilitates the computation. Before doing the integrals we have to specify an integration order, and this is given, as seen in last section, by the ordering of the Chan–Paton factors in the trace, before the kinematic function A^{π}. Without loss of generality we choose the ordering $\pi=(1,2,3,4)$. Then the function A^{π} is given

$$A^{(1,2,4,3)} = \frac{\Gamma(s)\ \Gamma(t)}{\Gamma(1+s+t)}\ [\ tu\ (\xi_1\xi_2)\ (\xi_3\xi_4) + su\ (\xi_1\xi_3)\ (\xi_2\xi_4) + st\ (\xi_1\xi_4)\ (\xi_2\xi_3) + \ldots\ ]\ , \tag{4.38}$$

where as expected, the result only depends on the four polarization vectors ξ^{μ} and the Mandelstam variables, defined in last chapter. For a more detailed computation, and also for slightly different scenarios, consult e.g. [82, 83]. First of all, the result

[5]Since in the superstring all the fields have superpartners, so does also the metric of the bosonic field. The ghost field associated with it is just ϕ.

presented in (4.38) is not the entire kinematics as the dots on the right hand side may suggest. We have just listed the first three contractions, namely those entirely between polarization vectors. The whole kinematics contains also mixed scalar products, i.e. scalar products between momenta and polarization vectors, as encoded in the $t8$–tensor, see [28]. Taking into account that also the Mandelstam variables depend on the momenta, the full string S–matrix, i.e. also the other five orderings of the Chan–Paton factors λ^i can be obtained by simply taking the corresponding permutations of the momenta and polarization. This, however, won't generate any new terms, the $t8$–tensor being invariant under such permutations, the only difference being in the factor of six which will multiply then the result. Further on is it worth to point on the beta–function which depends on the momenta and multiplies the $t8$–tensor. This function also encodes dependence on the momenta but much more important, all the poles are encoded solely in that function. So all the exchange diagrams cannot be seen until we series expand the gamma–functions. In that case although, this is not a problem, since every standard book on mathematics will furnish that expansion

$$\Gamma(x) = \frac{1}{x} - \gamma_E + \frac{\gamma_E^2 + \zeta(2)}{2}\, x - \frac{\gamma_E^3 + \gamma_E\, \zeta(2) + 2\zeta(3)}{6}\, x^2 + O(x^3). \tag{4.39}$$

This expansion can then be inserted into the equation (4.38) and then we can obtain the arbitrary momentum order of the four–point gluon interaction.

4.3 Same calculation, different technique

We should remember that we have chosen by chance one combination of the ghost–picture operators, i.e. one special manner which dictates which the four vertices is put into the (-1)–ghost picture. Since all choices are equivalent, for nobody gives us a prescription of how to do that, we are free to put every two vertices into that picture.

4.3.1 The ghost picture changing

Let's introduce first some notation: let $A^\pi(a, b, i_1, i_2, \ldots, i_{N-2})$ indicate that the first two vertices (a, b) are in the (-1)–ghost picture and that the other have been chosen to be in the zero–picture. While the ordering of the vertices is important we have named every vertex explicitly. It should be clear that this procedure is fully independent of the ordering of the Chan–Paton factors λ^i. Further we could calculate the S–matrix in just one ghost–picture $A^\pi(a, b, i_1, i_2, \ldots, i_{N-2})$ since all the ghost pictures are equivalent as already stressed; but relaxing this constraint will lead to highly non trivial relations.

Let's study thus all the other permutations of ghost–pictures, as depicted formal in the following formula, where no particular choice has been made about the permutation of vertices $(a, b, i, j) \in \overline{(1, 2, 3, 4)}$

$$\begin{aligned}\mathrm{Tr}(\lambda^{\pi(1)}\lambda^{\pi(2)}\dots\lambda^{\pi(N)})\ A^{\pi}(a,b,i,j)\\ = \int_{\mathcal{I}_\pi} d^2z\ \langle V^{(-1)}_{A^{a_a}}(z_a)\ V^{(-1)}_{A^{a_b}}(z_b)\ V^{(0)}_{A^{a_i}}(z_i)\ V^{(0)}_{A^{a_j}}(z_j)\rangle.\end{aligned} \tag{4.40}$$

If we now do the contractions between the fields as done before in order to obtain equation (4.38), with the only difference that we do not specify the vertices, we arrive at following result

$$\begin{aligned}A^{\pi}(a,b,i,j) &= A^{\pi}_2(a,b,i,j)\ (\xi_a\xi_b)\ (\xi_i\xi_j)\\ &+ A^{\pi}_1(a,b,i,j)\ (\xi_a\xi_i)\ (\xi_b\xi_j) + A^{\pi}_1(a,b,j,i)\ (\xi_a\xi_j)\ (\xi_b\xi_i)\\ &+ B^{\pi}_2(a,b,i,j)\ (\xi_a\xi_b) + B^{\pi}_3(i,j,a,b)\ (\xi_i\xi_j) + B^{\pi}_1(a,i,b,j)\ (\xi_a\xi_i)\\ &+ B^{\pi}_1(a,j,b,i)\ (\xi_a\xi_j) + B^{\pi}_1(b,i,a,j)\ (\xi_b\xi_i) + B^{\pi}_1(b,j,a,i)\ (\xi_b\xi_j)\end{aligned} \tag{4.41}$$

The functions A^{π}_i and B^{π}_j encode the different polynomials as well as the kinematic variables ξ^{μ} and k^{μ} coming from the contractions. It is worth to notice that no specification has been made about the Chan–Paton ordering π hence we are still free to chose one color permutation. However choosing one permutation $\pi \in (1,2,3,4)$ will uniquely specify the integration prescription $\mathcal{I}_\pi$. Finally, the polynomials have to be integrated and the integrals look like

$$\begin{aligned}A^{\pi}_1(a,b,i,j) &= \int_{\mathcal{I}_\pi} dz_4\ \langle c(z_1)c(z_2)c(z_3)\rangle\ \mathcal{E}\ \frac{k_ik_j}{z_{ai}\ z_{jb}\ z_{ij}}\ \frac{(-1)}{z_{ab}}\ ,\\ A^{\pi}_2(a,b,i,j) &= \int_{\mathcal{I}_\pi} dz_4\ \langle c(z_1)c(z_2)c(z_3)\rangle\ \mathcal{E}\ \frac{1}{z^2_{ab}\ z^2_{ij}}\ (1-k_ik_j)\ ,\\ B^{\pi}_1(a,i,b,j) &= \int_{\mathcal{I}_\pi} dz_4\ \langle c(z_1)c(z_2)c(z_3)\rangle\ \mathcal{E}\ \frac{1}{z_{ab}\ z_{ij}}\\ &\quad\times\left\{\frac{(\xi_bk_i)(\xi_jk_a)}{z_{aj}z_{bi}} + \frac{(\xi_bk_i)(\xi_jk_b)}{z_{ai}z_{bj}} - \frac{(\xi_bk_j)(\xi_jk_i)}{z_{ai}z_{bj}}\right\}\ ,\\ B^{\pi}_2(a,b,i,j) &= \int_{\mathcal{I}_\pi} dz_4\ \langle c(z_1)c(z_2)c(z_3)\rangle\ \mathcal{E}\ \frac{(-1)}{z^2_{ab}\ z^2_{ij}}\ \left\{(\xi_ik_b)(\xi_jk_b) - (\xi_ik_b)(\xi_jk_a)\ \frac{z_{bj}z_{ia}}{z_{aj}z_{ib}}\right.\\ &\quad\left. +(\xi_ik_a)(\xi_jk_a) + (\xi_ik_a)(\xi_jk_b)\ \frac{z_{aj}z_{ib}}{z_{bj}z_{ia}} - (\xi_ik_j)(\xi_jk_i)\right\}\ ,\\ B^{\pi}_3(i,j,a,b) &= \int_{\mathcal{I}_\pi} dz_4\ \langle c(z_1)c(z_2)c(z_3)\rangle\ \mathcal{E}\ \frac{1}{z_{ab}\ z_{ij}}\ \left\{\frac{(\xi_ak_i)(\xi_bk_j)}{z_{ai}z_{bj}} - \frac{(\xi_ak_j)(\xi_bk_i)}{z_{aj}z_{bi}}\right\}\ .\end{aligned} \tag{4.42}$$

As usual we encounter the bosonic ghost correlator $\langle c(z_1)c(z_2)c(z_3)\rangle$ which fixes the position of three vertices. We have decided to fix the first three vertices, whose positions will be explicitly specified in a while, and the only integral is performed over the fourth variable z_4. Further we recognize scalar products between momenta which will later be translated into Mandelstam variables and also products between polarizations and between polarizations and momenta. As usual z_{ij} means the difference $(z_i - z_j)$ and we have abbreviated $\mathcal{E} = \prod_{r<s} |z_{rs}|^{k_r k_s}$, those being the contractions between the exponents of the string function X^μ. Further analysis will reveal that specific kinematic contractions are always accompanied by the same function A_i or B_j, those being

$$A_1^\pi(a,b,i,j)\ (\xi_a\xi_i)(\xi_b\xi_j) \quad , \quad A_2^\pi(a,b,i,j)\ (\xi_a\xi_b)(\xi_i\xi_j)\ , \tag{4.43}$$

and further

$$B_1^\pi(a,i,b,j)\ (\xi_a\xi_i) \quad , \quad B_2^\pi(a,b,i,j)\ (\xi_a\xi_b) \quad , \quad B_3^\pi(i,j,a,b)\ (\xi_i\xi_j)\ . \tag{4.44}$$

A first major difference between the two functions A and B is that A contains just $(\xi \cdot \xi)$–contractions, i.e. no contractions between momenta and polarizations, on the other hand the B–function also captures those contractions. Simply said, the kinematic in equation (4.38) which can bee seen is fully captured by the A–functions, the rest of it being encoded in the B–functions.

Referring now to the the vertex ghost–picture a subtlety can be observed in the cases of A_1^π and A_2^π: in the second case just polarization vectors ξ from (-1)–ghost picture vertices are contracted with each other whereas in the first case we have only mixed contractions, i.e. polarizations from (-1)–ghost picture vertices contracted with those coming from zero–ghost picture vertices. Essentially the same difference exists in the case of the B–functions when we also observe the higher degree of freedom in combining the polarizations, since now we are also allowed to have contractions of the form $(\xi_i k_j)$, when also the special case occurs when both polarizations coming from (-1)–ghost picture vertices are contracted with momenta.

4.3.2 System of linear equations

Now a very interesting phenomenon emerges: when computing the string S–matrix with the functions A and B we obtain seemingly different expressions for the same result! First of all, let's concentrate on one specific functions, e.g. A_1^π. We should remember that this is nothing else than a specific choice for the distribution of the ghost–picture over the vertices. For sure, this function will give us all the kinematics made up solely by products between polarization vectors, i.e. $(\xi_1\xi_2)(\xi_3\xi_4)$, $(\xi_1\xi_3)(\xi_2\xi_4)$ and $(\xi_1\xi_4)(\xi_2\xi_3)$.

Now we still can take other choices and then for the **same** kinematic

$$(\xi_A\xi_B)\ (\xi_C\xi_D) \tag{4.45}$$

we have **three** expressions given by the functions

$$A_1^\pi(A,C,B,D) \quad , \quad A_1^\pi(A,D,B,C) \quad , \quad A_2^\pi(A,B,C,D) \; . \tag{4.46}$$

But they are equal. Thus we might try to equate them, and not only restrict to them, but take the whole S–matrix as given in one particular choice namely $A_2^\pi(1,2,3,4)$ in (4.38) and evaluate all other choices which might eventually lead to following system of equations:

$$\begin{aligned}
(\xi_1\xi_2)(\xi_3\xi_4): &\quad A_2^\pi(1,2,3,4) = A_1^\pi(1,3,2,4) = A_1^\pi(1,4,2,3) \; , \\
(\xi_1\xi_3)(\xi_2\xi_4): &\quad A_2^\pi(1,3,2,4) = A_1^\pi(1,2,3,4) = A_1^\pi(1,4,3,2) \; , \\
(\xi_1\xi_4)(\xi_2\xi_3): &\quad A_2^\pi(1,4,2,3) = A_1^\pi(1,2,4,3) = A_1^\pi(1,3,4,2) \; .
\end{aligned} \tag{4.47}$$

We can thus notice that every kinematic is produced three times, by the three different ghost choices and on the other hand every function produces each of the three kinematics in in the first column of the upper systems of equations once, thus showing that the generated system is complete. Sofar we have just given a look at the kinematics concerning the polarizations. In order to also calculate the momentum dependencies we have to integrate the polynomials which are encoded in the three functions A_j^π. For that we should make a decision what concerns the Chan–Paton ordering π, since this choice is in one to one correspondence with the integration prescription. Let us decide for the choice $\pi = (1,2,4,3)$. This translates into the trace factor multiplying the Beta–function in (4.38), i.e. $\mathrm{Tr}(\lambda^1\lambda^2\lambda^4\lambda^3)$. We have again to stress that this choice isn't constrictive since by just permuting the "names", i.e. all the indices in the result $(1,2,3,4)$, we can recover each Chan–Paton combination we desire.

Furthermore, since we have the ghost correlators which allow for fixing three positions of the involved vertices, we should also specify those positions to $z_1 \to -\infty$, $z_2 = 0$ and finally $z_3 = 1$. This choice is also equivalent to other positions but it turns out that this one is more convenient, simplifying the computations. Now we can formally integrate the polynomials and try to find the functions in mathematical books, this leading to

$$\begin{aligned}
A_2^\pi(1,2,3,4) &= (s-1)\; F_4 \quad , \quad A_1^\pi(1,3,2,4) = -t\; F_3 \quad , \quad A_1^\pi(1,4,2,3) = u\; F_1 \; , \\
A_2^\pi(1,3,2,4) &= (t-1)\; F_5 \quad , \quad A_1^\pi(1,2,3,4) = -s\; F_3 \quad , \quad A_1^\pi(1,4,3,2) = -u\; F_2 \; , \\
A_2^\pi(1,4,2,3) &= (u-1)\; F_0 \quad , \quad A_1^\pi(1,2,4,3) = s\; F_1 \quad , \quad A_1^\pi(1,3,4,2) = -t\; F_2 \; ,
\end{aligned} \tag{4.48}$$

where we have made the abbreviations

$$F_j = \int_0^1 dx\; P_j\; x^t (1-x)^s \; . \tag{4.49}$$

This integral is, as expected, different for each kinematic, since each kinematic was created by a unique contraction between some conformal fields from the operators, thus also the polynomials will be different, like shown in next equation:

$$\begin{aligned}
&P_0 = 1 \quad , \quad P_1 = \frac{1}{x-1} \quad , \quad P_2 = \frac{1}{x} \; , \\
&P_3 = \frac{1}{x(x-1)} \quad , \quad P_4 = \frac{1}{(x-1)^2} \quad , \quad P_5 = \frac{1}{x^2} \; .
\end{aligned} \tag{4.50}$$

We plug in now the functions from (4.48) into the system (4.47) which leads to

$$\begin{aligned}
(s-1)\; F_4 = u\; F_1 \quad &, \quad (s-1)\; F_4 = -t\; F_3 \\
(t-1)\; F_5 = -s\; F_3 \quad &, \quad (t-1)\; F_5 = -u\; F_2 \\
(u-1)\; F_0 = s\; F_1 \quad &, \quad (u-1)\; F_0 = -t\; F_2 \; .
\end{aligned} \tag{4.51}$$

This system allows now for a solution, or you may also call it parametrization, where we are free to chose each function as a parameter. For sure we will take the simplest one, namely the one where the integral over the corresponding polynomial is the simplest, for all the other functions will be related to that:

$$\begin{aligned}
&F_1 = \frac{u-1}{s}\; F_0 \quad , \quad F_2 = \frac{1-u}{t}\; F_0 \; , \\
&F_3 = \frac{u\;(1-u)}{st}\; F_0 \quad , \quad F_4 = \frac{u\;(1-u)}{s\;(1-s)}\; F_0 \quad , \quad F_5 = \frac{u\;(1-u)}{t\;(1-t)}\; F_0 \; .
\end{aligned} \tag{4.52}$$

After integrating the F_0–function to

$$F_0 = \frac{\Gamma(s+1)\;\Gamma(t+1)}{\Gamma(2+s+t)} \tag{4.53}$$

we have all other functions already given just in terms of one **single** function. We don't even have to integrate the other ones. For sure in this simple case, all the other integrals may as well be done for they are just different versions of the Beta–integral. But a much more important detail should be noticed: the function F_0 has been chosen such that it hasn't poles for the argument approaching zero[6]! This is of enormous importance since all the poles, as can be checked in the result (4.52) are factorized in front of F_0! This means, that we can simply expand our pole–free function F_0 and obtain automatically the poles of the other functions. Those relations aren't something "metaphysical" since they can be proved analytically and are true

[6]The Γ–function has infinitely many poles, for they occur at every negative integer. But we are concerned only of the poles at zero, since when doing field theory we have positive momenta, and can expand results in small positive Mandelstam variables!

mathematical identities. Beyond that they are stated in every mathematical book about Beta–functions. This will though dramatically change when dealing with the six–point function, where no such relations are established and also just little work has been done about substraction of poles from the functions involved.

We should remember that we considered solely the kinematics involving only $(\xi\xi)$–contractions. This should be now trivially generalized, since a similar system of equations can be established for the full $t8$–tensor, also involving kinematics as $(\xi\xi)(\xi k)(\xi k)$. Those contractions are captured by the six functions

$$\begin{aligned} &B_2^\pi(A,B,C,D) \quad , \quad B_1^\pi(A,B,C,D) \quad , \quad B_1^\pi(A,B,D,C)\ , \\ &B_1^\pi(B,A,C,D) \quad , \quad B_1^\pi(B,A,D,C) \quad , \quad B_3^\pi(A,B,C,D). \end{aligned} \tag{4.54}$$

Before solving the systems, some characterizing words about the new B_j^π–functions should be said: they symbolize the distribution of the ghost–pictures on the vertices as $(A,B),(A,C),(A,D),(B,C),(B,D)$ or (C,D), this meaning that exactly those pairs of vertices are put into the (-1)–ghost picture respectively.

In full analogy to the last system we set up a bigger system with the additional polynomials

$$\begin{aligned} &P_6 = \frac{1}{x(x-1)^2} \quad , \quad P_7 = \frac{x}{(x-1)^2} \quad , \quad P_8 = \frac{1}{x^2(x-1)} \\ &P_9 = \frac{x-1}{x} \quad , \quad P_{10} = \frac{x-1}{x^2} \quad , \quad P_{11} = \frac{x}{x-1}, \end{aligned} \tag{4.55}$$

of which solution (the relevant part) is

$$\begin{aligned} &F_6 = F_4 - F_3 \quad , \quad F_7 = F_1 + F_4 \quad , \quad F_8 = F_3 - F_5\ , \\ &F_9 = F_0 - F_2 \quad , \quad F_{10} = F_2 - F_5 \quad , \quad F_{11} = F_0 + F_2 + F_3\ . \end{aligned} \tag{4.56}$$

This again leads to a one–dimensional solution where all the functions can be related to just one, chosen at will, in our case the F_0–function. Again, the poles factorize in front of it in every case, such that just expansion of F_0 will suffice. Thus, formally, we may write

$$\boxed{F_j = \Lambda_j(s,t,u)\ \ F_0} \tag{4.57}$$

for the solution of the system with the function F_0 given by

$$F_0 = 1 - s - t + s^2 + 2\ s\ t - \zeta(2)\ s\ t + t^2 + \ldots \tag{4.58}$$

It is worth to say that the "basis function" F_0 remains the same, while the coefficients Λ_j depending on the Mandelstam variables will vary with the "unknowns" $F_j, j \neq 0$, in order to capture in every case the right pole behavior.

Chapter 5

Six gluon open superstring S–matrix

5.1 Setting up the amplitude

This chapter is dedicated to the main computation published in [26], which is the six–point open superstring amplitude. In principle we have acquired all the tools we need in order to attack the problem. Unfortunately the degree of complexity is that high that a lot of results cannot be integrally given here (they being to large for that purpose) and excerpts from them will be given in the appendix.

5.1.1 General expression of the six–gluon S–matrix

In accordance with the results achieved in the last chapter, our starting point will be the formula

$$\begin{aligned}&\mathcal{A}_6(k_1,\xi_1,a_1;k_2,\xi_2,a_2;k_3,\xi_3,a_3;k_4,\xi_4,a_4;k_5,\xi_5,a_5;k_6,\xi_6,a_6)\\&\quad=\prod_{r=4}^{6}\int d^2z_r\ \langle V^{(-1)}_{A^{a_1}}(z_1)\ V^{(-1)}_{A^{a_2}}(z_2)\ V^{(0)}_{A^{a_3}}(z_3)\ V^{(0)}_{A^{a_4}}(z_4)\ V^{(0)}_{A^{a_5}}(z_5)\ V^{(0)}_{A^{a_6}}(z_6)\rangle\ ,\end{aligned} \tag{5.1}$$

which describes a tree–level scattering amplitude of six open strings with vertices already presented in (4.32) for the (-1)–ghost picture respectively in (4.33) for the zero–ghost picture. The ordering of the distribution of the ghost pictures on the vertices as well as the ordering of the group factors λ^i in the trace are not specified, since in the former case, we are not going to calculate the S–matrix only in one specific picture but in all possible ones, in order to use the advantages presented in last chapter and in the latter case we just need one specific choice for the λs for performing the integrals.

The next step is to simply do the conformal Wick contractions between the fields, which might be a little lengthy but nevertheless straightforward. Before doing that

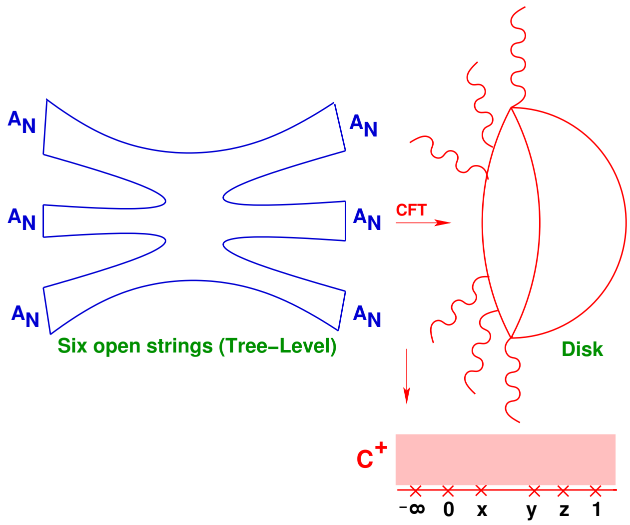


Figure 5.1: Six open string vertices inserted on the world–sheet and mapped to the disk (upper half complex plane)

though, let us look a little closer to the picture (5.1), where the string world–sheet with the six vertices inserted is depicted and also its conformal mapping to the disk is shown: on the left hand side we have the string world–sheet of six open external states, which in our case are chosen to be six gluons with $U(N)$ gauge group, denoted A_N in the picture. Exactly opposite to it, on the right hand side, we have the half sphere which is isomorphically equivalent to the string world–sheet. With a bit of fantasy we can deform the figure on the left and obtain the half sphere. Further, the half sphere is isomorphic to the upper half plane, at whose boundary the scattering states are attached. All those transformations can be done using the conformal symmetry, which characterizes the string action. We can see also pictorially, that the integral of the vertices over the world-sheet is equivalent to a real integral.

5.1.2 Kinematics and pictures

The contractions can be divided into classes according to the number of polarization vectors multiplied either with each other or with some momenta. Thus we differentiate between $(\xi\xi)(\xi\xi)(\xi\xi)$, $(\xi\xi)(\xi\xi)(\xi k)(\xi k)$ and $(\xi\xi)(\xi k)(\xi k)(\xi k)(\xi k)$ whose function multiplying them will be named $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ respectively. Those are to be considered as the analogues of the functions A and B in the four–point case. As a side remark it should be said that the last two kinematics can fully be inferred from the first one $(\xi\xi)(\xi\xi)(\xi\xi)$ by simple gauge invariance. Moreover, the full information about the Born–Infeld action, i.e. the Lagrangian at α'^4–order involving interaction of six gluons is fully settled by the first kinematic. The only reason the other kinematics are calculated is to capture all the possible functions which occur in the S–matrix and to involve all those functions into the complex system of equations, which eventually will be solved in terms of just few simple functions. So we invest now more energy doing more contractions, but later we collect the earnings in form of not being obliged to series expand some functions, not even being obliged to integrate them all.

Finally, the full string S–matrix is given in full analogy to (4.41) by

$$
\begin{aligned}
A^{\pi}(a\,,b\,,\,i\,,\,j\,,\,k\,,\,l) = \int_{\mathcal{I}_\pi} \prod_{r=4}^{6} d^2 z_r \sum_{\substack{(i_1,i_2,i_3,i_4)\\ \in(i,j,k,l)}} & \Big\{ \frac{1}{2}\, \mathcal{A}_1(a,b,i_1,i_2,i_3,i_4)\ (\xi_a\xi_{i_1})\ (\xi_b\xi_{i_4})\ (\xi_{i_2}\xi_{i_3}) \\
&+ \frac{1}{8}\, \mathcal{A}_2(a,b,i_1,i_2,i_3,i_4)\ (\xi_a\xi_b)\ (\xi_{i_1}\xi_{i_2})\ (\xi_{i_3}\xi_{i_4}) + \frac{1}{4}\, \mathcal{B}_2(a,b,i_1,i_2,i_3,i_4)\ (\xi_a\xi_b)\ (\xi_{i_1}\xi_{i_2}) \\
&+ \frac{1}{2}\, \mathcal{B}_1(a,b,i_1,i_2,i_3,i_4)\ (\xi_a\xi_{i_1})\ (\xi_b\xi_{i_2}) + \frac{1}{8}\, \mathcal{B}_4(i_1,i_2,i_3,i_4,a,b)\ (\xi_{i_1}\xi_{i_2})\ (\xi_{i_3}\xi_{i_4}) \\
&+ \frac{1}{2}\, \mathcal{B}_3(a,i_1,i_2,i_3,b,i_4)\ (\xi_a\xi_{i_1})\ (\xi_{i_2}\xi_{i_3}) + \frac{1}{2}\, \mathcal{B}_3(b,i_1,i_2,i_3,a,i_4)\ (\xi_b\xi_{i_1})\ (\xi_{i_2}\xi_{i_3}) \\
&+ \frac{1}{24}\, \mathcal{C}_1(a,b,i_1,i_2,i_3,i_4)\ (\xi_a\xi_b) + \frac{1}{4}\, \mathcal{C}_3(i_1,i_2,a,b,i_3,i_4)\ (\xi_{i_1}\xi_{i_2}) \\
&+ \frac{1}{6}\, \mathcal{C}_2(a,i_1,b,i_2,i_3,i_4)\ (\xi_a\xi_{i_1}) + \frac{1}{6}\, \mathcal{C}_2(b,i_1,a,i_2,i_3,i_4)\ (\xi_b\xi_{i_1}) \Big\}\,.
\end{aligned}
\tag{5.2}
$$

The first two terms are concerned solely with the kinematics $(\xi\xi)(\xi\xi)(\xi\xi)$, where the functions $\mathcal{A}_1$ and $\mathcal{A}_2$ capturing the momentum behavior are respectively given by

$$
\begin{aligned}
\mathcal{A}_1(a,b,i,j,k,l) &= -\frac{\mathcal{E}}{z_{ab}\ z_{ai}} \left\{ \frac{(k_ik_j)\ (k_kk_l)}{z_{lb}\ z_{jk}\ z_{ij}\ z_{kl}} - \frac{(k_ik_k)\ (k_jk_l)}{z_{lb}\ z_{jk}\ z_{ik}\ z_{jl}} - \frac{(k_ik_l)\ (1-k_jk_k)}{z_{lb}\ z_{il}\ z_{jk}^2} \right\}\ , \\
\mathcal{A}_2(a,b,i,j,k,l) &= -\frac{\mathcal{E}}{z_{ab}^2} \left\{ \frac{(1-k_ik_j)\ (1-k_kk_l)}{z_{ij}^2\ z_{kl}^2} - \frac{(k_ik_k)\ (k_jk_l)}{z_{ij}\ z_{kl}\ z_{ik}\ z_{jl}} + \frac{(k_ik_l)\ (k_jk_k)}{z_{ij}\ z_{kl}\ z_{il}\ z_{jk}} \right\}\ .
\end{aligned}
\tag{5.3}
$$

Here we have again made the abbreviation

$$
\mathcal{E} = \prod_{r<s}^{6} |z_{rs}|^{k_rk_s}\ , \tag{5.4}
$$

for the six exponentials $e^{ik_\mu\cdot X^\mu}$.

The rest of the functions multiplying the mixed contractions look similar to $\mathcal{A}_1$ and $\mathcal{A}_2$. However the expressions are much more lengthy, so for aesthetical reasons they are listed in the appendix, in equations (C.1–C.6).

We come now back to equation (5.2), where we analyze the first part of it, namely the ξ–contractions. Given six ξ–polarizations we have exactly $\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot2\cdot2\cdot2} = 15$ combinations of three pairs. This quantity arises from the total permutation of six different polarizations divided by the cyclic combinations and the pair redundancy of the three pairs. Those fifteen combinations are:

$$\begin{aligned}
\Xi_1 &:= (\xi_1\xi_2)\ (\xi_3\xi_4)\ (\xi_5\xi_6) \quad , \quad \Xi_2 := (\xi_1\xi_2)\ (\xi_3\xi_5)\ (\xi_4\xi_6) \quad , \quad \Xi_3 := (\xi_1\xi_2)\ (\xi_3\xi_6)\ (\xi_4\xi_5), \\
\Xi_4 &:= (\xi_1\xi_3)\ (\xi_2\xi_4)\ (\xi_5\xi_6) \quad , \quad \Xi_5 := (\xi_1\xi_3)\ (\xi_2\xi_5)\ (\xi_4\xi_6) \quad , \quad \Xi_6 := (\xi_1\xi_3)\ (\xi_2\xi_6)\ (\xi_4\xi_5), \\
\Xi_7 &:= (\xi_1\xi_4)\ (\xi_2\xi_3)\ (\xi_5\xi_6) \quad , \quad \Xi_8 := (\xi_1\xi_4)\ (\xi_2\xi_5)\ (\xi_3\xi_6) \quad , \quad \Xi_9 := (\xi_1\xi_4)\ (\xi_2\xi_6)\ (\xi_3\xi_5), \\
\Xi_{10} &:= (\xi_1\xi_5)\ (\xi_2\xi_3)\ (\xi_4\xi_6) \quad , \quad \Xi_{11} := (\xi_1\xi_5)\ (\xi_2\xi_4)\ (\xi_3\xi_6) \quad , \quad \Xi_{12} := (\xi_1\xi_5)\ (\xi_2\xi_6)\ (\xi_3\xi_4), \\
\Xi_{13} &:= (\xi_1\xi_6)\ (\xi_2\xi_3)\ (\xi_4\xi_5) \quad , \quad \Xi_{14} := (\xi_1\xi_6)\ (\xi_2\xi_4)\ (\xi_3\xi_5) \quad , \quad \Xi_{15} := (\xi_1\xi_6)\ (\xi_2\xi_5)\ (\xi_3\xi_4).
\end{aligned} \tag{5.5}$$

As in the four–point case they are multiplied by the two different $\mathcal{A}_j$–functions which distinguish weather the polarization vectors from the (-1)–ghost vertices are contracted among themselves, i.e. $(\xi_a\xi_b)\ (\xi_i\xi_j)\ (\xi_k\xi_l)$, those being multiplied by $\mathcal{A}_2(a,b,i,j,k,l)$, or if they are contracted with zero–ghost picture polarizations $(\xi_a\xi_i)\ (\xi_b\xi_l)\ (\xi_j\xi_k)$, those being consequently multiplied by $\mathcal{A}_1(a,b,i,j,k,l)$. Moreover, the functions $\mathcal{A}_j$ share also symmetries which can directly be extracted from the ξ–products. Those symmetries are exactly the interchange in indices which let the three scalar products invariant. For example, $\mathcal{A}_2$ has the symmetries $(a \to b)$, $(i \to j)$ and $(k \to l)$ and $\mathcal{A}_1$ has the symmetries $(j \to k)$ and $(a \to b, i \to l)$. Also pairs of indices can be exchanged, again according to the picture found in the ξ–products. From that it should be clear that when exchanging the ghost–picture exactly that way the appearance of the strings S–matrix (5.2) won't change. On the other hand side, when doing all the other permutations which are not exactly those symmetries, then we will obtain different functions in front of the **same** kinematic. And since the functions $\mathcal{A}$ (as well as $\mathcal{B}$ and $\mathcal{C}$) are complete, i.e. every possible combination out of the fifteen in eq. (5.5) are reproduced exactly once, we obtain different expressions which are identities and can be equated. So specializing to a given ξ–product $(\xi_A\xi_B)\ (\xi_C\xi_D)\ (\xi_E\xi_F)$ we have exactly $\binom{6}{2} = 15$ possibilities to express the function in front of that contraction, following the scheme

$$\begin{aligned}
(a,b) \in \{\ &(A,B),\ (A,C),\ (A,D),\ (A,E),\ (A,F),\ (B,C),\ (B,D),\ (B,E), \\
&(B,F),\ (C,D),\ (C,E),\ (C,F),\ (D,E),\ (D,F),\ (E,F)\ \}\ .
\end{aligned} \tag{5.6}$$

Here we have chosen every combination in which the (a,b)–pair of vertices is put in the (-1)–ghost picture. This way, we arrive at the fifteen different functions which show up in front of the same contraction $(\xi_A\xi_B)\ (\xi_C\xi_D)\ (\xi_E\xi_F)$, given by

$$\begin{aligned}
&A_1^\pi(A,C,B,E,F,D) \quad , \quad A_1^\pi(A,D,B,E,F,C) \quad , \quad A_1^\pi(A,E,B,C,D,F)\ , \\
&A_1^\pi(A,F,B,C,D,E) \quad , \quad A_1^\pi(B,C,A,E,F,D) \quad , \quad A_1^\pi(B,D,A,E,F,C)\ , \\
&A_1^\pi(B,E,A,C,D,F) \quad , \quad A_1^\pi(B,F,A,C,D,E) \quad , \quad A_1^\pi(C,E,D,A,B,F)\ , \\
&A_1^\pi(C,F,D,A,B,E) \quad , \quad A_1^\pi(D,E,C,A,B,F) \quad , \quad A_1^\pi(D,F,C,A,B,E)\ , \\
&A_2^\pi(A,B,C,D,E,F) \quad , \quad A_2^\pi(C,D,A,B,E,F) \quad , \quad A_2^\pi(E,F,A,B,C,D)\ .
\end{aligned} \tag{5.7}$$

We have to stress that upper expressions are different looking functions expressing the same object. This is again the consequence of the fact that no prescription is given for the possible distribution of the (-1)–ghost picture operators among the vertices in the correlation function. Thus all the fifteen arising "pictures" can be set equal, creating following system of linear equations:

$$\begin{aligned} &A_2^\pi(1,2,3,4,5,6) = A_2^\pi(3,4,1,2,5,6) = A_2^\pi(5,6,1,2,3,4) \\ &= A_1^\pi(1,3,2,5,6,4) = A_1^\pi(1,4,2,5,6,3) = A_1^\pi(1,5,2,3,4,6) = A_1^\pi(1,6,2,3,4,5) \\ &= A_1^\pi(2,3,1,5,6,4) = A_1^\pi(2,4,1,5,6,3) = A_1^\pi(2,5,1,3,4,6) = A_1^\pi(2,6,1,3,4,5) \\ &= A_1^\pi(3,5,4,1,2,6) = A_1^\pi(3,6,4,1,2,5) = A_1^\pi(4,5,3,1,2,6) = A_1^\pi(4,6,3,1,2,5)\ . \end{aligned} \tag{5.8}$$

In order to set up the exemplary equations we have explicitly chosen the kinematic Ξ_1, i.e. $(\xi_1\xi_2)(\xi_3\xi_4)(\xi_5\xi_6)$. The rest of the S–matrix, i.e. the kinematics of the type $(\xi\xi)$ $(\xi\xi)$ (ξk) (ξk) and $(\xi\xi)$ (ξk) (ξk) (ξk) (ξk), is treated in full analogy to the discussion so far. Again, for the sake of aesthetics we list the results for those kinematics in Appendix D.

5.2 Integrating the S–matrix

Until now, we treated only the functions $\mathcal{A}_j$, $\mathcal{B}_j$, $\mathcal{C}_j$, without concerning about the fact that they are actually polynomials under an integral, and thus have to be integrated. We cannot anymore neglect this, and have to look at the integrals

$$A_i^\pi = \int_{\mathcal{I}_\pi} \prod_{r=4}^{6} d^2z_r\ \mathcal{A}_i \quad , \quad B_i^\pi = \int_{\mathcal{I}_\pi} \prod_{r=4}^{6} d^2z_r\ \mathcal{B}_i \quad , \quad C_i^\pi = \int_{\mathcal{I}_\pi} \prod_{r=4}^{6} d^2z_r\ \mathcal{C}_i\ , \tag{5.9}$$

which we will develop partially in this section.

It should be clear, that since we have thousands of functions, we will not do every integral here. In fact we will just have to solve very few integrals, i.e. only six, as will become clear in a moment, in the next section. The reason for that, is the powerful system of equations which we have generated in last section. It relates all the integrals or functions with each other such that knowing a few of them we will know all!

5.2.1 Specific choice for the fixed operators

The conformal invariance of the superstring action allows for fixing three positions of the vertices. Those are chosen by convenience to be

$$z_1 = -z_\infty \quad , \quad z_2 = 0 \quad , \quad z_3 = 1\ , \tag{5.10}$$

similar to the case of the four–point function. The value of the z_1–coordinate should not worry since it will exactly be canceled by the ghost correlator

$< c(z_1)c(z_2)c(z_3) > \sim -\infty^2$. Now, according to (4.28) we should also define our kinematic invariants, which of course, will be more numerous, because of the higher number of particles. They are given by

$$\begin{aligned}
&s_1 = k_2k_4 \quad , \quad s_2 = k_2k_5 \quad , \quad s_3 = k_2k_6 \ , \\
&s_4 = k_3k_4 \quad , \quad s_5 = k_3k_5 \quad , \quad s_6 = k_3k_6 \ , \\
&s_7 = k_4k_5 \quad , \quad s_8 = k_4k_6 \ , \\
&s_9 = k_5k_6 \ .
\end{aligned} \tag{5.11}$$

Those invariants are found by finding the minimal number of parameters in the equations generated by momentum conservation for all scalar products between momenta. With those "parameters", which we call Mandelstam variables, we can express all other momentum products, and analyzing the structure of the polynomials encountered in the expressions for the $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$–functions, we arrive at a very first, rough picture of the integral:

$$\begin{aligned}
\widetilde{F}\begin{bmatrix} n_{24}, n_{25}, n_{26}, n_{34}, n_{35} \\ n_{36}, n_{45}, n_{46}, n_{56} \end{bmatrix} &:= \int_{\mathcal{I}_\pi} dz_4\ dz_5\ dz_6\ \ |z_4|^{\alpha_{24}}\ |z_5|^{\alpha_{25}}\ |z_6|^{\alpha_{26}} \qquad (5.12) \\
&\times |1-z_4|^{\alpha_{34}}\ |1-z_5|^{\alpha_{35}}\ |1-z_6|^{\alpha_{36}}\ |z_4-z_5|^{\alpha_{45}}\ |z_4-z_6|^{\alpha_{46}}\ |z_5-z_6|^{\alpha_{56}}.
\end{aligned}$$

We cannot say anything about that general integral, until we don't know the structure of the exponentials and also the region of integration. The first problem is solved immediately, for the exponentials can be captured by the formula

$$\begin{aligned}
&\alpha_{24} = s_1 + n_{24} \quad , \quad \alpha_{25} = s_2 + n_{25} \quad , \quad \alpha_{26} = s_3 + n_{26} \ , \\
&\alpha_{34} = s_4 + n_{34} \quad , \quad \alpha_{35} = s_5 + n_{35} \quad , \quad \alpha_{36} = s_6 + n_{36} \ , \\
&\alpha_{45} = s_7 + n_{45} \quad , \quad \alpha_{46} = s_8 + n_{46} \ , \\
&\alpha_{56} = s_9 + n_{56} \ ,
\end{aligned} \tag{5.13}$$

where we recognize the Mandelstam variables defined above and the natural numbers $n_{ij} \in \{\pm 2, \pm 1, 0\}$ being just the exponentials of the respective polynomials in the denominator.

If those numbers might appear strange, we should look at equation 4.30 where we find exactly this simplified situation. The exponentials are sums of Mandelstam variables and some negative integers, which come from the contractions (4.37). So they aren't that mysterious, for they mechanically arise when doing the Wick–contractions and then collecting all powers of the same polynomial.

The second major piece missing is the integration region. Again, we will focus on the Chan–Paton sequence $\pi = (1,2,3,4,5,6)$ which translates in the trace as $\mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5\lambda^6)$. We stress that this implies no loss of generality, since every other choice can be immediately obtained by just changing the numbers of all quantities labeled with indices ranging from one to six. It is worth to say that there are exactly $\frac{6!}{12} = 60$ independent choices, resulting from the total number of permutations divided by the cyclic redundancy 6 and also the equivalent inverse direction of the respective order. With that choice we have fixed our integration region completely since the Chan–Paton factors λ^a are in one to one correspondence with the integration order, as explained in section 4.3. The former is given by

$$\mathcal{I}_\pi = \{\ z_4, z_5, z_6 \in \mathbf{R} \mid 1 < z_4 < z_5 < z_6 < \infty\} \ , \tag{5.14}$$

as intuitively expected from the very first formula (5.1).

5.2.2 First explicit integrals

Having fully specified the integration range we are ready to attack the integral of the six–point function. This integral expression is given in following formula:

$$\begin{aligned}&\mathcal{A}_6(k_1,\xi_1,a_1;k_2,\xi_2,a_2;k_3,\xi_3,a_3;k_4,\xi_4,a_4;k_5,\xi_5,a_5;k_6,\xi_6,a_6) \\ &= \int_{\mathbf{R}} dx \int_{\mathbf{R}} dy \int_{\mathbf{R}} dz\ x^2 y\ \langle V^{(-1)}_{A^{a_1}}(-\infty)\ V^{(-1)}_{A^{a_2}}(0)\ V^{(0)}_{A^{a_3}}(1)\ V^{(0)}_{A^{a_4}}(x)\ V^{(0)}_{A^{a_5}}(xy)\ V^{(0)}_{A^{a_6}}(xyz)\rangle.\end{aligned} \tag{5.15}$$

In that integral we have made the change of variables

$$z_4 = x \quad , \quad z_5 = xy \quad , \quad z_6 = xyz \ , \tag{5.16}$$

which prove very convenient to later calculations.The change induces the Jacobian $\det\left(\frac{\partial(z_4,z_5,z_6)}{\partial(x,y,z)}\right) = x^2 y$. With the new variables we have thus the simple integration region $1 < x, y, z < \infty$ and our new integral with the Chan–Paton factors considered, is

$$\begin{aligned}&\mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5\lambda^6)\ A^{(1,2,3,4,5,6)} \equiv \mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5\lambda^6)\ A^{(1,2,3,4,5,6)}(1,2,3,4,5,6) \\ &= \int_1^\infty dx \int_1^\infty dy \int_1^\infty dz\ x^2 y\ \langle V^{(-1)}_{A^{a_1}}(-\infty)\ V^{(-1)}_{A^{a_2}}(0)\ V^{(0)}_{A^{a_3}}(1)\ V^{(0)}_{A^{a_4}}(x)\ V^{(0)}_{A^{a_5}}(xy)\ V^{(0)}_{A^{a_6}}(xyz)\rangle.\end{aligned} \tag{5.17}$$

One last change of variables, namely $(x \to 1/x), (y \to 1/y), (z \to 1/z)$ will bring us finally to our desired integral form

$$\widetilde{F}\begin{bmatrix} n_{24}, n_{25}, n_{26}, n_{34}, n_{35} \\ n_{36}, n_{45}, n_{46}, n_{56} \end{bmatrix} \longrightarrow F\begin{bmatrix} a, b, d, e, g \\ c, f, h, j \end{bmatrix} \ , \tag{5.18}$$

with

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} := \int_0^1 dx \int_0^1 dy \int_0^1 dz \tag{5.19}$$
$$\times\, x^a\, y^b\, z^c\, (1-x)^d\, (1-y)^e\, (1-z)^f\, (1-xy)^g\, (1-yz)^h\, (1-xyz)^j .$$

According to the variable transform the exponentials get shifted to

$$\begin{aligned}
a &= -4 - \alpha_{24} - \alpha_{25} - \alpha_{26} - \alpha_{34} - \alpha_{35} - \alpha_{36} - \alpha_{45} - \alpha_{46} - \alpha_{56}\ , \\
b &= -3 - \alpha_{25} - \alpha_{26} - \alpha_{35} - \alpha_{36} - \alpha_{45} - \alpha_{46} - \alpha_{56}\ , \\
c &= -2 - \alpha_{26} - \alpha_{36} - \alpha_{46} - \alpha_{56}\ , \\
d &= \alpha_{34} \quad , \quad e = \alpha_{45} \quad , \quad f = \alpha_{56} \\
g &= \alpha_{35} \quad , \quad h = \alpha_{46} \quad , \quad j = \alpha_{36}\ ,
\end{aligned} \tag{5.20}$$

which completely specifies the integrals, on which a lot more will be said in part II of the present work. They will be fully analyzed and solved there; we treat them here just as if they were already known. So we are in the position to give the expression for the homogeneous ξ–kinematics as

$$\begin{aligned}
A_1^\pi(a,b,i,j,k,l) &= \sigma_{lb}\sigma_{ai}\sigma_{ab}\ \Bigg\{\sigma_{jk}\sigma_{ij}\sigma_{kl}\ (k_ik_j)\ (k_kk_l)\ \widetilde{F}\begin{bmatrix} \overline{n}_{lb}, \overline{n}_{jk} = -1, \overline{n}_{ai} = -1 \\ \overline{n}_{ij}, \overline{n}_{kl} = -1, \overline{n}_{ab} = -1 \end{bmatrix} \\
&\quad - \sigma_{jk}\sigma_{ik}\sigma_{jl}\ (k_ik_k)\ (k_jk_l)\ \widetilde{F}\begin{bmatrix} \overline{n}_{lb}, \overline{n}_{jk} = -1, \overline{n}_{ai} = -1 \\ \overline{n}_{ik}, \overline{n}_{jl} = -1, \overline{n}_{ab} = -1 \end{bmatrix} \\
&\quad -\sigma_{il}\ (k_ik_l)\ (1-k_jk_k)\ \widetilde{F}\begin{bmatrix} \overline{n}_{lb}, \overline{n}_{il} = -1, \overline{n}_{ai} = -1 \\ \overline{n}_{jk} = -2, \overline{n}_{ab} = -1 \end{bmatrix}\Bigg\} , \\
A_2^\pi(a,b,i,j,k,l) &= (1-k_ik_j)\ (1-k_kk_l)\ \widetilde{F}\begin{bmatrix} \overline{n}_{ij} = -2 \\ \overline{n}_{kl} = -2 \end{bmatrix}, \overline{n}_{ab} = -2\Bigg] \\
&\quad - \sigma_{ij}\sigma_{kl}\sigma_{ik}\sigma_{jl}\ (k_ik_k)\ (k_jk_l)\ \widetilde{F}\begin{bmatrix} \overline{n}_{ij}, \overline{n}_{kl} = -1 \\ \overline{n}_{ik}, \overline{n}_{jl} = -1 \end{bmatrix}, \overline{n}_{ab} = -2\Bigg] \\
&\quad + \sigma_{ij}\sigma_{kl}\sigma_{il}\sigma_{jk}\ (k_ik_l)\ (k_jk_k)\ \widetilde{F}\begin{bmatrix} \overline{n}_{ij}, \overline{n}_{kl} = -1 \\ \overline{n}_{il}, \overline{n}_{jk} = -1 \end{bmatrix}, \overline{n}_{ab} = -2\Bigg]\ ,
\end{aligned} \tag{5.21}$$

with the useful conventions $\sigma_{ij} = \mathrm{sign}(i-j)$, and $\overline{n}_{ij} := \begin{cases} n_{ij}\,, i<j \\ n_{ji}\,, i>j \end{cases}$. The numbers $\overline{n_{ij}}$ are to be understood such that they are only of relevance if they show up in the definition equation (5.13). The number n_{12}, for example, can fully be neglected since it doesn't was defined as such, due to our specific choice for the fixed vertices positions.

We said that one specific trace combination of Chan–Paton factors, e.g. $\mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5\lambda^6)$ is invariant under cyclic permutations and thus should'n be counted twice. This implies that also the functions in front of every kinematics should also be invariant under cyclic permutation of indices,($j \to j+1$), which indeed is true. Under such a shift of indices the Mandelstam variables transform according the transformation of the momenta ($k_j \to k_{j+1}$). Furthermore, in order to show this invariance also some variable substitution has to be done, but when done it proves for the equation to be a mathematical identity. Beyond this, also some of the kinematics Ξ_j already given in (5.5) stay invariant under this permutation, especially Ξ_8 which is invariant under all five cyclic permutations. When looking more closely to that invariance we may notice that all the other Ξ_j could actually be generated from a minimal number of such products when performing permutations of their indices, which are exactly $\Xi_1, \Xi_2, \Xi_5, \Xi_7$ and Ξ_8. Thus they can be regarded as the generators of the whole string matrix, which consequently can be written as

$$\begin{aligned}
&-\left\{(1-s_4)(1-s_9)\widetilde{F}\begin{bmatrix}0,0,0,-2,0\\0,0,0,-2\end{bmatrix}+s_6s_7\,\widetilde{F}\begin{bmatrix}0,0,0,-1,0\\-1,-1,0,-1\end{bmatrix}-s_5s_8\,\widetilde{F}\begin{bmatrix}0,0,0,-1,-1\\0,0,-1,-1\end{bmatrix}\right\}\Xi_1\\
&-\left\{(1-s_5)(1-s_8)\widetilde{F}\begin{bmatrix}0,0,0,0,-2\\0,0,-2,0\end{bmatrix}-s_6s_7\,\widetilde{F}\begin{bmatrix}0,0,0,0,-1\\-1,-1,-1,0\end{bmatrix}-s_4s_9\,\widetilde{F}\begin{bmatrix}0,0,0,-1,-1\\0,0,-1,-1\end{bmatrix}\right\}\Xi_2\\
&-\left\{s_5(1-s_8)\widetilde{F}\begin{bmatrix}0,-1,0,0,-1\\0,0,-2,0\end{bmatrix}+s_6s_7\,\widetilde{F}\begin{bmatrix}0,-1,0,0,0\\-1,-1,-1,0\end{bmatrix}+s_4s_9\,\widetilde{F}\begin{bmatrix}0,-1,0,-1,0\\0,0,-1,-1\end{bmatrix}\right\}\Xi_5\\
&-\left\{-s_4(1-s_9)\,\widetilde{F}\begin{bmatrix}0,0,0,-1,0\\0,0,0,-2\end{bmatrix}+s_6s_7\,\widetilde{F}\begin{bmatrix}0,0,0,0,0\\-1,-1,0,-1\end{bmatrix}-s_5s_8\,\widetilde{F}\begin{bmatrix}0,0,0,0,-1\\0,0,-1,-1\end{bmatrix}\right\}\Xi_7\\
&+\left\{-s_7\,(1-s_6)\,\widetilde{F}\begin{bmatrix}0,-1,0,0,0\\-2,-1,0,0\end{bmatrix}-s_5s_8\,\widetilde{F}\begin{bmatrix}0,-1,0,0,-1\\-1,0,-1,0\end{bmatrix}+s_4s_9\,\widetilde{F}\begin{bmatrix}0,-1,0,-1,0\\-1,0,0,-1\end{bmatrix}\right\}\Xi_8\,.
\end{aligned}\tag{5.22}$$

Here we have already inserted the "integrated" functions and also the multiplicative momentum factors. Also is it worth to notice we are using here again the "old" functions $\widetilde{F}$ before the last variable transform. This is just a matter of convenience. After doing the integrals in chapter 6 we will of course replace at the end the functions $\widetilde{F}$ with F.

5.3 Solution of the equations and its basis

We can now move on, and look at the equations already generated from the S–matrix by permuting the ghost–operators.

5.3.1 Some simple examples

It turns out that those coming from the $\mathcal{C}$–system are the simplest and shortest ones. A typical example may be

$$\widetilde{F}\begin{bmatrix}0,-1,0,0,-1\\ -1,-1,-1,1\end{bmatrix}=\widetilde{F}\begin{bmatrix}0,0,-1,0,-1\\ -1,-1,-1,1\end{bmatrix}-\widetilde{F}\begin{bmatrix}0,-1,-1,0,-1\\ -1,-1,-1,2\end{bmatrix}\ . \tag{5.23}$$

which, because of its simplicity, may even be shown to hold with low mathematical input. When inserting the mathematically convenient functions F, given in (5.19), the relation boils down to

$$\begin{aligned}&\frac{y\ (1-z)}{(1-y)\ (1-xy)\ z\ (1-yz)\ (1-xyz)}\\&=\frac{y\ (1-z)}{(1-y)\ (1-xy)\ (1-yz)\ (1-xyz)}+\frac{y\ (1-z)^2}{(1-y)\ (1-xy)\ z\ (1-yz)\ (1-xyz)}.\end{aligned} \tag{5.24}$$

which can almost be "seen" to hold with partial fraction techniques. So this is the first example of identities obtained out of the system of equations which we have also proved mathematically. A little more involved example, but still manageable is the following one

$$\begin{aligned}\widetilde{F}\begin{bmatrix}-1,-1,-1,0,0\\ -1,0,0,0\end{bmatrix}&=\widetilde{F}\begin{bmatrix}-1,0,-1,0,-1\\ -1,-1,-1,0\end{bmatrix}-\widetilde{F}\begin{bmatrix}-1,0,-1,0,-1\\ -1,-1,0,1\end{bmatrix}-\widetilde{F}\begin{bmatrix}-1,0,0,2,-1\\ -1,-1,-1,0\end{bmatrix}\\&-\widetilde{F}\begin{bmatrix}1,-1,-1,-1,0\\ 0,-1,-1,0\end{bmatrix}+\widetilde{F}\begin{bmatrix}2,-1,-1,-1,0\\ 0,-1,-1,0\end{bmatrix}-2\ \widetilde{F}\begin{bmatrix}0,-1,-1,-1,1\\ -1,-1,0,0\end{bmatrix}\\&+2\ \widetilde{F}\begin{bmatrix}0,-1,-1,-1,0\\ -1,-1,1,0\end{bmatrix}-2\ \widetilde{F}\begin{bmatrix}0,-1,-1,-1,0\\ -1,-1,0,1\end{bmatrix}\\&-\widetilde{F}\begin{bmatrix}0,-1,-1,0,-1\\ -1,0,-1,1\end{bmatrix}+\widetilde{F}\begin{bmatrix}0,-1,-1,0,0\\ -1,-1,-1,1\end{bmatrix}\ .\end{aligned} \tag{5.25}$$

whose polynomial origin is

$$\begin{aligned}\frac{1}{1-xyz}&=-\frac{y}{(1-y)(1-yz)(1-x)}-\frac{y(1-x)^2}{x(1-y)(1-xy)(1-yz)(1-xyz)}\\&+\frac{1-yz}{(1-y)(1-xy)z(1-xyz)}-\frac{1-z}{(1-y)(1-xy)z(1-xyz)}\\&+\frac{y(1-z)}{(1-y)(1-yz)(1-xyz)}-\frac{y(1-z)}{(1-xy)(1-yz)(1-xyz)}+\frac{y}{x(1-y)(1-yz)(1-x)}\\&-2\ \frac{1-xy}{(1-y)(1-xyz)(1-x)}+2\ \frac{1-yz}{(1-y)z(1-xyz)(1-x)}-2\ \frac{1-z}{(1-y)z(1-xyz)(1-x)}.\end{aligned} \tag{5.26}$$

Moreover, as mentioned before, all the equation coming from the $\mathcal{C}$–system are short and of "polynomial" kind, i.e. thy all can be easily proved by partial fraction techniques, when the functions are inserted into the equations. No higher mathematical tools are needed, such as variable transform, partial integration, etc.

All the other examples are in principle similar to that one, with the difference that they are much more involved: on the one hand the identities contain much more functions, e.g. 20 or 30 functions, so in order to prove their equality laborious work has to be done; on the other hand side, also the techniques used may exceed those used before, so it is usually necessary to make complicated variable transforms and partial integrations which are by far not obvious, when considering also the fact that the functions involved may also have some poles! Some more involved equalities, which still can be proven by partial integration are given below

$$
\begin{aligned}
&(1-s_6)\left(\widetilde{F}\begin{bmatrix}-1,0,0,0,0\\-2,-1,0,0\end{bmatrix}-\widetilde{F}\begin{bmatrix}0,-1,0,0,0\\-2,-1,0,0\end{bmatrix}\right)-s_8\,\widetilde{F}\begin{bmatrix}-1,-1,0,0,0\\-1,0,-1,0\end{bmatrix}\\
&=s_9\widetilde{F}\begin{bmatrix}-1,-1,0,0,0\\-1,0,0,-1\end{bmatrix}+s_3\left(\widetilde{F}\begin{bmatrix}-1,0,-1,0,0\\-1,-1,0,0\end{bmatrix}-\widetilde{F}\begin{bmatrix}0,-1,-1,0,0\\-1,-1,0,0\end{bmatrix}\right),
\end{aligned}
$$

$$
\begin{aligned}
(1+s_4)\,\widetilde{F}\begin{bmatrix}0,-1,-1,-2,0\\0,0,0,0\end{bmatrix}&=s_1\left(\widetilde{F}\begin{bmatrix}-1,-1,0,-1,0\\0,0,0,-1\end{bmatrix}-\widetilde{F}\begin{bmatrix}-1,0,-1,-1,0\\0,0,0,-1\end{bmatrix}\right)\\
&-s_7\,\widetilde{F}\begin{bmatrix}0,-1,-1,-1,0\\0,-1,0,0\end{bmatrix}-s_8\widetilde{F}\begin{bmatrix}0,-1,-1,-1,0\\0,0,-1,0\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&s_2\,\widetilde{F}\begin{bmatrix}0,-1,0,-1,0,\\0,0,-1,0\end{bmatrix}= \qquad (5.27)\\
&-(s_4+s_5+s_6+s_7+s_8+s_9)\left(\widetilde{F}\begin{bmatrix}0,0,0,-1,0\\0,-1,0,-1\end{bmatrix}-\widetilde{F}\begin{bmatrix}0,0,0,-1,0\\0,0,-1,-1\end{bmatrix}\right)\\
&-s_3\,\widetilde{F}\begin{bmatrix}0,0,-1,-1,0\\0,-1,0,0\end{bmatrix}-(1+s_1+s_2+s_3+s_4+s_5+s_6+s_7+s_8+s_9)\,\widetilde{F}\begin{bmatrix}0,0,0,0,0\\0,-1,-1,0\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
s_2\,\widetilde{F}\begin{bmatrix}0,0,-1,0,-1\\0,-1,-1,1\end{bmatrix}&=s_3\,\widetilde{F}\begin{bmatrix}0,-1,-1,-1,0\\-1,-1,0,1\end{bmatrix}+(s_1+s_2-s_7)\,\widetilde{F}\begin{bmatrix}0,-1,-1,0,0\\-1,-1,-1,1\end{bmatrix}\\
&+(s_2-s_3-s_4+s_9)\,\widetilde{F}\begin{bmatrix}0,-1,-1,0,-1\\-1,-1,-1,2\end{bmatrix}.
\end{aligned}
$$

5.3.2 The final system of linear equations

We have produced a huge amount of such equations, i.e about $50,000$ of them, which qualitatively look exactly as (5.27) but are in general much more complex and thus involved. This system can be solved, by first plugging in the simplest equations, namely those polynomial equations coming from the $\mathcal{C}$–system; this way we are able to dramatically reduce the number of equations and also the number of functions.

Starting with $1,270$ $\widetilde{F}$–functions we end up at this stage with just 576 functions. Further solving of the system eventually leads to the conclusion that all the functions occurring there can be expressed in terms of just **six functions**! This we call the base of the system. This is of course to be seen as the complete analogy of the four–point case and its solution given in (4.57). Moreover, exactly as in the four–point case, we are still free to choose which functions we are willing to take as parameters. By the same arguments as in the section 4.3 we chose our six–dimensional basis to be made out of functions entirely without poles, this being a very good feature for the momentum expansion of the S–matrix.

Thus again as in the four–point case we can summarize the solution in the following abstract formula

$$\boxed{\begin{aligned}\widetilde{F}\begin{bmatrix} n_{24},n_{25},n_{26},n_{34},n_{35} \\ n_{36},n_{45},n_{46},n_{56}\end{bmatrix} &= \Lambda^1_{\{n_{ij}\}}(s_i)\ \widetilde{F}\begin{bmatrix} -1,-1,-2,0,0 \\ 0,0,0,0\end{bmatrix} + \Lambda^2_{\{n_{ij}\}}(s_i)\ \widetilde{F}\begin{bmatrix} -1,-1,0,0,0 \\ -2,0,0,0\end{bmatrix} \\ &+\Lambda^3_{\{n_{ij}\}}(s_i)\ \widetilde{F}\begin{bmatrix} -1,0,-2,0,-1 \\ 0,0,0,0\end{bmatrix} + \Lambda^4_{\{n_{ij}\}}(s_i)\ \widetilde{F}\begin{bmatrix} -1,0,-2,0,0 \\ -1,0,0,0\end{bmatrix} \\ &+\Lambda^5_{\{n_{ij}\}}(s_i)\ \widetilde{F}\begin{bmatrix} 0,-2,-1,0,0 \\ -1,0,0,0\end{bmatrix} + \Lambda^6_{\{n_{ij}\}}(s_i)\ \widetilde{F}\begin{bmatrix} -2,-1,-1,0,0 \\ -1,0,0,0\end{bmatrix}\ .\end{aligned}} \tag{5.28}$$

with the following important qualities: the six functions there, which will be treated with high precision in the mathematical section, are known and relatively simple to expand in their momenta. Furthermore, for they have no poles, the whole pole structure is encoded solely in the multiplicative coefficients $\Lambda^j_{\{n_{ij}\}}(s_i)$ which are highly complex polynomials in the momenta s_j. Furthermore, since the basis of functions $\widetilde{F}_j, j = 1, ..6$, is chosen once and for all, the coefficients are specific for each unknown functions $\widetilde{F}$ which will be expressed through the basis, and thus is on the left hand side of the equation (5.28). This is also symbolized by the fact the coefficients $\Lambda^j_{\{n_{ij}\}}(s_i)$ have also a dependency on the numbers n_{ij} which uniquely characterize the functions on the left hand side $\widetilde{F}\begin{bmatrix} n_{24},n_{25},n_{26},n_{34},n_{35} \\ n_{36},n_{45},n_{46},n_{56}\end{bmatrix}$.

One may wonder, in how far the choice for one particular Chan–Paton order $\mathrm{Tr}(\lambda^a\lambda^b\lambda^c\lambda^d\lambda^e\lambda^f)$ and thus, by direct implication, for one specific integration region $\mathcal{I}_\pi$ will influence the solution (5.28). Changing the integration region will just lead to another representation of the same function, as given in (5.19). Then exactly the same relations as shown in (5.28) will hold between exactly the same functions, the only difference being the various integral representation those function will have. Therefore we may conclude that our solution and thus the mathematical relations found are completely general, independent of any choice of integration, just depending on the functions they involve.

The basic functions on the right hand side of the system can now be given as

$$\begin{aligned}
\widetilde{F}\begin{bmatrix} -1,-1,-2,0,0 \\ 0,0,0,0 \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \mathcal{P}(x,y,z) \; , \\
\widetilde{F}\begin{bmatrix} -1,-1,0,0,0 \\ -2,0,0,0 \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{\mathcal{P}(x,y,z)}{(1-xyz)^2} \; , \\
\widetilde{F}\begin{bmatrix} -1,0,-2,0,-1 \\ 0,0,0,0 \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{\mathcal{P}(x,y,z)}{1-xy} \; , \\
\widetilde{F}\begin{bmatrix} -1,0,-2,0,0 \\ -1,0,0,0 \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{z\;\mathcal{P}(x,y,z)}{1-xyz} \; , \\
\widetilde{F}\begin{bmatrix} 0,-2,-1,0,0 \\ -1,0,0,0 \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{y\;\mathcal{P}(x,y,z)}{1-xyz} \; , \\
\widetilde{F}\begin{bmatrix} -2,-1,-1,0,0 \\ -1,0,0,0 \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{x\;\mathcal{P}(x,y,z)}{1-xyz} .
\end{aligned} \tag{5.29}$$

with

$$\begin{aligned}
\mathcal{P}(x,y,z) &= x^{-s_1-s_2-s_3-s_4-s_5-s_6-s_7-s_8-s_9} \; y^{-s_2-s_3-s_5-s_6-s_7-s_8-s_9} \; z^{-s_3-s_6-s_8-s_9} \\
&\quad \times (1-x)^{s_4} \; (1-y)^{s_7} \; (1-z)^{s_9} \; (1-xy)^{s_5} \; (1-yz)^{s_8} \; (1-xyz)^{s_6} \\
&= x^{k_2k_3} \; y^{k_2k_3+k_2k_4+k_3k_4} \; z^{k_1k_6} \; (1-x)^{k_3k_4} \; (1-y)^{k_4k_5} \; (1-z)^{k_5k_6} \\
&\quad \times (1-xy)^{k_3k_5} \; (1-yz)^{k_4k_6} \; (1-xyz)^{k_3k_6} ,
\end{aligned} \tag{5.30}$$

stemming from the contractions $\mathcal{E}$ of the exponentials defined in equation (5.4).

So one last piece missing is the momentum expansion of upper functions. This will enable us finally to expand the whole S–matrix in its momenta, since all the unknown functions are then series expanded, as a direct consequence of (5.28), and then they can be inserted in the appropriate expressions (5.21) or (5.22)

5.4 Momentum expansion of the S–matrix

Since we wish to express the whole S–matrix as an expanded relation in the nine Mandelstam invariants s_j, we now will give the expansion of our basis functions from

(5.28) and then be able to insert that expansion in the appropriate expression leading us directly to our goal. In order to series expand the basis we will use a different choice of kinematical variables given as

$$\begin{aligned} s_1 &= k_1 k_2 \quad , \quad s_2 = k_2 k_3 \quad , \quad s_3 = k_3 k_4 \; , \\ s_4 &= k_4 k_5 \quad , \quad s_5 = k_5 k_6 \quad , \quad s_6 = k_6 k_1 \; , \\ s_7 &= \frac{1}{2}\,(k_1+k_2+k_3)^2 \quad , \quad s_8 = \frac{1}{2}\,(k_2+k_3+k_4)^2 \quad , \quad s_9 = \frac{1}{2}\,(k_3+k_4+k_5)^2 \end{aligned} \tag{5.31}$$

This turns out to be more convenient for the calculation and doesn't have any consequences on the expansion, since the functions remain the same. The deeper reason is to be found in the cyclic symmetry, which now just acts as a simple permutation within the two sets $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $\{s_7, s_8, s_9\}$.

The basis, expanded in its small parameter, being the Mandelstam variables, reads

$$\begin{aligned}
\Phi_1 = \widetilde{F}\begin{bmatrix} -1,-1,-2,0,0 \\ 0,0,0,0 \end{bmatrix} &= 1 - 3\ s_1 - s_2 + s_3 + s_5 - s_6 + s_7 - s_8 + s_9 \\
&+ (s_1 - s_3 - s_4 - s_5)\ \zeta(2) + (s_1 + s_4 - s_7 - s_9)\ \zeta(3) + \ldots\ , \\
\Phi_2 = \widetilde{F}\begin{bmatrix} -1,-1,0,0,0 \\ -2,0,0,0 \end{bmatrix} &= (1 + s_1 + s_4 - s_7 - s_9)\ \zeta(2) \\
&- (2\ s_1 + s_2 + s_3 + 2\ s_4 + s_5 + s_6 - s_7 + s_8 - s_9)\ \zeta(3) + \ldots\ , \\
\Phi_3 = \widetilde{F}\begin{bmatrix} -1,0,-2,0,-1 \\ 0,0,0,0 \end{bmatrix} &= (1 - s_1 + s_2 + s_3 + 3\ s_4 + s_5 + s_7 - s_8 - s_9)\ \zeta(2) \\
&+ (s_1 - s_2 - 2\ s_3 - 4\ s_4 - s_5 - s_7 + 2\ s_8 + 2\ s_9)\ \zeta(3) + \ldots\ , \\
\Phi_4 = \widetilde{F}\begin{bmatrix} -1,0,-2,0,0 \\ -1,0,0,0 \end{bmatrix} &= -1 + \zeta(2) + s_1 - s_2 - s_4 + s_5 + 3\ s_6 + s_7 - s_8 - s_9 \\
&+ (s_2 - s_3 - s_4 - s_5 - 2\ s_6 + s_8 + 2\ s_9)\ \zeta(2) \\
&- (s_1 + s_2 - s_3 - 2\ s_4 + s_7 + s_8 + 2\ s_9)\ \zeta(3) \ldots\ , \\
\Phi_5 = \widetilde{F}\begin{bmatrix} 0,-2,-1,0,0 \\ -1,0,0,0 \end{bmatrix} &= -1 + \zeta(2) + s_1 - s_2 - 2\ s_3 - 2\ s_5 - s_6 + s_7 + 3\ s_8 + s_9 \\
&+ (s_2 + s_3 - s_4 + s_5 + s_6 - 2\ s_8)\ \zeta(2) \\
&- (s_1 + s_2 - s_4 + s_6 + s_7 + s_9)\ \zeta(3) + \ldots\ , \\
\Phi_6 = \widetilde{F}\begin{bmatrix} -2,-1,-1,0,0 \\ -1,0,0,0 \end{bmatrix} &= -1 + \zeta(2) + s_1 + 3\ s_2 + s_3 - s_4 - s_6 - s_7 - s_8 + s_9 \\
&- (2\ s_2 + s_3 + s_4 + s_5 - s_6 - 2\ s_7 - s_8)\ \zeta(2) \\
&- (s_1 - 2\ s_4 - s_5 + s_6 + 2\ s_7 + s_8 + s_9)\ \zeta(3) + \ldots\ .
\end{aligned} \tag{5.32}$$

This ad hoc result will be the extended subject of the next mathematics chapter and will especially be derived in section 6.4.

Now, we can regard our S–matrix as completely solved, since it is expanded in its momenta, and may be given in short notation by the following formula

$$\boxed{\mathcal{A}^{(1,2,3,4,5,6)} = \sum_{j=1}^{6} \mathcal{P}^j(s_1,\ldots,s_9)\ \ \Phi_j\ .} \tag{5.33}$$

where we have chosen again, by no loss of generality, the group structure $\mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5\lambda^6)$ in order to explicitly state upper result. $\mathcal{P}^j(s_1,\ldots,s_9)$ are products of the kinematics $(\xi\xi)(\xi\xi)(\xi\xi)$, $(\xi\xi)(\xi\xi)(\xi k)(\xi k)$ and $(\xi\xi)(\xi k)(\xi k)(\xi k)(\xi k)$ and the coefficients $\Lambda^j_{\{n_{ij}\}}(s_j)$ as given in (5.28). We expect those $\mathcal{P}^j$ to be of large complexity, and indeed they are.

At this point, we are in the position to extract the necessary information for the low energy effective action of N D-branes on top of each other, i.e. the Born–Infeld action. Also the rich pole structure will give important information about the reducible diagrams which have to be subtracted from the Born–Infeld action, but more on this will be said in chapter 7. At this point we will list the S–matrix expansion up to some order in the momentum k^μ. We start with the kinematic $(\xi\xi)\ (\xi k)\ (\xi k)\ (\xi k)\ (\xi k)$, the other two are given in the appendix, again for the sake of aesthetics. The contractions coming from the $\mathcal{C}$–system of equations are various, hence we will show some representative ones, without being able to list all of them, this requiring hundreds of pages! Thus after eliminating on shell some superfluous kinematics (this being done by using momentum conservation $\sum_i k_i^\mu = 0$ and $\xi_1^\mu k_\mu^1 = 0$) we obtain results of the form

$$\begin{aligned}(\xi_2\xi_3)(\xi_1k_3)(\xi_4k_1)(\xi_5k_1)(\xi_6k_1)\ &\left\{\frac{1}{s_2\ s_6\ s_8} - \left(\frac{s_4}{s_2\,s_6} + \frac{s_5}{s_2\,s_8} + \frac{s_3}{s_6\,s_8}\right)\ \zeta(2)\right.\\ &+\left(\frac{s_4}{s_2} + \frac{s_5}{s_2} + \frac{s_3}{s_6} + \frac{s_4}{s_6} + \frac{{s_4}^2}{s_2\,s_6} - \frac{s_7}{s_2} + \frac{{s_5}^2}{s_2\,s_8} + \frac{s_2\,s_3}{s_6\,s_8}\right.\\ &\left.\left.+\frac{{s_3}^2}{s_6\,s_8} + \frac{s_5\,s_6}{s_2\,s_8} + \frac{s_4\,s_8}{s_2\,s_6} - \frac{s_9}{s_6}\right)\ \zeta(3)\right\} + \mathcal{O}(k^6)\ .\end{aligned} \tag{5.34}$$

for the kinematics $(\xi_2\xi_3)(\xi_1k_3)(\xi_4k_1)(\xi_5k_1)(\xi_6k_1)$ up to order $O(k^6)$ in the momentum k^μ or

$$\begin{aligned}(\xi_1\xi_2)(\xi_3k_6)(\xi_4k_2)(\xi_5k_2)(\xi_6k_2)\ &\left\{\zeta(3) - \left(\frac{1}{4}\ s_1 + s_2 + \frac{3}{4}\ s_3 + \frac{1}{2}\ s_4 + \frac{3}{4}\ s_5 + s_6\right.\right.\\ &\left.\left.+\frac{1}{4}\ s_7 + s_8 + \frac{1}{4}\ s_9\right)\ \zeta(4)\right\} + \mathcal{O}(k^8)\ ,\end{aligned} \tag{5.35}$$

for the kinematics $(\xi_1\xi_2)(\xi_3k_6)(\xi_4k_2)(\xi_5k_2)(\xi_6k_2)$, up to the order $O(k^8)$. If one wishes to go higher in the momentum expansion, this will be just a matter of time and work since all the ingredients for obtaining those results are completely given in the next chapter. All the kinematics which occur in the S–matrix either from the $\mathcal{A}$–, $\mathcal{B}$– or the $\mathcal{C}$–system have been calculated and the only reason for not listing them entirely here is their overwhelming length! Nevertheless some more representative examples are given in the Appendix E.

Part II

Multiple Hypergeometric Functions and Euler–Zagier Sums

Chapter 6

Mathematical tools

As calculating the six–point open string amplitude in [26], we were faced with great mathematical problems, which could not be solved instantaneously. Thus we were forced to acquire a great amount of techniques and even develop some new results, in order to reach our goals. Those results, their full derivation and also introduction to the subject will be given in that part II. It is as such self contained and may be read independently of the previous work.

This chapter should serve as a good and complete introduction to the topic of hypergeometric functions. Beyond their definition and properties, there will also be presented different representation as the integral one or the sum representation. Furthermore, a lot will be said in general about sums and their evaluation, since they are indispensable tools for the analysis of hypergeometric functions. Finally, we should success in expanding the hypergeometric functions in their parameters (to be defined in short), fact which is by far non trivial and even not to be found in mathematical literature.

6.1 Special functions

In this subsection we will define all types of generalized functions, generalization thereof and connect them with other special function, arriving finally at the analytic function presented in (5.19), which is the very heart of every six point function.

6.1.1 Generalized hypergeometric functions ${}_pF_q$

Although the functions involved in the calculations done sofar, i.e. during the computation of the six–point S–matrix are much more involved than simple hypergeometric functions or even their generalization, we find it pedagogically worth to introduce first those ones and then gradually to generalize until we reach our goal.

Let's start then with the simplest case involving string amplitudes. As seen already in this case, the four–point scattering, we encounter there an integral, which, when evaluated, eventually leads to the Euler–Beta[1] function

$$\begin{aligned} B(a,b) &= \int_0^1 dx\ x^a\ (1-x)^b = \frac{1}{1+a}\ {}_2F_1\left[1+a,-b,2+a;1\right] \\ &= \frac{\Gamma(1+a)\ \Gamma(1+b)}{\Gamma(2+a+b)} \quad , \quad \mathrm{Re}\ a > -1\ ,\ \mathrm{Re}\ b > -1\ , \end{aligned} \tag{6.1}$$

with $\Gamma(x)$ being the usual Gamma–function, of which one possible definition is

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{6.2}$$

As seen on the right hand side of the first line in equation (6.1), the Euler–Beta integral is just a simplification, a special value of the underlying ${}_2F_1$ hypergeometric function. The family of such functions will be defined in a while, but we see that for the special value of one for the argument of this function we get the Beta–function.

Now a few more words on the Gamma function. Its definition is of course inspired from the corresponding integral for integer arguments n, I(n)=$\int_0^\infty e^{-t}t^{n-1}dt$ which obeys the functional equality $I(n) = n \cdot I(n-1)$ and is of course the definition of the faculty function for integers, $I(n) = n!$. A very comprehensive and complete treatment of the Γ–function can be found in [84, 85].

The very next step, is to generalize this Beta-function. As a motivation, this generalization especially shows up in the five–point scattering, where we have following integral

$$C(a,b,c,d,e) := \int_0^1 dx \int_0^1 dy\ x^a\ y^b\ (1-x)^c\ (1-y)^d\ (1-xy)^e\ . \tag{6.3}$$

In both cases (6.1) and (6.3), the parameters entering the functions are exactly the kinematic invariants involved in the corresponding physical problem, thus the number of polynomials which have to be integrated, is in one to one correspondence with the number of Mandelstam variables describing that physical process.

As stated already before, the Euler–Beta integral proves to be just a simple case of the more general hypergeometric function ${}_2F_1$

$${}_2F_1[-c,1+a,2+a+b;\ y\ ]\ \frac{\Gamma(1+a)\ \Gamma(1+b)}{\Gamma(2+a+b)} = \int_0^1\ dx\ x^a\ (1-x)^b\ (1-xy)^c\ , \tag{6.4}$$

[1]It should be clear that the Beta function is encountered all over in physics not just in string theory. Moreover, in general, the hypergeometric functions very often describe solutions to physical problems. It is also possible to describe all known "elementary" functions, like sin, log or Γ, by special parameter choices of the hypergeometric functions.

whose generalization is given by

$$ {}_{p+1}F_{q+1}\left[\begin{matrix}1+a,a_1,\ldots,a_p\\2+a+b,b_1,\ldots,b_q\end{matrix};\lambda\right]\ \frac{\Gamma(1+a)\,\Gamma(1+b)}{\Gamma(2+a+b)} = \int_0^1 dx\ x^a\ (1-x)^b\ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix};\lambda x\right], \tag{6.5} $$

with Re $a > -1$, Re $b > -1$, and $p \leq q+1$. For the case of a ${}_{p+1}F_p$ function the parameter λ has to obey the condition $|\lambda| \leq 1$. Also note that equation (6.5) is not just the simple generalization of the ${}_2F_1$ function, but instead it defines all the family of hypergeometric functions, $\{{}_pF_q \mid p\ ,q \in \mathbb{N}\ \}$. Thus the generic integral which arises in the five–point case turns out to be just a ${}_3F_2$ hypergeometric function,

$$ C(a,b,c,d,e) = \frac{\Gamma(1+a)\,\Gamma(1+b)\,\Gamma(1+c)\,\Gamma(1+d)}{\Gamma(2+a+c)\,\Gamma(2+b+d)}\ {}_3F_2\left[\begin{matrix}1+a,\ 1+b,\ -e\\2+a+c,\ 2+b+d\end{matrix}\ ;\ 1\right], \tag{6.6} $$

with $\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c), \mathrm{Re}(d) > -1$. Again, we have evaluated the hypergeometric function at the special argument value of one. The series representation of such a function[2] can easily be obtained from the integral: first we series expand the polynomials in their small variables $x, y, z, ...$, which have to be integrated. Of course, since those variables are ranging between zero and one the series will be infinite. After that we can interchange summation and integration, thus integrating the infinite series and obtaining a series solely given in terms of the exponentials of the original polynomials. This series looks like [86]:

$$ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\ ;\ x\right] = \sum_{s=0}^{\infty} \frac{1}{s!}\ \frac{(a_1,s)\cdot\ldots\cdot(a_p,s)}{(b_1,s)\cdot\ldots\cdot(b_q,s)}\ x^s\ . \tag{6.7} $$

The notation $(a,m) = \frac{\Gamma(a+m)}{\Gamma(a)}$ is usual an was introduced by Pochhammer, this being also the reason for which (a,m) is called the Pochhammer–symbol. As a consequence of the divergence of the Γ-function at negative integers, the sum 6.7 will be not defined for $b_j = 0, -1, -2, \ldots$. A good criteria of convergence for upper sum is

$$ \omega := \mathrm{Re}\left(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i\right)\ . \tag{6.8} $$

In the case of ${}_{p+1}F_p$ the series converges for $|x| < 1$ and in the case of $|x| = 1$ the series is absolute convergent for $\omega > 0$ and diverges for $\omega \leq -1$.

[2]It is worth mentioning that for the first time hypergeometric functions were discovered when studying hypergeometric differential equations, see [79], from where they also inherited their name. Thus they were first written in the series representation, and later on as integrals of polynomials.

A lot more can be said on the hypergeometric functions, we will restrict us however to that. The information collected above should be enough for an introduction and also for understanding the present work. More on that topic can be read in the special literature about it.

6.1.2 Triple hypergeometric functions

Unfortunately, the hypergeometric functions treated before, are not complex enough to capture the six–point function. Although our integral functions encountered in the six–point amplitude were given by a triple integral which is also the case for the generalized hypergeometric ${}_4F_3$ function, they prove not to be the same; the structure of the integrated polynomial in the ${}_4F_3$ function is simpler than the one we are faced with. Hypergeometric functions will have to be further generalized until we reach the desired degree of complexity, which eventually captures the physical problems. For that we will start with our complete general integral

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} := \int_0^1 dx \int_0^1 dy \int_0^1 dz \\ \times\ x^a\ y^b\ z^c\ (1-x)^d\ (1-y)^e\ (1-z)^f\ (1-xy)^g\ (1-yz)^h\ (1-xyz)^j . \tag{6.9}$$

Here we recognize the nine kinematic invariants which uniquely determine the number of the nine polynomials, the latter being under the integral to be done. This is also a sign that, since the nine invariants characterize all the six–point processes, this integral is the most general, thus occurring in all scatterings where six external states coming from open strings are involved. It is thus worth to study it well, for all results can also be used in similar processes!

Properties of the function can be already extracted from its integral representation. Thus we see for example that following symmetry is present

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} = F\begin{bmatrix} c,b,f,e,h \\ a,d,g,j \end{bmatrix} . \tag{6.10}$$

The proof of this exchange is very simple to establish, since we immediately see that the integral in (6.9) stays invariant under it. We would like now to obtain the series representation of that function. This is a straight forward task; first we have to expand following polynomials in their exponents:

$$(1-xy)^g\ (1-yz)^h = \sum_{m,n=0}^{\infty} \frac{(-g,m)\ (-h,n)}{(1,m)\ (1,n)}\ x^m\ z^n\ y^{m+n} \quad , \quad |x|<1\ \ ;\ \ |y|<1\ \ ;\ \ |z|<1\ . \tag{6.11}$$

We recognize them as the two polynomials with the respective exponentials in our integral (6.9). Note that this expansion is actually a product of the single expansions of the polynomials.We can now insert those expanded polynomials in our integral (6.9), which then will look something like

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} := \int_0^1 dx \int_0^1 dy \int_0^1 dz \sum_{m,n=0}^{\infty} \frac{(-g,m)\ (-h,n)}{(1,m)\ (1,n)} \times\ x^{a+m}\ y^{b+m+n}\ z^{c+n}\ (1-x)^d\ (1-y)^e\ (1-z)^f (1-xyz)^j\ . \tag{6.12}$$

The integrand can though now easily be integrated with the help of formula (6.5) to give

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} = \Gamma(1+d)\ \Gamma(1+e)\ \Gamma(1+f) \sum_{m,n=0}^{\infty} \frac{(-g,m)\ (-h,n)}{(1,m)\ (1,n)} \times \frac{\Gamma(1+m+n+b)\ \Gamma(1+m+a)\ \Gamma(1+n+c)}{\Gamma(2+m+n+b+e)\ \Gamma(2+m+a+d)\ \Gamma(2+n+c+f)} \times {}_4F_3\left[\begin{matrix} 1+m+n+b,\ 1+m+a,\ 1+n+c,\ -j \\ 2+m+n+b+e,\ 2+m+a+d,\ 2+n+c+f \end{matrix}\ ;\ 1\right]. \tag{6.13}$$

where the sum is only defined when the conditions $\mathrm{Re}(d), \mathrm{Re}(e), \mathrm{Re}(f) > -1$ and $m + \mathrm{Re}(a) > -1,\ n + \mathrm{Re}(c) > -1,\ \mathrm{Re}(b) + m + n > -1$ hold. It will prove that the function (6.13) has already a name in the literature although very less is known about it. Before concerning about it, it will prove a good idea to first study some simplified cases of it, this giving us the opportunity to learn more about this special function.

For the beginning let us set $h = g = 0$. In this case, the sums over the ${}_4F_3$ hypergeometric function vanish, since the identity $\frac{(-g,m)}{(1,m)} \to \delta_m$ for $g \to 0$ and similarly for the other sum, holds. This fact can also be seen in a different way. When we look at the integral (6.9), we can already there set the exponents g and h equal to zero such that the polynomials reduce to one, which confirms the result established with the sums. When the two sums cancel, we end up with the result

$$F\begin{bmatrix} a,b,d,e,0 \\ c,f,0,j \end{bmatrix} = \frac{\Gamma(1+a)\ \Gamma(1+b)\ \Gamma(1+c)\ \Gamma(1+d)\ \Gamma(1+e)\ \Gamma(1+f)}{\Gamma(2+b+e)\ \Gamma(2+a+d)\ \Gamma(2+c+f)} \times {}_4F_3\left[\begin{matrix} 1+b,\ 1+a,\ 1+c,\ -j \\ 2+b+e,\ 2+a+d,\ 2+c+f \end{matrix}\ ;\ 1\right]\ . \tag{6.14}$$

Those mathematical simplifications, which may be seen as games, show a very interesting fact about the physics, fact which is also expected: when, by some reason, the scattering matrix gets simplified, for example by fixing some particles through a special D–brane configuration, and the number of kinematic invariants gets reduced, automatically the integrals become easier, and eventually, in special cases they can be expressed by ”normal” hypergeometric functions, in contrast to the most general case.

Some other simplifying scenario, though more complex than the last one, might be $h = 0$, which translates in the language of functional representation as

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,0,j \end{bmatrix} = \Gamma(1+d)\,\Gamma(1+e)\,\Gamma(1+f)\,\sum_{m=0}^{\infty}\frac{(-g,m)}{(1,m)}$$
$$\times\frac{\Gamma(1+m+b)\,\Gamma(1+m+a)\,\Gamma(1+c)}{\Gamma(2+m+b+e)\,\Gamma(2+m+a+d)\,\Gamma(2+c+f)}$$
$$\times\,{}_4F_3\begin{bmatrix} 1+m+b,\ 1+m+a,\ 1+c,\ -j \\ 2+m+b+e,\ 2+m+a+d,\ 2+c+f \end{bmatrix};1\Bigg] . \quad (6.15)$$

We notice here that one sum is gone, as expected, but we still have one sum running over the hypergeometric function, matter which definitely complicates the problem when compared with just one hypergeometric function. The function represented in (6.15) can also be seen as a simplified version of the Kampé de Fériet function which is a function in N variables with following definition

$$F^{A:B}_{C:D}\left[\begin{matrix} a_1,\ldots,a_A : b_{1,1},\ldots,b_{1,B};\ b_{2,1},\ldots,b_{2,B};\ \ldots\ ;\ b_{N,1},\ldots,b_{N,B} \\ c_1,\ldots,c_C : d_{1,1},\ldots,d_{1,D};\ d_{2,1},\ldots,d_{2,D};\ \ldots\ ;\ d_{N,1},\ldots,d_{N,D} \end{matrix};\ x_1,\ldots,x_N\right]$$
$$= \sum_{m_1,\ldots,m_N=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j,m_1+\ldots m_N)\ \prod_{j=1}^{B}(b_{1,j},m_1)\cdot\ldots(b_{N,j},m_N)}{\prod_{j=1}^{C}(c_j,m_1+\ldots m_N)\ \prod_{j=1}^{D}(d_{1,j},m_1)\cdot\ldots(d_{N,j},m_N)}\ \frac{x_1^{m_1}\cdot\ldots\cdot x_N^{m_N}}{m_1!\cdot\ldots\cdot m_N!} . \quad (6.16)$$

The Kampé de Fériet function is a generalization of the four Lauricella functions which again are closely connected with Appell's hypergeometric functions. More on that topic can be found in [87, 88]. Thus our result may be rewritten in the language of a Kampé de Fériet function as

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,0,j \end{bmatrix} = \frac{\Gamma(1+a)\,\Gamma(1+b)\,\Gamma(1+c)\,\Gamma(1+d)\,\Gamma(1+e)\,\Gamma(1+f)}{\Gamma(2+a+d)\,\Gamma(2+b+e)\,\Gamma(2+c+f)}$$
$$\times\,F^{2:2}_{2:1}\left[\begin{matrix} 1+a,1+b\ :\ 1+c,\ -j\ ;\ -g,\ 1 \\ 2+a+d,2+b+e\ :\ 2+c+f\ ;\ 1 \end{matrix};\ 1,1\right] . \quad (6.17)$$

The last case, which we should mention is $g=0$; this is similar to the last one and may be treated in full analogy to that, as expected. Our goal though, is the full integral (6.9) or, equivalently the full sum (6.13). The latter reads, when also the series representation for the ${}_4F_3$ hypergeometric function is inserted:

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} = \frac{\Gamma(1+d)\,\Gamma(1+e)\,\Gamma(1+f)}{\Gamma(-g)\,\Gamma(-h)\,\Gamma(-j)}\sum_{m_i=0}^{\infty}\frac{\Gamma(-g+m_1)\,\Gamma(-h+m_2)\,\Gamma(-j+m_3)}{m_1!\ m_2!\ m_3!}$$
$$\times\frac{\Gamma(1+m_1+m_2+m_3+b)}{\Gamma(2+m_1+m_2+m_3+b+e)}\ \frac{\Gamma(1+m_1+m_3+a)}{\Gamma(2+m_1+m_3+a+d)}\ \frac{\Gamma(1+m_2+m_3+c)}{\Gamma(2+m_2+m_3+c+f)} . \quad (6.18)$$

Another convenient expression for the upper sum may be one also with three sums, two of which are though terminating:

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} = \frac{\Gamma(1+d)\,\Gamma(1+e)\,\Gamma(1+f)}{\Gamma(-g)\,\Gamma(-h)\,\Gamma(-j)}$$
$$\times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\sum_{n_3=n_1-n_2}^{n_1} \frac{\Gamma(-g+n_1-n_3)}{\Gamma(1+n_1-n_3)}\frac{\Gamma(-h+n_1-n_2)}{\Gamma(1+n_1-n_2)}\frac{\Gamma(-j-n_1+n_2+n_3)}{\Gamma(1-n_1+n_2+n_3)}$$
$$\times \frac{\Gamma(1+n_1+b)}{\Gamma(2+n_1+b+e)}\,\frac{\Gamma(1+n_2+a)}{\Gamma(2+n_2+a+d)}\,\frac{\Gamma(1+n_3+c)}{\Gamma(2+n_3+c+f)} \tag{6.19}$$

In order to express our triple sum (6.18) in terms of an existing function we may use the general triple hypergeometric function with three variables $F^3[x,y,z]$ introduced for the first time by Srivastava

$$F^{(3)}[x,y,z] \equiv F^{(3)}\left[\begin{matrix} (a) :: (b);(b');(b'') : (c);(c');(c'') \\ (e) :: (g);(g');(g'') : (h);(h');(h'') \end{matrix}\;;\; x,y,z\right]$$
$$= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p)\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!}\;, \tag{6.20}$$

where the coefficients $\Lambda(m,n,p)$ are defined as

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j,m+n+p)\;\prod_{j=1}^{B}(b_j,m+n)\;\prod_{j=1}^{B'}(b'_j,n+p)\;\prod_{j=1}^{B''}(b''_j,m+p)}{\prod_{j=1}^{E}(e_j,m+n+p)\;\prod_{j=1}^{G}(g_j,m+n)\;\prod_{j=1}^{G'}(g'_j,n+p)\;\prod_{j=1}^{G''}(g''_j,m+p)}$$
$$\times \frac{\prod_{j=1}^{C}(c_j,m)\;\prod_{j=1}^{C'}(c'_j,n)\;\prod_{j=1}^{C''}(c''_j,p)}{\prod_{j=1}^{H}(h_j,m)\;\prod_{j=1}^{H'}(h'_j,n)\;\prod_{j=1}^{H''}(h''_j,p)}\;. \tag{6.21}$$

Thus, this monstrous looking function is the simplest one, which may be used in order to cast our function into following closed form:

$$F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j \end{bmatrix} = \frac{\Gamma(1+a)\,\Gamma(1+b)\,\Gamma(1+c)\,\Gamma(1+d)\,\Gamma(1+e)\,\Gamma(1+f)}{\Gamma(2+a+d)\,\Gamma(2+b+e)\,\Gamma(2+c+f)}$$
$$\times F^{(3)}\left[\begin{matrix} 1+b :: 1;1+c;1+a : -g,1;-h,1;-j,1 \\ 2+b+e :: 1;2+c+f;2+a+d : 1;1;1 \end{matrix}\;;\;1,1,1\right]\;. \tag{6.22}$$

We have actually at the moment enough functional knowledge to start to look a little bit closer to the parameter expansion of our integral (6.9) or equivalently our series (6.13). It is important now not to mix up two different things: our function (6.13) is given as a triple infinite sum in (6.18), but what we need is a finite expansion in its momenta $a, b, c, \ldots$. This can be better visualized in the integral picture. Again our function is given as a triple infinite integral (6.9) (this being expected, since integrals are equivalent to sums, thus every integral being convertible into a sum and viceversa) of which we would like to have a finite expansion in the parameters $a, b, c, \ldots$, which show up in the exponentials of the integrated polynomials.

This task proves to be a very complicated one, and by far not to be found in the mathematical literature. Special functions as the lowest hypergeometric functions are well treated in reference books but the knowledge diminishes very fast with increasing complexity. In order to attack this series expansion a well–founded knowledge about sums is inevitable, fact which will be treated in the next section.

6.2 Some convergent infinite sums

The most natural way to get started with the problem is looking back at the series representation of a general hypergeometric function as given in (6.7), since the general six–point integral is expressed as an infinite double sum over a ${}_4F_3$ hypergeometric function. Thus we might try to expand each of the single Pochhammer symbols leading to

$$
{}_pF_q\left[\begin{matrix} a_1+\epsilon\ \alpha_1, \ldots, a_p+\epsilon\ \alpha_p \\ b_1+\epsilon\ \beta_1, \ldots, b_q+\epsilon\ \beta_q \end{matrix}\right] = \sum_{s=0}^{\infty} \frac{1}{s!} \frac{\prod\limits_{i=1}^{p} (a_i+\epsilon\ \alpha_i, s)}{\prod\limits_{j=1}^{q} (b_j+\epsilon\ \beta_j, s)}
$$

$$
\frac{(m+\alpha\ \epsilon, s)}{(m,s)} = e^{-\sum\limits_{k=1}^{\infty} \frac{(-\alpha\ \epsilon)^k}{k} \left(H_{m+s-1,k}-H_{m-1,k}\right)} \,. \qquad (6.23)
$$

In upper formula nothing mysterious has happened: we have just rewritten the function as a product of Pochhammer symbols which finally are series expanded in small parameters by standard procedures. We see, that for each term $\sum_s()$ in the overall sum a set of infinite sums appears, known as Euler sums which involve the so called harmonic numbers and their generalizations; those infinite sums are precisely the expansions of the single Pochhammer symbols encounterd. It is worth to stress here a very important fact: supposed we had the values of the various sums in (6.23), which we will indeed calculate thoroughly in this chapter, we still couldn't expand general hypergeometric functions. The reason for that is these functions have a lot of poles at different values of their parameters. So when using the technique presented above on a singular function, we will obtain genuinely divergent series which cannot be cured! The only possibility to evade this problem: one should subtract the poles, i.e. subtract the infinities and also encode them in terms of other objects[3].

[3]This technique will be also presented very thoroughly in this work, see section 6.5

6.2.1 Harmonic numbers and the polygamma function

Before going on with Euler sums, it is worth to define properly the harmonic numbers, which are given by following expression:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \quad , \quad H_{n,a} = \sum_{k=1}^{n} \frac{1}{k^a} \ , \tag{6.24}$$

where H_n is the harmonic number and $H_{n,a}$ is the generalized harmonic number of power a, obeying the functional identity

$$H_n = H_{n-1} + \frac{1}{n} \qquad H_{n,a} = H_{n-1,a} + \frac{1}{n^a}. \tag{6.25}$$

Thus, harmonic numbers are finite sums which have a definite value. Exactly as the Γ–function interpolates the factorial function $n!$ the polygamma functions interpolates the harmonic numbers and its generalization is thus defined as

$$\psi^n(z) = \frac{d^{n+1}}{dz^{n+1}} \log(\Gamma(z)) = \frac{d^n}{dz^n}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{d^n}{dz^n}\psi^0(z), \tag{6.26}$$

or alternatively, for $(n > 0)$ as

$$\psi^n(z) = (-1)^{n+1}\ n!\ \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}. \tag{6.27}$$

When evaluated on integers, we get immediate relations between the polygamma and harmonic function

$$\begin{aligned} \psi(n) &= -\gamma_E + H_{n-1} \ , \\ \psi^{(1)}(n) &= \zeta(2) - H_{n-1,2} \ , \end{aligned} \tag{6.28}$$

or for the general case

$$\psi^{(b-1)}(n) = (-1)^b\ (b-1)!\ [\ \zeta(b) - H_{n-1,b}\ ] \quad , \quad b > 1 \tag{6.29}$$

with γ_E being the Euler-Mascheroni constant defined by the famous equation $\gamma_E = \lim_{n\to\infty}(H_n - \log(n))$ and $\zeta(2)$ being the Riemann zeta function obeying the definition $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$, evaluated at integer 2.

6.2.2 Euler sums

Now we are prepared to attack the Euler sums, which we remember to appear in the expansion of the general hypergeometric functions (6.23). There is a whole rich family of Euler sums, which are more or less well understood; however we will be faced with two classes of them, which are relevant for us:

$$\begin{aligned} s_h(m,n) &= \sum_{k=1}^{\infty} \frac{H_k^m}{(k+1)^n} \ , \\ \sigma_h(m,n) &= \sum_{k=1}^{\infty} \frac{H_{k,m}}{(k+1)^n} \ . \end{aligned} \tag{6.30}$$

for $m \geq 1, \ n \geq 2$. With a little work invested we can establish following relations, which are obtained by shifting the corresponding denominator and then reevaluating the arizing sums:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{H_{k,m}}{k^n} &= \sigma_h(m,n) + \zeta(m+n) \ , \\ \sum_{k=1}^{\infty} \frac{H_k^m}{k^n} &= s_h(m,n) - \sum_{j=0}^{m-1} \binom{m}{j} \sum_{k=1}^{\infty} (-1)^{m-j} \, \frac{H_k^j}{k^{m+n-j}} \ . \end{aligned} \tag{6.31}$$

From the last equation, by plugging the appropriate expressions for the sums on the right hand side, we obtain the identity:

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^n} = s_h(2,n) + 2 \ s_h(1,n+1) + \zeta(2+n) \ . \tag{6.32}$$

Since we now want to evaluate those sums, in order to be able to write down the expansion for (6.23), we should worry about the method of evaluating such Euler sums. Unfortunately this is a matter of high proficiency and no general, mechanical method can be presented. However, recent work on that subject, see e.g. [89, 90] has lead to fruitful results, of which we will list some here:

$$\begin{aligned} s_h(2,2) &= \frac{3}{2} \ \zeta(4) + \frac{1}{2} \ \zeta(2)^2 = \frac{11}{4} \ \zeta(4) \ , \\ s_h(3,2) &= \frac{15}{2} \ \zeta(5) + \zeta(2) \ \zeta(3) \ , \\ s_h(2,4) &= \frac{2}{3} \ \zeta(6) - \frac{1}{3} \ \zeta(2) \ \zeta(4) + \frac{1}{3} \ \zeta(2)^3 - \zeta(3)^2 \ , \end{aligned} \tag{6.33}$$

as well as

$$\begin{aligned}
\sigma_h(1,2) &= \zeta(3) \ , \\
\sigma_h(1,3) &= \frac{3}{2}\ \zeta(4) - \frac{1}{2}\ \zeta(2)^2 \ , \\
\sigma_h(2,2) &= \frac{1}{2}\ \zeta(2)^2 - \frac{1}{2}\ \zeta(4) \ , \\
\sigma_h(1,4) &= 2\ \zeta(5) - \zeta(2)\ \zeta(3) \ , \\
\sigma_h(2,3) &= -\frac{11}{2}\ \zeta(5) + 3\ \zeta(2)\ \zeta(3) \ , \\
\sigma_h(2,4) &= -6\ \zeta(6) + \frac{8}{3}\ \zeta(2)\ \zeta(4) + \zeta(3)^2 \ , \\
\sigma_h(1,5) &= \frac{5}{2}\ \zeta(6) - \zeta(2)\ \zeta(4) - \frac{1}{2}\ \zeta(3)^2 \ , \\
\sigma_h(4,2) &= 5\ \zeta(6) - \frac{5}{3}\ \zeta(2)\ \zeta(4) - \zeta(3)^2 \ .
\end{aligned} \tag{6.34}$$

Another interesting and useful formula is

$$s_h(1,n) = \sigma_h(1,n) = \frac{1}{2}\ n\ \zeta(n+1) - \frac{1}{2}\ \sum_{k=1}^{n-2} \zeta(n-k)\ \zeta(k+1). \tag{6.35}$$

A so called *reflection formula* involving the Euler sum is also known

$$\sigma_h(s,t) + \sigma_h(t,s) = \zeta(s)\ \zeta(t) - \zeta(s+t) \quad , \quad s,t \geq 2 \ . \tag{6.36}$$

The latter formula derives its name from the fact that on the left hand side we have the same Euler sum, however with its two parameters reversed. An immediate consequence of this fact can be established when taking the same value for the arguments; we then obtain

$$\sigma_h(a,a) = \frac{1}{2}\ \zeta(a)^2 - \frac{1}{2}\ \zeta(2a) \quad , \quad a \geq 2 \ . \tag{6.37}$$

The definition for $\sigma_h(m,n)$ is given in (6.30)and we can plug it in upper formula, which leads us to

$$\sum_{n=1}^{\infty} \frac{H_{n,a}}{n^a} = \frac{1}{2}\ \zeta(a)^2 + \frac{1}{2}\ \zeta(2a) \quad , \quad a \geq 2 \ . \tag{6.38}$$

We have sofar collected some very useful formulas, which will constitute some basic tools in order to reach our goal. If we look back more closely at our initial expansion

(6.23), some sums will appear which we are already able to calculate with the tools collected until now. They are

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\,\zeta(3) \quad , \quad \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{11}{2}\,\zeta(4) - \frac{1}{2}\,\zeta(2)^2 = \frac{17}{4}\,\zeta(4) \quad , \quad \sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\,\zeta(4)\ ,$$
$$\sum_{n=1}^{\infty} \frac{H_n}{n^4} = 3\,\zeta(5) - \zeta(2)\,\zeta(3) \quad , \quad \sum_{n=1}^{\infty} \frac{H_n}{n^5} = \zeta(2)\,\zeta(4) - \frac{1}{2}\,\zeta(3)^2\ . \tag{6.39}$$

We can immediately derive more complex results from upper formulas, which will also find their contribution later on:

$$\sum_{n=1}^{\infty} \frac{H_{n,2}}{n^4} = \sigma_h(2,4) + \zeta(6) = -5\,\zeta(6) + \frac{8}{3}\,\zeta(2)\,\zeta(4) + \zeta(3)^2\ ,$$
$$\sum_{n=1}^{\infty} \frac{H_{n,2}}{n^3} = \sigma_h(2,3) + \zeta(5) = -\frac{9}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3)\ ,$$
$$\sum_{n=1}^{\infty} \frac{H_{n,4}}{n^2} = \sigma_h(4,2) + \zeta(6) = 6\,\zeta(6) - \frac{5}{3}\,\zeta(2)\,\zeta(4) - \zeta(3)^2\ . \tag{6.40}$$

A particulary and also for us interesting sum which we shall need later is

$$\sum_{k=1}^{\infty} \frac{1}{k^a}\,\psi^{(b)}(k) = (-1)^{b+1}\,b!\ [\ \zeta(a)\,\zeta(1+b) - \sigma_h(1+b,a)\ ]\,. \tag{6.41}$$

Here we recognize the former defined polygamma function, (6.26), which additionally obeys the relation

$$\psi^{(b)}(x) = (-1)^{b+1}\,b!\,\zeta(b+1,x)\ , \tag{6.42}$$

where we have introduced the generalized Riemann zeta function. This latter function was introduced by Hurwitz, and given by:

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}. \tag{6.43}$$

If we thus write down explicitly upper definition we encounter no troubles in establishing formula (6.42). If we now specialize to the case $a = 1 + b$ in equation (6.41) with the sums collected before, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^{1+b}}\ \psi^{(b)}(k) = \frac{1}{2}\ (-1)^{b+1}\ b!\ \left[\ \zeta(2b+2) + \zeta(b+1)^2\ \right]\ . \tag{6.44}$$

Another important series is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^{\alpha}} = \alpha - \sum_{i=2}^{\alpha} \zeta(i), \tag{6.45}$$

whose proof is not very hard; it might be done by induction. The first step is chosing $\alpha = 1$, which leads to

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \psi(1) + \gamma_E + 1 = 1, \quad \psi(1) = -\gamma_E\ . \tag{6.46}$$

This relation again, can be proven by partial fractions: We easily see then, that each term of the sum will be splited in two, according to the partial fraction technique. In each case infinitely many pairs of terms cancel except the first one, which is 1. The induction can be successfully be continued for every value of α. Another some more complex identity is

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)^{\alpha}} H_{n-1} = \frac{1}{2}\alpha(\alpha+1) - \frac{1}{2}\sum_{i=2}^{\alpha} i\ \zeta(i+1) - \sum_{i=2}^{\alpha}(\alpha - i + 1)\ \zeta(i) \\ + \frac{1}{2}\sum_{k=0}^{\alpha-3}\sum_{i\geq 0}^{k} \zeta(2+i)\ \zeta(2+k-i)\ .\end{aligned} \tag{6.47}$$

One way to prove it, is again induction, with the first step being $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} H_{n-1} = 1$. This, on his part, can be proven by partial fraction; we may however face the case when we have to shift the argument of the harmonic number, in order to be able to cancel the some terms. Special values of α and also partial fractionats, will enrich our colletion of relations with

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(n+1)^4} H_{n-1} &= -4 + \zeta(2) + \zeta(3) + \zeta(4) - \zeta(2)\ \zeta(3) + 2\ \zeta(5)\ , \\ \sum_{n=1}^{\infty} \frac{1}{n(n+1)^4} H_{n-1} &= 10 - 3\ \zeta(2) - 3\ \zeta(3) - \frac{5}{4}\ \zeta(4) + \zeta(2)\ \zeta(3) - 2\ \zeta(5)\ , \\ \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^4} H_{n-1} &= -20 + 6\ \zeta(2) + 7\ \zeta(3) + \frac{3}{2}\ \zeta(4) - \zeta(2)\ \zeta(3) + 2\ \zeta(5)\end{aligned} \tag{6.48}$$

Furthermore, also occurring in the expansion of the hypergeometric function and its generalization, are the following two sums

$$
\begin{aligned}
(i)\quad & \sum_{n=1}^{\infty} \frac{\psi^{(1)}(n)}{n\,(n+1)^3} = 10 - 3\,\zeta(2) - \frac{1}{2}\,\zeta(2)^2 - \zeta(3) - \frac{1}{2}\,\zeta(4) + 2\,\zeta(2)\,\zeta(3) - \frac{11}{2}\,\zeta(5)\;, \\
(ii)\quad & \sum_{n=1}^{\infty} \frac{H_{n-1}^2}{n\,(n+1)^3} = 10 - \frac{1}{2}\,\zeta(2)^2 - 5\,\zeta(3) - 2\,\zeta(4) - \zeta(2)\,\zeta(3) + \frac{3}{2}\,\zeta(5),
\end{aligned}
\tag{6.49}
$$

which can again be both proven by using partial fraction techniques, and especially using the identity

$$
\frac{1}{n(n+1)^3} = \frac{1}{n(n+1)} - \frac{1}{(n+1)^2} - \frac{1}{(n+1)^3}. \tag{6.50}
$$

More on their derivation will be said in a little wile. We can also prove with upper decomposition, assumed we accept them as correct, the following more involved series:

$$
\sum_{n=1}^{\infty} \frac{H_{n,2}}{n(n+1)^3} = \zeta(3) - \frac{1}{2}\,\zeta(2)^2 + \frac{1}{2}\,\zeta(4) + \frac{11}{2}\,\zeta(5) - 3\,\zeta(2)\,\zeta(3)\;, \tag{6.51}
$$

which splits up when using (6.50), into

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{H_{n,2}}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) H_{n,2} = \sum_{n=1}^{\infty} \left(\frac{H_{n,2}}{n} - \frac{H_{n-1,2}}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)\quad, \\
& \sum_{n=1}^{\infty} \frac{H_{n,2}}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{H_{n-1,2}}{n^2} = \sum_{n=1}^{\infty} \frac{H_{n,2}}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{2}\,\zeta(2)^2 - \frac{1}{2}\,\zeta(4)\;, \\
& \sum_{n=1}^{\infty} \frac{H_{n,2}}{(n+1)^3} = \sigma_h(2,3) = -\frac{11}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3)\;.
\end{aligned}
\tag{6.52}
$$

Now simply adding them, will resemble the result established in (6.51).

As promised, for the sake of completeness, we should prove the series in (6.49). The first of them, (i) may be transformed to

$$\sum_{n=1}^{\infty} \frac{\psi^{(1)}(n)}{n(n+1)^3} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^3} \sum_{k=0}^{\infty} \frac{1}{(n+k)^2} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^3} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n-1} \frac{1}{k^2} \right)$$
$$= \zeta(2) \sum_{n=1}^{\infty} \frac{1}{n(n+1)^3} - \sum_{n=1}^{\infty} \frac{H_{n-1,2}}{n(n+1)^3}$$
$$= \zeta(2)\ [3 - \zeta(2) - \zeta(3)] - \sum_{n=1}^{\infty} \frac{H_{n,2}}{n(n+1)^3} + \sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3}\ . \tag{6.53}$$

In the second line, the first series can be red off from equation (6.45), and the second has been rewritten as the last two terms in line three, using the functional identity (6.25). The last two terms are easy to calculate, the first one from (6.51) and the second with the relation

$$\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3} = 10 - \pi^2\ , \tag{6.54}$$

which on his part can easily be computed. We can now add all the pieces collected until now and arrive at the result established in (6.49) under (i). The second sum (ii) is even more tricky, we shall thus need following intermediary results:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)^3} = 3\ \zeta(3) - \frac{3}{2}\ \zeta(4) - \frac{1}{2}\ \zeta(2)^2 + \frac{3}{2}\ \zeta(5) - \zeta(2)\ \zeta(3). \tag{6.55}$$

The following sums are calculated more or less straightforward; partially we can also see their derivations below:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)\ H_n^2 = \sum_{n=1}^{\infty} \left(\frac{H_n^2}{n} - \frac{H_{n-1}^2}{n} \right)$$
$$= 2 \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3} = 3\ \zeta(3)\ \ ,$$
$$\sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = s_h(2,2) = \frac{3}{2}\ \zeta(4) + \frac{1}{2}\ \zeta(2)^2\ ,$$
$$\sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^3} = s_h(2,3). \tag{6.56}$$

The last of the sum involving $s_h(2,4)$ is not among the results collected until now. For that, we shall derive it using following identity

$$\sum_{n=1}^{\infty} \frac{1}{n^a} \; H_{n-1,b} \; H_{n-1,c} = \zeta(a,b,c) + \zeta(a,c,b) + \sigma_h(b+c,a). \tag{6.57}$$

The symbol $\zeta(a,b,c)$ which is new here, denotes the *triple Euler sum*, defined as

$$\zeta(a,b,c) = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} \frac{1}{n^a m^b k^c} \; . \tag{6.58}$$

For more values of the triple Euler function one may consult the reference [91], where also the specially useful formula $\zeta(3,1,1) = 2\;\zeta(5) - \zeta(2)\;\zeta(3)$ is derived. Therefore, we can write

$$s_h(2,3) = \sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^3} = \sum_{n=1}^{\infty} \frac{H_{n-1}^2}{n^3} = 2\;\zeta(3,1,1) + \sigma_h(2,3) = -\frac{3}{2}\;\zeta(5) + \zeta(2)\;\zeta(3) \; . \tag{6.59}$$

We have now calculated everything in order to establish the result in (6.55). We recover it from (6.56) and also using the decomposition (6.50). And this is the last step in order to prove the result (*ii*) in (6.49). We make first following conversion

$$\sum_{n=1}^{\infty} \frac{H_{n-1}^2}{n(n+1)^3} = \sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)^3} + \sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3} - 2\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)^3} \; . \tag{6.60}$$

We have already just calculated the first two sums, wile the last one is relatively easy to calculate and yields

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)^3} = -3\;\zeta(2) + 4\;\zeta(3) + \frac{1}{4}\;\zeta(4) \; . \tag{6.61}$$

As a last remark, we should use the relation (6.57) to derive following equality

$$\sum_{n=1}^{\infty} \frac{H_{n-1,b}^2}{n^a} = 2\;\zeta(a,b,b) + \sigma_h(2b,a) \; , \tag{6.62}$$

by simply setting $b = c$. With the additional value of the triple Euler function

$$\zeta(a,a,a) = \frac{1}{6}\;\zeta(a)^3 - \frac{1}{2}\;\zeta(a)\;\zeta(2a) + \frac{1}{3}\;\zeta(3a) \; , \tag{6.63}$$

we recover this last important identity

$$\sum_{n=1}^{\infty} \frac{H^2_{n-1,2}}{n^2} = 2\,\zeta(2,2,2) + \sigma_h(4,2) = \frac{1}{3}\,\zeta(2)^3 - \zeta(3)^2 - \frac{8}{3}\,\zeta(2)\,\zeta(4) + \frac{17}{3}\,\zeta(6)\ . \tag{6.64}$$

We have derived in this section a lot of very useful relations which will be used in the next section to series expand hypergeometric functions, their generalizations as well as triple hypergeometric functions. The main results were calculated here, however some particular values of special functions have been used since their derivation might have taken much more space. Nevertheless, they are partially derived in [91]. Also a very good lecture on that topic are the papers [89, 90], without claiming to give a complete list of refferences here.

6.3 Series expansion of higher hypergeometric functions

We are now in the position to write down series expansions of various generalized hypergeometric functions. In this section we will adopt following convention: all the parameters of the considered functions, denoted by $a, b, c, \ldots$, should be regarded as small, i.e. close to zero, such that we can expand the respective functions in those small parameters. The first function we want to look at, is the general ${}_3F_2$ occurring in the five–point integral (6.3):

$$\begin{aligned}
C(a,b,c,d,e) &= \frac{\Gamma(1+a)\,\Gamma(1+b)\,\Gamma(1+c)\,\Gamma(1+d)}{\Gamma(2+a+c)\,\Gamma(2+b+d)}\ {}_3F_2\left[\begin{matrix}1+a,\ 1+b,\ -e\\ 2+a+c,\ 2+b+d\end{matrix}\right]\\
&= 1-a-b-c-d-2\,e+e\,\zeta(2)+a^2+a\,b+b^2+b\,c+c^2+a\,d+c\,d+d^2\\
&+3\,a\,e+3\,b\,e+3\,c\,e+3\,d\,e+3\,e^2+2\,ac+2\,bd\\
&-(a\,c+b\,d+a\,e+b\,e+e^2)\,\zeta(2)-(a\,e+b\,e+2\,c\,e+2\,d\,e+e^2)\,\zeta(3)\\
&-a^3-a^2\,b-a\,b^2-b^3-3\,a^2\,c-2\,a\,b\,c-b^2\,c-3\,a\,c^2-b\,c^2-c^3-a^2\,d-2\,a\,b\,d-3\,b^2\,d\\
&-2\,a\,c\,d-2\,b\,c\,d-c^2\,d-a\,d^2-3\,b\,d^2-c\,d^2-d^3-4\,a^2\,e-4\,a\,b\,e-4\,b^2\,e-8\,a\,c\,e\\
&-4\,b\,c\,e-4\,c^2\,e-4\,a\,d\,e-8\,b\,d\,e-4\,c\,d\,e-4\,d^2\,e-6\,a\,e^2-6\,b\,e^2-6\,c\,e^2-6\,d\,e^2-4\,e^3\\
&+(a^2\,c+a\,b\,c+a\,c^2+a\,b\,d+b^2\,d+a\,c\,d+b\,c\,d+b\,d^2+a^2\,e+a\,b\,e+b^2\,e+3\,a\,c\,e+3\,b\,d\,e\\
&-c\,d\,e+2\,a\,e^2+2\,b\,e^2+e^3)\,\zeta(2)+(a^2\,c+a\,c^2+b^2\,d+b\,d^2+a^2\,e+a\,b\,e+b^2\,e+2\,a\,c\,e\\
&+2\,b\,c\,e+2\,a\,d\,e+2\,b\,d\,e+2\,a\,e^2+2\,b\,e^2+2\,c\,e^2+2\,d\,e^2+e^3)\,\zeta(3)\\
&+\left(a^2\,e+a\,b\,e+b^2\,e+3\,a\,c\,e+\frac{5\,b\,c\,e}{4}+3\,c^2\,e+\frac{5\,a\,d\,e}{4}+3\,b\,d\,e+\frac{17\,c\,d\,e}{4}+3\,d^2\,e+\frac{a\,e^2}{4}\right.\\
&\left.+\frac{b\,e^2}{4}+3\,c\,e^2+3\,d\,e^2+e^3\right)\,\zeta(4)+\ldots.
\end{aligned} \tag{6.65}$$

We may wonder, which benefit we have from series expanding the ${}_3F_2$ function, for it just represents the five–point scattering in string theory. First of all we put to work for the first time the sums we have calculated so far in the last section. However, this expansion is more than just a pedagogical exposition, since we will use (6.65) and (6.66) in section 6.5 in order to series expand the much more complex triple hypergeometric function, which will be related to the upper ${}_3F_2$ functions and the following one:

$$\begin{aligned}
C(a,b,c,d,e-1) &= \frac{\Gamma(1+a)\;\Gamma(1+b)\;\Gamma(1+c)\;\Gamma(1+d)}{\Gamma(2+a+c)\;\Gamma(2+b+d)}\;\;{}_3F_2\left[\begin{matrix}1+a,\;1+b,\;1-e\\ 2+a+c,\;2+b+d\end{matrix}\right]\\
&= \zeta(2) - (a+b+2\,c+2\,d+e)\,\zeta(3)\\
&+ \left(a^2+a\,b+b^2+\frac{a\,c}{2}+\frac{5\,b\,c}{4}+3\,c^2+\frac{5\,a\,d}{4}+\frac{b\,d}{2}+3\,d^2+\frac{a\,e}{4}+\frac{b\,e}{4}+3\,c\,e\right.\\
&\left.\frac{17\,c\,d}{4}+3\,d\,e+e^2\right)\;\zeta(4)+\ldots.
\end{aligned} \tag{6.66}$$

When we look at the last tow functions we notice immediately that they don't have any poles at zero[4]. This can be seen with difficulty when analyzing the right hand side in the first line of upper equations. The reason for that is that we are faced with complicated hypergeometric functions, whose pole behaviour is not easy to see at once. However, on the left hand side, the function denoted by $C(a,b,c,d,e)$, has its definition as exactly being the integral of polynomials with the powers (a,b,c,d,e). Thus, sice the integral defined in (6.3) is taken from zero to one, and the exponents are infinitesimal, we immediately see its convergence.

However, we are now able to relate these "harmless" functions to a hypergeometric ${}_3F_2$ function which has poles, namely to the ${}_3F_2\left[\begin{smallmatrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{smallmatrix}\right]$. Exactly as in the case of the four–point function and its exemplary system of equations, we will have all the pole structure factorized in front of the two functions (6.65) and (6.66). So, when switching to the notation of the integral presented in (6.3) or equivalently in (6.6) we have:

[4]For sure hypergeometric functions have infinitely many poles and not just those at zero parameters. But remember, we are just looking at the special region where the parameters are strictly positive and very small, i.e. that what is known in mathematics as being ϵ

$$
{}_3F_2\begin{bmatrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{bmatrix} = \frac{1}{\alpha_1\ \alpha_2\ (\alpha_1-\beta_1)\ (\alpha_2-\beta_2)\ (\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)}
$$
$$
= \Big\{\ (1-\alpha_3+\beta_1)\ (1-\alpha_3+\beta_2)\ [\ \beta_1\ \beta_2\ (\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)-\alpha_1\ \alpha_2\ \alpha_3\ ]
$$
$$
\times\ C(\alpha_1,\ \alpha_2,\ \beta_1-\alpha_1,\ \beta_2-\alpha_2,\ -\alpha_3)
$$
$$
-\alpha_3\ \big[\ \alpha_1^2\ (\alpha_2-\beta_1)\ (\alpha_2-\beta_2)-\beta_1\ \beta_2\ (\alpha_2+\alpha_3-\beta_1-\beta_2)\ (1-\alpha_2-\alpha_3+\beta_1+\beta_2)+
$$
$$
+\alpha_1\ \alpha_2\ \alpha_3-\alpha_1\ \alpha_2\ \alpha_3^2-\alpha_1\ \alpha_2^2\ \beta_1+\alpha_1\ \alpha_2\ \beta_1^2-\alpha_1\ \alpha_2^2\ \beta_2-\alpha_1\ \beta_1\ \beta_2
$$
$$
+3\ \alpha_1\ \alpha_2\ \beta_1\ \beta_2+2\ \alpha_1\ \alpha_3\ \beta_1\ \beta_2-2\ \alpha_1\ \beta_1^2\ \beta_2+\alpha_1\ \alpha_2\ \beta_2^2-2\ \alpha_1\ \beta_1\ \beta_2^2\ \big]
$$
$$
\times\ C(\alpha_1,\ \alpha_2,\ \beta_1-\alpha_1,\ \beta_2-\alpha_2,\ -\alpha_3-1)\ \Big\}\ \frac{\Gamma(\beta_1)\ \Gamma(\beta_2)}{\Gamma(\alpha_1)\ \Gamma(\alpha_2)\ \Gamma(\beta_1-\alpha_1)\ \Gamma(\beta_2-\alpha_2)}
$$
$$
= 1-\frac{\alpha_1\alpha_2\alpha_3}{\beta_1\ \beta_2\ (\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)}
$$
$$
\times\big\{1+\big[\ (\alpha_2-\beta_1)\ (\alpha_3-\beta_1)+\alpha_1\ (\alpha_2+\alpha_3-\beta_1-\beta_2)-(\alpha_2+\alpha_3-\beta_1)\ \beta_2+\beta_2^2\ \big]\ \zeta(2)
$$
$$
+\big[\ \alpha_1^2\,\alpha_2+\alpha_1\,\alpha_2^2+\alpha_1^2\,\alpha_3+4\,\alpha_1\,\alpha_2\,\alpha_3+\alpha_2^2\,\alpha_3+\alpha_1\,\alpha_3^2+\alpha_2\,\alpha_3^2-\alpha_1^2\,\beta_1-4\,\alpha_1\,\alpha_2\,\beta_1
$$
$$
-\alpha_2^2\,\beta_1-4\,\alpha_1\,\alpha_3\,\beta_1-4\,\alpha_2\,\alpha_3\,\beta_1-\alpha_3^2\,\beta_1+3\,\alpha_1\,\beta_1^2+3\,\alpha_2\,\beta_1^2+3\,\alpha_3\,\beta_1^2-2\,\beta_1^3-\alpha_1^2\,\beta_2
$$
$$
-4\,\alpha_1\,\alpha_2\,\beta_2-\alpha_2^2\,\beta_2-4\,\alpha_1\,\alpha_3\,\beta_2-4\,\alpha_2\,\alpha_3\,\beta_2-\alpha_3^2\,\beta_2+4\,\alpha_1\,\beta_1\,\beta_2+4\,\alpha_2\,\beta_1\,\beta_2
$$
$$
+4\,\alpha_3\,\beta_1\,\beta_2-3\,\beta_1^2\,\beta_2+3\,\alpha_1\,\beta_2^2+3\,\alpha_2\,\beta_2^2+3\,\alpha_3\,\beta_2^2-3\,\beta_1\,\beta_2^2-2\,\beta_2^3\ \big]\ \zeta(3)+\ldots. \tag{6.67}
$$

The relation (6.67) is emergent out of our system of equations (5.28). It enables us to expand a singular hypergeometric function in its parameters in a region where exactly those singularities occur. When we insert the expansion of the functions (6.65) and (6.66) we see that the singular function

$$
{}_3F_2\begin{bmatrix}\alpha_1,\ \alpha_2,\ \alpha_3\\ \beta_1,\ \beta_2\end{bmatrix} = \frac{\Gamma(\beta_1)\ \Gamma(\beta_2)}{\Gamma(\alpha_1)\ \Gamma(\alpha_2)\ \Gamma(\beta_1-\alpha_1)\ \Gamma(\beta_2-\alpha_2)}
$$
$$
\times\ C(\alpha_1-1,\alpha_2-1,\beta_1-\alpha_1-1,\beta_2-\alpha_2-1,-\alpha_3), \tag{6.68}
$$

has poles at β_1,β_2 and $\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2$. Furthermore, trying to simply expand (6.68) as shown in (6.23) would automatically lead to divergent sums. Those sums would exactly multiply the coefficients $\beta_1,\beta_2,\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2$, signalizing the singularities there. We also present the expansion of a nonsingular hypergeometric ${}_4F_3$ function in Appendix G.

6.4 Parameter expansion of the triple hypergeometric $F^{(3)}$ function

Until now we worried mainly about generalized hypergeometric functions. Those are, as we have seen, useful when analyzing four– and five–point scattering processes in string theory. However, it's time to present now the tools for the expansion of the six–point function, i.e. the triple hypergeometric function. All the derivations we made until now for different series are of great value since they will still be used, thus being the basis of our next discussion.

6.4.1 Multiple zeta sums and Euler–Zagier series

Looking back at equation (6.18), we might try to expand the Γ–functions there and finally to evaluate the three running infinite sums over the expansions, as we did it before. Though, we will encounter a new type of series, not known yet, which is a generalization of length k of the multiple zeta function:

$$\zeta(s_1,\ldots,s_k) = \sum_{n_1>\ldots>n_k>0} \prod_{j=1}^{k} \frac{1}{n_j^{s_j}} = \sum_{n_1,\ldots,n_k=1}^{\infty} \prod_{j=1}^{k} \left(\sum_{i=j}^{k} n_i\right)^{-s_j} , \tag{6.69}$$

with $s_1 \geq 2$, $s_2,\ldots,s_k \geq 1$. It turns out that these functions are closely related to another class of series, the polylogarithmic function, defined as, [92],

$$\mathcal{L}i_{s_1,\ldots,s_k}(x_1,\ldots,x_k) = \sum_{n_1>\ldots>n_k>0} \prod_{j=1}^{k} \frac{x_j^{n_j}}{n_j^{s_j}} , \tag{6.70}$$

for $x_j = 1$, *i.e.* $\zeta(s_1,\ldots,s_k) = \mathcal{L}i_{s_1,\ldots,s_k}(1,\ldots,1)$. As one can deduce from the definition, the polylogarithmic function reduces to the basic zeta function for $k = 1$, i.e. $\mathcal{L}i_a(1) = \zeta(a)$. Moreover, we had already to do with the special polylogarithmic function $\mathcal{L}i_{s_1,s_2,s_3}(x_1,x_2,x_3)$, also known as triple Euler sum, and defined in section 6.2. Thus, more numerical values can be inferred from the literature given throughout and at the end of that section. We shall need also an integral representation of the generalized multiple zeta function (6.69) as given in [93]:

$$\zeta(s_1,\ldots,s_k) = \int_1^{\infty} \frac{dx_1}{x_1} \ldots \int_1^{\infty} \frac{dx_k}{x_k} \prod_{j=1}^{k} \frac{1}{\Gamma(s_j)} \frac{(\ln x_j)^{s_j-1}}{\prod_{i=1}^{j} x_i - 1}, \tag{6.71}$$

which just reduces for $k = 3$, our case of interest[5], to

$$\zeta(s_1,s_2,s_3) = -\frac{(-1)^{s_1+s_2+s_3}}{\Gamma(s_1)\,\Gamma(s_2)\,\Gamma(s_3)} \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^2\, y\; \frac{(\ln x)^{s_1-1}\,(\ln y)^{s_2-1}\,(\ln z)^{s_3-1}}{(1-x)\,(1-xy)\,(1-xyz)} . \tag{6.72}$$

[5]We should remember that the functions we are interested in are triple integrals, whose integrands involve exactly polynomials as seen in (6.72)

The latter formula presented is very important since it is a major connection between the two representations of the triple hypergeometric function as shown in (6.9) and (6.18). Thus, when we expand the integral representation and integrate term by term we will exactly recover integrals of the type (6.71). But this is just the integral representation of the generalized multiple zeta function, whose series representation (6.69) will exactly emerge when expanding the triple hypergeometric function in the series representation (6.18). We see thus, there is a very deep and reach web of connections between the hypergeometric functions, which describe disk processes and the infinite sums which have their roots in number theory.

A slightly change in the form of (6.69) for the case $k = 3$ leads us to the class of sums known as *Euler–Zagier double series* or for short *Witten zeta function* and are of the type

$$W(r, s, t) = \sum_{m,n=1}^{\infty} \frac{1}{n^r \; m^s \; (m+n)^t}. \tag{6.73}$$

It derives its name also from its occurrence in quantum field theories, where special values of it calculate volumes of special moduli spaces of vector bundles over curves. More on their mathematical properties can be read in [94] and [95]. In [96] also the *Pascal triangle recurrence* relation can be found

$$W(r, s, t) = W(r-1, s, t+1) + W(r, s-1, t+1) \;, \tag{6.74}$$

as well as the useful relations

$$\begin{aligned} 2\; W(a-2,1,1) - W(1,1,a-2) = 2\; \zeta(a) \;, & \\ W(1,1,a-2) &= (a-1)\; \zeta(a) - \sum_{i=2}^{a-2} \zeta(i)\; \zeta(a-i) \;, \\ W(a-2,1,1) &= \frac{1}{2}\; W(1,1,a-2) + \zeta(a) \;, \\ W(1,0,a-1) &= \frac{1}{2}\; W(1,1,a-2) \;, \end{aligned} \tag{6.75}$$

which are fulfilled by the same function (6.73). The importance of upper relations can be immediately seen for special values for a; thus we can deduce some concrete equalities, like the following might be

$$W(1,1,1) = 2\; \zeta(3) \quad , \quad W(1,1,2) = \frac{1}{2}\; \zeta(4), \tag{6.76}$$

or a more involved relation

$$\sum_{m,n=1}^{\infty} \frac{1}{m\ (1+n)\ (1+m+n)^{\alpha}} = -\alpha + (\alpha+1)\ \zeta(\alpha+2) + \sum_{i=2}^{\alpha} \zeta(i) - \sum_{i=2}^{\alpha} \zeta(i)\ \zeta(\alpha+2-i), \tag{6.77}$$

whose proof is immediate using (6.45) and the relation

$$\sum_{m,n=1}^{\infty} \frac{1}{m\ (1+n)\ (1+m+n)^{\alpha}} = W(1,1,\alpha) - \sum_{m=1}^{\infty} \frac{1}{m\ (m+1)^{\alpha}}. \tag{6.78}$$

Shall we also use (6.35) in addition to (6.45) we are able to derive following formula

$$\begin{aligned}\sum_{m,n=1}^{\infty} \frac{1}{m\ (1+m+n)^{\alpha}} &= \sum_{\substack{m,n=1\\ n>m}}^{\infty} \frac{1}{m\ (n+1)^{\alpha}} = \sum_{n=1}^{\infty} \frac{H_{n-1}}{(n+1)^{\alpha}} = \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^{\alpha}} - \sum_{n=1}^{\infty} \frac{1}{n\ (n+1)^{\alpha}} \\ &= -\alpha + \frac{1}{2}\ \alpha\ \zeta(\alpha+1) + \sum_{i=2}^{\alpha} \zeta(i) - \frac{1}{2}\ \sum_{i=1}^{\alpha-2} \zeta(\alpha-i)\ \zeta(i+1)\ ,\end{aligned} \tag{6.79}$$

or similarly, with the same building blocks, we may deduce

$$\sum_{m,n=1}^{\infty} \frac{1}{(1+m)\ (1+m+n)^{\alpha}} = 1 + \frac{1}{2}\ \alpha\ \zeta(\alpha+1) - \zeta(\alpha) - \frac{1}{2}\ \sum_{i=1}^{\alpha-2} \zeta(\alpha-i)\ \zeta(i+1)\ . \tag{6.80}$$

The last two equations can now be subtracted from each other, using partial fraction techniques, and we arrive at

$$\sum_{m,n=1}^{\infty} \frac{1}{m\ (1+m)\ (1+m+n)^{\alpha}} = -\alpha - 1 + 2\ \zeta(\alpha) + \sum_{i=2}^{\alpha-1} \zeta(i)\ . \tag{6.81}$$

We now present a much more involved Euler sum, whose derivation we shall need the tools collected until now:

$$\sum_{m,n=1}^{\infty} \frac{1}{n\ (1+m)\ (1+m+n)\ (2+m+n)} = 2\ \zeta(3) - \frac{9}{4}\ . \tag{6.82}$$

Some few words to the proof of last forula: if we look at its last two factors in the denominator, we see that they can be partially decomposed to:

$$\frac{1}{(1+m+n)\ (2+m+n)} = \frac{1}{(1+m+n)} - \frac{1}{(2+m+n)} \ . \tag{6.83}$$

This will increase the numnber of sums to calculate to two, but decrease their degree of complexity; the two arising sums are

$$\sum_{m,n=1}^{\infty} \frac{1}{n\ (1+m)\ (1+m+n)} = -1 + 2\ \zeta(3)\ , \tag{6.84}$$
$$\sum_{m,n=1}^{\infty} \frac{1}{n\ (1+m)\ (2+m+n)} = \sum_{m,n=1}^{\infty} \frac{1}{n\ m\ (1+m+n)} - \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = 2 - \frac{3}{4} = \frac{5}{4}\ ,$$

and they can be proved with 6.77 and following relation

$$\sum_{m,n=1}^{\infty} \frac{1}{n\ m\ (1+m+n)} = \sum_{m,n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{1+m+n} \right)\ \frac{1}{m(m+1)} = \sum_{m=1}^{\infty} \frac{H_{m+1}}{m(m+1)} = 2\ . \tag{6.85}$$

Next identity to prove is

$$\sum_{m,n=1}^{\infty} \frac{1}{m\ n\ (1+m)\ (1+n)\ (1+m+n)} = 5 - 4\ \zeta(3)\ , \tag{6.86}$$

which can be done using again the partial decomposition, as done before:

$$\frac{1}{m\ n\ (1+m)\ (1+n)\ (1+m+n)} = \left(\frac{1}{m} - \frac{1}{1+m} \right)\ \left(\frac{1}{n} - \frac{1}{1+n} \right)\ \frac{1}{1+m+n}\ . \tag{6.87}$$

This has as consequence again, that we increase the number of sums to four (two of which are identical) but decrease their complexity; the sums are given by

$$\sum_{m,n=1}^{\infty} \frac{1}{m\ n\ (1+m+n)} = 2\ ,$$
$$\sum_{m,n=1}^{\infty} \frac{1}{m\ (1+n)\ (1+m+n)} = W(1,1,1) - \sum_{m=1}^{\infty} \frac{1}{m\ (m+1)} = 2\ \zeta(3) - 1\ ,$$
$$\sum_{m,n=1}^{\infty} \frac{1}{(1+m)\ (1+n)\ (1+m+n)} = \sum_{m,n=1}^{\infty} \frac{1}{n\ (n+1)} \left(\frac{1}{m+1} - \frac{1}{m+n+1} \right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n\ (n+1)}\ (H_{n+1} - 1) = 1. \tag{6.88}$$

The latter sums can easily be determined from the series already presented before. If we now use the partial fraction decomposition

$$\frac{1}{(1+m)\ n\ (1+n)\ (1+m+n)} = \left(\frac{1}{n} - \frac{1}{1+n}\right)\ \frac{1}{(1+m)\ (1+m+n)} \tag{6.89}$$

we find from (6.88)

$$\sum_{m,n=1}^{\infty} \frac{1}{(1+m)\ n\ (1+n)\ (1+m+n)} = 2\ \zeta(3) - 2\ , \tag{6.90}$$

and additionally two more equalities

$$\begin{aligned} \sum_{m,n=1}^{\infty} \frac{1}{n\ m\ (1+m)\ (1+m+n)^2} &= 8 - 3\ \zeta(2) - 2\ \zeta(3) - \frac{1}{2}\ \zeta(4)\ , \\ \sum_{m,n=1}^{\infty} \frac{1}{m\ (2+n)\ (1+m+n)^2} &= -\frac{3}{2}\ \zeta(2) + 3\ \zeta(3) - 1. \end{aligned} \tag{6.91}$$

The first of the two sums presented in equation (6.91) may be converted to

$$\begin{aligned} &\sum_{\substack{n=1 \\ m\geq 2}}^{\infty} \frac{1}{n\ m\ (m-1)\ (m+n)^2} = \\ &\sum_{m,n=1}^{\infty} \frac{1}{m\ n\ (1+m+n)^2} - \sum_{m,n=1}^{\infty} \frac{1}{m\ n\ (m+n)^2} + \sum_{n=1}^{\infty} \frac{1}{n\ (n+1)^2}\ . \end{aligned} \tag{6.92}$$

This last sum can be worked with, since the last two sums on the right hand side are $W(1,1,2) = \frac{1}{2}\zeta(4)$ and $2-\zeta(2)$ as stated in (6.73), (6.76) and respectively in (6.45), and the first sum on the right hand side is

$$\begin{aligned} \sum_{m,n=1}^{\infty} \frac{1}{m\ n\ (1+m+n)^2} &= \sum_{m,n=1}^{\infty} \left(\frac{1}{m} + \frac{1}{n}\right)\ \frac{1}{(m+n)\ (1+m+n)^2} \\ = 2 \sum_{m,n=1}^{\infty} \frac{1}{m\ (m+n)\ (1+m+n)^2} &= 2 \sum_{n=1}^{\infty} \frac{H_{n-1}}{n(n+1)^2} = 2\ [\ 3 - \zeta(3) - \zeta(2)\ ] \end{aligned} \tag{6.93}$$

The second of the two equations in (6.91) can be converted and evaluated to

$$\sum_{m,n=1}^{\infty} \frac{1}{m\ (2+n)\ (1+m+n)^2} = 2 \sum_{m,n=1}^{\infty} \frac{1}{m\ (m+n)\ (m+n-1)^2} - \sum_{m=1}^{\infty} \frac{1}{2\ m\ (1+m)^2}$$
$$- \sum_{m=1}^{\infty} \frac{1}{m^3} = -1 + \frac{1}{2}\ \zeta(2) - \zeta(3) + 2 \sum_{n=1}^{\infty} \frac{H_n}{n^2(1+n)} = -\frac{3}{2}\ \zeta(2) + 3\ \zeta(3) - 1\ . \tag{6.94}$$

Next identity on the list which we shall prove is

$$\sum_{m,n=1}^{\infty} \frac{1}{(1+n)\ m^2\ (m+n)^2} = -\frac{1}{2}\ \zeta(4) + \frac{1}{2}\ \zeta(2)^2 + 5\ \zeta(3) - 4\ \zeta(2)\ . \tag{6.95}$$

where we can rewrite the one in the numerator as $1 = 1 + n + m - m - n$ and regrouping it as $1 = (1+n) + m - (m+n)$; this leads us to following three partial results:

$$\begin{aligned}
\sum_{m,n=1}^{\infty} \frac{1}{m^2\ (m+n)^2} &= \sum_{m<n} \frac{1}{m^2\ n^2} = \zeta(2,2) = \frac{1}{2}\ \zeta(2)^2 - \frac{1}{2}\ \zeta(4)\ , \\
\sum_{m,n=1}^{\infty} \frac{1}{m\ (1+n)\ (m+n)^2} &= \sum_{m,n=1}^{\infty} \frac{1}{m\ (2+n)\ (1+m+n)^2} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m\ (1+m)^2} \\
&= -2\ \zeta(2) + 3\ \zeta(3)\ , \\
\sum_{m,n=1}^{\infty} \frac{1}{m^2\ (1+n)\ (m+n)} &= \sum_{m,n=1}^{\infty} \frac{1}{m^2}\ \frac{1}{n\ (1+n)} - \sum_{n=1}^{\infty} \frac{1}{n\ (1+n)^2} \\
&\quad - \sum_{m,n=1}^{\infty} \frac{1}{n\ (1+n)\ (1+m)\ (1+m+n)} = 2\ \zeta(2) - 2\ \zeta(3).
\end{aligned} \tag{6.96}$$

Those sums can be either proved by standard techniques or by using results already derived in this chapter.

We have now collected exactly as many results concerning infinite finite sums as to be able to series expand the triple hypergeometric functions, as they stand in (5.29) and thus arrive at (5.32). However, we have listed the expansion until the first order in the Mandelstam variables, i.e. until the second order in their momenta k^μ. And this is also what we are able to do with the sums collected until now. Should we wish to go higher into the momentum expansion we have to work harder and evaluate the real complicated sums

$$\sum_{m_i=1}^{\infty} \frac{m_3}{m_1\ m_2\ (m_1+m_3)\ (m_2+m_3)\ (m_1+m_2+m_3)} = \frac{7}{4}\,\zeta(4), \tag{6.97}$$

$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ m_2\ (1+m_1+m_3)\ (m_2+m_3)\ (m_1+m_2+m_3)} = \frac{19}{4}\,\zeta(4) - 4\,\zeta(3)\ ,$$

$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ m_2\ (m_1+m_3)\ (m_2+m_3)\ (1+m_1+m_2+m_3)} = \frac{17}{4}\,\zeta(4) + 2\,\zeta(3) - \zeta(2) - 5.$$

Those triple sums are really hard to do and very little is known about them. A lot of energy has to be invested in order to be able to write down the results on the right hand side of the equations (6.97). In order to compute them, some intermediary, more simpler triple sums and also double sums involving harmonic numbers have to be evaluated. The results are presented in Appendix H.

6.4.2 $F^{(3)}$ series expansion

All those sums were quite a hard job, however the results enable us to expand triple hypergeometric functions, as the one given in (6.9); moreover, those mathematical tools allow us to go even higher with the parameter expansion, provided that we are willing to invest more time for solving the new arising type of series. This is of course, as we have already stressed, equivalent to expanding the relation (6.18) in its small parameters $a, b, c, ...$, since the latter is just the series representation of the triple hypergeometric function. The effective difference is, when series expanding the formulas, we have to evaluate the corresponding sums and not the integrals, the latter however being every time convertible to sums, and viceversa. We shall show here in greater detail such an expansion, until second order in the Mandelstam variables s_i, though not complete because lacking of space. Thus we present some part of the expansion through second order in s_j without being complete, neither in the first order nor in the second one, the emphasis being on the principle of tackling such problems. We will pick up one exemplary triple hypergeometric function, namely $F\left[\begin{smallmatrix} a,b,d,e,g \\ c,f,h,j-2 \end{smallmatrix}\right]$. (Just a little later we will take care of the functions (5.29), they being the six dimensional basis of the equation system and thus vital for the S–matrix.) This function is to be regarded as having small entries $a, b, ..., j$, where the last power is shifted by -2. Thus, the only non infinitesimal quantity is the -2 in the exponential, and when we look at the integral definition of $F\left[\begin{smallmatrix} a,b,d,e,g \\ c,f,h,j-2 \end{smallmatrix}\right]$, we see that this -2 implies the squared polynomial $(1-x\,y\,z)$ in the denominator[6]. As we will notice in a while, every parameter is multiplied by

[6]It is maybe also worth noticing that this term won't induce poles in the integral: although we take the integration limits from zero to one, this squared polynomial integrated in the denominator has as result $\zeta(2)$. Since when series expanding the whole function, this polynomial will be the leading order, we expect our parameter expansion to be finite and go on with $\zeta(2)$ which is also the case, as can be seen in 6.98.

one special sum, as we have treated them before, and evaluating those sums will left us with the parameter multiplied by some (transcendental) number, the latter being the limit of its corresponding sum:

$$
\begin{aligned}
F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j-2 \end{bmatrix} &= \frac{\Gamma(1+d)\,\Gamma(1+e)\,\Gamma(1+f)}{\Gamma(-g)\,\Gamma(-h)\,\Gamma(2-j)} \sum_{m_i=0}^{\infty} \frac{\Gamma(-g+m_1)\,\Gamma(-h+m_2)\,\Gamma(2-j+m_3)}{m_1!\;m_2!\;m_3!} \\
&\times \frac{\Gamma(1+m_1+m_2+m_3+b)}{\Gamma(2+m_1+m_2+m_3+b+e)}\,\frac{\Gamma(1+m_1+m_3+a)}{\Gamma(2+m_1+m_3+a+d)}\,\frac{\Gamma(1+m_2+m_3+c)}{\Gamma(2+m_2+m_3+c+f)} \\
&= 1+\underbrace{\sum_{m_3=1}^{\infty}\frac{1}{(1+m_3)^2}}_{=\zeta(2)-1} -(g+h)\;[\underbrace{\sum_{m_1,m_3=1}^{\infty}\frac{1}{m_1\;(1+m_1+m_3)^2}}_{=-2+\zeta(2)+\zeta(3)}+\underbrace{\sum_{m_1=1}^{\infty}\frac{1}{m_1\;(1+m_1)^2}}_{=2-\zeta(2)}\;] \\
&+ gh\;\underbrace{\sum_{m_i=1}^{\infty}\frac{m_3}{m_1\;m_2\;(m_1+m_3)\;(m_2+m_3)\;(m_1+m_2+m_3)}}_{=\frac{7}{4}\;\zeta(4)} \\
&+ (g^2+h^2)\;[\underbrace{\sum_{m_1,m_3=1}^{\infty}\frac{H_{m_1-1}}{m_1\;(m_1+m_3)^2}}_{=\zeta(2,1,1)=\zeta(4)}\;] + (a,b,c,d,e,f,j) - \text{dependent terms} + \ldots \\
&= \zeta(2)-(g+h)\;\zeta(3)+\left(g^2+h^2+\frac{7}{4}\;gh\right)\;\zeta(4)+(a,b,c,d,e,f,j) - \text{dep. terms.}
\end{aligned}
\tag{6.98}
$$

Upper formula expresses maybe at best the expansion: we start with a special series as given in (6.18), which uniquely determines one triple hypergeometric function; this being done, we expand the encountered Γ–functions in the small parameters $a, b, ...$; after that, we still have to run the three infinite sums over the terms of expanded Γ–functions; this gives precisely rise to our sums treated in the last section, i.e. harmonic number series, Euler sums, triple and triple zeta sums; evaluating those sums will nicely lead to short, catchy numbers multiplying the parameters, as depicted in the last line of (6.98). Exactly that way, we also write down the missing coefficients of the function or go deeper in the order of the series expansion, the latter however being non trivial because of the gradually increasing complexity. In Appendix I we will list the complete needed expansion of the six base functions.

6.5 Series expansion of singular special functions

For the sake of mathematical completeness and also for reasons of convenience we shall address in this subsection the topic of a singular triple function. Although, strictly speaking, we don't have to expand such functions, since our basis (5.28) was chosen in order to be solely made up by nonsingular functions, it is worth knowing how to handle singular functions, for we can, e.g. prove some equations evolving from the picture changing operation and thus have mathematical control over it. Moreover, in order to address the question of field theory, it might prove more convenient to solve the system of equations in terms of basic functions which have some pole structure; this choice could throw more light on the procedure of disentangling Feynman diagrams.

6.5.1 First simple examples

We chose for that purpose following singular function

$$\begin{aligned} F\begin{bmatrix} a-1, b-1, d, e, g \\ c, f, h, j \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{a-1}\ y^{b-1}\ z^c \\ &\times (1-x)^d\ (1-y)^e\ (1-z)^f\ (1-xy)^g\ (1-yz)^h\ (1-xyz)^j, \end{aligned} \tag{6.99}$$

and, as we can see, the integral will diverge as the parameters a and b approach zero, since in that case, we will integrate $\frac{1}{x}$ and $\frac{1}{y}$ with the lower integration limit equal to zero. Thus, we encounter a genuine singularity problem, which doesn't have its origin in the chosen representation but is a structural part of the function. We want to emphasize, that since those poles are not some artifacts from an unlucky representation, they will show up in any circumstance. As a very consequence, we will encounter divergent sums, which are impossible to sum up: we are faced with a new problem, which is not anymore comparable with the ones encountered in previous sections. In order to be able to handle that, we will have to subtract the divergent part and encode it in an object whose expansion we can master, as shown in [97]. Thus we first rewrite our integral adding and subtracting the same singular piece. Further, we rearrange our expression as the sum of two expressions,

$$F\begin{bmatrix} a-1, b-1, d, e, g \\ c, f, h, j \end{bmatrix} = \mathcal{I}_I^a + \mathcal{I}_I^b\ , \tag{6.100}$$

where the first expression is the original integral minus the singular piece, and the second is just the singular piece:

$$\begin{aligned} \mathcal{I}_I^a &= \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{a-1}\ y^{b-1}\ z^c\ (1-x)^d\ (1-y)^e\ (1-z)^f\ (1-yz)^h \\ &\times \left\{\ (1-xy)^g\ (1-xyz)^j - 1\ \right\} \\ \mathcal{I}_I^b &= \underbrace{\left(\int_0^1 dx\ x^{a-1}\ (1-x)^d\right)}_{=\ B(a-1,d)} \underbrace{\left(\int_0^1 dy \int_0^1 dz\ y^{b-1}\ z^c\ (1-y)^e\ (1-z)^f\ (1-yz)^h\right)}_{=\ C(b-1,c,e,f,h)}. \end{aligned} \tag{6.101}$$

Taking now the limit $(a, b) \to 0$ the first piece $\mathcal{I}_I^a$ will stay finite since when x and y approach zero the singular term will also diverge but with an minus sign, thus canceling the infinity. This enables us to expand the first piece in small parameters $a, b, \ldots$. However, we have now shifted the singularities to the second integral expression $\mathcal{I}_I^b$. If we look though more attentive at the second piece, we recognize the first factorized out integral: it is the notorious Beta function, whose expansion we can obtain either by an classical computation or with every usual mathematics software. The second integral in the last line of (6.101) we recognize again as a generalized ${}_3F_2$ hypergeometric function whose expansion we can either obtain by the same subtraction trick as just described or just look it up in [26], where it has been expanded by using the picture changing trick. A third possibility to write down the expansion of $C(b-1, c, e, f, h)$ is to rewrite it as the sum:

$$C(a-1, b, c, d, e) = \frac{(1-a+b+d+e)\ (1+a+c+e)\ \Phi_1 - e\ (1+c+d+e)\ \Phi_2}{a\ (1-a+b+d)}. \tag{6.102}$$

Here, we have named Φ_1 and Φ_2 the two functions already expanded in equations (6.65) and (6.66). For sure, the relation (6.102) cannot be simply guessed, but was found in [26] when using the same picture changing trick in the case of the five gluon open superstring scattering on the disk. To this end, we arrive at:

$$\mathcal{I}_I^a = j\ (\zeta(2)-1) - (g+j)\ \zeta(3) + \ldots\ , \tag{6.103}$$

$$\mathcal{I}_I^b = \left(\frac{1}{a} - d\ \zeta(2) + \ldots \right) \left(\frac{1}{b} - \frac{c+f}{b} + h + \frac{(c+f)^2}{b} - \zeta(2)\ (e + \frac{cf}{b} + h) + \ldots \right),$$

and thus for (6.99) we add the two pieces obtaining:

$$F\begin{bmatrix} a-1, b-1, d, e, g \\ c, f, h, j \end{bmatrix} = \frac{1}{a\ b} - \frac{c+f}{a\ b} + \frac{(c+f)^2 + b\ h}{a\ b} - \left(\frac{d}{b} + \frac{c\ f}{a\ b} + \frac{e+h}{a} \right)\ \zeta(2) + \mathcal{O}(\epsilon)\ . \tag{6.104}$$

As expected, the latter function has a pole at each, a and b as they are the exponents of the singular integrated polynomials. The second example we want to present is again a singular triple hypergeometric function, this time with a single pole at a:

$$F\begin{bmatrix} a-1, b, d, e, g \\ c, f, h, j \end{bmatrix} = \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{a-1}\ y^b\ z^c \times (1-x)^d\ (1-y)^e\ (1-z)^f\ (1-xy)^g\ (1-yz)^h\ (1-xyz)^j. \tag{6.105}$$

We will treat it in the very same manner as the function in (6.99). Thus, the first step is to add and subtract the relevant singular piece and then rewrite the expression as a sum of two integrals, one finite and the other singular:

$$\begin{aligned}
\mathcal{I}_{II}^{a} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^{a-1}\; y^{b}\; z^{c}\; (1-x)^{d}\; (1-y)^{e}\; (1-z)^{f}\; (1-yz)^{h} \\
&\quad \times \left\{\; (1-xy)^{g}\; (1-xyz)^{j} - 1 \;\right\} \qquad (6.106) \\
\mathcal{I}_{II}^{b} &= \underbrace{\left(\int_0^1 dx\; x^{a-1}\; (1-x)^{d}\right)}_{=\;B(a-1,d)} \underbrace{\left(\int_0^1 dy \int_0^1 dz\; y^{b}\; z^{c}\; (1-y)^{e}\; (1-z)^{f}\; (1-yz)^{h}\right)}_{=\;C(b,c,e,f,h)} .
\end{aligned}$$

Here, we have again shifted the poles from the first expression $\mathcal{I}_{II}^{a}$ to the second one $\mathcal{I}_{II}^{b}$, hence the first can easily be expanded with the relations presented in sections 6.3 and 6.4. The second expression, however, is again factorized into a Beta-function, whose poles are given in every math book on special functions or may be obtained with usual mathematics software, and the second factor being again the generalized ${}_3F_2$ hypergeometric function already expanded and given in (6.65). We find so far

$$\begin{aligned}
\mathcal{I}_{II}^{a} &= g + 3\;j\; - (g + 2\;j)\;\zeta(2) + \ldots\;, \\
\mathcal{I}_{II}^{b} &= \left(\frac{1}{a} - d\;\zeta(2) + \ldots\right)\;(\;1 - b - c - e - f - 2\;h + \zeta(2)\;h + \ldots\;)\;, \qquad (6.107)
\end{aligned}$$

and finally, for (6.105)

$$F\begin{bmatrix} a-1, b, d, e, g \\ c, f, h, j \end{bmatrix} = \frac{1}{a} - \frac{b+c+e+f+2h}{a} + \frac{h}{a}\;\zeta(2) + \mathcal{O}(\epsilon)\;. \qquad (6.108)$$

6.5.2 More involved functions

As a last example, we want to analyze a function with a much more richer pole structure, and thus more complex. It is given by following integral expression:

$$\begin{aligned}
F\begin{bmatrix} a-1, b-1, d-1, e-1, g \\ c-1, f-1, h, j \end{bmatrix} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^{a-1}\; y^{b-1}\; z^{c-1}\; (1-x)^{d-1} \\
&\quad \times (1-y)^{e-1}\; (1-z)^{f-1}\; (1-xy)^{g}\; (1-yz)^{h}\; (1-xyz)^{j}\;.
\end{aligned} \qquad (6.109)$$

Before going on with the expansion, it is worth to say something about its structure. We again recognize the kind of poles encountered in the last two examples, where they occurred at the first and/or second polynomials. In this case, we have a, b and c

shifted by -1 and thus when taking the limit $(a, b, c) \to 0$ we will have to integrate $\frac{1}{x\,y\,z}$ with the lower integration limit being zero. This clearly will produce divergencies. The three poles could be handled exactly as shown in last two examples, since they are of the same kind. However, we also have the powers d, e and f shifted by -1 which will generate in the limit $(d, e, f) \to 0$ again poles, for we have to integrate the expression $\frac{1}{(1-x)\,(1-y)\,(1-z)}$ with upper integration limit one. Those pole will occur thus at one and not at zero as the other three did. This will surely mix up the pole structure of the singular piece in a non trivial way. We may help us however with the system of equations generated for the six–point function, and from there we will deduce the expansion of the latter function:

$$
\begin{aligned}
F\begin{bmatrix} a-1, b-1, d-1, e-1, g \\ c-1, f-1, h, j \end{bmatrix} &= \frac{(a+d)\ (c+f)}{a\ b\ c\ d\ f} + \frac{(a+d)\ (d+e)+d\ g}{a\ c\ d\ e\ (d+e+g)} + \frac{e+f}{a\ e\ f\ (e+f+h)} \\
&+ \frac{(d+e)\ (e+f)\ (d+e+f+g)+[(d+e)\ (e+f)+e\ g]\ h}{d\ e\ f\ (d+e+g)\ (e+f+h)\ (d+e+f+g+h+j)} \\
&+ \mathcal{O}(\epsilon^{-1})\ .
\end{aligned}
\tag{6.110}
$$

We should also notice, that this type of function has the maximal degree of freedom we can encounter in this case; although the three exponentials g, h, j are not shifted by negative integers, we haven't chosen a simplified version, since such a shift would't generate any poles as seen from the integral representation: the exponentials g, h, j belong to the polynomials $\frac{1}{(1-xy)}$, $\frac{1}{(1-yz)}$ and respective $\frac{1}{(1-xyz)}$. A possible shift with negative integers would place the polynomials in the denominator. However, the integral taken from zero to one over such polynomials stays finite, as can be proven by means of a simple computation. This involved expansion should conclude the actual chapter and we shall move to the last topic of this work, the low energy application of the six–point S–matrix we have treated so far.

Chapter 7

Reducible diagrams and contact interactions

Main part of this work was the calculation of the S–matrix involving six open strings on the disk. We have found this expression and also expanded it in its in α', as given in the first part of Appendix E. The S–matrix is written as an infinite power series in the Mandelstam variables s_j, of which we have of course given just a finite part, up to α'^4. This is sofar the analysis of the S–matrix from the string point of view. On the other hand, when looking at the same matter from the field theory point of view, we will have to deal with various Feynman diagrams, all of which have six external particles, since we are dealing with the six–point function, but are of different "topology". By that we mean that there are precisely two types of diagrams: contact diagrams and exchange diagram. We shall be just interested in the contact diagrams, since only those reveal new interactions at a given order. Nevertheless we still have to handle the exchange diagrams, since these are present in the S–matrix, thus we have to separate them from the others. Since the S–matrix is organized as power series in the momenta, we will get the information about the type of diagrams exactly from the momentum order and general shape of each term in the amplitude. To this end, the amplitude formally has the momentum expansion

$$\mathcal{A}_6(k) \sim k^{-2} + 0\ k^0 + \zeta(2)\ k^2 + \zeta(3)\ k^4 + \zeta(4)\ k^6 + \mathcal{O}(k^8). \qquad (7.1)$$

As we can deduce from upper formula, every order in the momentum is multiplied by a specific value of the Riemann zeta function, fact which also disentangles the diagrams, delivering a method of classification. Since we want to compute the Born–Infeld action at order α'^4 and we are going to extract those results from the six–point function, especially from the terms D^4F^4, D^2F^5 and F^6, we shall be interested solely in irreducible diagrams at order α'^4, since only those will contain nontrivial information, i.e. new interaction terms. We will turn our attention thus to the α'^4–order of the S–matrix given in Appendix E. It is worth to mention that the terms D^4F^4 and D^2F^5 are known from the four–point amplitude respectively from the five–point amplitude. They represent just higher momentum expansions of the S–matrix, as in our case the

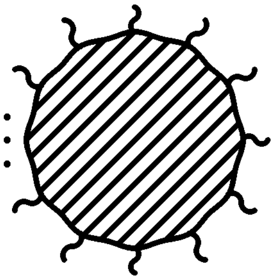


Figure 7.1: General Feynman diagram with n external legs

terms of order α'^5, α'^6, etc, would be. Thus the corresponding Feynman diagrams can be extracted from those interactions. However, the term F^6 is uniquely determined by our six–point function. The terms coming with the momentum powers k^{-2}, k^2 and k^4 in the six–point matrix, represent reducible diagrams of six gluons interactions. On the other hand side, the terms beyond the order k^6 stand for gluon processes such as D^2F^6, D^4F^5 and D^6F^4, which are only to be considered when analyzing the order α'^5 as explained before for the six–point case.

In order to better understand the terms k^6 we shall set up some elementary formulas concerning field theory diagrams based on dimensional analysis. For that purpose figure (7.1) may be quite useful. It depicts a very general Feynman diagram with an indefinite number of external lines N, representing the number of interacting particles. For sure, we will concentrate on the case $N = 6$. The shaded surface in the figure should hide the vertices and eventual propagators if we have a reducible diagram. So again the number of vertices and internal propagators is indefinite and will be fixed by reasonable requirements just in short.

Thus, given the general Feynman diagram depicted in (7.1), we shall denote with P the number of its internal propagators and V_n^k the number of vertices $\mathcal{V}_n^k$ with n legs and energy dimension k. The the number N of external particles of that general Feynman diagram is

$$N = \sum_{\substack{n\geq 3 \\ k\geq 0}} nV_n^k - 2P. \tag{7.2}$$

The sum should start at $n = 3$, since a vertex cannot have less than three legs. The factor in of 2 in front of the propagator number P has also a simple explanation: if we put to vertices together then each vertex will lose exactly one leg and the two legs will form a propagator. This is why we have to subtract the double number of Propagators per given number of vertices. A further constraint on the above sum is our consideration of solely tree–level diagrams, since we have computed our amplitude on the disk. This comes along with the relation

$$P = \sum_{\substack{n\geq 3\\ k\geq 0}} V_n^k - 1. \tag{7.3}$$

We can now plug in the relation from propagator into the relation for external particles (7.2) and find

$$N = \sum_{\substack{n\geq 3\\ k\geq 0}} (n-2)\ V_n^k + 2, \tag{7.4}$$

noticing that the sum runs over all possible kind of vertices. This is still very general, we shall impose one more constraint. Since we study six–point interactions on the disk, the corresponding reducible Feynman diagrams are composed out of vertices with no more than five legs. Also allowing for contact interactions, we will have vertices with six legs. Thus the sum (7.4) runs over all vertices with at most six legs. It is natural to also fix the number of external legs of the whole diagram to six, since we have computed a six–point amplitude. This further reduces the sum (7.4) to

$$6 = 2 + \sum_{k\geq 0} \left(V_3^k + 2V_4^k + 3V_5^k + 4V_6^k \right). \tag{7.5}$$

We have arrived now at a simple and useful relation, which we have to fulfil in order to see which diagrams contribute to our string S–matrix. It turns out that there are only five ways how we can satisfy equation (7.5). Those are summarized in Table 2.

	V_3^k	V_4^k	V_5^k	V_6^k	P
a	4	0	0	0	3
b	2	1	0	0	2
c	1	0	1	0	1
d	0	2	0	0	1
e	0	0	0	1	0

Table 2: *Number of vertices V_n^k to meet the condition 7.5.*

The table thus tells us, we can meet the condition in equation (7.5), for example, as in case a), by forming one reducible Feynman diagram, consisting of four vertices, each oh which has three legs. This is just the a–row of the table. In the second case, case b), we form our Feynman diagram out of two vertices, each with three legs and one additional vertex with four legs. The other cases are to be read from the table

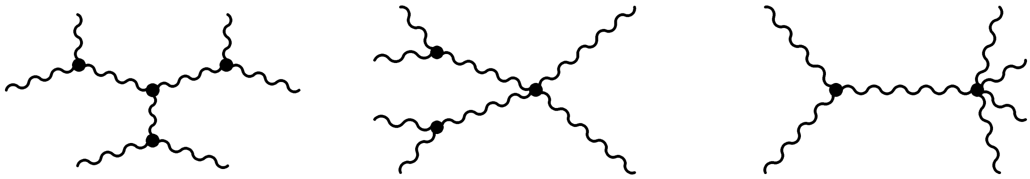


Figure 7.2: Reducible Feynman diagrams made up of vertices as described in cases $a), b)$ and $c)$ in table 7.

similarly. We have also depicted the involved diagrams in their order of appearance in figures 7.2 and 7.3.

Looking at the vertices in table 7 and at their definition we will notice fast that $\mathcal{V}_3^1$ and $\mathcal{V}_4^4$ are the "usual" Yang–Mills vertices as shown in every book on the subject, e.g. in [8]. A deeper analysis will also show that there could also be in principle possible to count the vertices $\mathcal{V}_3^3$, $\mathcal{V}_4^2$ and $\mathcal{V}_5^1$. Those vertices are extracted from the term $\mathrm{Tr}F^3$.

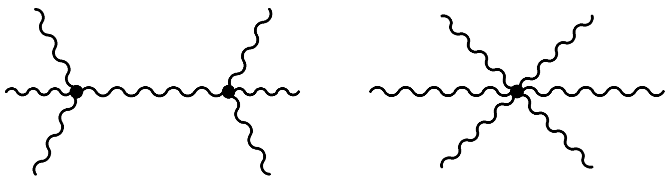


Figure 7.3: Reducible Feynman diagrams made up of vertices as described in cases $d)$ and $e)$ in table 7.

But this term is absent in the superstring theory, i.e. the respective amplitude is zero! Thus also the vertices presented above cannot be extracted out of it and consequently we cannot use them in building up Feynman diagrams.To conclude, we list all vertices $\mathcal{V}_n^k$ with at most six external legs following from the F^n– and D^mF^n–terms in the effective action (see Table 1) in Table 3:

1	α'^0	F^2	$\mathcal{V}_3^1$	$\mathcal{V}_4^0$	
$\zeta(2)$	α'^2	F^4	$\mathcal{V}_4^4$	$\mathcal{V}_5^3$	$\mathcal{V}_6^2$
$\zeta(3)$	α'^3	F^5	$\mathcal{V}_5^5$	$\mathcal{V}_6^4$	
$\zeta(3)$	α'^3	D^2F^4	$\mathcal{V}_4^6$	$\mathcal{V}_5^5$	$\mathcal{V}_6^4$
$\zeta(4)$	α'^4	F^6	$\mathcal{V}_6^6$		
$\zeta(4)$	α'^4	D^4F^4	$\mathcal{V}_4^8$	$\mathcal{V}_5^7$	$\mathcal{V}_6^6$
$\zeta(4)$	α'^4	D^2F^5	$\mathcal{V}_5^7$	$\mathcal{V}_6^6$	
$\zeta(5)$	α'^5	D^6F^4	$\mathcal{V}_4^{10}$	$\mathcal{V}_5^9$	$\mathcal{V}_6^8$
$\zeta(5)$	α'^5	D^2F^6	$\mathcal{V}_6^8$		

Table 3: *Possible vertices $\mathcal{V}_n^k$ with at most six external legs*

We have so far solely analyzed the topology of vertices. i.e. their number of legs, vertices and internal propagators. Nothing was said about their energy dimension. A similar computation as last few equation depict, we can done also for the energy. Thus, given the total energy K

$$K = \sum_{n\geq 3,\ k\geq 0} kV_n^k - 2P \ , \tag{7.6}$$

and using the relation (7.3) we find

$$K = \sum_{\substack{n\geq 3 \\ k\geq 0}} (k-2)\ V_n^k + 2\ . \tag{7.7}$$

As stated before we will only be concerned with diagrams of momentum order k^6, which is equivalent to specifying the value of th energy to $K = 6$. Furthermore, since we have just one single vertex with three external legs, the notorious vertex from the Yang–Mills theories, we can pull that one out of the sum, and obtain:

$$4 = 4\ V_6^6 - V_3^1 + \sum_{k\geq 0}(k-2)\ V_4^k + \sum_{k\geq 0}(k-2)\ V_5^k. \tag{7.8}$$

The next step is of course to use this energy constraint on equation (7.5) or equivalently on each of the five cases presented in Table 2. This is easily done for the cases where we have to deal with just one type of diagrams, as the two cases a) and e). So first to the case a): any Feynman diagram made up of four vertices $\mathcal{V}_3^1$ has a total energy of k^{-2}, because of the various internal propagators which lower the energy. Thus this case will never fulfil condition (7.8). However, this diagram contributes to the terms involving the poles in our S–matrix momentum expansion (7.1). On the other hand, case e) involves one single diagram composed of a $\mathcal{V}_6^6$–vertex, meets exactly the requirements of equation (7.5) and (7.8).

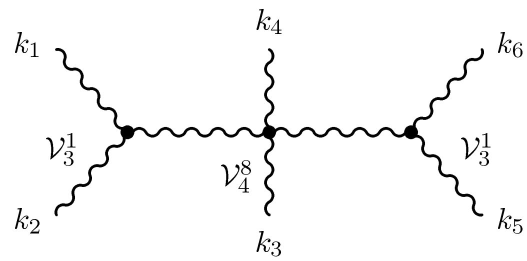


Figure 7.4: Diagram $\sim \zeta(4)\ k^6$ with one four–vertex $\mathcal{V}_4^8$ and two three–vertices $\mathcal{V}_3^1$

This should not be surprising since as we have already stressed, this vertex encodes new interactions in the effective action at order α'^4. The next case we should treat is case b). Here the conditions on the number of external states and total energy may be meet choosing $\mathcal{V}_4^8$ for the four–vertex. The corresponding diagram is depicted in figure 7.4. Analogously we can go on and determine case c). Here it turns out that we need $\mathcal{V}_5^7$ as our five–vertex. The corresponding diagram is depicted in figure 7.5.

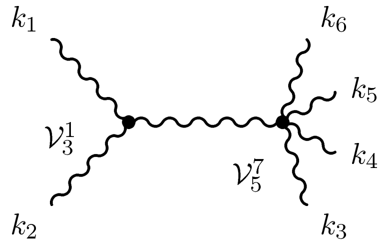


Figure 7.5: Diagram $\sim \zeta(4)\ k^6$ with one five–vertex $\mathcal{V}_5^7$ and one three–vertex $\mathcal{V}_3^1$

The last case we should deal with is case d). Here we see that the diagram should be made up of two vertices of the same topology, namely two vertices of the type $\mathcal{V}_4^k$, i.e. with four legs. The energy constraint imposed on the two vertices is of course $K = 8$, for the connecting propagator lowers the energy two units. Thus the most simple possibility is take twice the vertex $\mathcal{V}_4^4$. The corresponding diagram is presented in figure 7.6.

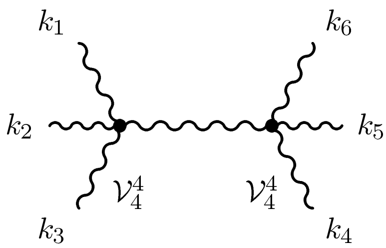


Figure 7.6: Diagram $\sim \zeta(2)^2\ k^6$ with two four–vertices $\mathcal{V}_4^4$ and one propagator

However this is not the only solution. We can as good as in the last case meet the energy condition $K = 6$ when we take one $\mathcal{V}_4^0$–vertex and one $\mathcal{V}_4^8$–vertex. They again have a total energy of $K = 8$ but the connecting propagator lowers the energy with two units, thus leading to the desired result. This diagram is depicted in figure 7.7.

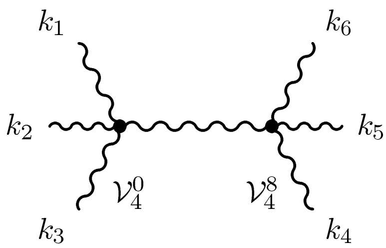


Figure 7.7: Diagram $\sim \zeta(4)\ k^6$ with the two four–vertices $\mathcal{V}_4^0$ and $\mathcal{V}_4^8$

In order not to lose overview about the facts collected until now, let us summarize what we have found out about the low energy action, its momentum expansion and the corresponding Feynman graphs, which reproduce exactly the terms in the action. When computing the superstring amplitude with six gluons as the external states one finds the result (7.1). However, this formula depicts the results only partially, since we have expanded the S–matric in its momenta. This expansion allows us to see more structure within the result, since we find that each momentum order is multiplied by a specific value of the Riemann zeta function, which makes it more comfortable to distinguish between different terms. This string result is now to be matched with the field theory ansatz. The latter is a non redundant sum of terms of the type $D^m F^n$. Here F denotes the field strength and D the covariant derivative

$$F_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + ig[A_\mu, A_\nu]. \tag{7.9}$$

Furthermore, we have introduced the coupling constant g and the the commutator between two polarizations ξ. The latter contain the representation of the lie algebra for the underling gauge group $U(N)$. The explicit notation can be found in great detail in Appendix J.

From those terms vertices can be extracted, such that the Feynman graphs made out of them reproduce the results shown in (7.1). In order to correctly write down the low energy field ansatz great care is required, since terms of the form $D^m F^n$ might look different but still be the same. The techniques of writing such field terms in different ways are partial integration, the equations of motion, Bianchi identities or the specific identity

$$[D_\mu, D_\nu] F_{\rho\sigma} = -[F_{\mu\nu}, F_{\rho\sigma}], \tag{7.10}$$

which occurs in non–abelian Yang–Mills theories. Considering those facts only some of the terms $D^m F^n$ have to be considered, in our case namely the ones emphasized in Table 1. They have to be equated with the string matrix giving rise to a system of linear equations for the coefficients which multiply the terms $D^m F^n$. The solution of this system allows for writing the exact α'^4–part of the effective interaction as given in (3.3). However before being able to do that one has to extract the vertices out of the terms in the effective action. This is done by just multiplying out the expression (7.9) and eventually the covariant derivatives. This will lead exactly to the vertices presented in Table 3. This technique was pioneered in ([98]). Since we are interested just in the new interactions occurring at order α'^4 we will not match the entire string matrix (7.1) but just the piece of interest, namely the one proportional to k^6. This restriction together with the fixed number of six external particles we want to consider and some basic deductions from the theory of Feynman graphs will lower the number of possibilities of diagrams. The possible cases are depicted in Table 2. Fulfilling those constraints eventually leads to the four type of diagrams presented in the four pictures above (7.4–7.7) which all have energy dimension $K = 6$. The four pictures exactly depict the cases b, c and d from presented in Table 2. In the last case e) we have to deal with the vertex $\mathcal{V}_6^6$ coming either from the term F^6 or D^4F^4. This is actually the case of interest for us, since it gives rise to new interactions at order α'^4. This information is surely comprised in the S–matrix (7.1). In order to extract that information we have to look at the part proportional to k^6 and subtract from it all the contributions shown in diagrams (7.4–7.7) such that just the interaction coming from $\mathcal{V}_6^6$ will remain. This is in principle simple but in reality is quite technical work. With the help of the vertices in Table 3 we have to calculate all diagrams of interest presented above. This calculation is a pure field calculation. In practice this means to sum over all permutations of gauge indices both when extracting the vertices from the terms in Table 1 and when multiplying them to obtain the diagrams. All those diagrams summed together account for the reducible part of the k^6–order in the S–matrix. The former is then to be subtracted from (7.1) leaving an expression without any poles and representing exactly the six–point contact interaction following from F^6.

Chapter 8

Conclusion and open problems

This work has been mainly dedicated to the calculation of the six–gluon superstring interaction on the disk and to the mathematical methods used to tackle this problem. A new computational method was introduced in order to make this calculation possible. A very efficient and systematic way was found to compute string disk amplitudes in general, by equating seemingly different expressions for the same S–matrix. Those different looking expressions are the result of the world–sheet supersymmetry which generates a system of non trivial linear equations for the superstring matrix. In more detail, it is the super diffeomorphism invariance of the string world-sheet which imposes strong conditions on the form of the superstring tree–level amplitude; we write thus the S–matrix in different but fully equivalent ways which delivers us with a system of equations whose solution determines the full string S–matrix. These equations represent algebraic identities between the analytic functions involved in the six–gluon amplitude, the triple hypergeometric functions. This technique is to be seen as similar to the one used in loop amplitudes, where again it is the world–sheet supersymmetry which imposes strong constraints on the S–matrix, this time in form of Riemann identities between the functions encountered there, the modular forms[1]. More on Riemann identities and their application to loop amplitudes in string theory see [103]. Finally the string S–matrix on the disk, as calculated in this work, is expressed by six triple hypergeometric functions, which encode the full momentum behavior at every order α'. The next part of the work concerns the series expansion of those functions, which is vital for the determination of the effective field action. Thus, in order to extract the interesting momentum order from the string matrix and thus from those triple hypergeometric function a powerful mathematical formalism including the treatment of special functions and sums is developed. The latter helps us to expand those functions in their small parameters, fact which is by far nontrivial for those functions have also poles in the parameters. However, this problem can be solved by either relating singular functions to a linear combination of nonsingular ones through the system of equations, where the poles are factorized in front of the nonsingular functions, or even directly expand the singular functions with the tools from the mathematical chapter.

[1]The subject of modular transformation is a very active topic in mathematics. A lot of good literature can be found about that, e.g. [99], [100], [101],

In greater detail the latter are a basic but complete introduction to special functions, beginning with the classical Beta integral, then generalizing this to generalized hypergeometric functions and finally introducing Kampé de Fériet and Lauricella functions which are cosely related to our triple hypergeometric functions. Further the different representations for the latter functions is studied, where the emphasize is on the series and integral representation. The former naturally leads to infinite series which have to be evaluated in order to obtain the series expansion of the desired functions. Those sums involve harmonic numbers, Euler sums, triple and generalized zeta functions, and finally the more general triple sums, which are quite complicated and at the same time are the key to the series expansion of the triple hypergeometric function. Also different representations of those series are studied where there is a strong relation between these and the ones for the functions. Finally we success in expanding the functions and writing an expanded expression of the S–matrix result. The last part of the work deals with the field theory in more details. We collect there all the diagrams needed for the effective action in order to be able to extract exactly the part of the S–matrix we are interested in. This is a quite logical and straightforward work but very technical and involved. It is still to be done in future projects.

Part III

Appendices

Appendix A

Supersymmetry algebra

This section is inserted for the sake of completeness, neither explanations nor derivations are given. In eq. (A.1) the full algebra of $N = 1$ supersymmetry in $D = 4$ can be contemplated.

$$
\begin{aligned}
[P_m, P_n] &= 0 \\
[M_{mn}, P_p] &= i(\eta_{np}P_m - \eta_{mp}P_n) \\
[M_{mn}, M_{pq}] &= i(\eta_{mp}M_{nq} - \eta_{mq}M_{np} - \eta_{np}M_{mq} + \eta_{nq}M_{mp}) \\
[P_k, R] &= 0 \\
[M_{mn}, R] &= 0 \\
\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\} &= 2\sigma^m_{\alpha\dot{\beta}}P_m \\
\{Q_\alpha, Q_\alpha\} &= 0 \\
\left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} &= 0 \\
\{Q_\alpha, P_m\} &= 0 \\
\left\{\bar{Q}_{\dot{\alpha}}, P_m\right\} &= 0 \\
\{Q_\alpha, M_{mn}\} &= \sigma_{mn\alpha}{}^{\beta}Q_\beta \\
\left\{\bar{Q}_{\dot{\alpha}}, M_{mn}\right\} &= \bar{\sigma}^{\dot{\alpha}}_{mn\dot{\beta}}\bar{Q}^{\dot{\beta}} \\
\{Q_\alpha, R\} &= RQ_\alpha \\
\left\{\bar{Q}_{\dot{\alpha}}, R\right\} &= -R\bar{Q}_{\dot{\alpha}}
\end{aligned}
\qquad \text{(A.1)}
$$

The first three rows, as the reader may recognize, represents the Poincaré algebra. So P_m is just the momentum operator, whereas M_{mn} is the boost operator. In the next two lines we introduced some generator R which generates some internal symmetry, whose commutator with the Poincaré generators is zero.

The rest of the formula is solely concerned with the supersymmetry generators Q_α respectively $\bar{Q}_{\dot{\alpha}}$. They have non trivial commutation relations with the boost operators M_{mn}, the internal symmetry R, and with themselves.

The generalization to higher supersymmetries $N > 1$ can be made, the algebra getting slightly more complicated.

Appendix B

Faddeev–Popov ghosts and Grassmann variables

In this appendix we follow very nearly the first volume of Polchinski's book on string theory [29].

As already explained in section 4.1.3, we have to insert a gauge fixing factor into the partition function 4.1 in order to make it finite, or equivalently to explicitly evaluate the formula 4.2. For this purpose we will insert the identity into the path integral, written as

$$1 = \Delta_{FP}(g) \int [d\mu]\ \delta(g - \hat{g}^{\mu}). \tag{B.1}$$

Here we formally integrate over all gauge transformations of one metric $g_{\mu\nu}$ with given integration measure $d\mu$. Since the argument of delta functional is a function of the integration variable, we have to take care of the variable transform and thus insert the corresponding Jacobian determinant, here denoted $\Delta_{FP}(g)$, after its discoverers Faddeev and Popov.

Since we need an explicit form of the determinant, we shall now derive it, by looking at some infinitesimal gauge transformations applied near the identity. Thus the infinitesimal version of (4.7) is

$$\begin{aligned} \delta\, g_{\mu\nu} &= 2\,\delta\,\omega\, g_{\mu\nu} - \nabla_a\,\delta\,\sigma_b - \nabla_b\,\delta\,\sigma_a \\ &= (2\,\delta\,\omega - \nabla_c\,\delta\,\sigma^c - 2(P\,\delta\,\sigma)_{ab}\ , \end{aligned} \tag{B.2}$$

where in the last line we have converted the infinitesimal transformation by introducing the operator P as $(P\,\delta\,\sigma)_{ab} = \frac{1}{2}(\nabla_a\,\delta\,\sigma_b + \nabla_b\,\delta\,\sigma_a - g_{ab}\,\nabla_c\,\delta\,\sigma^c)$.

We can now formally invert equation (B.1) to obtain

$$\Delta_{FP}(\hat{g})^{-1} = \int [\, d\delta\omega\, d\delta\sigma\,]\delta\left[-(2\,\delta\,\omega - \hat{\nabla}_c\,\delta\,\sigma^c)\hat{g} + 2\,\hat{P}\,\delta\,\sigma\right] \tag{B.3}$$

Here we have split the measure over gauge transformations into a measure for the Weyl– and one for diffeomorphism transformations. Again we denote by a hat over an operator $\hat{\mathcal{O}}$ the fact that we have made use of just the gauged metric $\hat{g}$.

As in the case of delta functions, where we simply can represent $\delta(t)$ as an integral of $\int dx e^{ixt}$, we will also introduce here the exponential of β and σ such that the integral over them will produce a delta functional , which is the desired effect:

$$\begin{aligned}\Delta_{FP}(\hat{g})^{-1} &= \int [\, d\delta\omega \, d\beta \, d\delta\sigma \,] \exp\left\{ 2\pi\, i \int d^2\sigma \hat{g}^{1/2}\, \beta^{ab} \delta \left[-(2\,\delta\,\omega - \hat{\nabla}_c\,\delta\,\sigma^c)\hat{g} + 2\,\hat{P}\,\delta\,\sigma \right]_{ab} \right\} \\ &= \int [\, d\beta' \, d\delta\sigma \,] \exp\left\{ 4\pi\, i \int d^2\sigma \hat{g}^{1/2}\, \beta'^{\,ab} (\hat{P}\,\delta\,\sigma)_{ab} \right\}\end{aligned} \tag{B.4}$$

The last line of upper equality is established if we integrate over $\delta\,\omega$. This will impose β to be traceless, since the prime on the beta, for from now on we will solely integrate over traceless ghosts β.

We might recognize in (B.4) the path continuation of the classical formula

$$\int d^n x \; e^{-\pi\, \vec{x}\cdot A \cdot \vec{x}} = \frac{1}{\det(A)} \;, \tag{B.5}$$

where $\vec{x}$ is a n–dimensional vector and A a $n \times n$–matrix. We would like now to invert the right hand side of upper equation and thus end up with the determinant of the matrix A. We will make for that a trick, and use the Berezin integral over Grassmann variables. For an explicit introduction to integrals over non commuting variables see for example the relevant chapters in [12, 29]. Given a set of non commuting variables $\theta_1, .., \theta_n$ and a $n \times n$–matrix A we have

$$\int d^n\theta \; e^{\vec{\theta}\cdot A \cdot \vec{\theta}} = \det(A) \;. \tag{B.6}$$

We have formed here an n–vector out of the collection of θ' s. Exactly as in the bosonic, we will take the continuum limit of the latter integral and thus be able to invert the determinant in (B.4). For that we will replace the integration variables β and $\delta\,\sigma$ with anticommuting Grassmann variables b and c. However, we would like to keep those variables as scalars and not as fermionic quantities, thus we will have anticommuting scalars, which are also known as "ghosts". We obtain then immediately

$$\Delta_{FP}(\hat{g}) = \int [\, db \; dc \,] \exp\left[-\frac{1}{2\pi} \int d^2\sigma \; \hat{g}^{1/2}\, b_{ab}\, \hat{\nabla}^a\, c^b \right] = \int [\, db \; dc \,]\, e^{-S_g} = \det(\hat{P}) \;. \tag{B.7}$$

Just a few more words on the Grassmann variables and on their integration: a set of such variables is defined as

$$\{\phi, \chi\} = 0 \tag{B.8}$$

and their integrals (by deeper means of linearity and translational properties) defined as

$$\int d\phi = 0 \ , \quad \int d\phi \ \phi = -\int \phi \ d\phi = 1 \ . \tag{B.9}$$

Further, every function of only one Grassmann variable ϕ has a Taylor expansion liner in that variable, since every higher power vanishes by anticommutativity,

$$f(\phi) = a + b\,\phi \ . \tag{B.10}$$

The last two relations lead us to the integral of a Grassmann valued function, which is then

$$\int d\phi \ f(\phi) = b \tag{B.11}$$

with b the corresponding Taylor coefficient. Taking now the derivative of the function $f(\phi)$ with respect to its Grassmann variable

$$\frac{\partial}{\partial\,\phi} f(\phi) = b \tag{B.12}$$

we obtain the some unexpected equality, which holds between operators of anticommuting variables:

$$\int d\phi = \frac{\partial}{\partial\,\phi} \ . \tag{B.13}$$

It is now just a matter of time to prove the correctness of (B.6), since the exponential can be Taylor expanded in an finite number of terms, which can finally be integrated with the techniques presented above.

Appendix C

String Wick contractions

$$
\begin{aligned}
\mathcal{B}_1(a,b,i,j,k,l) = -\mathcal{E}\ \Bigg\{ & \frac{z_{ai}(\xi_k k_i)(\xi_l k_i)}{z_{ab}^2 z_{ij} z_{ik} z_{il}} \left(\frac{z_{ai}}{z_{ak} z_{al} z_{ij}} - \frac{z_{ai} k_i k_j}{z_{ak} z_{al} z_{ij}} + \frac{k_j k_k}{z_{al} z_{jk}} + \frac{k_j k_l}{z_{ak} z_{jl}} \right) \\
& - \frac{(\xi_k k_l)(\xi_l k_j)}{z_{ab}^2 z_{ij} z_{jl} z_{kl}} \left(\frac{z_{aj}(1 - k_i k_j)}{z_{ak} z_{ij}} - \frac{k_i k_k}{z_{ik}} - \frac{z_{al} k_i k_l}{z_{ak} z_{il}} \right) + \frac{(\xi_k k_i)(\xi_l k_b)}{z_{ab} z_{al} z_{bl} z_{ij} z_{ik}} \left(\frac{z_{ai}(1 - k_i k_j)}{z_{ak} z_{ij}} + \frac{k_j k_k}{z_{jk}} \right) \\
& + \frac{(\xi_k k_i)(\xi_l k_k)}{z_{ab}^2 z_{ij} z_{ik} z_{kl}} \left(\frac{z_{ai}(1 - k_i k_j)}{z_{al} z_{ij}} + \frac{z_{ak} k_j k_k}{z_{al} z_{jk}} + \frac{k_j k_l}{z_{jl}} \right) + \frac{(\xi_k k_b)(\xi_l k_j)}{z_{ab} z_{ak} z_{bk} z_{ij} z_{jl}} \left(\frac{z_{aj}(1 - k_i k_j)}{z_{al} z_{ij}} - \frac{k_i k_l}{z_{il}} \right) \\
& + \frac{(\xi_k k_j)(\xi_l k_i)}{z_{ab}^2 z_{ij} z_{il} z_{jk}} \left(\frac{z_{ai} z_{aj}(1 - k_i k_j)}{z_{ak} z_{al} z_{ij}} - \frac{z_{ai} k_i k_k}{z_{al} z_{ik}} + \frac{z_{aj} k_j k_l}{z_{ak} z_{jk}} + \frac{k_k k_l}{z_{kl}} \right) + \frac{(\xi_k k_b)(\xi_l k_k)(1 - k_i k_j)}{z_{ab} z_{al} z_{bk} z_{ij}^2 z_{kl}} \\
& + \frac{(\xi_k k_j)(\xi_l k_b)}{z_{ab} z_{al} z_{bl} z_{ij} z_{jk}} \left(\frac{z_{aj}(1 - k_i k_j)}{z_{ak} z_{ij}} - \frac{k_i k_k}{z_{ik}} \right) + \frac{(\xi_k k_b)(\xi_l k_i)}{z_{ab} z_{ak} z_{bk} z_{ij} z_{il}} \left(\frac{z_{ai}(1 - k_i k_j)}{z_{al} z_{ij}} + \frac{k_j k_l}{z_{jl}} \right) \\
& + \frac{z_{aj}(\xi_k k_j)(\xi_l k_j)}{z_{ab}^2 z_{ij} z_{jk} z_{jl}} \left(\frac{z_{aj}(1 - k_i k_j)}{z_{ak} z_{al} z_{ij}} - \frac{k_i k_k}{z_{al} z_{ik}} + \frac{k_i k_l}{z_{ak} z_{il}} \right) + \frac{(\xi_k k_b)(\xi_l k_b)(1 - k_i k_j)}{z_{ak} z_{al} z_{bk} z_{bl} z_{ij}^2} \\
& + \frac{(\xi_k k_j)(\xi_l k_k)}{z_{ab}^2 z_{ij} z_{jk} z_{kl}} \left(\frac{z_{aj}(1 - k_i k_j)}{z_{al} z_{ij}} - \frac{z_{ak} k_i k_k}{z_{al} z_{ik}} - \frac{k_i k_l}{z_{il}} \right) - \frac{(\xi_k k_l)(\xi_l k_b)(1 - k_i k_j)}{z_{ab} z_{ak} z_{bl} z_{ij}^2 z_{kl}} \\
& + \frac{(\xi_k k_i)(\xi_l k_j)}{z_{ab}^2 z_{ij} z_{ik} z_{jl}} \left(\frac{z_{ai} z_{aj}(1 - k_i k_j)}{z_{ak} z_{al} z_{ij}} - \frac{z_{ai} k_i k_l}{z_{ak} z_{il}} + \frac{z_{aj} k_j k_k}{z_{al} z_{jk}} - \frac{k_k k_l}{z_{kl}} \right) \\
& - \frac{(\xi_k k_l)(\xi_l k_i)}{z_{ab}^2 z_{ij} z_{il} z_{kl}} \left(\frac{z_{ai}(1 - k_i k_j)}{z_{ak} z_{ij}} + \frac{k_j k_k}{z_{jk}} + \frac{z_{al} k_j k_l}{z_{ak} z_{jl}} \right) \Bigg\} ,
\end{aligned}
\tag{C.1}
$$

$$\begin{aligned}
\mathcal{B}_2(a,b,i,j,k,l) = -\mathcal{E}\ \Bigg\{ & \frac{(\xi_k k_j)(\xi_l k_i)}{z_{ab}z_{bj}z_{il}z_{jk}}\left(\frac{z_{aj}k_ik_j}{z_{ak}z_{al}z_{ij}} + \frac{k_ik_k}{z_{al}z_{ik}} - \frac{z_{aj}k_jk_l}{z_{ai}z_{ak}z_{jl}} - \frac{k_kk_l}{z_{ai}z_{kl}}\right) \\
& + \frac{(\xi_k k_j)(\xi_l k_b)}{z_{ai}z_{al}z_{bj}z_{bl}z_{jk}}\left(\frac{z_{aj}k_ik_j}{z_{ak}z_{ij}}+\frac{k_ik_k}{z_{ik}}\right) + \frac{(\xi_k k_l)(\xi_l k_i)}{z_{ab}z_{bj}z_{il}z_{kl}}\left(\frac{z_{al}k_jk_l}{z_{ai}z_{ak}z_{jl}} + \frac{k_jk_k}{z_{ai}z_{jk}} - \frac{k_ik_j}{z_{ak}z_{ij}}\right) \\
& + \frac{(\xi_k k_b)(\xi_l k_j)}{z_{ai}z_{ak}z_{bj}z_{bk}z_{jl}}\left(\frac{z_{aj}k_ik_j}{z_{al}z_{ij}} - \frac{k_ik_l}{z_{il}}\right) + \frac{(\xi_k k_i)(\xi_l k_i)}{z_{ab}z_{bj}z_{ik}z_{il}}\left(\frac{z_{ai}k_ik_j}{z_{ak}z_{al}z_{ij}} - \frac{k_jk_k}{z_{al}z_{jk}} - \frac{k_jk_l}{z_{ak}z_{jl}}\right) \\
& + \frac{(\xi_k k_b)(\xi_l k_i)}{z_{ak}z_{bj}z_{bk}z_{il}}\left(\frac{k_ik_j}{z_{al}z_{ij}} - \frac{k_jk_l}{z_{ai}z_{jl}}\right) + \frac{(\xi_k k_j)(\xi_l k_j)}{z_{ab}z_{ai}z_{bj}z_{jk}z_{jl}}\left(\frac{z_{aj}^2k_ik_j}{z_{ak}z_{al}z_{ij}} + \frac{z_{aj}k_ik_k}{z_{al}z_{ik}} + \frac{z_{aj}k_ik_l}{z_{ak}z_{il}}\right) \\
& - \frac{(\xi_k k_l)(\xi_l k_j)}{z_{ab}z_{ai}z_{bj}z_{jl}z_{kl}}\left(\frac{z_{aj}k_ik_j}{z_{ak}z_{ij}} + \frac{k_ik_k}{z_{ik}} + \frac{z_{al}k_ik_l}{z_{ak}z_{il}}\right) + \frac{(\xi_k k_i)(\xi_l k_b)}{z_{al}z_{bj}z_{bl}z_{ik}}\left(\frac{k_ik_j}{z_{ak}z_{ij}} - \frac{k_jk_k}{z_{ai}z_{jk}}\right) \\
& + \frac{(\xi_k k_i)(\xi_l k_j)}{z_{ab}z_{bj}z_{ik}z_{jl}}\left(\frac{z_{aj}k_ik_j}{z_{ak}z_{al}z_{ij}} + \frac{k_ik_l}{z_{ak}z_{il}} - \frac{z_{aj}k_jk_k}{z_{ai}z_{al}z_{jk}} + \frac{k_kk_l}{z_{ai}z_{kl}}\right)\frac{(\xi_k k_b)(\xi_l k_k)}{z_{ai}z_{al}z_{bj}}\frac{k_ik_j}{z_{bk}z_{ij}z_{kl}} \\
& + \frac{(\xi_k k_i)(\xi_l k_k)}{z_{ab}z_{bj}z_{ik}z_{kl}}\left(\frac{k_ik_j}{z_{al}z_{ij}} - \frac{z_{ak}k_jk_k}{z_{ai}z_{al}z_{jk}} - \frac{k_jk_l}{z_{ai}z_{jl}}\right) + \frac{(\xi_k k_b)(\xi_l k_b)}{z_{ai}z_{ak}z_{al}}\frac{z_{ab}k_ik_j}{z_{bj}z_{bk}z_{bl}z_{ij}} \\
& + \frac{(\xi_k k_j)(\xi_l k_k)}{z_{ab}z_{ai}z_{bj}z_{jk}z_{kl}}\left(\frac{z_{aj}k_ik_j}{z_{al}z_{ij}} + \frac{z_{ak}k_ik_k}{z_{al}z_{ik}} + \frac{k_ik_l}{z_{il}}\right) + \frac{(\xi_k k_l)(\xi_l k_b)}{z_{ai}z_{ak}z_{bj}}\frac{k_ik_j}{z_{bl}z_{ij}z_{kl}}\Bigg\}
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
\mathcal{B}_3(a,i,j,k,b,l) = -\mathcal{E}\ \Bigg\{ & \frac{(k_jk_k-1)}{z_{jk}^2}\left(\frac{(\xi_b k_i)(\xi_l k_b)}{z_{ai}z_{al}z_{bi}z_{bl}} + \frac{(\xi_b k_i)(\xi_l k_i)}{z_{ab}z_{al}z_{bi}z_{il}} + \frac{(\xi_b k_l)(\xi_l k_i)}{z_{ab}z_{ai}z_{bl}z_{il}}\right) \\
& + \frac{(\xi_b k_i)(\xi_l k_j)}{z_{ab}z_{ai}z_{jk}z_{jl}}\left(\frac{(z_{aj})(k_jk_k-1)}{z_{al}z_{bi}z_{jk}} + \frac{k_kk_l}{z_{bi}z_{kl}}\right) + \frac{(\xi_b k_k)(\xi_l k_i)}{z_{ab}z_{bk}z_{il}z_{jk}}\left(\frac{k_ik_j}{z_{al}z_{ij}} - \frac{k_jk_l}{z_{ai}z_{jl}}\right) \\
& + \frac{(\xi_b k_j)(\xi_l k_i)}{z_{ab}z_{bj}z_{il}z_{jk}}\left(\frac{k_kk_l}{z_{ai}z_{kl}} - \frac{k_ik_k}{z_{al}z_{ik}}\right) + \frac{(\xi_b k_i)(\xi_l k_k)}{z_{ab}z_{ai}z_{bi}z_{kl}}\left(\frac{(z_{ak})(k_jk_k-1)}{z_{al}z_{jk}} + \frac{k_jk_l}{z_{jl}}\right) \\
& + \frac{k_ik_j}{z_{ai}z_{ij}z_{jk}}\left(\frac{(\xi_b k_k)(\xi_l k_b)}{z_{al}z_{bk}z_{bl}} + \frac{z_{ak}(\xi_b k_k)(\xi_l k_k)}{z_{ab}z_{al}z_{bk}z_{kl}} + \frac{(\xi_b k_l)(\xi_l k_k)}{z_{ab}z_{bl}z_{kl}}\right) \\
& - \frac{k_ik_k}{z_{ai}z_{ik}z_{jk}}\left(\frac{(\xi_b k_j)(\xi_l k_b)}{z_{al}z_{bj}z_{bl}} + \frac{z_{aj}(\xi_b k_j)(\xi_l k_j)}{z_{ab}z_{al}z_{bj}z_{jl}} + \frac{(\xi_b k_l)(\xi_l k_j)}{z_{ab}z_{bl}z_{jl}}\right) \\
& + \frac{(\xi_b k_k)(\xi_l k_j)}{z_{ab}z_{ai}z_{bk}z_{jk}z_{jl}}\left(\frac{z_{aj}k_ik_j}{z_{al}z_{ij}} + \frac{k_ik_l}{z_{il}}\right) - \frac{(\xi_b k_j)(\xi_l k_k)}{z_{ab}z_{ai}z_{bj}z_{jk}z_{kl}}\left(\frac{z_{ak}k_ik_k}{z_{al}z_{ik}} + \frac{k_ik_l}{z_{il}}\right)\Bigg\}\ ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_4(i,j,k,l,a,b) = -\frac{\mathcal{E}}{z_{ab}\ z_{ij}\ z_{kl}}\ \Bigg\{ & \frac{(1-k_kk_l)}{z_{kl}}\left(\frac{(\xi_a k_i)\ (\xi_b k_j)}{z_{ai}\ z_{bj}} - \frac{(\xi_a k_j)\ (\xi_b k_i)}{z_{aj}\ z_{bi}}\right) \\
& + \frac{k_ik_k}{z_{ik}}\left(\frac{(\xi_a k_j)(\xi_b k_l)}{z_{aj}z_{bl}} - \frac{(\xi_a k_l)(\xi_b k_j)}{z_{al}z_{bj}}\right) + \frac{k_ik_l}{z_{il}}\left(\frac{(\xi_a k_k)(\xi_b k_j)}{z_{ak}z_{bj}} - \frac{(\xi_a k_j)(\xi_b k_k)}{z_{aj}z_{bk}}\right) \\
& + \frac{k_jk_k}{z_{jk}}\left(\frac{(\xi_a k_l)(\xi_b k_i)}{z_{al}z_{bi}} - \frac{(\xi_a k_i)(\xi_b k_l)}{z_{ai}z_{bl}}\right) + \frac{k_jk_l}{z_{jl}}\left(\frac{(\xi_a k_i)(\xi_b k_k)}{z_{ai}z_{bk}} - \frac{(\xi_a k_k)(\xi_b k_i)}{z_{ak}z_{bi}}\right) \\
& + \frac{(1-k_ik_j)}{z_{ij}}\left(\frac{(\xi_a k_k)(\xi_b k_l)}{z_{ak}z_{bl}} - \frac{(\xi_a k_l)(\xi_b k_l)}{z_{al}z_{bk}}\right)\Bigg\}.
\end{aligned} \tag{C.3}$$

Further, the three functions $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are given

$$
\begin{aligned}
\mathcal{C}_1(a,b,i,j,k,l) = &-\mathcal{E}\,\frac{(\xi_j k_b)(\xi_k k_b)(\xi_l k_b)z_{ab}}{z_{ai}z_{bj}z_{bk}z_{bl}}\left(\frac{(\xi_i k_b)z_{ab}}{z_{aj}z_{ak}z_{al}z_{bi}}-\frac{(\xi_i k_j)}{z_{ak}z_{al}z_{ij}}-\frac{(\xi_i k_k)}{z_{aj}z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{aj}z_{ak}z_{il}}\right) \\
&+\frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_b)}{z_{aj}z_{bk}z_{bl}z_{ij}}\left(\frac{(\xi_i k_b)z_{ab}}{z_{ak}z_{al}z_{bi}}-\frac{(\xi_i k_k)}{z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{ak}z_{il}}\right)+\frac{(\xi_j k_b)(\xi_k k_i)(\xi_l k_b)}{z_{ak}z_{bj}z_{bl}z_{ik}}\left(\frac{(\xi_i k_b)z_{ab}}{z_{aj}z_{al}z_{bi}}-\frac{(\xi_i k_j)}{z_{al}z_{ij}}-\frac{(\xi_i k_l)}{z_{aj}z_{il}}\right) \\
&+\frac{(\xi_j k_k)(\xi_k k_b)(\xi_l k_b)}{z_{ai}z_{bk}z_{bl}z_{jk}}\left(\frac{(\xi_i k_j)}{z_{al}z_{ij}}-\frac{(\xi_i k_b)z_{ab}}{z_{aj}z_{al}z_{bi}}+\frac{(\xi_i k_k)z_{ak}}{z_{aj}z_{al}z_{ik}}+\frac{(\xi_i k_l)}{z_{aj}z_{il}}\right)+\frac{(\xi_j k_i)(\xi_k k_i)(\xi_l k_b)z_{ai}}{z_{aj}z_{ak}z_{bl}z_{ij}z_{ik}}\left(\frac{(\xi_i k_b)}{z_{al}z_{bi}}-\frac{(\xi_i k_l)}{z_{ab}z_{il}}\right) \\
&+\frac{(\xi_j k_l)(\xi_k k_b)(\xi_l k_b)}{z_{ai}z_{bk}z_{bl}z_{jl}}\left(\frac{(\xi_i k_j)}{z_{ak}z_{ij}}-\frac{(\xi_i k_b)z_{ab}}{z_{aj}z_{ak}z_{bi}}+\frac{(\xi_i k_k)}{z_{aj}z_{ik}}+\frac{(\xi_i k_l)z_{al}}{z_{aj}z_{ak}}\right)+\frac{(\xi_j k_k)(\xi_k k_i)(\xi_l k_b)}{z_{aj}z_{bl}z_{ik}z_{jk}}\left(\frac{(\xi_i k_l)}{z_{ab}z_{il}}-\frac{(\xi_i k_b)}{z_{al}z_{bi}}\right) \\
&+\frac{(\xi_j k_l)(\xi_k k_i)(\xi_l k_b)}{z_{ak}z_{bl}z_{ik}z_{jl}}\left(\frac{(\xi_i k_j)}{z_{ab}z_{ij}}-\frac{(\xi_i k_b)}{z_{aj}z_{bi}}+\frac{(\xi_i k_l)z_{al}}{z_{ab}z_{aj}z_{il}}\right)+\frac{(\xi_j k_b)(\xi_k k_j)(\xi_l k_b)}{z_{ai}z_{bj}z_{bl}z_{jk}}\left(\frac{(\xi_i k_b)z_{ab}}{z_{ak}z_{al}z_{bi}}-\frac{(\xi_i k_j)z_{aj}}{z_{ak}z_{al}z_{ij}}-\frac{(\xi_i k_k)}{z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{ak}z_{il}}\right) \\
&+\frac{(\xi_j k_l)(\xi_k k_j)(\xi_l k_b)}{z_{ai}z_{bl}z_{jk}z_{jl}}\left(\frac{(\xi_i k_j)z_{aj}}{z_{ab}z_{ak}z_{ij}}-\frac{(\xi_i k_b)}{z_{ak}z_{bi}}+\frac{(\xi_i k_k)}{z_{ab}z_{ik}}+\frac{(\xi_i k_l)z_{al}}{z_{ab}z_{ak}z_{il}}\right)+\frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_i)z_{ai}}{z_{aj}z_{al}z_{bk}z_{ij}z_{il}}\left(\frac{(\xi_i k_b)}{z_{ak}z_{bi}}-\frac{(\xi_i k_k)}{z_{ab}z_{ik}}\right) \\
&+\frac{(\xi_j k_b)(\xi_k k_l)(\xi_l k_b)}{z_{ai}z_{bj}z_{bl}z_{kl}}\left(\frac{(\xi_i k_j)}{z_{ak}z_{ij}}-\frac{(\xi_i k_b)z_{ab}}{z_{aj}z_{ak}z_{bi}}+\frac{(\xi_i k_k)}{z_{aj}z_{ik}}+\frac{(\xi_i k_l)z_{al}}{z_{aj}z_{ak}z_{il}}\right)+\frac{(\xi_j k_i)(\xi_k k_l)(\xi_l k_b)}{z_{aj}z_{bl}z_{ij}z_{kl}}\left(\frac{(\xi_i k_k)}{z_{ab}z_{ik}}-\frac{(\xi_i k_b)}{z_{ak}z_{bi}}+\frac{(\xi_i k_l)z_{al}}{z_{ab}z_{ak}z_{il}}\right) \\
&+\frac{(\xi_j k_k)(\xi_k k_l)(\xi_l k_b)}{z_{ai}z_{bl}z_{jk}z_{kl}}\left(\frac{(\xi_i k_b)}{z_{aj}z_{bi}}-\frac{(\xi_i k_j)}{z_{ab}z_{ij}}-\frac{(\xi_i k_k)z_{ak}}{z_{ab}z_{aj}z_{ik}}-\frac{(\xi_i k_l)z_{al}}{z_{ab}z_{aj}z_{il}}\right)+\frac{(\xi_j k_b)(\xi_k k_b)(\xi_l k_i)}{z_{al}z_{bj}z_{bk}z_{il}}\left(\frac{(\xi_i k_b)}{z_{aj}z_{ak}z_{bi}}-\frac{(\xi_i k_j)}{z_{ak}z_{ij}}-\frac{(\xi_i k_k)}{z_{aj}z_{ik}}\right) \\
&+\frac{(\xi_j k_l)(\xi_k k_l)(\xi_l k_b)z_{al}}{z_{ai}z_{bl}z_{jl}z_{kl}}\left(\frac{(\xi_i k_b)}{z_{aj}z_{ak}z_{bi}}-\frac{(\xi_i k_j)}{z_{ab}z_{ak}z_{ij}}-\frac{(\xi_i k_k)}{z_{ab}z_{aj}z_{ik}}-\frac{(\xi_i k_l)z_{al}}{z_{ab}z_{aj}z_{ak}z_{il}}\right)+\frac{(\xi_j k_i)(\xi_k k_j)(\xi_l k_b)}{z_{ak}z_{bl}z_{ij}z_{jk}}\left(\frac{(\xi_i k_b)}{z_{al}z_{bi}}-\frac{(\xi_i k_l)}{z_{ab}z_{il}}\right) \\
&+\frac{(\xi_j k_k)(\xi_k k_b)(\xi_l k_i)}{z_{al}z_{bk}z_{il}z_{jk}}\left(\frac{(\xi_i k_j)}{z_{ab}z_{ij}}-\frac{(\xi_i k_b)}{z_{aj}z_{bi}}+\frac{(\xi_i k_k)z_{ak}}{z_{ab}z_{aj}z_{ik}}\right)+\frac{(\xi_j k_l)(\xi_k k_b)(\xi_l k_i)}{z_{aj}z_{bk}z_{il}z_{jl}}\left(\frac{(\xi_i k_k)}{z_{ab}z_{ik}}-\frac{(\xi_i k_b)}{z_{ak}z_{bi}}\right) \\
&+\frac{(\xi_j k_b)(\xi_k k_i)(\xi_l k_i)z_{ai}}{z_{ak}z_{al}z_{bj}z_{ik}z_{il}}\left(\frac{(\xi_i k_b)}{z_{aj}z_{bi}}-\frac{(\xi_i k_j)}{z_{ab}z_{ij}}\right)+\frac{(\xi_i k_b)(\xi_j k_i)(\xi_k k_i)(\xi_l k_i)z_{ai}^2}{z_{ab}z_{aj}z_{ak}z_{al}z_{bi}z_{ij}z_{ik}z_{il}}-\frac{(\xi_k k_i)(\xi_l k_i)z_{ai}}{z_{ab}z_{aj}z_{bi}z_{ik}z_{il}}\left(\frac{(\xi_i k_b)(\xi_j k_k)}{z_{al}z_{jk}}+\frac{(\xi_i k_b)(\xi_j k_l)}{z_{ak}z_{jl}}\right) \\
&+\frac{(\xi_j k_b)(\xi_k k_j)(\xi_l k_i)}{z_{al}z_{bj}z_{il}z_{jk}}\left(\frac{(\xi_i k_b)}{z_{al}z_{bi}}-\frac{(\xi_i k_j)z_{aj}}{z_{ab}z_{ak}z_{ij}}-\frac{(\xi_i k_k)}{z_{ab}z_{ik}}\right)+\frac{(\xi_k k_j)(\xi_l k_i)}{z_{ab}z_{ak}z_{bi}z_{il}z_{jk}}\left(\frac{(\xi_i k_b)(\xi_j k_i)z_{ai}}{z_{al}z_{ij}}-\frac{(\xi_i k_b)(\xi_j k_l)}{z_{jl}}\right) \\
&+\frac{(\xi_j k_b)(\xi_k k_l)(\xi_l k_i)}{z_{ak}z_{bj}z_{il}z_{kl}}\left(\frac{(\xi_i k_j)}{z_{ab}z_{ij}}-\frac{(\xi_i k_b)}{z_{aj}z_{bi}}\right)+\frac{(\xi_k k_l)(\xi_l k_i)}{z_{ab}z_{aj}z_{bi}z_{il}z_{kl}}\left(\frac{(\xi_i k_b)(\xi_j k_k)}{z_{jk}}-\frac{(\xi_i k_b)(\xi_j k_i)z_{ai}}{z_{ak}z_{ij}}+\frac{(\xi_i k_b)(\xi_j k_l)z_{al}}{z_{ak}z_{jl}}\right) \\
&+\frac{(\xi_j k_b)(\xi_k k_b)(\xi_l k_j)}{z_{aj}z_{bj}z_{bk}z_{jl}}\left(\frac{(\xi_i k_b)z_{ab}}{z_{ak}z_{al}z_{bi}}-\frac{(\xi_i k_j)z_{aj}}{z_{ak}z_{al}}-\frac{(\xi_i k_k)}{z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{ak}z_{il}}\right)+\frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_j)}{z_{al}z_{bk}z_{ij}z_{jl}}\left(\frac{(\xi_i k_b)}{z_{ak}z_{bi}}-\frac{(\xi_i k_k)}{z_{ab}z_{ik}}\right) \\
&+\frac{(\xi_j k_k)(\xi_k k_b)(\xi_l k_j)}{z_{ai}z_{bk}z_{jk}z_{jl}}\left(\frac{(\xi_i k_j)z_{aj}}{z_{ab}z_{al}z_{ij}}-\frac{(\xi_i k_b)}{z_{al}z_{bi}}+\frac{(\xi_i k_k)z_{ak}}{z_{ab}z_{al}z_{ik}}+\frac{(\xi_i k_l)}{z_{ab}z_{il}}\right)+\frac{(\xi_j k_b)(\xi_k k_i)(\xi_l k_j)}{z_{ak}z_{bj}z_{ik}z_{jl}}\left(\frac{(\xi_i k_b)}{z_{al}z_{bi}}-\frac{(\xi_i k_j)z_{aj}}{z_{ab}z_{al}z_{ij}}-\frac{(\xi_i k_l)}{z_{ab}z_{il}}\right) \\
&+\frac{(\xi_k k_i)(\xi_l k_j)}{z_{ab}z_{al}z_{bi}z_{ik}z_{jl}}\left(\frac{(\xi_i k_b)(\xi_j k_i)z_{ai}}{z_{ak}z_{ij}}-\frac{(\xi_i k_b)(\xi_j k_k)}{z_{jk}}\right)+\frac{(\xi_j k_b)(\xi_k k_l)(\xi_l k_j)}{z_{ai}z_{bj}z_{jl}z_{kl}}\left(\frac{(\xi_i k_j)z_{aj}}{z_{ab}z_{ak}z_{ij}}-\frac{(\xi_i k_b)}{z_{ak}z_{bi}}+\frac{(\xi_i k_k)}{z_{ab}z_{ik}}+\frac{(\xi_i k_l)z_{al}}{z_{ab}z_{ak}z_{il}}\right) \\
&+\frac{(\xi_j k_b)(\xi_k k_j)(\xi_l k_j)z_{aj}}{z_{ai}z_{bj}z_{jk}z_{jl}}\left(\frac{(\xi_i k_b)}{z_{ak}z_{al}z_{bi}}-\frac{(\xi_i k_j)z_{aj}}{z_{ab}z_{ak}z_{al}z_{ij}}-\frac{(\xi_i k_k)}{z_{ab}z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{ab}z_{ak}z_{il}}\right)+\frac{(\xi_i k_b)(\xi_j k_i)(\xi_k k_j)(\xi_l k_j)z_{aj}}{z_{ab}z_{ak}z_{al}z_{bi}z_{ij}z_{jk}z_{jl}} \\
&-\frac{(\xi_i k_b)(\xi_j k_i)(\xi_k k_l)(\xi_l k_j)}{z_{ab}z_{ak}z_{bi}z_{ij}z_{jl}z_{kl}}+\frac{(\xi_j k_b)(\xi_k k_b)(\xi_l k_k)}{z_{ai}z_{bj}z_{bk}z_{kl}}\left(\frac{(\xi_i k_b)z_{ab}}{z_{aj}z_{al}z_{bi}}-\frac{(\xi_i k_j)}{z_{al}z_{ij}}-\frac{(\xi_i k_k)z_{ak}}{z_{aj}z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{aj}z_{il}}\right) \\
&+\frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_k)}{z_{aj}z_{bk}z_{ij}z_{kl}}\left(\frac{(\xi_i k_b)}{z_{al}z_{bi}}-\frac{(\xi_i k_k)z_{ak}}{z_{ab}z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{ab}z_{il}}\right)+\frac{(\xi_j k_b)(\xi_k k_i)(\xi_l k_k)}{z_{al}z_{bj}z_{ik}z_{kl}}\left(\frac{(\xi_i k_b)}{z_{aj}z_{bi}}-\frac{(\xi_i k_j)}{z_{ab}z_{ij}}\right) \\
&+\frac{(\xi_j k_k)(\xi_k k_b)(\xi_l k_k)z_{ak}}{z_{ai}z_{bk}z_{ik}z_{kl}}\left(\frac{(\xi_i k_j)}{z_{ab}z_{al}z_{ij}}-\frac{(\xi_i k_b)}{z_{aj}z_{al}z_{bi}}+\frac{(\xi_i k_k)z_{ak}}{z_{ab}z_{aj}z_{al}z_{ik}}+\frac{(\xi_i k_l)}{z_{ab}z_{aj}z_{il}}\right)-\frac{(\xi_i k_b)(\xi_j k_l)(\xi_k k_i)(\xi_l k_j)}{z_{ab}z_{aj}z_{bi}z_{ik}z_{jl}z_{kl}} \\
&+\frac{(\xi_j k_l)(\xi_k k_b)(\xi_l k_k)}{z_{ai}z_{bk}z_{jl}z_{kl}}\left(\frac{(\xi_i k_j)}{z_{ab}z_{ij}}-\frac{(\xi_i k_b)}{z_{aj}z_{bi}}+\frac{(\xi_i k_k)z_{ak}}{z_{ab}z_{aj}z_{ik}}+\frac{(\xi_i k_l)z_{al}}{z_{ab}z_{aj}z_{il}}\right)+\frac{(\xi_i k_b)(\xi_k k_i)(\xi_l k_k)}{z_{ab}z_{aj}z_{al}z_{bi}z_{ik}z_{kl}}\left(\frac{(\xi_j k_i)z_{ai}}{z_{ij}}-\frac{(\xi_j k_k)z_{ak}}{z_{jk}}\right) \\
&+\frac{(\xi_j k_b)(\xi_k k_j)(\xi_l k_k)}{z_{ai}z_{bj}z_{jk}z_{kl}}\left(\frac{(\xi_i k_b)}{z_{al}z_{bi}}-\frac{(\xi_i k_j)z_{aj}}{z_{ab}z_{al}z_{ij}}-\frac{(\xi_i k_k)z_{ak}}{z_{ab}z_{al}z_{ik}}-\frac{(\xi_i k_l)}{z_{ab}z_{il}}\right)+\frac{(\xi_i k_b)(\xi_j k_i)(\xi_k k_j)(\xi_l k_k)}{z_{ab}z_{al}z_{bi}z_{ij}z_{jk}z_{kl}}
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
\mathcal{C}_2(a,i,b,j,k,l) = -\mathcal{E}\, \Bigg\{ & \frac{(\xi_k k_b)(\xi_l k_b) z_{ab}}{z_{ai} z_{aj} z_{bi} z_{bk} z_{bl}} \left(\frac{(\xi_b k_i)(\xi_j k_l)}{z_{ak} z_{jl}} + \frac{(\xi_b k_i)(\xi_j k_k)}{z_{al} z_{jk}} - \frac{(\xi_b k_i)(\xi_j k_b) z_{ab}}{z_{ak} z_{al} z_{bj}} \right) \\
& + \frac{(\xi_j k_l)(\xi_k k_i)(\xi_l k_b)}{z_{aj} z_{bl} z_{ik} z_{jl}} \left(\frac{(\xi_b k_i)}{z_{ak} z_{bi}} + \frac{(\xi_b k_k)}{z_{ai} z_{bk}} \right) + \frac{(\xi_b k_i)(\xi_k k_l)(\xi_l k_b)}{z_{ai} z_{aj} z_{bi} z_{bl} z_{kl}} \left(\frac{(\xi_j k_b) z_{ab}}{z_{ak} z_{bj}} - \frac{(\xi_j k_k)}{z_{jk}} - \frac{(\xi_j k_l)}{z_{ak} z_{jl}} \right) \\
& - \frac{(\xi_k k_i)(\xi_l k_b)}{z_{al} z_{bl} z_{ik}} \left(\frac{(\xi_b k_i)(\xi_j k_b) z_{ab}}{z_{aj} z_{ak} z_{bi} z_{bj}} + \frac{(\xi_b k_k)(\xi_j k_b) z_{ab}}{z_{ai} z_{aj} z_{bj} z_{bk}} + \frac{(\xi_b k_i)(\xi_j k_i) z_{ai}}{z_{aj} z_{ak} z_{bi} z_{ij}} + \frac{(\xi_b k_j)(\xi_j k_i)}{z_{bj} z_{ij}} + \frac{(\xi_b k_k)(\xi_j k_i)}{z_{aj} z_{bk} z_{ij}} \right) \\
& + \frac{(\xi_j k_k)(\xi_k k_i)(\xi_l k_b)}{z_{bl} z_{ik} z_{jk}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{al} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{al} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{ai} z_{aj} z_{al} z_{bk}} \right) + \frac{(\xi_j k_i)(\xi_k k_l)(\xi_l k_b)}{z_{ak} z_{bl} z_{ij} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{bj}} \right) \\
& - \frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_b) z_{ab}}{z_{ak} z_{al} z_{bk} z_{bl} z_{ij}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{bj}} \right) - \frac{(\xi_j k_i)(\xi_k k_j)(\xi_l k_b)}{z_{al} z_{bl} z_{ij} z_{jk}} \left(\frac{(\xi_b k_i)}{z_{ak} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ai} z_{ak} z_{bj}} + \frac{(\xi_b k_k)}{z_{ai} z_{bk}} \right) \\
& - \frac{(\xi_j k_b)(\xi_k k_b)(\xi_l k_i) z_{ab}}{z_{aj} z_{ak} z_{bj} z_{bk} z_{il}} \left(\frac{(\xi_b k_i)}{z_{al} z_{bi}} + \frac{(\xi_b k_l)}{z_{ai} z_{bl}} \right) + \frac{(\xi_j k_b)(\xi_k k_l)(\xi_l k_i)}{z_{aj} z_{bj} z_{il} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{ak} z_{bi}} + \frac{(\xi_b k_k)}{z_{ai} z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{ai} z_{ak} z_{bl}} \right) \\
& - \frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_i)}{z_{ak} z_{bk} z_{ij} z_{il}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{aj} z_{al} z_{bi}} + \frac{(\xi_b k_j)}{z_{al} z_{bj} z_{il}} + \frac{(\xi_b k_l)}{z_{aj} z_{bl} z_{il}} \right) + \frac{(\xi_j k_k)(\xi_k k_b)(\xi_l k_i)}{z_{aj} z_{bk} z_{il} z_{jk}} \left(\frac{(\xi_b k_i)}{z_{al} z_{bi}} + \frac{(\xi_b k_l)}{z_{ai} z_{bl}} \right) \\
& + \frac{(\xi_j k_l)(\xi_k k_b)(\xi_l k_i)}{z_{ak} z_{bk} z_{il} z_{jl}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{bj}} + \frac{(\xi_b k_l) z_{al}}{z_{ai} z_{aj} z_{bl}} \right) - \frac{(\xi_j k_b)(\xi_k k_j)(\xi_l k_i)}{z_{ak} z_{bj} z_{il} z_{jk}} \left(\frac{(\xi_b k_i)}{z_{al} z_{bi}} + \frac{(\xi_b k_l)}{z_{ai} z_{bl}} \right) \\
& - \frac{(\xi_j k_b)(\xi_k k_i)(\xi_l k_i)}{z_{aj} z_{bj} z_{ik} z_{il}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{ak} z_{al} z_{bi}} + \frac{(\xi_b k_k)}{z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{ak} z_{bl}} \right) + \frac{(\xi_b k_i)(\xi_k k_j)(\xi_l k_b)}{z_{ai} z_{ak} z_{bi} z_{bl} z_{jk}} \left(\frac{(\xi_j k_l)}{z_{jl}} - \frac{(\xi_j k_b) z_{ab}}{z_{al} z_{bj}} \right) \\
& - \frac{(\xi_j k_i)(\xi_k k_i)(\xi_l k_i) z_{ai}}{z_{ab} z_{ij} z_{ik} z_{il}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{aj} z_{ak} z_{al} z_{bi}} + \frac{(\xi_b k_j)}{z_{ak} z_{al} z_{bj}} + \frac{(\xi_b k_k)}{z_{aj} z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{aj} z_{ak} z_{bl}} \right) \\
& + \frac{(\xi_j k_k)(\xi_k k_i)(\xi_l k_i)}{z_{ab} z_{ik} z_{il} z_{jk}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{aj} z_{al} z_{bi}} + \frac{(\xi_b k_j)}{z_{al} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{aj} z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{aj} z_{bl}} \right) + \frac{(\xi_b k_i)(\xi_j k_k)(\xi_k k_b)(\xi_l k_j)}{z_{ai} z_{al} z_{bi} z_{bk} z_{jk} z_{jl}} \\
& + \frac{(\xi_j k_l)(\xi_k k_i)(\xi_l k_i)}{z_{aj} z_{ik} z_{il} z_{jl}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{aj} z_{ak} z_{bi}} + \frac{(\xi_b k_j)}{z_{ak} z_{bj}} + \frac{(\xi_b k_k)}{z_{aj} z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{aj} z_{ak} z_{bl}} \right) - \frac{(\xi_b k_i)(\xi_j k_b)(\xi_k k_b)(\xi_l k_j) z_{ab}}{z_{ai} z_{ak} z_{al} z_{bi} z_{bj} z_{bk} z_{jl}} \\
& - \frac{(\xi_j k_i)(\xi_k k_j)(\xi_l k_i)}{z_{ab} z_{ij} z_{il} z_{jk}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{ak} z_{al} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ak} z_{al} z_{bj}} + \frac{(\xi_b k_k)}{z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{ak} z_{bl}} \right) - \frac{(\xi_b k_i)(\xi_j k_b)(\xi_k k_j)(\xi_l k_j) z_{aj}}{z_{ai} z_{ak} z_{al} z_{bi} z_{bj} z_{jk} z_{jl}} \\
& + \frac{(\xi_j k_l)(\xi_k k_j)(\xi_l k_i)}{z_{ab} z_{il} z_{jk} z_{jl}} \left(\frac{(\xi_b k_i)}{z_{ak} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ai} z_{ak} z_{bj}} + \frac{(\xi_b k_k)}{z_{ai} z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{ai} z_{ak} z_{bl}} \right) + \frac{(\xi_b k_i)(\xi_j k_b)(\xi_k k_l)(\xi_l k_j)}{z_{ai} z_{ak} z_{bi} z_{bj} z_{jl} z_{kl}} \\
& + \frac{(\xi_j k_i)(\xi_k k_l)(\xi_l k_i)}{z_{ab} z_{ij} z_{il} z_{kl}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{aj} z_{ak} z_{bi}} + \frac{(\xi_b k_j)}{z_{ak} z_{bj}} + \frac{(\xi_b k_k)}{z_{aj} z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{aj} z_{ak} z_{bl}} \right) - \frac{(\xi_b k_i)(\xi_j k_b)(\xi_k k_b)(\xi_l k_k) z_{ab}}{z_{ai} z_{aj} z_{al} z_{bi} z_{bj} z_{bk} z_{kl}} \\
& - \frac{(\xi_j k_k)(\xi_k k_l)(\xi_l k_i)}{z_{ab} z_{il} z_{jk} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{ai} z_{aj} z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{ai} z_{aj} z_{bl}} \right) + \frac{(\xi_b k_i)(\xi_j k_k)(\xi_k k_b)(\xi_l k_k) z_{ak}}{z_{ai} z_{aj} z_{al} z_{bi} z_{bk} z_{jk} z_{kl}} \\
& - \frac{(\xi_j k_l)(\xi_k k_l)(\xi_l k_i) z_{al}}{z_{ab} z_{il} z_{jl} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{ak} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{ak} z_{bj}} + \frac{(\xi_b k_k)}{z_{ai} z_{aj} z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{ai} z_{aj} z_{ak} z_{bl}} \right) \\
& - \frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_j)}{z_{ak} z_{bk} z_{ij} z_{jl}} \left(\frac{(\xi_b k_i)}{z_{al} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ai} z_{al} z_{bj}} + \frac{(\xi_b k_l)}{z_{ai} z_{bl}} \right) - \frac{(\xi_j k_b)(\xi_k k_i)(\xi_l k_j)}{z_{al} z_{bj} z_{ik} z_{jl}} \left(\frac{(\xi_b k_i)}{z_{ak} z_{bi}} + \frac{(\xi_b k_k)}{z_{ai} z_{bk}} \right) \\
& - \frac{(\xi_j k_i)(\xi_k k_i)(\xi_l k_j)}{z_{ab} z_{ij} z_{ik} z_{jl}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{ak} z_{al} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ak} z_{al} z_{bj}} + \frac{(\xi_b k_k)}{z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{ak} z_{bl}} \right) + \frac{(\xi_b k_i)(\xi_j k_l)(\xi_k k_b)(\xi_l k_k)}{z_{ai} z_{aj} z_{bi} z_{bk} z_{jl} z_{kl}} \\
& + \frac{(\xi_j k_k)(\xi_k k_i)(\xi_l k_j)}{z_{ab} z_{ik} z_{jk} z_{jl}} \left(\frac{(\xi_b k_i)}{z_{al} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ai} z_{al} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{ai} z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{ai} z_{bl}} \right) - \frac{(\xi_b k_i)(\xi_j k_i)(\xi_k k_j)(\xi_l k_k)}{z_{ab} z_{al} z_{bi} z_{ij} z_{jk} z_{kl}} \\
& - \frac{(\xi_j k_i)(\xi_k k_j)(\xi_l k_j) z_{aj}}{z_{ab} z_{ij} z_{jk} z_{jl}} \left(\frac{(\xi_b k_i)}{z_{ak} z_{al} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ai} z_{ak} z_{al} z_{bj}} + \frac{(\xi_b k_k)}{z_{ai} z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{ai} z_{ak} z_{bl}} \right) \\
& + \frac{(\xi_j k_i)(\xi_k k_l)(\xi_l k_j)}{z_{ab} z_{ij} z_{jl} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{ak} z_{bi}} + \frac{(\xi_b k_j) z_{aj}}{z_{ai} z_{ak} z_{bj}} + \frac{(\xi_b k_k)}{z_{ai} z_{bk}} + \frac{(\xi_b k_l)}{z_{ai} z_{ak} z_{bl}} \right) - \frac{(\xi_b k_j)(\xi_j k_i)(\xi_k k_j)(\xi_l k_k)}{z_{ab} z_{ai} z_{al} z_{bj} z_{ij} z_{jk} z_{kl}} \\
& - \frac{(\xi_j k_i)(\xi_k k_b)(\xi_l k_k)}{z_{al} z_{bk} z_{ij} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{bj}} \right) - \frac{(\xi_b k_i)(\xi_j k_b)(\xi_k k_j)(\xi_l k_k)}{z_{ai} z_{al} z_{bi} z_{bj} z_{jk} z_{kl}} \\
& - \frac{(\xi_j k_b)(\xi_k k_i)(\xi_l k_k)}{z_{aj} z_{bj} z_{ik} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{al} z_{bi}} + \frac{(\xi_b k_k) z_{ak}}{z_{ai} z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{ai} z_{bl}} \right) - \frac{(\xi_b k_k)(\xi_j k_i)(\xi_k k_j)(\xi_l k_k)}{z_{ab} z_{ai} z_{al} z_{bk} z_{ij} z_{jk} z_{kl}} \\
& - \frac{(\xi_j k_i)(\xi_k k_i)(\xi_l k_k)}{z_{ab} z_{ij} z_{ik} z_{kl}} \left(\frac{(\xi_b k_i) z_{ai}}{z_{aj} z_{al} z_{bi}} + \frac{(\xi_b k_j)}{z_{al} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{aj} z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{aj} z_{bl}} \right) - \frac{(\xi_b k_l)(\xi_j k_i)(\xi_k k_j)(\xi_l k_k)}{z_{ab} z_{ai} z_{bl} z_{ij} z_{jk} z_{kl}} \\
& + \frac{(\xi_j k_k)(\xi_k k_i)(\xi_l k_k) z_{ak}}{z_{ab} z_{ik} z_{jk} z_{kl}} \left(\frac{(\xi_b k_i) z_{aj}}{z_{al} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{al} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{ai} z_{aj} z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{ai} z_{aj} z_{bl}} \right) \\
& + \frac{(\xi_j k_l)(\xi_k k_i)(\xi_l k_k)}{z_{ab} z_{ik} z_{jl} z_{kl}} \left(\frac{(\xi_b k_i)}{z_{aj} z_{bi}} + \frac{(\xi_b k_j)}{z_{ai} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{ai} z_{aj} z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{ai} z_{aj} z_{bl}} \right) ,
\end{aligned}
\qquad (C.5)
$$

$$
\begin{aligned}
\mathcal{C}_3(i,j,a,b,k,l) = -\frac{\mathcal{E}}{z_{ij}} \Bigg\{ & \frac{(\xi_a k_i)(\xi_b k_j)(\xi_k k_b)(\xi_l k_b) z_{ab}}{z_{ai} z_{ak} z_{al} z_{bj} z_{bk} z_{bl}} + \frac{(\xi_a k_j)(\xi_b k_i)(\xi_k k_b)(\xi_l k_b) z_{ab}}{z_{aj} z_{ak} z_{al} z_{bi} z_{bk} z_{bl}} \\
& - \frac{(\xi_a k_j)(\xi_b k_i)(\xi_k k_i)(\xi_l k_b) z_{ai}}{z_{aj} z_{ak} z_{al} z_{bi} z_{bl} z_{ik}} - \frac{(\xi_a k_j)(\xi_b k_k)(\xi_k k_i)(\xi_l k_b)}{z_{aj} z_{al} z_{bk} z_{bl} z_{ik}} + \frac{(\xi_a k_i + \xi_a k_k)(\xi_b k_j)(\xi_k k_i)(\xi_l k_b)}{z_{al} z_{bj} z_{bl} z_{ik}} \\
& - \frac{(\xi_a k_j + \xi_a k_k)(\xi_b k_i)(\xi_k k_j)(\xi_l k_b)}{z_{ak} z_{al} z_{bi} z_{bl} z_{jk}} + \frac{(\xi_a k_i)(\xi_b k_j)(\xi_k k_j)(\xi_l k_b) z_{aj}}{z_{ai} z_{ak} z_{al} z_{bj} z_{bl} z_{jk}} + \frac{(\xi_a k_i)(\xi_b k_k)(\xi_k k_j)(\xi_l k_b)}{z_{ai} z_{al} z_{bk} z_{bl} z_{jk}} \\
& + \frac{(\xi_a k_j)(\xi_b k_i)(\xi_k k_l)(\xi_l k_b)}{z_{aj} z_{ak} z_{bi} z_{bl} z_{kl}} + \frac{(\xi_a k_i)(\xi_b k_j)(\xi_k k_l)(\xi_l k_b)}{z_{ai} z_{ak} z_{bj} z_{bl} z_{kl}} + \frac{(\xi_a k_i + \xi_a k_k)(\xi_b k_j)(\xi_k k_b)(\xi_l k_i)}{z_{ak} z_{al} z_{bj} z_{bk} z_{il}} \\
& - \frac{(\xi_a k_j)(\xi_b k_l)(\xi_k k_b)(\xi_l k_i)}{z_{aj} z_{ak} z_{bk} z_{bl} z_{il}} + \frac{(\xi_k k_i)(\xi_l k_i) z_{ai}}{z_{ab} z_{ak} z_{al} z_{ik} z_{il}} \left(\frac{1}{z_{bj}} - \frac{z_{ai}}{z_{aj} z_{bi}} \right) - \frac{(\xi_a k_j)(\xi_b k_i)(\xi_k k_b)(\xi_l k_i) z_{ai}}{z_{aj} z_{ak} z_{al} z_{bi} z_{bk} z_{il}} \\
& + \frac{(\xi_a k_k + \xi_a k_l)(\xi_b k_j)(\xi_k k_i)(\xi_l k_i) z_{ai}}{z_{ab} z_{ak} z_{al} z_{bj} z_{ik} z_{il}} - \frac{(\xi_a k_j)(\xi_k k_i)(\xi_l k_i) z_{ai}}{z_{ab} z_{aj} z_{ik} z_{il}} \left(\frac{\xi_b k_k}{z_{al} z_{bk}} + \frac{\xi_b k_l}{z_{ak} z_{bl}} \right) \\
& - \frac{(\xi_a k_j + \xi_a k_k)(\xi_b k_i)(\xi_k k_j)(\xi_l k_i) z_{ai}}{z_{ab} z_{ak} z_{al} z_{bi} z_{il} z_{jk}} + \frac{(\xi_a k_i + \xi_a k_l)(\xi_b k_j)(\xi_k k_j)(\xi_l k_i) z_{aj}}{z_{ab} z_{ak} z_{al} z_{bj} z_{il} z_{jk}} \\
& + \frac{(\xi_a k_i + \xi_a k_l)(\xi_b k_k)(\xi_k k_j)(\xi_l k_i)}{z_{ab} z_{al} z_{bk} z_{il} z_{jk}} - \frac{(\xi_a k_j + \xi_a k_k)(\xi_b k_l)(\xi_k k_j)(\xi_l k_i)}{z_{ab} z_{ak} z_{bl} z_{il} z_{jk}} - \frac{(\xi_a k_j)(\xi_b k_i)(\xi_k k_i)(\xi_l k_k) z_{ai}}{zab z_{aj} z_{al} z_{bi} z_{ik} z_{kl}} \\
& + \frac{(\xi_a k_j)(\xi_b k_i)(\xi_k k_l)(\xi_l k_i) z_{ai}}{z_{ab} z_{aj} z_{ak} z_{bi} z_{il} z_{kl}} - \frac{(\xi_a k_i + \xi_a k_k + \xi_a k_l)(\xi_b k_j)(\xi_k k_l)(\xi_l k_i)}{z_{ab} z_{ak} z_{bj} z_{il} z_{kl}} \\
& + \frac{(\xi_a k_j)(\xi_k k_l)(\xi_l k_i)}{z_{ab} z_{aj} z_{il} z_{kl}} \left(\frac{\xi_b k_k}{z_{bk}} + \frac{\xi_b k_l}{z_{ak} z_{bl}} \right) + \frac{(\xi_k k_b)(\xi_l k_j)}{z_{ai} z_{ak} z_{bk} z_{jl}} \left(\frac{(\xi_a k_i)(\xi_b k_j) z_{aj}}{z_{al} z_{bj}} + \frac{(\xi_a k_i)(\xi_b k_l)}{z_{bl}} \right) \\
& - \frac{(\xi_a k_j + \xi_a k_l)(\xi_b k_i)(\xi_k k_b)(\xi_l k_j)}{z_{ak} z_{al} z_{bi} z_{bk} z_{jl}} + \frac{(\xi_a k_i)(\xi_l k_j) z_{aj}}{z_{ab} z_{ai} z_{ak} z_{jl}} \left(\frac{(\xi_b k_l)(\xi_k k_j) z_{bl}}{z_{jk}} - \frac{(\xi_b k_j)(\xi_k k_l)}{z_{bj} z_{kl}} \right) \\
& + \frac{(\xi_k k_i)(\xi_l k_j)}{z_{ab} z_{ak} z_{al} z_{ik} z_{jl}} \left(\frac{(\xi_a k_i)(\xi_b k_j) z_{aj}}{z_{bj}} - \frac{(\xi_a k_l)(\xi_b k_i) z_{ai}}{z_{bi}} - \frac{(\xi_a k_j)(\xi_b k_i) z_{ai}}{z_{bi}} \right) \; , \\
& + \frac{(\xi_k k_i)(\xi_l k_j)}{z_{ab} z_{ik} z_{jl}} \left(\frac{(\xi_a k_k)(\xi_b k_j) z_{aj}}{z_{ak} z_{al} z_{bj}} - \frac{(\xi_a k_j)(\xi_b k_k)}{z_{al} z_{bk}} - \frac{(\xi_a k_l)(\xi_b k_k)}{z_{al} z_{bk}} + \frac{(\xi_a k_i)(\xi_b k_l)}{z_{ak} z_{bl}} + \frac{(\xi_a k_k)(\xi_b k_l)}{z_{ak} z_{bl}} \right) \\
& + \frac{(\xi_b k_i)(\xi_k k_l)(\xi_l k_j)}{z_{ab} z_{ak} z_{bi} z_{jl} z_{kl}} ((\xi_a k_j) + (\xi_a k_k) + (\xi_a k_l)) - \frac{(\xi_a k_i)(\xi_k k_l)(\xi_l k_j)}{z_{ab} z_{ai} z_{jl} z_{kl}} \left(\frac{\xi_b k_k}{z_{bk}} + \frac{(\xi_b k_l) z_{al}}{z_{ak} z_{bl}} \right) \\
& + \frac{(\xi_k k_j)(\xi_l k_j) z_{aj}}{z_{ab} z_{al} z_{jk} z_{jl}} \left(\frac{(\xi_a k_i)(\xi_b k_k)}{z_{bk}} + \frac{(\xi_a k_i)(\xi_b k_j) z_{aj}}{z_{ai} z_{ak} z_{bj}} - \frac{(\xi_a k_l)(\xi_b k_i)}{z_{ak} z_{bi}} - \frac{(\xi_a k_k)(\xi_b k_i)}{z_{ak} z_{bi}} - \frac{(\xi_a k_j)(\xi_b k_i)}{z_{ak} z_{bi}} \right) \\
& + \frac{(\xi_k k_b)(\xi_l k_k)}{z_{al} z_{bk} z_{kl}} \left(\frac{(\xi_a k_i)(\xi_b k_j)}{z_{ai} z_{bj}} - \frac{(\xi_a k_j)(\xi_b k_i)}{z_{aj} z_{bi}} \right) + \frac{(\xi_b k_j)(\xi_k k_i)(\xi_l k_k)}{z_{ab} z_{al} z_{bj} z_{ik} z_{kl}} ((\xi_a k_i) + (\xi_a k_k) + (\xi_a k_l)) \\
& - \frac{(\xi_a k_j)(\xi_k k_i)(\xi_l k_k)}{z_{ab} z_{aj} z_{ik} z_{kl}} \left(\frac{(\xi_b k_k) z_{ak}}{z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{bl}} \right) - \frac{(\xi_b k_i)(\xi_k k_j)(\xi_l k_k)}{z_{ab} z_{al} z_{bi} z_{jk} z_{kl}} ((\xi_a k_j) + (\xi_a k_k) + (\xi_a k_l)) \\
& + \frac{(\xi_a k_i)(\xi_k k_j)(\xi_l k_k)}{z_{ab} z_{ai} z_{jk} z_{kl}} \left(\frac{(\xi_b k_j) z_{aj}}{z_{al} z_{bj}} + \frac{(\xi_b k_k) z_{ak}}{z_{al} z_{bk}} + \frac{(\xi_b k_l)}{z_{bl}} \right) \Bigg\} \; . \qquad \text{(C.6)}
\end{aligned}
$$

Appendix D

$\mathcal{B}$– and $\mathcal{C}$–equations

This appendix concerns with explaining how the system of equations will be created in case of the kinematics given in the titles of the respective sections.

D.1 *Kinematics* $(\xi\xi)\ (\xi\xi)\ (\xi k)\ (\xi k)$

This type of combination of polarizations and momentum vectors has $\frac{1}{2}\binom{6}{4}\binom{4}{2}\times 4^2 = 45\times 16$ representants, calculation being again based on combinatorial reasons. All of those are multiplied by the $\mathcal{B}_j$–functions introduced in (C.1) and (C.2). Again the difference between them, is the manner in which the polarization vectors coming from the (-1)–ghost picture operators are contracted with the other ones. More precisely, the $\mathcal{B}_j$–functions multiplied by their respective kinematics look like:

$$\begin{aligned}
B_2^\pi(a,b,i,j,k,l)\ (\xi_a\xi_b)\ (\xi_i\xi_j) &\rightarrow (\xi_a\xi_b)\ (\xi_i\xi_j)\ (\xi_k k_r)\ (\xi_l k_s)\ ,\\
B_1^\pi(a,b,i,j,k,l)\ (\xi_a\xi_i)\ (\xi_b\xi_j) &\rightarrow (\xi_a\xi_i)\ (\xi_b\xi_j)\ (\xi_k k_r)\ (\xi_l k_s)\ ,\\
B_3^\pi(a,i,j,k,b,l)\ (\xi_a\xi_i)\ (\xi_j\xi_k) &\rightarrow (\xi_a\xi_i)\ (\xi_j\xi_k)\ (\xi_b k_r)\ (\xi_l k_s)\ ,\\
B_4^\pi(i,j,k,l,a,b)\ (\xi_i\xi_j)\ (\xi_k\xi_l) &\rightarrow (\xi_i\xi_j)\ (\xi_k\xi_l)\ (\xi_a k_r)\ (\xi_b k_s)\ .
\end{aligned}\tag{D.1}$$

By the argument of completeness, i.e. considering that again, each kinematic of the type $(\xi\xi)\ (\xi\xi)\ (\xi k)\ (\xi k)$ occurs exactly once with each function $\mathcal{B}_j$, and considering all the non–symmetric possibilities of distributing the (-1)–ghost picture over the vertices, we get $\binom{6}{2}$ equations which are given in

$$\begin{array}{lll}
B_2^\pi(A,B,C,D,E,F)\ , & B_2^\pi(C,D,A,B,E,F)\ , & B_1^\pi(A,C,B,D,E,F)\ ,\\
B_1^\pi(A,D,B,C,E,F)\ , & B_1^\pi(B,C,A,D,E,F)\ , & B_1^\pi(B,D,A,C,E,F)\ ,\\
B_3^\pi(A,B,C,D,E,F)\ , & B_3^\pi(B,A,C,D,E,F)\ , & B_3^\pi(C,D,A,B,E,F)\ ,\\
B_3^\pi(D,C,A,B,E,F)\ , & B_3^\pi(A,B,C,D,F,E)\ , & B_3^\pi(B,A,C,D,F,E)\ ,\\
B_3^\pi(C,D,A,B,F,E)\ , & B_3^\pi(D,C,A,B,F,E)\ , & B_4^\pi(A,B,C,D,E,F)\ .
\end{array}\tag{D.2}$$

From that system we obtain again 14 equations. Moreover, since we also have the contractions $\xi_E k$ and $\xi_F k$ (which are included in the $\mathcal{B}$–functions), of which we have $4 \times 4 = 16$ on–shell, we obtain a total of $14 \times 16 = 224$ equations for that given kinematics. Hence in total, after taking into account all $\mathcal{B}$–kinematics we obtain $45 \times 224 = 10,080$ non–trivial relations, of which many might turn out to be the same. In fact, the length of the set boils down to $5,464$.

D.2 *Kinematics* $(\xi\xi)\,(\xi k)\,(\xi k)\,(\xi k)\,(\xi k)$

In this special case, we have $\binom{6}{2} \times 4^4 = 15 \times 256$ possibilities of combinations, which are multiplied by the three functions $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, given in (C.3,C.4,C.5,C.6).

Considering how the polarizations from different ghost pictures can be contracted we arrive at

$$\begin{aligned}
C_1^\pi(a,b,i,j,k,l)\ (\xi_a\xi_b) &\to (\xi_a\xi_b)\ (\xi_i k)\ (\xi_j k)\ (\xi_k k)\ (\xi_l k)\ ,\\
C_2^\pi(a,i,b,j,k,l)\ (\xi_a\xi_i) &\to (\xi_a\xi_i)\ (\xi_b k)\ (\xi_j k)\ (\xi_k k)\ (\xi_l k)\ ,\\
C_3^\pi(i,j,a,b,k,l)\ (\xi_i\xi_j) &\to (\xi_i\xi_j)\ (\xi_a k)\ (\xi_b k)\ (\xi_k k)\ (\xi_l k)\ .
\end{aligned} \tag{D.3}$$

We have 15 different expressions for each given kinematics, those expressions being given here:

$$\begin{aligned}
&C_1^\pi(A,B,C,D,E,F) \quad , \quad C_2^\pi(A,B,C,D,E,F) \quad , \quad C_2^\pi(A,B,D,C,E,F)\ ,\\
&C_2^\pi(A,B,E,C,D,F) \quad , \quad C_2^\pi(A,B,F,C,D,E) \quad , \quad C_2^\pi(B,A,C,D,E,F)\ ,\\
&C_2^\pi(B,A,D,C,E,F) \quad , \quad C_2^\pi(B,A,E,C,D,F) \quad , \quad C_2^\pi(B,A,F,C,D,E)\ ,\\
&C_3^\pi(A,B,C,D,E,F) \quad , \quad C_3^\pi(A,B,C,E,,D,F) \quad , \quad C_3^\pi(A,B,C,F,D,E)\ ,\\
&C_3^\pi(A,B,D,E,C,F) \quad , \quad C_3^\pi(A,B,D,F,C,E) \quad , \quad C_3^\pi(A,B,E,F,C,D)\ .
\end{aligned} \tag{D.4}$$

They give rise to 14 equations for each kinematics under consideration. More precisely, since the functions $\mathcal{C}_j$ contain the contractions $\xi_C k$, $\xi_D k$, $\xi_E k$ and $\xi_F k$, of which we have $4^4 = 256$ on–shell, we obtain a total of $14 \times 256 = 3,584$ equations. Hence in total, after taking into account all $\mathcal{C}$–kinematics we end up with $15 \times 3,584 = 53,760$ non–trivial relations, of which many are the same. In fact, the length of the set boils down to $6,727$. From the structure of the functions $\mathcal{C}_i$, namely that they do not involve self–contracted momenta, we deduce, that all those relations lead to polynomial identities.

Appendix E

Momentum expansion of selected S–matrix kinematics

E.1 *Kinematics* $(\xi\xi)\ (\xi\xi)\ (\xi\xi)$

We are able to list here the full homogeneous kinematics, since as shown in 5.22 those five kinematics are the generators of the full $\mathcal{A}$–system. This way, it will suffice to just list the five kinematics Ξ_1, Ξ_2, Ξ_5, Ξ_7, and Ξ_8, the rest being easily recoverable from those by permutation of the indices.

$$\begin{aligned}
\Xi_1\ \Bigg\{ & -\frac{s_9 s_2}{s_1 s_3 s_5} - \frac{s_2}{s_1 s_3} + \frac{s_2}{s_1 s_5} + \frac{s_2}{s_3 s_5} - \frac{s_2}{s_1 s_7} - \frac{s_4 s_2}{s_1 s_5 s_7} - \frac{s_6 s_2}{s_3 s_5 s_8} - \frac{s_2}{s_3 s_8} + \frac{s_4}{s_1 s_3} + \frac{s_6}{s_1 s_3} \\
& - \frac{s_6}{s_1 s_5} + \frac{s_6}{s_3 s_5} - \frac{s_6 s_7}{s_1 s_3 s_5} - \frac{s_7}{s_1 s_5} + \frac{s_7 s_8}{s_1 s_3 s_5} - \frac{s_4 s_8}{s_1 s_3 s_5} - \frac{s_8}{s_3 s_5} + \frac{s_7 s_9}{s_1 s_3 s_5} + \frac{s_8 s_9}{s_1 s_3 s_5} - \frac{s_9}{s_1 s_3} \\
& + \frac{1}{s_1} + \frac{1}{s_3} + \frac{s_4}{s_1 s_5} - \frac{s_4}{s_3 s_5} + \frac{1}{s_5} - \frac{s_4}{s_5 s_7} - \frac{1}{s_7} - \frac{s_6}{s_5 s_8} - \frac{1}{s_8} - \frac{s_4}{s_3 s_9} - \frac{s_4 s_6}{s_1 s_3 s_9} - \frac{s_6}{s_1 s_9} - \frac{1}{s_9} \\
& + \Bigg(\frac{s_2^2}{s_7} - \frac{s_2^2}{s_5} + \frac{s_4 s_2^2}{s_5 s_7} + \frac{s_6 s_2^2}{s_5 s_8} + \frac{s_2^2}{s_8} + \frac{s_9^2 s_2}{s_1 s_3} + \frac{s_3 s_2}{s_1} - \frac{s_4 s_2}{s_1} - \frac{s_5 s_2}{s_1} - \frac{s_5 s_2}{s_3} - \frac{s_6 s_2}{s_3} + \frac{s_1 s_6 s_2}{s_3 s_5} \\
& - \frac{s_6 s_2}{s_5} + \frac{s_7 s_2}{s_1} - \frac{s_3 s_7 s_2}{s_1 s_5} + \frac{s_8 s_2}{s_3} - \frac{s_1 s_8 s_2}{s_3 s_5} + \frac{s_5 s_9 s_2}{s_1 s_3} + \frac{s_7 s_9 s_2}{s_1 s_5} + \frac{s_8 s_9 s_2}{s_3 s_5} - \frac{s_9 s_2}{s_1} - \frac{s_9 s_2}{s_3} \\
& - \frac{s_9 s_2}{s_5} + \frac{s_1 s_2}{s_3} + \frac{s_3 s_4 s_2}{s_1 s_5} - \frac{s_4 s_2}{s_5} + \frac{s_4^2 s_2}{s_1 s_7} + \frac{s_1 s_2}{s_7} + \frac{s_4 s_5 s_2}{s_1 s_7} + \frac{s_1 s_4 s_2}{s_5 s_7} + \frac{s_6^2 s_2}{s_3 s_8} + \frac{s_3 s_2}{s_8} \\
& + \frac{s_5 s_6 s_2}{s_3 s_8} + \frac{s_3 s_6 s_2}{s_5 s_8} - 2 s_2 - \frac{s_4^2}{s_1} - \frac{s_6^2}{s_3} + \frac{s_6 s_7^2}{s_1 s_5} + \frac{s_3 s_7^2}{s_1 s_5} - \frac{s_7 s_8^2}{s_3 s_5} + \frac{s_4 s_8^2}{s_3 s_5} + \frac{s_1 s_8^2}{s_3 s_5} + \frac{s_5 s_9^2}{s_1 s_3} \\
& - \frac{s_7 s_9^2}{s_1 s_3} - \frac{s_8 s_9^2}{s_1 s_3} - \frac{s_1 s_4}{s_3} - 2 s_4 + \frac{s_4 s_5}{s_3} - \frac{s_3 s_6}{s_1} - \frac{s_4 s_6}{s_1} - \frac{s_4 s_6}{s_3} + \frac{s_4 s_5 s_6}{s_1 s_3} + \frac{s_5 s_6}{s_1} + \frac{s_1 s_6}{s_5} \\
& - \frac{s_3 s_6}{s_5} - 2 s_6 - \frac{s_3 s_7}{s_1} - \frac{s_6 s_7}{s_1} - \frac{s_6 s_7}{s_3} + \frac{s_3 s_6 s_7}{s_1 s_5} - \frac{s_6 s_7}{s_5} - \frac{s_3 s_7}{s_5} - \frac{s_3 s_4 s_7}{s_1 s_5} + \frac{s_4 s_7}{s_5} -
\end{aligned} \tag{E.1}$$

$$
\begin{aligned}
&- \frac{s_7^2 s_8}{s_1 s_5} - \frac{s_4 s_8}{s_1} - \frac{s_4 s_8}{s_3} - \frac{s_1 s_6 s_8}{s_3 s_5} + \frac{s_6 s_8}{s_5} + \frac{s_6 s_7 s_8}{s_3 s_5} + \frac{s_7 s_8}{s_1} + \frac{s_7 s_8}{s_3} + \frac{s_4 s_7 s_8}{s_1 s_5} + \frac{2 s_7 s_8}{s_5} \\
&- \frac{s_1 s_8}{s_3} - \frac{s_1 s_8}{s_5} + \frac{s_1 s_4 s_8}{s_3 s_5} - \frac{s_4 s_8}{s_5} - \frac{s_7^2 s_9}{s_1 s_5} - \frac{s_8^2 s_9}{s_3 s_5} + \frac{s_4 s_9}{s_3} - \frac{s_4 s_5 s_9}{s_1 s_3} - \frac{s_5 s_9}{s_1} - \frac{s_5 s_9}{s_3} \\
&- \frac{s_5 s_6 s_9}{s_1 s_3} + \frac{s_6 s_9}{s_1} + \frac{s_6 s_7 s_9}{s_1 s_3} + \frac{2 s_7 s_9}{s_1} + \frac{s_7 s_9}{s_3} + \frac{s_7 s_9}{s_5} + \frac{s_4 s_8 s_9}{s_1 s_3} - \frac{s_7 s_8 s_9}{s_1 s_3} - \frac{s_7 s_8 s_9}{s_1 s_5} \\
&- \frac{s_7 s_8 s_9}{s_3 s_5} + \frac{s_8 s_9}{s_1} + \frac{2 s_8 s_9}{s_3} + \frac{s_8 s_9}{s_5} - \frac{s_1 s_4}{s_5} + \frac{s_3 s_4}{s_5} + \frac{s_4^2}{s_7} + \frac{s_4 s_5}{s_7} + \frac{s_6^2}{s_8} + \frac{s_5 s_6}{s_8} + \frac{s_4^2}{s_9} \\
&+ \left. \frac{s_4 s_6^2}{s_3 s_9} + \frac{s_6^2}{s_9} + \frac{s_3 s_4}{s_9} + \frac{s_4^2 s_6}{s_1 s_9} + \frac{s_1 s_6}{s_9} + \frac{s_3 s_4 s_6}{s_1 s_9} + \frac{s_1 s_4 s_6}{s_3 s_9} \right) \zeta(2) \Bigg\} + \\
&+ \Xi_2 \left\{ \frac{s_2}{s_1 s_7} - \frac{1}{s_1} + \frac{1}{s_7} + \frac{s_6}{s_1 s_9} + \frac{1}{s_9} \right. \\
&+ \left(-\frac{s_2^2}{s_7} - \frac{s_3 s_2}{s_1} + \frac{s_5 s_2}{s_1} - \frac{s_1 s_2}{s_7} - \frac{s_4 s_5 s_2}{s_1 s_7} + s_2 + s_1 + s_4 + \frac{s_3 s_6}{s_1} - \frac{s_5 s_6}{s_1} + s_6 + \frac{s_3 s_7}{s_1} \right. \\
&\left. \left. - s_7 + \frac{s_4 s_8}{s_1} - \frac{s_7 s_8}{s_1} + \frac{s_5 s_9}{s_1} - \frac{s_8 s_9}{s_1} - s_9 - \frac{s_4 s_5}{s_7} - \frac{s_6^2}{s_9} - \frac{s_3 s_4}{s_9} - \frac{s_1 s_6}{s_9} - \frac{s_3 s_4 s_6}{s_1 s_9} \right) \zeta(2) \right\} \\
&+ \Xi_5 \left\{ -\frac{1}{s_7} + \left(\frac{s_2 s_1}{s_7} - s_1 - s_2 - s_4 - s_5 + s_7 + s_8 + s_9 + \frac{s_4 s_5}{s_7} \right) \zeta(2) \right\} \\
&+ \Xi_7 \left\{ -\frac{s_1}{s_2 s_7} - \frac{s_4 s_1}{s_2 s_5 s_7} + \frac{s_9}{s_2 s_5} + \frac{1}{s_2} + \frac{1}{s_5} - \frac{s_4}{s_5 s_7} - \frac{1}{s_7} - \frac{s_3}{s_2 s_8} - \frac{s_3 s_6}{s_2 s_5 s_8} - \frac{s_6}{s_5 s_8} - \frac{1}{s_8} \right. \\
&+ \left[\frac{s_4 s_1^2}{s_5 s_7} + \frac{s_1^2}{s_7} - \frac{s_4 s_1}{s_2} + \frac{s_6 s_1}{s_2} + \frac{s_6 s_1}{s_5} + \frac{s_4 s_8 s_1}{s_2 s_5} - \frac{s_8 s_1}{s_5} - \frac{s_9 s_1}{s_5} + \frac{s_3 s_1}{s_5} - \frac{s_4 s_1}{s_5} + \frac{s_4^2 s_1}{s_2 s_7} \right. \\
&+ \frac{s_2 s_1}{s_7} + \frac{s_4 s_5 s_1}{s_2 s_7} + \frac{s_2 s_4 s_1}{s_5 s_7} - s_1 - s_2 - s_3 + \frac{s_3 s_4}{s_2} - s_4 - s_5 - \frac{s_3 s_6}{s_2} + \frac{s_4 s_6}{s_2} - \frac{s_3 s_6}{s_5} - s_6 \\
&- \frac{s_6 s_7}{s_2} + \frac{s_3 s_6 s_7}{s_2 s_5} - \frac{s_3 s_7}{s_5} + s_7 - \frac{s_4 s_8}{s_2} + s_8 - \frac{s_4 s_9}{s_2} - \frac{s_5 s_9}{s_2} - \frac{s_6 s_9}{s_2} + \frac{s_7 s_9}{s_2} + \frac{s_7 s_9}{s_5} - \frac{s_7 s_8 s_9}{s_2 s_5} \\
&+ \frac{s_8 s_9}{s_2} + \frac{s_8 s_9}{s_5} - \frac{s_2 s_9}{s_5} - \frac{s_3 s_9}{s_5} + \frac{s_3 s_4}{s_5} + \frac{s_4^2}{s_7} + \frac{s_4 s_5}{s_7} + \frac{s_3^2}{s_8} + \frac{s_3 s_6^2}{s_2 s_8} + \frac{s_6^2}{s_8} + \frac{s_2 s_3}{s_8} + \frac{s_3 s_5 s_6}{s_2 s_8} \\
&\left. \left. + \frac{s_5 s_6}{s_8} + \frac{s_3^2 s_6}{s_5 s_8} + \frac{s_2 s_3 s_6}{s_5 s_8} \right] \zeta(2) \right\} \\
&+ \Xi_8 \left(s_1 + s_2 + s_3 + s_4 + s_5 + s_6 - s_7 - s_8 - s_9 \right) \zeta(2) + \mathcal{O}(k^4) .
\end{aligned}
$$

E.2 *Kinematics* $(\xi\xi)$ $(\xi\xi)$ (ξk) (ξk)

The $\mathcal{B}$–kinematics are again very numerous, thus we will just list some representatives of it:

$$
\begin{aligned}
&(\xi_1\xi_2)(\xi_3\xi_4)(\xi_5 k_2)(\xi_6 k_1)\\
&\quad\times\Bigg\{-\frac{s_2}{s_3s_5s_8}-\frac{s_2}{s_3s_6s_8}-\frac{s_7}{s_1s_3s_5}+\frac{1}{s_1s_3}-\frac{1}{s_1s_5}+\frac{1}{s_3s_5}+\frac{1}{s_3s_6}-\frac{1}{s_5s_8}-\frac{1}{s_6s_8}\\
&\quad-\frac{s_4}{s_1s_3s_9}-\frac{1}{s_1s_9}-\frac{s_4}{s_3s_6s_9}-\frac{1}{s_6s_9}\\
&\quad+\Bigg(\frac{s_2^2}{s_5s_8}+\frac{s_2^2}{s_6s_8}+\frac{s_9s_2}{s_3s_6}-\frac{s_2}{s_3}+\frac{s_1s_2}{s_3s_5}-\frac{s_2}{s_5}-\frac{s_2}{s_6}+\frac{s_5s_2}{s_3s_8}+\frac{s_6s_2}{s_3s_8}+\frac{s_3s_2}{s_5s_8}+\frac{s_3s_2}{s_6s_8}\\
&\quad+\frac{s_7^2}{s_1s_5}-\frac{s_3}{s_1}-\frac{s_4}{s_1}-\frac{s_4}{s_3}+\frac{s_4s_5}{s_1s_3}+\frac{s_5}{s_1}-\frac{s_6}{s_3}-\frac{s_7}{s_1}-\frac{s_7}{s_3}+\frac{s_3s_7}{s_1s_5}-\frac{s_7}{s_5}+\frac{s_7s_8}{s_3s_5}-\frac{s_1s_8}{s_3s_5}\\
&\quad+\frac{s_8}{s_5}+\frac{s_4s_8}{s_3s_6}+\frac{s_8}{s_6}-\frac{s_5s_9}{s_1s_3}+\frac{s_7s_9}{s_1s_3}-\frac{s_8s_9}{s_3s_6}+\frac{s_9}{s_1}+\frac{s_9}{s_6}+\frac{s_1}{s_5}-\frac{s_3}{s_5}-\frac{s_3}{s_6}-\frac{s_4}{s_6}+\frac{s_5}{s_8}\\
&\quad+\frac{s_6}{s_8}+\frac{s_4^2}{s_1s_9}+\frac{s_1}{s_9}+\frac{s_3s_4}{s_1s_9}+\frac{s_1s_4}{s_3s_9}+\frac{s_4s_6}{s_3s_9}+\frac{s_6}{s_9}+\frac{s_4^2}{s_6s_9}+\frac{s_3s_4}{s_6s_9}-2\Bigg)\zeta(2)\\
&\quad+\Bigg(-\frac{s_2^3}{s_5s_8}-\frac{s_2^3}{s_6s_8}-\frac{2s_3s_2^2}{s_5s_8}-\frac{2s_3s_2^2}{s_6s_8}-\frac{s_9^2s_2}{s_3s_6}+\frac{3s_7s_2}{s_5}+\frac{s_8s_2}{s_3}-\frac{s_1s_8s_2}{s_3s_5}+\frac{s_8s_2}{s_5}\\
&\quad+\frac{s_8s_2}{s_6}-\frac{s_8s_9s_2}{s_3s_6}+\frac{s_9s_2}{s_3}-\frac{s_9s_2}{s_6}+\frac{s_1s_2}{s_3}-\frac{s_1s_2}{s_5}+\frac{2s_3s_2}{s_5}-\frac{s_1^2s_2}{s_3s_5}+\frac{2s_3s_2}{s_6}\\
&\quad+\frac{3s_4s_2}{s_6}-\frac{s_5^2s_2}{s_3s_8}-\frac{s_6^2s_2}{s_3s_8}-\frac{2s_5s_6s_2}{s_3s_8}-\frac{s_3^2s_2}{s_5s_8}-\frac{s_3^2s_2}{s_6s_8}-2s_2-\frac{s_7^3}{s_1s_5}+\frac{s_3^2}{s_1}\\
&\quad-\frac{s_4s_5^2}{s_1s_3}-\frac{s_5^2}{s_1}+\frac{s_6^2}{s_3}-\frac{2s_3s_7^2}{s_1s_5}-\frac{s_7s_8^2}{s_3s_5}+\frac{s_1s_8^2}{s_3s_5}-\frac{s_8^2}{s_5}-\frac{s_4s_8^2}{s_3s_6}-\frac{s_8^2}{s_6}+\frac{s_5s_9^2}{s_1s_3}\\
&\quad-\frac{s_7s_9^2}{s_1s_3}+\frac{s_8s_9^2}{s_3s_6}-\frac{s_9^2}{s_1}-\frac{s_9^2}{s_6}+s_1-2s_3+\frac{2s_3s_4}{s_1}-2s_4-\frac{s_4s_5}{s_1}+\frac{s_4s_5}{s_3}-\frac{s_1s_5}{s_3}\\
&\quad+s_5+2s_6+\frac{2s_3s_7}{s_1}+\frac{3s_4s_7}{s_1}+\frac{s_5s_7}{s_1}+\frac{s_5s_7}{s_3}+\frac{2s_6s_7}{s_3}+\frac{s_1s_7}{s_3}-\frac{s_3^2s_7}{s_1s_5}+\frac{s_1s_7}{s_5}\\
&\quad+\frac{2s_3s_7}{s_5}+s_7+\frac{s_4s_8}{s_3}-\frac{s_1s_7s_8}{s_3s_5}-\frac{s_7s_8}{s_5}-\frac{s_1s_8}{s_5}+\frac{s_1^2s_8}{s_3s_5}-\frac{s_4s_8}{s_6}+2s_8+\frac{s_5^2s_9}{s_1s_3}\\
&\quad+\frac{s_8^2s_9}{s_3s_6}+\frac{s_4s_9}{s_1}+\frac{s_4s_9}{s_3}-\frac{s_4s_5s_9}{s_1s_3}-\frac{s_5s_9}{s_1}-\frac{s_5s_7s_9}{s_1s_3}-\frac{s_7s_9}{s_1}-\frac{s_8s_9}{s_3}-\frac{s_4s_8s_9}{s_3s_6}\\
&\quad-\frac{s_8s_9}{s_6}+\frac{s_4s_9}{s_6}+2s_9-\frac{s_1^2}{s_5}+\frac{s_3^2}{s_5}+\frac{s_3^2}{s_6}+\frac{2s_3s_4}{s_6}-\frac{s_5^2}{s_8}-\frac{s_6^2}{s_8}-\frac{2s_5s_6}{s_8}\\
&\quad-\frac{s_4^3}{s_1s_9}-\frac{s_1^2}{s_9}-\frac{2s_3s_4^2}{s_1s_9}-\frac{s_4s_6^2}{s_3s_9}-\frac{s_6^2}{s_9}-\frac{s_3^2s_4}{s_1s_9}-\frac{s_1^2s_4}{s_3s_9}-\frac{2s_1s_6}{s_9}-\frac{2s_1s_4s_6}{s_3s_9}\\
&\quad-\frac{s_4^3}{s_6s_9}-\frac{2s_3s_4^2}{s_6s_9}-\frac{s_3^2s_4}{s_6s_9}\Bigg)\zeta(3)\Bigg\}+\mathcal{O}(k^6)\ .
\end{aligned}
\tag{E.2}
$$

Appendix F

Some more useful evaluated series

We list in this Appendix, without proving, some more series which we shall need when expanding the ${}_3F_2$ hypergeometric function. Its relevance is given by its occurrence in the five–point tree amplitude. Thus, every five–point process on the disk, independent of the external states inserted on the world–sheet and of the brane setup is described by the function ${}_3F_2$. Thus the sums involved in its parameter series expansion are:

$$\begin{aligned}
(i) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ \psi^{(1)}(n+1) = 1\ , \\
(ii) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ \psi^{(1)}(n+2) = \zeta(2) - \zeta(3)\ , \\
(iii) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ \psi^{(1)}(n+3) = -3 + 2\ \zeta(2)\ , \\
(iv) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ H_{n+1}\ \psi(n+1) = (1-\gamma_E)\ \zeta(2) + \zeta(3)\ , \\
(v) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ H_{n+2}\ \psi(n+1) = 3 - 2\ \gamma_E\ , \\
(vi) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ H_{n+2}\ \psi(n+2) = -2\ \gamma_E + \zeta(2) + 2\ \zeta(3)\ , \\
(vii) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ \psi(n+1)^2 = (1-\gamma_E)^2 + \zeta(2)\ , \\
(viii) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ \psi(n+2)^2 = \gamma_E^2 - 2\ \gamma_E\ \zeta(2) + 3\ \zeta(3)\ , \\
(ix) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}\ \psi(n+3)^2 = 3 - 4\ \gamma_E + \gamma_E^2 + \zeta(2)\ . \qquad \text{(F.1)}
\end{aligned}$$

These identities may be proven by applying formulae shown in section 6.2. Furthermore, we list the three series, which again, can be proved with the tools collected in section 6.2:

$$\begin{aligned}
(i) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}\ \psi(n+1)^2 = \frac{11}{4}\ \zeta(4) - 2\ \gamma_E\ \zeta(3) + \gamma_E^2\ \zeta(2)\ , \\
(ii) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}\ \psi(n+2)^2 = \frac{17}{4}\ \zeta(4) - 4\ \gamma_E\ \zeta(3) + \gamma_E^2\ \zeta(2)\ , \\
(iii) \quad & \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}\ \psi^{(1)}(n+1)^2 = \frac{1}{3}\ \zeta(2)^3 - \zeta(3)^2 - \frac{5}{3}\ \zeta(2)\ \zeta(4) + \frac{17}{3}\ \zeta(6).
\end{aligned} \tag{F.2}$$

Appendix G

Expansion of the hypergeometric ${}_4F_3$ function

The same techniques as applied in section 6.3 and collected in section 6.2 are put in the same way to work and used to expand the following (in the positive parameter region about zero) finite ${}_4F_3$ hypergeometric function, the purpose of that being again the collection of such finite, nonsingular functions with the aim to find again relations between singular and nonsingular functions. Although the ${}_4F_3$ hypergeometric functions is not the most general function which encodes the behavior of any scattering process, the functions appears in the case of the six-point disk S–matrix when more special assumptions are made in order to restrict the momenta of the involved particles:

$$
\begin{aligned}
&\frac{\Gamma(1+d)\ \Gamma(1+e)\ \Gamma(1+f)\ \Gamma(1+b)\ \Gamma(1+a)\ \Gamma(1+c)}{\Gamma(2+b+e)\ \Gamma(2+a+d)\ \Gamma(2+c+f)}\ {}_4F_3\left[\begin{matrix}1+b,\ 1+a,\ 1+c,\ -j\\ 2+b+e,\ 2+a+d,\ 2+c+f\end{matrix}\right]\\
&= 1-a-b-c-d-e-f-3j+a^2+ba+ca+2da+ea+fa+4ja+b^2+c^2+d^2\\
&+e^2+f^2+6j^2+bc+bd+cd+2be+ce+de+bf+2cf+df+ef+4bj+4cj+4dj\\
&+4ej+4fj-a^3-ba^2-ca^2-3da^2-ea^2-fa^2-5ja^2-b^2a-c^2a-3d^2a-e^2a\\
&-f^2a-10j^2a-bca-2bda-2cda-2bea-cea-2dea-bfa-2cfa-2dfa-efa\\
&-5bja-5cja-10dja-5eja-5fja-b^3-c^3-d^3-e^3-f^3-10j^3-bc^2-bd^2-cd^2\\
&-3be^2-ce^2-de^2-bf^2-3cf^2-df^2-ef^2-10bj^2-10cj^2-10dj^2-10ej^2-10fj^2\\
&-b^2c-b^2d-c^2d-bcd-3b^2e-c^2e-d^2e-2bce-2bde-cde-b^2f-3c^2f-d^2f\\
&-e^2f-2bcf-bdf-2cdf-2bef-2cef-def-5b^2j-5c^2j-5d^2j-5e^2j-5f^2j\\
&-5bcj-5bdj-5cdj-10bej-5cej-5dej-5bfj-10cfj-5dfj-5efj\\
&+\zeta(2)\ \big\{j-2j^2-aj-bj-cj-ad-be-cf+3j^3+3aj^2+3bj^2+3cj^2+a^2j+b^2j\\
&+c^2j+abj+acj+bcj+4adj+4bej-dej+4cfj-dfj-efj+ad^2+be^2+cf^2+a^2d\\
&+abd+acd+b^2e+abe+bce+ade+bde+c^2f+acf+bcf+adf+cdf+bef+cef\big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\,\zeta(4)\;\big\{-j^2-4aj-4bj-4cj-5dj-5ej-5fj+5j^3+5aj^2+5bj^2+5cj^2+17dj^2\\
&+17ej^2+17fj^2+4a^2j+4b^2j+4c^2j+12d^2j+12e^2j+12f^2j+4abj+4acj+4bcj\\
&+2adj+5bdj+5cdj+5aej+2bej+5cej+17dej+5afj+5bfj+2cfj+17dfj+17efj\big\}\\
&+\zeta(3)\;\big\{j-2j^2-aj-bj-cj-2dj-2ej-2fj+3j^3+3aj^2+3bj^2+3cj^2+4dj^2+4ej^2\\
&+4fj^2+a^2j+b^2j+c^2j+abj+acj+bcj+2adj+2bdj+2cdj+2aej+2bej+2cej\\
&+2afj+2bfj+2cfj+ad^2+be^2+cf^2+a^2d+b^2e+c^2f\big\}\\
&+\zeta(2)\,\zeta(3)\;\big\{-j^3-aj^2-bj^2-cj^2+d^2j+e^2j+f^2j+adj-bdj-cdj-aej+bej-cej\\
&-dej-afj-bfj+cfj-dfj-efj\big\}\\
&+\frac{1}{2}\,\zeta(5)\;\big\{4j^3+4aj^2+4bj^2+4cj^2+dj^2+ej^2+fj^2+2a^2j+2b^2j+2c^2j-d^2j-e^2j\\
&-f^2j+2abj+2acj+2bcj-3adj+6bdj+6cdj+6aej-3bej+6cej+7dej+6afj\\
&+6bfj-3cfj+7dfj+7efj\big\}+\ldots\;.
\end{aligned}
\tag{G.1}
$$

Appendix H

Some intermediary, not very simple, triple sums

$$\sum_{m_i=1}^{\infty} \frac{1}{(1+m_1+m_3)\ (m_2+m_3)\ (m_1+m_2+m_3)^2} = -\frac{13}{4}\ \zeta(4) - 3\ \zeta(3) + 2\ \zeta(2) + \frac{3}{2}\ \zeta(2)^2\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ (1+m_1+m_3)\ (m_2+m_3)\ (m_1+m_2+m_3)} = -2\ \zeta(4) - 2\ \zeta(3) + \zeta(2) + \frac{3}{2}\ \zeta(2)^2\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ m_2\ (1+m_1+m_3)\ (m_2+m_3)} = 2\ \zeta(3) + \zeta(2)\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ m_2\ (m_2+m_3)\ (m_1+m_2+m_3)} = 8\ \zeta(4) - 2\ \zeta(2)^2\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ m_2\ (m_2+m_3)\ (m_1+m_3)} = \frac{11}{2}\ \zeta(4) - \zeta(2)^2\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{m_2\ (m_1+m_3)\ (m_2+m_3)\ (m_1+m_2+m_3)} = -\frac{5}{2}\ \zeta(4) + \frac{3}{2}\ \zeta(2)^2\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{(m_1+\ m_3)\ (m_2+m_3)\ (m_1+m_2+m_3)^2} = \frac{17}{4}\ \zeta(4) - 2\ \zeta(2)\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ m_2\ (m_1+m_3)\ (m_2+m_3)\ (1+m_1+m_2+m_3)} = \frac{17}{4}\ \zeta(4) + 2\ \zeta(3) - \zeta(2) - 5\ ,$$
$$\sum_{m_i=1}^{\infty} \frac{1}{m_1\ m_2\ (1+m_2)\ (m_2+m_3)\ (1+m_1+m_2+m_3)} = 5 - 2\ \zeta(3) - \zeta(2). \tag{H.1}$$

Also some double series involving harmonic numbers have to be computed. They read:

$$
\begin{aligned}
\sum_{m_i=1}^{\infty} \frac{H_{m_1+m_3}}{m_1\,(1+m_1+m_3)^2} &= -\zeta(2) + \frac{1}{2}\,\zeta(2)^2 + \zeta(3) + \frac{3}{2}\,\zeta(4)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_3}}{m_1\,(1+m_1+m_3)^2} &= \frac{9}{2}\,\zeta(4) - \zeta(2)^2\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_1-1}}{m_1\,(1+m_1+m_3)^2} &= -3 + \zeta(2) + \zeta(3) + \zeta(4)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_3}}{m_1\,(1+m_3)\,(1+m_1+m_3)\,(2+m_1+m_3)} &= 1 + \zeta(2) - 2\,\zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_1-1}}{m_1\,(1+m_3)\,(1+m_1+m_3)\,(2+m_1+m_3)} &= \frac{15}{4} - \zeta(2) - 2\,\zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_1+m_3}}{m_1\,(1+m_3)\,(1+m_1+m_3)\,(2+m_1+m_3)} &= -\frac{3}{2} - \frac{5}{2}\,\zeta(2) + \frac{11}{2}\,\zeta(4)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_1+m_2-1}}{m_1\,(m_1+m_2)^2} &= \frac{3}{2}\,\zeta(4) + \frac{1}{2}\,\zeta(2)^2\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_2}}{m_1\,(m_1+m_2)^2} &= 5\,\zeta(4) - \zeta(2)^2\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_2}}{m_1\,m_2\,(m_1+m_2)} &= \frac{11}{2}\,\zeta(4) - \frac{1}{2}\,\zeta(2)^2\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_2}}{m_2\,(m_1+m_2)^2} &= \frac{1}{2}\,\zeta(4) + \frac{1}{2}\,\zeta(2)^2\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_1+m_2-1}}{(1+m_1)\,(m_1+m_2)^2} &= \frac{7}{4}\,\zeta(4) + \frac{1}{2}\,\zeta(2)^2 - \zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_2}}{(1+m_1)\,(m_1+m_2)^2} &= 5\,\zeta(4) - \zeta(2)^2 + 2\,\zeta(3) - 2\,\zeta(2)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_3}}{m_2\,(2+m_3)\,(1+m_2+m_3)^2} &= -2\,\zeta(2) - \zeta(2)^2 + \frac{23}{4}\,\zeta(4)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_2-1}}{m_2\,(2+m_3)\,(1+m_2+m_3)^2} &= -\frac{3}{2} + \frac{1}{2}\,\zeta(2) - \zeta(2)^2 - \frac{5}{2}\,\zeta(3) + 6\,\zeta(4)\ , \\
\sum_{m_i=1}^{\infty} \frac{H_{m_2+m_3}}{m_2\,(2+m_3)\,(1+m_2+m_3)^2} &= \frac{23}{4}\,\zeta(4) - \frac{5}{2}\,\zeta(2) - \frac{3}{2}\,\zeta(3)\ ,
\end{aligned}
\tag{H.2}
$$

where the simplification from the reduction of the triple sum to the double one is compensated by the presence of the harmonic number. Finally again some triple sums:

$$\begin{aligned}
\sum_{m_i=1}^{\infty} \frac{1}{m_1\ (1+m_1+m_3)^3} &= -3 + \frac{3}{2}\ \zeta(4) - \frac{1}{2}\ \zeta(2)^2 + \zeta(2) + \zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{m_1\ (1+m_3)\ (1+m_1+m_3)^2} &= -2 + 3\ \zeta(4) - \zeta(2)^2 + \zeta(2)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{m_1\ (1+m_3)\ (1+m_1+m_3)\ (2+m_1+m_3)} &= -\frac{9}{4} + 2\ \zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{m_1\ (1+m_3)^2\ (1+m_1+m_3)\ (2+m_1+m_3)} &= \frac{11}{4} - \frac{\pi^2}{6} + \frac{\pi^4}{72} - 2\ \zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{m_1\ (1+m_3)\ (1+m_1+m_3)\ (2+m_1+m_3)^2} &= -\frac{29}{4} + \frac{\pi^2}{4} + 4\ \zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{m_1\ (1+m_3)\ (1+m_1+m_3)^2\ (2+m_1+m_3)} &= \frac{1}{4} + 3\ \zeta(4) + \zeta(2) - \zeta(2)^2 - 2\ \zeta(3)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{m_2\ (2+m_3)^2\ (1+m_2+m_3)^3} &= -\frac{1}{2}\ - \frac{15}{4}\ \zeta(2) + \frac{1}{2}\ \zeta(2)^2 + 4\ \zeta(3) + \frac{1}{2}\ \zeta(4)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{m_2\ (2+m_3)\ (1+m_1+m_2+m_3)^3} &= -\frac{3}{2} + \frac{5}{2}\ \zeta(2) - \frac{1}{2}\ \zeta(2)^2 - \frac{7}{2}\ \zeta(3) + \frac{11}{4}\ \zeta(4)\ , \\
\sum_{m_i=1}^{\infty} \frac{1}{(1+\ m_3)\ m_2\ (2+m_3)\ (1+m_2+m_3)^2} &= -1 + 3\ \zeta(4) + \frac{5}{2}\ \zeta(2) - 3\ \zeta(3) - \zeta(2)^2\ .
\end{aligned} \tag{H.3}$$

Appendix I

Full expansion of the six base functions

$$
\begin{aligned}
F\begin{bmatrix} a\,,\,b\,,\,d\,,\,e\,,\,g \\ c\,,\,f\,,\,h\,,\,j \end{bmatrix} &= 1 - a - b - c - d - e - f - 2\ g - 2\ h - 3\ j + (g + h + j)\ \zeta(2) + j\ \zeta(3) \\
&+ a^2 + a\,b + b^2 + a\,c + b\,c + c^2 + 2\,a\,d + b\,d + c\,d + d^2 + a\,e + 2\,b\,e + c\,e \\
&+ d\,e + e^2 + a\,f + b\,f + 2\,c\,f + d\,f + e\,f + f^2 + 3\,a\,g + 3\,b\,g + 2\,c\,g + 3\,d\,g \\
&+ 3\,e\,g + 2\,f\,g + 3\,g^2 + 2\,a\,h + 3\,b\,h + 3\,c\,h + 2\,d\,h + 3\,e\,h + 3\,f\,h + 5\,g\,h \\
&+ 3\,h^2 + 4\,a\,j + 4\,b\,j + 4\,c\,j + 4\,d\,j + 4\,e\,j + 4\,f\,j + 8\,g\,j + 8\,h\,j + 6\,j^2 \\
&- \big(a\,d + b\,e + c\,f + a\,g + b\,g + c\,g + f\,g + g^2 + a\,h + b\,h + c\,h + d\,h + h^2 \\
&+ a\,j + b\,j + c\,j + 3\,g\,j + 3\,h\,j + 2\,j^2\big)\ \zeta(2) \\
&- \big[\ g^2 + 2\ dg + 2\ eg + 4\ hg + h^2 + 2\ j^2 + 2\ eh + 2\ fh \\
&+ 2\ (d + e + f + g + h)\ j + a\ (g + j) + c\ (h + j) + b\ (g + h + j)\ \big]\ \zeta(3) \\
&- [\ j\ (a + b + c) + \frac{j}{4}\ (5\ d + 5\ e + 5\ f + 2\ g + 2\ h + j)\ ]\ \zeta(4) + \ldots \\
F\begin{bmatrix} a, b, d, e, g - 1 \\ c, f, h, j \end{bmatrix} &= \zeta(2) - (c + f - h + j)\ \zeta(2) \\
&- (a + b + 2\ d + 2\ e + g + 2\ h - j)\ \zeta(3) + \ldots\ , \\
F\begin{bmatrix} a, b, d, e, g \\ c + 1, f, h, j - 1 \end{bmatrix} &= -1 + \zeta(2) - a - b + 3\ c + 3\ f + 2\ h + j \\
&+ (a + b - 2\ c + d + e - f + 2\ g)\ \zeta(2) \\
&- (a + b + 2\ d + 2\ e + 2\ f + 3\ g + 2\ h + j)\ \zeta(3) + \ldots \\
F\begin{bmatrix} a, b + 1, d, e, g \\ c, f, h, j - 1 \end{bmatrix} &= -1 + \zeta(2) - a + 3\ b - c + 3\ e + 2\ g + 2\ h + j \\
&+ (a - 2\ b + c + d - e + f)\ \zeta(2) \\
&- (a + c + 2\ d + 2\ e + 2\ f + 2\ g + 2\ h + j)\ \zeta(3) + \ldots
\end{aligned}
\tag{I.1}
$$

$$
\begin{aligned}
F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j-2 \end{bmatrix} &= (1+j)\ \zeta(2) - (a+b+c+2\ d+2\ e+2\ f+g+h+2\ j)\ \zeta(3) \\
&+ j^2\ \zeta(2) - j\ (a+b+c+2\ d+2\ e+2\ f+g+h+2\ j)\ \zeta(3) \\
&+ \Big\{\ a^2+ba+ca+b^2+c^2+3\ d^2+3e^2+3\ f^2+g^2+h^2+3\ dg \\
&+3\ eg+3\ eh+3\ fh+bc+\frac{da}{2}+\frac{5ea}{4}+\frac{5fa}{4}+\frac{ga}{4}+\frac{ha}{2}+\frac{5ja}{4} \\
&+\frac{5j^2}{4}+\frac{5bd}{4}+\frac{5cd}{4}+\frac{be}{2}+\frac{5ce}{4}+\frac{17de}{4}+\frac{5bf}{4}+\frac{cf}{2}+\frac{17df}{4} \\
&+\frac{17ef}{4}+\frac{bg}{4}+\frac{cg}{2}+\frac{5fg}{2}+\frac{bh}{4}+\frac{ch}{4}+\frac{5dh}{2}+\frac{7gh}{4}+\frac{5bj}{4} \\
&+\frac{5cj}{4}+\frac{17dj}{4}+\frac{17ej}{4}+\frac{17fj}{4}+\frac{5gj}{2}+\frac{5hj}{2}+\frac{5j^2}{4}\ \Big\}\ \zeta(4)+\dots\ , \\
F\begin{bmatrix} a+1,b,d,e,g \\ c,f,h,j-1 \end{bmatrix} &= -1+\zeta(2)+3\ a-b-c+3\ d+2\ g+j \\
&-(2\ a-b-c+d-e-f-2\ h)\ \zeta(2) \\
&-(b+c+2\ d+2\ e+2\ f+2\ g+3\ h+j)\ \zeta(3) \\
&-6\ a^2+4\ (b+c-3\ d-2\ g-j)\ a-b^2-c^2-6\ d^2+\frac{31g^2}{8}-j^2 \\
&+4\ cd+3\ cg-8\ dg+4\ fg+cj-4\ dj+gj+b\ (-c+4\ d+2\ g+j) \\
&+\{\ 3\ a^2+[-3\ b-3\ c+5\ d-2\ (e+f-g+2\ h)+j]\ a+b^2+c^2 \\
&+d^2-g^2-2\ cd+ce-2\ de+2\ cf-2\ df-cg-2\ eg+2\ ch \\
&-4\ dh+b\ (c-2\ d+2\ e+f+2\ h)-(e+f-g+2\ h)\ j\ \}\ \zeta(2) \\
&+\{a^2+[b+c+4\ d+3e+3\ f+4\ (g+h)+2\ j]\ a-b^2-c^2+2\ d^2 \\
&-2\ e^2-2\ f^2-2\ g^2-3\ h^2-bc-bd-cd-3\ be-2\ ce-2\ bf-3\ cf \\
&-3\ ef-2\ bg-2\ cg+2\ dg-2\ eg-4\ fg-4\ bh-4\ ch+dh-6\ eh \\
&-6\ fh-4\ gh-bj-cj+dj-ej-fj-2\ gj\}\ \zeta(3) \\
&+\Big\{\ b^2+cb+\frac{5db}{4}+\frac{eb}{2}+\frac{5fb}{4}+\frac{gb}{2}+\frac{3hb}{2}+\frac{jb}{4}+c^2+3\ d^2+3\ e^2 \\
&+3\ f^2+\frac{7h^2}{2}+j^2-\frac{3ad}{4}+\frac{5cd}{4}+\frac{5ce}{4}+\frac{17de}{4}+\frac{cf}{2}+\frac{17df}{4}+\frac{17ef}{4} \\
&+\frac{5cg}{4}+6\ dg+6eg+\frac{5fg}{4}+\frac{7ah}{4}+\frac{3ch}{2}+\frac{11dh}{2}+\frac{29eh}{4}+\frac{29fh}{4} \\
&+\frac{19gh}{4}+\frac{cj}{4}+3\ dj+3\ ej+3\ fj+\frac{13hj}{4}\ \Big\}\ \zeta(4)+\dots \\
F\begin{bmatrix} a,b,d,e,g \\ c,f,h,j-1 \end{bmatrix} &= \zeta(3)-\frac{1}{4}\ (4\ a+4\ b+4\ c+5\ d+5\ e+5\ f+2\ g+2\ h+j)\ \zeta(4)+\dots
\end{aligned}
\tag{I.2}
$$

Appendix J

Basic formulas for non Abelian Yang–Mills theories

The notation in this Appendix follows the one presented in [8]. There, the field strength $F^a_{\mu\nu}$ is defined as:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{J.1}$$

with the structure constants f^{abc} for an $U(N)$ gauge group. Let us recall some basic facts about Lie Algebras. The commutation relations of a representation T^a of the Lie Algebra read:

$$[T^a, T^b] = i\ f^{abc}\ T^c, \tag{J.2}$$

with the structure constants f^{abc}. We impose the standard normalization condition

$$\mathrm{Tr}(T^a T^b) = C(r)\ \delta^{ab}\ , \tag{J.3}$$

with $C(r)$ being a constant for each representation r. Then we have the relation

$$f^{abc} = -\frac{i}{C(r)}\ \mathrm{Tr}\left(\ [T^a, T^b]\ T^c\ \right), \tag{J.4}$$

which implies, that f^{abc} is totally anti–symmetric.
The adjoint representation $r = G$ is given by the matrices

$$(T^a)_{bc} = -i\ f^{abc}\ , \tag{J.5}$$

which obviously satisfies (J.2) and (J.4). The covariant derivative $D^{\text{adj.}}_\mu$ acting on fields in the adjoint representation is introduced as:

$$(D_\lambda^{\text{adj.}})_{ab} = \partial_\lambda\ \delta_{ab} - i\ gA_\lambda^m\ (T^m)_{ab} = \partial_\lambda\ \delta_{ab} -\ g\ f_{mab}\ A_\lambda^m\ . \tag{J.6}$$

Hence, we have:

$$D_\lambda\ F_{\mu\nu}^a = \partial_\lambda\ F_{\mu\nu}^a - i\ g\ A_\lambda^m\ (T^m)_{ab}\ F_{\mu\nu}^b. \tag{J.7}$$

In addition, we derive:

$$\begin{aligned} D_\kappa D_\lambda\ F_{\mu\nu}^a &= \partial_\kappa \partial_\lambda\ F_{\mu\nu}^a + g\ f^{and}\ A_\kappa^n\ \partial_\lambda F_{\mu\nu}^d - g\ f^{mac}\ \partial_\kappa (A_\lambda^m\ F_{\mu\nu}^c) \\ &+ g^2\ f^{amc}\ f^{mnd}\ A_\kappa^n\ A_\lambda^d\ F_{\mu\nu}^c + g^2\ f^{amc}\ f^{cnd}\ A_\lambda^m\ A_\kappa^n\ F_{\mu\nu}^d. \end{aligned} \tag{J.8}$$

Finally, for $F_{\mu\nu} \equiv T^a F_{\mu\nu}^a$ in the adjoint representation, we may write (J.1) as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\ g\ [A_\mu, A_\nu]\ , \tag{J.9}$$

and (J.7) as:

$$D_\lambda = \partial_\lambda - i\ g\ [A_\lambda,\ \star\ ]\ . \tag{J.10}$$

Furthermore, equation (J.8) gives rise to:

$$\begin{aligned} D_\kappa(D_\lambda\ F_{\mu\nu}) &= \partial_\kappa\partial_\lambda\ F_{\mu\nu} - i\ g\ [A_\kappa,\ \partial_\lambda F_{\mu\nu}] - i\ g\ [\partial_\kappa A_\lambda,\ F_{\mu\nu}] \\ &- i\ g\ [A_\lambda,\ \partial_\kappa F_{\mu\nu}] - g^2\ [\ A_\kappa, [A_\lambda,\ F_{\mu\nu}]\ ] \end{aligned} \tag{J.11}$$

To this end, we may prove:

$$[D_\mu, D_\nu]\ F_{\rho\sigma} = -i\ g\ [F_{\mu\nu}, F_{\rho\sigma}]\ . \tag{J.12}$$

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