On the Interplay between the Tits Boundary and the Interior of Hadamard Spaces

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Deutsche Zusammenfassung

In dieser Arbeit geht es um nicht-positiv gekrümmte metrische Räume; wir untersuchen einige Aspekte des Zusammenspiels zwischen dem Rand im Unendlichen und dem Inneren von solchen Räumen.

Ein *Hadamard-Raum* (auch *CAT(0)-Raum* oder *nicht-positiv gekrümmter metrischer Raum*) X ist ein vollständiger metrischer Raum, in dem jedes Punktepaar durch eine kürzeste Kurve miteinander verbunden werden kann, deren Länge der Abstand der beiden Punkte ist; außerdem muss X den Dreiecksvergleich erfüllen: Für jedes Tripel von Punkten in X können wir das *euklidische Vergleichsdreieck* in der euklidischen Ebene betrachten: Es ist durch die Bedingung bestimmt, dass es die gleichen Seitenlängen hat wie das Dreieck in X. Jetzt fordern wir, dass jedes Dreieck in X *dünner* als sein Vergleichsdreieck ist.

Jeder Hadamard-Raum hat einen *Rand im Unendlichen* (auch *Tits-Rand* genannt): Er besteht aus Äquivalenzklassen von Strahlen: Zwei Strahlen sind asymptotisch zueinander, wenn sie nur beschränkten (Hausdorff-)Abstand haben. Die natürliche Metrik auf diesem Rand $\partial_\infty X$ oder $\partial_T X$ ist die *Tits Winkelmetrik* $\angle_{Tits}$, bezüglich der $\partial_T X$ ein CAT(1)-Raum wird (wieder über Dreiecksvergleich definiert, aber jetzt liegen die Vergleichsdreiecke in der runden Sphäre).

Wichtige Beispiele für Hadamard-Räume sind *symmetrische Räume von nicht-kompaktem Typ*, und ihr diskretes Analogon, *euklidische Gebäude*. Eine der vielen geometrischen Gemeinsamkeiten ist, dass der Rand im Unendlichen die Struktur eine *sphärischen Gebäudes* trägt.

In Kapitel I erläutern wir grundlegende Tatsachen über Hadamard-Räume und euklidische Gebäude, die für unsere Beweise benötigt werden.

In Proposition I.5.1 zeigen wir, dass für eine zusammenhängende, abgeschlossene Teilmenge C eines Hadamard-Raumes Konvexität eine lokale Eigenschaft ist. Dieses Ergebnis erinnert an das Theorem von Hadamard-Cartan; es ist aber keine unmittelbare Konsequenz, weil wir a priori noch nicht wissen, ob C ein geodätischer Raum ist.

Wir werden diese Proposition (in Kapitel III) verwenden, um die Konvexität der Vereinigung von (geeigneten) konvexen Mengen zu zeigen.

Kapitel II enthält die Ergebnisse aus [Bals06]: Wir ordnen einem Polygon in einem Hadamard-Raum eine Menge von gewichteten Konfigurationen auf seinem asymptotischen Rand zu (wir nennen diese Zuordnung *Gauß-Abbildung*; siehe Abbildung II.1). Jetzt ändern wir die Sichtweise und fragen: Gegeben eine gewichtete Konfiguration auf dem asymptotischen Rand eines Hadamard-Raumes, (wie) können wir entscheiden, ob sie zu einem Polygon gehört? Liegt dies nur an der Geometrie des Tits-Randes?

Aus der geometrischen Invariantentheorie gibt es den Begriff der (Semi-) Stabilität einer Konfiguration (siehe Abschnitt II.1).

Wir beantworten die erläuterte Frage, in dem wir ein Resultat von Kapovich, Leeb und Millson aus [KLM04c] verallgemeinern:

Theorem 1. *Sei X ein euklidisches Gebäude und c eine semi-stabile gewichtete Konfiguration auf seinem Rand im Unendlichen, oder sei X ein lokalkompakter Hadamard-Raum und c eine stabile gewichtete Konfiguration auf seinem Rand im Unendlichen. Dann hat die zugehörige schwache Kontraktion Φ_c einen Fixpunkt. Insbesondere gibt es ein Polygon p in X, so dass c eine Gauß-Abbildung für p ist.*

In Korollar II.5.3 findet sich eine etwas allgemeinere Aussage für Hadamard-Räume.[1]

Als unmittelbare Konsequenz können wir gewichtete Konfigurationen, die als Gauß-Abbildungen in euklidischen Gebäuden und symmetrischen Räumen auftreten, klassifizieren:

Korollar. *Sei X ein symmetrischer Raum nicht-kompakten Typs oder ein euklidisches Gebäude, und sei c eine gewichtete Konfiguration auf seinem Rand im Unendlichen.*

Dann gibt es ein Polygon, das diese Konfiguration als Gauß-Abbildung hat, genau dann wenn die Konfiguration semi-stabil im Gebäude-Fall und "nice semistable" im symmetrischen Raum-Fall ist.

Die Notwendigkeit der Semistabilität, und Theorem 1 und das Korollar im Fall, wenn X ein symmetrischer Raum oder ein lokalkompaktes euklidisches Gebäude ist, wurden in [KLM04a], [KLM04c] gezeigt. Wir erweitern ihre Ergebnisse durch die Verwendung von geeigneten Ultra-Limiten.

[1] Die Idee, diese Frage allgemein für Hadamard-Räume zu untersuchen, entstand in Diskussionen mit Vitali Kapovich und Viktor Schroeder auf einer Konferenz in Münster 2004.

Im dritten und vierten Kapitel diskutieren wir eine Frage ähnlicher Natur, bei der es um konvexe Teilmengen geht: Welche (abgeschlossenen) π-konvexen Teilmengen von $\partial_T X$ können als asymptotischer Rand einer konvexen Teilmenge von X auftreten?

Das ist wieder eine Frage der Art, inwiefern sich eine bekannte Implikation umdrehen lässt: Jede konvexe Teilmenge eines Hadamard-Raumes X induziert eine π-konvexe Teilmenge von $\partial_T X$.

Die untersuchte Frage wird zum Beispiel aus der Arbeit [KL06] motiviert, in der Bruce Kleiner und Bernhard Leeb klassifizieren, welche konvexen Teilmengen von $\partial_T X$ als asymptotischer Rand einer konvexen Teilmenge C von X auftreten, wobei C invariant unter einer Gruppe von Isometrien des symmetrischen Raumes X ist, die kokompakt auf C operiert.

Wenn $\partial_T X$ ein sphärisches Gebäude ist, und C eine π-konvexe Teilmenge von $\partial_T X$ der Dimension höchstens zwei ist, dann hat C ein Zentrum oder ist ein Untergebäude ([BL05]).

Bei den Teilmengen von $\partial_T X$ beschränken wir uns auf 0-dimensionale Untergebäude; d.h. wir betrachten Teilmengen $A \subset \partial_T X$, so dass für jedes Paar $\eta, \xi \in A$ gilt: $\angle_T(\eta, \xi) \geq \pi$.

Eine Teilmenge C eines Hadamard-Raums X ist eine *konvexe Rang 1-Teilmenge* wenn sie abgeschlossen und konvex ist, und ihr asymptotischer Rand $\partial_T C$ ein 0-dimensionales Untergebäude von $\partial_T X$ ist (siehe Def. III.3.1).

Wir werden feststellen, dass es für jedes Tripel η_1, η_2, η_3 von Punkten aus A ein ideales Dreieck geben muss; d.h., es ist notwendig, dass es Linien $l_{i,j} \subset X, i, j \in \{1, 2, 3\}$ gibt, die die Randpunkte η_i und η_j miteinander verbinden, und die paarweise stark asymptotisch sind (im gemeinsamen Endpunkt). Formal gesprochen ist es notwendig, dass die *Holonomie-Abbildung* des Tripels einen Fixpunkt hat. In Abschnitt I.1.2 wird Holonomie näher erläutert.

Die Idee, diese Frage zu untersuchen und mit Holonomie zu argumentieren, stammt von Bruce Kleiner und Bernhard Leeb, die die möglichen Ränder von konvexen Rang 1-Teilmengen von $\mathbb{R}H^2 \times \mathbb{R}H^2$ klassifizieren können.

Diese Frage hat auch Bezug zu der Arbeit [HLS00] von Hummel, Lang und Schroeder: Sie zeigen, dass in einem CAT(-1)-Raum die konvexe Hülle von endlich vielen abgeschlossenen konvexen Mengen in einer (endlichen) Tubenumgebung ihrer Vereinigung liegt.

In Kapitel III untersuchen wir, inwieweit man dies für euklidische Gebäude vom Typ A_2 verallgemeinern kann, wobei die Ausgangsmengen Tripoden sind (siehe Proposition III.3.4). (Man beachte, dass in einem CAT(-1)-Raum jede Teilmenge des asymptotischen Randes ein 0-dimensionales Untergebäude ist.)

Wir erhalten notwendige und hinreichende Bedingungen, wann ein 0-dimensionales Untergebäude von $\partial_T X$ im asymptotischen Rand einer konvexen Rang 1-Teilmenge liegt:

Theorem 2. *Sei X ein euklidisches Gebäude vom Typ A_2, und sei $A \subset \partial_T X$ ein endliches 0-dimensionales Untergebäude seines Randes. Dann gibt es eine konvexe Rang 1-Teilmenge $C \subset X$ mit $\partial_T C \supset A$ genau dann, wenn es zu jedem Tripel von Punkten aus A ein ideales Dreieck gibt. Wenn A unendlich ist, gilt die Behauptung unter einer weiteren notwendigen Voraussetzung (A muss "gut" sein, siehe Definition III.3.5).*

In Kapitel IV beantworten wir die gleiche Frage für den symmetrischen Raum $SL(3,\mathbb{R})/SO(3,\mathbb{R})$ (der auch vom Typ A_2 ist):

In unserer Notation ist $SO(n,\mathbb{C}) := SU(n)$, und die reelle und komplexe hyperbolische Ebene wird mit $\mathbb{R}H^2, \mathbb{C}H^2$ bezeichnet.

Zunächst untersuchen wir Tripel von Randpunkten und zeigen gleichzeitig für $M_{\mathbb{K}} := SL(3,\mathbb{K})/SO(3,\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$:

Theorem 3. *Sei C eine konvexe Rang 1-Teilmenge von $M_{\mathbb{K}}$ mit $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, und $\eta_i \in \partial_T C$ (für $i \in \{1,2,3\}$). Dann gibt es eine isometrische Einbettung (bis auf Reskalierung) von $\mathbb{K}H^2 \hookrightarrow M_{\mathbb{K}}$, so dass $\eta_i \in \partial_T \mathbb{K}H^2$ für alle i gilt.*

Für $\mathbb{K} = \mathbb{R}$ führt dies zur folgenden Klassifikation von konvexen Rang 1-Teilmengen von $M_{\mathbb{R}}$:

Theorem 4. *Sei C eine konvexe Rang 1-Teilmenge von $M_{\mathbb{R}}$. Dann gilt $\partial_T C \subset \partial_T \mathbb{R}H^2$ für ein geeignete isometrische Einbettung von $\mathbb{R}H^2 \hookrightarrow M_{\mathbb{R}}$ (bis auf Reskalierung).*

Obwohl wir Theorem 3 gleichzeitig für $\mathbb{R}$ und $\mathbb{C}$ zeigen können, benutzt unser Beweis von Theorem 4 ganz explizit die Geometrie von $\mathbb{R}P^2$, und lässt sich nicht unmittelbar verallgemeinern.

Das Hauptproblem bei der Verallgemeinerung ist *nicht*, dass $M_{\mathbb{C}}$ außer $\mathbb{C}H^2$ auch isometrische Kopien von $\mathbb{R}H^3$ enthält. Das Problem liegt vielmehr darin, dass sich die Ränder von verschiedenen Kopien von $\mathbb{C}H^2$ auf kompliziertere Arten schneiden können, als dies für $\mathbb{R}$ der Fall ist.

Dennoch formulieren wir aufgrund unserer Ergebnisse die folgende Vermutung:

Vermutung 5. *Sei C eine konvexe Rang 1-Teilmenge des symmetrischen Raumes X (von nicht-kompaktem Typ). Dann gibt es einen symmetrischen Unterraum $Y \subset X$ von Rang 1, so dass $\partial_T C \subset \partial_T Y$ gilt.*

Für euklidische Gebäude ist es nicht so leicht, eine Vermutung für den allgemeinen Fall zu formulieren:

Zunächst ist es so, dass man, wenn man Theorem 4 kennt und versucht, Theorem 2 zu erraten, vermuten würde, dass ein eingebetteter Baum existieren muss. Dies stellt sich als falsch heraus (aber ein Baum spielt dennoch eine wichtige Rolle, siehe Abschnitt III.3.4).

Für die drei anderen 2-dimensionalen Coxeter-Komplexe (B_2, G_2 und $A_1 \times A_1$) ist die Situation grundlegend anders: Im Gegensatz zu A_2 (wo jede Holonomie-Abbildung orientierungserhaltend ist), ist die Holonomie-Abbildung eines Paares antipodaler Punkte orientierungsumkehrend.

In diesen Fällen ist die Frage nach der Existenz von Tripoden essentiell: Wenn man zeigen könnte, dass Tripoden existieren (wie in Proposition III.3.4), würde die angesprochen Orientierungsumkehrung der Holonomie sofort zeigen, dass ein eingebetteter Baum existieren muss.

Introduction

This thesis is about non-positively curved metric spaces; in particular, we study certain aspects of the interplay between subsets of the asymptotic boundary of non-positively curved metric spaces and their interior.

A *Hadamard space* (also *CAT(0)-space* or *non-positively curved metric space*) X is a complete metric space in which every pair of points can be joined by a shortest curve whose length is precisely the distance of the points, and which satisfies triangle comparison: For every triple of points, we can consider the *Euclidean comparison triangle* in the Euclidean plane; it is a triangle with the same side lengths. Then we require that every triangle in X is *thinner* than its comparison triangle.

Every Hadamard space has a *boundary at infinity* (also called *Tits boundary* in our context): It consists of equivalence classes of rays: Two rays are asymptotic if they lie within bounded distance of each other. The natural metric on this boundary $\partial_\infty X$ or $\partial_T X$ is the *Tits angle metric* $\angle_{Tits}$, and it turns $\partial_T X$ into a CAT(1)-space (defined via triangle comparison as above, but this time, the comparison triangles lie in the standard round sphere).

Important examples of Hadamard spaces are *symmetric spaces of non-compact type*, and their discrete analogs, *Euclidean buildings.* One of the many geometric aspects these classes have in common is that their Tits boundary is a *spherical building.*

In chapter I, we introduce basic facts about Hadamard spaces and Euclidean buildings needed for the proofs of our theorems. In Proposition I.5.1, we show that for a connected, closed subset C of a Hadamard space, convexity is a local property. This result feels like a version of the Hadamard-Cartan theorem; however, it is not an immediate consequence, because we do not know that C is a geodesic space itself.

We will use this proposition (in Chapter III) to show convexity of certain unions of convex sets.

In chapter II, I present the results published in [Bals06]: We associate to a polygon in a Hadamard space a (set of) weighted configurations on its boundary (we call this assignment *Gauß map*; see Figure II.1). Then one looks in the other direction and asks: Given a weighted configuration on the boundary of a Hadamard space, (how) can we decide whether it belongs to a polygon? Is this possible purely in terms of the Tits geometry of $\partial_T X$?

Stemming from geometric invariant theory, there is a notion of (semi-) stability for weighted configurations on $\partial_T X$ (see Section II.1).

Generalizing a result of Kapovich, Leeb and Millson from [KLM04c], we give the following answer:

Theorem 1. *Let X be a Euclidean building and c a semistable weighted configuration on its boundary at infinity, or let X be a locally compact Hadamard space and c a stable weighted configuration on its boundary at infinity. Then the associated weak contraction Φ_c has a fixed point. In particular, there exists a polygon p in X such that c is a Gauß map for p.*

For a slightly more general statement in the case of a Hadamard space, see Corollary II.5.3.[2]

As an immediate consequence, we can formulate the following classification of configurations which can occur as Gauß maps on Euclidean buildings and symmetric spaces:

Corollary. *Let X be a symmetric space of non-compact type or a Euclidean building, and let c be a weighted configuration on its boundary at infinity.*

Then there exists a polygon having this configuration as a Gauß map if and only if the configuration is semistable in the building case and nice semistable in the case of a symmetric space.

Necessity of semistability, as well as Theorem 1 and its Corollary in the case where X is a symmetric space or a locally compact Euclidean building were shown in [KLM04a], [KLM04c].

We extend their ideas by suitable use of ultralimits.

In the third and fourth chapter, we discuss a question of similar nature, about convex sets: Which (closed) π-convex subsets of $\partial_T X$ can occur as the asymptotic boundary of a convex subset of X?

This is again a looking-the-other-way question, since every convex subset of a Hadamard space induces a π-convex subset of $\partial_T X$.

Examining this question arises (for example) from [KL06], where Bruce Kleiner and Bernhard Leeb classify which convex subsets of $\partial_T X$ arise as the asymptotic boundary of a convex subset C of X, which is invariant under a group of isometries of the symmetric space X acting cocompactly on C.

[2]The idea of examining this question for Hadamard spaces in general came up in discussions with Vitali Kapovich and Viktor Schroeder at the 2004 Münster conference.

If $\partial_T X$ is a spherical building, then π-convex subsets of $\partial_T X$ of dimension at most two have a center or are subbuildings ([BL05]).

In this thesis, we restrict our attention to 0-dimensional subbuildings. I.e. we consider subsets $A \subset \partial_T X$ such that every pair $\eta, \xi \in A$ satisfies $\angle_T(\eta, \xi) \geq \pi$.

A subset $C \subset X$ of a Hadamard space X is a *convex rank 1*-subset if it is closed, convex, and its asymptotic boundary $\partial_T C$ is a 0-dimensional subbuilding of $\partial_T X$ (see also definition III.3.1).

We will find that it is necessary for each triple of points η_1, η_2, η_3 of A to correspond to an ideal triangle: That is, it is necessary that there are lines $l_{i,j} \subset X, i, j \in \{1, 2, 3\}$ joining the boundary points η_i, η_j, which are pairwise strongly asymptotic (at the common endpoint). Formally speaking, it is necessary that the *holonomy map* of this triple has a fixed point. For more details on holonomy, see Section I.1.2.

The idea to examine this question, and the technique of using holonomy, are due to Bruce Kleiner and Bernhard Leeb, who can classify the possible boundaries of convex rank 1-subsets of $\mathbb{R}H^2 \times \mathbb{R}H^2$.

This topic is also related to the work [HLS00] of Hummel, Lang and Schroeder: They show that in a CAT(-1)-space, the convex hull of finitely many closed convex sets lies in a (finite) tubular neighbor of this union.

In chapter III, we examine how to generalize this to Euclidean buildings of type A_2, where the starting blocks may be considered lines, or rather tripods, see Prop. III.3.4 (Observe that every subset of the boundary of a CAT(-1)-space is a 0-dimensional subbuilding).

We give necessary and sufficient conditions, when a 0-dimensional subbuilding of $\partial_T X$ lies in the asymptotic boundary of a convex rank 1-set:

Theorem 2. *Let X be a Euclidean building of type A_2, and let $A \subset \partial_T X$ be a finite 0-dimensional subbuilding of its boundary. Then there exists a convex rank 1-subset $C \subset X$ such that $\partial_T C \supset A$ if and only if each triple of points of A corresponds to an ideal triangle.*

If A is infinite, the claim holds under an additional necessary assumption (A needs to be "good", see Definition III.3.5).

In chapter IV, we answer the same question for the symmetric space $SL(3, \mathbb{R})/SO(3, \mathbb{R})$ (which is also of type A_2):

In our notation, $SO(n, \mathbb{C}) := SU(n)$, and the real versus complex hyperbolic plane are denoted $\mathbb{R}H^2, \mathbb{C}H^2$ respectively.

First, we discuss triples of boundary points, and show simultaneously for $M_\mathbb{K} := SL(3, \mathbb{K})/SO(3, \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$:

Theorem 3. *Let C be a convex rank 1-subset of $M_{\mathbb{K}} := SL(3, \mathbb{K})/SO(3, \mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $\eta_1, \eta_2, \eta_3 \in \partial_T C$. Then there exists an isometric embedding (up to rescaling) $\mathbb{K}H^2 \hookrightarrow M_{\mathbb{K}}$ s.t. $\eta_i \in \partial_T \mathbb{K}H^2$ for all i.*

For $\mathbb{K} = \mathbb{R}$, this leads to the following classification of convex rank 1-subsets:

Theorem 4. *Let C be a convex rank 1-subset of $M_{\mathbb{R}} := SL(3, \mathbb{R})/SO(3, \mathbb{R})$. Then we have $\partial_T C \subset \partial_T \mathbb{R}H^2$ for a suitable isometric embedding (up to rescaling) $\mathbb{R}H^2 \hookrightarrow M_{\mathbb{R}}$.*

While Theorem 3 can be shown for both $\mathbb{R}$ and $fieldC$ simultaneously, our proof of Theorem 4 relies heavily on the geometry of $\mathbb{R}P^2$, and does not generalize immediately.

The key problem in generalizing our result is *not* that for $M_{\mathbb{C}}$, one has to take more symmetric subspaces of rank 1 into account (there is also a copy of $\mathbb{R}H^3 \subset M_{\mathbb{C}}$). The problem is that the boundaries of different copies of $\mathbb{C}H^2$ may intersect in more complicated ways than in the real case.

However, our results lead us to make the following conjecture:

Conjecture 5. *Let C be a convex rank 1-subset of the symmetric space X (of non-compact type).*
Then there exists a symmetric subspace $Y \subset X$ of rank 1, s.t. $\partial_T C \subset \partial_T Y$.

For Euclidean buildings, it is not as easy to make a guess for the general case:

First of all, if one tries to guess Theorem 2 after knowing Theorem 4, one would expect that a convex rank 1-subset of a Euclidean building of type A_2 is essentially a thickening of an embedded tree. This is wrong (although a tree does come in, but it is not embedded, see Section III.3.4).

For the three other 2-dimensional Coxeter complexes (B_2, G_2, and $A_1 \times A_1$), the situation is different: In contrast to A_2 (where every holonomy map is orientation preserving), the holonomy map of a pair of antipodal points is orientation reversing in the other cases.

In these cases, existence of tripods is the essential question: If one could show that tripods exist (as in III.3.4), then the orientation-reversing property of holonomy maps leads immediately to the conclusion that there exists an isometrically embedded tree.

Thanks goes to all the people who helped, supported and encouraged me during the evolution of this thesis.

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Chapter I

Preliminaries

I.1 Hadamard spaces

We will use the language of non-positively curved metric spaces, as developed in [Ball95].

Throughout, let X be a Hadamard space, unless otherwise stated. We will use the terms Hadamard space and CAT(0)-space synonymously; i.e., we impose completeness on every CAT(0)-space. Note that a Hadamard space need not be locally compact.

Recall that X has a *boundary at infinity* $\partial_\infty X$, which is given by equivalence classes of rays, where two (unit-speed) rays are equivalent if their distance is bounded.

In particular, we will use Busemann functions b_η associated to an asymptotic boundary point $\eta \in \partial_\infty X$. A Busemann function measures (relative) distance from a point at infinity, and is determined up to an additive constant only. Busemann functions are convex (along any geodesic) and 1-Lipschitz.

Geodesics, rays, and geodesic segments are always assumed to be parametrized by unit speed (i.e. they are isometric embeddings).

For a line l in X, there is the space P_l of parallel lines. P_l splits as a product $P_l \cong l \times CS(l)$, where $CS(l)$ is a Hadamard space again.

For points x, ξ with $x \in X$, $\xi \in X \cup \partial_\infty X =: \bar{X}$, and $t \geq 0$ (if $\xi \in X$, let $t \leq d(x, \xi)$), we let $\overline{x\xi}(t)$ denote the point on the segment/ray $\overline{x\xi}$ at distance t from x. When we denote a ray by $\overline{o\eta}$, we order the points such that $o \in X$ and $\eta \in \partial_\infty X$.

Definition I.1.1. For $\xi \in \partial_\infty X$ and $t \geq 0$, we define the map $\phi_{\xi,t} : X \to X$ defined by $\phi_{\xi,t}(x) := \overline{x\xi}(t)$. Observe that $\phi_{\xi,t}$ is a 1-Lipschitz map by convexity of a non-positively curved metric.

We state Lemma [BH99, II.8.3], since it will be of fundamental importance in the proof of Lemma II.2.1. It says that given one geodesic ray $\overline{o\eta}$ and another point $y \in X$, the ray $\overline{y\eta}$ can be approximated by segments $\overline{y\rho(t)}$ for t large enough, and the approximation can be controlled independently from the Hadamard space X.

Lemma I.1.2. *Given $\varepsilon > 0, m > 0$ and $c > 0$, there is a constant $K = K(\varepsilon, m, c) > 0$ such that: Let ρ be a ray $\overline{o\eta}$ in a Hadamard-space X. If $y \in X$ satisfies $d(y, o) \leq m$, then we have*

$$d(\overline{y\eta}(c), \overline{y\rho(K)}(c)) < \varepsilon. \qquad \square$$

In our notation, $B_r(o) := \{x \in X \mid d(x, o) \leq r\}$, for $o \in X, r \geq 0$; i.e. balls in Hadamard spaces are always assumed closed.

Whenever C is a closed convex subset of a Hadamard space X, then $\pi_C : X \to C$ denotes the nearest-point projection (see [BH99, II.2.4]).

I.1.1 Angles, spaces of directions, and Tits distance

Let $o \in X$ be a point in a Hadamard space, and let $\eta, \xi \in \partial_\infty X$. Let c, c' be the rays $\overline{o\eta}, \overline{o\xi}$. For points $c(t), c'(t')$, one can consider the *Euclidean comparison triangle* corresponding to the points $o, c(t), c'(t')$, i.e. the Euclidean triangle with side-lengths $d(o, c(t)), d(c(t), c'(t')), d(c'(t'), o)$ (which is well-defined up to isometries of the Euclidean plane). The *comparison angle* between $c(t)$ and $c'(t')$ at o is the angle of the comparison triangle at the point corresponding to o. It is denoted by $\tilde{\angle}_o(c(t), c'(t'))$.

We have the following monotonicity property:

$$0 < t \leq s \text{ and } 0 < t' \leq s' \text{ implies } \tilde{\angle}_o(c(t), c'(t')) \leq \tilde{\angle}_o(c(s), c'(s')).$$

From this, one can deduce a notion of angle between geodesic segments and rays:

$$\angle_o(\eta, \xi) := \lim_{t,t' \to 0} \tilde{\angle}_o(c(t), c'(t')) \in [0, \pi],$$

and an "angle at infinity", the *Tits angle* between boundary points

$$\angle(\eta, \xi) := \angle_{Tits}(\eta, \xi) := \lim_{t,t' \to \infty} \tilde{\angle}_o(c(t), c'(t')) \in [0, \pi].$$

It is easy to see that the Tits angle between η, ξ does not depend on the chosen basepoint o. The length metric induced on $\partial_\infty X$ by $\angle$ is called *Tits distance* Td, and makes $\partial_\infty X$ a CAT(1)-space. If one wants to emphasize that the Tits distance and corresponding topology on $\partial_\infty X$ is considered, this space is sometimes called $\partial_T X$. We will use these expressions synonymously (and we usually consider the Tits topology). If the Tits angle (between η, ξ) is less than π, there is a unique geodesic $\overline{\eta\xi} \subset \partial_\infty X$ connecting them.

Similarly, the space of directions $\Sigma_o(X)$, i.e. the completion of the space of starting directions of geodesic segments initiating in o (modulo the equivalence of directions enclosing a zero angle), can be regarded as a CAT(1)-space. For $o \in X, x \in \bar{X}$, we let $\overrightarrow{ox} \in \Sigma_o(X)$ be the starting direction of the segment $\overline{ox}$.

We call a subset $C \subset B$ of a CAT(1)-space B convex if it is π-convex, i.e. if all pairs of points of distance less than π can be joined by a geodesic.

I.1.2 Strong asymptote classes and holonomy

Two rays $\overline{o\eta}, \overline{x\eta}$ in a Hadamard space X are called strongly asymptotic if

$$d_\eta(\overline{o\eta}, \overline{x\eta}) := \lim_{t \to \infty} d(\overline{o\eta}(t), \overline{x\eta}) = 0.$$

This defines an equivalence relation on the set of rays asymptotic to η. The metric completion X_η of this set of equivalence classes is called the space of strong asymptote classes at η. It is a Hadamard space again (see [Kar67], [Lee00, sect. 2.1.3]).

Now assume that X is a symmetric space or a Euclidean building, and consider two antipodal points $\eta, \xi \in \partial_T X$. It is well known that the parallel set of (η, ξ), i.e. all the lines with asymptotic endpoints η, ξ, represents all the strong asymptote classes at η and at ξ.

This induces a natural isometry $h_{\eta,\xi} : X_\eta \to X_\xi$.

Such a map (and composition of such maps) is called a *holonomy map* (see [Lee00, ch. 3]).

I.2 Euclidean buildings

We will also need some Euclidean building geometry. For an introduction, we refer to [KL97, sect. 4]. A brief introduction of the notation we use can be found in [KLM04c, sect. 2.4]. Note that in particular, a Euclidean building is a Hadamard space.

A 1-dimensional Euclidean building is called a *tree*.

In a Euclidean building, we call a geodesic segment regular, if all its interior points are regular.

The boundary at infinity of a Euclidean building X of rank n is a spherical building of dimension $n-1$; we refer to [KL97, sect. 3] for an introduction.

We will use that a spherical building B is a spherical simplicial complex, where all the simplices are isometric to a spherical polytope Δ (in particular, Δ tesselates S^{n-1}), which is the *spherical Weyl chamber* of the building. Apartments (i.e. isometrically embedded copies S^{n-1}) intersect in (unions of) Weyl chambers. There is a natural map $B \to \Delta$, and the image of a point is called its *type*.

We prove some elementary lemmas which we will use later:

Lemma I.2.1. *Let X be a Euclidean building, l a line in X with $l(\infty) = \eta \in \partial_\infty X$, and c a ray asymptotic to η. Then c eventually coincides with a line parallel to l.*

Proof. Pick an apartment $A' \supset c$, and an apartment A containing $\eta^- := l(-\infty)$ in its boundary, which has the property that $\partial A = \partial A'$ near η (i.e.: let $S \subset \partial_\infty A'$ be the subset of $\partial_\infty A'$ consisting of the union of Weyl chambers containing η, and let A be an apartment containing S and η^- in its boundary).

We want to show that $c(t) \in A$ for large t, which finishes the proof.

We may assume that η is singular, since otherwise $c(t) \in A$ for large t by [KL97, L. 4.6.3].

Pick regular points $\xi_i \in S$ ($i \in \{1, 2\}$) such that η is the midpoint of $\overline{\xi_1\xi_2}$ (and $\angle_{Tits}(\xi_1, \xi_2) < \pi$).

Let c_i be the ray $\overline{c(0)\xi_i} \subset A'$. For some t_0, both $c_i(t_0) \in A \cap A'$ (again by [KL97, L. 4.6.3]). Then the midpoint of $\overline{c_1(t_0)c_2(t_0)}$ is also in $A \cap A'$; this midpoint is $c(T)$ for some T (since c_1, c_2 span a flat sector in A'), implying that $c(t)$ is in $A \cap A'$ for $t \geq T$. □

Observe that this shows in particular that the space X_η of strong asymptote classes at η is isometric to $CS(l)$.

Lemma I.2.2. *Consider a ray $\rho = \overline{o\xi}$ and a segment $\overline{op'}$ in a Euclidean building X. Then there is an apartment containing ρ and an initial part of $\overline{op'}$.*

Proof. The claim is clear if ρ and $\overline{op'}$ initially coincide or their initial directions are antipodal. So we assume that they do not. By [KL97, L. 4.1.2], there is a point $p \in \overline{op'}$, such that the triangle $D := \Delta(o, p, \xi)$ is flat (i.e. a flat half-strip). If X is discrete, the claim follows from [BL04, Prop. 1.3, Rem. 1.4]. We give a direct argument for our special situation here:

We show that D is contained in a half-plane: Let H be a flat half-strip containing D with $\partial H \supset \overline{op}$; assume that H cannot be enlarged under these conditions, and is not a half-plane. Since X is complete, we see that H is closed, i.e. of the form $\Delta(p_1, p_2, \xi)$. Now $\Sigma_{p_1} H$ is a geodesic segment, which can be prolonged to a geodesic of length π in the spherical building $\Sigma_{p_1}X$. By [KL97, L. 4.1.2], this yields a direction in which we can glue another flat half-strip to H, so H was not maximal.

Thus, D is contained in a half-plane, and this half-plane is contained in a plane by [Lee00, L. 5.2]. Finally, every plane in X is contained in an apartment by [Lee00, Cor. 5.4]. □

I.3 The geometry of spaces modeled on A_2

In chapter III, we will work with Euclidean buildings of type A_2, and in chapter IV, we examine the symmetric spaces $M_{\mathbb{K}} := SL(3, \mathbb{K})/SO(3, \mathbb{K})$ (for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), which are also of type A_2. Both kinds of spaces are geometric spaces modeled on the Coxeter complex A_2 in the sense of [KLM04b, sect. 4.2]; in particular their Tits boundaries are spherical buildings of type A_2.

The (spherical) Coxeter complex A_2 is the unit circle with the group of symmetries of an isosceles triangle acting on it (see Figure III.2). A (discrete) Euclidean Coxeter complex of type A_2 is the Euclidean plane, tesselated by isosceles triangles (see Figure III.1).

The most important example of a Euclidean building of type A_2 is the building associated to $SL(3, \mathbb{Q}_p)$; its geometry is described in detail in [Kre06].

I.3.1 The spherical building structure of $\partial_T X$

Let X be a space modeled on A_2. Then the boundary at infinity $\partial_T X$ carries the structure of a spherical building of type A_2. If X is a Euclidean building, then the space of directions $\Sigma_x(X)$ for any $x \in X$ carries such a structure as well.

Every apartment in such a building B consists of six Weyl chambers of length $\pi/3$. The vertices (the singular points of B, the ends of the Weyl chambers) have two different types (see Figure III.2).

I.3.2 Holonomy in spaces modeled on A_2

Let $\eta \in \partial_T X$ be a regular boundary point. By the remarks above, $X_\eta \simeq \mathbb{R}$. Since every apartment asymptotic to η represents all strong asymptote classes, we get an orientation on X_η (induced from a choice of orientation on Weyl chambers, determined by the two types of boundary points). Then every holonomy map $h_{\eta,\xi}$ is orientation preserving, and so is the composition

$$h_{\eta_1,\eta_2,\eta_3} := h_{\eta_3,\eta_1} \circ h_{\eta_2,\eta_3} \circ h_{\eta_1,\eta_2} : X_{\eta_1} \to X_{\eta_1}$$

for any triple $\eta_1, \eta_2, \eta_3 \in \partial_T X$ of pairwise antipodal regular boundary points. Such a holonomy map, as an orientation preserving isometry of $\mathbb{R}$, is just a translation. We will call the translation length of such a triple its *shift*.

Observe that for the other three 2-dimensional Coxeter complexes, holonomy maps are orientation-reversing. This major difference makes it hard to predict a general (2-dimensional) version of Theorem 2.

I.3.3 The space $M_{\mathbb{K}}$

We let $M_{\mathbb{K}} := SL(3, \mathbb{K})/SO(3, \mathbb{K})$ (for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).[1]

In general, $SL(n, \mathbb{K})/SO(n, \mathbb{K})$ is the space of positive definite quadratic forms of determinant 1 on $\mathbb{K}^n$. This is a symmetric space of non-compact type, so it is a Hadamard space (see [Ebe96], [BGS85], [Hel78]).

We argue using metric geometry, so let us describe the metric on $M_{\mathbb{K}}$ in geometric terms:

Given two quadratic forms $q, r \in M_{\mathbb{K}}$, it is always possible to diagonalize one with respect to the other; i.e. there is a basis v_i of $\mathbb{K}^n$ s.t. $q(v_i, v_j) = \delta_{ij}$ and $r(v_i, v_j) = b_{ij}$, where $b_{ij} = 0$ if $i \neq j$ (and $q(\cdot,\cdot), r(\cdot,\cdot)$ are the inner products on $\mathbb{K}^3$ associated to q, r, resp). Then

$$d^2(q, r) := \sum_i \left(\frac{1}{2} \log(|b_{ii}|) \right)^2 . \tag{I.1}$$

The geometry of $M_{\mathbb{K}}$ from a metric point of view is described in detail in [Lee05].

Every element g of $SL(3, \mathbb{K})$ acts by an isometry on $M_{\mathbb{K}}$ via $p(g^{-1}\cdot)$.

The singular vertices of $\partial_T M_{\mathbb{K}}$ of the one type represent 1-dimensional subspaces of $\mathbb{K}^3$, and vertices of the other type represent 2-dimensional subspaces (see [Lee05]). Two vertices span a Weyl chamber if the corresponding subspaces are incident.

The set of Weyl chambers of $\partial_T M_{\mathbb{K}}$ (also known as the *Fürstenberg boundary*) may be visualized in the projective plane $\mathbb{K}P^2$: A Weyl chamber corresponds to a point-line pair (v, p), i.e. a 1-dimensional subspace v lying inside a 2-dimensional subspace p of $\mathbb{K}^3$.

Let $r(t)$ be a ray in $M_{\mathbb{K}}$. Let v be the eigenspace of the smallest eigenvalue of $r(t)$ (note that this eigenspace does not change for t large enough; however, the eigenvalue shrinks to 0).

If v is 2-dimensional, then $r(t)$ is asymptotic to the corresponding singular vertex. Otherwise, let p be the direct sum of v with the eigenspace

[1] In our notation, $SO(n, \mathbb{C}) := SU(n), SO(2, 1, \mathbb{C}) := SU(2, 1), \ldots$, to have a uniform way of describing the groups relevant for $M_{\mathbb{K}}$.

corresponding to the second-smallest eigenvalue. If $p = \mathbb{K}^3$, then $r(t)$ is asymptotic to the singular vertex corresponding to v again. Otherwise, the asymptotic endpoint of $r(t)$ is regular and lies inside the Weyl chamber determined by (v, p). An element $g \in SL(3, \mathbb{K})$ acts on $\partial_T M_{\mathbb{K}}$ by sending the Weyl chamber (v, p) to the Weyl chamber (gv, gp). Again, this is described in more detail in [Lee05].

I.4 Ultralimits and ultraproducts

I.4.1 Ultralimits

This section introduces the notion of ultralimit, and the special cases ultraproduct and asymptotic tube, which play an important role in our proof.

We keep the general discussion of ultralimits brief and refer the interested reader to [BH99, pp. 77-80] and [KL97, sect. 2.4] for more details.

Definition I.4.1. Let ω be a (fixed) non-principal ultrafilter[2], and consider a sequence $(X_i, d_i, o_i)_i$ of metric spaces X_i with metrics d_i and basepoints o_i.

Then $X_\omega := \lim_\omega (X_i, d_i, o_i)$ is the *ultralimit* of this sequence, a space consisting of equivalence classes of sequences (x_i) with $x_i \in X_i$ and $d(x_i, o_i)$ bounded. The distance between two such sequences $(x_{i,n})_n$ (for $i \in \{1, 2\}$) is $\lim_\omega d(x_{1,n}, x_{2,n})$, the accumulation point of $(d(x_{1,n}, x_{2,n}))_n$ picked by ω. The equivalence classes consist of sequences having distance zero.

If all X_i are CAT(0), then their ultralimit is a Hadamard space; if all X_i are (additionally) geodesically complete, then every geodesic segment, ray and line in X_ω arises as ultralimit of geodesic segments, rays, and lines respectively ([KL97, 2.4.2, 2.4.4]).

If all X_i are Euclidean buildings with isometric spherical Weyl chamber, then their ultralimit is also a Euclidean building with the same spherical Weyl chamber ([KL97, sect. 5.1]).

Let us assume for the rest of this section that $(X_i, d_i)_i = (X, d)_i$ is a constant sequence, and X is a Hadamard space; so only the basepoint varies in the construction of the ultralimit X_ω.

Then there is a natural map $* : \partial_\infty X \to \partial_\infty X_\omega$, obtained by assigning to $\xi \in \partial_\infty X$ the equivalence class of rays in X_ω which has finite distance from the ray defined by the sequence of rays $\overline{o_i \xi}$. We denote the image of ξ by ξ_*.

[2] In our context, a non-principal ultrafilter is a means of (consistently) choosing an accumulation point for any bounded sequence of real numbers.

Now we can push a weighted configuration c on $\partial_\infty X$ forward to a weighted configuration c_* on $\partial_\infty X_\omega$ by mapping the ξ_i to $\xi_{i,*}$ and keeping the weights.

Lemma I.4.2. *Under the assumptions above, let Φ_* denote the weak contraction associated to the pushed forward configuration. Then Φ_* has the form*

$$\Phi_* \left((x_i)_i\right) = (\Phi(x_i))_i$$

Proof. It suffices to show that for any $\xi \in \partial_\infty X$ and a real number $m > 0$, pushing towards ξ_* by $\phi_{\xi_*,m}$ has the form given above. So let $x = (x_i)_i \in X_\omega$. Recall that by definition, the distances $d(x_i, o_i)$ are bounded. Hence, the ray $\overline{x\xi_*}$ can be represented by the ultralimit of the rays $\overline{x_i\xi}$, which implies the claim. □

I.4.2 Ultraproducts

Definition I.4.3. For a metric space X let the *ultraproduct* of X be the ultralimit of the constant sequence $(X_i, d_i, o_i) := (X, d, o)$; i.e. $X^\omega := \lim_\omega(X, d, o)$ (where we have chosen a basepoint o for X, which has no influence on the isometry type of X^ω).

There is a canonic isometric embedding $X \to X^\omega$ sending x to $(x, x, \dots)$.

Observe that if X is proper (e.g. a locally compact CAT(0)-space), the ultraproduct X^ω is isometric to X.

For details on ultraproducts, see [Lyt05, sect. 11].

I.5 Convexity is a local property in CAT(0)-spaces

Let $\varepsilon > 0$. A subset C of a CAT(0)-space X is called ε-locally convex, if for all $x \in C$, the set $B_\varepsilon(x) \cap C$ is convex. Note that a convex set is ε-locally convex for all $\varepsilon > 0$. Since an ε-locally convex set is locally path-connected, path-connectedness and connectedness are equivalent for ε-locally convex sets.

We show that if C is closed and connected, then one ε suffices to make sure that C is convex:

Proposition I.5.1. *Let $\varepsilon > 0$, and X be a CAT(0)-space. Let $C \subset X$ be a connected, closed, ε-locally convex set. Then C is convex.*

Observe that this claim is similar in nature to the Hadamard-Cartan theorem, saying that a geodesic space which is simply connected and locally CAT(0), is actually globally CAT(0). In our case, we do not know whether C is a geodesic space, so this proposition is not an immediate consequence of Hadamard-Cartan.

Proof. Since C is ε-locally connected, for every point $x \in C$, the set of points of C which can be joined to x by a *rectifiable* curve is a path component of C, hence all of C. So every pair of points of C can be joined by a rectifiable curve.

For $x, y \in C$, let $l(x, y)$ be the infimum of possible lengths of curves in C joining x and y.

We argue by induction on n and show: if $l(x, y) < n\varepsilon$, then $\overline{xy} \subset C$ (and $l(x, y) = d(x, y)$). For $n = 1$, the claim is trivial.

Assume the claim to be true for n, and let x, y be such that $l(x, y) \in [n\varepsilon, (n+1)\varepsilon)$. Let $g_m : [0, l(x, y)] \to C$ be curves of constant speed, such that $l(g_m) < (n + 1)\varepsilon$ and $l(g_m) \searrow l(x, y)$.

Let $p_m := g_m(t)$ be such that $d(p_m, y) = \varepsilon$ (such a point exists; otherwise, the claim were trivial). We have $d(x, p_m) \leq l(x, p_m) \leq l(g_m|_{[0,l(x,y)-\varepsilon]}) < n\varepsilon$, so by induction hypothesis, we have $\overline{xp_m} \cup \overline{p_m y} \subset C$, and we may assume that g_m is a parametrization of these two segments. Let $q_m := \overline{p_m x}(\varepsilon)$ (as above, q_m has to exist in order for the claim to be non-trivial: if q_m does not exist, then $\{x, y\} \subset B_\varepsilon(p_m)$, so $\overline{xy} \subset C$ by ε-local-convexity).

We examine the comparison angle $\tilde{\angle}_{p_m}(q_m, y)$: Since C is ε-locally convex around p_m, we have $\overline{q_m y} \subset C$. Therefore, the comparison angle has to be large when m is large: $d(x, q_m) + d(q_m, y) \geq l(x, y) \swarrow l(g_m)$, implies $d(q_m, p_m) + d(p_m, y) - d(q_m, y) = 2\varepsilon - d(q_m, y) \to 0$.

Hence, $\tilde{\angle}_{p_m}(q_m, y) \to \pi$. Since $q_m \in \overline{xp_m}$, we have $\tilde{\angle}_{p_m}(q_m, y) \leq \tilde{\angle}_{p_m}(x, y)$. So for large m, the union $\overline{xp_m} \cup \overline{p_m y} \subset C$ is almost a geodesic; in particular, we have $l(x, y) = d(x, y)$ and it is now immediate that the g_m converge to $\overline{xy}$, finishing the proof. □

Remark I.5.2. Let C be a closed connected subset of X, and ∂C be the (usual) boundary of C as a topological subset of X. Assume that for some $\varepsilon > 0$ and every $x \in \partial C$ we have convexity of $B_\varepsilon(x) \cap C$. Then C is $\varepsilon/2$-locally convex, hence convex.

Similarly, we have the following lemma:

Lemma I.5.3. *Let C_1, C_2 be two closed convex subsets of X and $\varepsilon > 0$. If $C_1 \cup C_2$ is connected, and $B_\varepsilon(x) \cap (C_1 \cup C_2)$ is convex for every $x \in \partial C_1 \cap \partial C_2$, then $C_1 \cup C_2$ is convex.*

Proof. First, we will show that for every $x \in C_1 \cap C_2$, we have convexity of $B_{\varepsilon/2}(x) \cap (C_1 \cup C_2)$. Then we show that this is sufficient.

So let $x \in C_1 \cap C_2$. We show directly that for $y, y' \in B_{\varepsilon/2}(x) \cap (C_1 \cup C_2)$, we have $\overline{yy'} \subset C_1 \cup C_2$. Assume that this is not the case (for some x, y, y' as above). Then by assumption, we have $\partial C_1 \cap \partial C_2 \cap B_{\varepsilon/2}(x) = \emptyset$.

Without loss of generality, we have $y \in C_1 \backslash C_2$ and $y' \in C_2 \backslash C_1$. Let $y_t := \overline{yy'}(t)$, and let z_t be the endpoint of the segment $\overline{xy_t} \cap C_1$. Let $T \geq 0$ be the minimal real number such that for the interval $(T, d(y, y')]$, we have $z_t \neq y_t$. For (the closure of) this interval, we have $z_t \in \partial C_1$.

Similarly, define z'_t, and obtain an interval $[0, T')$ such that in this interval, $z'_t \in \partial C_2$. By assumption, there is a point $y_{T''} \notin C_1 \cup C_2$, so the two intervals introduced above intersect.

Consider the functions $d(x, z_t)/d(x, y_t)$ and $d(x, z'_t)/d(x, y_t)$ on $[T, T']$. Both are continuous (observe that $x \notin \overline{yy'}$), and by the intermediate value theorem, they are equal at some point. But then, we have found a point of $\partial C_1 \cap \partial C_2 \cap B_{\varepsilon/2}(x)$, in contradiction to the assumption that $B_{\varepsilon/2}(x) \cap (C_1 \cup C_2)$ is not convex.

Now, we want to show that for any $x \in C_1 \cup C_2$, the set $B_{\varepsilon/8}(x) \cap (C_1 \cup C_2)$ is convex.

Assume that this is not the case for some x, y, y' as above. Note that by the discussion above, we have $d(x, C_1 \cap C_2) > 3\varepsilon/8$. Hence, we have

$$\min(d(y, C_1 \cap C_2), d(y', C_1 \cap C_2)) > \varepsilon/4, \text{ and trivially } d(y, y') \leq \varepsilon/4.$$

Let $p := \pi_{C_1 \cap C_2}(y)$. The comparison angle satisfies $\tilde{\angle}_p(y, y') < \pi/3$.

Let $z := \overline{py}(\varepsilon/8), z' := \overline{py'}(\varepsilon/8)$. By the remark about the comparison angle above, $d(z, z') < \varepsilon/8$ and $d(y, q) < d(y, p)$ for every $q \in \overline{zz'}$. Note that $z \in C_1, z' \in C_2$. Convexity of $B_{\varepsilon/2}(p) \cap (C_1 \cup C_2)$ implies that $\overline{zz'} \cap C_1 \cap C_2 \neq \emptyset$. This is a contradiction to the definition of p. □

We will use the following reformulation quite often:

Corollary I.5.4. *Let C_1, C_2, K be closed convex subsets of X, and let $\varepsilon > 0$. Assume that $(C_1 \cup C_2) \cap K$ is connected, and that for every $x \in \partial C_1 \cap \partial C_2 \cap K$, we have convexity of $B_\varepsilon(x) \cap K \cap (C_1 \cup C_2)$.*

Then $(C_1 \cup C_2) \cap K$ is convex.

Proof. Since K is a CAT(0)-space itself, this is just a redraft of the previous lemma. □

Chapter II

Polygons with prescribed Gauß map in Hadamard spaces and Euclidean buildings

In this chapter, we present the results from [Bals06]; in particular, we prove Theorem 1.

We show that given a stable weighted configuration on the asymptotic boundary of a locally compact Hadamard space, there is a polygon with Gauß map prescribed by the given weighted configuration. Moreover, the same result holds for semistable configurations on arbitrary Euclidean buildings.

In the first section, we introduce the concepts needed to state and prove our Theorems. In particular, we define stability for weighted configurations on the boundary at infinity of a Hadamard space.

In the second section, we introduce asymptotic tubes, a special case of ultralimits, which are an important tool in our proofs.

In sections 3-5, we prove our results, and in the last section, we discuss relations to algebra.

II.1 Weighted configurations at infinity

Let us recall some notions from [KLM04a] and [KLM04c] needed to discuss the relationship of configurations on $\partial_\infty X$ and polygons in X.

Definition II.1.1. Let X be a Hadamard space. A *weighted configuration c on $\partial_\infty X$* is an n-tuple of points $(\xi_1, \ldots, \xi_n)$ in $\partial_\infty X$ together with a weight function $m : \{1, \ldots, n\} \to \mathbb{R}_{>0}$.

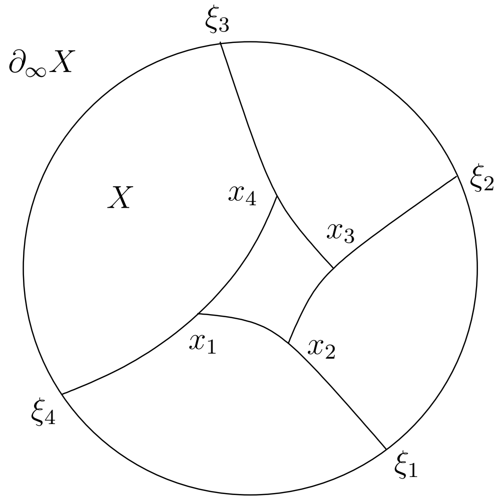

Figure II.1: Gauß maps

There is a *weighted Busemann function* associated to a weighted configuration c. It is given by

$$b_c := \sum_{i=1}^{n} m_i b_{\xi_i};$$

weighted Busemann functions are convex, asymptotically linear, Lipschitz-continuous, and well-defined up to an additive constant. As for any convex, asymptotically linear Lipschitz-function on a Hadamard space, we can associate a function $\text{slope}_{b_c} : \partial_\infty X \to \mathbb{R}$ to a weighted Busemann function, which is given by assigning the asymptotic slope of b_c on a ray $\overline{o\xi}$ to the point ξ. Since two rays asymptotic to the same boundary point have bounded distance and b_c is Lipschitz, the slope does not depend on the choice of o, so slope_{b_c} is well-defined (see also [KLM04a, sect. 3]).

We have

$$\text{slope}_c(\xi) := \text{slope}_{b_c}(\xi) = -\sum_{i=1}^{n} m_i \cos\angle(\xi_i, \xi).$$

The configuration c is called *semistable* if $\text{slope}_c \geq 0$, and it is called *stable* if $\text{slope}_c > 0$.

Observe that (semi-)stability is defined purely in terms of the Tits-geometry of $\partial_\infty X$, without reference to X itself.

Now we discuss the relation between polygons and weighted configurations:

Consider a polygon p in X, which is determined by an n-tuple of points $(x_1, \ldots, x_n)$ (with $x_i \neq x_{i+1}$ for all $i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ [1]). We can associate a set of weighted configurations $\mathcal{G}(p)$ on $\partial_\infty X$ to p, by choosing ξ_i such that $x_{i+1} \in \overline{x_i\xi_i}$, and setting $m_i := d(x_i, x_{i+1})$. Then all $c \in \mathcal{G}(p)$ are semistable by [KLM04c, Lemma 4.3] (their proof generalizes without problems). Observe that (if X is not geodesically complete) it may happen that $\mathcal{G}(p) = \emptyset$.

An element $c \in \mathcal{G}(p)$ is called a *Gauß map* for p (since this construction, in the case of the hyperbolic plane, was mentioned in a letter from Gauß to Bolyai, [Gau63]).

On the other hand, consider a weighted configuration c.

Let

$$\Phi_c := \phi_{\xi_n,m_n} \circ \cdots \circ \phi_{\xi_1,m_1}.$$

Since a composition of 1-Lipschitz maps is 1-Lipschitz, Φ_c is 1-Lipschitz, i.e. a weak contraction. Every fixed point of Φ_c is a first vertex of a polygon p with $c \in \mathcal{G}(p)$.

A more general discussion of measures on $\partial_\infty X$ (if X is a symmetric space or Euclidean building) can be found in [KLM04a], [KLM04c].

II.2 Asymptotic tubes

One of the main ideas in the proof of Theorem 1 is that the weak contraction Φ_c associated to a weighted configuration asymptotically moves a ray to a parallel ray.

We make this idea precise by using particular ultra-limits.

Throughout this section, X will be a Hadamard space and $\rho = \overline{o\eta}$ will be a ray in X.

Let $\xi \in \partial_\infty X$. The following lemma says that pushing towards η and ξ asymptotically commutes when moving out along ρ.

Lemma II.2.1. *Let $m, c > 0$ and $\xi \in \partial_\infty X$. Then $\lim_{t\to\infty} d(\phi_{\xi,m}\circ\phi_{\eta,c}\circ\rho(t), \phi_{\eta,c}\circ \phi_{\xi,m} \circ \rho(t)) = 0$.*

Proof. Let $o_t := \rho(t), x_t := \phi_{\eta,c}(o_t) = \rho(t + c), y_t := \phi_{\xi,m}(o_t)$, and $\hat{\alpha} := \angle(\eta, \xi)$. We may assume $\hat{\alpha} \neq 0$, since otherwise $\eta = \xi$, and there is nothing to show.

If we set $z_t := \phi_{\eta,c}(y_t)$, then the claim is $d(z_t, y_{t+c}) \underset{t\to\infty}{\longrightarrow} 0$.

Let $\varepsilon > 0$ be given.

[1] For notational convenience, we consider the indices modulo n.

1. Let $K = K(\varepsilon, m, c)$ be the constant from Lemma I.1.2. We may assume $K \geq c$.

   Let $z'_t := \overline{y_t\rho(t+K)}(c)$. We have $d(z'_t, z_t) \leq \varepsilon$, so we try to get information about $d(z'_t, y_{t+c})$.

2. Let $\bar{\alpha} < \hat{\alpha}$ be such that for a Euclidean triangle ABC with sides AC, AB of length $K-c, m$ respectively, the length of the third side varies by at most ε when the angle at A varies in the interval $[\bar{\alpha}, \hat{\alpha}]$.

   Let l be the maximal length of the third side (occurring when the angle is equal to $\hat{\alpha}$).

3. Observe that in a Euclidean triangle ABC with sides AC, AB of length K, m respectively, and angle at A in the interval $[\bar{\alpha}, \hat{\alpha}]$, the third side has length at least $c + (l - 2\varepsilon)$.

   Since the constant K from Lemma I.1.2 is independent from the Hadamard space (so we may choose $X = \mathbb{R}^2$ here), the claim follows from (2).

4. Finally, let $T > 0$ be such that for $t > T$, we have $\bar{\alpha} \leq \alpha_t := \angle_{o_t}(\eta, \xi) \leq \hat{\alpha}$ (observe that the second inequality is trivial).

Now we consider the triangle $\Delta(y_t y_{t+c} \rho(t+K))$ for $t > T$.
Since the angle corresponding to α in the comparison triangle is in the interval $[\bar{\alpha}, \hat{\alpha}]$, (2) implies $d(y_{t+c}, \rho(t+K)) \leq l$; and since $\phi_{\xi,m}$ is 1-Lipschitz, we have $d(y_t, y_{t+c}) \leq c$. On the other hand, we have $d(y_t, \rho(t+K)) \geq c + l - 2\varepsilon$ by (3).

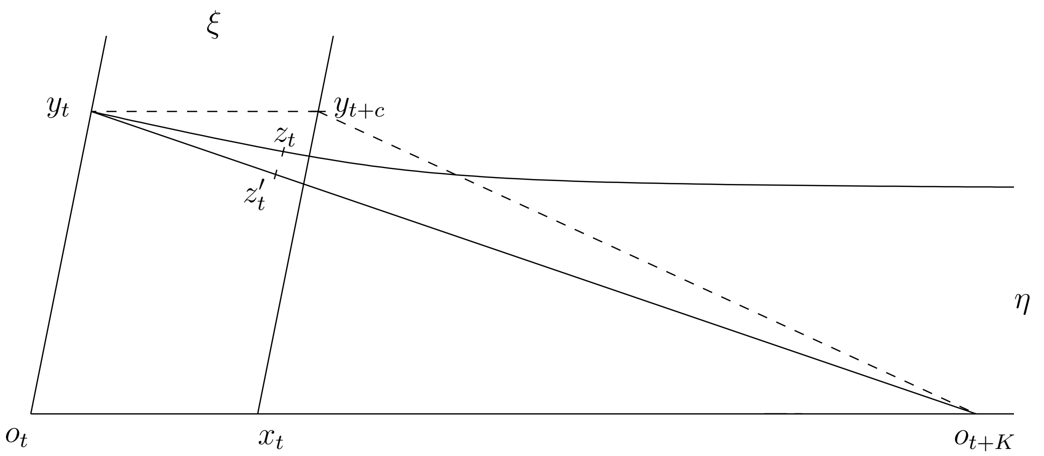

Figure II.2: The points from the proof of Lemma II.2.1

Considering the Euclidean comparison triangle, this shows that we have control over $d(z'_t, y_{t+c})$, and this quantity becomes arbitrarily small as ε goes to zero. With (1), this finishes the proof. $\square$

Definition II.2.2. In the situation described above, define the CAT(0)-space

$$X_\omega := \lim_\omega(X, d, \rho(i)).$$

Observe that in X_ω, the image of ρ is a line l. Let $T_\eta = T_\rho := P_l$, and call this space the *asymptotic tube* of η (it is easy to see that $T_\rho \cong T_{\rho'}$ if ρ and ρ' are rays asymptotic to η).

Consider the map $* : \partial_\infty X \to \partial_\infty X_\omega$ introduced at the end of section I.4.1.

Lemma II.2.3. *We have* $* : \partial_\infty X \to \partial_\infty T_\eta$,
and for any $\xi \in \partial_\infty X$*, we have* $\angle(\xi, \eta) = \angle(\xi_*, \eta_*)$.

Proof. Let $\xi \in \partial_\infty X$ and $m > 0$. We claim that the map

$$l_{\xi,m} : t \mapsto (\phi_{\xi,m} \circ \rho(i+t))_i$$

defines a line parallel to l in X_ω (for given t, we set the coordinates with $i + t < 0$ arbitrarily; since these are finitely many, they have no influence on the point defined in X_ω): Indeed, by the Lemma above, the following equality holds in X_ω (for $t' > t$):

$$l_{\xi,m}(t') = (\phi_{\xi,m} \circ \phi_{\eta,t'-t} \circ \rho(i+t))_i = (\phi_{\eta,t'-t} \circ \underbrace{\phi_{\xi,m} \circ \rho(i+t)}_{\text{defining } l_{\xi,m}(t)})_i.$$

The right hand side shows $d(l_{\xi,m}(t'), l_{\xi,m}(t)) = t' - t$ for $t' \geq t$; hence, $l_{\xi,m}$ is a geodesic line. Clearly, $l_{\xi,m}$ stays within bounded distance of l, so it is parallel to l (by [BH99, II.2.13]).

For given t, we have $\angle_{\rho(i+t)}(\xi, \eta) \underset{i\to\infty}{\to} \angle(\xi, \eta)$ and $\angle_{\rho(i+t)}(\rho(0), \xi) \to \pi - \angle(\xi, \eta)$ (by [Ball95, Prop. 4.2]), so we find $d(l_{\xi,m}, l) = m \sin\angle(\xi, \eta)$.

It is clear that the flat strip spanned by $l_{\xi,m'}$ and l contains $l_{\xi,m}$ for $m' > m > 0$, so ξ determines a half-plane in P_l if $\angle(\eta, \xi) \neq 0, \pi$. In the other cases, $l = l_{\xi,m}$. $\square$

The following observation is immediate from the previous lemma:

Lemma II.2.4. *Let c be a weighted configuration on $\partial_\infty X$, and consider the map $* : \partial_\infty X \to \partial_\infty T_\rho$. Then* $\mathrm{slope}_c(\eta) = \mathrm{slope}_{c_*}(\eta_*)$. $\square$

Remark II.2.5. One can show that $*$ also has the following properties:

The half-planes determined by ξ, ξ' agree if the geodesic segments $\overline{\eta\xi}, \overline{\eta\xi'}$ start in the same direction. The induced map between the spaces of directions $\Sigma_\eta(\partial_\infty X) \to \Sigma_{\eta_*}(\partial_\infty T)$ is 1-Lipschitz, but not an isometric embedding in general.

We show below that in a Euclidean building, one even gets a map (with the properties we need) $* : \partial_\infty X \to \partial_\infty P_l$ for a line l containing ρ. The same result holds for symmetric spaces of noncompact type.

The question arises whether in a general Hadamard space, one can get a suitable map to the boundary of $\mathbb{R} \times X_\eta$, the space of parametrized strong asymptote classes at η (see [Kar67], [Lee00, sect. 2.1.3], [KLM04a, sect. 3.1.2]).

However, consider the following subset of the Euclidean plane:

$$X = \{(x, y) \,|\, x \geq 1, y \geq \log x\}$$

With the induced length metric, X becomes a Hadamard space; the boundary at infinity is an arc of length $\frac{\pi}{2}$. Consider the boundary point η corresponding to the ray ρ in X which is given by parametrizing the graph of the logarithm with unit speed. Then X_η consists of one point only (every ray asymptotic to η eventually lies on the graph of the logarithm), but T_η is a half-plane.

II.2.1 Asymptotic tubes in Euclidean buildings

In the case where X is a Euclidean building or a symmetric space, the construction described above specializes to the folding map described in [KLM04a, sect. 3.2.5]. We discuss the building case:

Lemma II.2.6. *Let X be a Euclidean building, $\rho = \overline{o\eta}$ a ray in X, and l a line extending ρ. Let T be the asymptotic tube associated to ρ. Then there is a natural isometric embedding $\iota : P_l \to T$, and we have $Im(*) \subset \partial_\infty(\iota(P_l))$.*

Proof. We state an explicit formula for ι: We map $p \in P_l$ to $(\phi_{\eta,i}(p))_i$.

Since $\phi_{\eta,t}|_{P_l}$ is an isometry of P_l for every $t \geq 0$, the first claim holds.

Let $\xi \in \partial_\infty X$ be a boundary point of X. For t large enough, the rays $\overline{\rho(t)\eta}$ and $\overline{\rho(t)\xi}$ bound a Euclidean sector (by discreteness of the angle, see [KL97, Axiom 4.1.2.EB2]). This shows that $\phi_{\xi,m}$ eventually maps the ray ρ to a parallel ray. Since this ray eventually coincides with a line parallel to l by Lemma I.2.1, the claim follows. □

An immediate consequence is:

Lemma II.2.7. *Let c be a weighted configuration on the boundary of the Euclidean building X. Let l be a line with $\lim_{t\to\pm\infty} l(t) = \xi_\pm$. Let c_* denote the weighted configuration on $\partial_\infty P_l$ obtained from c via Lemma II.2.6. Then there exists $T > 0$ such that*

$$\forall t > T : \Phi_c \circ l(t) = \Phi_{c_*} \circ l(t).$$

Proof. In the proof of Lemma II.2.6, we showed that the definition of ι implies that the claim holds for configurations consisting of a single point, i.e. for maps $\phi_{\xi,m}$.

Since Φ_c, Φ_{c_*} are finite compositions of such maps, the lemma follows. □

For Euclidean buildings, we obtain the following refinement of Lemma II.2.4:

Lemma II.2.8. *Let X be a Euclidean building, and let c be a weighted configuration on its boundary at infinity. Let $\eta \in \partial_\infty X$, and l a line asymptotic to η. Consider the measure c_* on $\partial_\infty P_l$ obtained via Lemma II.2.6. Then*

$$\mathrm{slope}_c = \mathrm{slope}_{c_*}$$

on a neighborhood of η.

Proof. Let U be the neighborhood of η consisting of points lying in a common Weyl chamber with η, and let $\xi \in U, \xi' \in \partial_\infty X$. It follows from the proof of Lemma II.2.6 that $\angle(\xi, \xi') = \angle(\xi_*, \xi'_*)$, since the triangles $\xi\eta\xi'$ and $\xi_*\eta_*\xi'_*$ are isometric (both are spherical, have two sides of the same length, and have the same angle at $\eta_{(*)}$). □

Lemma II.2.9. *Let X be a Euclidean building, and c a semistable configuration on its boundary at infinity. Let $\eta \in \partial_\infty X$ be a point with $\mathrm{slope}_c(\eta) = 0$, and l a line asymptotic to η. Consider the measure c_* on $\partial_\infty P_l$ obtained via Lemma II.2.6. Then c_* is semistable on P_l.*

Proof. The measure c_* is supported on the product $l \times CS(l)$, and

$$\mathrm{slope}_{c_*}(\eta_*) = \mathrm{slope}_c(\eta) = 0.$$

Thus for the antipode η_*^- of η_*, we have $\mathrm{slope}_{c_*}(\eta_*^-) = -\mathrm{slope}_{c_*}(\eta_*) = 0$.

For a point ξ on $\partial_\infty P_l$ which has distance less than π from η_*, the claim $\mathrm{slope}_{c_*}(\xi) \geq 0$ follows from (strict) convexity of the zero-sublevel set of slope_{c_*} ([KLM04a, Prop. 3.1.(ii)], together with Lemma II.2.8. □

II.3 Projecting rays to subspaces

We examine how rays project to a subspace of a Hadamard space:

Proposition II.3.1. *Let X' be a Hadamard space and $X \subset X'$ a closed convex subset. Consider $\eta \in \partial_\infty X'$ such that $\angle(\eta, \partial_\infty X) < \frac{\pi}{2}$. Let $o \in X$, $\rho := \overline{o\eta}$, and $\pi : X' \to X$ be the nearest point projection. Then the segments $\overline{o\,(\pi \circ \rho(t))}$ converge to the ray $\overline{o\xi}$ (in the cone topology), where $\xi \in \partial_\infty X$ is the unique point with $\angle(\eta, \xi) = \angle(\eta, \partial_\infty X)$.*

Proof. Observe that $\partial_\infty X$ is a closed convex subset of $\partial_\infty X'$ (it is even closed in the cone topology); since $\angle(\eta, \partial_\infty X) < \frac{\pi}{2}$, the projection ξ of η exists and is unique ([BH99, II.2.6]).

Let $\bar{\alpha} := \angle(\eta, \xi)$, $c_t := \rho(t)$, $p_t := \pi(c_t)$, and $\alpha_t := \tilde{\angle}_o(c_t, p_t)$.

By considering triangles D of the form $\Delta(o, c_t, \overline{o\xi}(t))$, we conclude

$$d(c_t, p_t) \leq t \sin \bar{\alpha}$$

(since the comparison triangle of D has angle at most $\bar{\alpha}$ at o, the CAT(0)-condition gives the upper bound on $d(c_t, p_t)$); this implies that $\alpha_t \leq \bar{\alpha}$ for all $t > 0$.

Since $d(c_t, p_t) \leq t \sin \bar{\alpha}$, we have $d(o, p_t) \geq t(1 - \sin \bar{\alpha})$. Thus, for $s(1 - \sin \bar{\alpha}) \geq t$, the same argument as for the boundedness of α_t shows $\alpha_t \leq \alpha_s$ $(*)$.

Let $t_n := (1 - \sin \bar{\alpha})^{-n}$ for $n \in \mathbb{N}$ (observe that $\bar{\alpha} \geq \alpha_t > 0$ as soon as $c_t \notin X$). By what we have shown, α_{t_n} is an increasing bounded sequence, which converges to some $\hat{\alpha} \leq \bar{\alpha}$.

Given $\varepsilon > 0$, let N be such that $\alpha_{t_N} \geq \hat{\alpha} - \varepsilon$. Then for $t \geq t_{N+1}$ (so $t \in [t_n, t_{n+1}]$ for some $n > N$), we have $\hat{\alpha} - \varepsilon \leq \alpha_{t_N} \leq \alpha_t \leq \alpha_{t_{n+2}} \leq \hat{\alpha}$ by $(*)$. Hence $\alpha_t \underset{t\to\infty}{\to} \hat{\alpha}$.

We will show next that $d(p_t, \overline{op_s})/t \to 0$ for s, t large; since $d(p_t, o) \geq t(1 - \sin \bar{\alpha})$, this implies that the segments $\overline{op_t}$ converge to a ray.

For $s(1 - \sin \bar{\alpha}) \geq t$, let $p_{s,t}$ be the projection of c_t to the segment $\overline{op_s}$. For $\varepsilon > 0$, there exists T such that $t \geq T$ implies $\sin \alpha_t \geq \sin \hat{\alpha} - \varepsilon$. Then for $s(1 - \sin \bar{\alpha}) \geq t \geq T$, we have $d(c_t, p_t) \geq t(\sin \alpha_t) \geq t(\sin \hat{\alpha} - \varepsilon)$ and $d(c_t, p_{s,t}) \leq t \sin \alpha_s \leq t \sin \hat{\alpha}$.

Consider the comparison triangle $\Delta(c_t, p_t, p_{s,t})$. Since p_t is the projection of c_t to X, its angle at p_t is at least $\frac{\pi}{2}$. Hence for the comparison angle $\gamma_{s,t} := \tilde{\angle}_{c_t}(p_t, p_{s,t})$, we have $\cos \gamma_{s,t} \geq \frac{\sin \hat{\alpha} - \varepsilon}{\sin \hat{\alpha}} \underset{\varepsilon \to 0}{\to} 1$.

Thus

$$d(p_t, p_{s,t})/t \underset{\varepsilon \to 0, s(1-\sin \bar{\alpha}) \geq t \geq T_\varepsilon}{\longrightarrow} 0.$$

This shows that the segments $\overline{op_t}$ converge to a ray $\overline{o\hat{\xi}}$ for some $\hat{\xi} \in \partial_\infty X$.

By [KL97, L. 2.3.1], we have $\angle(\eta, \hat{\xi}) \leq \liminf_{t\to\infty} \tilde{\angle}(c_t, p_t) = \hat{\alpha} \leq \bar{\alpha}$. Hence, $\hat{\alpha} = \bar{\alpha}$ and $\hat{\xi} = \xi$. □

Proposition II.3.2. *Let X' be a Hadamard space and $X \subset X'$ a closed convex subset. Consider $\eta \in \partial_\infty X'$, and assume that for some $o \in X$, the projection of the ray $\overline{o\eta}$ to X is bounded, i.e. there is m s.t. $d(o, \pi \circ \overline{o\eta}(t)) < m$ for all $t > 0$.*

Then there exists a point $p \in X$ s.t. $\pi \circ \overline{p\eta}(t) = p$ for all $t > 0$.

Proof. Let $c_t := \overline{o\eta}(t)$ and $p_t := \pi(c_t)$.

Let $t_1 := 1$, and define t_n inductively by $t_n := K(\frac{1}{n}, m, t_{n-1})$, where K is the constant from Lemma I.1.2. Observe that t_n is strictly increasing and unbounded.

Observe that $\pi(\overline{p_{t_n} c_{t_n}}(t_{n-1})) = p_{t_n}$. Since π is 1-Lipschitz, we get from Lemma I.1.2 that $d(p_{t_n}, \pi(\overline{p_{t_n}\eta}(t_{n-1}))) < \frac{1}{n}$.

We consider the ultraproducts $X^\omega \subset (X')^\omega$. Let $\pi_{X^\omega} : (X')^\omega \to X^\omega$ be the projection. Note that π_{X^ω} can be given in the form

$$\pi_{X^\omega}(x_n)_n = (\pi(x_n))_n.$$

Then $p' := (p_{t_n})_n$ is a point in X^ω which satisfies $\pi_{X^\omega}(\overline{p'\eta}(t)) = p'$ for all $t > 0$.

Now let p be the projection of p' to X. By the above, we have $\pi_{X^\omega}|_{X'} = \pi$, so $\pi_{X^\omega}(\overline{p\eta}(t)) \in X$. On the other hand, $d(\pi_{X^\omega}(\overline{p\eta}(t)), p') \leq d(p, p') = d(p', X)$, so the projection of the ray $\overline{p\eta}$ is constant. □

Remark II.3.3. Observe that a point with the properties from the Lemma above is a global minimum of the Busemann function $b_\eta|_X$.

Note also, that the example from Remark II.2.5 shows that the assumption of the proposition above needs not be fulfilled if $\angle(\eta, \partial_\infty X) \geq \frac{\pi}{2}$.

II.4 Persistence of semistability

Now persistence of semistability follows easily:

Proposition II.4.1. *Let $X \subset X'$, where X is a closed convex subset of the Hadamard space X', and let c be a weighted configuration on the asymptotic boundary of X. If c is semistable on X, then c is semistable on X'.*

Proof. Assume there is $\eta \in \partial_\infty X'$ with $\operatorname{slope}_c(\eta) = -c < 0$. From the formula for the slope, we conclude that there must be some ξ_i in the support of c which satisfies $\angle(\eta, \xi_i) < \frac{\pi}{2}$. Hence Proposition II.3.1 applies.

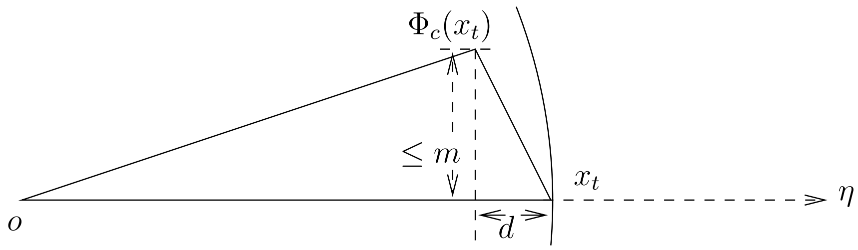

Figure II.3: For t large, $d(o, \Phi_c(x_t)) < d(o, x_t)$.

From this point, we obtain a contradiction as in the end of the proof of [KLM04a, L. 3.10.ii]:

Use the notation of the proof above, and for $s \geq t$, let $\bar{p}_{s,t} := \overline{op_s}(\frac{t}{s} d(o, p_s))$. We may normalize b_c such that $b_c(o) = 0$. Then by convexity, we have $b_c(c_s) \leq -cs$. As in the proof of [KLM04a, L. 3.10], we have $b_c \geq b_c \circ \pi$ (where π is the projection $X' \to X$). In particular, $b(p_s) \leq -cs$.

For $s \geq t$, we conclude from convexity that $b_c(p_{s,t}) \leq -ct$. Fixing t and letting $s \to \infty$, this shows $b_c(\overline{o\xi}(t \cos \hat{\alpha})) \leq -ct$, implying $\text{slope}_c(\xi) \leq -c/\cos\hat{\alpha} < 0$. This is the desired contradiction. □

Remark II.4.2. Observe that we cannot expect stability to be preserved under general embeddings, as one sees e.g. by embedding X into $X \times \mathbb{R}$.

We will only use the above proposition for the inclusion $X \subset X^\omega$. However, we may not expect stability to be preserved in this case either, as the following example shows:

Consider the disjoint union of copies of $\mathbb{H}^2 \times [-n, n]$ for $n \in \mathbb{N}$, identified along $\mathbb{H}^2 \times \{0\}$. This is a Hadamard space by [BH99, II.11.3]. Its boundary is precisely the boundary of $\mathbb{H}^2$, but its ultraproduct contains a copy of $\mathbb{H}^2 \times \mathbb{R}$.

II.5 Existence of Polygons

In this section we present the proof of Theorem 1.

We will need a lemma about fixed points of weak contractions, which we recall without proof:

Lemma II.5.1 ([KLM04c, Lemma 4.5]). *Let X be a Hadamard space of finite diameter. Then every weak contraction $\Phi : X \to X$ has a fixed point.* □

The following lemma was essentially contained in an earlier version of [KLM04c]:

Lemma II.5.2. *Let c be a weighted configuration on the boundary of a Hadamard space of the form $l \times Y$, where l is a line with endpoints η, η_-, and Y is a Hadamard space.*

If $\mathrm{slope}_c(\eta) > 0$, *then there exists $T > 0$ such that*

$$d(\Phi_c((l(t), y)), (l(0), y)) < t \text{ for all } t > T \text{ and } y \in Y.$$

Proof. The configuration c can be split into configurations c_1, c_2 on $\{\eta, \eta_-\}$, $\partial_\infty Y$ respectively, and this splitting is compatible with the action of Φ (see [KLM04a, L. 3.12]). In particular, we have $(b_\eta \circ \Phi_c - b_\eta) \equiv \mathrm{slope}_c \eta =: d > 0$.

Let $o := (l(0), y)$ and $x_t := (l(t), y)$. The triangle $\Delta(o, x_t, \Phi_c(x_t))$ is Euclidean, so the claim follows from the fact that the displacement of Φ_c is bounded (by $m := \sum_{i=1}^n m_i$); see figure II.5. □

Now we have all ingredients for the proof of Theorem 1; we start with the building case:

Theorem 6. *Let X be a Euclidean building, and let c be a semistable weighted configuration on its boundary at infinity. Then the associated weak contraction Φ_c has a fixed point. In particular, there exists a polygon p in X such that c is a Gauß map for p.*

Proof. Fix a basepoint $o \in X$. If we find a ball $B(o, R) \subset X$ which is preserved by Φ, we are done by Lemma II.5.1.

We argue by contradiction: Assume that for each $i \in \mathbb{N}$, there exists a point $x_i \in X$ such that $d(o, x_i) \geq i$ and $d(\Phi(x_i), o) \geq d(x_i, o)$ $(*)$. Observe that $(*)$ holds for each $x \in \overline{ox_i}$ since Φ is a weak contraction.

The segments $\overline{ox_i}$ define a ray $\rho = \overline{o\eta}$ in the ultraproduct X^ω (for some $\eta \in \partial_\infty X^\omega$): We have

$$\rho(t) = (\overline{ox_i}(t))_i$$

where we set $\overline{ox_i}(t) := o$ for $i < t$ (clearly, these finitely many points have no influence on the point defined in X^ω).

Let c_* be the configuration c considered as a configuration on $\partial_\infty X^\omega$, and let Φ_* be the associated weak contraction. Now ρ satisfies $d(\Phi_*(\rho(t)), o) \geq d(\rho(t), o) = t$ for all t, since we have

$$d(\Phi_*(\rho(t)), o) = \lim\nolimits_\omega \underbrace{d(\Phi(\overline{ox_i}(t)), o)}_{\geq t \text{ if } i \geq t} \geq t = d(\rho(t), o). \qquad (\dagger)$$

By Proposition II.4.1, there are two cases to be considered:

Case 1: $\text{slope}_{c_*}(\eta) > 0$*:* We consider the asymptotic tube T_η, and the pushed forward configuration, which we denote by c_{**}; the associated weak contraction will be denoted by Φ_{**}.

Let l be the line which is obtained from ρ when passing to the asymptotic tube. By Lemma II.2.4 and Lemma II.5.2, we have $d(\Phi_{**} \circ l(t), l(0)) < t$ for large t. This implies that for large t and ω-almost all i, we have $d(\Phi_* \circ \rho(i + t), \rho(i)) < t$.

By the triangle inequality, this implies $d(\Phi_* \circ \rho(i + t), o) < i + t$, in contradiction to (†).

Case 2: $\text{slope}_{c_*}(\eta) = 0$*:* We argue by induction on $\text{rank}(X)$:

Let l be a line extending ρ; we pass to a configuration c_{**} on $\partial_\infty P_l$ (via Lemma II.2.6). Then c_{**} is semistable by Lemma II.2.9. Since $P_l = l \times CS(l)$, c_{**} splits, and we obtain a semistable configuration on $\partial_\infty l$ and a semistable configuration on $\partial_\infty CS(l)$.

A semistable configuration on the boundary of a flat Euclidean space (i.p. a line) yields a constant map Φ; a semistable configuration on $\partial_\infty CS(l)$ has a fixed point by the induction hypothesis.

Thus, we have a line of fixed points for c_{**} in X^ω. This line of fixed points yields a ray of fixed points for Φ_* by Lemma II.2.7.

So let $p \in X^\omega$ be a fixed point of Φ_*. There is a unique point $p' \in X$ which is closest to p. Since Φ_* is 1-Lipschitz, it has to fix p'. Now the observation $\Phi_*|_X = \Phi$ finishes the proof. □

Corollary II.5.3. *Let X be a Hadamard space, and c a weighted configuration on its boundary at infinity, which is stable on X^ω. Then the associated weak contraction Φ_c has a fixed point. In particular, there exists a polygon p in X such that c is a Gauß map for p.*

Proof. By assumption, case 2 in the proof of Theorem 6 above does not occur; hence the proof works exactly the same (observe that building geometry was used only in the second case). □

In the locally compact case, $X^\omega \cong X$; hence Corollary II.5.3 finishes the proof of the Theorem 1.

Observe that we cannot expect Theorem 6 to fully generalize to Hadamard spaces, since in the case of symmetric spaces, nice semistability of the configuration is necessary.

II.6 Relations to Algebra

Here, we discuss the relevance of Theorem 1 to problems from algebra. Such problems were studied e.g. in [KLM04b].

In the algebraic problems, one only fixes the *type* of a configuration, i.e. the projection of the points ξ_i to the spherical Weyl chamber Δ. Taking the weights m_i into account, such a type of a configuration may be viewed as an element of Δ^n_{euc}, n copies of the Euclidean Weyl chamber (the Euclidean cone over the spherical Weyl chamber Δ). Consider the following theorem:

Theorem 7 ([KLM04c, Thm. 1.2])**.** *Let X be a Euclidean building. Then for $h \in \Delta^n_{euc}$ there exists an n-gon in X with Δ-side lengths h if and only if there exists a semistable weighted configuration on $\partial_\infty X$ of type h.* □

Our results from this chapter give a natural proof, and may in fact be seen as a refinement, since the proof in [KLM04c] does not provide explicit configurations for which there exists a fixed point. This indicates that there will eventually be more applications to algebra.

Chapter III

Convex rank 1 subsets of Euclidean buildings of type A_2

Throughout this chapter, we will often deal with points $\eta_i, \eta_{i,j}, \xi_{i,j} \in \partial_T X$ (for a CAT(0)-space X). To simplify notation, we will call the corresponding Busemann functions $b_i, b_{i,j}, b'_{i,j}$ respectively (instead of the standard notation b_{η_i}, $b_{\eta_{i,j}}$, $b_{\xi_{i,j}}$).

The aim of this chapter is to prove Theorem 2.

In section III.1, we examine under which conditions Busemann functions agree, and in which cases unions of horoballs are (locally) convex. The lemmas in this section are formulated generally for Euclidean buildings, and we hope that they will be useful in the study of convex rank 1-subsets of higher-dimensional buildings.

Section III.2 contains geometric lemmas about buildings of type A_2: We exclude the existence of triangles $\Delta(a, b, c)$ with certain properties. To formulate it positively, we show that under certain circumstances, the starting direction $\overrightarrow{xc}$ (for $x \in \overline{ab}$) always points in "roughly the same direction".

In section III.3, we examine necessary conditions for $A \subset \partial_T X$ to lie in the boundary of a convex rank 1-set. In particular, we show that for every triple of boundary points, a tripod has to exist (Proposition III.3.4). We call $A \subset \partial_T X$ an S-set if it satisfies this condition.

If one knows (or expects) Theorem 4, one might also expect that for a building of type A_2, every convex rank 1-subset is essentially a tree. This turns out to be wrong. However, in section III.3.4, we obtain a tree $\mathcal{T}$ as a quotient of a subset of X naturally associated to the S-set $A \subset \partial_T X$.

In section III.4, we "thicken" tripods; i.e. we search for convex rank 1-subsets of X containing a given tripod. This motivates the definition of the convex set $\mathcal{K}$ in the section which follows, and introduces the techniques for proving convexity.

The last section, finally, presents the proof of Theorem 2: Given a good S-set $A \subset \partial_T X$, we consider the associated tree $\mathcal{T}$. For every point $[x] \in \mathcal{T}$, we define a closed convex subset $\mathcal{K}_{[x]}$ of X, and we show that the (closure of the) union $\mathcal{K}$ of these sets is convex. In a last step, we obtain a subset $\bar{\mathcal{C}}$ of $\bar{\mathcal{K}}$ which can easily be seen to be convex rank 1, and contain A in its boundary.

III.1 Remarks on Busemann functions and horoballs in Euclidean buildings

Setting: Let X be a Euclidean building without flat de Rham factor, $\eta_1, \eta_2 \in \partial_T X$ be two boundary points of the same type (not necessarily regular), and $p \in X$. We normalize the corresponding Busemann functions b_1, b_2 such that $b_1(p) = b_2(p) = 0$.

Consider η_1 as a point in the (spherical) model apartment S. Let $\alpha > 0$ be the maximal angle such that $\angle(\eta_1, \eta) \leq \alpha$ implies that η_1 and η lie in a common Weyl chamber of the Coxeter complex (S, W). Since X has no flat de Rham factor, we have $\alpha \leq \pi/2$.

For the first two lemmas, assume there exists a ray $\overline{p\xi}$ such that

$$\angle_p(\eta_1, \xi) = \angle_p(\eta_2, \xi) = \pi.$$

Since the set of singular points of the Coxeter complex (S, W) is invariant under the map sending every point to its antipode, we have: Whenever $\angle(\xi, \xi') \leq \alpha$ for some $\xi' \in \partial_T X$, then the points ξ and ξ' lie in a common Weyl chamber of $\partial_T X$.

Note that this implies in particular: If $\xi' \neq \xi$ has the same type as ξ, then $\angle(\xi, \xi') \geq 2 \cdot \alpha$.

Lemma III.1.1. *Let $\overline{p\xi'}$ be a ray with $\angle_p(\xi, \xi') \leq \alpha$. Then*

$$b_1|_{\overline{p\xi'}} = b_2|_{\overline{p\xi'}}.$$

Proof. Note that any Busemann function b_η is piecewise linear and convex along any ray $\overline{p\xi'}$, the slope in $x \in \overline{p\xi'}$ being $-\cos(\angle_x(\eta, \xi'))$ (this is well known; it follows from [KL97, 4.1.2]).

Now the possible values of $\angle_x(\eta_i, \xi')$ form a finite set (determined by the types of η_i, ξ'), and if $\overrightarrow{x\xi''}$ is of the same type as $\overrightarrow{x\xi'}$, then $\angle_x(\hat{\xi}, \xi'') \geq \angle_x(\xi, \xi')$ for every antipode $\hat{\xi}$ of η_1 (if $\overrightarrow{x\xi''}$ does not lie in a common Weyl chamber with $\overrightarrow{x\hat{\xi}}$, then $\angle_x(\hat{\xi}, \xi'') \geq \alpha$).

So $\angle_p(\eta_1, \xi') = \angle_p(\eta_2, \xi') = \pi - \angle_p(\xi, \xi') \in [\pi - \alpha, \pi]$ is maximal. Since the slope of b_i is increasing along $\overline{p\xi'}$, it is constant, and the claim follows from our assumption $b_1(p) = b_2(p)$. $\square$

We continue working in the setting introduced above.

Lemma III.1.2. *Let $R > 0$, and $D \geq \max(R, R/\tan\alpha)$. Then*

$$B_R(\overline{p\xi}) \cap \{b_1 \leq D\} = B_R(\overline{p\xi}) \cap \{b_2 \leq D\}.$$

Proof. Let $x \in B_R(\overline{p\xi})$. If $\angle_p(x, \xi) \leq \alpha$, then $b_1(x) = b_2(x)$ by the previous lemma, so x is either contained in both sets, or in none of them.

So it suffices to show: if $\angle_p(x, \xi) > \alpha$, then $b_i(x) \leq D$ for both i.
Let $x' := \pi_{\overline{p\xi}}(x)$. We may assume $x' \neq p$ because of $D \geq R$. Consider a point $y \in \overline{xx'}$ such that $\angle_p(y, \xi) = \alpha$. Then $b_1(y) = b_2(y) = d(p, y) \cdot \cos\alpha$, and $d(y, \overline{p\xi}) = d(y, x') \geq d(p, y) \cdot \sin\alpha$. We have

$$\begin{aligned} b_i(x) &\leq b_i(y) + (R - d(y, \overline{p\xi})) \leq R \cdot \left(1 + \frac{d(p, y)}{R} \cdot (\cos\alpha - \sin\alpha)\right) \\ &= R \cdot \left(1 + \frac{d(p, y) \cdot \sin\alpha}{R} \cdot (\operatorname{cotan}\alpha - 1)\right) \end{aligned}$$

If $\alpha \geq \pi/4$, we have $\operatorname{cotan}\alpha \leq 1$ so the inequality above implies $b_i(x) \leq R \leq D$.

If $\alpha \leq \pi/4$, we have $\operatorname{cotan}\alpha \geq 1$, and we use $d(p, y) \leq R/\sin\alpha$: Then the inequality above becomes $b_i(x) \leq R \cdot (1 + \operatorname{cotan}\alpha - 1) \leq D$. □

From now on, we do not require the existence of a common antipode $\overrightarrow{p\xi}$ of the two $\overrightarrow{p\eta_i}$ in $\Sigma_p(X)$ anymore.

Lemma III.1.3. *Let $D > R \cdot \cos\alpha > 0$. Then the set*

$$C := B_R(p) \cap (\{b_1 \leq D\} \cup \{b_2 \leq D\})$$

is convex.

Proof. We want to apply Corollary I.5.4: It suffices to find an $\varepsilon > 0$ such that for any point $x \in C$ with $b_1(x) = b_2(x) = D$, we have convexity of $B_\varepsilon(x) \cap C$.

Let us first choose the ε, depending only on the type of D, R, and α (but not on a specific point $x \in C$):

Let $\delta := (D - R \cdot \cos\alpha)/2$, and choose $\hat{\alpha} < \alpha$ such that $R \cdot \cos\hat{\alpha} \leq D - \delta$. Now consider a Euclidean triangle A, B, C with $d(A, B) \geq \delta$ and $\angle_A(B, C) \geq \alpha - \hat{\alpha}$. Let ε' be such that $d(B, C) \geq \max(\varepsilon', \varepsilon'/\tan\alpha)$. Set $\varepsilon := \min(\delta, \varepsilon')$.

Now let $x \in C$ be a point with $b_1(x) = b_2(x) = D$.
There is a finite subdivision $(x_0 = x, x_1, \ldots, x_m = p)$ of $\overline{xp}$ such that

$\mathrm{Conv}(x_j, x_{j+1}, \eta_i)$[1] is isometric to a flat half-strip (for $0 \leq j < m, 1 \leq i \leq 2$) (see [KL97, 4.1.2]).

Now $D/R > \cos\alpha$ implies that both b_i have maximal slope on the segment $\overline{x_1 x}$.

In fact, since $R \cdot \cos\alpha < D - \delta$, we have $d(x_1, x) > \delta$ (recall that if the slope of b_i is not maximal, then it is at most $\cos\alpha$).

For a point $y \in \overline{x_1 x} \backslash \{x_1\}$, we have $\angle_y(x_1, \eta_i) \leq \hat{\alpha}$ (since the slope of b_i along $\overline{x_1 x}$ has to be larger than $\cos\hat{\alpha}$), hence $\angle_y(\eta_1, \eta_2) < 2\alpha$, and $\overrightarrow{y\eta_1} = \overrightarrow{y\eta_2}$.

Let $x' \in \overline{x\eta_1} \cap \overline{x\eta_2}$ be such that $\overrightarrow{x'\eta_1} \neq \overrightarrow{x'\eta_2}$. We have $\angle_{x_1}(x', \eta_i) \leq \pi - \alpha$ (otherwise, we would obtain $\overrightarrow{x'\eta_1} = \overrightarrow{x'\eta_2}$ as above), and $\angle_{x_1}(x, \eta_i) \geq \pi - \hat{\alpha}$. This implies that we have $d(x', x) \geq \varepsilon$ by construction.

Of course, $b_1(x') = b_2(x') = b_i(x) - d(x, x') = D - d(x, x')$. By Lemma III.1.2, we have

$$B_\varepsilon(x) \cap \{b_1 \leq D\} = B_\varepsilon(x) \cap \{b_2 \leq D\}.$$

Hence, $B_\varepsilon(x) \cap C$ is convex, and Corollary I.5.4 applies. □

III.2 Geometry of buildings of type A_2

In this section, we collect some geometric properties of Euclidean buildings of type A_2 that will be useful later.

Lemma III.2.1. *Let X be a building of type A_2, $p \in X$ and η_1, η_2, η_3 be three singular boundary points of the same type, such that the $\overrightarrow{p\eta_i}$ span a flat in $\Sigma_p(X)$. Normalize such that $b_i(p) = 0$. Let $q \in X$ be such that $\angle_p(q, \eta_i) = 2\pi/3$ for all i and $R := d(q, p) > 0$ (so $b_i(q) = R/2$).*

Let $C := \{x \,|\, \text{at least two } b_i(x) \leq R/2\}$. Then $K := C \cap B_{R\cdot\sqrt{3}/2}(q)$ is convex. More specifically, there exist i, j such that $K = B_{R\cdot\sqrt{3}/2}(q) \cap \{b_i \leq R/2\} \cap \{b_j \leq R/2\}$.

Proof. Pick an $x \in B_{R\cdot\sqrt{3}/2}(q)$, and observe that $\angle_p(x, q) \leq \tilde{\angle}_p(x, q) \leq \pi/3$ (by triangle comparison).

We distinguish two cases: The first case is that the initial directions of $\overline{\overrightarrow{pq}\overrightarrow{p\eta_i}} \subset \Sigma_p X$ are all distinct. Then, there are two i such that $\angle_p(x, \eta_i) \geq 2\pi/3$. So $x \in C$ if and only if all three Busemann functions are at most $R/2$. (In this case, we can choose i, j arbitrarily.)

[1] Throughout this chapter, Conv always denotes the convex hull of its arguments. We use it with a variety of different kinds of arguments, but no confusion should arise.

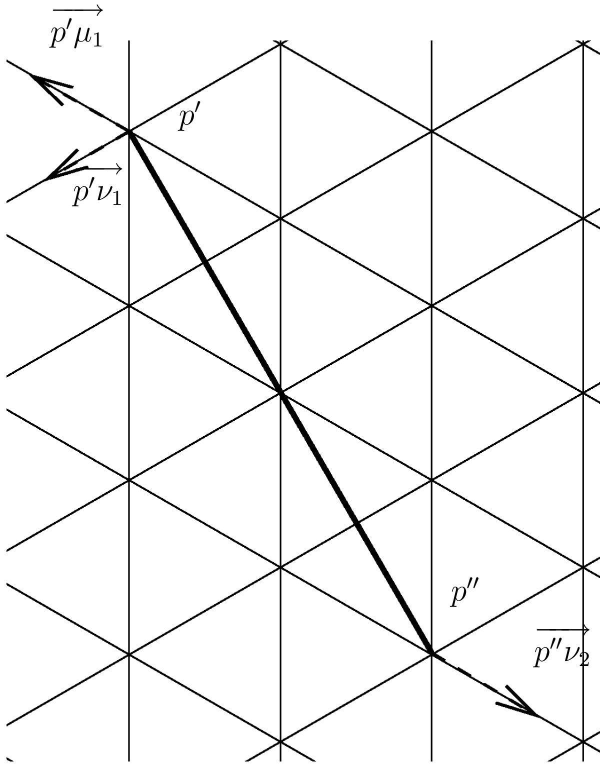

Figure III.1: The setting of the lemmas

Otherwise, (exactly) two of the above-mentioned initial directions agree (without loss of generality, those corresponding to $1, 2$; these correspond to the i, j in the claim).

We claim that $K = B_{R\cdot\sqrt{3}/2}(q) \cap \{b_1 \leq R/2\} \cap \{b_2 \leq R/2\}$.
Indeed, if $b_3(x) \leq R/2$ for $x \in K$, then either $\angle_p(x, \eta_3) \geq 2\pi/3$ and $b_3(x) \leq \min(b_1(x), b_2(x))$, or $\angle_p(x, \eta_1) = \angle_p(x, \eta_2) > 2\pi/3$. In the last case, we have $b_1(x) = b_2(x)$, hence the claim follows. $\square$

For the next two lemmas, we need a setting which will be introduced in more detail later:

Let η_1, η_2 be antipodal centers of Weyl chambers in the boundary of a Euclidean building X of type A_2, and let $F_{1,2}$ be the flat containing η_1, η_2 in its boundary. Let $\nu_1, \eta_{1,2}, \nu_2, \mu_2, \xi_{1,2}, \mu_1$ be the singular points in $\partial_T F_{1,2}$ as in Figure III.2. Figure III.1 shows a part of the flat $F_{1,2}$, with the boundary being aligned as in Figure III.2 (with $i = 1$, $j = 2$).

Lemma III.2.2. *Let $p', p'' \in F_{1,2}$ such that $b_1(p') \leq b_1(p'')$.*

Let $0 \leq \hat{\alpha} < \pi/3$. *Then there is no* $x \in X$ *with*

$$\begin{aligned} \angle_{p'}(x,\nu_1) &< \pi/3+\hat{\alpha}, & \angle_{p''}(x,\nu_2) &< \pi/3+\hat{\alpha}, \\ \textit{but } \angle_{p'}(x,\eta_{1,2}) &> \hat{\alpha}, & \angle_{p''}(x,\eta_{1,2}) &> \hat{\alpha}. \end{aligned}$$

Proof. If $p' = p''$, there is nothing to show.

Without loss of generality, we assume $b_{1,2}(p'') \leq b_{1,2}(p')$.
Let $\beta := \angle_{p''}(p', \xi_{1,2}) = \angle_{p'}(p'', \eta_{1,2}) \leq \pi/2$.

If $\pi/3 \leq \beta$ ($\leq \pi/2$), one obtains a contradiction to the sum of angles in a triangle: Indeed, we have

$$\begin{aligned} \angle_{p''}(p',x) &> \min(\pi+\hat{\alpha}-\beta, \pi/3+\beta-\hat{\alpha}) \\ \angle_{p'}(p'',x) &> \min(\beta+\hat{\alpha}, \frac{4\pi}{3}-\beta-\hat{\alpha}). \end{aligned}$$

We see that if $\beta \geq \pi/3$, the sum of these two angles is greater than π. Therefore, we have $\beta < \pi/3$. Then

$$\angle_{p''}(p',x) > \pi/3+\beta-\hat{\alpha}, \tag{III.1}$$

and

$$\angle_{p'}(p'',x) > \hat{\alpha} + \angle_{p'}(p'',\eta_{1,2}) = \hat{\alpha}+\beta \tag{III.2}$$

Let $(p_0 = p'', p_1, \ldots, p_n = p')$ be a finite subdivision of $\overline{p''p'}$ such that each triangle $\Delta(p_i, p_{i+1}, x)$ is flat.

Let $i_0 > 0$ be such that the initial directions of $\overline{\overrightarrow{p_{i_0}p''}\overrightarrow{p_{i_0}x}}$ and $\overline{\overrightarrow{p_{i_0}p''}\overrightarrow{p_{i_0}\nu_2}}$ agree $(*)$.

We will show by induction that every $1 \leq i_0 \leq n$ has this property, and obtain a contradiction for $i_0 = n$.

Base case: $i_0 = 1$ has Property $(*)$.

If this is not the case, then the starting direction of $\overline{\overrightarrow{p''p_1}\overrightarrow{p''x}}$ has to be different from the starting direction of $\overline{\overrightarrow{p''p_1}\overrightarrow{p''\nu_2}}$ (since the triangle $\Delta(p'', p_1, x)$ is flat). If this is the case, we have $\angle_{p''}(p', x) = \angle_{p''}(p_1, x) > \pi - (\hat{\alpha}+\beta)$. This is a contradiction to (III.2).

Claim: If $i_0 < n$ has property $(*)$, then the initial directions of $\overline{\overrightarrow{p_{i_0}p'}\overrightarrow{p_{i_0}\nu_2}}$ and $\overline{\overrightarrow{p_{i_0}p'}\overrightarrow{p_{i_0}x}}$ agree as well.

Observe that $\angle_{p_{i_0}}(p'', x) < 2\pi/3 + \hat{\alpha} - \beta$ (by (III.1)). Assume that the claim is false: Then

$$\angle_{p_{i_0}}(p', x) > \pi - (\underbrace{(\pi/3 + \hat{\alpha})}_{> \angle_{p_{i_0}}(\nu_2, x)} - \underbrace{(\pi/3 - \beta)}_{\angle_{p_{i_0}}(p'', \nu_2)}) = \pi - \hat{\alpha} + \beta.$$

Together with (III.2), this is a contradiction.

Now this claim implies $(*)$ for $i_0 + 1$ (by the same argument as in the base case, with p_{i_0} taking the place of p''), and it follows by induction that property $(*)$ holds for all $1 \leq i_0 \leq n$.

For $i_0 = n$, we get $\angle_{p'}(p'', x) > \pi - \hat{\alpha} + (\pi/3 - \beta) = 4\pi/3 - (\hat{\alpha} + \beta)$.

If this is less than π, we continue our calculation:

$$\angle_{p'}(p'', x) + \angle_{p''}(p', x) > 5\pi/3 - 2\hat{\alpha}.$$

This is a contradiction, since $\hat{\alpha} < \pi/3$. □

Remark III.2.3. In the statement of the lemma, we can replace the directions $\overrightarrow{p'\nu_1}$ and $\overrightarrow{p''\nu_2}$ by any other directions antipodal to $\overrightarrow{p'\mu_2}, \overrightarrow{p''\mu_1}$ resp. Of course, we also have to adjust the assumptions after the "but". We will usually take care of this by showing that $\angle_{p''}(\xi_{1,2}, x) < \pi - \hat{\alpha}$ and $\angle_{p'}(\xi_{1,2}, x) < \pi - \hat{\alpha}$.

Lemma III.2.4. *Let $p', p'' \in F_{1,2}$ such that $b_1(p') \leq b_1(p'')$.*

Assume that $\angle_{p''}(p', \eta_{1,2}) \geq \pi/3$. Let $0 \leq \hat{\alpha} \leq \pi/6$. Then there is no $x \in X$ with

$$\begin{aligned} \angle_{p'}(x, \mu_1) &< \pi/3 + \hat{\alpha}, & \angle_{p''}(x, \nu_2) &< \pi/3 + \hat{\alpha} \\ \textit{but } \angle_{p'}(x, \xi_{1,2}) &> \hat{\alpha}, & \angle_{p''}(x, \eta_{1,2}) &> \hat{\alpha}. \end{aligned}$$

Proof. As above, we may assume $p' \neq p''$.

We distinguish two cases: The first case is $\angle_{p''}(p', \xi_{1,2}) =: \beta \in [\pi/3, 2\pi/3]$. In this case, we have

$$\begin{aligned} \angle_{p''}(p', x) &> \min(\pi - \beta + \hat{\alpha}, \underbrace{\beta + \pi/3 - \hat{\alpha}}_{\geq \pi/2}) \\ \angle_{p'}(p'', x) &> \min(\pi - \beta + \hat{\alpha}, \underbrace{\beta + \pi/3 - \hat{\alpha}}_{\geq \pi/2}). \end{aligned}$$

Adding these angles, the only case in which we do not get a contradiction to the sum of angles in a triangle is, if $\overrightarrow{p''x} \in \overrightarrow{p''\eta_{1,2}}\overrightarrow{p''\nu_2}$, $\overrightarrow{p'x} \in \overrightarrow{p'\xi_{1,2}}\overrightarrow{p'\mu_1}$ and $\beta > \pi/2 + \hat{\alpha}$.

Now this case can be finished as in the lemma above: The deciding inequalities are

$$\angle_{p_{i_0}}(p'', x) < \beta - \hat{\alpha}, \qquad \angle_{p_{i_0}}(p', x) < \beta - \hat{\alpha},$$
$$\angle_{p_{i_0}}(p'', \nu_2) = \beta - \pi/3.$$

The second case is $\beta < \pi/3$. Now we have $\angle_{p'}(p'', x) > \beta + (\pi/3 - \hat{\alpha})$ and $\angle_{p''}(p', x) > (\pi/3 - \hat{\alpha})$. Again, we have to have $\overrightarrow{p''x} \in \overline{\overrightarrow{p''p'}\overrightarrow{p''\nu_2}}$ and $\overrightarrow{p'x} \in \overline{\overrightarrow{p'p''}\overrightarrow{p'\mu_1}}$.

To be able to "switch sides", we would need a p_{i_0} with $\angle_{p_{i_0}}(p'', x)$ at least $\underbrace{\pi/3 - \beta}_{\angle_{p_{i_0}}(p'', \nu_2)} + 2\pi/3 - \hat{\alpha} = \pi - \beta - \hat{\alpha}$, which is impossible (because $\hat{\alpha} \leq \pi/6$). □

Remark III.2.5. Again, this lemma remains true if we replace $\overrightarrow{p''\nu_2}$ and/or $\overrightarrow{p'\mu_1}$ by other directions antipodal to $\overrightarrow{p''\mu_1}, \overrightarrow{p'\nu_2}$ resp. (and again, we also have to adjust the assumptions after the "but").

III.3 Necessary conditions: S-sets

Let us state the precise definition of a convex rank 1-set:

Definition III.3.1. A subset $C \subset X$ of a Hadamard space X is called *convex rank 1*, if it is closed, convex, has at least 3 boundary points at infinity and satisfies: $\partial_T C$ is a 0-dimensional building (i.e.: for all $\eta, \xi \in \partial_T C$ with $\eta \neq \xi$, we have $\angle_T(\eta, \xi) \geq \pi$).

Observe that the restriction $|\partial_T C| \geq 3$ is not serious: Every pair of antipodal points in $\partial_T X$ (for $X = M_{\mathbb{K}}$ or a Euclidean building) can be joined by a geodesic.

From now on, we focus on a special class of buildings: In the remainder of this chapter, X will always stand for a building of type A_2.

In this section we examine necessary conditions for points $\eta_i \in \partial_T X$ to be in the boundary of a convex rank 1 set. The most important necessary condition is that there has to be a tripod for every triple of asymptotic boundary points (Proposition III.3.4). We also examine the structure of the set of singular points of these tripods, and we will obtain a metric tree which is closely related.

Lemma III.3.2. *If there are at least three points η_i, then to be pairwise antipodal, it is necessary that each η_i is the center of a Weyl chamber.*

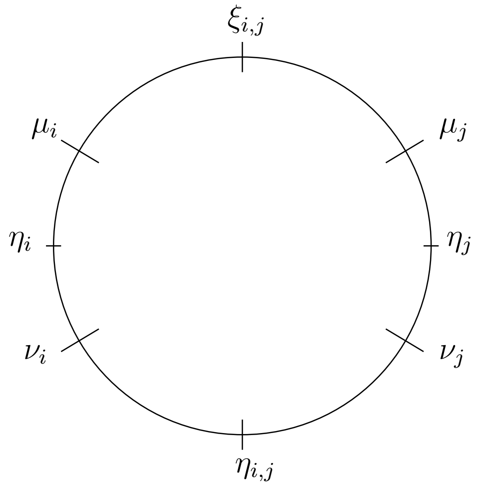

Figure III.2: The apartment $\partial_T F_{i,j} \subset \partial_T X$ with its singular points $\nu_i, \eta_{i,j}, \nu_j, \mu_j, \xi_{i,j}, \mu_i$ and the two regular points η_i, η_j.

Proof. In the Coxeter complex A_2, the centers of Weyl chambers are the only points which have the following property: An antipode has the same type. This property is necessary, since the points η_i have to be pairwise antipodal. □

There is another obvious necessary condition: Let $A = \partial_T C$ be the asymptotic boundary of a convex rank 1-set. Consider a triple of boundary points. Then the corresponding holonomy map has to have a fixed point (see Section I.1.2 and the proof of Corollary IV.3.2). Since our boundary points are regular, this holonomy map is an isometry of $\mathbb{R}$ to itself. This map is also orientation preserving, so it is just a translation, which is determined by its translation length, which we call its shift.

So the necessary condition is: For each triple of points of A, their shift has to be 0 (i.e. the holonomy map has to be the identity map); a more detailed explanation of this condition can be found in the proof of Corollary IV.3.2.

III.3.1 Notation

Definition III.3.3. A subset $A \subset \partial_T X$ with $|A| \geq 3$ is an *S-set*, if the points of A are pairwise antipodal (i.e. A is a 0-dimensional subbuilding), and for each triple of points of A, the shift is 0.

In what follows, $A = \{\eta_i \,|\, i \in I\}$ will always be an S-set (see also the definition of a *good S-set*, III.3.5).

A *tripod* is a metric tree with three asymptotic boundary points. It may also be viewed as the Euclidean cone over a set of cardinality three. A tripod in X is determined by a (singular) point p and three boundary points ξ, ν, μ. This data determines a tripod $(p, \xi, \nu, \mu) = \mathrm{Conv}(p, \xi, \nu, \mu)$ if and only if $\angle_p(\xi, \nu) = \angle_p(\nu, \mu) = \angle_p(\mu, \xi) = \pi$.

In our setting, a *tripodal point* $p_{i',j',k'}$ (for three distinct $i', j', k' \in I$) is a point such that $(p_{i',j',k'}, \eta_{i'}, \eta_{j'}, \eta_{k'})$ determines a tripod. When a tripodal point is given, then $T_{i,j,k}$ denotes the corresponding tripod. If the tripodal point is to be emphasized, we also say that $(p_{i',j',k'}, \eta_{i'}, \eta_{j'}, \eta_{k'}) \in X \times (\partial_T X)^3$ is a tripod.

For $i \in I$, let ν_i, μ_i be the endpoints of the Weyl chamber spanned by η_i, such that all the ν_i have the same type.

For a pair $i, j \in I$, let $\eta_{i,j}$ be the center of the geodesic $\overline{\nu_i \nu_j} \subset \partial_T X$; similarly define $\xi_{i,j}$ (see Figure III.2). Let $F_{i,j}$ denote the unique flat in X such that $\eta_i, \eta_j \in \partial_T F_{i,j}$; so the singular vertices of $\partial_T F_{i,j}$ are $\nu_i, \eta_{i,j}, \nu_j, \mu_j, \xi_{i,j}, \mu_i$.

For a triple $i, j, k \in I$, let $l_{i,j,k} := F_{i,j} \cap F_{j,k} \cap F_{i,k}$. By definition of tripods, $l_{i,j,k}$ is precisely the set of tripodal points for (η_i, η_j, η_k). It follows from the flat strip theorem ([BH99, II.2.13]), that for every pair of tripods for a given triple of boundary points, they are parallel to each other, and their convex hull splits as a product “tripod × interval”.

We will see below that $l_{i,j,k}$ is a non-empty line segment (which may degenerate to a point, a ray or a geodesic line). We will say that p is the “lower endpoint” of $l_{i,j,k}$ if p minimizes $b_{i,j}|_{l_{i,j,k}}$ (it follows from Proposition III.3.4 that such a point exists). Analogously, we define the “upper endpoint” of $l_{i,j,k}$.

III.3.2 Existence of tripods

Proposition III.3.4. *Let X be a Euclidean building of type A_2, and let $\eta_1, \eta_2, \eta_3 \in \partial_T X$ be three pairwise antipodal points. If their shift is 0, then there exists a tripod $(p, \eta_1, \eta_2, \eta_3)$.*

Note that the proposition can also be phrased as follows: Every S-set of cardinality 3 is the asymptotic boundary of a convex rank 1-set, and this rank 1-set can be chosen to be a tripod.

Proof. Observe that $F_{1,2} \cap F_{1,3} =: S$ is non-empty, closed and convex (by [KL97, 4.6.3]). The Busemann function b_1 is bounded above on S, since otherwise we have $\angle_T(\eta_2, \eta_3) < \pi$ (note that ∂S is a polygonal curve consisting of at most three line segments/rays/lines, see Figure III.3).

Let p be an extremal point for $b_1|_S$. We claim that $(p, \eta_1, \eta_2, \eta_3)$ is a tripod.

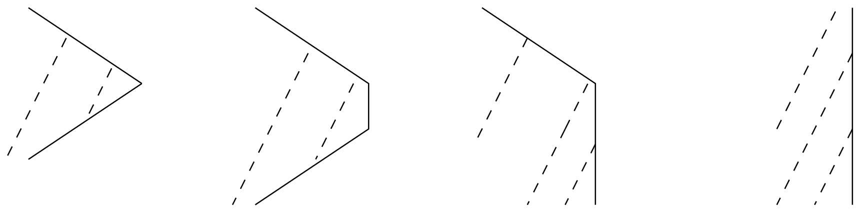

Figure III.3: the possibilities for $F_{1,2} \cap F_{1,3}$ (where η_1 is lying "on the left")

Assume that this is not the case. Then $\angle_p(\eta_2, \eta_3) \geq \pi/3$ (since both directions are the centers of a Weyl chamber, the smallest non-zero value for their angle is $\pi/3$).

Since the shift is zero, we obtain points p', p'' in $\overline{p\eta_2}, \overline{p\eta_3}$ resp. such that $\overline{p'\eta_2} \cup \overline{p'p''} \cup \overline{p''\eta_3}$ is a geodesic line.

Let us choose p', p'' (as p) such that $\angle_{p'}(\eta_1, \eta_3) \neq 0 \neq \angle_{p''}(\eta_1, \eta_2)$. Then each of these angles is at least $\pi/3$, so $\Delta(p, p', p'')$ is a flat isosceles triangle (by triangle rigidity in CAT(0)-spaces, see [BH99, II.2.9]).

Let ν be the midpoint of the geodesic $\overrightarrow{pp'}\overrightarrow{pp''} \subset \Sigma_p(X)$. Then $\angle_p(\eta_2, \nu) = \angle_p(\eta_3, \nu) = \pi/6$; Observe that $\mu \in \Sigma_p(X), \angle_p(\eta_j, \mu) \leq \pi/6$ implies $\mu \in \Sigma_p(F_{1,j})$ for $j \in \{2, 3\}$ (because μ and $\overrightarrow{p\eta_j}$ lie in a common Weyl chamber of $\Sigma_p(X)$). So either

$$\nu = \overrightarrow{p\nu_2} = \overrightarrow{p\nu_3} \text{ or } \nu = \overrightarrow{p\mu_2} = \overrightarrow{p\mu_3}.$$

In both cases, we have $\overrightarrow{p\nu} \in \Sigma_p(F_{1,2}) \cap \Sigma_p(F_{1,3}) = \Sigma_p(F_{1,2} \cap F_{1,3})$. Since $\angle_p(\eta_1, \nu) = 2\pi/3$, this is a contradiction to the construction of p. $\square$

The proof also shows that a convex rank 1-subset of X has to contain a tripod for every triple of boundary points (because for a strong asymptote class which does not correspond to a tripod, we obtain a flat isosceles triangle in the convex hull, and its center is a tripodal point).

Hence, the following condition is also necessary for an S-set A to be in the asymptotic boundary of a convex rank 1-set:

Definition III.3.5. An S-set A is called *good*, if it satisfies the following condition: We can choose tripodal points $p_{i,j,k}$ (for every triple $i, j, k \in I$) such that for all $i' \in I$, the convex hull of (all) the strong asymptote classes $[\overline{p_{i',j,k}\eta_{i'}}]$ is bounded.

Example III.3.6. Let us give an example of a 4-point S-set which does not lie in the boundary of an embedded tree:

We start with two antipodal centers of Weyl chambers, η_1, η_2, and pick a singular vertex p in $F_{1,2}$. Choose a ray $\overline{p\eta_3}$, such that $(p, \eta_1, \eta_2, \eta_3)$ is a tripod. Let us choose η_3 such that the intersection $F_{1,2} \cap F_{2,3}$ is a flat sector (this corresponds to the left-most set drawn in Figure III.3).[2]

Now pick an inner point p' of $F_{1,2} \cap F_{2,3}$ satisfying $b_{1,2}(p') \neq b_{1,2}(p)$. As above, pick a ray $\overline{p'\eta_4}$, such that $(p', \eta_1, \eta_2, \eta_4)$ is a tripod and $F_{1,2} \cap F_{1,4}$ is a flat sector. By construction, we have

$$\overrightarrow{p'\eta_1} = \overrightarrow{p'\eta_3},$$

so we also have a tripod $(p', \eta_3, \eta_2, \eta_4)$. Similarly, our construction implies that p lies in the interior of $F_{1,2} \cap F_{1,4}$, so we also have the tripod $(p, \eta_1, \eta_3, \eta_4)$. Our choices imply that p, p' are the unique tripodal points (at least when considered as $p_{1,2,3}, p_{1,2,4}$ resp.), so $b_{1,2}(p) \neq b_{1,2}(p')$ implies that there is no embedded tree with the given four boundary points.

A similar situation is depicted in Figure III.4; in the next section, we are going to show that the general situation is always similar to the one described here.

Applying the construction above to obtain an S-set with infinitely many boundary points, we see that an S-set needs not be good.

III.3.3 S-sets with 4 points: relative position of their tripodal points

In this section, we examine S-sets A of cardinality 4: We show that we can always do with at most 2 tripodal points: If there is no 4-pod (i.e. a Euclidean cone over A) embedded in X, then we construct two points, each of which is tripodal for two triples of points of A.

All of the Lemmas in this section are formulated such that the assumptions rule out existence of a 4-pod; only Proposition III.3.12 is formulated to make sense even in this case.

We also discuss the possible choices for the tripodal points in question, and the relative position of the two (sets of) points to each other. These are technical results needed in the sequel.

Lemma III.3.7. *Let* $\{\eta_1, \eta_2, \eta_3, \eta_4\} \subset \partial_T X$ *be an S-set of cardinality 4. Assume there are tripods* $(\bar{p}, \eta_1, \eta_2, \eta_3)$ *and* $(\bar{p}', \eta_1, \eta_2, \eta_4)$ *with* $b_1(\bar{p}) < b_1(\bar{p}')$. *Then there are points* $p, p' \in X$ *such that we have tripods*

$$(p, \eta_1, \eta_2, \eta_3), (p, \eta_1, \eta_3, \eta_4), \text{ and } (p', \eta_1, \eta_2, \eta_4), (p', \eta_2, \eta_3, \eta_4).$$

[2] This is possible in "most" Euclidean buildings of type A_2; pick the building associated to $SL(3, \mathbb{Q}_5)$ for example.

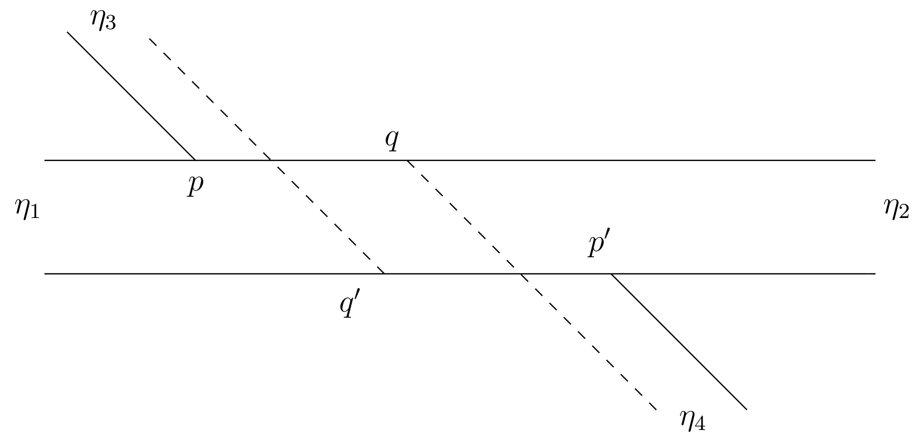

Figure III.4: The situation from Lemma III.3.7. Observe that q, q' are not tripodal points (unless $q = p'$ and $q' = p$).

In particular,

$$\overline{pp'} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}.$$

and $\angle_p(\eta_{1,2}, p') \in [\pi/3, 2\pi/3]$.

Proof. Let us choose tripodal points $p \in l_{1,2,3}, p' \in l_{1,2,4}$ such that $d(p, p')$ is minimal; note that $b_1(p') - b_1(p) = b_1(\bar{p}') - b_1(\bar{p}) > 0$. Note that this implies in particular that there is no 4-pod with the given four boundary points embedded in X.

If $b_{1,2}(p) = b_{1,2}(p')$, then we have found an isometrically embedded tree having $\eta_1, \eta_2, \eta_3, \eta_4$ as asymptotic boundary points. The claim of the lemma is now trivial.

So we may assume $b_{1,2}(p) \neq b_{1,2}(p')$, and without loss of generality that $b_{1,2}(p) > b_{1,2}(p')$ (by exchanging the $\eta_{i,j}$ and the $\xi_{i,j}$, if necessary); note that under these assumptions, p is the lower endpoint of $l_{1,2,3}$, and p' is the upper endpoint of $l_{1,2,4}$. We normalize such that $b_{1,2}(p) = b_1(p) = b_2(p) = 0$.

First, we want to show $p' \in F_{2,3}$. Assume that this is not the case.

In $F_{1,2}$, consider the line $l_{1,2}$ passing through p with endpoints μ_1, ν_2. Then the ray $l_{1,2} \cap \{b_2 \leq 0\}$ is a boundary segment of $F_{1,2} \cap F_{2,3}(\dagger)$.

Similarly, consider the line $l'_{1,2}$ passing through p' with endpoints ν_1, μ_2. Then the ray $l'_{1,2} \cap \{b_2 \leq b_2(p')\}$ is a boundary segment of $F_{1,2} \cap F_{2,4}(\ddagger)$.

The two lines $l_{1,2}, l'_{1,2}$ bound a sector $S \subset F_{1,2}$ with tip p'', containing η_2 in its asymptotic boundary. Let $\rho \subset l_{1,2}, \rho' \subset l'_{1,2}$ be its bounding rays.

We are assuming that $p' \notin F_{1,2} \cap F_{2,3}$. Since $b_{1,2}(p) > b_{1,2}(p')$, this implies that p' lies "below" l (otherwise, $p' \in \mathrm{Conv}(p, \eta_2, \nu_2) \subset F_{1,2} \cap F_{2,3}$), see Figure III.5.

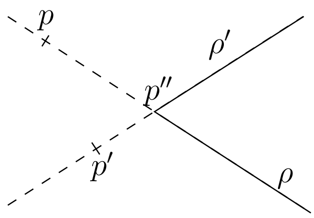

Figure III.5: The relative position of p, p', p''.

In this case, we claim $S = F_{2,3} \cap F_{2,4}$: The relation $\subset$ is clear (because $\{p''\} = l_{1,2} \cap l'_{1,2} \subset F_{1,2} \cap F_{2,3} \cap F_{2,4}$ by † and ‡, and $\overline{\nu_2\mu_2} \subset \partial_T F_{2,j}$ for all j). For the other inclusion, observe: near ρ', the flat $F_{2,3}$ agrees with $F_{1,2}$, while this is not true for $F_{2,4}$. Similarly near ρ, the flat $F_{2,4}$ agrees with $F_{1,2}$, but the flat $F_{2,3}$ does not.

So $S = F_{2,3} \cap F_{2,4}$ as claimed. Then $(p'', \eta_2, \eta_3, \eta_4)$ is a tripod by our assumptions and our discussion showing existence of tripods.

However, we see immediately that $\angle_{p''}(\eta_3, \eta_4) \leq 2\pi/3$: Indeed, we have

$$\begin{aligned} \angle_{p''}(\eta_3, p) &= \pi/6 \text{ (by †, we have } \overrightarrow{p''p} \in \Sigma_{p''}(F_{2,3})), \\ \angle_{p''}(p, p') &= \pi/3, \\ \angle_{p''}(p', \eta_4) &= \pi/6 \text{ (by ‡, we have } \overrightarrow{p''p'} \in \Sigma_{p''}(F_{2,4})). \end{aligned}$$

This contradiction shows $p' \in F_{1,2} \cap F_{2,3}$. At the same time, this shows $\angle_p(\eta_{1,2}, p') \in [\pi/3, \pi/2]$; this is the last claim (the angle can be bigger than $\pi/2$ if we have exchanged $\eta_{i,j}$ and $\xi_{i,j}$ before; we still need to verify that p, p' have the other desired properties).

If $p' \in \operatorname{int}(F_{1,2} \cap F_{2,3})$, then it is immediate that $(p', \eta_2, \eta_3, \eta_4)$ is a tripod (because $\overline{p'\eta_1} \cap \overline{p'\eta_3} \supsetneq \{p'\}$).

If $p' \in \partial(F_{1,2} \cap F_{2,3})$, we still have (since $p' \in F_{2,3}$ by the above and $p' \in F_{2,4}$ by definition)

$$\angle_{p'}(\eta_2, \eta_4) = \pi = \angle_{p'}(\eta_2, \eta_3).$$

If p' is not tripodal for this triple, we would have to have $\angle_{p'}(\eta_3, \eta_4) \in \{0, \pi/3\}$ (since the shift of the triple is zero by assumption; see the proof of Proposition III.3.4).

However, $\angle_{p'}(\eta_1, \eta_4) = \pi$ by construction and $\angle_{p'}(\eta_1, \eta_3) \leq \pi/3$ since p' cannot be tripodal for (η_1, η_2, η_3). Therefore, $\angle_{p'}(\eta_3, \eta_4) \geq 2\pi/3$, showing that p' is tripodal for the triple (η_2, η_3, η_4).

Similarly, we see $p \in F_{1,2} \cap F_{1,4}$ and that $(p, \eta_1, \eta_3, \eta_4)$ is a tripod. □

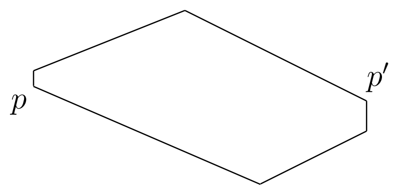

Figure III.6: $F_{1,2} \cap F_{3,4} = F_{1,4} \cap F_{2,3}$

The lemma above shows in particular that $F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4} \neq \emptyset$. Let us examine this set in more detail, and give some more interpretation to the results from the previous lemma:

Lemma III.3.8. *In the situation as in the previous lemma, we have*

$$F_{1,2} \cap F_{3,4} = F_{1,4} \cap F_{2,3} = F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}.$$

Proof. Let us introduce a set C, drawn in Figure III.6: The left vertical boundary is

$$s_1 := l_{1,2,3} \cap l_{1,3,4} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4},$$

which we have just shown to be non-empty. Observe that every point in s_1 is tripodal for (η_1, η_2, η_3) and for (η_1, η_3, η_4). Every *interior* point x of s_1 satisfies $\overrightarrow{x\eta_2} = \overrightarrow{x\eta_4}$.

Similar properties hold for the vertical boundary on the right, which is defined as

$$\emptyset \neq s_2 := l_{1,2,4} \cap l_{2,3,4} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}.$$

Now we set C to be the smallest convex polygon in $F_{1,2}$ containing $s_1 \cup s_2$ and such that all the boundary segments are singular. (Observe that C may degenerate to a segment.)

By definition (and the two inclusions above), the set C is a subset of both $F_{1,2} \cap F_{3,4}$ and of $F_{1,4} \cap F_{2,3}$.

Let us explain the relations to the previous lemma: There, we have found that if $b_{1,2}(s_1) \cap b_{1,2}(s_2) \neq \emptyset$, then there is an isometrically embedded tree in X with the given 4 asymptotic endpoints. If this is not the case, then we have made the assumption that $b_{1,2}(s_1) > b_{1,2}(s_2)$, and our choice of p, p' was such that p is the lower endpoint of s_1 and p' is the upper endpoint of s_2.

To finish the proof of our current lemma, we want to show that every boundary segment of C lies in the boundary of both $F_{1,2} \cap F_{3,4}$ and $F_{1,4} \cap F_{2,3}$.

This is immediate for the vertical segments s_1 and s_2.

Observe that $b_1(s_2) > b_1(s_1)$ by the assumptions of Lemma III.3.7. Therefore, there are non-degenerate angular segments bounding C. The argument for the angular boundary components are similar to each other, let us give one in detail:

Let $\bar{p}$ be the upper endpoint of s_2, and consider the segment $s := \overline{\bar{p}\mu_1} \cap C = \overline{\bar{p}\mu_3} \cap C$. Let us assume that s is non-degenerate (i.e. $s \neq \{\bar{p}\}$ and let x be an interior point of s.

Note that $\bar{p}$ is the upper endpoint of $l_{1,2,4}$ or of $l_{2,3,4}$.[3] If $\bar{p}$ is the upper endpoint of $l_{1,2,4}$, then s lies in the boundary of $F_{1,2} \cap F_{1,4}$; in particular, we have

$$\Sigma_x(F_{1,2}) \cap \Sigma_x(F_{2,3}) \ni \overrightarrow{x\mu_2} \neq \overrightarrow{x\mu_4} \in \Sigma_x(F_{3,4}) \cap \Sigma_x(F_{1,4}).$$ [4]

Otherwise, $\bar{p}$ is the upper endpoint of $l_{2,3,4}$, so s lies in the boundary of $F_{2,3} \cap F_{3,4}$; then the equation above holds as well.

The equation above shows that x (and hence all of s) lies in the boundary of both sets, $F_{1,2} \cap F_{3,4}$ and $F_{1,4} \cap F_{2,3}$. Similar arguments hold for the other segments bounding C. □

Remark III.3.9. Let us examine what the last argument shows about $\bar{p}$ (using the notation of the previous lemma): Since $x \in C \subset F_{2,3} \cap F_{3,4}$, the last footnote shows that there cannot be a tripod (η_2, η_3, η_4) at $b_{2,3}$-level higher than $b_{2,3}(\bar{p})$. Hence, $\bar{p}$ is the upper endpoint of $l_{2,3,4}$. Similarly, $\bar{p}$ is the upper endpoint of $l_{1,2,4}$.

This shows that $l_{1,2,4} = l_{2,3,4}$ if the two angular boundary segments of C starting from the upper and lower endpoints of s_2 have different slope.

The same statement holds for $l_{1,2,3}$ and $l_{1,3,4}$.

Therefore, if C has four angular boundary segments, then we have $l_{1,2,4} = l_{2,3,4}$ and $l_{1,2,3} = l_{1,3,4}$.

Remark III.3.10. Let us also give a description of C in terms of Busemann functions: Normalize such that

$$b_1(p) = b_2(p) = b_3(p) = b_4(p) = 0.$$

Then we have for all $i \in \{1, 2, 3, 4\}$ (taking the indices modulo 4)

$$(b_i + b_{i+1})|_{F_{i,i+1}} = const = b_i(p) + b_{i+1}(p) = 0.$$

Now $x \in C$ if and only if $x \in F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}$, as we have shown above. Let $f := b_1 + b_2 + b_3 + b_4$. Then it follows that $x \in C$ implies

$$f(x) = \frac{1}{2}((b_1(x)+b_2(x))+(b_2(x)+b_3(x))+(b_3(x)+b_4(x))+(b_1(x)+b_4(x))) = 0.$$

[3]In fact, the following argument shows that “s non-degenerate implies that $\bar{p}$ is the upper endpoint of both segments; see the remark below the lemma.

[4]Observe that we have $\overrightarrow{xp} = \overrightarrow{x\nu_2} = \overrightarrow{x\nu_4}$; hence, we have $\angle_x(\eta_2, \eta_4) = \pi/3$.

Since $x \in F_{i,i+1}$ if and only if $b_i(x) + b_{i+1}(x) = 0$, and $x \notin F_{i,i+1}$ implies $b_i(x) + b_{i+1}(x) > 0$, we have

$$x \in C \Leftrightarrow f(x) = 0,$$

so C is the set of minima of f.

Lemma III.3.11. *In the situation as in the previous lemmas, we have*

$$F_{1,3} \cap F_{2,4} = \emptyset.$$

Proof. Choose points p, p' as in Lemma III.3.7, and normalize (as above) such that $b_1(p) = b_2(p) = b_3(p) = b_4(p) = 0$. Note that on $F_{i,j}$, we have $b_i + b_j = const.$

Let $x \in \overline{p'\eta_1}$. Oberserve $\pi_{F_{2,4}}(x) = p'$. This implies $b_1|_{F_{2,4}} \geq b_1(p') > b_1(p) = 0$.

Arguing similarly for $x \in \overline{p'\eta_3}$, we find $b_3|_{F_{2,4}} \geq b_3(p') = -b_2(p') = b_1(p') > 0$. Hence, $b_1 + b_3|_{F_{2,4}} > 0$, implying the claim (since $b_1 + b_3|_{F_{1,3}} \equiv 0$). □

Let us phrase a version of Lemma III.3.7 which is valid for every S-set of cardinality 4:

Proposition III.3.12. *Let $\{\xi_i \,|\, i \in \{1,2,3,4\}\} \subset \partial_T X$ be an S-set of cardinality four. Set $\eta_1 := \xi_1$. Then there are points $p, p' \in X$ and a numbering $\{\eta_2, \eta_3, \eta_4\} = \{\xi_2, \xi_3, \xi_4\}$, such that we have tripods*

$$(p, \eta_1, \eta_2, \eta_3), (p, \eta_1, \eta_3, \eta_4), (p', \eta_1, \eta_2, \eta_4), \text{ and } (p', \eta_2, \eta_3, \eta_4).$$

In particular, we have

$$\overline{pp'} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}.$$

Proof. If possible, we choose the identification $\{\eta_2, \eta_3, \eta_4\} = \{\xi_2, \xi_3, \xi_4\}$ such that

$$b_1(p_{1,2,3}) \neq b_1(p_{1,2,4}). \tag{III.3}$$

Let us first assume that this is possible: then by exchanging η_3, η_4 if necessary, we may assume that $b_1(p_{1,2,3}) < b_1(p_{1,2,4})$. Now Lemma III.3.7 applies (and finishes the proof).

We still need to consider the case that a choice as in (III.3) is not possible: So pick an arbitrary identification $\{\eta_2, \eta_3, \eta_4\} = \{\xi_2, \xi_3, \xi_4\}$, and assume that $b_1(p_{1,j,k})$ is independent of j, k.

We claim that in this case, there exists a 4-pod. By our assumptions, we have $b_1(l_{1,2,3}) = b_1(l_{1,2,4}) = b_1(l_{1,3,4})$. If there is a point

$$p \in l_{1,2,3} \cap l_{1,2,4} \cap l_{1,3,4},$$

then p is the singular point of a 4-pod: This only means that $p \in l_{2,3,4}$ as well, which follows immediately from the other three inclusions. If this is the case (i.e. if a 4-pod exists), then we set $p' = p$, and we are done.

If the three sets above have pairwise non-empty intersection, then they share a point.Hence, we may assume that $l_{1,2,3} \cap l_{1,2,4} = \emptyset$.

If this were the case, the shift of (η_2, η_3, η_4) cannot be 0. The argument for this is the same as the one showing $p' \in F_{2,3}$ in the proof of Lemma III.3.7 (one can produce the tip p'' of $F_{2,3} \cap F_{2,4}$, which should be a tripodal point, but one can show that it cannot be). □

Let us summarize what we have achieved in this section:

Given a 4-point S-set A, there is either a 4-pod in X, or there is a $2+2$ partition $A_1 = \{a_1, a_1'\}, A_2 = \{a_2, a_2'\}$ of A, such that

$$s_1 := l_{a_1,a_1',a_2} \cap l_{a_1,a_1',a_2'} \neq \emptyset, \text{ and } s_2 := l_{a_1,a_2,a_2'} \cap l_{a_1',a_2,a_2'} \neq \emptyset.$$

In this case, the sets s_1 and s_2 can be joined to each other "almost horizontally" (this is the statement about the angle in Lemma III.3.7).

III.3.4 S-sets and trees

Let $A := \{\eta_i \,|\, i \in I\}$ be an S-set.

Let us examine the set $\mathcal{F} := \bigcup_{i,j \in I} F_{i,j}$. We are going to construct a "vertical" quotient of $\mathcal{F}$ which is a metric tree.

Let $x \in F_{i,j}$, and consider some point $\eta_k \in A$. Define

$$B_{k,i,j}(x) := b_k(\pi_{T_{i,j,k}}(x)) = \min\left(b_k(\{y \in F_{i,j} \,|\, b_i(y) = b_i(x)\})\right).$$

Remark III.3.13.
1. If $k = i$ or $k = j$, the right-most definition still makes sense (and $B_{k,i,j}(x) = b_k(x)$).
2. Note that the value of $B_{k,i,j}(x)$ does not depend on the choice of $p_{i,j,k}$.
3. If $y \in F_{i,j} \cap \{b_i = b_i(x)\}$, then $B_{k,i,j}(y) = B_{k,i,j}(x)$. We will say that such a y "represents" x. Using our convention for drawing flats $F_{i,j}$, this means that "vertical" lines all represent one point (in the space $\mathcal{T}$ which is defined below).

4. Assume that $b_i(x) \leq b_i(p_{i,j,k})$. Then for any $y \in F_{i,j} \cap F_{i,k} \cap \{b_i = b_i(x)\}$, we have $B_{k,i,j}(x) = b_k(y)$.

We will see that the definition of $B_{k,i,j}(x)$ depends on x, η_k only (Lemma III.3.16), so we can define $B_k(x)$ (for $x \in \mathcal{F}$).

Now for every $k \in I$ the definition

$$D_k(x, y) := |B_k(x) - B_k(y)|$$

defines a pseudometric on $\mathcal{F}$; indeed, the triangle inequality follows immediately from the inequality for real numbers.

We will see below that $D_k(x, y) \leq d(x, y)$. Hence, the following is also a pseudometric on $\mathcal{F}$:

$$D(x, y) := \sup_{k \in I} D_k(x, y)$$

Consider the metric space $(\mathcal{T}, D) := (\mathcal{F}/\{D = 0\}, D)$. In this section, we prove:

Theorem 8. *$(\mathcal{T}, D)$ is a metric tree.*

We start with some lemmas:

Lemma III.3.14. *Let i_0, i_1, j_0, j_1 be four distinct elements of I. Then*

$$F_{i_0,i_1} \cap F_{j_0,j_1} \subset F_{i_0,j_0} \cup F_{i_0,j_1}.$$

Proof. The claim is trivial if the intersection is empty. Otherwise, it is an immediate consequence of Lemma III.3.8. □

Lemma III.3.15. *If $x \in F_{i,j} \cap F_{i,j'}$, then*

$$B_{k,i,j}(x) = B_{k,i,j'}(x).$$

Proof. If $k = i$, the claim is trivial.

If $k = j$ (or analogously $k = j'$), we have $B_{k,i,j}(x) = b_j(x) = B_{j,i,j'}(x)$ by Remark III.3.13.4.

So we may assume that i, j, j', k are all distinct; we consider the situation discussed in Lemma III.3.7, and assume that $\{i, j, j', k\} = \{1, 2, 3, 4\}$. We may assume $k = 4$, and need to examine two cases: $i = 1$ and $i = 2$ (since $i = 3$ is equivalent to $i = 1$).

If $i = 2$, we have $x \in F_{1,2} \cap F_{2,3}$, and $b_4(\pi_{T_{1,2,4}}(x)) = b_4(\pi_{T_{2,3,4}}(x))$ follows: If $\pi_{T_{1,2,4}}(x) \in \overline{p'\eta_2}$, the two projections are equal, and the claim follows. If not, we have $b_2(x) \in [b_2(p'), 0]$, and we can represent x in $F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}$ (see Lemma III.3.8). Now the claim follows from Remark III.3.13.4.

For $i = 1$, we have $x \in F_{1,2} \cap F_{1,3}$; by Lemma III.3.7, we have $\overline{p\eta_1} \subset F_{1,2} \cap F_{1,3} \cap F_{1,4}$. Clearly, we can represent x in this ray, and the claim follows again from Remark III.3.13.4. □

Lemma III.3.16. *If* $x \in F_{i,j} \cap F_{i',j'}$, *then*

$$B_{k,i,j}(x) = B_{k,i',j'}(x) =: B_k(x).$$

Proof. If $\{i,j\} \cap \{i',j'\} \neq \emptyset$, then the claim follows from the previous lemma. Otherwise, we may assume $x \in F_{i,j} \cap F_{i,j'}$ by Lemma III.3.14. Hence, we can apply the previous lemma twice:

$$B_{k,i,j}(x) = B_{k,i,j'}(x) = B_{k,i',j'}(x). \qquad \square$$

We have shown that for every $k \in I$ the definition

$$D_k(x,y) := |B_k(x) - B_k(y)|$$

makes sense; so it is indeed a pseudometric on $\mathcal{F}$ as claimed above.

Hence, the following is also a pseudometric on $\mathcal{F}$ (possibly with value ∞):

$$D(x,y) := \sup_{k \in I} d_k(x,y).$$

Let $[x]$ denote the equivalence class of $x \in \mathcal{F}$. Recall that points $x, y \in F_{i,j}$ with $b_i(x) = b_i(y)$ satisfy $[x] = [y]$.

Lemma III.3.17. *Given points* $x \in F_{i,j}, y \in F_{i',j'}$, *there exist points* $x', y' \in F_{i'',j''}$ *such that* $\{i'',j''\} \subset \{i,j,i',j'\}$ *and* $[x] = [x'], [y] = [y']$, *and*

$$D(x,y) = d(x',y').$$

Proof. If $|\{i,j,i',j'\}| \leq 3$, we can project x, y to a tripod or line. In particular, we can represent x, y by points x', y' on a line in a flat $F_{i'',j''}$.

If $|\{i,j,i',j'\}| = 4$, let us consider only the corresponding boundary points. We may enumerate these as in Proposition III.3.12. Then we can represent x and y (uniquely) by points x'', y'' in $\overline{p\eta_1} \cup \overline{p\eta_3} \cup \overline{pp'} \cup \overline{p'\eta_2} \cup \overline{p'\eta_4}$. Every two points in this set lie in a common flat, so let $x'', y'' \in F_{i'',j''}$.

Choose

$$x' \in F_{i'',j''} \cap \{b_{i''} = b_{i''}(x'')\}, \text{ and } y' \in F_{i'',j''} \cap \{b_{i''} = b_{i''}(y'')\},$$

such that

$$d(x',y') = |b_{i''}(y'') - b_{i''}(x'')|.$$

Now for all $k \in I$, we have $D_k(x,y) = D_k(x',y') \leq d(x',y')$ (because projection to $T_{i'',j'',k}$ is 1-Lipschitz). Furthermore, we have $D_{i''}(x,y) = D_{i''}(x',y') = d(x',y')$. $\square$

Hence, we have a metric space $(\mathcal{T}, D) := (\mathcal{F}/\{D = 0\}, D)$. We claim that $\mathcal{T}$ is a metric tree.

Lemma III.3.18. *If the cardinality of A is 4, then $\mathcal{T}$ is a metric tree.*

Proof. This is almost immediate from the previous lemma: The discussion there shows that $\mathcal{T}$ has the topological structure of the set $\overline{p\eta_1} \cup \overline{p\eta_3} \cup \overline{pp'} \cup \overline{p'\eta_2} \cup \overline{p'\eta_4}$ (assuming that the elements of A are named such that Proposition III.3.12 and Lemma III.3.7 apply). Now D is almost the length metric on this graph: we just have to shorten $\overline{pp'}$ to have length $b_1(p') - b_1(p)$. □

Proof of Theorem 8. We put together the pieces collected above:

- For two points $x, y \in \mathcal{F}$, we can find a flat $F_{i,j}$ and points $x', y' \in F_{i,j}$, such that $[x] = [x'], [y] = [y']$, and $d(x', y') = D([x], [y])$ (Lemma III.3.17).

  Then the segment $\overline{x'y'}$ represents a geodesic $\overline{[x][y]}$ (of unit speed).

- From Lemma III.3.18, we conclude that $\mathcal{T}$ has extendible geodesics, and

- Since for every $z \in \mathcal{F}$, the geodesics between x', y', z (of the form introduced above) lie in a tree (again by Lemma III.3.18), every triangle in $\mathcal{T}$ is degenerate.

- This implies that geodesic segments are unique, and that $\mathcal{T}$ is 0-hyperbolic.

So $\mathcal{T}$ is indeed a tree. □

Let $\pi : \mathcal{F} \to \mathcal{T}$ be the projection, and observe that the asymptotic endpoints of $\mathcal{T}$ correspond to the points η_i. We let $\hat{\eta}_i$ denote the point of $\partial_T(\mathcal{T})$ corresponding to η_i. Then $B_i(x) = b_{\hat{\eta}_i}([x])$.[5]

Lemma III.3.19. *Assume that A is a good S-set, let $[x] \in \mathcal{T}$, and let $\mathcal{T}_{[x]} := \{(i,j) \mid [x] \in \pi(F_{i,j}) = \overline{\hat{\eta}_i\hat{\eta}_j}\}$. Set*

$$C_{[x]} := \bigcap_{(i,j) \in \mathcal{T}_{[x]}} F_{i,j}.$$

Then $C_{[x]}$ is non-empty, closed and convex, and $[x] \in \pi(C_{[x]})$.

[5]Abusing notation, we will sometimes also write $B_i([x]) := b_{\hat{\eta}_i}([x])$.

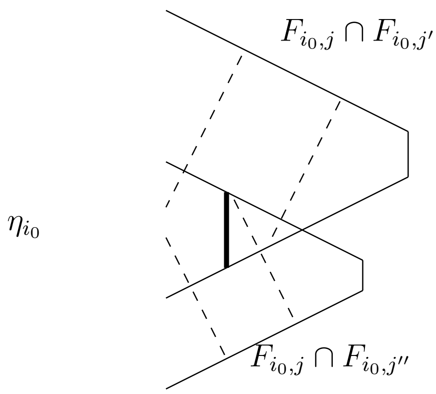

Figure III.7: The situation from Lemma III.3.19. The fat line is $l_{\{j,j',j''\}}$. The tip of the inner sector is $p_{i_0,j',j''}$.

Proof. Let $(i_0, j_0) \in \mathcal{T}_{[x]}$.

Consider $J := \{j \mid (i_0, j) \in \mathcal{T}_{[x]}\}$.

For every $j, j' \in J$, we have

$$s_{j,j'} := \sup\{b_{i_0}(F_{i_0,j} \cap F_{i_0,j'})\} = b_{i_0}(p_{i_0,j,j'}) = B_{i_0}(p_{i_0,j,j'}) \geq B_{i_0}([x]).$$

The last inequality is due to the fact that both (i_0, j) and (i_0, j') are in $\mathcal{T}_{[x]}$, hence $[x] \in \overline{[p_{i_0,j,j'}]\hat{\eta}_{i_0}}$.

By induction, one shows:

For every finite $U \subset J$, the set $\{B_{i_0} = B_{i_0}([x])\} \cap \bigcap_{j \in U} F_{i_0,j}$ is a non-empty geodesic segment l_U. Otherwise, we could find $j, j' \in J$ such that $s_{j,j'} < B_{i_0}([x])$ (by the argument for S in the proof of Lemma III.3.7). See Figure III.7.

If we can find $j, j' \in J$ such that $l_{\{j,j'\}}$ is compact, we use this compactness to conclude that $l_J \neq \emptyset$: The sets $l_{\{j,j'\}} \backslash l_{\{j,j',j''\}}$ form an open cover of $l_{\{j,j'\}}$, so finitely many j'' suffice, in contradiction to the above.

If such a choice of j, j' is not possible, then the assumption that A is good implies that l_J is a ray or a geodesic line.

Similarly, let $J' := \{i \mid (i, j_0) \in \mathcal{T}_{[x]}\}$, and obtain $l'_{J'} := \{B_{j_0} = B_{j_0}([x])\} \cap \bigcap_{i \in J'} F_{i,j_0}$.

If $l_J \cap l'_{J'}$ were empty, we could find $j \in J, i \in J'$ such that $\{i_0, j_0, i, j\}$ contradict Proposition III.3.12.

Now $l_J \cap l'_{J'} \subset C_{[x]}$ by Lemma III.3.8 (in fact, $l_J \cap l'_{J'} = C_{[x]} \cap \{B_{i_0} = B_{i_0}([x])\}$). □

Remark III.3.20. If I is finite or X is discrete, we can describe $C_{[x]}$ in detail as follows:

We find $j', j'' \in J$ such that we have $F_{i_0,j'} \cap F_{i_0,j''} \cap \{B_{i_0} = B_{i_0}([x])\} = l_J$.

Then we can similarly find $i', i'' \in J'$ such that

$$l_J \cap l'_{J'} = F_{i',j'} \cap F_{i'',j''} \cap \{B_{j'} = B_{j'}([x])\}$$

(see also Lemma III.3.8 and Figure III.6). To cover a "vertical cut-off", we may have to introduce third indices i''', j''' in J', J resp., such that

$$C_{[x]} = F_{i',j'} \cap F_{i'',j''} \cap F_{i''',j'''}.$$

In the general case, we can find sequences $i'_n, i''_n, i'''_n, j'_n, j''_n, j'''_n$ such that

$$F_{i'_n,j'_n} \cap F_{i''_n,j''_n} \cap F_{i'''_n,j'''_n}$$

is a descending sequence with $C_{[x]}$ as its limit.

Remark III.3.21. Although the tree $\mathcal{T}$ is not isometrically embedded in X, the lemma above shows that we can almost embed $\mathcal{T}$, and that intersection of vertical lines in $\mathcal{F}$ is an equivalence relation. One may think of the almost-embedded $\mathcal{T}$ in terms of sets as in Lemma III.3.8.

The following lemma follows immediately from the definition:

Lemma III.3.22. *Let $[x], [y] \in \mathcal{T}$, and $C_{[x]}, C_{[y]}$ from the lemma above. If $[x], [y]$ and $\overline{[x][y]}$ are regular, then $C_{[x]} = C_{[y]}$.* □

At one point, we will need the following technical observation:

Remark III.3.23. Let $x \in C_{[x]}$, and $y \in C_{[y]}$. Assume that x minimizes

$$b_{i',j'}|_{C_{[x]} \cap \{b_{i'} = b_{i'}(x)\}}$$

for some $(i', j') \in \mathcal{T}_{[x]} \cap \mathcal{T}_{[y]}$. Then

$$\angle_x(y, \xi_{i',j'}) \leq 2\pi/3.$$

Reason: We may assume that $b_{j'}(y) \leq b_{j'}(x)$. Let l be the line joining $\nu_{j'}$ to $\mu_{i'}$ passing through y. Let x' be the point in l satisfying $b_{j'}(x') = b_{j'}(x)$. Then it is easy to see that $b_{i',j'}(x') \geq b_{i',j'}(x)$.

III.4 Thickening tripods

Let $(p, \eta_1, \eta_2, \eta_3)$ be a tripod in X. We want to find convex rank 1-sets containing the tripod, other than a tubular neighborhood.

The results from this section are not used in the proof of Theorem 2; however, the techniques we introduce here are important for the proof (and will be generalized later on). Moreover, we get a feeling for the kinds of sets we use to build our convex set later on.

We normalize the Busemann functions to satisfy $b_{i,j}(p) = 0$ for all i, j. Let us agree to view the indices modulo 3. Note that for the singular vertices $\eta_{i,j}$, the α used in section III.1 is $\pi/3$.

Let us list some useful properties of the lower endpoint of $l_{1,2,3}$:

Lemma III.4.1. *Assume that p is the lower endpoint of $l_{1,2,3}$. Then (we will list only one version, but permuting the indices leaves the statement intact, of course):*

1. *All the $\overrightarrow{p\eta_{i,j}}$, $i, j \in \{1, 2, 3\}$ are distinct.*
2. *The $\overrightarrow{p\eta_{i,j}}$ span a flat in $\Sigma_p X$. The singular directions of this flat are in the directions of the $\eta_{i,j}$ and the ν_i. In particular, $\overline{p\nu_1} \cup \overline{p\eta_{2,3}}$ is a geodesic in X.*
3. *Let $x \in X \backslash \{p\}$. Then there exist i', j' such that $\angle_p(x, \eta_{i',j'}) \geq 2\pi/3$. It follows that $b_{i',j'}(x) \geq b_{i,j}(x)$ for all i, j.*
4. *If $b_{i,j}(x) \leq D$ for all i, j, then $d(x, p) \leq 2D$.*
5. *If $b_{1,2}(x) > \max(b_{1,3}(x), b_{2,3}(x))$, then $\angle_p(x, \nu_3) < \pi/3$.*
6. *If $b_{1,2}(x) > \max(b_{1,3}(x), b_{2,3}(x))$, then $\angle_q(x, \nu_3) < \pi/3$ for all $q \in l_{1,2,3}$.*
7. *If $\angle_p(x, \eta_1) \leq \pi/2$, we may distinguish two cases:*
   (a) *$\angle_p(x, \nu_1) \leq \pi/3$, which implies $b_{2,3}(x) \geq b_{i,j}(x)$ for all $i, j \in \{1, 2, 3\}$.*
   (b) *otherwise $b_{1,2}(x) = b_{1,3}(x) > d(x, p)/2$, and $\overrightarrow{x\eta_{1,2}} = \overrightarrow{x\eta_{1,3}}$.*

Proof. 1: If two of the $\overrightarrow{p\eta_{i,j}}$ agree, then all three have to be equal to each other; but then, p is not the lower endpoint of $l_{1,2,3}$.

Now 2 is clear.

3: Suppose $\angle_p(x, \eta_{1,j}) < 2\pi/3$ for both j.
This is only possible if $\angle_p(x, \nu_1) < \pi/3$, and by 2, we have $\angle_p(x, \eta_{2,3}) > 2\pi/3$. The second part of the claim is clear, since $b_{i',j'}$ increases at maximal slope along $\overline{px}$ (in the sense of section III.1).

4: By 3, at least one of the $b_{i,j}$ increases at maximal slope (at least $1/2 = -\cos(2\pi/3)$) along $\overline{px}$.

5: It follows from property 3, that

$$\angle_p(x, \eta_{1,2}) \geq 2\pi/3 > \max(\angle_p(x, \eta_{1,3}), \angle_p(x, \eta_{2,3})).$$

The claim follows as in the proof of 3.

6: We may assume $q \neq p$. Observe that $\angle_q(x, \eta_{1,2}) = \angle_q(x, \eta_{1,3})$, so this angle is less than $2\pi/3$ (otherwise, $b_{1,3}(x) = b_{1,2}(x)$).

If $\angle_q(x, \eta_{1,2}) = \angle_q(x, \eta_{1,3}) = 0$, let q' be the first point along $\overline{qp}$ where $\angle_{q'}(x, \eta_{1,3}) \neq 0$; if such a point does not exist, set $q' = p$. Then $\alpha := \angle_{q'}(x, \eta_{1,3}) = 2\pi/3$, or $q' = p$ and $\alpha = 0$ (since the type of $\overrightarrow{q''x}$ does not change along $\overline{qx}$). If α is $2\pi/3$, this is a contradiction to the above. If $\alpha = 0$, then $\angle_p(x, \eta_{2,3}) = 2\pi/3$, a contradiction again.

So there is a direction $\nu \in \Sigma_q(X)$ of the same type as $\overrightarrow{q\nu_3}$ with $\angle_q(\nu, \eta_{1,2}) = \pi/3$ and $\angle_q(\nu, x) < \pi/3$. If $\nu \neq \overrightarrow{q\nu_3}$, this is a contradiction to Lemma III.2.2. Hence, we have $\nu = \overrightarrow{q\nu_3}$.

7: First observe that $B_{\pi/2}(\overrightarrow{p\eta_1}) = B_{\pi/3}(\overrightarrow{p\nu_1}) \cup B_{\pi/3}(\overrightarrow{p\mu_1})$.
If $\angle_p(x, \nu_1) \leq \pi/3$, case (a) follows immediately (via 2 & 3).
If this is not the case, then $\alpha := \angle_p(x, \eta_{1,2}) = \angle_p(x, \eta_{1,3}) > 2\pi/3$, implying the first part of the claim. Since $\angle_x(\eta_{1,2}, \eta_{1,3}) \leq 2(\pi - \alpha) < 2\pi/3$, the second claim follows (because the two directions are singular and of the same type). □

Pick $R > 0$, and $D > R/2$. We define convex sets as follows: Let $T_{1,2,3} := \mathrm{Conv}(p, \eta_1, \eta_2, \eta_3)$ be the tripod, recall that we consider the indices modulo 3, and let

$$C_i := B_R(T_{1,2,3}) \cap \{b_{i,i+1} \leq D\} \cap \{b_{i,i+2} \leq D\}.$$

Proposition III.4.2. *$C_1 \cup C_2 \cup C_3$ is convex.*

We could prove this proposition directly; however, it can also be derived from Proposition III.4.4, so we omit a direct proof here.

To better understand the sets C_i, let us explain the relation to the sets $\tilde{C}_i$, which come to mind (more) naturally; define

$$\tilde{C}_i := B_R(\overline{p\eta_i}) \cap \{b_{i,i+1} \leq D\} \cap \{b_{i,i+2} \leq D\} = C_i \cap B_R(\overline{p\eta_i}).$$

Lemma III.4.3. *If p is the lower endpoint of $l_{1,2,3}$, we have*

$$\bigcup C_i = \bigcup \tilde{C}_i.$$

Proof. The two sets in question are obviously equal on $B_R(p)$, but the set on the left-hand side is potentially larger. Consider a point $x \in X$ with

$\pi_{T_{1,2,3}}(x) \in \overline{p\eta_1}\backslash\{p\}$; note that points of $B_R(\overline{p\eta_1})\backslash B_R(p)$ have this property. We have $\angle_p(x,\eta_1) < \pi/2$, so the cases from property III.4.1.7 apply.

We see that in both cases, $x \in \bigcup C_i$ implies $x \in \tilde{C}_1$. Since the conditions are symmetric, we are done. □

It will turn out that a convex rank 1-set as in Proposition III.4.2 is not quite good enough, so we need a more sophisticated approach:

In a first step, we show that we can do without tubular neighborhoods, by imposing conditions on $b'_{i,j}$:

Normalize such that $b_{i,j}(p) = b'_{i,j}(p) = 0$, let $D > 0$ and $D' \in (D/2, 2D)$. Consider the convex sets

$$\begin{aligned} K_i := &\{b_{i,i+1} \leq D\} \cap \{b_{i,i+2} \leq D\} \\ &\cap \{b'_{i,i+1} \leq D'\} \cap \{b'_{i,i+2} \leq D'\} \end{aligned}$$

Proposition III.4.4. $\bigcup K_i$ *is convex.*

We start with an elementary observation:

Lemma III.4.5.

$$\begin{aligned} \bigcup K_i &= \{x \mid \text{at least two } b_{i,j}(x) \leq D\} \\ &\quad \cap \{x \mid \text{at least two } b'_{i,j}(x) \leq D'\} \\ &=: \tilde{K}_1 \cap \tilde{K}_2 \end{aligned}$$

Proof. If the claim is not true, then there is (without loss of generality) $x \in X$ with

$$b_{2,3}(x) > D \geq \max(b_{1,2}(x), b_{1,3}(x)) \text{ and } b'_{1,3}(x) > D' \geq \max(b'_{1,2}(x), b'_{2,3}(x)).$$

This implies that $l_{1,2,3}$ has a lower endpoint p' and an upper endpoint p''. We obtain

$$\angle_{p'}(x,\nu_1) < \pi/3 \text{ and } \angle_{p''}(x,\mu_2) < \pi/3$$

by III.4.1.5. This is a contradiction to Lemma III.2.4. □

Proof of Proposition III.4.4. Note that $K_1 \cap K_2 = K_2 \cap K_3 = K_3 \cap K_1$.

We bring in the description $\bigcup K_i = \tilde{K}_1 \cup \tilde{K}_2$ from above: We show convexity of $B_\varepsilon(x) \cap \tilde{K}_i$ for every $x \in K_1 \cap K_2$, $i \in \{1,2\}$, and an $\varepsilon > 0$ that we will construct in an instant. Via Lemma I.5.3 and the lemma above, this shows the claim.

It suffices to show convexity of $\tilde{K}_1$ near x, since the proof is the same for $\tilde{K}_2$ (possibly with a different ε, but then Lemma I.5.3 applies to the smaller one).

We construct ε:

- Pick $\varepsilon, \hat{\alpha}$ such that in a Euclidean triangle $\Delta(A, B, C)$ with $d(A, B) = 2D$, $d(A, C) \in [2D - \varepsilon, 2D]$, and $\angle_A(B, C) < \hat{\alpha}$, we have $d(B, C) < D \cdot \sqrt{3}/2$; note that $\hat{\alpha} < \pi/3$.

- We decrease ε (if necessary), such that $\varepsilon < \min(D \cdot \sqrt{3}/2, (2D - D')/2)$.

- By decreasing $\hat{\alpha}$, we may assume that $(2D - \varepsilon) \cdot \underbrace{\cos \hat{\alpha}}_{\approx 1} > D'$.

- By decreasing ε again, we can require $(2D - \varepsilon) \cdot \underbrace{(-\cos(2\pi/3 + \hat{\alpha}))}_{>1/2} > D$.

This is the ε we work with.

Let p' be the lower endpoint of $l_{1,2,3}$ (if p' does not exist, the claim for $\tilde{K}_1$ is trivial); then $b_{1,2}(p') \leq 0 \leq b'_{1,2}(p')$, and we set $R' := 2 \cdot (D - b_{1,2}(p')) \geq 2D$. Note that $K_1 \cap K_2 \subset B_{R'}(p')$ (by III.4.1.4). Lemma III.1.3 shows convexity of $B_{R'}(p') \cap \tilde{K}_1$.

So it suffices to consider a point $x \in K_1 \cap K_2$ with $R' - \varepsilon \leq d(x, p') \leq R'$. Now for i', j' from III.4.1.3, the construction of ε (last item) and $b_{i',j'}(x) \leq D$ imply

$$\angle_{p'}(x, \eta_{i',j'}) \in [2\pi/3, 2\pi/3 + \hat{\alpha}].$$

On the other hand, $b'_{i,j}(x) \leq D'$ implies that $\angle_{p'}(x, \eta_{i,j}) > \hat{\alpha}$ for all i, j (by construction of $\hat{\alpha}$), so $\angle_p(x, \eta_{i,j}) \in [2\pi/3 - \hat{\alpha}, 2\pi/3 + \hat{\alpha}]$ for all i, j.

This implies that there is a direction $\nu \in \overrightarrow{p'\eta_{i',j'}}\overrightarrow{p'x}$ such that $\angle_{p'}(\nu, \eta_{i,j}) = 2\pi/3$ for all i, j.

We can extend the flat half-strip $\mathrm{Conv}(x, p', \eta_{i',j'})$ to a flat sector F with tip p', and inside this sector, we find a point x' with $d(x', p') = R'$ and $\overrightarrow{p'x'} = \nu$. By construction of $\hat{\alpha}$, we have $d(x, x') < R' \cdot \sqrt{3}/4$. Now Lemma III.2.1 applies to x'. We have $B_\varepsilon(x) \subset B_{R' \cdot \sqrt{3}/2}(x')$, so this shows the claim. □

This convex rank 1-set may have more asymptotic boundary points than just the η_i. We shrink it by putting in (large) tubular neighborhoods again:

Consider consistent (i.e. corresponding to each other under holonomy) compact subsets W_i of X_{η_i}, such that $[\overline{p\eta_i}] \in W_i$. Normalize such that $b_{i,j}(W_i) = [-S, S] = b'_{i,j}(W_i)$. Let $S_{i,j}$ be the flat strip in $F_{i,j}$ "spanned by" W_i, i.e. $S_{i,j} := \mathrm{Conv}(W_i, W_j) \subset F_{i,j}$. Let $R > 10S$.

Let $\tilde{C}_i = S_{i,i+1} \cap S_{i,i+2}$, and let[6]

$$\begin{aligned} C_i &:= B_R(\tilde{C}_i) \cap \{b_{i,i+1} \le 4S\} \cap \{b_{i,i+2} \le 4S\} \\ &\qquad \cap \{b'_{i,i+1} \le 4S\} \cap \{b'_{i,i+2} \le 4S\} \\ &= B_R(\tilde{C}_i) \cap K_i, \end{aligned}$$

where the K_i are defined as before (with $D = 4S + b_{1,2}(p)$, $D' = 4S - b_{1,2}(p) \in [3S, 5S]$, due to our new normalization).

Proposition III.4.6. *$C := \bigcup C_i$ is convex.*

Proof. It suffices to show that $C' := C_1 \cup C_2$ is convex.

The last sentence above this proposition shows that Proposition III.4.4 applies; hence $C' \cap B_R(\tilde{C}_1) \cap B_R(\tilde{C}_2)$ is convex.

If some endpoint of $l_{1,2,3}$ lies in $S_{1,2}$, then $C_1 \cap C_2 \subset B_{10S}(l \cap S_{1,2})$ (by III.4.1.4), and since $R > 10S$, we are done by Lemma I.5.3.

So we assume that no endpoint of $l_{1,2,3}$ lies in $S_{1,2}$. Then the following lemma shows (in a precise way) that near $C_1 \backslash B_R(\tilde{C}_2)$, the points in C' lie in K_1, and C' is convex in these points. Again, the claim follows via Lemma I.5.3. □

Lemma III.4.7. *Assume that no endpoint of $l_{1,2,3}$ lies in $S_{1,2}$. Then there exists an $\varepsilon > 0$ such that if $x \in C_2$ and $y \in B_{2\varepsilon}(x) \cap C_1 \backslash B_R(\tilde{C}_2)$, then $x \in K_1$, and $\overline{xy} \subset C_1 \cup C_2$.*

Proof. Let us first construct the ε:

- We pick $0 < \hat{\alpha} < \pi/6$ such that $(R + 10S)/2 \cdot \underbrace{(-\cos(2\pi/3 - \hat{\alpha}))}_{<1/2} > 5S$.
- Now pick $\varepsilon < (R - 10S)/4$ such that in a Euclidean triangle $\Delta(A, B, C)$ with

  $$d(A, B) \in [R - 4\varepsilon, R], d(A, C) \in [R - 2\varepsilon, R + 2\varepsilon], \text{ and } d(B, C) \in [0, 2\varepsilon],$$

  we have $\angle_A(B, C) < \hat{\alpha}$.

This is the ε (and $\hat{\alpha}$) we work with.

Now consider points x, y as in the statement of the lemma.

The important step is the following observation:

[6] instead of $4S$, we could choose any value $D > 3S$; the corresponding condition on R would be $R > 2(D + S)$.

There exists a point $q \in \tilde{C}_1 \cap \tilde{C}_2 = l \cap S_{1,2} = S_{1,2} \cap \{b_1 = b_1(p)\}$ such that $\angle_q(x, \eta_2) \geq \pi/2 - \hat{\alpha}$.

Reason: If $\pi_{S_{1,2}}(x) \in \tilde{C}_1$, it is easy to pick $q \in \tilde{C}_1 \cap \tilde{C}_2$ suitably: set $q := \pi_{\tilde{C}_2} \circ \pi_{\tilde{C}_1}(x)$, and observe $\angle_q(x, \eta_2) > \pi/2$ (because $\overline{\pi_{\tilde{C}_1}(x)} \cup \overline{q\eta_2}$ is a geodesic ray).

So assume that $\pi_{S_{1,2}}(x) \notin \tilde{C}_1$. By definition, the point $y \in B_{2\varepsilon}(x) \cap C_1$ satisfies $\pi_{S_{1,2}}(y) \in \tilde{C}_1$. Let $x' \in \overline{xy}$ such that $q := \pi_{S_{1,2}}(x') \in \tilde{C}_1 \cap \tilde{C}_2$. Then $d(q, x') \in [R - 4\varepsilon, R]$, $d(q, x) \in [R - 2\varepsilon, R + 2\varepsilon]$, and $d(x, x') \leq 2\varepsilon$. Since $\angle_q(x', \eta_2) \geq \pi/2$ by definition, q has the desired property by construction of $\varepsilon, \hat{\alpha}$.

Now assume that the claim is wrong. Then there is a point $x \in C_2$ as above with $\max(b_{1,2}(x), b_{2,3}(x)) \leq 4S < b_{1,3}(x)$ (this is without loss of generality, maybe we need to exchange η_2, η_3 to get this inequality).

As usual, we pick the lower endpoint p' of $l_{1,2,3}$, for which we find

$$\angle_{p'}(x, \nu_2) < \pi/3$$

by III.4.1.5. In particular, $\angle_{p'}(q, x) > \pi/3$.

Note that $\angle_q(p', x) > \pi/3 + \hat{\alpha}$ (otherwise, we get $b'_{1,2}(x) > 4S$, because $b'_{1,2}(q) \geq -S$ and the second item in the construction of ε). Now, if $\angle_q(x, \eta_2) \in [\pi/2 - \hat{\alpha}, \pi/2]$, we get $\angle_q(p', x) > 2\pi/3 - \hat{\alpha}$, implying $b_{1,2}(x) > 4S$.

So $\angle_q(x, \eta_2) > \pi/2$. Since $\angle_{p'}(q, x) > \pi/3$, we have $\angle_q(p', x) < 2\pi/3$. By the discussion above, there is a direction ν of the same type as $\overrightarrow{q\nu_1}$, but neither $\overrightarrow{q\xi_{1,2}}$ nor $\overrightarrow{q\nu_2}$, such that

$$\angle_q(x, \nu) \leq \pi/3.$$

But this is a contradiction to Lemma III.2.2 (resp. Remark III.2.3).

We have shown $x \in K_1$, so we have $x \in K_1 \cap K_2$. If $x \in B_R(\tilde{C}_1)$, then $x \in C_1$ and the second claim is immediate. If $x \notin B_R(\tilde{C}_1)$, then the argument from above, applied to y, shows that $\{x, y\} \subset K_1 \cap K_2$. Since $B_R(\tilde{C}_1 \cup \tilde{C}_2)$ is convex, the second claim follows. □

III.5 Existence of convex rank 1-sets

In this section, we prove Theorem 2.

III.5.1 Setting

Let $A := \{\eta_i \,|\, i \in I\}$ be a good S-set.

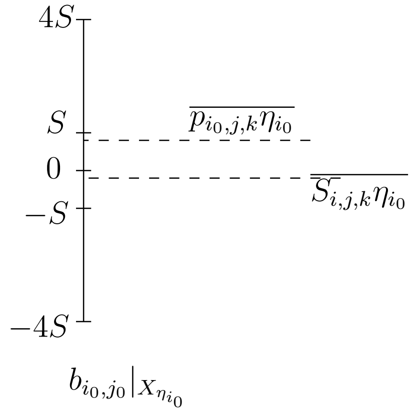

Figure III.8: $X_{\eta_{i_0}}$ and the different kinds of elements of K_{i_0}.

For every triple $i, j, k \in A$, we pick a tripodal point $p_{i,j,k}$. Let $S_{i,j,k} \in X_{\eta_i}$ be the strong asymptote class at η_i represented by $\overline{p_{i,j,k}\eta_i}$. Since order of the indices does not matter here, we can similarly define $S_{j,i,k} \in X_{\eta_j}$ and so on.

Since all the shifts are 0, we can pick a particular $i_0 \in I$, and join all the strong asymptote classes $S_{i,j,k}$ to η_{i_0}, where we obtain corresponding strong asymptote classes.

Let K_{i_0} be the closed convex hull of all these strong asymptote classes at η_{i_0}. Since all the shifts are 0, we similarly obtain isometric sets $K_i \subset X_{\eta_i}$ for all $i \in I$.

Because A is good, we may assume that we have chosen the $p_{i,j,k}$ such that the K_i are compact.

We normalize the Busemann functions such that

$$b_{i,j}(K_i) = [-S, S] = b'_{i,j}(K_i)$$

(so we have $(b_{i,j} + b'_{i,j})|_{F_{i,j}} = 0$.)

Recall the set $\mathcal{F} = \bigcup_{(i,j)\in I} F_{i,j}$, and its quotient tree $\mathcal{T}$ from section III.3.4; as usual, we let $\pi : \mathcal{F} \to T$ be the projection.

Also recalling the sets $\mathcal{T}_{[x]} = \{(i,j) \mid [x] \in \widehat{\eta_i\eta_j}\}$, we set

$$\mathcal{K}_{[x]} := \bigcap_{(i,j)\in\mathcal{T}_{[x]}} \{b_{i,j} \leq 4S\} \cap \{b'_{i,j} \leq 4S\}.$$

In our choice of the limit $4S$, the important property is the following: For every $p_{i,j,k}$, we have

$$4S - b_{i,j}(p_{i,j,k}) \leq 5S < 6S \leq 2(b_{i,j}(p_{i,j,k}) - (-4S)).$$

Of course, the same inequality holds for $b'_{i,j}$. These conditions corresponds to the condition $D' \in (D/2, 2D)$ in Proposition III.4.4. Actually, one can extend both results to the limit case where the inequality above is not strict; however, this is not needed for the purpose of this paper.

Lemma III.5.1. *For every $[x] \in \mathcal{T}$, the set $\mathcal{K}_{[x]}$ is non-empty, closed, convex and $[x] \in \pi(\mathcal{K}_{[x]})$.*

Proof. Let $(i_0, j_0) \in \mathcal{T}_{[x]}$.

We will use the notation of Lemma III.3.19.

Clearly, it suffices to show that

$$\hat{C}_{[x]} := \underbrace{C_{[x]} \cap \{B_{i_0} = B_{i_0}([x])\}}_{= l_J \cap l_{J'}} \cap \{b_{i_0,j_0} \in [-S, S]\} \neq \emptyset.$$

If we have $b_{i_0,j_0}(l_J) < -S$, there is $j, j' \in J$ such that $b_{i_0,j_0}(l_{\{i_0,j,j'\}}) < -S$ (by Remark III.3.20), in contradiction to the construction of K_{i_0}. Thus, we obtain $b_{i_0,j_0}(l_J) \cap [-S, S] \neq \emptyset$ and $b_{i_0,j_0}(l_{J'}) \cap [-S, S] \neq \emptyset$. Now the claim follows because $l_J, l_{J'}$ are intervals and have non-empty intersection. □

Let

$$\mathcal{K} := \bigcup_{[x] \in \mathcal{T}} \mathcal{K}_{[x]}$$

Lemma III.5.2. *$\mathcal{K}$ is connected.*

Proof. Let $x \in \mathcal{K}_{[x]}, y \in \mathcal{K}_{[y]}$, and pick $(i', j') \in \mathcal{T}_{[x]} \cap \mathcal{T}_{[y]}$. We can join x to $\hat{C}_{[x]} \subset S_{i',j'}$ and y to $\hat{C}_{[y]} \subset S_{i',j'}$. By construction, we have $S_{i',j'} \subset \mathcal{K}$, so the claim follows. □

We are going to show that $\bar{\mathcal{K}}$ is convex. Since it is hard to show that $\bar{\mathcal{K}}$ is of rank 1, we introduce tubular neighborhoods again: Pick $R > 10S$. For $[x] \in \mathcal{T}$, let

$$\begin{aligned} \tilde{C}_{[x]} &:= \mathcal{K}_{[x]} \cap B_R(\hat{C}_{[x]}), \\ \mathcal{C} &:= \bigcup_{[x] \in T} \tilde{C}_{[x]}. \end{aligned}$$

Exactly as for $\mathcal{K}$, we find that $\mathcal{C}$ is connected. After showing that $\bar{\mathcal{K}}$ is convex, we also show that $\bar{\mathcal{C}}$ is convex. Observe the analogon of moving from $\mathcal{K}$ to $\mathcal{C}$ and from Proposition III.4.4 to Proposition III.4.6. For the new closed convex set $\bar{\mathcal{C}}$, it is easy to show that it is of rank 1; this was obvious in both propositions mentioned above, because they were finite unions. Thus, the proof of Theorem 2 is complete after these steps.

III.5.2 The proof of Theorem 2

As a first step, we construct an $\varepsilon > 0$, and show that $\bar{\mathcal{K}}$ is δ-locally convex for every $\delta < \varepsilon/2$.

Construction III.5.3.

- Pick $0 < \hat{\alpha} < \pi/6$ such that

$$3S/\underbrace{\left(-\cos(2\pi/3+\hat{\alpha})\right)}_{>1/2} \geq 11S/2.$$

- By decreasing $\hat{\alpha}$ if necessary, we also require that $11S/2 \cdot \cos(\hat{\alpha}) > 5S$.
- Let $\varepsilon > 0$ be such that in a Euclidean triangle $\Delta(A, B, C)$ with

$$d(A, B) \geq 3S \text{ and } d(B, C) \leq \varepsilon,$$

  we have $\angle_A(B, C) < \hat{\alpha}/2$.

We introduce some more notation for this section: Consider points $[x_0] \neq [x_1] \in \mathcal{T}$.
Pick $(i', j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]}$ with $B_{i'}([x_0]) < B_{i'}([x_1])$.

Let $I_0 := \{i \in I \mid (i, j') \in \mathcal{T}_{[x_0]}\}$. Analogously, define $J_0 := \{j \in I \mid (i', j) \in \mathcal{T}_{[x_0]}\}$ and I_1, J_1 (see Figure III.9). Let $L := J_0 \cap I_1$.

Note that $\mathcal{T}_{[x_0]} \subset I_0 \times J_0$, and $\mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]} = I_0 \times J_1$.

Set $\mathcal{K}_0 := \mathcal{K}_{[x_0]}, \mathcal{K}_1 := \mathcal{K}_{[x_1]}$.

Let us start with a general lemma:

Lemma III.5.4. *Assume that $q \in \mathcal{K}_{[x_0]} \cap \mathcal{K}_{[x_1]}$ for $[x_0], [x_1] \in \mathcal{T}$. Then $q \in \mathcal{K}_{[x]}$ for all $[x] \in \overline{[x_0][x_1]}$.*

Proof. Let $[x] \in \overline{[x_0][x_1]}$, and $(k, k') \in \mathcal{T}_{[x]}$ (see Figure III.9, with $[x] = [p_x]$). If one of k, k' lies either in I_0 or in J_1, then $b_{k,k'}(q) \leq 4S$ follows by assumption. So we may assume $k, k' \in L$.

It suffices to show $b_{k,k'}(q) \leq 4S$ for all $(k, k') \in L$, since the claim for $b'_{k,k'}(q)$ follows analogously.

Assume that $b_{k,k'}(q) > 4S$ for some $k, k' \in L$. Consider the lower endpoint p of $l_{k,k',j'}$ and the lower endpoint p' of $l_{k,k',i'}$. By construction, we have $B_{i'}(p') < B_{i'}(p)$.

We have $b_{j',k}(q) \leq 4S$, $b_{j',k'}(q) \leq 4S$ and $b_{k,k'}(q) > 4S$. So we obtain $\angle_p(q, \nu_{j'}) < \pi/3$ from III.4.1.5. Similarly, we have $\angle_{p'}(q, \nu_{i'}) < \pi/3$. By Lemma III.2.2 (and Remark III.2.3), this is a contradiction. □

Proposition III.5.5. *Consider $x, y \in \mathcal{K}$ with $d(x,y) < \varepsilon$ (for the ε from Construction III.5.3) and $x \in \mathcal{K}_0, y \in \mathcal{K}_1$ for some $[x_0], [x_1] \in \mathcal{T}$.*

Let $[q] \in \overline{[x_0][x_1]}$. Then

$$\mathcal{K}_{[q]} \cap \{x, y\} \neq \emptyset.$$

Proof. We assume that $x \notin \mathcal{K}_{[q]}$, and show that this implies $y \in \mathcal{K}_{[q]}$. Without loss of generality, there is $(k_x, k'_x) \in \mathcal{T}_{[q]}$ with $b_{k_x,k'_x}(x) > 4S$ (note that neither k_x nor k'_x lies in I_0, because $x \in \mathcal{K}_0$). Pick $(i', j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]}$ as above.

Let p_x be the lower endpoint of l_{i',k_x,k'_x}.

We are going to show $y \in \mathcal{K}_{[p_x]}$, which implies $y \in \mathcal{K}_{[q]}$ by Lemma III.5.4 (since $B_{i'}([p_x]) \leq B_{i'}([q])$ by construction).

We have $b_{k_x,i'}(x) \leq 4S$, $b_{k'_x,i'}(x) \leq 4S$ and $b_{k_x,k'_x}(x) > 4S$. So we have

$$\angle_{p_x}(x, \nu_{i'}) < \pi/3 \quad \text{by } III.4.1.5. \tag{III.4}$$

Let p' be the lower endpoint of $C_{[p_x]} \cap \{B_{i'} = B_{i'}(p_x)\}$. By (the proof of) Lemma III.5.1, p' exists and satisfies $b_{i,j}(p') \leq S$ for all $(i,j) \in \mathcal{T}_{[p_x]}$. In particular, we have $d(x, p') \geq 3S$. This implies

$$\angle_{p'}(x, y) < \hat{\alpha}/2 \text{ (by construction of } \varepsilon). \tag{III.5}$$

We claim that we also have

$$\angle_{p'}(x, \xi_{i',j'}) < \pi - \hat{\alpha}. \tag{III.6}$$

Assume that this is not the case, and we have $\angle_{p'}(x, \xi_{i',j'}) \geq \pi - \hat{\alpha} (*)$. Further, we have $\angle_{p'}(x, \eta_{k_x,k'_x}) < 2\pi/3 + \hat{\alpha}$. Now $b_{k_x,k'_x}(x) > 4S$ and $b_{k_x,k'_x}(p') = b_{k_x,k'_x}(p') \leq S$ (the equality follows from $p' \in C_{[p_x]} \subset F_{i',j'} \cap F_{k_x,k'_x}$); this implies $d(p', x) > 11S/2$ (by construction of $\hat{\alpha}$).

Taking $b'_{i',j'}(p') = -b_{i',j'}(p') \geq -S$ into account, $(*)$ and $d(p', x) > 11S/2$ imply $b'_{i',j'}(x) > 4S$ (by construction of $\hat{\alpha}$), in contradiction to $x \in \mathcal{K}_0$. Thus, (III.6) is proven.

Let us phrase the next steps as Lemmas:

Lemma III.5.6. *We have $b_{k,k'}(y) \leq 4S$ for all $(k, k') \in \mathcal{T}_{[p_x]}$.*

Proof. Assume that the claim is false, i.e. there are $(k, k') \in \mathcal{T}_{[p_x]}$ with $b_{k,k'}(y) > 4S$. Observe that neither k nor k' lie in J_1, since $y \in \mathcal{K}_1$. Let p_y be the lower endpoint of $l_{k,k',j'}$. We have

$$\angle_{p_y}(y, \nu_{j'}) < \pi/3 \tag{III.7}$$

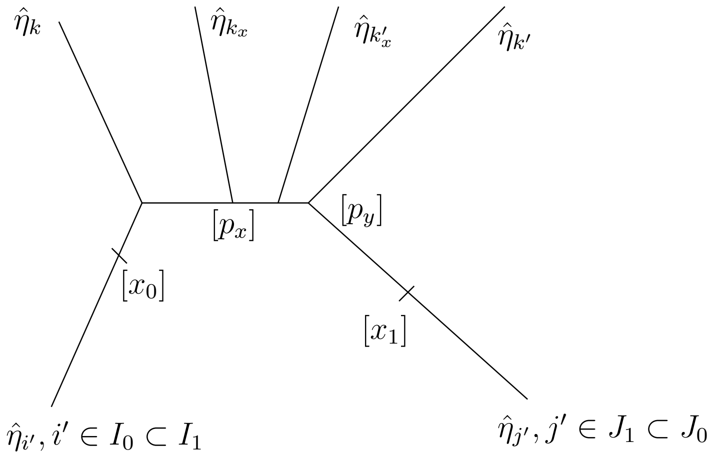

Figure III.9: the relative position of the points in the tree $\mathcal{T}$.

by III.4.1.5 (as in (III.4)). As for (III.6), we obtain

$$\max(\angle_{p_y}(y, \xi_{j',k}), \angle_{p_y}(y, \xi_{j',k'})) < \pi - \hat{\alpha}.$$

Note that either (j', k) or (j', k') lie in $\mathcal{T}_{[p_x]}$ (so $p' \in F_{j',k}$ or $p' \in F_{j',k'}$), and that $B_{j'}(p_y) \leq B_{j'}(p') = B_{j'}(p_x)$. So Lemma III.2.2 and Remark III.2.3 yield a contradiction (for p', p_y, y and $\hat{\alpha}/2$). □

Lemma III.5.7. *We have $b'_{k,k'}(y) \leq 4S$ for all $(k, k') \in \mathcal{T}_{[p_x]}$.*

Proof. Assume that this is not the case, i.e. there are $(k, k') \in \mathcal{T}_{[p_x]}$ with $b'_{k,k'}(y) > 4S$. Observe that neither k nor k' lie in J_1, since $y \in \mathcal{K}_1$. This time, let p_y be the upper endpoint of $l_{k,k',j'}$. We have

$$\angle_{p_y}(y, \mu_{j'}) < \pi/3 \tag{III.8}$$

by III.4.1.5. As for (III.6), we obtain

$$\max(\angle_{p_y}(y, \eta_{j',k}), \angle_{p_y}(y, \eta_{j',k'})) < \pi - \hat{\alpha}.$$

Note that (at least) one of (j', k) or (j', k') lie in $\mathcal{T}_{[p_x]}$, and that $B_{j'}(p_y) \leq B_{j'}(p')$. We may assume that $(j', k) \in \mathcal{T}_{[p_x]}$ (by exchanging k, k' if necessary).

Since p_y is the *upper* endpoint of $l_{k,k',j'}$, and $p' \in F_{k,k'} \cap F_{j',k} \supset C_{[p_x]}$, we have $\angle_{p_y}(p', \xi_{i',j'}) \geq \pi/3$ (by Remark III.3.23).

So we have a contradiction to Lemma III.2.4. □

This finishes the proof of Proposition III.5.5. □

Proposition III.5.8. *Let $x, y \in \mathcal{K}$ with $d(x, y) < \varepsilon$ (for the ε from Construction III.5.3). Then there exists $[q] \in \mathcal{T}$ such that $\overline{xy} \subset \mathcal{K}_{[q]}$.*

Proof. As usual, let $x \in \mathcal{K}_0, y \in \mathcal{K}_1$. By Lemma III.5.4, we know that the sets

$$I_x := \{[z] \in \overline{[x_0][x_1]} \,|\, x \in \mathcal{K}_{[z]}\},$$

and I_y similarly for y, are intervals. By Proposition III.5.5, $I_x \cup I_y$ covers $\overline{[x_0][x_1]}$. We want to show that $I_x \cap I_y \neq \emptyset$.[7]

When we assume that this is not the case, then we may assume that $I_x = \{[x_0]\}, I_y = \overline{[x_0][x_1]} \backslash \{[x_0]\}$.

Essentially, we want to show that I_x is open; more specifically, we will show that if $[b] \in \overline{[x_0][x_1]}$ is close enough to $[x_0]$, then $x \in \mathcal{K}_{[b]}$.

Pick $(i', j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]}$ such that $B_{j'}([x_1]) < B_{j'}([x_0])$.

Since $y \notin \mathcal{K}_0$, there exist (without loss of generality) $(k, k') \in \mathcal{T}_{[x_0]}$ such that $b_{k,k'}(y) > 4S$. We have $[p_{k,k',j'}] \in \overline{[x_0]\hat{\eta}_{j'}}$, so by (the proof of) Proposition III.5.5, $x \in \mathcal{K}_{[p_{k,k',j'}]}$ holds, implying $[p_{k,k',j'}] = [x_0]$.

Let p be the lower endpoint of $\hat{C}_{[x_0]}$. By III.4.1.6, we have $\angle_p(y, \nu_{j'}) < \pi/3$. By [KL97, 4.1.2], there exists a point $a' \in \overline{py} \backslash \{p\}$ such that $S' := \mathrm{Conv}(p, a', \xi_{i',j'})$ is a flat half-strip and $\overline{a'\xi_{i',j'}} \cap \overline{p\mu_{j'}} \neq \emptyset$.

Similarly, there exists a point $a'' \in \overline{py} \backslash \{p\}$ such that $S'' := \mathrm{Conv}(p, a'', \nu_{j'})$ is a flat half-strip.

Pick $a \in \mathrm{int}(\overline{pa'} \cap \overline{pa''})$, Then by construction, we have

$$\angle_a(y, \nu_{j'}) < \pi/3.$$

As for (III.6), we find $\angle_p(y, \xi_{i',j'}) < \pi - \hat{\alpha}$. Since $\angle_a(y, \xi_{i',j'}) = \angle_p(y, \xi_{i',j'})$, the point a has the same property (which we will need in order to apply Lemma III.2.2).

Now let $\{b\} := \overline{a\xi_{i',j'}} \cap \overline{p\mu_{j'}}$. This point exists by construction and lies in $F_{i',j'}$. So $\mathcal{K}_{[b]}$ is defined.

Observe that $B_{j'}(b) < B_{j'}(p) = B_{j'}([x_0])$ by construction, so $x \notin \mathcal{K}_{[b]}$.

We claim that $x \notin \mathcal{K}_{[b]}$ leads to a contradiction, which finishes the proof.

Step 1: $b_{k_x,k'_x}(x) \leq 4S$ for all $(k_x, k'_x) \in \mathcal{T}_{[b]}$.

Assume that $b_{k_x,k'_x}(x) > 4S$ for some $(k_x, k'_x) \in \mathcal{T}_{[b]}$. Let p' be the lower endpoint of $\hat{C}_{[p_{k_x,k'_x,i'}]} \subset F_{i',j'}$. By construction, we have

$$B_{j'}(b) \leq B_{j'}(p') < B_{j'}(p)$$

[7] If X is discrete or A is finite, then the tree $\mathcal{T}$ is discrete. In this case, it is easy to see that both I_x and I_y are open, so the claim follows.

(the last inequality follows from $(k_x, k'_x) \notin \mathcal{T}_{[x_0]}$), and by III.4.1.6, we have

$$\angle_{p'}(x, \nu_{i'}) < \pi/3.$$

We claim that Lemma III.2.2 leads to a contradiction (for a, p', y and $\hat{\alpha}/2$; as in the proof of Lemma III.5.6). This is clear if $a \in F_{i',j'}$.
If $a \notin F_{i',j'}$, then $\angle_p(a, \xi_{i',j'}) > 2\pi/3$, but $\angle_p(p', \xi_{i',j'}) \le 2\pi/3$ (if $b_{i',j'}(p) = -S$, this is trivial, because $p' \in S_{i',j'}$; otherwise, it follows from III.3.23). Therefore, $p' \in S'$ (because $B_{j'}(p') \ge B_{j'}(a)$), so we can apply Lemma III.2.2 as claimed.

Step 2: $b'_{k_x,k'_x}(x) \le 4S$ for all $(k_x, k'_x) \in \mathcal{T}_{[b]}$.

Assume that $b'_{k_x,k'_x}(x) > 4S$ for some $(k_x, k'_x) \in \mathcal{T}_{[b]}$. This time, let p' be the upper endpoint of $\hat{C}_{[p_{k_x,k'_x,i'}]} \subset F_{i',j'}$. As before, we have $B_{j'}(b) \le B_{j'}(p') < B_{j'}(p)$, and by III.4.1.6, we have

$$\angle_{p'}(x, \mu_{i'}) < \pi/3.$$

We have $\angle_p(p', \xi_{i',j'}) \le 2\pi/3$ as above. If $\angle_{p'}(a, \xi_{i',j'}) \ge \pi/3$, we can apply Lemma III.2.4 (as in the proof of Lemma III.5.7). Otherwise, we have $a \in F_{i',j'}$ and $\overrightarrow{ay} \in \overline{\overrightarrow{a\mu_{j'}}\overrightarrow{a\nu_{j'}}} \subset \Sigma_a(X)$. In this case, Lemma III.2.2 applies as in Step 1 (after exchanging the ν_j with the μ_j).

Together, steps 1 and 2 show that $x \in \mathcal{K}_{[b]}$, the desired contradiction. □

Proposition III.5.8 says: Whenever we consider $x, y \in \mathcal{K}$ with $d(x, y) < \varepsilon$, then $\overline{xy} \subset \mathcal{K}$. This property is inherited by the closure $\bar{\mathcal{K}}$. This implies that $\bar{\mathcal{K}}$ is δ-locally convex for every $\delta < \varepsilon/2$. From Proposition I.5.1, we obtain:

Theorem 9. *$\bar{\mathcal{K}}$ is convex.* □

It is hard to decide whether $\bar{K}$ is of rank 1. Hence, we bring in additional conditions again: Pick $R > 10S$. For $[x] \in \mathcal{T}$, recall the set $\hat{C}_{[x]}$ from Lemma III.5.1, and let

$$\tilde{C}_{[x]} := \mathcal{K}_{[x]} \cap B_R(\hat{C}_{[x]}),$$
$$\mathcal{C} := \bigcup_{[x] \in T} \tilde{C}_{[x]}.$$

As for $\mathcal{K}$, we find that $\mathcal{C}$ is connected.

We want to show that $\bar{\mathcal{C}}$ is convex, by the same tools as for $\bar{\mathcal{K}}$:

Proposition III.5.9. *There exists $\varepsilon > 0$ such that for $x, y \in \mathcal{C}$ with $d(x, y) < \varepsilon$, we have $\overline{xy} \subset \mathcal{C}$.*

Proof. We pick $\varepsilon, \hat{\alpha}$ such that they satisfy the conditions from the proof of Lemma III.4.7 as well as those from Construction III.5.3; this is possible, because in both constructions, we first impose conditions on $\hat{\alpha}$, and afterwards, we require $\varepsilon > 0$ to be small enough.

Assume that $x \in \tilde{C}_{[x_0]}, y \in \tilde{C}_{[x_1]}$. We know from Proposition III.5.8 that there is $[q] \in \overline{[x_0][x_1]}$ with $\overline{xy} \subset \mathcal{K}_{[q]}$. If $\{x, y\} \subset \tilde{C}_{[q]}$, there is nothing to show.

Assume that $x \notin \tilde{C}_{[q]}$: We know that $x \in \mathcal{K}_{[z]}$ for all $[z] \in \overline{[x_0][q]}$ by Lemma III.5.4. Hence, we have

$$\{[z] \in \overline{[x_0][q]} \,|\, x \in \tilde{C}_{[z]}\} = \{[z] \in \overline{[x_0][q]} \,|\, x \in B_R(\hat{C}_{[z]})\}. \tag{III.9}$$

Lemmas III.5.1 and III.3.19 imply that $\hat{C}_{[z]}$ varies continuously along $\overline{[x_0][q]}$.

Therefore (by pushing $[x_0]$ towards $[q]$ as far as possible), we may assume $x \notin \tilde{C}_{[z]}$ for every $[z] \in \overline{[x_0][q]} \backslash \{[x_0]\}$, and $d(x, \hat{C}_{[x_0]}) = R$ $(*)$.

Let $(i', j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]}$ such that $B_{j'}([x_1]) < B_{j'}([x_0])$ (as usual). For every singular $[z] \in \overline{[x_0][q]}$, we have

$$b_{i',j'}(\hat{C}_{[z]}) = [-S, S], \tag{III.10}$$

since otherwise, $d(x, \hat{C}_{[z]}) \leq 10S < R$ (by III.4.1.4), implying $x \in \tilde{C}_{[z]}$.

Similarly, we may assume $y \notin \tilde{C}_{[z]}$ for every $[z] \in \overline{[q][x_1]} \backslash \{[x_1]\}$, and we get (III.10) for every singular $[z] \in \overline{[q][x_1]}$.

Recalling from Figure III.6 what the sets $\mathcal{F}_{[z]}$ look like, we may conclude that $\bigcup_{[z]\in\overline{[x_0][x_1]}} \hat{C}_{[z]}$ is convex (a convex subset of the strip $S_{i',j'} = \mathrm{Conv}(K_{i'}, K_{j'})$; not necessarily a rectangle, if $[x_0]$ and/or $[x_1]$ are not singular), and so is

$$\bigcup_{[z]\in\overline{[x_0][x_1]}} B_R(\hat{C}_{[z]}) = B_R(\bigcup_{[z]\in\overline{[x_0][x_1]}} \hat{C}_{[z]}).$$

Along the lines of Lemma III.4.7, we obtain $y \in \mathcal{K}_{[x_0]}$ and similarly $x \in \mathcal{K}_{[x_1]}$ (see below). Then it is immediate (from Lemma III.5.4 and convexity of the metric) that $\overline{xy} \subset \mathcal{C}$.

Let us explain the argument for $y \in \mathcal{K}_{[x_0]}$:

Assume that $y \notin \mathcal{K}_{[x_0]}$, so without loss of generality, we have $b_{k,k'}(y) > 4S$ for some $(k, k') \in \mathcal{T}_{[x_0]}$.

Consider the lower endpoint p' of $l_{k,k',j'}$. It satisfies $\angle_{p'}(y, \nu_{j'}) < \pi/3$ by III.4.1.6.

Let $x' := \pi_{\hat{C}_{[x_0]}}(x)$, and $\{y'\} := \{\pi_{\hat{C}_{[p']}}(x')\} = \overline{x'\eta_{j'}} \cap \hat{C}_{[p']}$.

If $\angle_{y'}(y, \eta_j) > \pi/2$, we get a contradiction to either the sum of angles in a triangle, or Lemma III.2.2.

Observe that $d(y', x) \geq R$ and $\angle_{y'}(x, \eta_{j'}) \geq \pi/2$ (because $\angle_{x'}(x, \eta_{j'}) \geq \pi/2$ by $(*)$ and (III.9)). This implies $d(y', y) \geq R - \varepsilon$ and $\angle_{y'}(y, \eta_j) \geq \pi/2 - \hat{\alpha}$. Now we obtain a contradiction as in the proof of Lemma III.4.7. □

Just as for $\bar{\mathcal{K}}$, we now obtain that $\bar{\mathcal{C}}$ is convex. We claim that it is also of rank 1.

Theorem 10. *$\bar{\mathcal{C}}$ is a convex rank 1-subset of X.*

Proof. If $\partial_T \bar{\mathcal{C}}$ is not a 0-dimensional subbuilding, then there exists (without loss of generality) a point $\xi_{i,j} \in \partial_T \bar{\mathcal{C}}$. In fact, by [BL05], $\partial_T \bar{\mathcal{C}}$ is a subbuilding or has a center. So either all $\eta_{i,j}, \xi_{i,j}$ are in the asymptotic boundary, or all $\xi_{i,j}$ agree (again without loss of generality; it could also be the $\eta_{i,j}$ that agree).

So consider a point $x \in F_{i,j}$ with $b_{i,j}(x) = S + 2R + 3\varepsilon$ (for some $\varepsilon > 0$). Let $x' := \pi_{\hat{C}_{[x]}}(x)$. Then $d(x, x') \geq 2R + 3\varepsilon$.

To finish the proof, it suffices to lead the following assumption to a contradiction: There exists $[x''] \in \mathcal{T}$ such that $x \in B_\varepsilon(\tilde{C}_{[x'']})$.

Assume the contrary, and set $x'' := \pi_{\hat{C}_{[x'']}}(x)$. Obviously, $d(x', x'') \geq R + 2\varepsilon$. Pick i' such that $(i', j) \in \mathcal{T}_{[x]} \cap \mathcal{T}_{[x'']}$ (such an i' exists, after exchanging i, j if necessary).

Since $x', x'' \in S_{i',j}$, the inequality $2S < (R + 2\varepsilon)/2$ implies that

$$\angle_{x'}(x'', \xi_{i,j}) = \angle_{x'}(x'', x) \geq \pi/3.$$

Now triangle comparison yields $d(x'', x) \geq R + 2\varepsilon$, the desired contradiction. □

By definition and Lemma III.5.1, we have $S_{i,j} \subset \bar{\mathcal{C}}$ for all $i, j \in I$. Hence, we have $A \subset \partial_T \bar{\mathcal{C}}$. We have just shown that $\bar{\mathcal{C}} \subset B_{2R}(\bigcup_{i,j \in I} S_{i,j})$. Therefore, $\partial_T \bar{\mathcal{C}}$ is precisely the closure of $A \subset \partial_\infty X$ in the cone topology. The proof of Theorem 2 is now finished.

Chapter IV

Convex rank 1 subsets of $SL(3,\mathbb{C})/SU(3)$ and $SL(3,\mathbb{R})/SO(3)$

In this chapter, we discuss convex rank 1-subsets C of the symmetric spaces $M_{\mathbb{K}} := SL(3,\mathbb{K})/SO(3,\mathbb{K})$ (for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$); convex rank 1-sets are defined in Definition III.3.1. In section I.3.3, we have presented important properties of the geometry of $M_{\mathbb{K}}$.

We show that every triple of asymptotic boundary points of such a set C is contained in the asymptotic boundary of an isometrically embedded (up to rescaling) hyperbolic plane $\mathbb{K}H^2$ (Theorem 3). In case $\mathbb{K} = \mathbb{R}$, we show that all of these hyperbolic planes agree, i.e. $\partial_T C \subset \partial_T \mathbb{R}H^2 \subset \partial_T M_{\mathbb{R}}$ (Theorem 4).

In order to show Theorem 3, we use a direct computational approach: We normalize triples of pairwise antipodal Weyl chambers, and calculate which of these can occur in the boundary of an embedded $\mathbb{K}H^2$ (Lemmas IV.2.2 and IV.2.3).

Then we exclude most of the other triples by showing that their holonomy is non-zero (Cor. IV.3.2); finally, we exclude the remaining triples by a geometric argument (Proposition IV.3.3).

These results imply Theorem 3. To move on and obtain Theorem 4, we use the geometric interpretation of the asymptotic boundary of copies of $\mathbb{R}H^2$ as a subset of $\mathbb{R}P^2$; this is discussed in the following section.

It is note-worthy that apart from $\mathbb{C}H^2$, there is another symmetric subspace of rank 1 of $M_{\mathbb{C}}$, namely a copy of $\mathbb{R}H^3$, which parametrizes the parallel set of a singular geodesic. However, this is not a problem in the proof of Theorem 3, since every triple of asymptotic boundary points of such a copy of $\mathbb{R}H^3$ is contained in some copy of $\mathbb{R}H^2$, which in turn is (also) contained in a copy of $\mathbb{C}H^2$ (see Lemma IV.2.3).

IV.1 The Tits boundary of $\mathbb{K}H^2 \subset M_{\mathbb{K}}$

Observe that every isometric embedding $\mathbb{K}H^2 \hookrightarrow M_{\mathbb{K}}$ is a convex rank 1-subset. Hence, the boundary points are centers of Weyl chambers by Lemma III.3.2. As discussed in section I.3.3, Weyl chambers in the Tits boundary of $M_{\mathbb{K}}$ can be thought of as a tuple (v,p), where v is a 1-dimensional subspace of $\mathbb{K}^3$, and $p \supset v$ is a 2-dimensional subspace of $\mathbb{K}^3$.

Observe that there are two natural ways of embedding $\mathbb{R}H^2$ into $M_{\mathbb{R}}$: We have $\mathbb{R}H^2 \cong SO(2,1,\mathbb{R})/SO(2,\mathbb{R})$ and $\mathbb{R}H^2 \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

It is well known that these are the only ways of isometrically embedding (up to rescaling) $\mathbb{R}H^2$ into $M_{\mathbb{R}}$, but this also follows from our proof.

For $\mathbb{K} = \mathbb{C}$, it suffices to consider $\mathbb{C}H^2 \cong SU(2,1)/(SU(2) \times SU(1))$; however, observe that the parallel set of a singular geodesic is parametrized by $\mathbb{R}H^3$.

Let us first consider the natural embedding $X = SL(2,\mathbb{R})/SO(2,\mathbb{R}) \cong \mathbb{R}H^2 \subset M_{\mathbb{R}}$. By definition, there is a 1-dimensional subspace W of vectors which have constant length for all elements of X, and there is a 2-dimensional subspace O of $M_{\mathbb{R}}$ such that $p(W,O) = 0$ precisely for all $p \in X$.

This implies that $\eta \in \partial_T X$ precisely if η has distance $\frac{\pi}{2}$ from the vertices of $\partial_T M$ determined by W, O respectively. Hence, X parametrizes the parallel set of a geodesic line with endpoints corresponding to W and O. The Weyl chambers corresponding to points of $\partial_T X$ are of the form (v,p) with $W \subset p$ and $v = O \cap p$.

Now we let $X = SO(2,1,\mathbb{K})/(SO(2,\mathbb{K}) \times SO(1,\mathbb{K})) \subset M_{\mathbb{K}}$, and determine its asymptotic boundary points.

We start at $Id \cdot (SO(2,\mathbb{K}) \times SO(1,\mathbb{K}))$.

Recall the Lie algebras (see [Hel78, §IX.4])

$$\mathfrak{so}(2,1,\mathbb{K}) = \left\{ \begin{pmatrix} 0 & \bar{c} & \bar{a} \\ -c & 0 & \bar{b} \\ a & b & 0 \end{pmatrix} \mid a,b,c \in \mathbb{K} \right\}$$

and

$$\mathfrak{so}(2,\mathbb{K}) = \left\{ \begin{pmatrix} 0 & \bar{c} \\ -c & 0 \end{pmatrix} \mid c \in \mathbb{K} \right\}.$$

Hence, every tangent vector of X in $Id \cdot (SO(2,\mathbb{K}) \times SO(1,\mathbb{K}))$ can be represented uniquely by a matrix of the form

$$t \cdot Y := t \begin{pmatrix} 0 & 0 & \bar{a} \\ 0 & 0 & \bar{b} \\ a & b & 0 \end{pmatrix},$$

with $a\bar{a} + b\bar{b} = 1$ and $t > 0$.

We compute the boundary points of X by examining the quadratic forms occurring along the ray $\exp(tY)$ for Y fixed.

The eigenvectors of Y are $(b, -a, 0), (\pm\bar{a}, \pm\bar{b}, 1)$ for the eigenvalues $0, \pm 1$ respectively. So, as described in [Lee05], the boundary points at infinity of X are the centers of the Weyl chambers

$$(\left\langle(-\bar{a}, -\bar{b}, 1)\right\rangle, \left\langle(-\bar{a}, -\bar{b}, 1), (b, -a, 0)\right\rangle).$$

In the real case, we may view $\partial_T X$ as a quadric together with its tangent lines (in $\mathbb{R}P^2$); a point in $\partial_T X$ is the center of a Weyl chamber (v, p), where v is a point in the quadric and p is the tangent line to the quadric passing through v.

In the complex case, $\partial_T X$ is a hypersurface (in $\mathbb{C}H^2$) of real dimension 3, together with the unique complex tangent lines: Let q be the hypersurface, and consider a Weyl chamber (v, p) corresponding to a point of $\partial_T X$. Then v is a point of q, and p is the complex tangent line, determined by $T_v(q) \cap i \cdot T_v(q)$.

IV.2 Normalizing triples of Weyl chambers

In preparation for the proof of Theorem 3, we start by examining triples of pairwise antipodal Weyl chambers:

So consider an arbitrary triple of pairwise antipodal Weyl chambers: $\eta_i \simeq (v_i, p_i)$ $(i \in \{1, 2, 3\})$, where v_i is a 1-dimensional vector subspace, and $p_i \supset v_i$ is a 2-dimensional vector subspace of $\mathbb{K}^3$. Antipodality (only) means that $v_i \not\subset p_j$ if $i \neq j$.

Set $g_i := p_{i+1} \cap p_{i+2}$ (where the indices are to be taken modulo 3). In terms of our notation from the previous chapter, the asymptotic boundary points corresponding to v_1, p_1, g_3, p_2, v_2 were called $\mu_1, \nu_1, \eta_{1,2}, \nu_2, \mu_2$ respectively (see Figure III.2).

There are two types to be considered:

- **Type I**: $g_i \neq g_j$ if $i \neq j$ (this means that the 2d-subspaces p_i lie in general position).
- **Type II**: $g_i = g_j$ for all i, j (in this case, all the planes p_i intersect in the same line).

We consider triples of type I first:

We have $p_i = g_{i+1} \oplus g_{i+2}$, so antipodality implies that $g_1 \oplus g_2 \oplus g_3 = \mathbb{K}^3$.

Since every element of $SL(3, \mathbb{K})$ acts on $M_{\mathbb{K}}$ as an isometry (see section I.3.3), we are free to pick a standard basis; so we may assume $g_i = \langle e_i \rangle$.

Because $v_i \subset p_j$ if and only if $i = j$, we have $g_j \neq v_i$ for all i, j. Therefore, $v_i = \langle e_{i+1} + \lambda_i e_{i+2} \rangle$ for some $\lambda_i \in \mathbb{K}\backslash\{0\}$.

Applying a suitable isometry (namely $\mathrm{diag}(y/\lambda_2, y/(\lambda_2\lambda_3), y)$, where y is a solution of $y^3 = \lambda_2^2\lambda_3$), we may assume $\lambda_2 = \lambda_3 = 1$. Hence, triples of type I are parametrized by a parameter $\lambda \in \mathbb{K}\backslash\{0\}$.

Observe that the λ describing our triple is just $\lambda_1\lambda_2\lambda_3$. This number is known as the *product of ratios* (and clearly is a projective invariant), since each λ_i describes the ratio in which v_i separates $\langle e_{i+1} \rangle, \langle e_{i+2} \rangle$.

In the real case, Ceva's theorem states that the three lines in $\mathbb{R}P^2$ corresponding to the 2d-subspaces $\langle v_i, e_i \rangle \subset \mathbb{K}^3$ intersect if and only if $\lambda = 1$. It is also known that one can inscribe a quadric into a triangle such that the quadric touches the triangle in three given points, if and only if $\lambda = 1$. Actually, we will re-prove this in the course of this paper.

Remark IV.2.1. It makes sense to say that a triple is non-generic if it is of type II, or if it is of type I with $\lambda = -1$: For in type II, the planes p_i have a line in common, and in type I with $\lambda = -1$, the lines v_i are collinear (as points in $\mathbb{K}P^2$).

We proceed to examine which (normalized) triples can occur in the boundary of an embedded $\mathbb{K}H^2$:

Lemma IV.2.2. *If a normalized triple of type I satisfies* $|\lambda| = 1, \lambda \neq -1$, *then the triple lies in the boundary of a subspace* $\mathbb{K}H^2 \subset M_{\mathbb{K}}$.

Proof. From the previous section, we use the description of asymptotic boundary points of $SO(2,1,\mathbb{K})/(SO(2,\mathbb{K}) \times SO(1,\mathbb{K}))$.

Normalize a triple η_1, η_2, η_3 of its asymptotic boundary points. We can specify a boundary point by a tuple $(a, b) \in \mathbb{K}^2$ with $\|(a, b)\| = 1$. Since $SO(2,1,\mathbb{K})$ acts transitively on pairs of distinct boundary points, we may fix $\eta_{1/2} \simeq (\pm 1, 0)$, and we let $\eta_3 \simeq (a, b)$. If $b = 0$ (and $a \neq \pm 1$), (observe that this can only happen when $\mathbb{K} = \mathbb{C}$), we obtain a triple of type II; so we assume $|a| < 1$ (and consequently $|b| > 0$).

Let $\eta_i \simeq (v_i, p_i)$ as usual.[1] Then $g_1 = \left\langle (1, -\frac{a+1}{b}, 1) \right\rangle$, $g_2 = \left\langle (-1, \frac{a-1}{b}, 1) \right\rangle$, and $g_3 = \langle (0, 1, 0) \rangle$. Let $V_1 = (\frac{b}{a+1}, -1, \frac{b}{a+1})$, $V_2 = (\frac{b}{a-1}, -1, \frac{-b}{a-1})$, and $V_3 = (0, 1, 0)$.

Now $g_i = \langle V_i \rangle$, and to find the λ parametrizing our triple, we need only solve (since $v_i = \langle V_{i+1} + V_{i+2} \rangle$ for $i \in \{1, 2\}$)

$$V_1 + \lambda V_2 = \mu(-\bar{a}, -\bar{b}, 1) \in v_3.$$

[1] Here, we are abusing notation: We have $\eta_i \simeq (v_i, p_i)$ (for $v_i \subset p_i \subset \mathbb{K}^3$), and we also have $\eta_i \simeq (a, b)$ (for $a, b \in \mathbb{K}$). The context will make a confusion impossible.

The equation in the second coordinate is

$$-1-\lambda = -\mu\bar{b}, \text{ hence } \mu = \frac{1+\lambda}{\bar{b}}$$

Now we solve the equation for the third coordinate:

$$\begin{aligned}
\frac{b}{a+1} - \lambda\frac{b}{a-1} &= \frac{1+\lambda}{\bar{b}} \\
\lambda\left(-\frac{1}{\bar{b}} - \frac{b}{a-1}\right) &= \frac{1}{\bar{b}} - \frac{b}{a+1} \\
\lambda\cdot\frac{a}{\bar{b}}\cdot\frac{\bar{a}-1}{a-1} &= \frac{a}{\bar{b}}\cdot\frac{\bar{a}+1}{a+1} \\
\lambda = \frac{a-1}{\bar{a}-1}\cdot\frac{\bar{a}+1}{a+1} &= \frac{(a-1)^2(\bar{a}+1)^2}{(a-1)(\bar{a}-1)(a+1)(\bar{a}+1)} \\
\lambda &= \frac{(a\bar{a}-1+2i\,\mathrm{Im}(a))^2}{|a-1|^2|a+1|^2}
\end{aligned}$$

First, we observe $|\lambda|^2 = \lambda\bar{\lambda} = 1$ (from the second-to-last line). We also see immediately that if a is real, then $\lambda = 1$.

Since $|a| \neq 1$, we see (from the last line) that λ is not real unless a is real; in particular, $\lambda \neq -1$ in any case.

Letting $a \in (-i, i)$ be purely imaginary, we consider the numerator only (since the denominator is real): The complex number which is squared lies in the left half-plane (i.e. it has negative real part, since $|a| < 1$), and the argument of this number takes all values in $(\pi/2, 3\pi/2)$.

This implies that all $|\lambda| = 1, \lambda \neq -1$ can occur. □

Now we turn to triples of type II:

The triple η_1, η_2, η_3 is of type II if the corresponding planes p_i have the line $g_1 = g_2 = g_3$ in common.

We claim that such a triple than can occur in the boundary of a convex rank 1-set only if the lines v_i are collinear (as points in $\mathbb{K}P^2$); i.e. if $\langle v_1, v_2, v_3\rangle \neq \mathbb{K}^3$.

Let us state how to normalize a triple of type II: We can pick a standard basis as we like, so we may assume $g_i = \langle e_3\rangle$ and $v_i = \langle e_i\rangle$ for $i \in \{1, 2\}$.

Now $v_3 = \langle \sum \lambda_i e_i\rangle$ with $\lambda_1 \neq 0 \neq \lambda_2$ (because $v_3 \not\subset p_1 \cup p_2$).
Applying the isometry $\mathrm{diag}(1/\lambda_1, 1/\lambda_2, \lambda_1\lambda_2)$, we may assume $\lambda_1 = \lambda_2 = 1$. For $\lambda_3 \neq 0$, we could even assume $\lambda_3 = 1$, but we will not need this in the sequel.

Lemma IV.2.3. *A triple of type II in which the lines v_i are collinear (as points in $\mathbb{K}P^2$) lies in the asymptotic boundary of a copy of $\mathbb{K}H^2$.*

Proof. Note that with the normalization made above, the triples in question correspond to $\lambda_3 = 0$.

In the real case, the claim is immediate from our calculations above and the discussion of asymptotic boundary points of $SL(2,\mathbb{R})/SO(2)$: All of its asymptotic boundary points (v,p) satisfy $p \supset W = \langle e_3 \rangle$ and $v \subset O = \langle e_1, e_2 \rangle$.

In the complex case, we just need to exhibit a suitable triple of type II in the boundary of a copy of $\mathbb{C}H^2$. In the proof of IV.2.2, we have observed that the triple corresponding to $(\pm 1, 0)$ and $(i, 0)$ is of type II. Its 2d-subspaces p_i all contain the 1d-subspace $\langle e_2 \rangle$, and the three 1d-subspaces v_i all lie in the 2d-subspace $\langle e_1, e_3 \rangle$. □

IV.3 Proof of Theorem 3

From now on, let C be a convex rank 1-subset of $M_{\mathbb{K}}$.

The first step towards proving Theorem 4 is to exclude triples of pairwise antipodal Weyl chambers which do not satisfy the sufficient conditions discussed in the previous section:

Theorem (Theorem 3)**.** *Let $\eta_i \in \partial_T C$ (for $i \in \{1,2,3\}$). Then there exists an isometric embedding $\mathbb{K}H^2 \hookrightarrow M_{\mathbb{K}}$ s.t. $\eta_i \in \partial_T \mathbb{K}H^2$ for all i.*

By the results of the previous section, it suffices to exclude the triples which satisfy $\lambda = -1$ or $|\lambda| \neq 1$ (for type I), resp. $\lambda_3 \neq 0$ (for type II). To do this, we use holonomy (see Section I.1.2):

The holonomy map of a triple of pairwise antipodal points η_1, η_2, η_3 is an orientation-preserving isometry of $\mathbb{R}$, hence a translation (as in the previous chapter). Its translation length is called the *shift invariant* of the triple. Up to sign, this shift is independent of permutations of the triple.

Proposition IV.3.1. *Consider a triple of type I with parameter $\lambda \in \mathbb{K}\backslash\{0\}$. Then its shift-invariant is directly proportional to $|\log(|\lambda|)|$; in particular, the shift is zero if and only if $|\lambda| = 1$.*

Proof. As in the section about normalization, we may assume that for $i \in \{2,3\}$, we have $\eta_i \simeq (\langle e_{i+1} + e_{i+2} \rangle, \langle e_{i+1}, e_{i+2} \rangle)$. Then the point η_1 corresponds to $(\langle e_2 + \lambda e_3 \rangle, \langle e_2, e_3 \rangle)$.

We know that the shift invariant for $\lambda = 1$ is 0, since then the triple is contained in the asymptotic boundary of some $\mathbb{K}H^2$.

Let η_0 be the boundary point corresponding to $\lambda = 1$ (given η_2, η_3).

Let l_2 be an arbitrary line joining η_0 to η_2, and l_3 be a line joining η_0 to η_3, which is strongly asymptotic to l_2 at η_0 (i.e. $d_{\eta_0}(l_2, l_3) = 0$). Now there is a line l_1 joining $\eta_{2/3}$ which is strongly asymptotic to $l_{2/3}$ at its ends (because the shift of the triple η_0, η_2, η_3 is zero).

Let y be a third root of λ, and consider the isometries[2]

$$g := \operatorname{diag}(y, 1/y^2, y), \qquad h := \operatorname{diag}(1/y, 1/y, y^2)$$

We have $g\eta_2 = \eta_2, h\eta_3 = \eta_3, g\eta_0 = \eta_1, h\eta_0 = \eta_1$.

We consider

$$\begin{aligned} d_2 &:= d_{\eta_2}(l_1, gl_2) = d_{\eta_2}(l_2, gl_2) \\ d_3 &:= d_{\eta_3}(l_1, hl_3) = d_{\eta_3}(l_3, hl_3) \\ d_1 &:= d_{\eta_1}(gl_2, hl_3) = d_{\eta_0}(l_2, g^{-1}hl_3) = d_{\eta_0}(l_3, g^{-1}hl_3). \end{aligned}$$

If we had $d_2 = d_3 = 0$, the shift of η_1, η_2, η_3 would just be d_1. In our situation, the shift is of the form

$$\text{shift} = \pm d_1 \pm d_2 \pm d_3.$$

Using the formulas from the introduction, we calculate (below):

$$d_1 = d_2 = d_3,$$

and $d_i = 0$ if and only if $|\lambda| = 1$. This will finish the proof.

Let us give the argument for d_2 in detail:

For every quadratic form in the line l_2, the basis $(e_2 + e_3, e_3, e_1 + e_3)$ is orthogonal. In this basis, we may choose $l_2(t)$ to have the form $\operatorname{diag}(T^2, 1, T^{-2})$ for $T = e^t$ (then l_2 has unit speed and converges to η_2 for $t \to \infty$, see [Lee05]). A general element of the flat determined by η_0, η_2 can be written (in the same basis) as $l^A(t) := \operatorname{diag}(T^2/A, A^2, T^{-2}/A)$, for $A \in (0, \infty)$. Observe that for fixed t, the image of $A \mapsto l^A(t)$ is a geodesic orthogonal to l_2, and hence

$$d^2(l_2, l^A) = d^2(l_2(t), l^A(t)) = 3/2(\log A)^2 \quad \text{by (I.1).} \tag{IV.1}$$

We claim that $d_{\eta_2}(gl_2, l^{1/|y|}) = 0$ (with $|y| = \sqrt[3]{|\lambda|}$ as before): Since we know that $gl_2(t) \underset{t\to\infty}{\longrightarrow} \eta_2$, this follows from

$$\|e_3\|^2_{gl_2(t)} = \|e_3/y\|^2_{l_2(t)} = |y|^{-2}.$$

So $d_2 = d_{\eta_2}(l_2, gl_2) = d_{\eta_2}(l_2, l^{1/|y|}) = d(l_2(t), l^{1/|y|}(t)) = \sqrt{3/2}\,|\log(|y|)|$ by (IV.1). The same calculation shows that

$$d_1 = d_3 = d_2 = \sqrt{3/2}\,|\log(|y|)| = \frac{\sqrt{3/2}}{3}\,|\log(|\lambda|)|\,. \qquad \square$$

---

[2]These matrices are given in the (standard) basis (e_1, e_2, e_3).

Corollary IV.3.2. *Consider a triple of type I with product of ratios* $\lambda \in \mathbb{K}\backslash\{0\}$*. If* $|\lambda| \neq 1$*, the triple cannot occur in the boundary of a convex rank 1 set.*

Proof. Let η lie in the asymptotic boundary of a convex rank 1-set C. Then the set of strong asymptote classes $C_\eta \subset (M_\mathbb{K})_\eta$ at η which are representable in C has to be compact and invariant under the holonomy map of every triple (η, ξ_1, ξ_2) for $\xi_1, \xi_2 \in \partial_T C$.

This is only possible if the shift is 0 for every such triple, implying that $|\lambda| = 1$ is a necessary condition. □

To finish the case of type I, we need to exclude triples described by $\lambda = -1$. We discuss these triples together with triples of type II:[3]

Proposition IV.3.3. *A non-generic triple occurs in the asymptotic boundary of a copy of a convex rank 1-set if and only if it lies in the asymptotic boundary of a parallel set.*

Proof. Recall that a non-generic triple is a triple of type II, or a triple of type I with $\lambda = -1$.

Every such triple (η_1, η_2, η_3) has the property that there is a point $\xi \in \partial_T M_\mathbb{K}$ with $\angle(\xi, \eta_i) = \pi/2$ for all i (the point ξ is the vertex corresponding to $g_1 = g_2 = g_3$ for type II, and the vertex corresponding to $\langle v_1, v_2, v_3 \rangle$ for type I, $\lambda = -1$).

Now let Y be a convex rank 1-set containing all the η_i in its boundary. We claim that there exists a point p minimizing $b_\xi|_Y$: For every point $q \in Y$, the function b_ξ is decreasing along $\overline{q\eta_1}$. The strong asymptote class represented by $\overline{q\eta_1}$ can be represented by a line in the flat $F_{1,2}$, and every point p' in this line satisfies $b_\xi(p') = \inf(b_\xi(\overline{q\eta_1}))$.

This shows that we find a point p, where $b_\xi|_Y$ attains its minimum. This property of p implies $\angle_p(\xi, \eta_i) \geq \pi/2$ for all i. But $\angle(\xi, \eta_i) = \pi/2$ implies the inequality in the other direction, showing $\angle_p(\xi, \eta_i) = \pi/2$. Therefore, we have three flat sectors $\mathrm{Conv}(p, \xi, \eta_i)$.

Let l be the unique line in $M_\mathbb{K}$ extending the ray $\overline{p\xi}$. Basic properties of symmetric spaces imply that l bounds flat half-planes H_i, with $\eta_i \in \partial_T H_i$.[4] In particular, $\eta_i \in \partial_T P(l)$.

The other direction is immediate from Lemma IV.2.3. □

[3]Here, I use an idea due to B. Leeb, which replaces my own more technical argument.

[4]Indeed: Let q be an interior point of $\overline{p\xi}$, and let ζ be the other endpoint of l. Consider the singular point μ in $\overline{\xi\eta_1}$. Then $\mathrm{Conv}(q, \xi, \mu)$ is a flat sector of opening angle $\pi/3$, and $\angle_q(\mu, \zeta) = \angle(\mu, \zeta) = 2\pi/3$. Hence, $\mathrm{Conv}(q, \mu, \zeta)$ is a flat sector, too. The union of these two flat sectors is a flat half-plane. This half-plane contains the flat sector $\mathrm{Conv}(q, \xi, \eta_1)$, because geodesics do not branch in a symmetric space.

The proposition above excludes triples of type I with $\lambda = -1$, and triples of type II with $\lambda_3 \neq 0$.

Hence, the proof of Theorem 3 is finished.

Remark IV.3.4. Here, we have found an important difference between the building and the smooth case: We have just shown (using the notation introduced in the previous chapter) that there are good S-sets (consisting of three points) which do not bound a convex rank 1-set.

For the real case, we will proceed to show: Let A be an S-set containing only triples corresponding to embedded hyperbolic planes. Then A bounds a convex rank 1-set (which may be chosen as a subset of a hyperbolic plane).

It turns out that we do not need to assume that A is good (another contrast to the building case, see Example III.3.6).

IV.4 Deducing Theorem 4 from Theorem 3

For $\mathbb{K} = \mathbb{R}$, we can classify all the possible asymptotic boundaries of convex rank 1-sets. Our argument uses the geometry of $\mathbb{R}P^2$, so it does not generalize easily. Also, for $\mathbb{K} = \mathbb{C}$, one has to take more symmetric subspaces of rank 1 into account (there is also a copy of $\mathbb{R}H^3 \subset M_{\mathbb{C}}$).

In order to derive Theorem 4 from Theorem 3, it suffices to show:

Proposition IV.4.1. *Consider 4 points η_i, $i \in \{1,2,3,4\}$ in $\partial_T M_{\mathbb{R}}$, and suppose that each three of them are contained in the asymptotic boundary of a copy of $\mathbb{R}H^2$. Then all η_i lie in the asymptotic boundary of the same copy of $\mathbb{R}H^2$.*

Proof. As usual, let each point η_i be the center of the Weyl chamber determined by the tuple (v_i, p_i) of incident 1- resp. 2-dimensional vector subspaces of $\mathbb{R}^3$. Note that we may assume the η_i to be distinct (since otherwise, the claim is trivial).

We split the discussion in the following two cases:

- There are (at least) **two triples of type II**.
- Otherwise, there are (at least) **three triples of type I**.

Let us discuss show the case of type II first:
Without loss of generality, we assume that (η_1, η_2, η_3) and (η_1, η_2, η_4) are triples of type II. So $\langle v_1, v_2, v_3 \rangle = p$ is 2-dimensional, and since the η_i are distinct (and antipodal), $p = \langle v_1, v_2 \rangle$. By assumption, $\langle v_1, v_2, v_4 \rangle$ is 2-dimensional, too, so it is equal to p.

Similarly, one derives that for $i \neq j$, $p_i \cap p_j$ is independent from i, j, finishing the case of type II.

Now we consider the case of type I, and without loss of generality we assume that the triples (η_1, η_2, η_3), (η_1, η_2, η_4) and (η_1, η_3, η_4) are of type I. By our calculations above, this means that these triples lie on quadrics (in the sense described above) q_4, q_3, q_2.

We know that q_4, q_3 meet tangentially in two points $p_1, p_2 \in \mathbb{R}P^2$ (corresponding to the Weyl chambers determined by η_1, η_2 resp.).

It is easy to see that the set of quadrics touching in p_1, p_2 is a 1-dimensional family; in particular, two quadrics touching in two points agree if they have a third point of intersection.

Observe that a quadric splits $\mathbb{R}P^2$ into two connected components.

It follows that if two distinct quadrics have two Weyl chambers in common, then one of them lies entirely inside the other (except for the two points where they touch).

Now q_4, q_3, q_2 each divide $\mathbb{R}P^2$ into two components A_i, B_i. By the arguments above, we may assume $A_4 \subset A_3 \subset A_2$ (and hence $B_2 \subset B_3 \subset B_4$).

If $q_4 = q_3$, then q_2 touches this quadric in three points, so $q_2 = q_4 = q_3$. So we assume that the q_i are pairwise distinct, and the inclusions named above are strict. In this case, q_2 and q_4 could only touch in the point p_1, not in two points as required.

This contradiction finishes the proof. □

The proposition above, together with Theorem 3, implies Theorem 4.

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