Gauge-Field Theories and Gravity on Noncommutative Spaces

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Für Walkiria

Zusammenfassung

In dieser Doktorarbeit werden Eichtheorien und Gravitation auf nichtkommutativen Räumen studiert. Am Anfang werden die der Konstruktion von Feldtheorien auf nichtkommutativen Räumen zu Grunde liegenden Konzepte eingeführt. Unter einem nichtkommutativen Raum verstehen wir eine nichtkommutative Algebra, welche die Algebra der Funktionen auf gewöhnlicher Raumzeit ersetzt. Wir konstruieren Ableitungen und deformierte Symmetrien ("Quanten Gruppen" Symmetrien), welche auf nichtkommutative Räume wirken. Aus Konsistenzgründen muss die Wirkung auf ein Produkt von Darstellungen abgeändert werden ("deformierte Koprodukte"); dies führt insbesondere zu deformierten Leibnizregeln. Außerdem zeigen wir wie ein nichtkommutativer Raum und die Generatoren von deformierten Symmetrien, die auf diesen Raum wirken, auf der gewöhnlichen Funktionenalgebra dargestellt werden können; das punktweise Produkt wird durch ein nichtkommutatives, ein sogenanntes "Sternprodukt", ersetzt.

Eine Möglichkeit Eichtheorien auf nichtkommutativen Räumen zu konstruieren ist gegeben durch sogenannte "Seiberg–Witten Abbildungen". Dieser Zugang macht es möglich, alle nichtkommutativen Felder durch ihre kommutativen Entsprechungen auszudrücken. Wir veranschaulichen dieses Verfahren an zwei Beispielen, der zweidimensionale q-deformierte Euklidische Raum und die κ-deformierte Minkowski Raumzeit. Darüber hinaus werden Eichtheorien auf "fuzzy" $S^2 \times S^2$ als Multi-Matrixmodell studiert. Wir zeigen, dass das vorgestellte Modell in einem bestimmten Limes zur Eichtheorie auf dem nichtkommutativen $\mathbb{R}^4$ übergeht. Auch ein neuer Zugang zu deformierten Eichtheorien mittels "twist"-deformierten Eichtransformationen wird eingeführt. In diesem Zusammenhang treten zusätzlich zu den gewöhnlichen Eichfeldern neue Felder auf. Die Einführung dieser Felder ist notwending, um konsistente Bewegungsgleichungen und erhaltene Ströme zu erhalten. Dies ist das erste Mal, dass Erhaltungsgesetze von verallgemeinerten Symmetrien gegeben durch Quanten Gruppen abgeleitet werden konnten.

Ein Hauptteil dieser Arbeit befasst sich mit der Konstruktion von deformierten infinitesimalen Diffeomorphismen. Darauf basierend führen wir deformierte Gravitation als eine hinsichtlich dieser Diffeomorphismen kovariante Theorie ein. Dies führt zu deformierten Einsteingleichungen. Für "kanonisch" deformierte Räume kann sogar eine Deformation der Einstein–Hilbert Wirkung gefunden werden. Diese geht im kommutativen Grenzfall zur gewönlichen Einstein–Hilbert Wirkung über. Zudem werden alle relevanten Größen bis zur zweiten Ordnung im Deformationsparameter expandiert.

Abstract

In this thesis gauge-field theories and gravity on noncommutative spaces are studied. We start with an introduction to the concepts underlying the construction of field theories on noncommutative spaces. By a noncommutative space we mean a noncommutative algebra, which replaces the algebra of functions on ordinary space. We construct derivatives and deformed symmetries ("Quantum Group" symmetries) acting on noncommutative spaces. Consistency requires us to change the action on a product of representations ("deformed coproducts"); this gives rise in particular to deformed Leibniz rules. We also show how a noncommutative space and the generators of deformed symmetries acting on it can be represented on the ordinary algebra of functions; the commutative, point-wise product is substituted by a noncommutative one ("star-product").

One possible way to define gauge-field theories on noncommutative spaces is to construct "Seiberg–Witten maps". In this approach it is possible to express all noncommutative quantities in terms of their commutative counterparts. We illustrate this by two examples, the two-dimensional q-deformed Euclidean plane and the κ-deformed Minkowski space-time. In addition gauge-field theory on "fuzzy" $S^2 \times S^2$ is discussed as a multi-matrix model. We show that this model reduces in an appropriate limit to gauge-field theory on noncommutative $\mathbb{R}^4$. We also present a new approach to deformed gauge theories, which is based on "twisted" gauge transformations. In this setting new fields occur in addition to the usual gauge fields. Consistent equations of motion and conserved currents are obtained. This is the first time that conservation laws have been derived from a generalized, Quantum Group symmetry.

We discuss in detail how to construct deformed infinitesimal diffeomorphisms by deformations via generic "twists". Then we construct gravity as a theory, which is covariant with respect to these diffeomorphisms. This leads to a deformation of Einstein's equations. For canonically deformed spaces, a deformed Einstein–Hilbert action can be even defined. It reduces to the usual Einstein–Hilbert action in the commutative limit. All relevant quantities are expanded in terms of the usual, commutative fields up to second order in the deformation parameter.

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Contents

Chapter 1

Introduction

It is plausible to assume that space-time at very short distances changes its nature in a fundamental way. Our experience with the developments in theoretical physics suggests this point of view. In Newton's mechanics time and metric are absolute quantities. In quantum mechanics time is no longer an observable and general relativity teaches us that the space-time metric itself is a dynamical variable. We observe that absolute quantities are more and more replaced by relative or dynamical variables. However, the concept of a differentiable space-time structure remains preserved in all established theories (see also the table below[1]). It is compelling to assume that the idea of space-time described as a differentiable manifold does not survive in physics beyond Einstein's theory of gravity. A quantum theory of gravity should only be compatible with a space-time of quantum nature.

	Time	Metric	Topology, Differentiable Structure
Newtonian Mechanics	+	+	+
Quantum Mechanics	-	+	+
General Relativity	-	-	+
Quantum Gravity	-	-	-

Another argument supporting this conclusions was brought forward in [1]. The authors argue that trying to measure very short distances implies using test

[1]This table is taken from a talk given by J. Ehlers in Oviedo at the Spanish Relativity Meeting 2005 (as mentioned there it was first used by Trautman in 1972).

particles with very short wave lengths, i.e. very high energies. If the amount of energy in a very small region of space-time becomes too high, a black hole will be created and measurement below the horizon of this black hole will be impossible. A differentiable space-time looses its operational meaning below Planck-scale. It is therefore reasonable to assume that the nature of space-time at very short distances should change in a way, which reflects this fact. This is a very old idea. It is amazing that already Riemann argued in this direction. In his famous inaugural lecture 1854 he said[2]:

"Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypothesis of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena...

... An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to ensure that this work is not hindered by too restricted concepts, and that the progress in comprehending the connection of things is not obstructed by traditional prejudices".

One way to change the nature of space-time is to make it noncommutative. The concept of a differential space-time manifold is substituted by the algebra generated by noncommutative coordinates $\hat{x}^\mu$, which are subject to commutation relations of the type

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}(\hat{x}) \,. \tag{1.1}$$

This proposal has a long history. In 1930 Heisenberg wrote in a letter to Peierls [2] that he considers the assumption of non-commuting coordinates and the "corresponding uncertainty relations for quite reasonable". However, he was not able to give such relations a mathematical meaning and in this letter he therefore asked Peierls and Pauli for help. The ideas of non-commuting coordinates was worked out in detail for the first time in 1947 by Snyder, a student of Oppenheimer. He used noncommutative coordinates in order to regularize the divergent electron self-energy [3]. The fact that mathematical tools to treat such spaces were still not at hand and the progress in the development of Quantum Field Theories and their renormalization made the idea of noncommutativity disappear

[2]We would like to thank Jose Adolfo de Azcarraga for drawing this reference to our attention.

for quite some time. Developments in mathematics during the last twenty years and the persisting absence of a consistent formulation of quantum gravity revived the concept of noncommutativity and noncommutative geometry [4–6]. Recently, even a connection between noncommutative geometry and string theory could be found [7–9]. In this scenario the endpoints of open strings on D-branes in a background B-field behave in a noncommutative way. This raised the interest in noncommutative gauge-field theories on canonically deformed spaces where in (1.1) $\theta^{\mu\nu} \in \mathbb{R}$.

The approach studied in this thesis relies mainly on two recent developments in mathematics. Around twenty years ago, Drinfel'd and Jimbo constructed q-deformations of the universal enveloping algebras of semisimple Lie algebras [10,11]. In this way they showed that a continuous deformation of Lie algebras in the category of Hopf algebras is possible. Physicists interpreted these deformed Hopf algebras (also called Quantum Groups) as generalized, deformed symmetries, and it was found that they consistently act on noncommutative spaces [12–15]. This thesis does not only deal with the construction of such deformed symmetries acting on particular or even whole classes of noncommutative spaces, but we shall also show how to construct field theories and gauge-field theories in the noncommutative setting. The second important input comes from deformation quantization [16]. The construction of star-products ($\star$-products) enables us to study the effects of noncommutativity perturbatively in orders of a deformation parameter. Star-products are associative deformations of the usual, commutative point-wise product of functions on a manifold. The zeroth order in the deformation parameter reproduces the ordinary point-wise product and higher orders are given in terms of bidifferential operators. Star-products can be used in order to represent noncommutative space-time algebras on the space of ordinary functions on a manifold by equipping it with a new, noncommutative product, the star-product. In this way it is possible to construct noncommutative field theories using the fields of the commutative theory; noncommutative quantities can be expressed in terms of their commutative counterparts by constructing order by order an explicit map, called Seiberg–Witten map, between commutative and noncommutative gauge theories [9]. In this context, noncommutative gauge transformations are induced by ordinary gauge transformations applied to the gauge fields of the commutative theory. Seiberg–Witten maps were first constructed for canonically deformed spaces [17–19]. The field content remains unchanged but additional terms in the action lead to new phenomena. Such contributions can be determined order by order in the deformation parameter by expanding the star-products. In the framework of a noncommutative standard model [20] new phenomena such as usually forbidden couplings and decays have been studied intensively [21–30]. Recently, star-products were obtained for arbitrary Poisson manifolds [31] and Seiberg–Witten maps were constructed in this general setting based on the concept of covariant coordinates [32, 33]. However, the solutions for the gauge field, for instance, are proportional to the Poisson structure itself, and therefore vanish in the commutative limit. In order to obtain solutions with the right commutative limit, derivatives instead of coordinates have to be gauged. In the case when $\theta^{\mu\nu} \neq$ const. the problem arises that in general derivatives act via a deformed Leibniz rule and are not derivations on the star-product algebra. As a consequence derivative valued gauge fields have to be introduced as we shall see when we construct Seiberg–Witten maps for κ-deformed spaces in Section 5.3, [34]. A possibility to avoid this conclusion is by gauging derivations of the noncommutative algebra induced by commuting frames. This was first proposed in [35, 36] (Section 5.1–5.2) where gauge-field theories via this kind of Seiberg–Witten map are constructed in detail for the q-deformed Euclidean plane (see also [37]). A generalization of these concepts to star-product algebras, which possess a commuting frame, can be found in [38].

Another, new approach to gauge theories on noncommutative spaces is given by twisted gauge transformations. We introduce this approach in Section 5.5, [39], see also [40, 41]. By twisting the coproduct for gauge transformations, we can construct gauge invariant quantities. This kind of gauge transformations are compatible with the deformed diffeomorphisms constructed in [42, 43]. We derive consistent equations of motions and conserved currents. This is the first time that conservation laws for theories invariant with respect to deformed, generalized symmetries in the framework of Quantum Groups could be obtained. The usual derivation via Noether does not work in these cases since a deformed Leibniz rule has to be taken into account.

There is also a class of noncommutative spaces, which retain the undeformed symmetries of their classical analogues. Those spaces, known as fuzzy spaces, arise when quantizing symplectic manifolds given by coadjoint orbits of Lie groups. This leads to fuzzy manifolds, which in case of semi-simple Lie groups are given by finite dimensional algebras [44]. The most famous examples for fuzzy spaces are the fuzzy sphere S^2_N [45–47] and fuzzy CP^n [48–50]. The description of fuzzy spheres in terms of finite dimensional matrix algebras makes it possible to construct gauge theories on fuzzy spheres as random matrix models [51]. In this thesis we propose a four-dimensional model and construct gauge theories on fuzzy $S^2 \times S^2$. This finite matrix model reduces in the commutative limit to Yang-Mills theories on $S^2 \times S^2$. Moreover, a double scaling limit is found such that gauge theories on fuzzy $S^2 \times S^2$ reduce to noncommutative gauge theories on the four-dimensional canonically deformed space $\mathbb{R}^4_\theta$. Therefore we can use this model as a regularization of noncommutative gauge theories. A class of topologically non-trivial solutions (instantons) on fuzzy $S^2 \times S^2$ reduces in this limit to instantons on $\mathbb{R}^4_\theta$. The recovered instantons correspond to a generalization of the instantons constructed for noncommutative gauge theories in two dimensions [52–54].

A main part of this thesis is devoted to the formulation of gravity on noncommutative spaces and to deformations of Riemannian geometry. Noncommutative spaces break diffeomorphism invariance. We propose a new approach to noncommutative gravity based on a deformation of diffeomorphisms. For related approaches see [4, 6, 55–62]. A first construction is given for canonically deformed spaces. In this case translational invariance is not broken and integration can be defined without further problems. We obtain a deformation of the Einstein–Hilbert action [42]. In [43, 63] the construction is generalized to deformations of the universal enveloping algebra of vector fields by generic twists. This work is of particular interest since, as we shall see in Chapter 4, the class of noncommutative spaces defined by twists is very rich. It contains many interesting examples as, for instance, noncommutative spaces, which possess indeed a lattice-like structure. It is in particular for these discrete noncommutative space-times that we expect a regularization behaviour. The quantum properties of gravity on such lattice-like spaces still has to be investigated. What we end up with in this thesis is a deformation of gravity for a large class of noncommutative spaces. If it was possible to study the quantum behaviour for this whole class of gravity theories, renormalizable points may be found within this class. In this way renormalizability may select out particular examples of noncommutative spaces.

This thesis is organized as follows: In the following three chapters we introduce the fundamental concepts associated with noncommutativity and present the tools necessary to construct physical theories on noncommutative spaces. In Chapter 5 and 6 models for gauge-field theories and gravity theories on noncommutative spaces are constructed in detail. These two chapters contain the main part of this thesis. They consist of the publications [34–36, 39, 42, 43, 63–66]. The publications [37, 67, 68] were also written in the framework of this thesis. Their content is basically covered by the publications already included here. The aim of Chapters 2–4 is to provide the non-expert among the readers with a detailed introduction, which may help to understand better the arguments and derivations brought forward in Chapters 5–6.

In the second chapter we give an algebraic definition of noncommutative spaces and provide some examples. We also construct derivatives on noncommutative spaces and introduce deformed symmetries (Quantum Groups) [69–73]. They are constructed by deforming universal enveloping algebras of Lie algebras that possess a natural Hopf algebra structure. Noncommutative spaces are interpreted as module algebras with respect to such Quantum Groups [12, 13]. The necessary mathematical tools are presented in detail. We end this section discussing examples of Quantum Group symmetries, which act on the noncommutative spaces introduced in the first section of Chapter 2.

The third chapter is devoted to a detailed study of star products ($\star$-products) and $\star$-representations of operators. We give explicit examples for star-products in the canonical case, κ-deformed case and for the q-deformed Euclidean plane. The construction of star-products corresponding to normal ordering and symmetric ordering is presented in all detail for κ-deformed spaces.

The forth chapter is devoted to deformations by twists [70, 74, 75]. Twists provide a convenient way to deform Hopf algebras and their corresponding module algebras. We shall often consider the algebra of functions on a manifold as module algebra with respect to an appropriate Quantum Group. Using a twist, its product can be deformed by introducing a star-product. The cocycle condition satisfied by the twist guarantees associativity of this star-product. By twisting a Hopf algebra acting on the usual algebra of functions we obtain a deformed symmetry, which acts consistently on the twisted noncommutative space of functions. This procedure can be applied in order to construct a deformed Poincaré symmetry acting on canonically deformed spaces. By deforming the universal algebra of vector fields we construct deformed diffeomorphisms.

The fifth chapter treats various models of gauge-field theories on noncommutative spaces. Section 5.1–5.2 are devoted to the construction of gauge-field theories on the q-deformed two-dimensional Euclidean plane [35, 36]. The formalism is based on a commuting frame. A Seiberg–Witten map is constructed and all relevant quantities including the action are expanded up to first non-trivial order in the deformation parameter. Section 5.3 and 5.4 contain the construction of gauge theories on κ-deformed space-time and were published in [34, 64]. We admit derivative valued gauge fields. In this way Seiberg–Witten maps can be constructed and the noncommutative fields can be expressed in terms of their commutative counterparts. The physical field strength can be defined by projecting out the curvature like contribution of the commutator of two covariant derivatives and neglecting torsion like terms. Consistency with κ-Poincaré transformations is also shown. Section 5.5, [39], treats a different approach to deformed gauge theories. We consider the canonically deformed space and deform gauge transformations by twisting the coproduct. The Lie algebra generating gauge transformations is extended by translations. Gauge invariant quantities and gauge covariant Lagrangians are constructed. The field equations are consistent if we choose the gauge-field in the enveloping algebra. This leads to new fields in addition to the usual gauge field. Their number is finite if we use finite dimensional representations of the enveloping algebra. These fields reduce in the commutative limit to free fields and couple only weakly via the deformation parameter θ. The new fields depend on the representation chosen. We also derive conserved currents. In the last section we present a model for gauge theories on fuzzy $S^2 \times S^2$ [65]. We construct $U(n)$ gauge theories where fluctuations of the covariant coordinates correspond to gauge fields. The action reduces to Yang-Mills theories on ordinary $S^2 \times S^2$ in the commutative limit. Moreover, we present a gauge fixed action with BRST symmetry. The quantization of the model is given by convergent integrals over the matrix degrees of freedom. We explicitly define a double scaling limit, in which gauge theory on fuzzy $S^2 \times S^2$ reduces to gauge theory on the noncommutative, θ-deformed four-dimensional space. A class of topologically non-trivial solutions on fuzzy $S^2 \times S^2$, which can be interpreted as $U(1)$ instantons, reduces to $U(1)$ instantons on $\mathbb{R}^4_\theta$.

The sixth chapter is devoted to gravity on noncommutative spaces. We start this chapter in Section 6.1 with a lecture given at the I Modave Summer School in Mathematical Physics [63] in order to introduce to the publications [42,43] (see also [67, 68]). In the following section, [42], we present in detail the construction of a gravity theory on canonically deformed spaces. It is based on deformed infinitesimal diffeomorphisms, which are constructed by deforming the universal enveloping algebra of vector fields. The coproduct is deformed. With respect to these deformed infinitesimal diffeomorphisms, a whole tensor calculus is established. Based on these structures we construct a theory of gravity via the Einstein formalism. This leads to a deformation of the Einstein–Hilbert action. It reduces to the usual Einstein–Hilbert action in the commutative limit. The dynamical field is the vierbein or the metric as in the commutative theory. All relevant quantities are expanded up to second order in the deformation parameter. The action is constructed as an invariant under deformed infinitesimal diffeomorphisms. In Section 6.3, [43], we generalize the construction of [42]. By means of generic twists, the universal enveloping algebra of vector fields is deformed such that it acts consistently on twisted spaces. Tensors and forms are introduced as module algebras of this twisted Hopf algebra. Einstein equations, which are covariant with respect to twisted infinitesimal diffeomorphisms are formulated. The last section provides a short summary of our results about deformed gravity.

We finish this thesis in Chapter 7 with a short summary and discuss some open questions and current as well as future research projects.

Chapter 2

Deformed Spaces and Deformed Symmetries

2.1 Noncommutative Spaces

In field theories one usually considers differential space-time manifolds. In the noncommutative realm, the notion of a point is no longer well-defined and we have to give up the concept of differentiable manifolds. However, the space of functions on a manifold is an algebra. A generalization of this algebra can be considered in the noncommutative case. We take the algebra freely generated by the *noncommutative coordinates* $\hat{x}^\mu$, which respect commutation relations of the type

$$[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{2.1}$$

We don't want to care too much about convergence and therefore take the space of formal power series in the coordinates $\hat{x}^\mu$ and divide by the ideal generated by the above relations

$$\hat{\mathcal{A}} = \mathbb{C}\langle\langle \hat{x}^0, \dots, \hat{x}^n \rangle\rangle / ([\hat{x}^\mu, \hat{x}^\nu] - C^{\mu\nu}(\hat{x}))\,.$$

The function $C^{\mu\nu}(\hat{x})$ is unknown. For physical reasons it should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments. However, we can consider a power-series expansion

$$C^{\mu\nu}(\hat{x}) = i\theta^{\mu\nu} + iC^{\mu\nu}{}_\rho \hat{x}^\rho + (R^{\mu\nu}{}_{\rho\sigma} - \delta^\nu_\rho \delta^\mu_\sigma)\hat{x}^\rho \hat{x}^\sigma + \dots\,,$$

where $\theta^{\mu\nu}$, $C^{\mu\nu}{}_{\rho}$ and $R^{\mu\nu}{}_{\rho\sigma}$ are constants, and study cases where the commutation relations are constant, linear or quadratic in the coordinates. At very short distances these cases may provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$. We are led to the following three structures

1. canonical or θ-deformed case:
$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,. \tag{2.2}$$

2. Lie algebra case:
$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}{}_{\rho}\hat{x}^\rho. \tag{2.3}$$

3. Quantum Spaces:
$$\hat{x}^\mu\hat{x}^\nu = R^{\mu\nu}{}_{\rho\sigma}\hat{x}^\rho\hat{x}^\sigma. \tag{2.4}$$

We denote the algebra generated by noncommutative coordinates $\hat{x}^\mu$, which are subject to relations of the above type, by $\hat{\mathcal{A}}$. We shall often refer to it as the *algebra of noncommutative functions.* Commutative functions will be denoted by $\mathcal{A}$. In the following we want to give some examples for noncommutative spaces. Later we shall study deformed symmetries, which act consistently on these noncommutative spaces.

2.1.1 θ-deformed spaces

The easiest example of a noncommutative space is given by commutation relations of the type
$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \tag{2.5}$$
where $\theta^{\mu\nu}$ is an arbitrary anti-symmetric, constant tensor. θ-deformed spaces can be defined for any dimension. However, we shall often consider the case of four dimensions. Then, using suitable rotations, $\theta^{\mu\nu}$can always be cast in the following form:
$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\theta} \\ 0 & 0 & -\tilde{\theta} & 0 \end{pmatrix}.$$
We see that the algebra (2.5) in four dimensions is nothing but two copies of the Heisenberg algebra. This implies that the eigenvalue spectrum of the noncommutative coordinates is continuous. Hence, this noncommutative space does not

provide a lattice-like structure of space-time. However, the commutation relations (2.5) express an uncertainty in the measurement of two space-time coordinates. The assumption of such an uncertainty leads to new physical implications. The interest in θ-deformed spaces rose when it was discovered that gauge theories on this particular noncommutative space appear in an appropriate limit of String Theory [7–9]. It is still subject of intense research, see for instance [17, 19, 42, 76–78] and references therein. For its simplicity it is very-well suited to study many features of noncommutativity.

2.1.2 κ-deformed spaces

There is a well-known example for noncommutative spaces of the Lie algebra type (2.3): κ-deformed spaces. For a long time they were believed to be the only example of this type, which admit the action of a quantum group symmetry (the so-called κ-deformed Poincaré algebra [79–81], see also Section 2.1.2). Recent investigations, however, have given rise to a whole class of noncommutative spaces obtained by generic twists [43]. We shall learn about this in Chapter 4. Among this class also new examples for noncommutative spaces of Lie-type could be found [82]. These spaces still have to be studied in more detail and we focus our considerations on κ-deformed spaces. This example is of particular interest since it also appears as a low energy limit of loop quantum gravity [83].

κ-deformed spaces are generated by coordinates, which are subject to the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu), \qquad \mu, \nu = 0, 1, \ldots, n\,, \tag{2.6}$$

where a^μ are constants. Indices are raised and lowered with the usual Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$ and its inverse. The constant vector a^μ can be transformed by convenient linear transformations on the coordinates to the form $a^\mu = \delta^{\mu n} a$ such that the algebra becomes

$$\begin{aligned} [\hat{x}^n, \hat{x}^i] &= ia\hat{x}^j \\ [\hat{x}^i, \hat{x}^j] &= 0, \quad i, j = 0, \ldots, n-1\,. \end{aligned} \tag{2.7}$$

Note that we label the n commuting coordinates $\hat{x}^i$ $(i = 0, \ldots, n-1)$ by Latin letters $i, j, k, \ldots$. The noncommutative coordinate is always taken to be $\hat{x}^n$. Greek indices are meant to run always over all $(n+1)$-coordinates, $\mu, \nu, \ldots = 0, \ldots, n$. The deformation parameter a is related to the frequently used parameter κ by $a = \frac{1}{\kappa}$.

2.1.3 The fuzzy sphere

The fuzzy sphere was introduced about 15 years ago by John Madore [6, 46]. It is another example for a noncommutative space of Lie-type (2.3). The algebra of functions on the fuzzy sphere is the finite algebra S_N^2 generated by Hermitian operators $x_i = (x_1, x_2, x_3)$ satisfying the defining relations

$$[x_i, x_j] = i\Lambda_N \epsilon_{ijk} x_k \,, \tag{2.8}$$

$$x_1^2 + x_2^2 + x_3^2 = R^2 \,. \tag{2.9}$$

They are obtained from the N-dimensional representation of $su(2)$ with generators λ_i $(i = 1, 2, 3)$ and commutation relations

$$[\lambda_i, \lambda_j] = i\epsilon_{ijk}\lambda_k, \qquad \sum_{i=1}^{3} \lambda_i \lambda_i = \frac{N^2 - 1}{4} \tag{2.10}$$

by identifying

$$x_i = \Lambda_N \, \lambda_i, \qquad \Lambda_N = \frac{2R}{\sqrt{N^2 - 1}} \,. \tag{2.11}$$

The noncommutativity parameter Λ_N is of dimension length. The algebra of functions S_N^2 therefore coincides with the simple matrix algebra $\mathrm{Mat}(N, \mathbb{C})$. The normalized integral of a function $f \in S_N^2$ is given by the trace

$$\int_{S_N^2} f = \frac{4\pi R^2}{N} \mathrm{tr}(f) \,. \tag{2.12}$$

The functions on the fuzzy sphere can be mapped to functions on the commutative sphere S^2 using the decomposition into harmonics under the action

$$J_i f = [\lambda_i, f] \tag{2.13}$$

of the rotation group $SU(2)$. One obtains analogues of the spherical harmonics up to a maximal angular momentum $N - 1$. Therefore S_N^2 is a regularization of S^2 with an UV cutoff, and the commutative sphere S^2 is recovered in the limit $N \to \infty$.

The fuzzy sphere is of particular interest since it retains the rotational symmetry of the classical sphere. It is fuzzy in the sense that a precise localization of points is not possible. The construction of more general fuzzy spaces relies on the fact that coadjoint orbits of Lie groups (for semisimple Lie groups these are the same as adjoint orbits) are symplectic manifolds. These manifolds can be quantized under certain conditions, giving rise to fuzzy spaces, see [44] and references therein. The most famous examples are the fuzzy sphere and fuzzy $\mathbb{C}P^N$.

2.1.4 q-deformed Euclidean space

Examples of noncommutative spaces with commutation relations of type (2.4) are provided by q-deformed spaces. We want to introduce a simple example of such a space. Let us consider the algebra generated by coordinates $\hat{z}, \hat{\bar{z}}$, which are subject to the commutation relations

$$\hat{z}\hat{\bar{z}} = q^2 \hat{\bar{z}}\hat{z}\,, \tag{2.14}$$

where $q \in \mathbb{R}$. This noncommutative space is called the q-deformed two-dimensional Euclidean space [35, 84–86]. Defining real coordinates $\hat{x}, \hat{y}$ by $\hat{z} = \hat{x} + i\hat{y}$ and $\hat{\bar{z}} = \hat{x} - i\hat{y}$ the commutation relation (2.14) yields

$$[\hat{x}, \hat{y}] = i\frac{(q^2-1)}{(q^2+1)}(\hat{x}^2 + \hat{y}^2)\,. \tag{2.15}$$

2.1.5 The problem of broken symmetries

It is important to make the following observation: *Noncommutative spaces break symmetries.* The commutation relations respected by the noncommutative coordinates are in general not invariant with respect to symmetries, which are an invariance of the underlying undeformed space. The above examples for deformed spaces (except for the fuzzy sphere, which retains undeformed rotational symmetry) illustrate this. For instance, the usual Minkowski space is invariant with respect to Poincaré transformations. However, the noncommutative space

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,,$$

where e.g.

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\theta} \\ 0 & 0 & -\tilde{\theta} & 0 \end{pmatrix}$$

breaks Poincaré invariance down to invariance with respect to translations together with $SO(1,1) \times SO(2)$ (respectively $SO(2) \times SO(2)$ for Euclidean signature). Analogue considerations hold for the other examples. The construction of field theories is based on symmetries. Dealing with noncommutative spaces there are basically two possible ways to proceed: The first is to construct field theories in the usual way but based on a smaller symmetry group, which remains an invariance group of the noncommutative space [78]. The second way to proceed is to look for possible deformations of symmetries considered in the commutative case. A deformed symmetry is required to reduce to the original, undeformed one in the commutative limit and has to act consistently on the noncommutative space. Such deformations exist indeed and will be introduced in the following sections.

2.2 Derivatives

In order to study the dynamics of fields we need derivatives acting on the noncommutative algebra $\hat{\mathcal{A}}$. Derivatives are maps on the deformed coordinate space [87–89]

$$\hat{\partial}_\mu : \hat{\mathcal{A}} \to \hat{\mathcal{A}} .$$

This means that they have to be consistent with the commutation relations of the coordinates or, differently said, they have to map the ideal generated by the commutation relations of the coordinates to itself ("zero" has to be mapped to "zero"). Since we made the experience that it is often not clear to physicists what this is supposed to mean, we shall illustrate this in detail. Consider a commutative space in two dimensions

$$[x, y] = 0 .$$

We can introduce partial derivatives ∂_x, ∂_y by the commutation relations

$$\begin{aligned} [\partial_x, x] &= 1 \qquad [\partial_y, x] = 0 \\ [\partial_x, y] &= 0 \qquad [\partial_y, y] = 1 \\ [\partial_x, \partial_y] &= 0 \quad . \end{aligned} \tag{2.16}$$

These are consistent commutation relations. If we commute, for instance, a partial derivative with $0 = xy - yx$ using (2.16)

$$\begin{aligned} \partial_x(xy - yx) &= y + x\partial_x y - y\partial_x x \\ &= y + xy\partial_x - y - yx\partial_x \\ &= (xy - yx)\partial_x \end{aligned}$$

we end up with an expression proportional to zero. We can also see this on the level of the action. The commutation relations (2.16) imply the following action of the derivatives on coordinates[1]

$$\begin{aligned} (\partial_x x) &= 1 \qquad (\partial_y x) = 0 \\ (\partial_x y) &= 0 \qquad (\partial_y y) = 1 \end{aligned} \tag{2.17}$$

[1] We use brackets to distinguish the action of a differential operator from the multiplication in the algebra of differential operators.

together with the Leibniz rule

$$\begin{aligned} \partial_x(fg) &= (\partial_x f)g + f(\partial_x g) \\ \partial_y(fg) &= (\partial_y f)g + f(\partial_y g)\,. \end{aligned} \tag{2.18}$$

This action is consistent with the commutation relations $[x, y] = 0$ since for example

$$\begin{aligned} 0 &= \partial_x(xy - yx) = (\partial_x x)y + x(\partial_x y) - (\partial_x y)x - y(\partial_x x) \\ &= 1y - y1 = 0\,, \end{aligned}$$

where we used (2.17) and (2.18). The ideal generated by $xy-yx$ is mapped to itself by ∂_x ("zero" is mapped to "zero"). Now consider for instance a noncommutative space defined by

$$\hat{x}\hat{y} = q\hat{y}\hat{x}\,.$$

It is called the Manin plane [13]. Defining derivatives $\hat{\partial}_x, \hat{\partial}_y$ by the action on coordinates as in (2.17) and the Leibniz rule as in (2.18) is not consistent. We would have

$$\begin{aligned} 0 &= \hat{\partial}_x(\hat{x}\hat{y} - q\hat{y}\hat{x}) \\ &= (\hat{\partial}_x\hat{x})\hat{y} + \hat{x}(\hat{\partial}_x\hat{y}) - q(\hat{\partial}_x\hat{y})\hat{x} - \hat{y}(\hat{\partial}_x\hat{x}) \\ &= 1\hat{y} - q\hat{y}1 = (1-q)\hat{y}\,, \end{aligned}$$

which is only satisfied if $q = 1$, i.e. in the commutative limit. In order to obtain a consistent calculus for the case $q \neq 1$ we have to modify the definitions of the action and the Leibniz rule. We have to find consistent commutation relations for coordinates and derivatives.

To construct such a calculus for an arbitrary noncommutative space one usually proceeds as follows: Partial derivatives $\hat{\partial}_\mu$ are introduced as new elements by defining commutation relations with the noncommutative coordinates and among themselves. These commutation relations are required to satisfy the following two conditions:

1. They have to be consistent with the commutation relations of the noncommutative space.

2. They should reduce in the commutative limit to the undeformed commutation relations

$$\begin{aligned} [\hat{\partial}_\mu, \hat{x}^\nu] &\overset{\text{comm. limit}}{\longrightarrow} [\partial_\mu, x^\nu] = \delta_\mu^\nu \\ [\hat{\partial}_\mu, \hat{\partial}_\nu] &\longrightarrow [\partial_\mu, \partial_\nu] = 0 \end{aligned}$$

such that the derivatives constructed are indeed a deformation of the usual, undeformed ones.

To find commutation relations satisfying the above conditions, one usually starts with a general ansatz

$$\begin{aligned} [\hat{\partial}_\mu, \hat{x}^\nu] &= \delta^\nu_\mu + \sum_n A^{\mu\rho_1\cdots\rho_n}_\nu \hat{\partial}_{\rho_1} \cdots \hat{\partial}_{\rho_n} \\ [\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0 + \sum_m B^{\sigma_1\cdots\sigma_m}_{\mu\nu} \hat{\partial}_{\sigma_1} \cdots \hat{\partial}_{\sigma_m} \,, \end{aligned} \tag{2.19}$$

where the coefficients $A^{\mu\rho_1\cdots\rho_n}_\nu, B^{\sigma_1\cdots\sigma_m}_{\mu\nu}$ may also depend on the noncommutative coordinates but have to vanish in the commutative limit in order to meet the second condition. In general, the solution is not unique. By restricting the ansatz (2.19) looking, for example, for solutions, which contain at most terms linear in the derivatives, it is possible to find finitely many classes of solutions. Different solutions can often be mapped to each other [87, 90]

$$E : \hat{\partial}_\mu \to \hat{\partial}'_\mu = E(\hat{\partial}_\nu)^\rho_\mu \hat{\partial}_\rho \,,$$

where E^ρ_μ are functions, which reduce in the commutative limit to the Kronecker delta. So far, we have simply enlarged the algebra $\hat{\mathcal{A}}$ by adding in a consistent way new elements $\hat{\partial}_\mu$, which reduce to usual derivatives in the commutative limit. We still have to retrieve from the commutation relations (2.19) the action of derivatives on coordinates respectively functions and the way they act on products of functions in the noncommutative algebra $\hat{\mathcal{A}}$ (Leibniz rule). The action on a coordinate is given by the terms on the right-hand side of the commutator $[\hat{\partial}_\mu, \hat{x}^\nu]$, which do not contain derivatives. Comparing with (2.19) we read off

$$(\hat{\partial}_\mu \hat{x}^\nu) = \delta^\nu_\mu + A^\nu_\mu \,.$$

Here A^ν_μ vanishes in the commutative limit and implements a possible deformation of the action. The remaining coefficients in (2.19), $A^{\nu\rho}_\mu, A^{\nu\rho_1\rho_2}_\mu, \ldots$, determine the deformed Leibniz rule. It can be deduced by calculating the commutator first on the product of coordinates

$$[\hat{\partial}_\rho, \hat{x}^\mu \hat{x}^\nu]$$

and then on the product of ordered monomials. The results can be generalized to the product of two functions. What we obtain will be in general a deformed Leibniz rule

$$\hat{\partial}_\rho(\hat{f}\hat{g}) = (\hat{\partial}_\rho \hat{f})\hat{g} + \hat{f}(\hat{\partial}_\rho \hat{g}) + \sum_{n,m} C^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}_\rho (\hat{\partial}_{\alpha_1} \cdots \hat{\partial}_{\alpha_n} \hat{f})(\hat{\partial}_{\beta_1} \cdots \hat{\partial}_{\beta_m} \hat{g}) \,,$$

where again $C^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}_\rho$ are coefficients that vanish in the commutative limit.

2.2.1 θ-deformed spaces

In the θ-constant case consistent derivatives can be defined very easily by

$$\begin{aligned} &[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu \qquad (\hat{\partial}_\mu \hat{x}^\nu) = \delta_\mu^\nu \\ &[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0\,. \end{aligned} \tag{2.20}$$

These equations are undeformed. Let us show in a short calculation that these derivatives are indeed consistent with the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,.$$

We compute using (2.20)

$$\begin{aligned} \hat{\partial}_\rho([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}) &= \hat{\partial}_\rho(\hat{x}^\mu\hat{x}^\nu - \hat{x}^\nu\hat{x}^\mu - i\theta^{\mu\nu}) \\ &= (\delta_\rho^\mu + \hat{x}^\mu\hat{\partial}_\rho)\hat{x}^\nu - (\delta_\rho^\nu + \hat{x}^\nu\hat{\partial}_\rho)\hat{x}^\mu - i\theta^{\mu\nu}\hat{\partial}_\rho \\ &= \delta_\rho^\mu\hat{x}^\nu - \delta_\rho^\nu\hat{x}^\mu - i\theta^{\mu\nu}\hat{\partial}_\rho + \hat{x}^\mu(\delta_\rho^\nu + \hat{x}^\nu\hat{\partial}_\rho) - \hat{x}^\nu(\delta_\rho^\mu + \hat{x}^\mu\hat{\partial}_\rho) \\ &= ([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu})\hat{\partial}_\rho\,. \end{aligned}$$

An analogue calculation shows that

$$\hat{x}^\rho([\hat{\partial}_\mu, \hat{\partial}_\nu]) = ([\hat{\partial}_\mu, \hat{\partial}_\nu])\hat{x}^\rho$$

and consistency is shown. The above definitions (2.20) yield the usual Leibniz rule for the derivatives $\hat{\partial}_\mu$

$$(\hat{\partial}_\mu \hat{f}\hat{g}) = (\hat{\partial}_\mu \hat{f})\hat{g} + \hat{f}(\hat{\partial}_\mu \hat{g})\,. \tag{2.21}$$

This is a special feature of the fact that $\theta^{\mu\nu}$ are constants. In more complicated cases this undeformed Leibniz-rule cannot be preserved and has to be substituted by a deformed one [88], as we shall see in the following examples.

2.2.2 κ-deformed spaces

Differential calculi on κ-deformed space with commutation relations, which are at most linear in the derivatives, are constructed and classified in detail in [90–93]. One possible solution for the derivatives is given by

$$\begin{aligned} [\hat{\partial}_i, \hat{x}^j] &= \delta_i^j \\ [\hat{\partial}_i, \hat{x}^n] &= ia\hat{\partial}_i \\ [\hat{\partial}_n, \hat{x}^i] &= 0 \\ [\hat{\partial}_n, \hat{x}^n] &= 1 \end{aligned} \tag{2.22}$$

and

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0\,. \tag{2.23}$$

Let us show how to check consistency of these commutation relations with (2.7) by exhibiting one example: We calculate

$$\begin{aligned}
\hat{\partial}_j \hat{x}^n \hat{x}^i &= ia\hat{\partial}_j \hat{x}^i + \hat{x}^n \hat{\partial}_j \hat{x}^i \\
&= (ia + \hat{x}^n)\delta^i_j + (ia\hat{x}^i + \hat{x}^n \hat{x}^i)\hat{\partial}_j \\
\hat{\partial}_j \hat{x}^i \hat{x}^n &= (\delta^i_j + \hat{x}^i \hat{\partial}_j)\hat{x}^n \\
&= \delta^i_j \hat{x}^n + \hat{x}^i \hat{x}^n \hat{\partial}_j + ia\hat{x}^i \hat{\partial}_j \\
\hat{\partial}_j ia\hat{x}^i &= ia(\delta^i_j + \hat{x}^i \hat{\partial}_j)\,,
\end{aligned}$$

which yields

$$\hat{\partial}_j(\hat{x}^n \hat{x}^i - \hat{x}^i \hat{x}^n - ia\hat{x}^i) = (\hat{x}^n \hat{x}^i - \hat{x}^i \hat{x}^n - ia\hat{x}^i)\hat{\partial}_j\,.$$

From the commutation relations (2.22) we see that the action on single coordinates remains undeformed

$$(\hat{\partial}_\mu \hat{x}^\nu) = \delta^\nu_\mu$$

but the Leibniz rule, the way we have to act on a product of function, is deformed. We obtain

$$\begin{aligned}
\hat{\partial}_n(\hat{f}\hat{g}) &= (\hat{\partial}_n \hat{f})\hat{g} + \hat{f}(\hat{\partial}_n \hat{g}) \\
\hat{\partial}_i(\hat{f}\hat{g}) &= (\hat{\partial}_i \hat{f})\hat{g} + (e^{ia\hat{\partial}_n}\hat{f})(\hat{\partial}_i \hat{g})\,.
\end{aligned} \tag{2.24}$$

2.2.3 q-deformed Euclidean plane

Differential calculi on q-deformed spaces are studied in [88]. A detailed derivation of a differential calculus for the q-deformed Euclidean plane is given in [35, 86, 94]. There derivatives are defined by the following commutation relations

$$\begin{aligned}
\partial_{\hat{z}} \hat{z} &= 1 + q^{-2}\hat{z}\partial_{\hat{z}} \qquad & \partial_{\hat{z}} \hat{\bar{z}} &= q^{-2}\hat{\bar{z}}\partial_{\hat{z}} \\
\partial_{\hat{\bar{z}}} \hat{z} &= q^{2}\hat{z}\partial_{\hat{\bar{z}}} \qquad & \partial_{\hat{\bar{z}}} \hat{\bar{z}} &= 1 + q^{2}\hat{\bar{z}}\partial_{\hat{\bar{z}}}
\end{aligned}$$

and

$$\partial_{\hat{z}}\partial_{\hat{\bar{z}}} = q^2 \partial_{\hat{\bar{z}}}\partial_{\hat{z}}\,.$$

Also in this case the derivatives act via a deformed Leibniz rule

$$\begin{aligned}
\partial_{\hat{z}}(\hat{f}\hat{g}) &= (\partial_{\hat{z}}\hat{f})\hat{g} + f(q^{-2}\hat{z}, q^{-2}\hat{\bar{z}})(\partial_{\hat{z}}\hat{g}) \\
\partial_{\hat{\bar{z}}}(\hat{f}\hat{g}) &= (\partial_{\hat{\bar{z}}}\hat{f})\hat{g} + f(q^{2}\hat{z}, q^{2}\hat{\bar{z}})(\partial_{\hat{\bar{z}}}\hat{g})\,.
\end{aligned}$$

2.3 Deformed Symmetries

We have seen in the previous section that in general deformed spaces break symmetries. The question arises whether we can *deform* a symmetry in such a way that it acts consistently on the deformed space and such that it reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie algebras can be deformed in the category of Hopf algebras [69–73] (deformations of Hopf algebras coming from a Lie algebra are also called Quantum Groups)[2]. Throughout this thesis an important question will be how to construct such deformed symmetries. In a second step we propose models of field theories, which are covariant with respect to these generalized symmetries. In this section we introduce the fundamental concepts and mathematics and present useful examples.

2.3.1 Hopf algebras and Quantum Groups

This section is devoted to some important facts about Hopf algebras and Quantum Groups and their interplay with physics. The reader who is less interested in mathematical facts and their derivations may concentrate on the last paragraph of this section where the most important statements are summarized.

The Hopf algebra axioms

A Hopf algebra H over the field $\mathbb{C}$ is an algebra over $\mathbb{C}$ together with the algebra homomorphisms

$$\Delta \ : \quad H \to H \otimes H, \quad \varepsilon \ : \quad H \to \mathbb{C}, \tag{2.25}$$

i.e. Δ, ε are well-defined $\mathbb{C}$-linear maps satisfying for all $\xi, \zeta \in H$

$$\Delta(\xi\zeta) = \Delta(\xi)\Delta(\zeta) \ ; \quad \Delta(1) = 1 \otimes 1 \tag{2.26}$$

$$\varepsilon(\xi\zeta) = \varepsilon(\xi)\varepsilon(\zeta) \ ; \quad \varepsilon(1) = 1, \tag{2.27}$$

and with the $\mathbb{C}$-linear map

$$S : H \to H, \tag{2.28}$$

satisfying the following properties $\forall \xi, \zeta \in H$

$$(\Delta \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \Delta)\Delta(\xi) \tag{2.29}$$

[2]To be more precise the universal enveloping algebra of a semisimple Lie algebra can be deformed. Since semisimple Lie algebras form a discrete set, a continuous deformation is not possible within this set.

$$(\varepsilon \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \varepsilon)\Delta(\xi) = \xi \tag{2.30}$$

$$\mu(S \otimes \mathrm{id})\Delta(\xi) = \mu(\mathrm{id} \otimes S)\Delta(\xi) = \varepsilon(\xi)1 \tag{2.31}$$

where μ is the multiplication map $\mu(\xi \otimes \zeta) = \xi\zeta$. From these axioms we deduce [70]:

$$S(\xi\zeta) = S(\zeta)S(\xi)\ ;\ \ \Delta[S(\xi)] = \sigma(S\otimes S)\Delta(\xi)\ ;\ \ \varepsilon[S(\xi)] = \varepsilon(\xi)\ ;\ \ S(1) = 1 \tag{2.32}$$

where $\sigma(\xi\otimes\zeta) := \zeta\otimes\xi$ is the flip map. Written in Sweedler notation (see e.g. [72]), where we write $\xi_1 \otimes \xi_2$ as a symbolic notation for $\Delta(\xi)$, the above axioms and properties reed

$$(\xi\eta)_1 \otimes (\xi\eta)_2 = \xi_1\eta_1 \otimes \xi_2\eta_2 \tag{2.33}$$

$$\xi_{1_1} \otimes \xi_{1_2} \otimes \xi_2 = \xi_1 \otimes \xi_{2_1} \otimes \xi_{2_2} \equiv \xi_1 \otimes \xi_2 \otimes \xi_3 \tag{2.34}$$

$$\varepsilon(\xi_1)\xi_2 = \xi = \xi_1\varepsilon(\xi_2) \tag{2.35}$$

$$S(\xi_1)\xi_2 = \varepsilon(\xi)1 = \xi_1 S(\xi_2) \tag{2.36}$$

$$S(\xi)_1 \otimes S(\xi)_2 = S(\xi_2) \otimes S(\xi_1). \tag{2.37}$$

The following lemma is very useful for checking whether an algebra is a Hopf algebra.

Lemma 1. *Let H be an algebra. If $X \subset H$ is a set of algebra generators and if $\Delta : H \to H \otimes H$, $\varepsilon : H \to \mathbb{C}$ are algebra homomorphisms and $S : H \to H$ is an anti-algebra homomorphism, then H is a Hopf algebra if for all $x \in X$ holds:*

$$\Delta(x_1) \otimes x_2 = x_1 \otimes \Delta(x_2) \tag{2.38}$$

$$\varepsilon(x_1)x_2 = x_1\varepsilon(x_2) \tag{2.39}$$

$$x_1 S(x_2) = \varepsilon(x)1 = S(x_1)x_2\,, \tag{2.40}$$

where again $x_1 \otimes x_2$ is a short hand notation for $\Delta(x)$.

Proof. Note that

$$(\Delta \otimes \mathrm{id})\Delta : H \to H \otimes H \otimes H \tag{2.41}$$

and

$$(\mathrm{id} \otimes \Delta)\Delta : H \to H \otimes H \otimes H \tag{2.42}$$

are both algebra homomorphisms since id is an algebra homomorphism and Δ is an algebra homomorphism by assumption. Moreover, two algebra homomorphisms

$\phi_1 : A \to A'$ and $\phi_2 : A \to A'$ are equal if and only if they are equal on a set of algebra generators of the algebra A. Therefore

$$(\Delta \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \Delta)\Delta(\xi) \tag{2.43}$$

for all $\xi \in H$ since the equality holds by assumption on all generators $x \in X$. Analogue arguments show that

$$(\varepsilon \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \varepsilon)\Delta(\xi) = \xi \tag{2.44}$$

is satsified for all $\xi \in H$. It remains to show that S is an antipode. This can be seen as follows: We define

$$H' := \{h \in H \,|\, h_1 S(h_2) = \varepsilon(h)1\}, \quad H'' := \{h \in H \,|\, S(h_1)h_2 = \varepsilon(h)1\} \tag{2.45}$$

and first show that H' and H'' are subalgebras of H. Then the claim follows since we have $X \subset H'$ respectively $X \subset H''$ such that we conclude $H = H' = H''$ (X generates H), which means that S is an antipode. To see that H' is indeed a subalgebra of H we have to show that $yz \in H'$ if $y, z \in H'$:

$$y_1 z_1 S(y_2 z_2) = y_1 z_1 S(z_2) S(y_2) = y_1 \varepsilon(z) 1 S(y_2) = \varepsilon(y)\varepsilon(z)1 = \varepsilon(yz)\,, \tag{2.46}$$

where we used in the last step that ε is an algebra homomorphism. The same way one shows that H'' is a subalgebra of H. □

This lemma means in practise that in order to show whether H is a well-defined Hopf algebra it is enough to check the axioms (2.29-2.31) on the generators of the algebra H ones one has verified that Δ, ε are algebra homomorphisms and that S is an anti-algebra homomorphism. In the next subsection we will use this to show that the universal enveloping algebra of a Lie algebra is indeed a Hopf algebra.

Universal Enveloping Algebras of Lie Algebras

In our context, a special class of Hopf algebras is of particular interest: The universal enveloping algebra of any Lie algebra $\mathfrak{g}$ over the field $\mathbb{C}$ is a Hopf algebra equipped with a natural Hopf algebra structure. Deformations of such Hopf algebras give rise to what are often called deformed symmetries in physics, see next subsection. First let us recall how the universal enveloping algebra of a Lie algebra is defined. Given a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with Lie bracket

$$\begin{aligned} [\;] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (g, g') &\mapsto [g\, g'] \end{aligned} \tag{2.47}$$

and basis g_i, $i \in J$, where $[g_i\, g_j] = \sum_{l\in I} \alpha_{ij}^l g_l$ with structure constants $\alpha_{ij}^l \in \mathbb{C}$, its universal enveloping algebra $U\mathfrak{g}$ is given by the algebra $\mathbb{C}\langle g_i | i \in J\rangle$ freely generated by the basis elements g_i modulo the ideal I generated by the set of elements

$$\{g_i g_j - g_j g_i - \sum_{l\in I} \alpha_{ij}^l g_l \,|\, i,j \in J\}, \tag{2.48}$$

i.e.

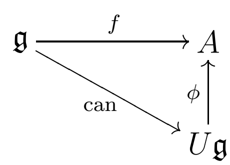
$$U\mathfrak{g} := \mathbb{C}\langle g_i | i \in J\rangle / I. \tag{2.49}$$

The universal enveloping algebra $U\mathfrak{g}$ realizes the Lie bracket of $\mathfrak{g}$ as a commutator: In $U\mathfrak{g}$ we have

$$[g\, g'] = gg' - g'g = [g, g'] \tag{2.50}$$

for all $g, g' \in \mathfrak{g}$. It is called "universal" since it has the following universal property [95]:

UNIVERSAL PROPERTY OF $U\mathfrak{g}$:
Let $\mathfrak{g}$ be a Lie algebra and $U\mathfrak{g}$ its universal enveloping algebra. Then for all algebras A and any Lie algebra homomorphism $f : \mathfrak{g} \to A$ (here any algebra A is a Lie algebra via the Lie structure given by the usual commutator) there exists a unique algebra homomorphism $\phi : U\mathfrak{g} \to A$ such that the following diagram commutes:



Here, can : $\mathfrak{g} \to U\mathfrak{g}$ denotes the canonical map $g \mapsto \bar{g}$. Usually we identify g with $\bar{g}$.

We claim that $U\mathfrak{g}$ possesses a natural Hopf algebra structure induced by

$$\begin{aligned} \Delta g &= g \otimes 1 + 1 \otimes g \\ \varepsilon(g) &= 0 \\ S(g) &= -g \end{aligned} \tag{2.51}$$

for all $g \in \mathfrak{g}$. The maps Δ, ε and S satisfy the following relations

$$\begin{aligned} \Delta(g)\Delta(h) - \Delta(h)\Delta(g) &= [g,h] \otimes 1 + 1 \otimes [g,h] = \Delta([g,h])\ , \\ \varepsilon(g)\varepsilon(h) - \varepsilon(h)\varepsilon(g) &= \varepsilon([g,h])\ , \\ S(h)S(g) - S(g)S(h) &= hg - gh = S([g,h]) \end{aligned} \tag{2.52}$$

for all $g, h \in \mathfrak{g}$. This allows us to extend Δ and ε as algebra homomorphisms and S as anti algebra homomorphism to the full enveloping algebra[3], $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$, $\varepsilon : U\mathfrak{g} \to \mathbb{C}$ and $S : U\mathfrak{g} \to U\mathfrak{g}$,

$$\begin{aligned} \Delta(uv) &:= \Delta(u)\Delta(v) \\ \varepsilon(uv) &:= \varepsilon(u)\varepsilon(v) \\ S(uv) &:= S(v)S(u)\,, \end{aligned} \tag{2.53}$$

for all $u, v \in U\mathfrak{g}$. To show that (2.51) induces indeed a Hopf algebra structure on $U\mathfrak{g}$ we can use Lemma 1 such that we have to show the remaining Hopf algebra axioms only on a set of generators, e.g. $\mathfrak{g}$. We have for all $g \in \mathfrak{g}$

$$\begin{aligned} (\Delta \otimes \mathrm{id})\Delta g &= (\Delta \otimes \mathrm{id})(g \otimes 1 + 1 \otimes g) \\ &= g \otimes 1 \otimes 1 + 1 \otimes g \otimes 1 + 1 \otimes 1 \otimes g \\ &= (\mathrm{id} \otimes \Delta)\Delta g, \end{aligned}$$

$$\begin{aligned} (\varepsilon \otimes \mathrm{id})\Delta g &= (\varepsilon \otimes \mathrm{id})(g \otimes 1 + 1 \otimes g) = g \\ &= (\mathrm{id} \otimes \varepsilon)\Delta g \end{aligned}$$

and

$$\begin{aligned} \mu(S \otimes \mathrm{id})\Delta g &= \mu(-g \otimes 1 + 1 \otimes g) = 0 \\ &= \varepsilon(g) = \mu(\mathrm{id} \otimes S)\Delta g\,, \end{aligned}$$

which proves that $U\mathfrak{g}$ is a Hopf algebra.

Module algebras

In this subsection we review the definition of module algebras. Module algebras are representations of Hopf algebras. In the framework of this thesis we will study several examples of module algebras, which are interesting for physical applications.

Let H be an algebra. Then a vector space A is a left H-module if the algebra H acts (from the left) on A, i.e. for all $a \in A$ and $\xi, \xi' \in H$ we have

$$\begin{aligned} (\xi\xi')(a) &= \xi(\xi'(a)) \\ 1(a) &= a\,. \end{aligned}$$

[3]This follows for example from the universal property of $U\mathfrak{g}$.

If H is a Hopf algebra and A is an algebra, then A is called a *H-module algebra* if the module structure is compatible with the algebra structure of A and the Hopf algebra structure of H. This means that for all $\xi \in H$ and $a, b \in A$

$$\xi(ab) = \mu \circ \Delta(\xi)(a \otimes b) = \xi_1(a)\xi_2(b) \tag{2.54}$$

and

$$\xi(1) = \varepsilon(\xi)1\,.$$

Equation (2.54) is very important: It tells us that the coproduct defines the way we act on a product of representations. This is usually known as the Leibniz rule. In Section 2.2 we have already seen examples for deformed Leibniz rules. Later we will see that such deformed Leibniz rules can be derived from non-trivial coproducts.

It is useful to give an example. Consider the Poincaré algebra as Lie algebra $\mathfrak{g}$. Let us denote the generators of Lorentz transformations by δ_ω, where $\omega \equiv \omega_{\mu\nu}$ are antisymmetric 4×4-matrices, and the generators of translations by ∂_μ. They enjoy the algebra relations

$$\begin{aligned} [\delta_\omega, \delta_{\omega'}] &= \delta_{[\omega,\omega']} \\ [\partial_\mu, \delta_\omega] &= -\omega_\mu{}^\nu \partial_\nu \\ [\partial_\mu, \partial_\nu] &= 0\,, \end{aligned} \tag{2.55}$$

where $[\omega, \omega']$ denotes the commutator of matrices. From above we know that the universal enveloping algebra corresponding to the Poincaré algebra is a Hopf algebra with canonical structure maps induced by

$$\begin{aligned} \Delta\delta_\omega = \delta_\omega \otimes 1 + 1 \otimes \delta_\omega, \quad \varepsilon(\delta_\omega) = 0, \quad S(\delta_\omega) = -\delta_\omega\,, \\ \Delta\partial_\mu = \partial_\mu \otimes 1 + 1 \otimes \partial_\mu, \quad \varepsilon(\partial_\mu) = 0, \quad S(\partial_\mu) = -\partial_\mu\,. \end{aligned} \tag{2.56}$$

An example for a module algebra with respect to this Hopf algebra is the algebra of functions $f(x)$. The action of the generators on functions is defined by

$$\begin{aligned} \delta_\omega f &= -x^\mu \omega_\mu{}^\nu (\partial_\nu f) \\ \partial_\mu f &= (\partial_\mu f)\,. \end{aligned}$$

The Leibniz rules of the differential operators $x^\mu \omega_\mu{}^\nu \partial_\nu$ respectively ∂_μ reflect the coproducts defined in (2.56)

$$\begin{aligned} \delta_\omega(fg) &= -x^\mu \omega_\mu{}^\nu (\partial_\nu fg) = -x^\mu \omega_\mu{}^\nu (\partial_\nu f)g - f x^\mu \omega_\mu{}^\nu (\partial_\nu g) = \mu \circ \Delta\delta_\omega(f \otimes g) \\ \partial_\mu(fg) &= (\partial_\mu f)g + f(\partial_\mu g) = \mu \circ \Delta\partial_\mu(f \otimes g)\,. \end{aligned}$$

In the next section we shall see how to deform Lie algebras in the category of Hopf algebras.

Let us summarize what we have learned so far

- Hopf algebras H are algebras with additional structures. For instance, they possess in addition to the algebra product also a coproduct map denoted by $\Delta : H \to H \otimes H$.

- Important examples for Hopf algebras are given by universal enveloping algebras of Lie algebras. Deformations of these Hopf algebras are also called Quantum Groups.

- In this setting the coproduct defines how the symmetry generators (e.g. generators of Poincaré transformations) act on a product of representations. It is the generalization of the Leibniz rule.

- This is why the mathematical structures of Hopf algebras are always present in physics. Always when we multiply representations of a Lie algebra, for instance dynamical fields, we act on this product via the usual Leibniz rule. This can be interpreted as coming from the coproduct corresponding to a Hopf algebra, the universal enveloping algebra of the considered Lie algebra.

- Noncommutative spaces are in general not covariant with respect to undeformed symmetries where the action of the symmetry generators and the way acting on products, the Leibniz rule, is undeformed.

- As we shall see in the next section it is possible to deform action and Leibniz rule in such a way that a consistent action can be defined. This is done by deforming the underlying Hopf algebra.

- Based on these generalized, deformed symmetries we shall construct field theories.

2.3.2 Deformations of Hopf algebras

In the previous section we have learned that the universal enveloping algebra of a Lie algebra acting on commutative functions is a Hopf algebra. Here we shall see how to deform this setting. Noncommutative spaces will arise as module algebras with respect to deformed universal enveloping algebras of Lie algebras.

Important examples of such deformations are q-deformations: Drinfel'd and Jimbo showed that there exists a q-deformation of the universal enveloping algebra of an arbitrary semisimple Lie algebra [10, 11]. Module algebras of these q-deformed universal enveloping algebras give rise to noncommutative spaces with commutation relations of type (2.4) [12–15]. There exists also a so-called κ-deformation of the Poincaré algebra [34, 79, 80, 91, 96], which leads to a noncommutative space of Lie- type (2.3). A Hopf algebra symmetry acting on the θ-deformed space was for a long time unknown. But recently also a θ-deformation of the Poincaré algebra acting on the space (2.2) was constructed [40,42,87,97,98]. Deformation by generic twists [42, 70, 75] leads to a large class of deformed Hopf algebras and their corresponding noncommutative spaces. We shall treat twist deformation separately in Chapter 4. In the following we list important examples of Quantum Groups, which underly the constructions of gauge theories and field theories in Chapter 5–6.

θ-deformed Poincaré algebra

The Hopf algebra defined in (2.55) and (2.56) can be deformed as follows: We denote the generators of the θ-deformed Poincaré algebra [42,87,97,98] by $\hat{\delta}_\omega, \hat{\partial}_\mu$. The algebra relations remain undeformed

$$\begin{aligned} [\hat{\delta}_\omega, \hat{\delta}_{\omega'}] &= \hat{\delta}_{[\omega,\omega']} \\ [\hat{\partial}_\mu, \hat{\delta}_\omega] &= -\omega_\mu{}^\nu \hat{\partial}_\nu \\ [\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0 \end{aligned} \tag{2.57}$$

but the coalgebra sector is deformed

$$\begin{aligned} \Delta\hat{\delta}_\omega &= \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega + \frac{i}{2}\theta^{\rho\sigma}(\omega_\rho{}^\nu \hat{\partial}_\nu \otimes \hat{\partial}_\sigma + \hat{\partial}_\rho \otimes \omega_\sigma{}^\nu \hat{\partial}_\nu) \\ \Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes 1 + 1 \otimes \hat{\partial}_\mu \,. \end{aligned} \tag{2.58}$$

Antipode and counit map remain undeformed, too

$$\begin{aligned} \varepsilon(\hat{\delta}_\omega) = 0 \quad &, \quad S(\hat{\delta}_\omega) = -\hat{\delta}_\omega \\ \varepsilon(\hat{\partial}_\mu) = 0 \quad &, \quad S(\hat{\partial}_\mu) = -\hat{\partial}_\mu \,. \end{aligned}$$

In analogy to what we have learned in Section 2.3.1 it is not difficult to show that the above definitions induce a well-defined Hopf algebra. This Hopf algebra acts on the θ-deformed space $\hat{\mathcal{A}}_\theta$ defined by (2.5), i.e. the θ-deformed space is a module algebra with respect to the θ-deformed Poincaré algebra. The operators $\hat{\partial}_\mu$ can be represented on the algebra of noncommutative functions $\hat{f} \in \hat{\mathcal{A}}_\theta$ by the undeformed derivatives introduced in (2.20)

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu \,. \tag{2.59}$$

This is obviously consistent with the undeformed coproduct and commutation relations for the derivatives in (2.57) and (2.58). The operators $\hat{\delta}_\omega$ are represented by the differential operator

$$-\hat{X}_\omega := -\hat{x}^\mu {\omega_\mu}^\nu \hat{\partial}_\nu + \frac{i}{2}\theta^{\rho\sigma}{\omega_\rho}^\nu \hat{\partial}_\nu \hat{\partial}_\sigma \,. \tag{2.60}$$

It is straightforward to check that the operators $\hat{X}_\omega$ indeed satisfy the Lie algebra (2.57), i.e.

$$[\hat{X}_\omega, \hat{X}_{\omega'}] = \hat{X}_{[\omega,\omega']} \,.$$

Also, they act on a product of functions via a deformed Leibniz rule, which reflects the deformed coproduct (2.58)

$$\hat{X}_\omega(\hat{f}\hat{g}) = (\hat{X}_\omega \hat{f})\hat{g} + \hat{f}(\hat{X}_\omega \hat{g}) + \frac{i}{2}\theta^{\rho\sigma}\{{\omega_\rho}^\nu(\hat{\partial}_\nu \hat{f})(\hat{\partial}_\sigma \hat{g}) + {\omega_\sigma}^\nu(\hat{\partial}_\rho \hat{f})(\hat{\partial}_\nu \hat{g})\} \,.$$

This can be seen as follows: First note that $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ implies

$$[\hat{x}^\mu, \hat{f}(\hat{x}^\rho)] = i\theta^{\mu\nu}(\hat{\partial}_\nu \hat{f}(\hat{x}^\rho)) \,.$$

Moreover, we conclude from (2.59) that $[\hat{\partial}_\mu, \hat{f}] = (\hat{\partial}_\mu \hat{f})$ (the derivatives are undeformed). Using this we calculate the commutator with the product of two functions

$$\begin{aligned}
[\hat{x}^\mu {\omega_\mu}^\nu \hat{\partial}_\nu, \hat{f}\hat{g}] &= [\hat{x}^\mu {\omega_\mu}^\nu \hat{\partial}_\nu, \hat{f}]\hat{g} + \hat{f}[\hat{x}^\mu {\omega_\mu}^\nu \hat{\partial}_\nu, \hat{g}] \\
&= \hat{x}^\mu {\omega_\mu}^\nu [\hat{\partial}_\nu, \hat{f}]\hat{g} + [\hat{x}^\mu {\omega_\mu}^\nu, \hat{f}]\hat{\partial}_\nu \hat{g} \\
&\quad + \hat{f}\hat{x}^\mu {\omega_\mu}^\nu [\hat{\partial}_\nu, \hat{g}] + \hat{f}[\hat{x}^\mu {\omega_\mu}^\nu, \hat{g}]\hat{\partial}_\nu \\
&= \hat{x}^\mu {\omega_\mu}^\nu (\hat{\partial}_\nu \hat{f})\hat{g} + \hat{f}\hat{x}^\mu {\omega_\mu}^\nu (\hat{\partial}_\nu \hat{g}) \\
&\quad + i\theta^{\mu\rho}{\omega_\mu}^\nu (\hat{\partial}_\rho \hat{f})(\hat{\partial}_\nu \hat{g}) \\
&\quad + i\theta^{\mu\rho}{\omega_\mu}^\nu \{(\hat{\partial}_\rho \hat{f})\hat{g}\hat{\partial}_\nu + \hat{f}(\hat{\partial}_\rho \hat{g})\hat{\partial}_\nu\}
\end{aligned}$$

and

$$\begin{aligned}
[-\frac{i}{2}\theta^{\mu\rho}{\omega_\mu}^\nu \hat{\partial}_\nu \hat{\partial}_\rho, \hat{f}\hat{g}] &= -\frac{i}{2}\theta^{\mu\rho}{\omega_\mu}^\nu \{(\hat{\partial}_\nu \hat{\partial}_\rho \hat{f})\hat{g} + \hat{f}(\hat{\partial}_\nu \hat{\partial}_\rho \hat{g}) \\
&\quad + (\hat{\partial}_\nu \hat{f})(\hat{\partial}_\rho \hat{g}) + (\hat{\partial}_\rho \hat{f})(\hat{\partial}_\nu \hat{g}) + (\hat{\partial}_\nu \hat{f}\hat{g})\hat{\partial}_\rho + (\hat{\partial}_\rho \hat{f}\hat{g})\hat{\partial}_\nu\}
\end{aligned}$$

such that we obtain altogether the deformed Leibniz rule

$$(\hat{X}_\omega \hat{f}\hat{g}) = (\hat{X}_\omega \hat{f})\hat{g} + \hat{f}(\hat{X}_\omega \hat{g}) + \frac{i}{2}\theta^{\mu\rho}\{\omega_\mu{}^\nu(\hat{\partial}_\rho \hat{f})(\hat{\partial}_\nu \hat{g}) + \omega_\rho{}^\nu(\hat{\partial}_\nu \hat{f})(\hat{\partial}_\mu \hat{g})\}\,. \qquad (2.61)$$

Comparing with (2.58) this yields that the θ-deformed Poincaré algebra as defined in (2.57) and (2.58) is represented on functions $\hat{f} \in \hat{\mathcal{A}}$ by the differential operators $\hat{X}_\omega$ and $\hat{\partial}_\mu$. The noncommutative space (2.2) is a module algebra with respect to θ-deformed Poincaré transformations defined in (2.57) and (2.58).

κ-deformed Poincaré algebra

The κ-deformed Poincaré algebra was first introduced in [79, 80]. It acts consistently on κ-deformed space-time, which we introduced in Section 2.1.2 (meaning that κ-deformed space-time is a module algebra with respect to the κ-deformed Poincaré algebra). Field theories on κ-deformed spaces have been studied extensively [34, 91–93, 96, 99–103]. The generators of Lorentz transformations are denoted by $\hat{M}^{\mu\nu}$; the generators of translations are denoted by $\hat{\partial}_\mu$. Again we use the convention that Latin letters $i, j, \ldots$ run from 0 to $n-1$, Greek letters $\mu, \nu, \ldots$ run from 0 to n and $\eta^{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$ denotes the usual Minkowski metric. The defining algebra relations are

$$\begin{aligned}
[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] &= \eta^{\mu\sigma}\hat{M}^{\nu\rho} + \eta^{\nu\rho}\hat{M}^{\mu\sigma} - \eta^{\mu\rho}\hat{M}^{\nu\sigma} - \eta^{\nu\sigma}\hat{M}^{\mu\rho} \\
[\hat{M}^{ij}, \hat{\partial}_\mu] &= \eta^j_\mu \hat{\partial}^i - \eta^i_\mu \hat{\partial}^j \\
[\hat{M}^{in}, \hat{\partial}_j] &= \eta^i_j \frac{e^{2ia\hat{\partial}_n} - 1}{2ia} - \frac{ia}{2}\eta^i_j \hat{\partial}^l \hat{\partial}_l + ia\hat{\partial}^i \hat{\partial}_j \\
[\hat{M}^{in}, \hat{\partial}_n] &= \hat{\partial}^i \\
[\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0\,.
\end{aligned} \qquad (2.62)$$

We see that the Lorentz algebra part remains undeformed. However, the cosector is deformed also for the Lorentz generators $\hat{M}^{in}$

$$\begin{aligned}
\Delta \hat{M}^{ij} &= \hat{M}^{ij} \otimes 1 + 1 \otimes \hat{M}^{ij} \\
\Delta \hat{M}^{in} &= \hat{M}^{in} \otimes 1 + e^{ia\hat{\partial}_n} \otimes \hat{M}^{in} + ia\hat{\partial}_k \otimes \hat{M}^{ik} \\
\Delta \hat{\partial}_i &= \hat{\partial}_i \otimes 1 + e^{ia\hat{\partial}_n} \otimes \hat{\partial}_i \\
\Delta \hat{\partial}_n &= \hat{\partial}_n \otimes 1 + 1 \otimes \hat{\partial}_n\,.
\end{aligned} \qquad (2.63)$$

Counit and antipode are given by

$$\varepsilon(\hat{M}^{\mu\nu}) = \varepsilon(\hat{\partial}_\mu) = 0$$

$$
\begin{aligned}
S(\hat{M}^{ij}) &= -\hat{M}^{ij} \\
S(\hat{M}^{in}) &= -\hat{M}^{in} e^{-ia\hat{\partial}_n} + ia\hat{M}^{ik}\hat{\partial}_k e^{-ia\hat{\partial}_n} + ia(n-1)\hat{\partial}^i e^{-ia\hat{\partial}_n} \\
S(\hat{\partial}_n) &= -\hat{\partial}_n \\
S(\hat{\partial}_j) &= -\hat{\partial}_j e^{-ia\hat{\partial}_n} .
\end{aligned}
$$

In order to verify that this defines a Hopf algebra the axioms (2.29)–(2.31) have to be checked on the generators as explained in Section 2.3.1.

The generators $\hat{\partial}_\mu$ can be represented by the derivatives introduced in (2.22) and (2.23). The generators $\hat{M}^{\mu\nu}$ can be represented as differential operators with respect to these derivatives. We find [34, 91]

$$
\begin{aligned}
\hat{M}^{ij} &= \hat{x}^i \hat{\partial}^j - \hat{x}^j \hat{\partial}^i \\
\hat{M}^{in} &= \hat{x}^i \frac{1 - e^{2ia\hat{\partial}_n}}{2ia} - \hat{x}^n \hat{\partial}^i + \frac{ia}{2} \hat{x}^i \hat{\partial}^l \hat{\partial}_l .
\end{aligned} \tag{2.64}
$$

They have a well-defined action on the κ-deformed space (2.7). Using the commutation relations for coordinates and derivatives, (2.7), (2.22) and (2.23), it is possible to check that the differential operators defined in (2.64) satisfy the algebra relations (2.62). Moreover, we can derive their action on functions by calculating the commutator with ordered monomials and obtain

$$
\begin{aligned}
(\hat{M}^{ij} \hat{f} \hat{g}) &= (\hat{M}^{ij} \hat{f}) \hat{g} + \hat{f} (\hat{M}^{ij} \hat{g}) \qquad (2.65) \\
(\hat{M}^{in} \hat{f} \hat{g}) &= (\hat{M}^{in} \hat{f}) \hat{g} + (e^{ia\hat{\partial}_n} \hat{f})(\hat{M}^{in} \hat{g}) + ia(\hat{\partial}_k \hat{f})(\hat{M}^{ik} \hat{g}) .
\end{aligned}
$$

We see that these deformed Leibniz rules reproduce the coproducts (2.63) such that the differential operators given in (2.64) represent indeed the generators $\hat{M}^{\mu\nu}$ on functions on κ-deformed spaces; κ-deformed space-time is a module algebra with respect to the κ-Poincaré algebra.

q-deformed Euclidean algebra

The only q-deformed Quantum Group that we shall present is the q-deformed Euclidean algebra in two dimensions. It is a rather simple example and very illustrative. It was first constructed in [84] as the dual Hopf algebra to the quantum $E(2)$ group introduced by Woronowicz [104].

The Quantum Group $U_q(e(2))$ is a deformation of the universal enveloping algebra of the Euclidean Lie algebra in two dimensions. It is generated by $T, \overline{T}, J$ with the following commutation relations and structure maps

$$
T\overline{T} = q^2 \overline{T} T \qquad [J, T] = iT \qquad [J, \overline{T}] = -i\overline{T}
$$

$$\Delta(T) = T \otimes q^{2iJ} + 1 \otimes T \qquad \Delta(\overline{T}) = \overline{T} \otimes q^{2iJ} + 1 \otimes \overline{T} \tag{2.66}$$

$$\Delta(J) = J \otimes 1 + 1 \otimes J \qquad \varepsilon(T) = \varepsilon(\overline{T}) = \varepsilon(J) = 0$$

$$S(T) = -Tq^{-2iJ} \qquad S(\overline{T}) = -\overline{T}q^{-2iJ} \qquad S(J) = -J \ ,$$

where $q \in \mathbb{R}$. The commutation relations reduce to the Lie algebra relations for the Euclidean algebra in two dimensions in the limit $q \to 1$. This quantum group acts from the right on functions on the two-dimensional q-deformed space, which we introduced in Section 2.2.3 (the action is denoted by the symbol $\triangleleft$)

$$\begin{array}{ll} \hat{z} \triangleleft T = 1 & \hat{z} \triangleleft \overline{T} = 0 \\ \hat{z} \triangleleft J = iz & \hat{\overline{z}} \triangleleft J = -i\overline{z} \\ \hat{\overline{z}} \triangleleft T = 0 & \hat{\overline{z}} \triangleleft \overline{T} = -q^2 \, . \end{array}$$

For further details see [35, 86].

Chapter 3

Star-Products and Star-Representations

The constructions given in the previous chapters involve abstract operators. It is, however, our aim to develop theories, which lead to physical predictions. It is therefore necessary to study representations of the introduced operators. A convenient way to do this is provided by star-products ($\star$-products) [16, 31, 105–107]. As we shall see, it is possible to equip the commutative algebra of functions with a new, noncommutative product, which is a deformation of the usual point-wise product. This product is called $\star$-product and implements the noncommutativity on the algebra of functions depending on commutative coordinates. For a nice introduction to deformation quantization and $\star$-products see [108, 109] and references therein. In a second step it is also possible to represent the operators acting on the noncommutative space (e.g. the generators of the corresponding Quantum Group) by means of differential operators acting on the ordinary functions on a manifold.

The construction of $\star$-products starting from a noncommutative space-time algebra $\hat{\mathcal{A}}$ is always possible if the algebra $\hat{\mathcal{A}}$ has the Poincaré-Birkhoff-Witt (PBW) property. The PBW-property states that the space of polynomials in noncommutative coordinates of a given degree is isomorphic to the space of polynomials in commutative coordinates of the same degree. Such an isomorphism between polynomials of a fixed degree is given by an ordering prescription. An example is the *symmetric ordering* W (symmetric ordering is also called Weyl ordering since it corresponds to Weyl's quantization prescription [110]), which assigns to every monomial the totally symmetric ordered one

$$W : \mathcal{A} \;\rightarrow\; \hat{\mathcal{A}}$$

$$\begin{array}{rcl} x^\mu & \mapsto & \hat{x}^\mu \\ x^\mu x^\nu & \mapsto & \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu) \\ & \cdots & . \end{array} \tag{3.1}$$

Later we shall also study other ordering prescriptions. By means of this vector space isomorphism it is possible to "pull back" the noncommutative product of $\hat{\mathcal{A}}$ to $\mathcal{A}$. We define a new product by

$$\begin{array}{rcl} \star : \mathcal{A} \times \mathcal{A} & \to & \mathcal{A} \\ (f, g) & \mapsto & f \star g := W^{-1}(W(f)W(g))\,. \end{array}$$

Associativity of the $\star$-product follows then from the associativity of the algebra product in $\hat{\mathcal{A}}$:[1]

$$\begin{array}{rcl} f \star (g \star h) & = & f \star (W^{-1}(W(g)W(h))) \\ & = & W^{-1}(W(f)(W(g)W(h))) \\ & = & W^{-1}((W(f)W(g))W(h)) \\ & = & (f \star g) \star h\,. \end{array}$$

In the following we give explicit examples of $\star$-products corresponding to the noncommutative spaces introduced in Section 2.1.

3.1 θ-Deformed Spaces

Star-products for θ-deformed spaces

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$$

are well-known and their construction can be found in many places [17,86,108,111]. The $\star$-product corresponding to symmetric ordering is also know as the *Moyal–Weyl product* [112]

$$f \star g = \mu \circ e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta} f \otimes g\,, \tag{3.2}$$

where $\mu(f \otimes g) := fg$ is the multiplication map. Here, we want to do without a derivation of the Moyal–Weyl product. It can be found, for instance, in the

[1] In general, $\star$-products can be constructed for any Poisson-manifold [31]. In this framework it is usually quite difficult to show associativity of the $\star$-product. If we introduce a $\star$-product by an ordering prescription, associativity follows immediately.

above references. In the next chapter we shall exhibit in detail the construction of star-products for κ-deformed spaces instead.

The Moyal-Weyl product has the convenient property that usual complex conjugation is an involution (we also say that the Moyal-Weyl product is a *hermitian* $\star$-product)

$$\overline{f \star g} = \bar{g} \star \bar{f}\,,$$

which follows immediately from the definition.

3.2 κ-Deformed Spaces

In this section we shall construct explicitly the normal ordered and Weyl-ordered $\star$-product for κ-deformed spaces, see also [90,92,93]. The construction is presented in all detail and is quite elucidating.

3.2.1 The normal ordered star-product I

Let us consider the two-dimensional algebra generated by the coordinates $\hat{x}, \hat{y}$ satisfying the commutation relation

$$[\hat{y}, \hat{x}] = ia\hat{x}\,. \tag{3.3}$$

We want to derive the normal-ordered star-product. Normal ordering is defined by all $\hat{y}$'s standing on the left.

We obtain from (3.3) for all $k \geq 1$

$$\hat{x}^k\hat{y} = \hat{y}\hat{x}^k + [\hat{x}^k, \hat{y}] = \hat{y}\hat{x}^k + [\hat{x}^{k-1}, \hat{y}]\hat{x} + \hat{x}^{k-1}[\hat{x}, \hat{y}] = \hat{y}\hat{x}^k - iak\hat{x}^k\,. \tag{3.4}$$

From this we immediately deduce:

$$\begin{aligned}
\hat{x}^k\hat{y}^2 &= \hat{y}^2\hat{x}^k + [\hat{x}^k, \hat{y}]\hat{y} + \hat{y}[\hat{x}^k, \hat{y}] \\
&= \hat{y}^2\hat{x}^k - 2kia\hat{y}\hat{x}^k + k^2(ia)^2\hat{x}^k && (3.5)\\
\hat{x}^k\hat{y}^3 &= \hat{y}^3\hat{x}^k - 3iak\hat{y}^2\hat{x}^k + 3(ia)^2k^2\hat{y}\hat{x}^k - k^3(ia)^3\hat{x}^k\,. && (3.6)
\end{aligned}$$

This suggests the following claim:

Claim 1.

$$\hat{x}^k\hat{y}^l = \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-iak)^{l-m}\hat{y}^m\hat{x}^k\,. \tag{3.7}$$

Proof. We will prove this claim by induction over l: Equations (3.4), (3.5) and (3.6) show that the claim is true for $l = 1, 2, 3$. Let us now suppose that equation (3.7) is true up to an arbitrary $l \geq 1$. We have

$$\begin{aligned}
\hat{x}^k \hat{y}^{l+1} &= (\hat{x}^k \hat{y}^l)\hat{y} \\
&= \Big(\sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-iak)^{l-m} \hat{y}^m \hat{x}^k\Big)\hat{y} \\
&\overset{(3.4)}{=} \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-iak)^{l-m} \hat{y}^{m+1} \hat{x}^k + \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-iak)^{l-m+1} \hat{y}^m \hat{x}^k \\
&= \sum_{m=1}^{l+1} \begin{pmatrix} l \\ m-1 \end{pmatrix} (-iak)^{l-m+1} \hat{y}^m \hat{x}^k + \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-iak)^{l-m+1} \hat{y}^m \hat{x}^k \\
&= \sum_{m=1}^{l} \Big(\begin{pmatrix} l \\ m-1 \end{pmatrix} + \begin{pmatrix} l \\ m \end{pmatrix}\Big)(-iak)^{l+1-m} \hat{y}^m \hat{x}^k \\
&\quad + \begin{pmatrix} l \\ l \end{pmatrix} \hat{y}^{l+1} \hat{x}^k + \begin{pmatrix} l \\ 0 \end{pmatrix} (-iak)^{l+1} \hat{x}^k \\
&= \sum_{m=0}^{l+1} \begin{pmatrix} l+1 \\ m \end{pmatrix} (-iak)^{l-m+1} \hat{y}^m \hat{x}^k \,,
\end{aligned}$$

where the last line follows with $\begin{pmatrix} j \\ 0 \end{pmatrix} = 1$ for all j $(0! := 1)$ and with

$$\begin{pmatrix} l \\ m-1 \end{pmatrix} + \begin{pmatrix} l \\ m \end{pmatrix} = \begin{pmatrix} l+1 \\ m \end{pmatrix} .$$

Thus, equation (3.7) is proven by induction.

□

Using the above formula we can now deduce the explicit expression for the normal ordered $\star$-product of arbitrary monomials. Let us write for the normal ordered $\star$-product $\star_n$ in order to distinguish it from the Weyl ordered one. By definition, the $\star$-product is the algebra product pulled back to the space of commutative functions

$$f \star_n g := \rho_n^{-1}(\rho_n(f)\rho_n(g)) \,,$$

where ρ_n denotes normal ordering. It is defined on monomials as follows

$$\rho_n : x^i y^j \longmapsto \hat{y}^j \hat{x}^i \,.$$

This yields

$$\begin{aligned}
(x^s y^t) \star_n (x^k y^l) &:= \rho_n^{-1}(\hat{y}^t \hat{x}^s \hat{y}^l \hat{x}^k) \\
&\overset{(3.7)}{=} \rho_n^{-1}(\hat{y}^t \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-ias)^{l-m} \hat{y}^m \hat{x}^s \hat{x}^k) \\
&= \rho_n^{-1}(\sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-ias)^{l-m} \hat{y}^{t+m} \hat{x}^{s+k}) \\
&= \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-ias)^{l-m} x^{s+k} y^{t+m} \, . \qquad (3.8)
\end{aligned}$$

The expression for the normal ordered $\star$-product given as a bidifferential operator has to reproduce (3.8) when applied to monomials $x^s y^t$ and $x^k y^l$. On the other hand, such a bidifferential operator is uniquely determined by its action on arbitrary monomials in each argument. We easily check that

$$f \star_n g(x,y) := \sum_{m=0}^{\infty} \frac{(-ia)^m}{m!} (x\partial_x)^m (f) (\partial_y)^m (g) = \mu \circ e^{-iax\partial_x \otimes \partial_y} f \otimes g \qquad (3.9)$$

reproduces (3.8) if we substitute $f = x^s y^t$ and $x^k y^l$:

$$\begin{aligned}
& \sum_{m=0}^{l} \frac{(-ia)^m}{m!} (x\partial_x)^m (x^s y^t) (\partial_y)^m (x^k y^l) \\
= & \sum_{m=0}^{l} \frac{(-ia)^m}{m!} s^m (x^s y^t) l(l-1)\cdots(l-m+1)(x^k y^{l-m}) \\
= & \sum_{m=0}^{l} (-ias)^m \begin{pmatrix} l \\ m \end{pmatrix} x^{s+k} y^{t+l-m} \\
= & \sum_{n=0}^{l} (-ias)^{l-n} \begin{pmatrix} l \\ l-n \end{pmatrix} x^{s+k} y^{t+n} = \sum_{n=0}^{l} (-ias)^{l-n} \begin{pmatrix} l \\ n \end{pmatrix} x^{s+k} y^{t+n} \, .
\end{aligned}$$

It therefore provides an expression for the normal ordered $\star$-product, which is valid for arbitrary functions.

This expression can be generalized without problems to the case of n commuting and one non-commuting coordinates

$$[\hat{x}^n, \hat{x}^i] = ia\hat{x}^i \, ,$$

where $i = 0, \ldots, n-1$. Then the normal ordered $\star$-product reads

$$f \star_n g(x) = \sum_{m=0}^{\infty} \frac{(-ia)^m}{m!} (x^i \partial_{x^i})^m (f) (\partial_n)^m (g) = \mu \circ e^{-iax^i \partial_i \otimes \partial_n} f \otimes g \,. \tag{3.10}$$

3.2.2 The normal ordered star-product II

Another way to map a commutative function to the noncommutative algebra of functions is to consider its Fourier transformation

$$f(x) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int d^{n+1}p \, e^{ip_\mu x^\mu} \tilde{f}(p)$$

and to order with respect to $e^{ip_\mu x^\mu}$. We defined normal ordering as the ordering prescription where all $\hat{x}^n$ stand on the left. Thus, the noncommutative function corresponding to $f(x)$ is given by

$$\hat{f}(\hat{x}) := \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int d^{n+1}p \, e^{ip_n \hat{x}^n} e^{ip_j \hat{x}^j} \tilde{f}(p)$$

in this ordering. Let us calculate the product of two arbitrary functions in order to deduce the normal ordered $\star$-product:

$$\begin{aligned} \hat{f}(\hat{x})\hat{g}(\hat{x}) &= \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p \, d^{n+1}q \, e^{ip_n \hat{x}^n} e^{ip_j \hat{x}^j} e^{iq_n \hat{x}^n} e^{iq_k \hat{x}^k} \tilde{f}(p) \tilde{g}(q) \qquad (3.11) \\ &= \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p \, d^{n+1}q \, e^{ip_n \hat{x}^n} e^{iq_n \hat{x}^n} e^{-iq_n \hat{x}^n} e^{ip_j \hat{x}^j} e^{iq_n \hat{x}^n} e^{iq_k \hat{x}^k} \tilde{f}(p) \tilde{g}(q) \,. \end{aligned}$$

The commutation relations of the κ-deformed space

$$\begin{aligned} [\hat{x}^n, \hat{x}^j] &= ia\hat{x}^j \\ [\hat{x}^i, \hat{x}^j] &= 0 \end{aligned}$$

yield the following result for the adjoint action of $e^{-iq_n \hat{x}^n}$ on $\hat{x}^j$:

$$e^{-iq_n \hat{x}^n} \hat{x}^j e^{iq_n \hat{x}^n} = e^{-iq_n [\hat{x}^n, \cdot]} \hat{x}^j = e^{aq_n} \hat{x}^j \,.$$

From this we obtain

$$e^{-iq_n \hat{x}^n} e^{ip_j \hat{x}^j} e^{iq_n \hat{x}^n} = e^{ip_j e^{aq_n} \hat{x}^j} \,.$$

If we insert this result in expression (3.11) we find

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p \, d^{n+1}q \, e^{i(p_n+q_n)\hat{x}^n} e^{i(p_j+q_j)\hat{x}^j} e^{ip_k (e^{aq_n}-1)\hat{x}^k} \tilde{f}(p) \tilde{g}(q) \,.$$

Now we can read off the expression for the normal ordered $\star$-product explicitly because all $\hat{x}^n$ stand on the left. This yields

$$\begin{aligned} f \star_n g(x) &= \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p\, d^{n+1}q\, e^{i(p_\mu + q_\mu)x^\mu} e^{ip_k(e^{aq_n}-1)x^k} \tilde{f}(p)\tilde{g}(q) \\ &= \exp(x^k \partial_{y^k}(e^{-ia\partial_{z^n}} - 1)) f(y)g(z)|_{y,z\to x} \,. \end{aligned} \tag{3.12}$$

We obtained two expressions for the normal ordered star-product, equation (3.9) and (3.12). Both expressions correspond to normal ordering and are therefore equal although they may look quite different at first glance. This is clear by construction. However, we shall show explicitly that (3.9) and (3.12) define equal $\star$-products. This may make things more transparent for the reader and also provides a good check for our calculations.

Since all coordinates $\hat{x}^i$ commute, we can consider without loss of generality the two-dimensional case with coordinates x, y, where $[y \stackrel{\star}{,} x] = iax$ for both $\star$-products. In order to distinguish both $\star$-products, let us denote the $\star$-product defined in (3.9) by $\star_n$ and the $\star$-product defined in (3.12) by $\star_n'$. We apply both star products to two arbitrary monomials and compare the results in both cases.

1. For $\star_n$:

$$\begin{aligned} (x^s y^t) \star_n (x^k y^l) &= \sum_{n=0}^{l} \frac{(-ia)^n}{n!} (x\partial_x)^n (x^s y^t)(\partial_y)^n (x^k y^l) \\ &= \sum_{n=0}^{l} \frac{(-ias)^n}{n!} l(l-1)\dots(l-n+1) x^s y^t x^k y^{l-n} \\ &= \sum_{n=0}^{l} (-ias)^n \begin{pmatrix} l \\ n \end{pmatrix} x^s y^t x^k y^{l-n} \,. \end{aligned} \tag{3.13}$$

2. For $\star_n'$:

$$\begin{aligned} (x^s y^t) \star_n' (x^k y^l) &= \exp(x\partial_z(e^{-ia\partial_u} - 1))(z^s y^t)(x^k u^l)|_{(z,u)\to(x,y)} \\ &= y^t(\exp(x\partial_z(e^{-ia\partial_u} - 1))(z^s)(u^l)|_{(z,u)\to(x,y)})x^k \\ &= y^t(x^s \star_n' y^l)x^k \,. \end{aligned} \tag{3.14}$$

From the definition for $\star_n'$ (3.12) it follows immediately that

$$x^s = \underbrace{x \star_n' \dots \star_n' x}_{s} \,,$$

such that we obtain ($\star_n'$ is associative)

$$x^s \star_n' y^l = \underbrace{x \star_n' \ldots \star_n' x}_{s} \star_n' y^l \,.$$

Moreover, the explicit expression for $\star_n'$ yields that for an arbitrary function $h(x,y)$ the following holds

$$x \star_n' h(x,y) = x e^{-ia\partial_y} h(x,y) \,,$$

which implies

$$\underbrace{x \star_n' \ldots \star_n' x}_{s} \star_n' h(x,y) = x^s (e^{-ia\partial_y})^s h(x,y) \,.$$

Thus, we conclude that

$$\begin{aligned} x^s \star_n' y^l &= x^s (e^{-ias\partial_y})(y^l) \\ &= x^s \sum_{n=0}^{l} \frac{1}{n!}(-ias)^n (\partial_y)^n (y^l) \\ &= x^s \sum_{n=0}^{l} \begin{pmatrix} l \\ n \end{pmatrix} (-ias)^n y^{l-n} \,. \end{aligned} \tag{3.15}$$

Finally, we obtain from (3.14), (3.15) and (3.13)

$$\begin{aligned} (x^s y^t) \star_n' (x^k y^l) &= \sum_{n=0}^{l} (-ias)^n \begin{pmatrix} l \\ n \end{pmatrix} x^s y^t x^k y^{l-n} \\ &= (x^s y^t) \star_n (x^k y^l) \,. \end{aligned}$$

Hence, both $\star$-products are equal applied to arbitrary monomials, and thus $\star_n = \star_n'$.

3.2.3 The normal ordered $\star$-product for the generic κ-deformed space

Let us now consider the algebra generated by coordinates $\hat{x}^\mu$ satisfying the defining commutation relations of the generic κ-deformed space

$$[\hat{x}^\mu, \hat{x}^\nu] = ia^\mu \hat{x}^\nu - ia^\nu \hat{x}^\mu \,. \tag{3.16}$$

If we make the following redefinitions

$$\begin{aligned}\hat{U} &:= \frac{1}{a} a^\alpha \eta_{\alpha\mu} \hat{x}^\mu \\ \hat{X}^\rho &:= P^\rho{}_\nu \hat{x}^\nu ,\end{aligned}$$

where $a^2 := a^\alpha \eta_{\alpha\mu} a^\mu$, $a := \sqrt{a^2}$ and

$$P^\rho{}_\nu := \delta^\rho{}_\nu - \frac{a^\rho a^\mu}{a^2} \eta_{\mu\nu} , \tag{3.17}$$

then the commutation relations (3.16) become

$$\begin{aligned}[\hat{U}, \hat{X}^\rho] &= ia\hat{X}^\rho \\ [\hat{X}^\rho, \hat{X}^\sigma] &= 0 .\end{aligned}$$

It is easy to check that P is indeed a projector, i. e. $P^2 = P$ and $P^\rho{}_\nu a^\nu = 0$. Moreover, we recover the coordinates $\hat{x}^\rho$ by

$$\hat{x}^\rho = \hat{X}^\rho + \frac{a^\rho}{a} \hat{U}. \tag{3.18}$$

Thus, the coordinates $\hat{U}, \hat{X}^\rho$ generate the same algebra (3.16). We can define normal ordering in the generic κ-deformed setting by ordering all $\hat{U}$ to the left. In order to deduce the normal ordered $\star$-product we proceed in analogy to Section 3.2.2: We Fourier-transform a function depending on x

$$f(x) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int d^{n+1}p \, e^{ip_\mu x^\mu} \tilde{f}(p) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int d^{n+1}p \, e^{ip_\mu (X^\mu + \frac{a^\rho}{a} U)} \tilde{f}(p) .$$

The noncommutative analogue of $f(x)$, i.e. its image under the normal ordering prescription, is given by:

$$\hat{f}(\hat{x}) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int d^{n+1}p \, e^{ip_\rho \frac{a^\rho}{a} \hat{U}} e^{ip_\rho \hat{X}^\rho} \tilde{f}(p) .$$

Multiplication of two functions leads to

$$\begin{aligned}\hat{f}(\hat{x})\hat{g}(\hat{x}) &= \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p \, d^{n+1}q \, e^{ip_\rho \frac{a^\rho}{a} \hat{U}} e^{ip_\rho \hat{X}^\rho} e^{iq_\sigma \frac{a^\sigma}{a} \hat{U}} e^{iq_\sigma \hat{X}^\sigma} \tilde{f}(p)\tilde{g}(q) \\ &= \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p \, d^{n+1}q \, e^{ip_\rho \frac{a^\rho}{a} \hat{U}} e^{iq_\sigma \frac{a^\sigma}{a} \hat{U}} \\ &\qquad \times e^{-iq_\sigma \frac{a^\sigma}{a} \hat{U}} e^{ip_\rho \hat{X}^\rho} e^{iq_\sigma \frac{a^\sigma}{a} \hat{U}} e^{iq_\sigma \hat{X}^\sigma} \tilde{f}(p)\tilde{g}(q) .\end{aligned}$$

Again, we can calculate the adjoint action

$$e^{-iq_\sigma \frac{a^\sigma}{a}\hat{U}} \hat{X}^\rho e^{iq_\sigma \frac{a^\sigma}{a}\hat{U}} = e^{aq_\sigma \frac{a^\sigma}{a}} \hat{X}^\rho = e^{q_\sigma a^\sigma} \hat{X}^\rho \,,$$

which yields

$$e^{-iq_\sigma \frac{a^\sigma}{a}\hat{U}} e^{ip_\rho \hat{X}^\rho} e^{iq_\sigma \frac{a^\sigma}{a}\hat{U}} = e^{ip_\rho e^{q_\sigma a^\sigma} \hat{X}^\rho} \,.$$

By inserting this in the above equation we obtain

$$\begin{aligned} & \hat{f}(\hat{x})\hat{g}(\hat{x}) \\ = \; & \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p\, d^{n+1}q\, e^{i(p_\rho \frac{a^\rho}{a} + q_\sigma \frac{a^\sigma}{a})\hat{U}} e^{i(p_\rho + q_\rho)\hat{X}^\rho} e^{ip_\rho (e^{q_\sigma a^\sigma} - 1)\hat{X}^\rho} \tilde{f}(p)\tilde{g}(q) \\ = \; & \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p\, d^{n+1}q\, e^{i(p_\rho \frac{a^\rho}{a} + q_\sigma \frac{a^\sigma}{a})\hat{U}} e^{i(p_\rho + q_\rho)\hat{X}^\rho} e^{ip_\rho (e^{q_\sigma a^\sigma} - 1)P^\rho{}_\nu \hat{x}^\nu} \tilde{f}(p)\tilde{g}(q) \,. \end{aligned}$$

We have ordered the product of two functions in the noncommutative algebra by commuting $\hat{U}$ to the left. To obtain the expression for the normal ordered $\star$-product we have to apply ρ_n^{-1} (ρ_n denotes normal ordering) to the above expression. We find

$$\begin{aligned} f \star_n g \;&=\; \rho_n^{-1}(\hat{f}(\hat{x})\hat{g}(\hat{x})) \\ &=\; \frac{1}{(2\pi)^{n+1}} \int d^{n+1}p\, d^{n+1}q\, e^{i(p_\mu + q_\mu)x^\mu} e^{ip_\rho (e^{q_\sigma a^\sigma} - 1)P^\rho{}_\nu x^\nu} \tilde{f}(p)\tilde{g}(q) \\ &=\; \exp(P^\rho{}_\nu x^\nu \partial_{y^\rho}(e^{-ia^\sigma \partial_{z^\sigma}} - 1)) f(y)g(z)|_{y,z\to x}. \end{aligned} \tag{3.19}$$

This is the normal ordered $\star$-product for the generic κ-deformed space-time. As a short consistency check, we note that we recover indeed the expression (3.12) for the special choice $a^\mu = a\delta_n^\mu$. Moreover, we obtain from the above expression

$$x^\mu \star_n x^\nu = x^\mu x^\nu - ia^\nu P^\mu{}_\alpha x^\alpha$$

Hence, we indeed reproduce the algebra relations defining the generic κ-deformed space-time

$$[x^\mu \stackrel{\star_n}{,} x^\nu] = ia^\mu x^\nu - ia^\nu x^\mu \,.$$

3.2.4 The Weyl-ordered $\star$-product for the generic κ-deformed space

Let us now construct the Weyl ordered $\star$-product for the generic κ-deformed space, which we want to denote by $\star_W$. We will proceed in the following way: First we determine

$$x^\mu \star_W g(x) \tag{3.20}$$

for an arbitrary function g. The star-product is fully determined by (3.20): If f is a polynomial then $f \star_W g$ follows from $x^\sigma \star_W g$ by induction[2] over the degree of f. To derive the expression for $\star_W$ we use the following convenient form of the Baker-Campbell-Hausdorff formula (BCH) [113]

$$e^A e^B = e^{A+\mathcal{L}_{\frac{A}{2}}\cdot B+(\mathcal{L}_{\frac{A}{2}}\coth(\mathcal{L}_{\frac{A}{2}}))\cdot B+\cdots}\,. \tag{3.21}$$

This expression contains all contributions linear in B. By $\mathcal{L}_A$ we denote the Lie derivate $\mathcal{L}_A \cdot B \equiv (A \cdot B) = [A,B]$ and the hyperbolic cotangent has to be understood in terms of its power series expansion, where

$$c\mathcal{L}_{A^n}\cdot B \equiv c(A^n \cdot B) := c\,\underbrace{[A,[A,[\ldots,[A,B]}_{n}\ldots]]]$$

for $n \geq 1$.

In order to calculate $x^\mu \star_W g$ for an arbitrary function g, we consider

$$A = iq_\mu \hat{x}^\mu \quad , \quad B = ip_\nu \hat{x}^\nu\,.$$

By inverting the formula (3.21) we obtain

$$e^B e^A = e^{A+\mathcal{L}_{-\frac{A}{2}}\cdot B+(\mathcal{L}_{-\frac{A}{2}}\coth(\mathcal{L}_{-\frac{A}{2}}))\cdot B}\,,$$

which contains all terms linear in B. Since by Fourier-transformation p_ν becomes $-i\partial_\nu$ this is all we need in order to calculate $x^\mu \star_W g(x)$ to all orders (we shall comment on this below). The commutation relations in the algebra (3.16) lead to

$$-\frac{A}{2}\cdot B = [-\frac{A}{2},B] = (\frac{1}{2}q_\mu a^\mu)B + (\frac{-1}{2}p_\nu a^\nu)A = c_1 B + c_2 A\,,$$

where $c_1 = \frac{1}{2}q_\mu a^\mu$ and $c_2 = \frac{-1}{2}p_\nu a^\nu$. Applying $-\frac{A}{2}$ n-times to B we find

$$(-\frac{A}{2})^n \cdot B = c_1^n B + c_1^{n-1}c_2 A\,, \quad n \geq 1\,. \tag{3.22}$$

By definition 1 acts trivially, i.e.

$$1 \cdot B := B\,. \tag{3.23}$$

[2]This is true since $W(x^{\sigma_1}\cdots x^{\sigma_n}) = \frac{1}{n!}\hat{x}^{(\sigma_1}\cdots\hat{x}^{\sigma_n)}$, where W denotes symmetric ordering, and therefore $x^{\sigma_1}\cdots x^{\sigma_n} = \frac{1}{n!}x^{(\sigma_1}\star\cdots\star x^{\sigma_n)}$.

Let f be a function of $-\frac{A}{2}$, which can be expanded in a Taylor-series

$$f(-\frac{A}{2}) = \sum_{n=0}^{\infty} u_n(-\frac{A}{2})^n = u_0 + \sum_{n=1}^{\infty} u_n(-\frac{A}{2})^n =: u_0 + \tilde{f}(-\frac{A}{2})\,.$$

Then equations (3.22) and (3.23) yield

$$\begin{aligned} f(-\frac{A}{2}) \cdot B &= f(c_1)B + \tilde{f}(c_1)c_1^{-1}c_2A \\ &= f(c_1)B + f(c_1)c_1^{-1}c_2A - u_0c_1^{-1}c_2A\,. \end{aligned} \tag{3.24}$$

Using this result we can calculate

$$(\mathcal{L}_{-\frac{A}{2}}\coth(\mathcal{L}_{-\frac{A}{2}})) \cdot B = c_1\coth(c_1)B + c_2\coth(c_1)A - \frac{c_2}{c_1}A\,, \tag{3.25}$$

where we note that formula (3.24) is applicable because the function $x\coth(x) = 1 + \frac{x^2}{3} + \cdots$ can be expanded in a Taylor-series.

Let us gather all results in order to determine e^Be^A explicitly

$$\begin{aligned} e^Be^A &= e^{A+\mathcal{L}_{-\frac{A}{2}}\cdot B + (\mathcal{L}_{-\frac{A}{2}}\coth(\mathcal{L}_{-\frac{A}{2}}))\cdot B} \\ &\overset{(3.22),(3.25)}{=} \exp(A + c_1B + c_2A + c_1\coth(c_1)B + c_2\coth(c_1)A - \frac{c_2}{c_1}A) \\ &= \exp(A + B + (c_1B + c_2A)(1 + \coth(c_1)) - \frac{c_2}{c_1}A - B)\,. \end{aligned}$$

By inserting the explicit expressions for A, B, c_1 and c_2 given above we obtain

$$e^{ip_\nu\hat{x}^\nu}e^{iq_\mu\hat{x}^\mu} = \exp(i\hat{x}^\nu(p_\nu + q_\nu + (\frac{1}{2}a^\mu q_\mu p_\nu - \frac{1}{2}a^\mu p_\mu q_\nu)(1 + \coth(\frac{1}{2}a^\mu q_\mu)) + \frac{a^\mu p_\mu}{a^\rho q_\rho}q_\nu - p_\nu))\,. \tag{3.26}$$

This equation contains all contributions linear in p_ν. This is sufficient in order to calculate $x^\sigma \star g$ because we have that

$$f \star_W g(x) = \exp(\frac{i}{2}x^\nu h_\nu(-i\partial_y, -i\partial_z))f(y)g(z)|_{y,z\to x}\,,$$

where the function h_ν is defined by

$$e^{ip_\nu\hat{x}^\nu}e^{iq_\mu\hat{x}^\mu} = e^{ix^\nu(p_\nu + q_\nu + \frac{1}{2}h_\nu(p,q))}\,.$$

If the function f is linear in x^μ, only terms, which are at most linear in p_μ, will contribute. Equation (3.26) yields

$$\begin{aligned}
x^\sigma \star_W g(x) &= \exp(\frac{i}{2}x^\nu h_\nu(-i\partial_y, -i\partial_z))y^\sigma g(z)|_{y,z\to x} \\
&= x^\sigma g(x) + (\frac{-i}{2}x^\sigma a^\mu\partial_\mu + \frac{i}{2}a^\sigma x^\mu\partial_\mu)(1+\coth(\frac{-i}{2}a^\mu\partial_\mu))g(x) \\
&\quad + x^\nu\partial_\nu\frac{a^\sigma}{a^\rho\partial_\rho}g(x) - x^\sigma g(x) \\
&= (\frac{-i}{2}x^\sigma a^\mu\partial_\mu + \frac{i}{2}a^\sigma x^\mu\partial_\mu)(1+\coth(\frac{-i}{2}a^\mu\partial_\mu))g(x) + x^\nu\partial_\nu\frac{a^\sigma}{a^\rho\partial_\rho}g(x) \\
&= x^\sigma\frac{ia^\mu\partial_\mu}{e^{ia^\rho\partial_\rho}-1}g(x) + a^\sigma(\frac{x^\nu\partial_\nu}{a^\rho\partial_\rho} - \frac{ix^\mu\partial_\mu}{e^{ia^\rho\partial_\rho}-1})g(x)\,. \qquad (3.27)
\end{aligned}$$

Here, the last line follows because $1+\coth(\frac{x}{2}) = \frac{-2}{e^{-x}-1}$.

Since the $\star$-product is entirely determined by (3.27) we just have to find an explicit expression for $f\star_W g$, which recovers equation (3.27) for $f = x^\sigma$ and which is associative. Both requirements are satisfied by

$$f\star_W g = \exp\{x^\mu(\partial_{z^\mu} - \partial_{y^\mu}\frac{a^\alpha\partial_{z^\alpha}}{a^\beta\partial_{y^\beta}})(\frac{a^\nu(\partial_{y^\nu}+\partial_{z^\nu})}{e^{-ia^\gamma(\partial_{y^\gamma}+\partial_{z^\gamma})}-1}\frac{e^{-ia^\sigma\partial_{z^\sigma}}-1}{a^\rho\partial_{z^\rho}} - 1)\}f(y)g(z)|_{y,z\to x}\,. \qquad (3.28)$$

To prove the associativity of this $\star$-product is not obvious. In the next section we give an explicit expression for an equivalence transformation from the normal ordered $\star$-product to the Weyl ordered $\star$-product as defined in (3.28). Associativity of the product (3.28) follows then from associativity of the normal ordered $\star$-product. In [92], equation (A.2), another expression for the symmetric ordered $\star$-product in the generic κ-deformed case, which equals ours given above in (3.28), can be found. The simpler expression (3.28) can be obtained from (A.2) in [92] using that

$$\begin{aligned}
& x^\mu\partial_{y^\mu}\frac{-ia^\nu(\partial_{y^\nu}+\partial_{z^\nu})}{e^{-ia^\gamma(\partial_{y^\gamma}+\partial_{z^\gamma})}-1}e^{-ia^\lambda\partial_{z^\lambda}}\frac{e^{-ia^\sigma\partial_{y^\sigma}}-1}{-ia^\rho\partial_{y^\rho}} \\
&= x^\mu\partial_{y^\mu}\frac{a^\alpha\partial_{z^\alpha}}{a^\beta\partial_{y^\beta}}\frac{-ia^\nu(\partial_{y^\nu}+\partial_{z^\nu})}{e^{-ia^\gamma(\partial_{y^\gamma}+\partial_{z^\gamma})}-1}e^{-ia^\lambda\partial_{z^\lambda}}\frac{e^{-ia^\sigma\partial_{y^\sigma}}-1}{-ia^\rho\partial_{z^\rho}} \\
&= x^\mu\partial_{y^\mu}\frac{a^\alpha\partial_{z^\alpha}}{a^\beta\partial_{y^\beta}}\frac{-ia^\nu(\partial_{y^\nu}+\partial_{z^\nu})}{e^{-ia^\gamma(\partial_{y^\gamma}+\partial_{z^\gamma})}-1}\frac{e^{-ia^\sigma(\partial_{y^\sigma}+\partial_{z^\sigma})}-1+1-e^{-ia^\lambda\partial_{z^\lambda}}}{-ia^\rho\partial_{z^\rho}} \\
&= x^\mu\partial_{y^\mu}\frac{a^\alpha\partial_{z^\alpha}}{a^\beta\partial_{y^\beta}}(\frac{-ia^\nu(\partial_{y^\nu}+\partial_{z^\nu})}{-ia^\rho\partial_{z^\rho}} + \frac{-ia^\nu(\partial_{y^\nu}+\partial_{z^\nu})}{e^{-ia^\gamma(\partial_{y^\gamma}+\partial_{z^\gamma})}-1}\frac{1-e^{-ia^\lambda\partial_{z^\lambda}}}{-ia^\rho\partial_{z^\rho}})
\end{aligned}$$

$$= \quad x^\mu \partial_{y^\mu} \frac{a^\alpha \partial_{z^\alpha}}{a^\beta \partial_{y^\beta}} (1 + \frac{a^\nu \partial_{y^\nu}}{a^\rho \partial_{z^\rho}} - \frac{a^\nu (\partial_{y^\nu} + \partial_{z^\nu})}{e^{-ia^\gamma (\partial_{y^\gamma} + \partial_{z^\gamma})} - 1} \frac{e^{-ia^\lambda \partial_{z^\lambda}} - 1}{a^\rho \partial_{z^\rho}}) \, .$$

See also [92] for an explicit calculation showing that equation (3.28) reproduces (3.27).

3.2.5 Equivalence between $\star_n$ and $\star_W$

In the previous sections we derived the explicit and closed expressions for the normal ordered and Weyl-ordered $\star$-products in the generic κ-deformed case. Both $\star$-products are equivalent, i.e. there exists an *equivalence transformation* S such that

$$f \star_W g = S^{-1}(S(f) \star_n S(g)) \, .$$

The corresponding equivalence transformation is given by

$$S = \exp(P^\mu{}_\nu y^\nu \partial_{x^\mu} (\frac{1 - e^{-ia^\rho \partial_{x^\rho}}}{ia^\sigma \partial_{x^\sigma}} - 1))|_{y \to x} \, .$$

Noting that $[a^\rho \partial_\rho, P^\mu{}_\nu x^\nu] = 0$ it is straightforward to show that S indeed mediates between the normal ordered star product in (3.19) and the Weyl-ordered star product in (3.28). The associativity of $\star_W$ in (3.28) then follows from the associativity of $\star_n$. This result for S for the generic κ-deformed case is a generalization of the equivalence transformation T^{-1} found in [92], equation (2.40), for the algebra (2.7). There, also an explicit calculation showing that T^{-1} is an equivalence transformation can be found.

3.3 q-Deformed Euclidean Space

Star-products for the q-deformed Euclidean space defined in (2.14) were studied in [86]. For completeness we repeat some expressions here: The normal ordered $\star$-product reads

$$f \star_n g = \mu \circ e^{-2h(\overline{z}\partial_{\overline{z}} \otimes z\partial_z)} (f \otimes g) \, , \tag{3.29}$$

where $q = e^h$. Another $\star$-product corresponding to a q-symmetric ordering is defined by

$$f \star_q g := \mu \circ e^{h(z\partial_z \otimes \overline{z}\partial_{\overline{z}} - \overline{z}\partial_{\overline{z}} \otimes z\partial_z)} (f \otimes g) \, . \tag{3.30}$$

This $\star$-product is hermitian. In the next section we will see how these $\star$-products and those presented before can be treated in a more general framework.

3.4 Star-products from Commuting Vector Fields

It is possible to construct a class of $\star$-product, which, as we shall see, contains the $\star$-products of the previous sections. In [114, 115] it was shown that given a set of commuting vector fields X_a,

$$[X_a, X_b] = 0\,, \tag{3.31}$$

$\star$-products are obtained by

$$f \star g := \mu \circ e^{\sigma^{ab} X_a \otimes X_b} f \otimes g\,, \tag{3.32}$$

where σ^{ab} are arbitrary constants. Associativity follows using (3.31). Although the requirement of commuting vector fields is quite restrictive, this class contains many interesting examples. The Moyal-Weyl product (3.2), for instance, is obviously obtained from the commuting vector fields $X_\alpha \equiv \partial_\alpha$ by setting in the above definition $\sigma^{\alpha\beta} \equiv \frac{i}{2}\theta^{\alpha\beta}$. Furthermore the normal ordered $\star$-product for κ-deformed space-time (3.10) is a $\star$-product obtained from commuting vector fields[3], where

$$X_1 = x^i \partial_i,\ X_2 = -ia\partial_n \tag{3.33}$$

and $\sigma^{ab} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The $\star$-products (3.29) and (3.30) defined for the q-deformed Euclidean plane are also given in terms of commuting vector fields. In this case they read

$$X_1 = z\partial_z,\ X_2 = \overline{z}\partial_{\overline{z}}\,. \tag{3.34}$$

Commuting vector fields yield an elegant way to construct $\star$-products corresponding to many noncommutative spaces [82, 114, 115]. However, given a $\star$-product coming from commuting vector fields representing a given noncommutative space, it is in general not clear, to which ordering prescription it corresponds. Knowing the ordering prescription can be useful, see for instance Section 3.5. In Section 4 we shall see how it is possible to deform gravity even without knowing the ordering prescription by starting from quite a large class of $\star$-products, which contains $\star$-products coming from commuting vector fields.

3.5 Star-representations

Given an ordering prescription underlying the construction of a $\star$-product corresponding to a noncommutative space $\hat{\mathcal{A}}$, we can also construct $\star$-representations of

[3] If we consider the expression (3.12), which, as we have seen, provides a different way to write the normal ordered $\star$-product, this conclusion is not obvious.

differential operators acting on the noncommutative space. The $\star$-representation is nothing but the representation of these operators as pseudo-differential operators acting on the algebra of functions depending on commutative coordinates $\mathcal{A}$. It can be obtained as follows: Let $\hat{D}$ be a differential operator acting on $\hat{\mathcal{A}}$ (this could for instance be a partial derivative $\hat{\partial}_\mu$ or a generator of a Quantum Group symmetry acting on $\hat{\mathcal{A}}$) and ρ an ordering prescription used to construct a $\star$-product for $\hat{\mathcal{A}}$ (often this will be the Weyl ordering). Then we define the $\star$-representation $D^\star$ of $\hat{D}$ by

$$\begin{array}{ccc} f \in \mathcal{A} & \xrightarrow{\quad\rho\quad} & f \in \hat{\mathcal{A}} \\ {\scriptstyle D^\star}\downarrow & & \downarrow{\scriptstyle \hat{D}} \\ (D^\star f) \in \mathcal{A} & \xrightarrow{\quad\rho\quad} & (\hat{D}\hat{f}) \in \hat{\mathcal{A}} \end{array}$$

where $\hat{f} = \rho(f) \in \hat{\mathcal{A}}$. Hence

$$(D^\star f) := \rho^{-1} \circ \hat{D} \circ \rho(f)\,.$$

For instance, let us determine the $\star$-representation for the partial derivatives $\hat{\partial}_\mu$ acting on the θ-deformed space as defined in (2.20). In this case we simply have

$$\partial^\star_\mu = \partial_\mu\,,$$

which is nothing but another way to say that the derivatives defined for the θ-deformed space are undeformed. In order to give a non-trivial example, let us consider the derivatives defined in (2.22) for κ-deformed spaces. In this case the derivatives are indeed deformed and we end up with [91]

$$\begin{aligned} \partial^\star_n &= \partial_n \\ \partial^\star_i &= \frac{e^{ia\partial_n} - 1}{ia\partial_n}\partial_i\,. \end{aligned}$$

The derivatives $\partial^\star_\mu$ (we also call them $\star$-derivatives) act on a $\star$-product of functions via the Leibniz rule (2.24)

$$\begin{aligned} \partial^\star_n(f \star g) &= (\partial^\star_n f) \star g + f \star (\partial^\star_n g) \\ \partial^\star_i(f \star g) &= (\partial^\star_i f) \star g + (e^{ia\partial^\star_n} f) \star (\partial^\star_i g)\,. \end{aligned} \tag{3.35}$$

In [34, 90, 91] the $\star$-representations of all generators of the κ-deformed Poincaré algebra can be found.

Chapter 4

Deformation by Twists

This chapter is devoted to deformation by twists [42, 70, 74, 75, 116], which underlies the construction of gauge and gravity theories on noncommutative spaces presented in Chapter 6 and Section 5.5, see also [39,42,43,63,87,97]. We shall see that by means of generic twists we obtain quite a large class of deformed spaces. At the same time Quantum Group symmetries acting on these spaces can be constructed. We shall furthermore see that the formalism presented here generalizes in some sense the constructions of the previous chapters: all the noncommutative spaces introduced in Section 2.1 can be obtained by twists and we will give explicit examples for such twists.

In particular, the construction of gravity on canonically deformed spaces [42], which we present in Section 6.2, can be understood in the framework of twists. This makes it possible to generalize this construction to a large class of noncommutative spaces, those defined by a generic twist, see Section 6.3, [43].

Deformation by twists

Let H be a Hopf algebra and A an H-module algebra (see Section 2.3.1). In this thesis H will always be the universal enveloping algebra of a Lie algebra. This is a Hopf algebra as we saw in Section 2.3.1. A can be for instance the algebra of functions (but also other module algebras as the algebra of forms or tensor fields are of interest for physical applications). A twist $\mathcal{F}$ is defined as follows

Definition 1. *A twist $\mathcal{F}$ is an element $\mathcal{F} \in H \otimes H$, which is invertible and that satisfies*

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F}\,, \tag{4.1}$$

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = 1 = (\mathrm{id} \otimes \varepsilon)\mathcal{F}\ , \tag{4.2}$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ *and* $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$.

In our context we in addition require

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\lambda) \ . \tag{4.3}$$

Property (4.1) states that $\mathcal{F}$ is a two cocycle, and it will turn out to be responsible for the associativity of the $\star$-products to be defined. Property (4.2) is just a normalization condition. From (4.3) it follows that $\mathcal{F}$ can be formally inverted as a power series in λ.

Given a twist we can deform the algebra A. Let us denote by $A_\star$ the algebra with the new, twisted product

$$a \star b \equiv \mu_\star(a \otimes b) := \mu \circ \mathcal{F}^{-1}(a \otimes b) \tag{4.4}$$

for all $a, b \in A$. The associativity of (4.4) follows from the cocycle condition (4.1), see also [43]. The algebra A is a module algebra with respect to the Hopf algebra H. This is not the case anymore for the algebra $A_\star$. However, it is possible to twist the Hopf algebra H such that $A_\star$ becomes a module algebra with respect to this twisted one. The resulting twisted Hopf algebra is denoted by $H^\mathcal{F}$ and is defined as follows:

- As algebra $H^\mathcal{F} = H$, i.e. no deformation takes place on the level of the multiplication in H.
- The counit ε remains unchanged, $\varepsilon^\mathcal{F} = \varepsilon$.
- The coproduct is deformed by conjugation with $\mathcal{F}$

$$\begin{aligned} \Delta^\mathcal{F} : H^\mathcal{F} &\rightarrow H^\mathcal{F} \otimes H^\mathcal{F} \\ h &\mapsto \Delta^\mathcal{F}(h) := \mathcal{F}\Delta(h)\mathcal{F}^{-1} \, . \end{aligned} \tag{4.5}$$

- The antipode S is deformed by conjugation with the invertible element $\chi := \mu \circ (\mathrm{id} \otimes S)\mathcal{F}$

$$S^\mathcal{F}(h) := \chi S(h) \chi^{-1} \, . \tag{4.6}$$

It is a standard proof to show that $H^\mathcal{F}$ defined as above is a Hopf algebra and we present it in detail in Section 6.3, [43].

Module algebras

For us the following theorem is very important:

Theorem 1. *Let A be an H-module algebra. Then $A_\star$ is a $H^{\mathcal{F}}$-module algebra, where $A_\star$ and $H^{\mathcal{F}}$ are defined as above.*

Since this basic theorem underlies our constructions in [42, 43], and in order to make the reader familiar with the formalism, we present a proof.

Proof. From the definition of a module algebra given in Subsection 2.3.1 we recall that $A_\star$ is a $H^{\mathcal{F}}$-module algebra if

$$h(a \star b) = \mu_\star \circ \Delta^{\mathcal{F}}(h)(a \otimes b)$$

and if

$$h(1) = \varepsilon^{\mathcal{F}}(h)1$$

for all $a, b \in A_\star$, $h \in H^{\mathcal{F}}$, and where $\mu_\star(a \otimes b) = a \star b$. The second condition follows immediately since A is a H-module algebra and $\varepsilon^{\mathcal{F}} = \varepsilon$. In order to see that the first requirement is satisfied we make a short calculation

$$\begin{aligned}
h(a \star b) &= h(\mu \circ \mathcal{F}^{-1}(a \otimes b)) \\
&= \mu \circ \Delta(h)\mathcal{F}^{-1}(a \otimes b) \\
&= \mu \circ \mathcal{F}^{-1}\mathcal{F}\Delta(h)\mathcal{F}^{-1}(a \otimes b) \\
&= \mu_\star \circ \mathcal{F}\Delta(h)\mathcal{F}^{-1}(a \otimes b) \\
&= \mu_\star \circ \Delta^{\mathcal{F}}(h)(a \otimes b) \, .
\end{aligned}$$

□

This formalism gives rise to a quite general construction of Quantum Group symmetries for noncommutative spaces: Let us assume that we are interested out of physical reasons in studying a noncommutative space $\hat{\mathcal{A}}$ defined by commutation relations for the noncommutative coordinates $\hat{x}^\mu$. Let us denote a star-product representing the noncommutative product of $\hat{\mathcal{A}}$ on the algebra of commutative functions $\mathcal{A}$ by $\star$. If this $\star$-product is defined by a twist $\mathcal{F}$, we may use this twist in order to construct a Quantum Group symmetry acting on $\hat{\mathcal{A}}$. Using Theorem 1 this is always possible if the twist lives in the tensor product of an interesting Hopf algebra, for instance the universal enveloping algebra of a Lie algebra acting on $\mathcal{A}$. Let us assume for example that the $\star$-product corresponding to $\hat{\mathcal{A}}$ is given by a twist

$$\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g} \, ,$$

where $U\mathfrak{g}$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then Theorem 1 tells us that we obtain a deformed symmetry acting on $\hat{\mathcal{A}}$ by twisting $U\mathfrak{g}$ to the Hopf algebra $U\mathfrak{g}^{\mathcal{F}}$.

This may seem quite restrictive but we will see how actually some of the cases discussed in Section 2.1 fit in this context. As first example let us consider the Moyal-Weyl $\star$-product (3.2). It can be interpreted as coming from a twist

$$f \star g = \mu \circ \mathcal{F}^{-1}(f \otimes g)\,,$$

where

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\mu}\,.$$

This twist involves only generators of translations ∂_μ. Therefore we have that $\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g}$ for any Lie algebra $\mathfrak{g}$, which contains translations. Such a Lie algebra is for example the Poincaré algebra and if we look carefully at the equations in (2.57) and (2.58), which define what we called in Section 2.3.2 the θ-deformed Poincaré algebra, then we see that the θ-deformed Poincaré algebra is obtained from the universal enveloping algebra of the Poincaré algebra by twisting with $\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\mu}$ [97].

Translations are also contained in the Lie algebra of vector fields Ξ such that we can use the twist $\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\mu}$ also in order to deform the universal enveloping algebra of vector fields $U\Xi$. As vector fields generate general coordinate transformations, we can construct a θ-deformed theory of gravity based on this deformed Hopf algebra [42]. Actually, it is not difficult to see that

$$\mathcal{F} := e^{\sigma^{ab} X_a \otimes X_b} \in U\Xi \otimes U\Xi\,,$$

where X_a is a set of commuting vector fields $[X_a, X_b] = 0$, is a twist [43]. Consequently all $\star$-products constructed using commuting vector fields provide us with a twist that can be used to twist $U\Xi$ in order to obtain a deformed Hopf algebra, which acts consistently on the $\star$-product algebra. This class of $\star$-products was introduced in Section 3.4 and we saw there that $\star$-products for κ-deformed spaces such as defined in (3.9) as well as $\star$-products for the q-deformed Euclidean plane (3.30) are contained in this class. Thus, we could take the twists $\mathcal{F} = e^{\sigma^{ab} X_a \otimes X_b}$ corresponding to the sets of commuting vector fields (3.33) and (3.34) in order to construct a deformation of $U\Xi$, which acts consistently on κ-deformed spaces respectively the q-deformed space (2.14). More examples for twist deformations can be found in [82, 115].

Theorem 1 suggests also another conclusion: It offers a possible way to generalize the concepts proposed and studied in Section 2.1. We may detach ourselves from considering noncommutative spaces defined as coset spaces only, i.e. those, which are given by taking the algebra generated by the noncommutative coordinates $\hat{x}^\mu$ modulo the ideal generated by their commutation relations. Instead we may consider the algebra of functions equipped with an arbitrary $\star$-product as the fundamental object. Generic twists $\mathcal{F} \in U\Xi \otimes U\Xi$ define already quite a large class of such spaces (which, as we have seen, contain many of our noncommutative spaces from Section 2.1). The twisted Hopf algebra $U\Xi^{\mathcal{F}}$ enables us to construct deformed gravity for this whole class of noncommutative algebras of functions [43]. The next step would be to treat even arbitrary Kontsevich $\star$-products [31] for the algebra of functions on a Poisson manifold. This is work in progress and in fact it turns out that many steps towards a deformation of gravity can be done even in this general setting [117].

Chapter 5

Gauge Theories on Noncommutative Spaces

In the previous chapters we have introduced in detail the concepts underlying the construction of physical theories on noncommutative spaces. In this chapter we shall see how these concepts can be applied in order to construct gauge theories.

It consists of four publications: The first one treats gauge theories on the q-deformed plane and was published together with Harold Steinacker in the International Journal of Modern Physics A [35]. The first part of this publication, the algebraic construction of a covariant differential calculus together with a set of commuting frames and the construction of an integral that is invariant with respect to the action of the q-deformed Euclidean algebra is already contained in [86]. New results, which are not contained in [86], are additional results concerning integration and the construction of an invariant metric. Moreover, gauge theories on the q-deformed plane are constructed by gauging the basis elements of a commuting frame of one forms. In this way Seiberg–Witten maps are constructed, which reduce in the commutative limit to the usual gauge fields. In general this is a subtle point, since solutions to Seiberg–Witten maps for the gauge fields proposed for arbitrary Poisson structures [33] are proportional to the Poisson structure itself and therefore do not reduce to the commutative gauge fields when the noncommutativity vanishes. If the Poisson structure is non-constant it would have to be inverted in a complicated way without spoiling gauge covariance [86]. By constructing Seiberg–Witten maps gauging the commuting frame we circumvent this problem. The commuting frames give rise to derivations on the noncommutative space and derivative valued gauge fields as in [34] respectively Section 5.3 can be avoided. It is the first time that the construction of gauge theories on non-trivial noncommutative spaces such as q-deformed space via gauging the commuting frame was proposed and we exhibit this procedure in detail. All noncommutative fields are expressed in terms of their commutative counterparts and in the commutative limit all fields reduce to the commutative ones. Furthermore, we propose an $U(1)$-gauge invariant action, which reduces in the classical limit to the usual, undeformed action[1]. The fields and the action are expanded up to the first non-trivial order in the deformation parameter. A generalization of these ideas to arbitrary noncommutative spaces, which possess a commuting frame, can be found in [38, 114, 118].

As second publication we present a contribution to the Proceedings of the 9-th Adriatic Meeting 2003 in Dubrovnik, Croatia, published together with Harold Steinacker in the Springer Proceedings series. It summarizes a talk given by the author about gauge theories on the q-deformed Euclidean plane [36].

The third publication treats gauge theories on κ-deformed spaces. It was published together with Marija Dimitrijević, Lutz Möller and Julius Wess in the European Physical Journal C [34]. There, a Seiberg–Witten map is constructed for non-abelian gauge theories on κ-deformed space-time. It is based on the concept of derivative valued gauge fields. All noncommutative quantities are expanded up to first order in the deformation parameter a. At the end, consistency of the gauge transformations with κ-Poincaré transformations is shown.

The fourth publication deals with gauge theories on fuzzy $S^2 \times S^2$ and was published together with Wolfgang Behr and Harold Steinacker in the Journal of High Energy Physics [65]. We have seen in Section 2.1.3 that fuzzy spaces retain undeformed rotational symmetry. Hence, this is an example of gauge theories on a noncommutative space, which possesses an undeformed symmetry. Gauge theories on fuzzy S^2 have been studied in detail in recent years [44, 51, 119–122]. In order to construct a four-dimensional model, we investigate gauge theories on fuzzy $S^2 \times S^2$. We define $U(n)$ gauge theory as a multi-matrix model. Moreover we show that our model reduces to noncommutative gauge theories on the θ-deformed space (2.5) in a double scaling limit. This model can therefore be used as a regularization of gauge theories on the θ-deformed plane. Monopole solutions are constructed on fuzzy $S^2 \times S^2$, which are mapped in the double scaling limit to instanton solutions of gauge theories on θ-deformed spaces.

[1] In the case of q-deformed spaces, it turns out that a measure function has to be introduced in order to guarantee cyclicity of the integral and this way gauge invariance of the action. This measure survives in the commutative limit. We show that it is possible to add a term in the action, which cancels the measure function in the commutative limit but which does not break gauge invariance.

In [39] we propose gauge theory on noncommutative spaces based on twisted gauge transformations. We construct gauge invariant Lagrangians and derive consistent equations of motions. This leads to conserved currents. Consistency of the equations of motion requires us to choose the gauge field in the enveloping algebra. This gives rise to additional, new fields, which reduce in the commutative limit to free fields and which couple weakly via the deformation parameter θ to the usual gauge fields. The number of the new fields is finite if we choose a finite dimensional representation of the enveloping algebra.

5.1 Gauge-Field Theory on the $E_q(2)$-Covariant Plane

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GAUGE FIELD THEORY ON THE $E_q(2)$-COVARIANT PLANE

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Gauge theory on the q-deformed two-dimensional Euclidean plane $\mathbb{R}_q^2$ is studied using two different approaches. We first formulate the theory using the natural algebraic structures on $\mathbb{R}_q^2$, such as a covariant differential calculus, a frame of one-forms and invariant integration. We then consider a suitable star product, and introduce a natural way to implement the Seiberg–Witten map. In both approaches, gauge invariance requires a suitable "measure" in the action, breaking the $E_q(2)$-invariance. Some possibilities to avoid this conclusion using additional terms in the action are proposed.

Keywords: Noncommutative gauge theory; Seiberg–Witten map; quantum groups.

1. Introduction

Gauge theories provide the best known descriptions of the fundamental forces in nature. At very short distances however, physics is not known, and it is plausible that space–time is quantized below some scale. This idea has been contemplated for quite some time, and gauge theory on noncommutative spaces has been the subject of much research activity, see e.g. Ref. 8 for a review.

There are several different approaches to gauge theories on noncommutative spaces: First, one can formulate the theory in terms of the algebraic structures which define the noncommutative space, such as the noncommutative algebra of functions, its modules, and differential calculi. Gauge transformations can then be defined by unitary elements of the algebra of functions. Examples of noncommutative gauge theories using this formulation can be found in Refs. 14, 7 and 8. While it is certainly very natural, this approach seems to be restricted to unitary gauge groups, and the set of admissible representations of the associated matter fields is also quite restricted.

Another approach has been developed following the discovery that string theory leads to noncommutative gauge theories under suitable conditions, as explained in Ref. 21. This lead to a technique expressing the noncommutative fields in terms of commutative ones, and writing the Lagrangians in terms of ordinary (commutative) fields and star products. It allows to formulate models with general gauge groups and representations, including the standard model.[3] However, the Lagrangians become increasingly complicated at each order in the deformation parameter, and there is generally a large amount of arbitrariness in these actions. Moreover, the formulation of gauge theories on general noncommutative spaces with nonconstant Poisson structure is less clear. In particular, no satisfactory formulation of gauge theory on spaces with quantum group symmetry has been given; see e.g. Ref. 18 for a clear manifestation of this problem. It seems that in general, a satisfactory implementation of generalized symmetries (quantum group symmetries) in noncommutative field theory is yet to be found.

In the present paper, we apply these different approaches to gauge theory on the Euclidean quantum plane $\mathbb{R}^2_q$, which is covariant under the q-deformed two-dimensional Euclidean group $E_q(2)$. This is one of the simplest quantum spaces with a nontrivial quantum group symmetry, and scalar field theory on $\mathbb{R}^2_q$ has already been studied in Ref. 5. It seems therefore well suited to gain some insights into gauge theory on spaces with quantum group symmetry.

We first try to formulate (Abelian) gauge theory on $\mathbb{R}^2_q$ using an algebraic approach, taking advantage of the covariant differential calculus on $\mathbb{R}^2_q$. This leads very naturally to a definition of gauge fields and their field strength, with gauge transformations being the unitaries of the algebra of functions. This field strength reduces to the usual one in the commutative limit. However, the definition of an invariant action turns out to be less clear: if one uses the natural invariant integral on $\mathbb{R}^2_q$, one must add a nontrivial "measure function" in order to obtain a gauge invariant action. This measure function explicitly breaks translation invariance, which seems to be a generic feature of gauge theory on spaces with quantum group symmetry. Hence gauge invariance appears to be in conflict with quantum group symmetry. However, we point out some ways to avoid this conclusion. We propose a model with an additional scalar ("Higgs") field with a suitable potential, which is manifestly gauge invariant and restores the formal $E_q(2)$-invariance while spontaneously breaking gauge invariance.

In the second part of this paper, we apply the star product approach to gauge theory on $\mathbb{R}^2_q$, expressing all fields in terms of commutative ones. We first construct a suitable star product, and study its properties and the relation with the integral. The gauge theory is then formulated using this star product in close analogy to the algebraic approach. In particular, the noncommutative calculus suggests a definition of the field strength in terms of a "frame," which ensure the correct classical limit. This is somewhat different from other approaches proposed in the literature.[19] The corresponding Seiberg–Witten maps are solved up to first order. The formulation of a gauge invariant action requires again a nontrivial measure function, which is essentially the same as in the algebraic approach. While it cannot be canceled as in the algebraic approach by introducing a Higgs field, we show how the action can be modified in order to obtain the correct commutative limit.

2. The q-Deformed Two-Dimensional Euclidean Group and Plane

2.1. *The dual symmetry algebras $E_q(2)$ and $U_q(e(2))$*

We start by reviewing the quantum group $E_q(2)$, which is a deformation of the (Hopf) algebra of functions on the two-dimensional Euclidean group $E(2)$. It is generated by the "functions" n, v, $\bar{n}$, $\bar{v}$ with the following relations and structure maps[20]

$$\begin{aligned}
&v\bar{v} = \bar{v}v = 1\,, && n\bar{n} = \bar{n}n\,, && vn = qnv\,,\\
&n\bar{v} = q\bar{v}n\,, && v\bar{n} = q\bar{n}v\,, && \bar{n}\bar{v} = q\bar{v}\bar{n}\,,\\
&\Delta(n) = n\otimes\bar{v} + v\otimes n\,, && \Delta(v) = v\otimes v\,, && \Delta(\bar{n}) = \bar{n}\otimes v + \bar{v}\otimes\bar{n}\,,\\
&\Delta(\bar{v}) = \bar{v}\otimes\bar{v}\,, && \varepsilon(n) = \varepsilon(\bar{n}) = 0\,, && \varepsilon(v) = \varepsilon(\bar{v}) = 1\,,\\
&S(n) = -q^{-1}n\,, && S(v) = \bar{v}\,, &&\\
&S(\bar{n}) = -q\bar{n}\,, && S(\bar{v}) = v\,, &&
\end{aligned} \tag{1}$$

where $q \in \mathbb{R}$. This is a star-Hopf algebra with the conjugation

$$n^* = \bar{n}\,, \qquad v^* = \bar{v}\,. \tag{2}$$

In terms of the operators θ, t, $\bar{t}$ defined by[20]

$$v = e^{\frac{i}{2}\theta}\,, \quad t = nv\,, \quad \bar{t} = \bar{v}\bar{n} \tag{3}$$

(note that v is unitary and can therefore be parametrized by a Hermitian element $\theta^* = \theta$), the coproduct of t and $\bar{t}$ reads

$$\Delta(t) = t\otimes 1 + e^{i\theta}\otimes t\,, \qquad \Delta(\bar{t}) = \bar{t}\otimes 1 + e^{-i\theta}\otimes\bar{t}\,. \tag{4}$$

It is often convenient to consider also the dual quantum group. The dual Hopf algebra $U_q(e(2))$ of $E_q(2)$ is generated by T, $\bar{T}$, J with the following commutation relations and structure maps[20,a]

$$\begin{aligned}
&T\bar{T} = q^2\bar{T}T\,, \quad [J,T] = iT\,, \quad [J,\bar{T}] = -i\bar{T}\,,\\
&\Delta(T) = T\otimes q^{2iJ} + 1\otimes T\,, \quad \Delta(\bar{T}) = \bar{T}\otimes q^{2iJ} + 1\otimes\bar{T}\,,\\
&\Delta(J) = J\otimes 1 + 1\otimes J\,, \quad \varepsilon(T) = \varepsilon(\bar{T}) = \varepsilon(J) = 0\,,\\
&S(T) = -Tq^{-2iJ}\,, \quad S(\bar{T}) = -\bar{T}q^{-2iJ}\,, \quad S(J) = -J\,,
\end{aligned} \tag{5}$$

where the dual pairing on the generators is given by

$$\langle T, \theta^i t^j \bar{t}^k\rangle = \delta_{0i}\delta_{1j}\delta_{0k}\,, \quad \langle\bar{T}, \theta^i t^j \bar{t}^k\rangle = -q^2\delta_{0i}\delta_{0j}\delta_{1k}\,, \quad \langle J, \theta^i t^j \bar{t}^k\rangle = \delta_{1i}\delta_{0j}\delta_{0k}\,. \tag{6}$$

This is again a star-Hopf algebra with the conjugation

$$J^* = -J\,, \quad T^* = \bar{T}\,.$$

[a]Our generators are related to the generators μ, ν, ξ in Ref. 20 by $\mu \equiv T$, $-q^2\nu \equiv \bar{T}$, $\xi \equiv J$.

2.2. *The $E_q(2)$-covariant Euclidean plane* $\mathbb{R}^2_q$

Hopf algebras can be used to define generalized symmetries. There are two equivalent, dual notions. A Hopf algebra $\mathcal{H}$ coacts on an algebra $\mathcal{A}$ if $\mathcal{A}$ is a left (or right) $\mathcal{H}$-comodule algebra (see App. A) via a left coaction $\rho : \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$. In particular, every Hopf algebra $\mathcal{H}$ admits a comodule structure on itself in virtue of the comultiplication

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \,. \tag{7}$$

Observing that the subalgebra of $E_q(2)$ generated by t, $\bar{t}$ is a $E_q(2)$-module subalgebra, we can obtain the $E_q(2)$-symmetric plane by renaming $t \to z$, $\bar{t} \to \bar{z}$. Hence $\mathbb{R}^2_q$ is the $E_q(2)$-comodule algebra with generators z, $\bar{z}$ and commutation relations

$$z\bar{z} = q^2 \bar{z} z \,. \tag{8}$$

We will also allow formal power series, and define the algebra of functions on the $E_q(2)$-*covariant plane*[5]

$$\mathbb{R}^2_q := \mathbb{R}\langle\!\langle z, \bar{z} \rangle\!\rangle / (z\bar{z} - q^2 \bar{z} z) \,. \tag{9}$$

By construction, it is covariant under the following left $E_q(2)$-coaction

$$\begin{aligned} \rho(z) &= e^{i\theta} \otimes z + t \otimes 1 \,, \\ \rho(\bar{z}) &= e^{-i\theta} \otimes \bar{z} + \bar{t} \otimes 1 \,. \end{aligned} \tag{10}$$

More formally, we have a coaction $\rho : \mathbb{R}^2_q \to E_q(2) \otimes \mathbb{R}^2_q$. From now on, functions are considered to be elements of this algebra.

In general, a left comodule algebra $\mathcal{A}$ under $\mathcal{H}$ is also a right $\mathcal{H}'$-module algebra, using the dual pairing between $\mathcal{H}$ and its dual $\mathcal{H}'$. Explicitly, the right[b] action $\triangleleft : \mathcal{A} \otimes \mathcal{H}' \to \mathcal{A}$ of $\mathcal{H}'$ on $\mathcal{A}$ is given by

$$f \triangleleft X := (\langle X, \cdot \rangle \otimes \mathrm{id}) \circ \rho(f) = \langle X, f_{(-1)} \rangle f_{(0)} \,, \quad X \in \mathcal{H}' \,, \quad f \in \mathcal{A} \,. \tag{11}$$

Applied to the present situation using the coaction (10) and the dual pairing (6), we obtain an action of $U_q(e(2))$ on $\mathbb{R}^2_q$. It is compatible with the conjugation $z^* = \bar{z}$ in the sense

$$(f \triangleleft X)^* = f^* \triangleleft S^{-1}(X^*) \tag{12}$$

for any $f \in \mathbb{R}^2_q$ and $X \in U_q(e(2))$. To calculate the action of $U_q(e(2))$ on formal power series in z, $\bar{z}$, it is useful to note that any formal power series $f(z, \bar{z})$ can be written as

$$f(z, \bar{z}) = \sum_{m \in \mathbb{Z}} z^m f_m(z\bar{z}) \,. \tag{13}$$

[b]Similarly one gets a left action via a dual pairing from a right coaction.

The action on terms of this form is calculated in App. A:

$$\begin{aligned}
z^k f(z\bar{z}) \triangleleft T &= \frac{z^{k-1}}{1-q^{-2}} \left(f(q^2 z\bar{z}) - q^{-2k} f(z\bar{z})\right),\\
z^k f(z\bar{z}) \triangleleft \bar{T} &= \frac{q^4}{1-q^2} z^{k+1} \frac{f(z\bar{z}) - f(q^{-2}z\bar{z})}{z\bar{z}},\\
z^k f(z\bar{z}) \triangleleft J &= i^k z^k f(z\bar{z}),
\end{aligned} \tag{14}$$

which has again the above form.

2.3. *Covariant differential calculus on* $\mathbb{R}^2_q$

A differential calculus is useful to write down Lagrangians. A covariant differential calculus over $\mathbb{R}^2_q$ is a graded bimodule $\Omega^*_q = \oplus_n \Omega^n_q$ over $\mathbb{R}^2_q$ which is a $U_q(e(2))$-module algebra, together with an exterior derivative d which satisfies $d^2 = 0$ and the usual graded Leibniz rule. Its construction[23,6] is reviewed here for convenience, in order to establish the notation. We start by introducing variables dz and $d\bar{z}$, which are the q-differentials of z and $\bar{z}$. These are noncommutative differentials which do not commute with the space coordinates z, $\bar{z}$. Covariance and $d(1) = 0$ implies the coaction

$$\begin{aligned}
\rho(dz) &= e^{i\theta} \otimes dz,\\
\rho(d\bar{z}) &= e^{-i\theta} \otimes d\bar{z},
\end{aligned} \tag{15}$$

and the commutation relations between coordinates and their differentials must be

$$\begin{aligned}
z dz &= q^{-2} dz\, z, & \bar{z}\, dz &= q^{-2} dz\, \bar{z},\\
z\, d\bar{z} &= q^2\, d\bar{z}\, z, & \bar{z}\, d\bar{z} &= q^2\, d\bar{z}\, \bar{z}.
\end{aligned} \tag{16}$$

To see that $d : \mathbb{R}^2_q \to \Omega^1_q$ is well-defined, we have to verify that it respects the commutation relations of the algebra, i.e.

$$d(z\bar{z} - q^2 \bar{z} z) \stackrel{!}{=} 0, \tag{17}$$

which is easy to see. To obtain a higher order differential calculus, we apply d on the commutation relations (16), which gives

$$dz\, d\bar{z} = -q^2\, d\bar{z}\, dz \tag{18}$$

and

$$(dz)^2 = (d\bar{z})^2 = 0.$$

This defines a star-calculus (i.e. with a reality structure), where the star of forms and derivatives is defined in the obvious way. One can now introduce *q-deformed partial derivatives* by

$$d =: dz^i\, \partial_i = dz\, \partial_z + d\bar{z}\, \partial_{\bar{z}}, \tag{19}$$

as in the commutative case. This defines the action of ∂_z and $\partial_{\bar{z}}$ on functions. One can also introduce the algebra of *differential operators* with generators ∂_z, $\partial_{\bar{z}}$, z, $\bar{z}$. In order to distinguish the generators $\partial_z, \partial_{\bar{z}}$ in this algebra from their action on a function, we denote the latter by

$$\partial_z(f) \quad \text{and} \quad \partial_{\bar{z}}(f)\,,$$

whereas we will not use brackets if ∂_z, $\partial_{\bar{z}}$ are interpreted as part of the algebra of differential operators.

The derivatives ∂_z, $\partial_{\bar{z}}$ satisfy a modified Leibniz rule. It can be derived from the Leibniz rule of the exterior differential together with the commutation relations of differentials and coordinates as follows: On the one hand, we have

$$\begin{aligned} d(fg) &= (df)g + f(dg) \\ &= dz^i\,\partial_i(f)g + f\,dz^i\,\partial_i(g) \\ &= dz^i\,\partial_i(f)g + dz\,f(q^{-2}z, q^{-2}\bar{z})\partial_z(g) \\ &\quad + d\bar{z}\,f(q^2 z, q^2\bar{z})\partial_{\bar{z}}(g) \end{aligned} \tag{20}$$

using the commutation relations

$$\begin{aligned} f(z,\bar{z})dz &= dz\,f(q^{-2}z, q^{-2}\bar{z})\,, \\ f(z,\bar{z})d\bar{z} &= d\bar{z}\,f(q^2 z, q^2\bar{z})\,, \end{aligned} \tag{21}$$

which follow from (16). On the other hand, we have

$$d(fg) = dz\,\partial_z(fg) + d\bar{z}\,\partial_{\bar{z}}(fg)\,,$$

and together with (20) we obtain the *q-Leibniz rule*

$$\partial_z(fg) = \partial_z(f)g + f(q^{-2}z, q^{-2}\bar{z})\partial_z(g)\,, \tag{22}$$

$$\partial_{\bar{z}}(fg) = \partial_{\bar{z}}(f)g + f(q^2 z, q^2\bar{z})\partial_{\bar{z}}(g)\,. \tag{23}$$

Applying this to the functions zf resp. $\bar{z}f$, one obtains the following commutation relations:

$$\begin{aligned} \partial_z z &= 1 + q^{-2}z\partial_z\,, & \partial_z\bar{z} &= q^{-2}\bar{z}\partial_z\,, \\ \partial_{\bar{z}} z &= q^2 z\partial_{\bar{z}}\,, & \partial_{\bar{z}}\bar{z} &= 1 + q^2\bar{z}\partial_{\bar{z}}\,. \end{aligned} \tag{24}$$

Furthermore, applying $\partial_z\partial_{\bar{z}}$ on the function $z\bar{z}$ we find

$$\partial_z\partial_{\bar{z}} = q^2\partial_{\bar{z}}\partial_z\,. \tag{25}$$

For completeness we also give the commutation relations for differentials and derivatives:

$$\begin{aligned} \partial_z\,dz &= q^2\,dz\,\partial_z\,, & \partial_z\,d\bar{z} &= q^{-2}\,d\bar{z}\,\partial_z\,, \\ \partial_{\bar{z}}\,dz &= q^2\,dz\,\partial_{\bar{z}}\,, & \partial_{\bar{z}}\,d\bar{z} &= q^{-2}\,d\bar{z}\,\partial_{\bar{z}}\,. \end{aligned} \tag{26}$$

Clearly, the q-differentials and q-derivatives become the classical differentials resp. derivatives in the limit $q \to 1$.

2.3.1. *The frame*

On many noncommutative spaces,[15,4] there exists a particularly convenient basis of one-forms (a "frame") $\theta^a \in \Omega^1$, which commute with all functions. They are easy to find here: consider the elements

$$\theta \equiv \theta^z := z^{-1}\bar{z}\,dz\,, \qquad \bar{\theta} \equiv \theta^{\bar{z}} := d\bar{z}\,z\bar{z}^{-1}\,. \tag{27}$$

Then the following holds:

Lemma 1.

$$[\theta, f] = [\bar{\theta}, f] = 0 \tag{28}$$

for all functions $f \in \mathbb{R}^2_q$, and

$$\theta\bar{\theta} = -q^2\bar{\theta}\theta\,. \tag{29}$$

Proof. Easy verification using the above commutation relations. □

It is even possible to find a one-form Θ which generates the exterior differential: consider the following "duals" of the frame,

$$\lambda_z := \frac{1}{1-q^{-2}}\bar{z}^{-1}\,, \tag{30}$$

$$\lambda_{\bar{z}} := -\frac{1}{1-q^{-2}}z^{-1} \tag{31}$$

and define

$$\Theta := \theta^i\lambda_i\,.$$

Then we have

Lemma 2. *The anti-Hermitian one-form $\Theta^* = -\Theta$ generates the exterior differential by*

$$df = [\Theta, f] = [\lambda_i, f]\theta^i \tag{32}$$

for all $f \in \mathbb{R}^2_q$. Similarly,

$$d\alpha = \{\Theta, \alpha\} \tag{33}$$

for any one-form α. Here $\{\cdot,\cdot\}$ denotes the anticommutator. Furthermore,

$$d\Theta = \Theta^2 = 0\,. \tag{34}$$

Proof. Equations (32) and (34) are shown in App. A.3. Equation (33) then follows easily noting that $\{\Theta, \alpha f\} = \{\Theta, \alpha\}f - \alpha[\Theta, f]$ and $\{\Theta, f\alpha\} = [\Theta, f]\alpha + f\{\Theta, \alpha\}$ for arbitrary functions f and one-forms α. □

2.4. *Invariant metric*

A relation between the algebra, the differential calculus and the geometry on noncommutative spaces was proposed in Ref. 15. We briefly address this issue here, arguing that $\mathbb{R}_q^2$ is flat. This can be seen as follows. According to Ref. 15, "local" line elements must have the form

$$ds^2 = \theta^i \otimes \theta^j g_{ij} \,, \tag{35}$$

where g_{ij} must be a central (i.e. numerical, here) tensor, and θ^i is the frame introduced above. The symmetry of g_{ij} is expressed in the equation

$$g_{ij} P^{(-)}{}^{ij}_{kl} = 0 \,, \tag{36}$$

where $P^{(-)}{}^{ij}_{kl}$ is the antisymmetrizer defined by the calculus

$$\theta^k \theta^l P^{(-)}{}^{ij}_{kl} = \theta^i \theta^j \,. \tag{37}$$

If we require furthermore that ds^2 be invariant under $E_q(2)$, it follows that

$$ds^2 = \theta \otimes \bar{\theta} + q^2 \bar{\theta} \otimes \theta = q^{-2}\, dz \otimes d\bar{z} + q^4\, d\bar{z} \otimes dz \,. \tag{38}$$

This is certainly a flat metric, and for $q \to 1$ reduces to the usual Euclidean metric on $\mathbb{R}^2$.

2.5. *Representations of* $\mathbb{R}_q^2$

In the following we will only need representations of the algebra $\mathbb{R}_q^2$, not including derivatives or forms. They are easy to find:[5] Since $r^2 = z\bar{z}$ is formally Hermitian, we assume that it can be diagonalized. The commutation relations then imply that z and $\bar{z}$ are rising resp. lowering operators which are invertible,

$$\begin{aligned} r^2 \, |n\rangle_{r_0} &= r_0^2 q^{2n} \, |n\rangle_{r_0} \,, \\ \bar{z} \, |n\rangle_{r_0} &= r_0 q^n \, |n+1\rangle_{r_0} \,, \\ z \, |n\rangle_{r_0} &= r_0 q^{n-1} \, |n-1\rangle_{r_0} \,. \end{aligned} \tag{39}$$

We will denote this irreducible representation with L_{r_0}, where r_0 can be either positive or negative. The representations with r_0 and $-r_0$ are equivalent. The irreducible representations are labeled by $r_0 \in [1, q)$. It follows that z^{-1} and $\bar{z}^{-1}$ are well-defined on L_{r_0} unless $r_0 = 0$.

3. Invariant Integration

3.1. *Integral of functions*

In order to define an invariant action, we need an integral on $\mathbb{R}_q^2$ which is invariant under $E_q(2)$. In general, an integral (i.e. a linear functional) is called invariant with respect to the right action of $U_q(e(2))$ if it satisfies the following invariance condition

$$\int^q f(z,\bar{z}) \triangleleft X = \varepsilon(X) \int^q f(z,\bar{z}) \tag{40}$$

for all $f \in \mathbb{R}^2_q$ and $X \in U_q(e(2))$. Here $\varepsilon(X)$ is the counit. Such an integral was found in Ref. 13; however, we want to determine the most general invariant integral here. Since ε is an algebra homomorphism, it is sufficient to check the condition (40) for the generators T, $\bar{T}$ and J. Let us first consider functions of the type

$$z^m f(z\bar{z}) \,, \tag{41}$$

where $f(r^2)$, $r^2 = z\bar{z}$ can be considered as a classical function in one variable. We can choose it such that the integral will be well defined. Invariance under the action (14) of J implies

$$\int^q z^m f(z\bar{z}) = \delta_{m,0} \langle f(r^2) \rangle_r \,, \tag{42}$$

where $\langle f(r^2)\rangle_r$ is a "radial" integral to be determined. Invariance under the action of T and $\bar{T}$ then leads to the following algebraic condition

$$\langle f(q^2 r^2) - q^{-2} f(r^2) \rangle_r = 0 \tag{43}$$

on the radial part of the integral. This condition is satisfied for

$$\langle f(r^2) \rangle_{r_0} := r_0^2 (q^2 - 1) \sum_{k=-\infty}^{\infty} q^{2k} f(q^{2k} r_0^2) \,, \tag{44}$$

for any $r_0 \in \mathbb{R}$. Notice that the integral can then be written as "quantum trace" (or Jackson-sum) over the irreducible representation L_{r_0} defined in (39):

$$\int^{q,(r_0)} f(z,\bar{z}) := (q^2 - 1)\, \mathrm{Tr}_{r_0}(r^2 f(z,\bar{z})) \,, \tag{45}$$

where Tr_{r_0} is the ordinary trace on L_{r_0}; note that $\mathrm{Tr}_{r_0}(z^m f(r^2)) = 0$ for $m \neq 0$. If we allow superpositions of this basic integral (resp. direct sums of irreps of $\mathbb{R}^2_q$), then we can take an arbitrary superposition of the form

$$\langle f(r^2) \rangle_r = \int_1^q dr_0\, \mu(r_0) \langle f(r^2) \rangle_{r_0} \tag{46}$$

with arbitrary (positive) "weight" function $\mu(r) > 0$. If $\mu(r)$ is a delta-function, this is simply the above Jackson-sum. For $\mu(r_0) = \frac{1}{r_0(q^2-1)}$, one obtains the classical radial integral

$$\int^q f(z,\bar{z}) = \int_1^q dr_0 \frac{1}{r_0(q^2-1)} \int^{q,(r_0)} f_0(z\bar{z}) = \int_0^\infty dr\, r f_0(r^2) \tag{47}$$

for $f(z,\bar{z}) = \sum_m z^m f_m(r^2)$, assuming $q > 1$. Any of these integrals reduces to the usual (Riemann) integral on $\mathbb{R}^2$ for $q \to 1$, using the obvious mapping from $\mathbb{R}^2_q$ to $\mathbb{R}^2$ induced by (13).

It is quite remarkable that the classical radial integral is indeed invariant, cf. Ref. 22. This will be useful in the star product approach in Sec. 6. Nevertheless, the invariant integrals are not cyclic in the ordinary sense:

Lemma 3. *For any invariant integral* (40) *the following cyclic property holds*:

(i) *For any functions f, g, we have*

$$\int^q fg = \int^q g\mathcal{D}(f)\,, \tag{48}$$

where $\mathcal{D}$ is the algebra homomorphism defined by

$$\mathcal{D}(z^m) := q^{-2m} z^m\,, \qquad \mathcal{D}(\bar{z}^m) := q^{2m}\bar{z}^m\,. \tag{49}$$

(ii) *$\mathcal{D}$ is an inner automorphism*:

$$\mathcal{D}(f(z,\bar{z})) = z\bar{z}f(z,\bar{z})\bar{z}^{-1}z^{-1}\,. \tag{50}$$

Proof. Easy verification using the commutation relation (8). □

A similar cyclic property for invariant integrals on a $\mathrm{SO}_q(N)$-covariant space was found in Ref. 22.

3.2. *Integral of forms*

Since any two-form $\alpha^{(2)} \in \Omega_q^2$ can be written as $\alpha^{(2)} = f\theta\bar{\theta}$ and $\theta\bar{\theta}$ is invariant, we define

$$\int^q \alpha^{(2)} = \int^q f\theta\bar{\theta} := \int^q f\,. \tag{51}$$

For one-forms α, β we then obtain the following cyclic property:

$$\int^q \alpha\beta = -\int^q \beta\mathcal{D}(\alpha)\,, \tag{52}$$

where $\mathcal{D}$ is defined on forms as above. Noting that $\mathcal{D}(\Theta) = \Theta$, this immediately yields Stokes theorem:

Theorem 1. *Let α be a one-form. Then*

$$\int^q d\alpha = 0\,. \tag{53}$$

Proof. Since $d\alpha = \{\Theta, \alpha\}$ due to (33), we get with (52)

$$\int^q d\alpha = \int^q \{\Theta, \alpha\} = 0\,.$$

□

4. Gauge Transformations, Field Strength and Action

We consider matter fields as functions in $\mathbb{R}^2_q$. An infinitesimal noncommutative gauge transformation of a matter field ψ is defined as[16]

$$\delta\psi = i\Lambda\psi \tag{54}$$

while of course $\delta z^i = 0$. We introduce the "covariant derivative" (or rather a covariant one-form)

$$D := \Theta - iA\,, \tag{55}$$

which should be an anti-Hermitian one-form. Requiring that $D\psi(x)$ transforms covariantly, i.e.

$$\delta D\psi = i\Lambda\, D\psi$$

leads to

$$\delta D = i[\Lambda, D]\,, \tag{56}$$

which using (32) implies the following gauge transformation property for the gauge field A

$$\delta A = [\Theta, \Lambda] + i[\Lambda, A] = d\Lambda + i[\Lambda, A]\,. \tag{57}$$

This suggests to define the noncommutative field strength F as

$$F := iD^2 = F_{ij}\theta^i\theta^j\,,$$

which is a two-form transforming as

$$\delta_\Lambda F = i[\Lambda, F]\,. \tag{58}$$

Since $\Theta^2 = 0$ and $\{\Theta, A\} = dA$ we obtain the familiar form

$$F = dA - iA^2\,, \tag{59}$$

which shows that F reduces to the classical field strength in the limit $q \to 1$. To write it in terms of components, it is most natural to expand the one-forms in the frame basis $\theta^i = (\theta, \bar\theta)$, because then no ordering prescription is needed. Hence we can write

$$A = A_i\theta^i = \theta^i A_i\,, \tag{60}$$

and the field strength is

$$\begin{aligned} F &= (\lambda_i A_j + A_i\lambda_j - iA_iA_j)\theta^i\theta^j \\ &= (\lambda_1 A_2 - q^{-2}A_2\lambda_1 - q^{-2}\lambda_2 A_1 \\ &\quad + A_1\lambda_2 - iA_1A_2 + iq^{-2}A_2A_1)\theta\bar\theta\,, \end{aligned} \tag{61}$$

where $\lambda_i = (\lambda_z, \lambda_{\bar{z}})$. Notice that this is written in terms of the components of the frame, not of the differentials $dz, d\bar{z}$. In order to understand its classical limit, it is better to write[c]

$$A = \tilde{A}_z \, dz + \tilde{A}_{\bar{z}} \, d\bar{z} \,, \tag{62}$$

and we recover from (59) the classical field strength

$$F \xrightarrow{q \to 1} (\partial_z \tilde{A}_{\bar{z}} - \partial_{\bar{z}} \tilde{A}_z) dz \, d\bar{z} \,. \tag{63}$$

In order to write down a Lagrangian for a Yang–Mills theory, we also need the *Hodge dual* $*_H F$ of F. This is easy to find: since any two-form F can be written as

$$F = f\theta\bar{\theta} = q^{-2} f \, dz \, d\bar{z}$$

for some function f, we define $*_H$ on two-forms as

$$*_H F := \frac{1}{2} f \,. \tag{64}$$

This is the correct definition because $dzd\bar{z}$ is invariant under $U_q(e(2))$ transformations, hence the Hodge dual satisfies

$$(*_H F) \triangleleft u = *_H (F \triangleleft u) \tag{65}$$

for all $u \in U_q(e(2))$. We can now write down the following action using one of the invariant integrals found in Subsec. 3.1:

$$S := \int^q F(*_H F) \bar{z}^{-1} z^{-1} = \int^q \frac{1}{2} f^2 \bar{z}^{-1} z^{-1} \theta\bar{\theta} \,. \tag{66}$$

The factor $\bar{z}^{-1} z^{-1}$ is required by gauge invariance under (57), using the property

$$\int^q f g \bar{z}^{-1} z^{-1} = \int^q g f \bar{z}^{-1} z^{-1} \,, \tag{67}$$

which follows from Lemma 3. In the classical limit we obtain

$$S \xrightarrow{q \to 1} \int \frac{1}{2} (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)^2 z^{-1} \bar{z}^{-1} \, dz \, d\bar{z} \,.$$

The "measure factor" $\bar{z}^{-1} z^{-1}$ breaks the $E_q(2)$-invariance explicitly. Unfortunately, it is required by gauge invariance. In other words, the invariant integral seems incompatible with this kind of gauge invariance, and one is faced with the choice of giving up either gauge invariance or $E_q(2)$-invariance.[d] In this paper, we will insist on gauge invariance.

There are several possibilities how this problem might be avoided. One may try to modify the gauge transformation, e.g. by using some kind of q-deformed gauge invariance as in Ref. 10. Unfortunately we were not able to find a satisfactory

[c] This is not natural for $q \neq 1$, since then $dz, d\bar{z}$ do not commute with functions.

[d] In the classical limit, the measure function can be written as $\bar{z}^{-1} z^{-1} \, dz \, d\bar{z} = \frac{1}{r^2}(r \, dr \, d\varphi) = d(\ln r) d\varphi$, which is the volume-form on a cylinder. Therefore this action could be interpreted as Yang–Mills action on a quantum cylinder. However this is not the aim of this paper.

prescription here.[17] Alternatively, we will propose in the next section a mechanism using spontaneous symmetry breaking, which yields an $E_q(2)$-invariant action for low energies. In any case, the above action is certainly appealing because of its simplicity, and the gauge transformations (56) are very natural. This problem may also be a hint that the quantum group space–time-symmetry has not been correctly implemented in the field theory, beyond a formal level. A proper treatment would presumably require a second quantization, such that the $E_q(2)$-symmetry acts on a many-particle Hilbert space and the quantum fields, as in Ref. 9.

Let us briefly discuss the critical points of the above action. The absolute minima are given by solutions of the zero curvature condition $F = 0$. In terms of the coordinates $D = D_i\theta^i$ this leads to

$$D_2 D_1 = q^2 D_1 D_2 \,.$$

This is the defining relation of the deformed Euclidean plane with *opposite* multiplication. One solution is of course $D = \Theta$, and we get all possible solutions in terms of the automorphisms of $\mathbb{R}^2_q$.

5. Restoring $E_q(2)$-Invariance Through Spontaneous Symmetry Breaking

The explicit "weight" factor $\bar{z}^{-1}z^{-1}$ in (66) is rather unwelcome, because it explicitly breaks the $E_q(2)$-invariance of the action, which was the starting point for our considerations. One could in principle interpret it as some kind of additional "metric" term in the action, which is required by gauge invariance. However, it is also possible to cancel it by the vacuum-expectation value (VEV) of a suitable scalar field: Consider the action

$$S_1 := \int^q F(*_H F)e^{\phi}\bar{z}^{-1}z^{-1} \,. \tag{68}$$

This is gauge invariant if ϕ transforms in the adjoint:

$$\phi \to i[\Lambda, \phi] \,. \tag{69}$$

We can then add an action for ϕ, such as

$$S_2 = \int^q V(\phi)\bar{z}^{-1}z^{-1} \,, \tag{70}$$

where $V(x)$ is an ordinary function, which is again gauge invariant. If we could find a potential $V(\phi)$ which has $e^{\phi} = z\bar{z}$ as solution, we would obtain the following "low-energy" action

$$S_1[A, \langle\phi\rangle] = \int^q F(*_H F) \tag{71}$$

replacing ϕ by its VEV $\langle\phi\rangle$. This is formally invariant under $E_q(2)$, while the gauge invariance is spontaneously broken rather than explicitly. To find such a potential V, consider the equation of motion

$$\delta S_2[\phi] = \int^q \delta\phi\, V'(\phi)\bar{z}^{-1}z^{-1} = 0 \tag{72}$$

using the cyclic property of the integral, where V' denotes the ordinary derivative of the power series $V(x)$. We therefore need a potential $V(x)$ such that $V'(\ln(z\bar{z})) = 0$. For a given irrep L_{r_0} labeled by r_0 as in (39), the eigenvalues of $z\bar{z}$ are $r_0^2 q^{2n} = e^{2n\ln(q)+2\ln(r_0)}$ for $n \in \mathbb{Z}$. Therefore

$$V'_{r_0}(2n\ln(q) + 2\ln(r_0)) = 0\,, \quad n \in \mathbb{Z}\,. \tag{73}$$

This certainly holds for $V'_{r_0}(x) \propto \sin\left(2\pi\frac{x-2\ln(r_0)}{2\ln(q)}\right)$, thus

$$V_{r_0}(x) = -V_0\cos\left(2\pi\frac{x-2\ln(r_0)}{2\ln(q)}\right) \tag{74}$$

is a possible potential. Hence we will use the representation L_{r_0}, and the quantum trace $\int^{q,(r_0)}$ on L_{r_0} as invariant integral for the action. Note furthermore that

$$\delta_\phi S_1 = 0 \tag{75}$$

for $F = 0$, therefore $e^\phi = z\bar{z}$, $F = 0$ is indeed a possible "vacuum" of the combined action

$$S = S_1 + S_2 = \int^{q,(r_0)} (F(*_H F)e^\phi + V(\phi))\bar{z}^{-1}z^{-1}\,. \tag{76}$$

Replacing $\phi \to \langle\phi\rangle = \ln(z\bar{z})$, it reduces to

$$\int^{q,(r_0)} F(*_H F) + \text{const}\,, \tag{77}$$

as desired. The fluctuations in ϕ are suppressed if V_0 is chosen large enough. Of course there are other solutions for ϕ, which would give a nontrivial "effective metric" $e^{\langle\phi\rangle}\bar{z}^{-1}z^{-1}$ in the action. This is somewhat reminiscent of the low-energy effective actions in string theory, where the dilaton enters in a similar way.

For reducible representations of $\mathbb{R}^2_q$ one could still find such potentials, but if we take continuous superpositions as in (47) in order to have the classical integral (as in the Seiberg–Witten approach below), this is no longer possible.

6. Star Product Approach

We now want to study gauge theory on $\mathbb{R}^2_q$ using the star product approach, which was developed in Refs. 1 and 16. We will denote classical variables on $\mathbb{R}^2$ by greek letters ζ, $\bar{\zeta}$ in this section, in order to distinguish them from the generators z, $\bar{z}$ of the algebra $\mathbb{R}^2_q$.

A star product corresponding to $\mathbb{R}_q^2$ is defined as the pull-back of the product in $\mathbb{R}_q^2$ via an invertible map

$$\rho : \mathbb{R}[[\zeta, \bar{\zeta}]][[h]] \to \mathbb{R}_q^2 \tag{78}$$

of vector spaces,

$$f \star g := \rho^{-1}(\rho(f)\rho(g)), \tag{79}$$

where

$$q = e^h. \tag{80}$$

For example, the star product corresponding to normal ordering in $\mathbb{R}_q^2$ (i.e. commuting all z to the left and all $\bar{z}$ to the right) reads[16]

$$f \star_n g = \mu \circ e^{-2h(\bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta)}(f \otimes g). \tag{81}$$

For our purpose the following star product will be more useful

$$\begin{aligned} f \star_q g &:= \mu \circ e^{h(\zeta\partial_\zeta \otimes \bar{\zeta}\partial_{\bar{\zeta}} - \bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta)}(f \otimes g) \\ &= fg + h\zeta\bar{\zeta}(\partial_\zeta f \partial_{\bar{\zeta}} g - \partial_{\bar{\zeta}} f \partial_\zeta g) + \mathcal{O}(h^2), \end{aligned} \tag{82}$$

because it is Hermitian, i.e. $\overline{f \star_q g} = \bar{g} \star_q \bar{f}$, and satisfies other nice properties as shown in Lemma 4 (see Eq. (88) below). The corresponding Poisson structure reads[e]

$$\theta^{ij} = -2i\zeta\bar{\zeta}\varepsilon^{ij}. \tag{83}$$

This star product is equivalent to the normal ordered one (81) via the equivalence transformation

$$T := e^{-h\zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}}}.$$

To see this, we first note that

$$\zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}} \circ \mu = \mu \circ (\zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}} \otimes \mathrm{id} + \mathrm{id} \otimes \zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}} + \zeta\partial_\zeta \otimes \bar{\zeta}\partial_{\bar{\zeta}} + \bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta).$$

This leads to

$$\begin{aligned} T(f \star_q g) &= e^{-h\zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}}} \circ \mu \circ e^{h(\zeta\partial_\zeta \otimes \bar{\zeta}\partial_{\bar{\zeta}} - \bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta)}(f \otimes g) \\ &= \mu \circ e^{-h(\zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}} \otimes \mathrm{id} + \mathrm{id} \otimes \zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}} + \zeta\partial_\zeta \otimes \bar{\zeta}\partial_{\bar{\zeta}} + \bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta)} \\ &\quad \circ e^{h(\zeta\partial_\zeta \otimes \bar{\zeta}\partial_{\bar{\zeta}} - \bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta)}(f \otimes g) \\ &= \mu \circ e^{-2h(\bar{\zeta}\partial_{\bar{\zeta}} \otimes \zeta\partial_\zeta)}\left(e^{-h\zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}}} f \otimes e^{-h\zeta\partial_\zeta\bar{\zeta}\partial_{\bar{\zeta}}} g\right) \\ &= T(f) \star_n T(g), \end{aligned}$$

hence T is indeed an equivalence transformation from $\star_n$ to $\star_q$. If we denote the normal ordering by ρ_n, this new star product can be obtained by $f \star_q g := \rho_q^{-1}(\rho_q(f)\rho_q(g))$ in terms of an "ordering prescription" ρ_q given by

$$\rho_q := \rho_n \circ T.$$

[e]The Poisson structure is given by $[f \overset{\star_q}{,} g] = ih\theta^{ij}\partial_i f \partial_j g + O(h^2)$.

For illustration we give the image of ρ_q of some simple polynomials:

$$\begin{aligned} \zeta^n &\mapsto z^n\,, \\ \bar\zeta^n &\mapsto \bar z^n\,, \\ (\zeta\bar\zeta)^n &\mapsto q^{-n}(z\bar z)^n\,. \end{aligned} \tag{84}$$

Moreover, $\star_q$ is compatible with J:

$$(f\star_q g)\triangleleft J = (f\triangleleft J)\star_q g + f\star_q (g\triangleleft J)\,, \tag{85}$$

where the action of J on $\mathbb{R}^2$ is the obvious one.

One can easily extend the star product formalism to include differential forms, which will be useful in Subsec. 6.3. We simply use the invertible map

$$\begin{aligned} \Omega^* &\to \Omega_q^*\,, \\ f = f(\zeta,\bar\zeta) &\mapsto \rho_q(f)\,, \\ \zeta^{-1}\bar\zeta d\zeta &\mapsto \theta\,, \\ \zeta\bar\zeta^{-1}d\bar\zeta &\mapsto \bar\theta \end{aligned} \tag{86}$$

(extended in the obvious way) from the differential forms on $\mathbb{R}^2$ to the calculus Ω_q^* defined in Subsec. 2.3, and define the "star-wedge" $\wedge_q$ on Ω^* as the pull-back of Ω_q^*. Using the same notation $\theta = \zeta^{-1}\bar\zeta\, d\zeta$, $\bar\theta = \zeta\bar\zeta^{-1}\, d\bar\zeta$ as in the noncommutative algebra, one has for example $\theta\wedge_q\bar\theta = -q^2\bar\theta\wedge_q\theta$ in Ω^*, as in Ω_q^*. Clearly $\theta\star_q f = f\star_q\theta$ in self-explanatory notation, and we will omit the star in this case from now on.

6.1. $E_q(2)$-*invariance of the Riemann integral*

Since there exists an integral on the commutative space, it is natural to use the isomorphism ρ (78) corresponding to the star product, and define

$$\int^\rho f(z,\bar z) := \int \rho^{-1}(f)(\zeta,\bar\zeta)d\zeta\, d\bar\zeta\,. \tag{87}$$

In general, one should not expect that the integral defined in this way is invariant under $E_q(2)$. Nevertheless, for the star product $\star_q$ defined by ρ_q, this integral is indeed invariant, i.e. (40) is satisfied. We want to explain this in detail. Consider

$$f(z,\bar z) = \sum_{n=-\infty}^{\infty} z^n f_n(z\bar z) \in \mathbb{R}_q^2\,.$$

Applying ρ_q^{-1} gives

$$\rho_q^{-1}(f) = \sum_{n=-\infty}^{\infty} \zeta^n \star_q \rho_q^{-1}(f_n(r^2))\,.$$

On the other hand, we can write the function $\rho_q^{-1}(f)$ in polar coordinates, and expand it in a Fourier series with r-dependent coefficients

$$\rho_q^{-1}(f) = \sum_{n=-\infty}^{\infty} e^{in\phi} a_n(r) \,.$$

Then

$$a_0(r) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \rho_q^{-1}(f)(\phi, r) \,.$$

Since $\zeta = re^{i\phi}$ and $\rho_q^{-1}(f_n(r^2)) = f_n(qr^2)$ is a function of r^2 by (84) and using the fact (85) that $\star_q$ is compatible with J, it follows that

$$a_0(r) = \rho_q^{-1}(f_0)(r^2) = f_0(qr^2) \,.$$

Therefore

$$\int \rho_q^{-1}(f)(\zeta, \bar{\zeta}) d\zeta \, d\bar{\zeta} = 2\pi \int dr \, r f_0(qr^2) \,.$$

This agrees essentially with (47), which is indeed invariant under $U_q(e(2))$ transformations as was shown there.

From now on, we will use the Riemann integral (87) in this context, and omit the superscript $\rho = \rho_q$ for brevity.

6.2. *Trace property and measure*

The Riemann integral does not possess the trace property, i.e. star multiplication is not commutative under the integral. However the trace property is necessary to obtain a gauge invariant action. We therefore look for a measure $\mu(\zeta, \bar{\zeta})$ such that

$$\int f(\zeta, \bar{\zeta}) \star_q g(\zeta, \bar{\zeta}) \mu(\zeta, \bar{\zeta}) d\zeta \, d\bar{\zeta} = \int g(\zeta, \bar{\zeta}) \star_q f(\zeta, \bar{\zeta}) \mu(\zeta, \bar{\zeta}) d\zeta \, d\bar{\zeta} \,.$$

Such a measure function can indeed be found.

Lemma 4. *Let f, g be two arbitrary functions which vanish sufficiently fast at infinity. Then*

$$\begin{aligned} \int f(\zeta, \bar{\zeta}) \star_q g(\zeta, \bar{\zeta}) \frac{1}{\zeta\bar{\zeta}} d\zeta \, d\bar{\zeta} &= \int g(\zeta, \bar{\zeta}) \star_q f(\zeta, \bar{\zeta}) \frac{1}{\zeta\bar{\zeta}} d\zeta \, d\bar{\zeta} \\ &= \int f(\zeta, \bar{\zeta}) g(\zeta, \bar{\zeta}) \frac{1}{\zeta\bar{\zeta}} d\zeta \, d\bar{\zeta} \,. \end{aligned} \tag{88}$$

Proof. See App. A.4. □

Equation (88) has also an analog on the canonical quantum plane $\mathbb{R}^2_\theta$, see e.g. Ref. 8.

A small puzzle arises here: since the Riemannian integral is invariant under $E_q(2)$ as we argued above, we also have the following cyclic property

$$\int f(\zeta,\bar\zeta) \star_q g(\zeta,\bar\zeta) \star_q \bar\zeta^{-1} \star_q \zeta^{-1}\, d\zeta\, d\bar\zeta = \int g(\zeta,\bar\zeta) \star_q f(\zeta,\bar\zeta) \star_q \bar\zeta^{-1} \star_q \zeta^{-1}\, d\zeta\, d\bar\zeta \tag{89}$$

because of Lemma 3. These two cyclic properties are in fact equivalent, because

$$\int G(\zeta,\bar\zeta) \star_q (\bar\zeta^{-1} \star_q \zeta^{-1}) d\zeta\, d\bar\zeta = q^{-1} \int G(\zeta,\bar\zeta) \frac{1}{\zeta\bar\zeta} d\zeta\, d\bar\zeta\,. \tag{90}$$

To see this, note that the second equality in (88) implies

$$\int G(\zeta,\bar\zeta) \star_q \bar\zeta^{-1} \star_q \zeta^{-1}\, d\zeta\, d\bar\zeta = \int ((G(\zeta,\bar\zeta) \star_q \bar\zeta^{-1} \star_q \zeta^{-1}) \star_q \zeta\bar\zeta)\, \frac{1}{\zeta\bar\zeta}\, d\zeta\, d\bar\zeta\,. \tag{91}$$

With $\zeta\bar\zeta = q^{-1}\zeta \star_q \bar\zeta$ which is easy to verify, it follows that

$$\int G(\zeta,\bar\zeta) \star_q \bar\zeta^{-1} \star_q \zeta^{-1}\, d\zeta\, d\bar\zeta = q^{-1} \int G(\zeta,\bar\zeta)\, \frac{1}{\zeta\bar\zeta}\, d\zeta\, d\bar\zeta \tag{92}$$

using the associativity of the star product. This shows the equivalence of the cyclic properties (88) and (89).

6.3. *Seiberg–Witten map*

The map ρ_q defines a one-to-one correspondence between noncommutative and commutative functions, and we can identify f with $\rho_q^{-1}(f)$. We construct a Seiberg–Witten map for the noncommutative fields expressing them by their commutative counterparts:[21]

$$\begin{aligned} \Lambda &= \Lambda_\alpha[a_i]\,, \\ A_i &= A_i[a_i]\,, \\ \Psi &= \Psi[\psi, a_i]\,. \end{aligned}$$

Here a_i is the classical gauge field, α the classical gauge parameter and ψ a classical matter field. The noncommutative gauge transformations are defined as in Sec. 4 and will be spelled out below. We assume that it is possible to expand in orders of h

$$\begin{aligned} \Lambda_\alpha[a_i] &= \alpha + h\Lambda^1_\alpha[a_i] + h^2\Lambda^2_\alpha[a_i] + \cdots\,, \\ A_i[a_i] &= A^0_i + hA^1_i[a_i] + h^2 A^2_i[a_i] + \cdots\,, \\ \Psi[\psi, a_i] &= \psi + h\Psi^1[\psi, a_i] + h^2\Psi^2[\psi, a_i] + \cdots\,. \end{aligned} \tag{93}$$

The explicit dependence on the commutative fields can be obtained by requiring the following *consistency condition*[16]

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\Psi = \delta_{-i[\alpha,\beta]}\Psi \Leftrightarrow i\delta_\alpha\Lambda_\beta - i\delta_\beta\Lambda_\alpha + [\Lambda_\alpha \stackrel{\star_q}{,} \Lambda_\beta] = i\Lambda_{-i[\alpha,\beta]}\,, \tag{94}$$

which amounts to requiring that the noncommutative gauge transformations are induced by the commutative gauge transformations of the commutative fields:

$$\begin{aligned} A_i[a_i] + \delta_\Lambda A_i[a_i] &= A_i[a_i + \delta_\alpha a_i]\,, \\ \Psi[\psi, a_i] + \delta_\Lambda \Psi[\psi, a_i] &= \Psi[\psi + \delta_\alpha \psi, a_i + \delta_\alpha a_i]\,. \end{aligned} \tag{95}$$

The consistency condition has the well-known solution[12]

$$\Lambda_\alpha[a_i] = \alpha + h\frac{1}{2}\theta^{ij}\partial_i \alpha a_j + \mathcal{O}(h^2)\,. \tag{96}$$

This solution is Hermitian for real gauge parameters α and for gauge fields a_i corresponding to the Hermitian connection form $a = a_\zeta\, d\zeta + a_{\bar\zeta}\, d\bar\zeta$. As usual, this solution is not unique. Solutions to the homogeneous part of the corresponding Seiberg–Witten equation may be added leading to field redefinitions.[11]

The crucial point of our approach is that we will essentially work with one-forms and their components A_i w.r.t the frame $\theta^i = (\theta, \bar\theta)$,

$$A = A_i \theta^i = \theta^i A_i = \tilde{A}_i\, dz^i\,, \tag{97}$$

and that we are gauging the one-form Θ as in Sec. 4. In this way we naturally obtain a noncommutative gauge field and field strength, with the correct classical limit. This is not the case if one introduces covariant coordinates to define gauge fields and field strengths,[12,17] because θ^{ij} is not constant here. Using $[\Theta, f] = df = [\lambda_i, f]\theta^i$, this led to the gauge transformation law in the noncommutative algebra

$$\delta_\Lambda A_i = [\lambda_i, \Lambda] + i[\Lambda, A_i]\,, \tag{98}$$

where

$$\lambda_z = \frac{1}{1-q^{-2}}\bar z^{-1} \quad \text{and} \quad \lambda_{\bar z} = \frac{-1}{1-q^{-2}} z^{-1}\,.$$

Since the commutator with λ_i satisfies the usual Leibniz rule we do not have to introduce a "vielbein" field that transforms under gauge transformations as in Ref. 19.

In order to translate the above gauge transformation law to the star product approach, we simply have to apply ρ_q^{-1}. This leads to

$$\delta_\Lambda A_i = [\lambda_i \stackrel{\star_q}{,} \Lambda] + i[\Lambda \stackrel{\star_q}{,} A_i]\,, \tag{99}$$

where we note that

$$\rho_q^{-1}(z_i^{-1}) = \zeta_i^{-1}\,.$$

Furthermore, we remark that

$$\frac{1}{1-q^{-2}} = \frac{1}{2h}(1 + h + \mathcal{O}(h^2))$$

such that to zeroth order we have for the gauge field

$$\delta_\alpha A_1^0 = \zeta\bar\zeta^{-1}\partial_\zeta \alpha\,. \tag{100}$$

An analogous calculation for A_2^0 leads to the solution

$$A_i^0 = c_i a_i \,, \tag{101}$$

where

$$c_\zeta = \zeta\bar{\zeta}^{-1} \quad \text{and} \quad c_{\bar{\zeta}} = \zeta^{-1}\bar{\zeta} \,. \tag{102}$$

This is the solution for the gauge field, written in the basis $(\theta, \bar{\theta}) = (c_\zeta^{-1} d\zeta, c_{\bar{\zeta}}^{-1} d\bar{\zeta})$ of one-forms (cf. (27)). To obtain the components in the more familiar basis $(d\zeta, d\bar{\zeta})$ we have to multiply the above solution by c_i^{-1}, and we indeed obtain the classical gauge field a_i in zeroth order:

$$\tilde{A}_i^0 = a_i \,. \tag{103}$$

Defining $c_i =: \frac{1}{1-q^{-2}} l_i$, i.e. $l_\zeta := \bar{\zeta}^{-1}$ and $l_{\bar{\zeta}} := -\zeta^{-1}$, we obtain to first order the equation

$$\delta_\alpha A_i^1 = \frac{1}{2}\theta^{kl}\partial_k l_i \partial_l \Lambda_\alpha^1 - \theta^{kl}\partial_k \alpha \partial_l (c_i a_i) + \frac{1}{2}\,\theta^{kl}\partial_k l_i \partial_l \alpha \,, \tag{104}$$

which admits the solution

$$A_i^1 = c_i \left(\frac{-1}{2}\theta^{kl} a_k (\partial_l a_i + F_{li}^0) \right) - \frac{1}{2}\,\theta^{kl} a_k \partial_l (c_i) a_i + c_i a_i \,, \tag{105}$$

where

$$F_{ij}^0 := \partial_i a_j - \partial_j a_i$$

is the usual, commutative field strength. This solution satisfies $\overline{A_1^1} = A_2^1$, $\overline{A_2^1} = A_1^1$.

We now define the noncommutative field strength as in Sec. 4,

$$F = (\lambda_i \star_q A_j + A_i \star_q \lambda_j - i A_i \star_q A_j)\theta^i \wedge_q \theta^j = f\theta \wedge_q \bar{\theta} \tag{106}$$

using the "star-calculus" defined by (86), because it satisfies the correct transformation law

$$\delta f = i[\Lambda \overset{\star_q}{,} f] \,. \tag{107}$$

The above solution then leads to

$$\begin{aligned} f = F_{12}^0 + h\{ & F_{12}^0 + \theta^{12}(F_{12}^0 F_{12}^0 - a_\zeta \partial_{\bar{\zeta}} F_{12}^0 + a_{\bar{\zeta}} \partial_\zeta F_{12}^0) \\ & + \partial_\zeta \theta^{12}(a_\zeta \partial_{\bar{\zeta}} a_{\bar{\zeta}} + a_{\bar{\zeta}} \partial_{\bar{\zeta}} a_\zeta + 2 a_{\bar{\zeta}} \partial_\zeta a_{\bar{\zeta}}) \\ & + \partial_{\bar{\zeta}} \theta^{12}(a_\zeta \partial_\zeta a_{\bar{\zeta}} + a_{\bar{\zeta}} \partial_\zeta a_\zeta + 2 a_\zeta \partial_{\bar{\zeta}} a_\zeta)\} + \mathcal{O}(h^2) \,. \end{aligned} \tag{108}$$

We can now write down the following action using the classical integral:

$$S := \frac{1}{2} \int f \star_q f \, \frac{1}{\zeta\bar{\zeta}} \, d\zeta \, d\bar{\zeta} \,. \tag{109}$$

Recall that the measure function $\mu(\zeta,\bar\zeta)=\frac{1}{\zeta\bar\zeta}$ is necessary to ensure gauge invariance of the action, using the trace property of the integral by Lemma 4. This action can be written in terms of commutative fields using the above result:

$$\begin{aligned} S = \int d\zeta\, d\bar\zeta \frac{1}{\zeta\bar\zeta}\bigg\{ & \frac{1}{2}F^0_{12}F^0_{12} + h\big(F^0_{12}F^0_{12} \\ & + \theta^{12}(F^0_{12}F^0_{12}F^0_{12} - a_\zeta F^0_{12}\partial_{\bar\zeta}F^0_{12} + a_{\bar\zeta}F^0_{12}\partial_\zeta F^0_{12}) \\ & + \partial_{\bar\zeta}\theta^{12}F^0_{12}(2a_\zeta\partial_{\bar\zeta}a_\zeta + a_\zeta\partial_\zeta a_{\bar\zeta} + a_{\bar\zeta}\partial_\zeta a_\zeta) \\ & + \partial_\zeta\theta^{12}F^0_{12}(2a_{\bar\zeta}\partial_\zeta a_{\bar\zeta} + a_{\bar\zeta}\partial_{\bar\zeta}a_\zeta + a_\zeta\partial_{\bar\zeta}a_{\bar\zeta})\big)\bigg\} + \mathcal{O}(h^2)\,. \end{aligned} \tag{110}$$

Observe that this action is also the Seiberg–Witten form of (66), because

$$S = \frac{1}{2}\int f\star_q f\frac{1}{\zeta\bar\zeta}d\zeta\, d\bar\zeta = \frac{q}{2}\int f\star_q f\star_q(\bar\zeta^{-1}\star_q\zeta^{-1})d\zeta\, d\bar\zeta \tag{111}$$

using (90). We see that as in the algebraic approach of Sec. 4, gauge invariance requires a measure function $\mu(\zeta,\bar\zeta)=\frac{1}{\zeta\bar\zeta}d\zeta\, d\bar\zeta$ which breaks translation invariance. However, one should realize that even without this measure function, this "classical" action would not be invariant under $E(2)$, because the star product is not compatible with the symmetry (only for rotations (85) holds). This would only be the case if one could find a star product on $\mathbb{R}^2_q$ which is compatible with the coproduct of $E_q(2)$, cf. Refs. 2 and 9.

6.4. *The classical limit and the measure function*

The measure function $\mu(\zeta,\bar\zeta)=\frac{1}{\zeta\bar\zeta}d\zeta\, d\bar\zeta$ survives in the classical limit $q\to 1$. If we want a deformation of the classical theory, this should not be the case. We therefore would like to get rid of this measure function in the classical limit. This can be achieved by multiplying the action with a gauge-covariant expression,[f] which in the classical limit exactly cancels the measure function μ. For this purpose we introduce covariant coordinates:[16]

$$\mathcal{Z}_i := \zeta_i + \mathcal{A}_i\,. \tag{112}$$

Here $\mathcal{A}_i$ should not be confused with A_i. The one-form $A_i\theta^i$ is a noncommutative analog of the classical gauge field, because its gauge transformation law (99) is the noncommutative generalization of the classical gauge transformation law. Indeed, we recovered the classical gauge field a_i with respect to the basis $d\zeta$, $d\bar\zeta$ (103) in zeroth order of h. In contrast, the covariant coordinates are used here just as a quantity which transforms covariantly and reduces to the usual coordinates in the classical limit, in order to cancel the measure function. We will see that $\mathcal{A}_i$ does not

[f] This was suggested by Peter Schupp.

reduce to the classical gauge field for $q \to 1$. Requiring the covariant transformation rule $\delta \mathcal{Z}_i = i[\Lambda \stackrel{\star_q}{,} \mathcal{Z}_i]$ leads to the following gauge transformation rule for $\mathcal{A}_i$

$$\delta \mathcal{A}_i = i[\zeta_i \stackrel{\star_q}{,} \Lambda] + i[\Lambda \stackrel{\star_q}{,} \mathcal{A}_i] \,. \tag{113}$$

As before we can express $\mathcal{A}_i$ in terms of commutative fields by solving the corresponding Seiberg–Witten equations. This gives[12]

$$\mathcal{A}^i = h\theta^{ij} a_j + h^2 \frac{1}{2}\theta^{kl} a_l (\partial_k(\theta^{ij} a_j) - \theta^{ij} F^0_{jk}) + \mathcal{O}(h^3) \,. \tag{114}$$

In principle, covariant coordinates may be used to define noncommutative gauge fields and covariant expressions such as field strength.[16,17] However, the above equation shows that gauge fields and field strengths defined in that way do not lead to the classical gauge field a_i and field strength F^0_{ij} in the limit $h \to 0$ whenever the Poisson-structure is not constant and not invertible, as is the case here.[g] Nevertheless they are a convenient tool for our purpose, because they satisfy

$$\mathcal{Z} \star_q \bar{\mathcal{Z}} \to \zeta\bar{\zeta} \tag{115}$$

for $q \to 1$, and

$$\delta(\mathcal{Z} \star_q \bar{\mathcal{Z}}) = i[\Lambda \stackrel{\star_q}{,} \mathcal{Z} \star_q \bar{\mathcal{Z}}] \,. \tag{116}$$

Now we can define a gauge-invariant action with the correct classical limit:

$$S' := \frac{1}{2} \int f \star_q f \star_q \mathcal{Z} \star_q \bar{\mathcal{Z}} \, \frac{1}{\zeta\bar{\zeta}} d\zeta \, d\bar{\zeta} \,. \tag{117}$$

Expanded up to first order of h we obtain

$$\begin{aligned} S' = \int d\zeta \, d\bar{\zeta} \, \frac{1}{2} \, F^0_{12} F^0_{12} + h\Big(& F^0_{12} F^0_{12} + \theta^{12} (F^0_{12} F^0_{12} F^0_{12} - a_\zeta F^0_{12} \partial_{\bar{\zeta}} F^0_{12} + a_{\bar{\zeta}} F^0_{12} \partial_\zeta F^0_{12}) \\ & + \partial_{\bar{\zeta}} \theta^{12} F^0_{12} (2 a_\zeta \partial_{\bar{\zeta}} a_\zeta + a_\zeta \partial_\zeta a_{\bar{\zeta}} + a_{\bar{\zeta}} \partial_\zeta a_\zeta) + \partial_\zeta \theta^{12} F^0_{12} (2 a_{\bar{\zeta}} \partial_\zeta a_{\bar{\zeta}} + a_{\bar{\zeta}} \partial_{\bar{\zeta}} a_\zeta + a_\zeta \partial_{\bar{\zeta}} a_{\bar{\zeta}}) \\ & + \frac{1}{\zeta\bar{\zeta}} \theta^{12} F^0_{12} F^0_{12} (\bar{\zeta} a_{\bar{\zeta}} - \zeta a_\zeta) - F^0_{12} F^0_{12} + \zeta \partial_\zeta (F^0_{12} F^0_{12}) - \bar{\zeta} \partial_{\bar{\zeta}} (F^0_{12} F^0_{12}) \Big) + \mathcal{O}(h^2) \,. \end{aligned} \tag{118}$$

This reduces indeed to a Yang–Mills theory in the classical limit. However, choosing $\mathcal{Z} \star_q \bar{\mathcal{Z}}$ is only one possibility to cancel $\frac{1}{\zeta\bar{\zeta}}$. There are other expressions which are gauge-covariant, and lead to the same classical limit. Our choice is motivated by simplicity.

[g] To obtain in the classical limit the classical gauge field a_i we have to invert θ^{ij} and write $\frac{1}{h}\theta^{-1}_{ij}\mathcal{A}^j$. This is only defined if θ is invertible, and even then it spoils the covariant transformation property whenever θ is not constant. To maintain covariance one has to "invert θ covariantly" as done in Ref. 17, leading to complicated expressions. The approach that we propose in Subsec. 6.3 does not have these problems. Gauging the one-form Θ instead of the coordinates leads very naturally to a noncommutative gauge-field (99) and field strength (106). Compare also with Ref. 19, where a different approach using a "vielbein" is discussed.

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Appendix A. Mathematical Appendix

A.1. *Coaction and action*

Definition A.1. A left coaction of a Hopf algebra $\mathcal{H}$ on an algebra $\mathcal{A}$ is a linear mapping

$$\rho : \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}\,, \tag{A.1}$$

which satisfies

$$\begin{aligned} (\mathrm{id} \otimes \rho) \circ \rho &= (\Delta \otimes \mathrm{id}) \circ \rho\,, \quad & (\varepsilon \otimes \mathrm{id}) \circ \rho &= \mathrm{id}\,, \\ \rho(ab) &= \rho(a)\rho(b)\,, & \rho(1) &= 1 \otimes 1\,. \end{aligned} \tag{A.2}$$

In Sweedler notation, one writes

$$\rho(a) =: a_{(-1)} \otimes a_{(0)}\,.$$

$\mathcal{A}$ is then called a left $\mathcal{H}$-comodule algebra.

Definition A.2. A Hopf algebra $\mathcal{H}$ is acting on an algebra $\mathcal{A}$ from the right if $\mathcal{A}$ if there is an action $\triangleleft : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A}$ which satisfies

$$ab \triangleleft h = (a \otimes b) \triangleleft \Delta(h) = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)})\,, \quad \text{and} \quad 1 \triangleleft h = \varepsilon(h)1 \tag{A.3}$$

for any $h \in \mathcal{H}$ and $a,\, b \in \mathcal{A}$. $\mathcal{A}$ is then called a right $\mathcal{H}$-module algebra.

By (11), these two notions are dual to each other. There are obvious analogs replacing left with right everywhere.

For the action of J, T and $\bar{T}$ on the generators $z, \bar{z}$, we obtain

$$\begin{aligned} z \triangleleft T &= 1\,, \quad & z \triangleleft \bar{T} &= 0\,, \quad & z \triangleleft J &= iz\,, \\ \bar{z} \triangleleft T &= 0\,, & \bar{z} \triangleleft \bar{T} &= -q^2\,, & \bar{z} \triangleleft J &= -i\bar{z}\,. \end{aligned}$$

The action on arbitrary functions is calculated in the following subsection.

A.2. *The right action of $U_q(e(2))$ on $\mathbb{R}^2_q$*

Knowing the structure maps (5) for J, T, $\bar{T} \in U_q(e(2))$ and their action on z, $\bar{z}$ given above, we can determine the action of J, T, $\bar{T}$ on arbitrary functions using $(xy) \triangleleft U = (x \triangleleft U_{(1)})(y \triangleleft U_{(2)})$ for arbitrary x, $y \in \mathbb{R}^2_q$, $U \in U_q(e(2))$. Since an arbitrary function $f(z,\bar{z}) \in \mathbb{R}^2_q$ can be written as $f(z,\bar{z}) = \sum_{k\in\mathbb{Z}} z^k f_k(z\bar{z})$, it is sufficient to know the action on the terms

$$z^k f(z\bar{z})\,,$$

where f is a formal power series in $z\bar{z}$. We will derive the formulas even for negative powers of $z\bar{z}$, i.e. $f(z\bar{z}) = \sum_{l\in\mathbb{Z}} a_l (z\bar{z})^l$. We start with the action on z^k:

Claim A.1. *For $k \in \mathbb{Z}$ we have*

$$\begin{aligned} z^k \triangleleft T &= \frac{1-q^{-2k}}{1-q^{-2}} z^{k-1}\,, \\ z^k \triangleleft \bar{T} &= 0\,, \\ z^k \triangleleft J &= i^k z^k\,. \end{aligned} \tag{A.4}$$

Proof. The first equation can be shown by induction, using $z \triangleleft T = 1$ and $z^{-1} \triangleleft T = -q^2 z^{-2}$, which follows from

$$0 = 1 \triangleleft T = (z^{-1}z) \triangleleft T = (z^{-1} \triangleleft T)(z \triangleleft q^{2iJ}) + z^{-1}(z \triangleleft T) = (z^{-1} \triangleleft T) q^{-2} z + z^{-1}\,.$$

The last two equations finally follow immediately with $z \triangleleft \bar{T} = 0$, $z \triangleleft J = iz$ and $\Delta(\bar{T}) = \bar{T} \otimes q^{2iJ} + 1 \otimes \bar{T}$. □

The action on $f(z\bar{z}) = \sum_{l\in\mathbb{Z}} a_l (z\bar{z})^l$ follows from

Claim A.2. *For $l \in \mathbb{Z}$ we have*

$$\begin{aligned} (z\bar{z})^l \triangleleft T &= q^2 \frac{1-q^{-2l}}{1-q^{-2}} (z\bar{z})^{l-1} \bar{z}\,, \\ (z\bar{z})^l \triangleleft \bar{T} &= -q^2 \frac{1-q^{2l}}{1-q^{2}} (z\bar{z})^{l-1} z\,, \\ (z\bar{z})^l \triangleleft J &= (z\bar{z})^l\,. \end{aligned} \tag{A.5}$$

Proof. The last equation follows immediately with $z \triangleleft J = iz$, $\bar{z} \triangleleft J = -i\bar{z}$. The first equation follows again by induction, starting with $(z\bar{z}) \triangleleft T = (z \triangleleft T)(\bar{z} \triangleleft q^{2iJ}) + z(\bar{z} \triangleleft T) = q^2 \bar{z}$, and concluding inductively

$$\begin{aligned} (z\bar{z})^{l+1} \triangleleft T &= ((z\bar{z})^l \triangleleft T)((z\bar{z}) \triangleleft q^{2iJ}) + (z\bar{z})^l ((z\bar{z}) \triangleleft T) \\ &= q^2 \frac{1-q^{-2l}}{1-q^{-2}} (z\bar{z})^{l-1} \bar{z}(z\bar{z}) + (z\bar{z})^l q^2 \bar{z} \\ &= q^2 \frac{1-q^{-2l-2}}{1-q^{-2}} (z\bar{z})^l \bar{z} \end{aligned}$$

for $l > 0$. If $l = 0$, then $1 \triangleleft T = 0$, which is consistent with the claim. To derive the action of T on $(z\bar{z})^{-1}$ we calculate

$$\begin{aligned} 0 &= ((z\bar{z})^{-1}(z\bar{z})) \triangleleft T \\ &= ((z\bar{z})^{-1} \triangleleft T) z\bar{z} + (z\bar{z})^{-1}((z\bar{z}) \triangleleft T) \\ &= ((z\bar{z})^{-1} \triangleleft T) z\bar{z} + (z\bar{z})^{-1} q^2 \bar{z} \end{aligned}$$

hence

$$(z\bar{z})^{-1} \triangleleft T = -(z\bar{z})^{-2} \bar{z}\,,$$

consistent with (A.5). For $l < 0$ the claim follows similarly by induction, and the second equation follows also inductively. □

Putting these results together and using $f(z\bar{z}) = \sum_{l\in\mathbb{Z}} a_l (z\bar{z})^l$ we obtain

$$\begin{aligned} z^k f(z\bar{z}) \triangleleft T &= (z^k \triangleleft T)(f(z\bar{z}) \triangleleft q^{2iJ}) + z^k (f(z\bar{z}) \triangleleft T) \\ &= \frac{1-q^{-2k}}{1-q^{-2}} z^{k-1} f(z\bar{z}) + z^{k-1} \sum_{l\in\mathbb{Z}} a_l q^2 \, \frac{1-q^{-2l}}{1-q^{-2}} \, q^{2(l-1)} (z\bar{z})^l \\ &= \frac{z^{k-1}}{1-q^{-2}} (f(q^2 z\bar{z}) - q^{-2k} f(z\bar{z})) \,. \end{aligned}$$

A similar calculation finally leads to (14).

A.3. *Proof of Lemma* 2

Proof. Since the θ^i commute with all functions, we have

$$[\Theta, f] = \theta \left[\frac{1}{1-q^{-2}} \bar{z}^{-1}, f \right] - \bar{\theta} \left[\frac{1}{1-q^{-2}} z^{-1}, f \right].$$

Plugging in the explicit expressions (27) for θ^i we find

$$[\Theta, f] = dz\, z^{-1}\bar{z} \left[\frac{1}{1-q^{-2}} \bar{z}^{-1}, f \right] - d\bar{z}\, z\bar{z}^{-1} \left[\frac{1}{1-q^{-2}} z^{-1}, f \right],$$

using the commutation relations (16). Taking $f = z$ and $f = \bar{z}$ we get

$$z^{-1}\bar{z} \left[\frac{1}{1-q^{-2}} \bar{z}^{-1}, z \right] = \frac{1}{1-q^{-2}} - \frac{q^{-2}}{1-q^{-2}} = 1 \tag{A.6}$$

and

$$z\bar{z}^{-1} \left[\frac{1}{1-q^{-2}} z^{-1}, z \right] = 0 \,. \tag{A.7}$$

Thus $[\Theta, z] = dz$, and similarly $[\Theta, \bar{z}] = d\bar{z}$. Hence the claim is true on the generators of the algebra of functions, and since $[\Theta, \cdot]$ is a derivation we can conclude that

$$df = [\Theta, f]$$

for all functions f.

To show $d\Theta = \Theta^2 = 0$, consider

$$\begin{aligned} ((1-q^{-2})\Theta)^2 &= (\theta\bar{z}^{-1} - \bar{\theta} z^{-1})^2 = (q^{-2} z^{-1} dz - \bar{z}^{-1}\, d\bar{z})^2 \\ &= -q^{-2} z^{-1} dz\, \bar{z}^{-1}\, d\bar{z} - \bar{z}^{-1}\, d\bar{z}\, q^{-2} z^{-1}\, dz \\ &= -q^{-4} z^{-1} \bar{z}^{-1}\, dz\, d\bar{z} - \bar{z}^{-1} z^{-1}\, d\bar{z}\, dz = 0 \,, \end{aligned}$$

using the commutation relations (10), (16) and (18). Furthermore,

$$(1-q^{-2}) d\Theta = d(q^{-2} z^{-1}\, dz - \bar{z}^{-1}\, d\bar{z}) = -q^{-4} z^{-2}\, dz\, dz + q^2 \bar{z}^{-2}\, d\bar{z}\, d\bar{z} = 0 \,,$$

where we used $d(z^{-1}) = -q^{-2} z^{-2}\, dz$ and $d(\bar{z}^{-1}) = -q^2 \bar{z}^{-2}\, d\bar{z}$, which follows from the q-Leibniz rule applied to $0 = d1 = d(zz^{-1}) = d(\bar{z}\bar{z}^{-1})$. □

A.4. *Proof of Lemma* 4

Proof. We have

$$\int \frac{d\zeta\, d\bar{\zeta}}{\zeta\bar{\zeta}}\, f \star_q g = \int \frac{d\zeta\, d\bar{\zeta}}{\zeta\bar{\zeta}}\, fg + \int \frac{d\zeta\, d\bar{\zeta}}{\zeta\bar{\zeta}}\, \mu \circ \sum_{n=1}^{\infty} \frac{h^n}{n!}$$
$$\times \left(\sum_{i_1,j_1=1}^{2} \varepsilon^{i_1 j_1} \zeta^{i_1} \frac{\partial}{\partial \zeta^{i_1}} \otimes \zeta^{j_1} \frac{\partial}{\partial \zeta^{j_1}} \right)$$
$$\times \left(\sum_{i_2,j_2=1}^{2} \varepsilon^{i_2 j_2} \zeta^{i_2} \frac{\partial}{\partial \zeta^{i_2}} \otimes \zeta^{j_2} \frac{\partial}{\partial \zeta^{j_2}} \right) \cdots$$
$$\times \left(\sum_{i_n,j_n=1}^{2} \varepsilon^{i_n j_n} \zeta^{i_n} \frac{\partial}{\partial \zeta^{i_n}} \otimes \zeta^{j_n} \frac{\partial}{\partial \zeta^{j_n}} \right) (f \otimes g)\,.$$

Consider the nth term of the sum on the right-hand side:

$$\int \frac{d\zeta\, d\bar{\zeta}}{\zeta\bar{\zeta}} \frac{h^n}{n!} \mu \circ \left(\sum_{i_1,j_1=1}^{2} \varepsilon^{i_1 j_1} \zeta^{i_1} \frac{\partial}{\partial \zeta^{i_1}} \otimes \zeta^{j_1} \frac{\partial}{\partial \zeta^{j_1}} \right)$$
$$\times \left(\sum_{i_2,j_2=1}^{2} \varepsilon^{i_2 j_2} \zeta^{i_2} \frac{\partial}{\partial \zeta^{i_2}} \otimes \zeta^{j_2} \frac{\partial}{\partial \zeta^{j_2}} \right) \cdots \left(\sum_{i_n,j_n=1}^{2} \varepsilon^{i_n j_n} \zeta^{i_n} \frac{\partial}{\partial \zeta^{i_n}} \otimes \zeta^{j_n} \frac{\partial}{\partial \zeta^{j_n}} \right) (f \otimes g)\,.$$

Introducing the short hand notation

$$f' \otimes g' := \left(\sum_{i_2,j_2=1}^{2} \varepsilon^{i_2 j_2} \zeta^{i_2} \frac{\partial}{\partial \zeta^{i_2}} \otimes \zeta^{j_2} \frac{\partial}{\partial \zeta^{j_2}} \right) \cdots$$
$$\times \left(\sum_{i_n,j_n=1}^{2} \varepsilon^{i_n j_n} \zeta^{i_n} \frac{\partial}{\partial \zeta^{i_n}} \otimes \zeta^{j_n} \frac{\partial}{\partial \zeta^{j_n}} \right) (f \otimes g)\,,$$

the nth term of the sum can be written as

$$\int \frac{d\zeta\, d\bar{\zeta}}{\zeta\bar{\zeta}} \frac{h^n}{n!} \mu \circ \left(\sum_{i_1,j_1=1}^{2} \varepsilon^{i_1 j_1} \zeta^{i_1} \frac{\partial}{\partial \zeta^{i_1}} \otimes \zeta^{j_1} \frac{\partial}{\partial \zeta^{j_1}} \right) (f' \otimes g')$$
$$= \frac{h^n}{n!} \int d\zeta\, d\bar{\zeta} \sum_{i_1,j_1=1}^{2} \varepsilon^{i_1 j_1} \frac{\partial}{\partial \zeta^{i_1}} (f') \frac{\partial}{\partial \zeta^{j_1}} (g')\,.$$

For $n > 0$, this leads after partial integration (assuming that the functions vanish at infinity) to

$$-\frac{h^n}{n!} \int d\zeta\, d\bar{\zeta} \sum_{i_1,j_1=1}^{2} \varepsilon^{i_1 j_1} f' \frac{\partial}{\partial \zeta^{i_1}} \frac{\partial}{\partial \zeta^{j_1}} (g') = 0\,.$$

This is valid for any summand corresponding to $n > 0$, so that only the zeroth order term does not vanish. Hence we find indeed

$$\int \frac{d\zeta\, d\bar{\zeta}}{\zeta\bar{\zeta}}\, f \star_q g = \int \frac{d\zeta\, d\bar{\zeta}}{\zeta\bar{\zeta}}\, fg\,. \qquad \square$$

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5.2 Noncommutative Gauge Theory on the q-Deformed Euclidean Plane

by

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Noncommutative Gauge Theory on the q-deformed Euclidean Plane

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1 Introduction

The physical nature of space at very short distances is still not known. Already Heisenberg proposed in a letter to Peierls [1] that spacetime is quantized below some scale, suggesting that this could help to resolve the problem of infinities in quantum field theories. With this motivation, there has been a lot of work and progress in the formulation of quantum field theory on quantized or noncommutative spaces. Noncommutativity is implemented by replacing a differentiable space-time manifold by an algebra of noncommutative coordinates

$$[x^i, x^j] = \theta^{ij}(x) \neq 0\,. \tag{1}$$

The simplest case is the so-called canonical quantum plane $\mathbb{R}^n_\theta$, where θ^{ij} is a constant tensor independent of x. This is the space which is usually considered in the literature [2]. However, most of the rotational symmetry is lost on $\mathbb{R}^n_\theta$. On the other hand, there exist quantum spaces which admit a generalized notion of symmetry, being covariant under a quantum group. Not much is known about field theory on this type of spaces.

One of the simplest spaces with quantum group symmetry is the Euclidean quantum plane $\mathbb{R}^2_q$. It is covariant with respect to the q-deformed two-dimensional Euclidean group $E_q(2)$. We report here on our work [3], proposing a formulation of gauge theory based on the natural algebraic structures on this spaces and using a suitable star product.

2 The $E_q(2)$-Symmetric Plane

The $E_q(2)$-Symmetric Plane is generated by the complex coordinates $z, \bar{z}$ with the commutation relation

$$z\bar{z} = q^2 \bar{z} z\,. \tag{2}$$

We consider formal power series in these variables $z, \bar{z}$ as functions on this space,

$$\mathbb{R}^2_q := \mathbb{R}\langle\langle z, \bar{z}\rangle\rangle/(z\bar{z} - q^2\bar{z}z)\,. \tag{3}$$

Notice that the simple commutation relation (2) are inconsistent with the usual formulas for differentiation and integration. We should therefore first discuss the appropriate differential calculus and invariant integration. Finally, to get physical predictions i.e. real numbers from the abstract algebra, we also need either a representation of the noncommutative algebra, or a realization of the algebra using a star product.

2.1 Covariant Differential Calculus

It is natural to require that there exist deformed spaces of k-forms Ω_q^k which are covariant with respect to $E_q(2)$, and that the exterior differential $d : \Omega_q^k \to \Omega_q^{k+1}$ satisfies the usual Leibniz-rule as well as $d^2 = 0$. One can show that there exists a unique covariant differential calculus with these properties [4]:

$$\begin{aligned} z dz &= q^{-2} dz z, & \overline{z} dz &= q^{-2} dz \overline{z} \\ z d\overline{z} &= q^2 d\overline{z} z, & \overline{z} d\overline{z} &= q^2 d\overline{z}\, \overline{z} \end{aligned} \tag{4}$$

The following result is particularly useful for the construction of gauge field theories on $\mathbb{R}_q^2$:

Lemma 1. *Consider the one-forms*

$$\theta \equiv \theta^z := z^{-1} \overline{z} dz \quad , \quad \overline{\theta} \equiv \theta^{\overline{z}} := d\overline{z} z \overline{z}^{-1} \tag{5}$$

and define

$$\Theta := \theta^i \lambda_i, \quad \textit{where} \quad \lambda_z := \frac{1}{1-q^{-2}} \overline{z}^{-1}, \; \lambda_{\overline{z}} := -\frac{1}{1-q^{-2}} z^{-1}. \tag{6}$$

Then for all functions $f \in \mathbb{R}_q^2$ and one-forms α, the following holds:

$$[\theta, f] = [\overline{\theta}, f] = 0, \tag{7}$$

$$df = [\Theta, f] = [\lambda_i, f]\theta^i \tag{8}$$

$$d\alpha = \{\Theta, \alpha\} \tag{9}$$

denoting with $\{\cdot,\cdot\}$ the anti-commutator.

2.2 Invariant Integral

In order to define an invariant action, we need an integral on $\mathbb{R}_q^2$ which is invariant under quantum group transformations. This means that

$$\int^q f(z,\overline{z}) \triangleleft X = \varepsilon(X) \int^q f(z,\overline{z}) \tag{10}$$

for all $f \in \mathbb{R}_q^2$ and $X \in U_q(e(2))$. Here $U_q(e(2))$ is the q-deformed universal enveloping Lie algebra of the two-dimensional Euclidean group, $\triangleleft$ denotes the right action of $U_q(e(2))$ on $\mathbb{R}_q^2$ and $\varepsilon(X)$ is the counit. Now any function in $\mathbb{R}_q^2$ can be decomposed as $f(z,\overline{z}) = \sum_{m\in\mathbb{Z}} z^m f_m(z\overline{z})$. It can be shown that (10) is satisfied for the following discrete quantum traces [5]

$$\int^{q,(r_0)} f(z,\overline{z}) := r_0^2(q^2-1) \sum_{k=-\infty}^{\infty} q^{2k} f_0(q^{2k} r_0^2), \tag{11}$$

where $r_0 \in \mathbb{R}$ labels the irreducible representations of $\mathbb{R}_q^2$. The most general invariant integral is given by superpositions of these integrals,

$$\int^q f(z,\overline{z}) = \int_1^q dr_0 \mu(r_0) \int^{q,(r_0)} f_0(z\overline{z}) \tag{12}$$

with arbitrary "weight" function $\mu(r) > 0$. It is quite remarkable and useful that for the special choice $\mu(r_0) = \frac{1}{r_0(q^2-1)}$, one recovers the usual Riemannian integral, which is therefore also invariant under $U_q(e(2))$ [3].

3 Star Product Approach

3.1 The Star Product

The noncommutative algebra $\mathbb{R}_q^2$ can be realized on the algebra of commutative functions on $\mathbb{R}^2$ using a new, noncommutative product, called star product. Let us denote the commutative variables on $\mathbb{R}^2$ by greek letters $\zeta, \bar{\zeta}$ to distinguish them from the generators $z, \bar{z}$, and let $q =: e^h$. Then a hermitian star product for $\mathbb{R}_q^2$ is given by

$$f \star g := \mu \circ e^{h(\zeta\partial_\zeta \otimes \overline{\zeta}\partial_{\overline{\zeta}} - \overline{\zeta}\partial_{\overline{\zeta}} \otimes \zeta\partial_\zeta)}(f \otimes g) = fg + h\zeta\overline{\zeta}(\partial_\zeta f \partial_{\overline{\zeta}} g - \partial_{\overline{\zeta}} f \partial_\zeta g) + \mathcal{O}(h^2) \,. \tag{13}$$

3.2 Noncommutative Gauge Transformations

The formalism of covariant coordinates was established in [6] for an arbitrary Poisson structure. This leads to problems in the semi-classical limit[3]. Therefore we propose the following approach, taking advantage of the frame $\theta, \bar{\theta}$ which commutes with all functions and the generator Θ of the exterior differential. We define infinitesimal gauge transformations of a matter field as

$$\delta\psi = i\Lambda \star \psi, \qquad \delta\zeta^i = 0 \,. \tag{14}$$

Let us introduce the "covariant derivative" (or covariant one-form) as

$$D := \Theta - iA \,. \tag{15}$$

Then requiring that $D\psi$ transforms covariantly, i.e.

$$\delta D \star \psi \stackrel{!}{=} i\Lambda \star D \star \psi \tag{16}$$

leads to the following gauge transformation property for the gauge field A:

$$\delta A = [\Theta \stackrel{\star}{,} \Lambda] + i[\Lambda \stackrel{\star}{,} A] = d\Lambda + i[\Lambda \stackrel{\star}{,} A] \,. \tag{17}$$

We define the field strength as the two-form

[3] To obtain in the classical limit the classical gauge field a_i we have to invert θ^{ij}. This is only well-defined if θ is invertible, and even then it spoils the covariant transformation property whenever θ is not constant. To maintain covariance one has to "invert θ covariantly" as done in [7], leading to complicated expressions.

$$F := D \wedge_q D \tag{18}$$

where $\wedge_q$ the star-wedge [3]. Then

$$\delta F = i[\Lambda \stackrel{\star}{,} F]. \tag{19}$$

As a two-form, the field strength can be written as $F = f\theta \wedge_q \bar{\theta}$. Since the frame $\theta, \bar{\theta}$ commutes with functions, f transforms covariantly as well:

$$\delta f = i[\Lambda \stackrel{\star}{,} f]\,. \tag{20}$$

We note that all transformations have the correct classical limit as $q \to 1$.

3.3 Seiberg-Witten Map

The Seiberg-Witten map [8] allows to express the noncommutative gauge fields in terms of the commutative ones. Hence the noncommutative theory can be interpreted as a deformation of the commutative theory. Its physical predictions can be explicitly obtained by expanding in the deformation parameter h, and the commutative theory is reproduced in the limit $h \to 0$. The Seiberg-Witten map is based on the following requirement:

– The *consistency condition:*

$$\begin{aligned}(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\Psi &= \delta_{-i[\alpha,\beta]}\Psi \\ \Leftrightarrow i\delta_\alpha\Lambda_\beta - i\delta_\beta\Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] &= i\Lambda_{-i[\alpha,\beta]},\end{aligned} \tag{21}$$

– Noncommutative gauge transformations are related to commutative ones:

$$\begin{aligned}A_i[a_i] + \delta_\Lambda A_i[a_i] &= A_i[a_i + \delta_\alpha a_i] \\ \Psi[\psi, a_i] + \delta_\Lambda \Psi[\psi, a_i] &= \Psi[\psi + \delta_\alpha\psi, a_i + \delta_\alpha a_i]\,.\end{aligned} \tag{22}$$

Solving these conditions in our case, we obtain the following expression for the field strength expanded in powers of h:

$$\begin{aligned}f = F^0_{12} + h\big\{&F^0_{12} + \theta^{12}(F^0_{12}F^0_{12} - a_\zeta\partial_{\bar\zeta}F^0_{12} + a_{\bar\zeta}\partial_\zeta F^0_{12}) + \partial_\zeta\theta^{12}(a_\zeta\partial_{\bar\zeta}a_{\bar\zeta} \\ &+ a_{\bar\zeta}\partial_{\bar\zeta}a_\zeta + 2a_{\bar\zeta}\partial_\zeta a_{\bar\zeta}) + \partial_{\bar\zeta}\theta^{12}(a_\zeta\partial_\zeta a_{\bar\zeta} + a_{\bar\zeta}\partial_\zeta a_\zeta + 2a_\zeta\partial_{\bar\zeta}a_\zeta)\big\} + \mathcal{O}(h^2)\,,\end{aligned} \tag{23}$$

where a_{ζ_i} is the classical gauge field and $F^0_{ij} = \partial_{\zeta_i}a_j - \partial_{\zeta_j}a_i$ is the classical field strength.

3.4 The Action

To define a gauge-invariant action, we need an integral which is cyclic with respect to the star product, since the field strength transforms in the adjoint. The invariant integrals (11) do not have this property, which can be restored by introducing a measure function μ. It can be shown that for the measure function $\mu := \frac{1}{\zeta\bar\zeta}$, we can even drop the star under the integral:

$$\int d\zeta d\bar\zeta \mu f \star g = \int d\zeta d\bar\zeta \mu f g = \int d\zeta d\bar\zeta \mu g \star f \tag{24}$$

Putting all this together, we can now define a gauge invariant action:

$$S := \frac{1}{2} \int d\zeta d\bar{\zeta} \mu f \star f . \tag{25}$$

Gauge invariance is guaranteed because of (20). Moreover, the classical action for abelian gauge field theory is reproduced in the classical limit $h \to 0$ because of (23).

Another possibility is to replace the measure function μ by a scalar "Higgs" field φ, which transforms such that it restores the gauge invariance of the action. One can in fact find a suitable potential for ϕ which admits a solution $\langle \phi \rangle = \mu = \frac{1}{\bar{\zeta}\zeta}$, leading to a $E_q(2)$-invariant action through spontaneous symmetry breaking.

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5.3 Gauge Theories on the κ-Minkowski Spacetime

by

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Gauge theories on the κ-Minkowski spacetime

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Abstract. This study of gauge field theories on κ-deformed Minkowski spacetime extends previous work on field theories on this example of a non-commutative spacetime. We construct deformed gauge theories for arbitrary compact Lie groups using the concept of enveloping algebra-valued gauge transformations and the Seiberg–Witten formalism. Derivative-valued gauge fields lead to field strength tensors as the sum of curvature- and torsion-like terms. We construct the Lagrangians explicitly to first order in the deformation parameter. This is the first example of a gauge theory that possesses a deformed Lorentz covariance.

1 Introduction

The best known description of the fundamental forces of nature is given by gauge theories. Nevertheless intrinsic difficulties arise in these theories at very high energies or very short distances. Physics is not very well known in this limit. This has lead to the idea of modifying the structure of spacetime at very short distances and to introduce uncertainty relations for the coordinates to provide a natural cut-off (for reviews of this wide field see [1,2]). It is expected that gauge theories still play a vital role in this regime.

We expect especially interesting new features of gauge field theories formulated on spaces with a deformed spacetime symmetry. Here we concentrate on the κ-deformed Poincaré algebra (introduced in [3–5][1]). The spacetime which is covariant with respect to this deformed symmetry algebra is called the κ-deformed quantum space.

In a previous paper [6] deformed field theories on a κ-deformed quantum space have been constructed. The techniques necessary for such a construction have been thoroughly discussed there. In this paper we show how the deformation concept can be applied to a gauge field theory. We construct deformed gauge theories for arbitrary compact Lie groups. "Deformed" does not mean that the Lie groups will be deformed, however, the transformation parameters will depend on the elements of the κ-deformed coordinate space. This implies that Lie algebra gauge transformations are generalized to enveloping algebra-valued gauge transformations.

This is possible by making use of the Seiberg–Witten map [7–9]. This is a map that allows one to express all elements of the non-commutative gauge theory by their commutative analogs. It follows that a deformed gauge theory can be constructed with exactly the same number of fields (gauge fields, matter fields) as the standard gauge theory on undeformed space.

Of special interest is the interplay of the gauge transformations with the κ-deformed Lorentz transformations. Gauge theories are based on the concept of covariant derivatives constructed with gauge fields. Covariance now refers both to the gauge transformations and to the κ-Lorentz transformations.

Theories like the one presented here can be used to deform the standard model (compare with the approach in [10, 11]). For example, new coupling terms in the Lagrangian arise. This has experimental consequences and the model can be tested phenomenologically. We exhibit these terms for an arbitrary gauge group to first order in the deformation parameter. These models should be understood in such an expansion; it renders an infrared cutoff. We do not assume that these models should be used to describe physics at large distances.

To obtain phenomenologically interesting results we develop the theory on a space-like κ-deformed spacetime with Minkowski signature (in [6] κ-deformed Euclidean spacetime was discussed).

This paper is organized as follows: In the first section we present a compilation of all relevant formulae for κ-Minkowski spacetime. To derive and understand these formulae, [6] is essential. In the second section we investigate covariant derivatives for enveloping algebra-valued gauge transformations. Attention is given to the κ-Lorentz covariance as well as to gauge covariance. For this purpose the enveloping algebra-valued gauge formalism is devel-

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[1] For additional references concerning this model see [6].

oped and the transformation property of the gauge field is derived. This leads to the new concept of derivative-valued gauge fields. The field strength tensors, defined by commutators of covariant derivatives, are derivative dependent as well. We expand them in terms of covariant derivatives and show that each expansion coefficient is a tensor by itself; we call them torsion-like tensors. The derivative-independent term is a deformation of $F^0_{\mu\nu}$ and is used in the construction of Lagrangians.

In the third section we construct the Seiberg–Witten map. We use the $\star$-product formalism and expand in the deformation parameter. All gauge and matter fields of the deformed theory can be expressed in terms of the standard Lie algebra gauge fields and the standard matter fields. This allows for the construction of a Lagrangian in terms of the standard fields.

In the fourth section we discuss the interplay of κ-Lorentz and gauge transformations. We show that gauge transformations and κ-Lorentz transformations commute and that the κ-Lorentz transformed gauge transformation reproduces again the algebra. This can be implemented in a more abstract setting, but we discuss this issue explicitly in order to become familiar with the comultiplication rules and their consequences.

2 The κ-Minkowski space

In a previous paper we discussed the κ-Euclidean space [6] (introduced in [3,4]) and argued that the generalization to a Minkowski version is straightforward. We introduce here this space-like κ-deformed Minkowski spacetime, which is more interesting for phenomenological applications. First we present the relevant formulae.

Coordinate space

We start from the same algebra as in [6]:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu), \qquad \mu,\nu = 0,1,\ldots,n, \tag{1}$$

but the metric $\eta^{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$ and its inverse are used to raise and lower indices. Space-like deformation will be achieved by assuming a^μ to be space-like. The n-direction is rotated in the direction of a^μ, $a^n = a$, $a^j = 0$. We label the n commuting coordinates with $\hat{x}^i$ ($i = 0,\ldots,n-1$) as opposed to $\hat{x}^n$ and obtain the following commutation relations:

$$[\hat{x}^n, \hat{x}^j] = \mathrm{i}a\hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0, \quad i,j = 0,1,\ldots,n-1. \tag{2}$$

The parameter a is related to the frequently used parameter κ:

$$a = \frac{1}{\kappa}. \tag{3}$$

The κ-deformed Lorentz algebra

The formulae for the transformation of the spacetime coordinates are the same as for the Euclidean space, replacing $\delta^{\mu\nu}$ with $\eta^{\mu\nu}$ and paying attention to upper and lower indices:

$$\begin{aligned} [M^{ij}, \hat{x}^\mu] &= \eta^{\mu j}\hat{x}^i - \eta^{\mu i}\hat{x}^j, \\ [M^{in}, \hat{x}^\mu] &= \eta^{\mu n}\hat{x}^i - \eta^{\mu i}\hat{x}^n + \mathrm{i}aM^{i\mu}, \end{aligned} \tag{4}$$

$\mu = 0,1,\ldots,n$. These relations are consistent with the algebra (2) and they lead to the undeformed algebra relations of the generators $M^{\mu\nu}$ of the Lorentz algebra $so(1,n)$:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}. \tag{5}$$

As in [6], this is again the undeformed algebra [5]. However, the generators act in a deformed way on products of functions (i.e. they have a deformed coproduct)

$$\begin{aligned} M^{ij}(\hat{f}\cdot\hat{g}) &= (M^{ij}\hat{f})\cdot\hat{g} + \hat{f}\cdot(M^{ij}\hat{g}), \\ M^{in}(\hat{f}\cdot\hat{g}) &= (M^{in}\hat{f})\cdot\hat{g} + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(M^{in}\hat{g}) \\ &\quad + \mathrm{i}a(\hat{\partial}_k\hat{f})\cdot(M^{ik}\hat{g}). \end{aligned} \tag{6}$$

In this paper we adopt the convention that over double Latin indices should be summed from 0 to $n-1$ and over double Greek from 0 to n.

2.1 Derivatives

We introduce derivatives in such a way that they are consistent with (2):

$$\begin{aligned} [\hat{\partial}_n, \hat{x}^\mu] &= \eta_n{}^\mu, \\ [\hat{\partial}_i, \hat{x}^\mu] &= \eta_i{}^\mu - \mathrm{i}a\eta^{\mu n}\hat{\partial}_i. \end{aligned} \tag{7}$$

Derivatives naturally carry a lower index. It is possible to derive from (7) the Leibniz rule (i.e. the coproduct):

$$\begin{aligned} \hat{\partial}_n\,(\hat{f}\cdot\hat{g}) &= (\hat{\partial}_n\hat{f})\cdot\hat{g} + \hat{f}\cdot(\hat{\partial}_n\hat{g}), \\ \hat{\partial}_i\,(\hat{f}\cdot\hat{g}) &= (\hat{\partial}_i\hat{f})\cdot\hat{g} + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{\partial}_i\hat{g}). \end{aligned} \tag{8}$$

The derivatives are a κ-Lorentz algebra module:

$$\begin{aligned} [M^{ij}, \hat{\partial}_\mu] &= \eta^j{}_\mu\hat{\partial}^i - \eta^i{}_\mu\hat{\partial}^j, \\ [M^{in}, \hat{\partial}_n] &= \hat{\partial}^i, \\ [M^{in}, \hat{\partial}_j] &= \eta^i{}_j\,\frac{\mathrm{e}^{2\mathrm{i}a\hat{\partial}_n} - 1}{2\mathrm{i}a} - \frac{\mathrm{i}a}{2}\eta^i{}_j\hat{\partial}^l\hat{\partial}_l + \mathrm{i}a\hat{\partial}^i\hat{\partial}_j. \end{aligned} \tag{9}$$

The part of $M^{\mu\nu}$ that acts on the coordinates and derivatives can be expressed in terms of the coordinates and the derivatives:

$$\begin{aligned} \hat{M}^{ij} &= \hat{x}^i\hat{\partial}^j - \hat{x}^j\hat{\partial}^i, \\ \hat{M}^{in} &= \hat{x}^i\,\frac{1 - \mathrm{e}^{2\mathrm{i}a\hat{\partial}_n}}{2\mathrm{i}a} - \hat{x}^n\hat{\partial}^i + \frac{\mathrm{i}a}{2}\hat{x}^i\hat{\partial}^l\hat{\partial}_l. \end{aligned} \tag{10}$$

Dirac operator

The κ-deformed Dirac operator has the components:

$$\hat{D}_n = \frac{1}{a}\sin(a\hat{\partial}_n) - \frac{\mathrm{i}a}{2}\hat{\partial}^l\hat{\partial}_l\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n},$$
$$\hat{D}_i = \hat{\partial}_i\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}, \tag{11}$$
$$[M^{\mu\nu},\hat{D}_\rho] = \eta^\nu{}_\rho\hat{D}^\mu - \eta^\mu{}_\rho\hat{D}^\nu. \tag{12}$$

It can be seen as a derivative as well and satisfies the Leibniz rule:

$$\hat{D}_n(\hat{f}\cdot\hat{g}) = (\hat{D}_n\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{D}_n\hat{g})$$
$$-\mathrm{i}a(\hat{D}_i\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{D}^i\hat{g}), \tag{13}$$
$$\hat{D}_i\,(\hat{f}\cdot\hat{g}) = (\hat{D}_i\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + \hat{f}\cdot(\hat{D}_i\hat{g}).$$

That the Dirac operator really acts as a derivative follows from the commutation relations, when we expand the square root:

$$[\hat{D}_n,\hat{x}^j] = -\mathrm{i}a\hat{D}^j,$$
$$[\hat{D}_n,\hat{x}^n] = \sqrt{1+a^2\hat{D}^\mu\hat{D}_\mu},$$
$$[\hat{D}_i,\hat{x}^j] = \eta_i{}^j\left(-\mathrm{i}a\hat{D}_n + \sqrt{1+a^2\hat{D}^\mu\hat{D}_\mu}\right), \tag{14}$$
$$[\hat{D}_i,\hat{x}^n] = 0.$$

$\star$-product

In the $\star$-product formulation (explained in detail in [6]) all the elements of the coordinate algebra can be realized as functions of commuting variables. Derivatives and the κ-Lorentz algebra generators can be realized in terms of commuting variables and derivatives. On the $\star$-product of functions they act with their comultiplication rules without seeing the x and derivative dependence of the $\star$-product.

Here we present for convenience a compilation of the relevant formulae used in the rest of this paper.

The κ-Minkowski spacetime (2) can be considered as a Lie algebra with $C^{\mu\nu}_\lambda = a(\eta^\mu{}_n\eta^\nu{}_\lambda - \eta^\nu{}_n\eta^\mu{}_\lambda)$ as structure constants. These structure constants appear also in the expansion of the symmetric $\star$-product:

$$f\star g(x)$$
$$= \lim_{\substack{y\to x\\ z\to x}}\exp\Bigg(\frac{\mathrm{i}}{2}x^\lambda C^{\mu\nu}_\lambda\partial_\mu\otimes\partial_\nu$$
$$-\frac{a}{12}x^\lambda C^{\mu\nu}_\lambda(\partial_n\partial_\mu\otimes\partial_\nu - \partial_\mu\otimes\partial_n\partial_\nu)+\ldots\Bigg)$$
$$\times f(y)\otimes g(z)$$
$$= f(x)g(x) + \frac{\mathrm{i}a}{2}x^j(\partial_n f(x)\partial_j g(x) - \partial_j f(x)\partial_n g(x))$$
$$+\ldots. \tag{15}$$

The derivatives

$$\partial^*_n f(x) = \partial_n f(x),$$
$$\partial^*_i f(x) = \frac{\mathrm{e}^{\mathrm{i}a\partial_n}-1}{\mathrm{i}a\partial_n}\partial_i\, f(x) \tag{16}$$

have the Leibniz rule

$$\partial^*_n(f(x)\star g(x))$$
$$= (\partial^*_n f(x))\star g(x) + f(x)\star(\partial^*_n g(x)),$$
$$\partial^*_i(f(x)\star g(x)) \tag{17}$$
$$= (\partial^*_i f(x))\star g(x) + (\mathrm{e}^{\mathrm{i}a\partial^*_n}f(x))\star(\partial^*_i g(x)).$$

The Dirac operator

$$D^*_n f(x) = \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{\mathrm{i}a\partial_n^2}\partial_j\partial^j\right)f(x),$$
$$D^*_i f(x) = \frac{\mathrm{e}^{-\mathrm{i}a\partial_n}-1}{-\mathrm{i}a\partial_n}\partial_i f(x) \tag{18}$$

has the following Leibniz rule:

$$D^*_n(f(x)\star g(x))$$
$$= (D^*_n f(x))\star(\mathrm{e}^{-\mathrm{i}a\partial^*_n}g(x)) + (\mathrm{e}^{\mathrm{i}a\partial^*_n}f(x))\star(D^*_n g(x))$$
$$-\mathrm{i}a\left(D^*_j\mathrm{e}^{\mathrm{i}a\partial^*_n}f(x)\right)\star(D^{j*}g(x)),$$
$$D^*_i(f(x)\star g(x))$$
$$= (D^*_i f(x))\star(\mathrm{e}^{-\mathrm{i}a\partial^*_n}g(x)) + f(x)\star(D^*_i g(x)). \tag{19}$$

The generators of κ-Lorentz transformations, acting on coordinates and derivatives,

$$M^{*in}f(x) \tag{20}$$
$$= \Bigg(x^i\partial^n - x^n\partial^i + x^i\partial_\mu\partial^\mu\frac{\mathrm{e}^{\mathrm{i}a\partial_n}-1}{2\partial_n}$$
$$+x^\mu\partial_\mu\partial^i\frac{a\partial_n + \mathrm{i}(\mathrm{e}^{\mathrm{i}a\partial_n}-1)}{a\partial_n^2}\Bigg)f(x),$$
$$M^{*ij}f(x) = \left(x^i\partial^j - x^j\partial^i\right)f(x),$$

have the following coproduct:

$$M^{*in}\left(f(x)\star g(x)\right)$$
$$= \left(M^{*in}f(x)\right)\star g(x) + \left(\mathrm{e}^{\mathrm{i}a\partial^*_n}f(x)\right)\star\left(M^{*in}g(x)\right)$$
$$+\mathrm{i}a\left(\partial^*_j f(x)\right)\star\left(M^{*ij}g(x)\right),$$
$$M^{*ij}\left(f(x)\star g(x)\right)$$
$$= (M^{*ij}f(x))\star g(x) + f(x)\star(M^{*ij}g(x)). \tag{21}$$

Thus, the entire calculus developed in the abstract algebra can be formulated in the $\star$-product setting. For applications in physics this is of advantage because functions of commuting variables x are suitable representations of physical objects like fields.

3 Covariant derivatives

Gauge theories will be formulated with the help of covariant derivatives. We shall demand covariance under the κ-Lorentz algebra as well as covariance under the gauge group. Gauge fields have to be vector fields that transform like the Dirac operator under the deformed Lorentz algebra to render a covariant theory.

κ-Lorentz covariance

We start from the definition of a scalar field. In an undeformed theory this would be $\phi'(x') = \phi(x)$. For noncommuting variables we try however to avoid $\hat{x}'$. Note that

$$\hat{x}'^{\mu}\hat{x}'^{\nu} \neq (1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{x}^{\mu}\hat{x}^{\nu}. \tag{22}$$

Therefore we replace $\hat{\phi}'(\hat{x}')$ with $(1+\epsilon_{\mu\nu}\hat{M}^{\mu\nu})\phi'(\hat{x})$, where $\hat{M}^{\mu\nu}$ acts on the coordinates and the derivatives and has been defined in (10). The defining equation for a scalar field will take the form

$$(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{\phi}'(\hat{x}) = \hat{\phi}(\hat{x}), \tag{23}$$

with the immediate consequence

$$\hat{\phi}'(\hat{x}) = \hat{\phi}(\hat{x}) - \epsilon_{\mu\nu}\hat{M}^{\mu\nu}\hat{\phi}(\hat{x}). \tag{24}$$

To compute the transformation law of a derivative of a scalar field we calculate $(1+\epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{D}_\rho\hat{\phi}'(\hat{x})$ that replaces $\hat{D}'_\rho\hat{\phi}'(\hat{x}')$:

$$\begin{aligned}&(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{D}_\rho\hat{\phi}'(\hat{x}) \\ &= \hat{D}_\rho(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{\phi}'(\hat{x}) + \epsilon_{\mu\nu}[\hat{M}^{\mu\nu}, \hat{D}_\rho]\hat{\phi}'(\hat{x}) \\ &= \hat{D}_\rho\hat{\phi}(\hat{x}) + \epsilon_{\mu\nu}(\eta^{\nu}{}_{\rho}\hat{D}^{\mu} - \eta^{\mu}{}_{\rho}\hat{D}^{\nu})\hat{\phi}(\hat{x}).\end{aligned} \tag{25}$$

We have used (23) to obtain this result and the fact that the second part on the right-hand side is already ϵ-linear.

The transformation law of a derivative of a scalar field is used to define the transformation law of a vector field:

$$(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{V}'_\rho(\hat{x}) = \hat{V}_\rho(\hat{x}) + \epsilon_{\mu\nu}(\eta^{\nu}{}_{\rho}\hat{V}^{\mu} - \eta^{\mu}{}_{\rho}\hat{V}^{\nu}). \tag{26}$$

Thus, the derivative

$$\hat{\mathcal{D}}_\rho = \hat{D}_\rho - \mathrm{i}\hat{V}_\rho \tag{27}$$

is κ-Lorentz covariant.

Gauge covariant derivatives

Gauge theories are based on a gauge group. This is a compact Lie group with generators T^a:

$$[T^a, T^b] = \mathrm{i}f^{ab}{}_{c}T^c. \tag{28}$$

Fields are supposed to span linear representations of this group. Infinitesimal transformations with constant parameters α_a are

$$\delta_\alpha\psi = \mathrm{i}\alpha\psi, \qquad \alpha := \sum_a \alpha_a T^a = \alpha_a T^a. \tag{29}$$

As usual, α is Lie algebra-valued and the commutator of two such transformations closes into a Lie algebra-valued transformation:

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = [\alpha, \beta]\psi = \mathrm{i}\alpha_a\beta_b f^{ab}{}_{c}T^c\psi \equiv \delta_{\alpha\times\beta}\psi. \tag{30}$$

The symbol $\alpha \times \beta$ is defined by this equation and it is independent of the representation of the generators T^a.

We generalize the gauge transformations (29) by considering $\hat{x}$-dependent parameters $\hat{\alpha}_a(\hat{x})$. Whereas for commuting coordinates Lie algebra-valued transformations close in the Lie algebra, this will not be true for non-commuting coordinates. This effect of non-commutativity leads to a generalization of Lie algebra-valued gauge transformations [8, 9].

There are exactly two representation-independent concepts based on the commutation relations (28). These are the Lie algebra and the enveloping algebra. The enveloping algebra of the Lie algebra is defined as the free algebra generated by the elements T^a and then divided by the ideal generated by the commutation relations (28). It is infinite-dimensional and consists of all the (abstract) products of the generators modulo the relations (28)[2]. Two elements of the enveloping algebra are identified if they can be transformed into each other by the use of the commutation relations (28).

A basis can be chosen for the enveloping algebra; we use the symmetrized products as such a basis and denote elements of the basis with colons:

$$\begin{aligned} :T^a: &= T^a, \\ :T^aT^b: &= \frac{1}{2}(T^aT^b + T^bT^a), \\ :T^{a_1}\dots T^{a_l}: &= \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\dots T^{\sigma(a_l)}). \end{aligned} \tag{31}$$

Any formal product of the generators can be expressed in the above basis by using the commutation relations (28), e.g.

$$T^aT^b = \frac{1}{2}\{T^a, T^b\} + \frac{1}{2}[T^a, T^b] =: T^aT^b: + \frac{\mathrm{i}}{2}f^{ab}{}_{c} :T^c:. \tag{32}$$

The new concept is to allow gauge transformations that are enveloping algebra-valued:

$$\begin{aligned}\hat{\Lambda}_{\{\alpha\}}(\hat{x}) &= \sum_{l=1}^{\infty}\sum_{\text{basis}} \alpha^{l}_{a_1\dots a_l}(\hat{x}) :T^{a_1}\dots T^{a_l}: \\ &= \alpha_a(\hat{x}) :T^a: + \alpha^2_{a_1a_2} :T^{a_1}T^{a_2}: + \dots\end{aligned} \tag{33}$$

[2] Note that the product is not the matrix product of the generators in a particular representation.

With this definition we write this generalized transformation law as follows:

$$\delta_{\{\alpha\}}\hat{\psi}(\hat{x}) = \mathrm{i}\hat{\Lambda}_{\{\alpha\}}(\hat{x})\hat{\psi}(\hat{x}), \tag{34}$$

where $\{\alpha\}$ denotes the set of the coefficient functions. It is clear that the commutator of two enveloping algebra-valued transformations will be enveloping algebra-valued again. The price we have to pay are the infinitely many parameters $\{\alpha\}$. This is too expensive. But in the next chapter we will get a price reduction. We will find in the next section that we may define the enveloping algebra-valued transformation such that there will only be as many independent parameters as there are in the Lie algebra-valued case. Therefore it is worthwhile to pursue this idea.

It can be seen that under these generalized gauge transformations a covariant derivative

$$\delta_{\{\alpha\}}\left(\hat{\mathcal{D}}_\mu\hat{\psi}(\hat{x})\right) = \mathrm{i}\hat{\Lambda}_{\{\alpha\}}(\hat{x})\hat{\mathcal{D}}_\mu\hat{\psi}(\hat{x}) \tag{35}$$

has to become enveloping algebra-valued as well, by adding an enveloping algebra-valued gauge field:

$$\hat{\mathcal{D}}_\mu = \hat{D}_\mu - \mathrm{i}\hat{V}_\mu,$$
$$\hat{V}_\mu = \sum_{l=1}^{\infty}\sum_{\text{basis}} V^l_{\mu a_1\ldots a_l} : T^{a_1}\ldots T^{a_l} : . \tag{36}$$

Comparing with (27), the gauge field $\hat{V}_\mu$ has to be a vector field under κ-Lorentz transformations. Therefore each gauge field $\hat{V}^l_{\mu\, a_1\ldots a_l}$ has to transform vector-like to guarantee (26).

A new feature arises due to the deformed coproducts (13) of the Dirac operator which we used to define the covariant derivative. We write (35) more explicitly:

$$\begin{aligned}
&\delta_{\{\alpha\}}\left((\hat{D}_\mu - \mathrm{i}\hat{V}_\mu)\hat{\psi}(\hat{x})\right)\\
&= \mathrm{i}\hat{D}_\mu\left(\hat{\Lambda}_{\{\alpha\}}(\hat{x})\hat{\psi}(\hat{x})\right) + \hat{V}_\mu\hat{\Lambda}_{\{\alpha\}}\hat{\psi}(\hat{x})\\
&\qquad -\mathrm{i}\left(\delta_{\{\alpha\}}\hat{V}_\mu\right)\hat{\psi}(\hat{x})\\
&\overset{!}{=} \mathrm{i}\hat{\Lambda}_{\{\alpha\}}(\hat{x})(\hat{D}_\mu - \mathrm{i}\hat{V}_\mu)\hat{\psi}(\hat{x}).
\end{aligned} \tag{37}$$

Both $\hat{D}_n$ and $\hat{D}_i$ act in a non-trivial way on products of functions. For example, (35) will be satisfied for $\hat{\mathcal{D}}_i$ if

$$(\delta_{\{\alpha\}}\hat{V}_i)\hat{\psi} = (\hat{D}_i\hat{\Lambda}_{\{\alpha\}})\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{\psi} - \mathrm{i}[\hat{V}_i, \hat{\Lambda}_{\{\alpha\}}]\hat{\psi}. \tag{38}$$

If we want to use this formula in such a way that it is independent of $\hat{\psi}$, we see from (38) that the gauge field has to be derivative-valued. Only then the transformation

$$\delta_{\{\alpha\}}\hat{V}_i = (\hat{D}_i\hat{\Lambda}_{\{\alpha\}})\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} - \mathrm{i}[\hat{V}_i, \hat{\Lambda}_{\{\alpha\}}], \tag{39}$$

will lead to (35). For $\hat{V}_n$ we can proceed in the same way and find

$$\begin{aligned}
\delta_{\{\alpha\}}\hat{V}_n &= (\hat{D}_n\hat{\Lambda}_{\{\alpha\}})\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} + \left((\mathrm{e}^{\mathrm{i}a\hat{\partial}_n} - 1)\hat{\Lambda}_{\{\alpha\}}\right)\hat{D}_n\\
&\quad -\mathrm{i}a(\hat{D}_j\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{\Lambda}_{\{\alpha\}})\hat{D}^j - \mathrm{i}[\hat{V}_n, \hat{\Lambda}_{\{\alpha\}}].
\end{aligned} \tag{40}$$

The gauge fields have to be derivative-valued to accommodate the first three terms on the right-hand side of (40) (first term on the right-hand side of (39)). That the gauge fields appear as derivative-valued is a new feature and is a direct consequence of the coproduct rules. We will discuss more details in the next section.

The commutator of two covariant derivatives defines a covariantly transforming object:

$$\hat{\mathcal{F}}_{\mu\nu} = \mathrm{i}[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu]. \tag{41}$$

It will be enveloping algebra- and derivative-valued as it is the case for the gauge field. Its transformation properties are tensor-like:

$$\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu}\hat{\psi} = (\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu})\hat{\psi} + \hat{\mathcal{F}}_{\mu\nu}\delta_{\{\alpha\}}\hat{\psi}. \tag{42}$$

From the definition of the covariant derivative follows

$$\begin{aligned}
\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu}\hat{\psi} &= \mathrm{i}\hat{\Lambda}_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu}\hat{\psi}\\
&= \mathrm{i}(\hat{\Lambda}_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu} - \hat{\mathcal{F}}_{\mu\nu}\hat{\Lambda}_{\{\alpha\}})\hat{\psi} + \mathrm{i}\hat{\mathcal{F}}_{\mu\nu}\hat{\Lambda}_{\{\alpha\}}\hat{\psi}.
\end{aligned} \tag{43}$$

Comparing this with (42) and (34) shows the covariant transformation property of $\hat{\mathcal{F}}_{\mu\nu}$:

$$\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu} = \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{\mathcal{F}}_{\mu\nu}]. \tag{44}$$

The tensor $\hat{\mathcal{F}}_{\mu\nu}$ is derivative-valued. Instead of expanding it in terms of the derivatives $\hat{\partial}_\mu$, we can expand it in terms of covariant derivatives $\hat{\mathcal{D}}_\mu$.

First we express $\hat{\partial}_n$ by $\hat{D}_\mu$ [6]:

$$\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} = -\mathrm{i}a\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}. \tag{45}$$

Next we replace $\hat{D}_\mu$ by $\hat{\mathcal{D}}_\mu$ and subtract the additional terms introduced that way:

$$\hat{D}_n = \hat{\mathcal{D}}_n + \mathrm{i}\hat{V}_n. \tag{46}$$

Each $\hat{V}_n$ will be derivative-valued again, but each derivative carries a factor a and thus contributes to the next order in a. To first order in a we obtain from (45) and (46):

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} &\to 1 - \mathrm{i}a\hat{\partial}_n = 1 - \mathrm{i}a\hat{D}_n\\
&= 1 - \mathrm{i}a\hat{\mathcal{D}}_n + a\hat{V}_n.
\end{aligned}$$

To lowest order in a (compare with (40)), $\hat{V}_n$ is not derivative-valued and contributes to the term in $\hat{\mathcal{F}}_{\mu\nu}$ that has no derivatives. Finally we arrive at the expression

$$\hat{\mathcal{F}}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{T}^\rho_{\mu\nu}\hat{\mathcal{D}}_\rho + \ldots + \hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu} : \hat{\mathcal{D}}_{\rho_1}\ldots\hat{\mathcal{D}}_{\rho_l} : + \ldots \tag{47}$$

The colons denote a basis in the free algebra of covariant derivatives. To each finite order in a this expansion will have a finite number of terms. The individual terms will transform like tensors as well:

$$\hat{\mathcal{F}}_{\mu\nu} \to \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{\mathcal{F}}_{\mu\nu}]$$

$$= \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{F}_{\mu\nu}] + \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{T}^{\rho}_{\mu\nu}\hat{\mathcal{D}}_\rho] + \ldots + \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu} : \hat{\mathcal{D}}_{\rho_1}\ldots\hat{\mathcal{D}}_{\rho_l} :] + \ldots \quad (48)$$

When we apply this to a field $\hat{\psi}$ we find as before in (42) to (44)

$$\delta_{\{\alpha\}}\hat{F}_{\mu\nu} = \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{F}_{\mu\nu}], \quad (49)$$

$$\delta_{\{\alpha\}}\hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu} = \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu}]. \quad (50)$$

Thus, $\hat{\mathcal{F}}_{\mu\nu}$ can be expanded in terms of a full series of derivative-independent tensors. For the dynamics (Lagrangian) we are only going to use the curvature-like term $\hat{F}_{\mu\nu}$. It transforms like a tensor and reduces to the usual field strength $F^0_{\mu\nu}$ for $a \to 0$. To first order we get one torsion-like contribution $\hat{T}^{\rho}_{\mu\nu}$.

4 Seiberg–Witten map

In the previous chapter we saw that an enveloping algebra-valued gauge transformation depends on an infinite set of parameters. The same is true for the enveloping algebra-valued gauge field; it depends on an infinite set of vector fields. This gauge theory would feature an infinite number of independent degrees of freedom.

This unphysical situation can be avoided if we make the additional assumption that the transformation parameters $\Lambda_{\{\alpha\}}$ depend on the usual, Lie algebra-valued gauge field $A^0_{\mu a}$ [7,8]. We will find that this dependence reduces the infinite number of degrees of freedom of the deformed gauge theory to the finite number of degrees of freedom of the Lie algebra gauge theory.

To find this dependence, known as the Seiberg–Witten map, we start from the gauge transformation:

$$\delta_{\{\alpha\}}\hat{\psi} = \mathrm{i}\hat{\Lambda}_{\{\alpha\}}\hat{\psi}. \quad (51)$$

The condition that this is actually a gauge transformation reads

$$(\delta_{\{\alpha\}}\delta_{\{\beta\}} - \delta_{\{\beta\}}\delta_{\{\alpha\}})\hat{\psi} = \delta_{\{\alpha\times\beta\}}\hat{\psi}. \quad (52)$$

We now introduce Λ_α as opposed to $\Lambda_{\{\alpha\}}$ referring to solutions of the Seiberg–Witten map. We have to replace all parameters in (33) by

$$\alpha^l_{a_1\ldots a_l}(\hat{x}) \longrightarrow \alpha^l_{a_1\ldots a_l}(x;\alpha_a(x), A^0_{\mu a}(x)). \quad (53)$$

The parameters are functions of x, the parameters $\alpha_a(x) \equiv \alpha^1_a(x)$ and the gauge field $A^0_{\mu a}(x)$ as well as of their derivatives. Since we define that the non-commutative gauge parameters have a functional dependence only on commuting variables, we have to use the ⋆- product formalism. We choose as a starting point [10]

$$\delta_\alpha\psi = \mathrm{i}\Lambda_\alpha \star \psi \quad \text{with} \quad (\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = \delta_{\alpha\times\beta}\psi. \quad (54)$$

The Lie algebra-valued gauge field A^0_μ (in the following we omit all explicit dependence on coordinates x):

$$A^0_\mu = A^0_\mu(x) = \sum_a A^0_{\mu a}(x)T^a \quad (55)$$

has the transformation property

$$\delta_\alpha A^0_\mu = \partial_\mu\alpha - \mathrm{i}[A^0_\mu, \alpha], \quad (56)$$

where $\alpha = \alpha_a(x)T^a$ is Lie algebra-valued as well.

The gauge parameter Λ_α depends on A^0_μ and because of (56) $\delta_\alpha\Lambda_\beta$ is not zero. We take this into account when we write (54) more explicitly

$$\begin{aligned}&(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi \\ &= (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi + \mathrm{i}(\delta_\alpha\Lambda_\beta - \delta_\beta\Lambda_\alpha) \star \psi \\ &= \delta_{\alpha\times\beta}\psi. \end{aligned} \quad (57)$$

That (57) has a solution can be seen on more general grounds [12] (also [13–15]). Here we construct a solution by a power series expansion in the deformation parameter a:

$$\Lambda_\alpha = \alpha + a\Lambda^1_\alpha + \ldots + a^k\Lambda^k_\alpha + \ldots\ . \quad (58)$$

In this paper we will consider only the first-order term in a to make the formalism transparent. Assuming a to be small, only the leading term will be of relevance for phenomenological applications. We have however calculated all quantities also to second order and have checked the validity of the statements made here.

We expand (57) to first order in a:

$$\begin{aligned}&\Lambda^0_\alpha\Lambda^1_\beta + \Lambda^1_\alpha\Lambda^0_\beta + \Lambda^0_\alpha \star \Lambda^0_\beta|_{\mathcal{O}(a)} - \Lambda^0_\beta\Lambda^1_\alpha \\ &- \Lambda^1_\beta\Lambda^0_\alpha - \Lambda^0_\beta \star \Lambda^0_\alpha|_{\mathcal{O}(a)} + \mathrm{i}(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) = \mathrm{i}\Lambda^1_{\alpha\times\beta},\end{aligned} \quad (59)$$

or, using $\Lambda^0_\alpha = \alpha, \Lambda^0_\beta = \beta$ and the explicit form of the ⋆-product,

$$[\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] + \frac{\mathrm{i}}{2}x^\lambda C^{\mu\nu}_\lambda\{\partial_\mu\alpha, \partial_\nu\ \beta\} + \mathrm{i}(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) = \mathrm{i}\Lambda^1_{\alpha\times\beta}\,. \quad (60)$$

To first order in a the non-commutative structure contributes a term from the ⋆-product, which forbids Λ^1_α equal zero. Equation (60) is an inhomogeneous linear equation in Λ^1_α, with the solution

$$\Lambda^1_\alpha = -\frac{1}{4}x^\lambda C^{\mu\nu}_\lambda\{A^0_\mu, \partial_\nu\ \alpha\}, \quad (61)$$

where $C^{\mu\nu}_\lambda$ are the structure constants of the coordinate algebra. More explicitly this is

$$\Lambda^1_\alpha = \frac{a}{4}x^j\left(\{A^0_j, \partial_n\alpha\} - \{A^0_n, \partial_j\alpha\}\right). \quad (62)$$

This solution is hermitean for real fields $A^0_{\mu a}(x)$ and real parameters $\alpha_a(x)$. That this specific solution of the inhomogeneous equation is not unique and that it is possible to add to it solutions of the homogeneous equation

$$[\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] + \mathrm{i}(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) = \mathrm{i}\Lambda^1_{\alpha\times\beta} \quad (63)$$

has been discussed thoroughly in many places (e.g. [10,15]). There are no new aspects to this question in first order in a in this particular model.

With a solution for Λ^1_α at our disposition, it is possible to express a "matter" field ψ (i.e. field in the fundamental representation) in terms of A^0_μ and a matter field ψ^0 of the standard gauge theory

$$\delta_\alpha \psi^0 = \mathrm{i}\alpha\psi^0. \tag{64}$$

Up to first order in a, (54) is solved by

$$\psi = \psi^0 - \frac{1}{2} x^\lambda C^{\mu\nu}_\lambda A^0_\mu \partial_\nu \psi^0 + \frac{\mathrm{i}}{8} x^\lambda C^{\mu\nu}_\lambda [A^0_\mu, A^0_\nu]\psi^0. \tag{65}$$

The same way as we express ψ in terms of A^0_μ and ψ^0, we can define the Seiberg–Witten map for gauge fields (they are in the adjoint representation of the enveloping algebra). When we derived the respective formulae in the previous section, we discovered that the gauge fields have to be derivative-valued. Therefore we have to discuss solutions of the Seiberg–Witten map for the following relations:

$$\delta_\alpha V_i = (D^*_i \Lambda_\alpha)\mathrm{e}^{-\mathrm{i}a\partial_n} - \mathrm{i}[V_i \stackrel{*}{,} \Lambda_\alpha], \tag{66}$$

and

$$\begin{aligned}\delta_\alpha V_n = {} & (D^*_n \Lambda_\alpha)\mathrm{e}^{-\mathrm{i}a\partial_n} + \left((\mathrm{e}^{\mathrm{i}a\partial_n} - 1)\Lambda_\alpha\right) D^*_n \\ & -\mathrm{i}a(D^*_j \mathrm{e}^{\mathrm{i}a\partial_n}\Lambda_\alpha) D^{*j} - \mathrm{i}[V_n \stackrel{*}{,} \Lambda_\alpha]. \end{aligned} \tag{67}$$

It is technically not simple to solve these two equations (especially since the second one is a sum of several terms with different dependence on derivatives), but conceptually there are no further problems. Without going into details we present the solution up to first order in a:

$$\begin{aligned} V_i = {} & A^0_i - \mathrm{i}aA^0_i\partial_n - \frac{\mathrm{i}a}{2}\partial_n A^0_i - \frac{a}{4}\{A^0_n, A^0_i\} \\ & + \frac{1}{4} x^\lambda C^{\mu\nu}_\lambda \left(\{F^0_{\mu i}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_i\}\right), \end{aligned} \tag{68}$$

$$\begin{aligned} V_n = {} & A^0_n - \mathrm{i}aA^{0j}\partial_j - \frac{\mathrm{i}a}{2}\partial_j A^{0j} - \frac{a}{2} A^0_j A^{0j} \\ & + \frac{1}{4} x^\lambda C^{\mu\nu}_\lambda \left(\{F^0_{\mu n}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_n\}\right). \end{aligned} \tag{69}$$

Here $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - \mathrm{i}[A^0_\mu, A^0_\nu]$ is the field strength of the undeformed gauge theory. We emphasize the dependence on derivatives in the terms $A^0_i\partial_n$ and $A^{0j}\partial_j$.

From the covariant derivative $\mathcal{D}_\mu = D^*_\mu - \mathrm{i}V_\mu$ we can calculate $\mathcal{F}_{\mu\nu} = \mathrm{i}[\mathcal{D}_\mu \stackrel{*}{,} \mathcal{D}_\nu]$ to first order in a. As discussed in the previous section, it will be of first order in the derivatives, the sum of a curvature-like term and a torsion-like term:

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + T^\rho_{\mu\nu}\mathcal{D}_\rho. \tag{70}$$

The result is (up to first order in a)

$$\begin{aligned} F_{ij} = {} & F^0_{ij} - \mathrm{i}a\mathcal{D}^0_n F^0_{ij} \\ & + \frac{1}{4} x^\lambda C^{\mu\nu}_\lambda \left(2\{F^0_{\mu i}, F^0_{\nu j}\} + \{\mathcal{D}^0_\mu F^0_{ij}, A^0_\nu\} \right. \\ & \left. -\{A^0_\mu, \partial_\nu F^0_{ij}\}\right), \end{aligned} \tag{71}$$

$$T^\mu_{ij} = -2\mathrm{i}a\eta^\mu{}_n F^0_{ij}, \tag{72}$$

$$\begin{aligned} F_{nj} = {} & F^0_{nj} - \frac{\mathrm{i}a}{2}\mathcal{D}^{\mu 0} F^0_{\mu j} \\ & + \frac{1}{4} x^\lambda C^{\mu\nu}_\lambda \left(2\{F^0_{\mu n}, F^0_{\nu j}\} + \{\mathcal{D}^0_\mu F^0_{nj}, A^0_\nu\} \right. \\ & \left. - \{A^0_\mu, \partial_\nu F^0_{nj}\}\right), \end{aligned} \tag{73}$$

$$T^\mu_{nj} = -\mathrm{i}a\eta^{\mu l} F^0_{lj} - \mathrm{i}a\eta^\mu{}_n F^0_{nj}. \tag{74}$$

These quantities transform covariantly:

$$\delta_\alpha F_{\mu\nu} = \mathrm{i}[\Lambda_\alpha \stackrel{*}{,} F_{\mu\nu}], \tag{75}$$

$$\delta_\alpha T^\rho_{\mu\nu} = \mathrm{i}[\Lambda_\alpha \stackrel{*}{,} T^\rho_{\mu\nu}]. \tag{76}$$

Now we have all the ingredients to construct to first order in a a gauge theory based on the non-commutative spaces defined by (1) in terms of the usual fields A^0_μ and ψ^0.

The dynamics of the gauge field can be formulated with the tensor $F^{\mu\nu}$

$$\mathcal{L}_{\text{gauge}} = c \ \mathrm{Tr}\left(F^{\mu\nu} \star F_{\mu\nu}\right). \tag{77}$$

Note, however, that $\mathrm{Tr}\left(F_{\mu\nu} \star F^{\mu\nu}\right)$ is not invariant because the coordinates do not commute. The Lagrangian $\mathcal{L}_{\text{gauge}}$ will render an action gauge invariant if it is formulated with an integral with the trace property[3]. The trace will also depend on the representation of the generators T^a because higher products of the generators will enter through the enveloping algebra (for a detailed discussion of this issue, see [17]).

To first order in a, when written in terms of A^0_μ, we obtain the following expression for the gauge part of the Lagrangian (choosing in analogy to the undeformed theory $c = -\frac{1}{4}$):

$$\begin{aligned} \mathcal{L}_{\text{gauge}}|_{\mathcal{O}(a)} = {} & -\frac{\mathrm{i}}{8} x^\lambda C^{\rho\sigma}_\lambda \\ & \times \mathrm{Tr}\left(\mathcal{D}^0_\rho F^{0\mu\nu}\mathcal{D}^0_\sigma F^0_{\mu\nu} + \frac{\mathrm{i}}{2}\{A^0_\rho, (\partial_\sigma + \mathcal{D}^0_\sigma)(F^{0\mu\nu}F^0_{\mu\nu})\} \right. \\ & \left. -\mathrm{i}\{F^{0\mu\nu}, \{F^0_{\mu\rho}, F^0_{\nu\sigma}\}\}\right) \\ & + \frac{\mathrm{i}a}{4}\mathrm{Tr}\left(\mathcal{D}^0_n(F^{0\mu\nu}F^0_{\mu\nu}) - \{\mathcal{D}^0_\mu F^{0\mu j}, F^0_{nj}\}\right), \end{aligned} \tag{78}$$

where $\mathcal{D}^0_\mu = \partial_\mu - \mathrm{i}A^0_\mu$ (or adjoint $\mathcal{D}^0_\mu \cdot = \partial_\mu \cdot - \mathrm{i}[A^0_\mu, \cdot]$ acting on $F^0_{\mu\nu}$). Cyclicity of the trace allows for several simplifications on the terms on the right-hand side.

[3] To attain the trace property, a measure function can be introduced (compare [6]). Since the measure function does in general not vanish in the limit $a \to 0$, it should be compensated without spoiling the gauge invariance of the action. This is possible, leading however to additional first-order terms in the action (compare e.g. [16]).

The matter part of the theory will be the gauge covariant version of the free Lagrangian as it was developed in [6]

$$\mathcal{L}_{\mathrm{matter}} = \bar{\psi} \star (\mathrm{i}\gamma^\mu \mathcal{D}_\mu - m)\,\psi. \tag{79}$$

To first order in a, when written in terms of A^0_μ and ψ^0, we obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{matter}}|_{\mathcal{O}(a)} = {} & \frac{\mathrm{i}}{2} x^\nu C_\nu^{\rho\sigma} \overline{\mathcal{D}^0_\rho \psi^0}\; \mathcal{D}^0_\sigma \left(\mathrm{i}\gamma^\mu \mathcal{D}^0_\mu - m\right) \psi^0 \\ & - \frac{\mathrm{i}}{2} x^\nu C_\nu^{\rho\sigma} \bar{\psi}^0 \gamma^\mu F^0_{\mu\rho} \mathcal{D}^0_\sigma \psi^0 \\ & + \frac{a}{2} \bar{\psi}^0 \gamma^j \mathcal{D}^0_n \mathcal{D}^0_j \psi^0 + \frac{a}{2} \bar{\psi}^0 \gamma^n \mathcal{D}^0_j \mathcal{D}^{0j} \psi^0. \end{aligned} \tag{80}$$

These are the Lagrangians which define the dynamics on the κ-deformed Minkowski space.

5 Gauge transformations and the κ-Lorentz algebra

Our concept of gauge transformations on non-commutative spaces rests on the Seiberg–Witten map. With the help of this map gauge transformations can be realized in the enveloping algebra of the Lie algebra

$$\begin{aligned} \phi' = \phi + \delta_\alpha \phi = \phi + \mathrm{i}\Lambda_\alpha \star \phi \\ \text{with} \quad (\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\phi = \delta_{\alpha\times\beta}\phi. \end{aligned} \tag{81}$$

To find such a realization it turned out to be necessary that Λ_α depends on the standard Lie algebra-valued gauge field A^0_μ and its derivatives. Therefore under a gauge transformation Λ_α will transform as well and (81) leads to

$$\begin{aligned} & \mathrm{i}(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) \star \phi + (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \phi \\ & \quad = \mathrm{i}\Lambda_{\alpha\times\beta} \star \phi. \end{aligned} \tag{82}$$

In this section we want to see how these equations behave under the κ-deformed Lorentz transformations. Only M^{*in} has a deformed coproduct rule (compare with (21)):

$$\begin{aligned} & M^{*in}\,(f \star g) \\ & = \left(M^{*in} f\right) \star g + \left(\mathrm{e}^{\mathrm{i}a\partial^*_n} f\right) \star \left(M^{*in} g\right) \\ & \qquad + \mathrm{i}a \left(\partial^*_j f\right) \star \left(M^{*ij} g\right) \end{aligned} \tag{83}$$

and therefore we will restrict our discussion to $N_\epsilon = \epsilon_i M^{in}$.

A scalar field transforms as follows:

$$\tilde{\phi} = \phi - N^*_\epsilon \phi, \tag{84}$$

where N^*_ϵ acts on the coordinates; compare with (23). This transformation can be inverted, to first order in ϵ:

$$\phi = \tilde{\phi} + N^*_\epsilon \phi. \tag{85}$$

We assume that Λ_α transforms like a scalar field.

First we compute $\widetilde{\phi'}$, by applying (84) to (81):

$$\widetilde{\phi'} = \phi + \mathrm{i}\Lambda_\alpha \star \phi - (N^*_\epsilon \phi) - \mathrm{i}N^*_\epsilon(\Lambda_\alpha \star \phi). \tag{86}$$

For evaluating the last term in (86), the coproduct (83) has to be used.

Next we compute $\tilde{\phi}\,'$ by applying (81) to (84):

$$\tilde{\phi}\,' = \phi - (N^*_\epsilon \phi) + \mathrm{i}\Lambda_\alpha \star \phi - \mathrm{i}N^*_\epsilon(\Lambda_\alpha \star \phi). \tag{87}$$

This shows that the two transformations commute. When we use (85), the gauge transformation (86) can be written as a gauge transformation on $\tilde{\phi}$:

$$\delta_\alpha \tilde{\phi} = \mathrm{i}\Lambda_\alpha \star \tilde{\phi} + \mathrm{i}\Lambda_\alpha \star (N^*_\epsilon \tilde{\phi}) - \mathrm{i}N^*_\epsilon(\Lambda_\alpha \star \tilde{\phi}). \tag{88}$$

We draw the commuting diagram to illustrate the result

$$\begin{array}{ccc} \phi & \xrightarrow{\alpha} & \phi' \\ \epsilon \downarrow & & \downarrow \epsilon \\ \tilde{\phi} & \xrightarrow{\alpha} & \widetilde{\phi'} \equiv \tilde{\phi}\,'. \end{array} \tag{89}$$

The gauge transformations on $\tilde{\phi}$ – the κ-Lorentz transformed scalar field – is now defined by (88). It remains to be shown that (88) realizes the gauge group as well:

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\tilde{\phi} = \delta_{\alpha\times\beta}\tilde{\phi}. \tag{90}$$

It is easier to compute $\delta_\beta \delta_\alpha \tilde{\phi}$ from (87) and to use (81). We make use of (83) and after some rearrangements we obtain

$$\begin{aligned} & (\delta_\beta \delta_\alpha - \delta_\alpha \delta_\beta)\tilde{\phi} \\ & = \left(\mathrm{i}(\delta_\beta \Lambda_\alpha - \delta_\alpha \Lambda_\beta) - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\right) \star \phi \\ & \quad - N^*_\epsilon \left(\mathrm{i}(\delta_\beta \Lambda_\alpha - \delta_\alpha \Lambda_\beta) \right. \\ & \qquad \left. - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\right) \star \phi \\ & \quad - \mathrm{e}^{\mathrm{i}a\partial^*_n} \left(\mathrm{i}(\delta_\beta \Lambda_\alpha - \delta_\alpha \Lambda_\beta) \right. \\ & \qquad \left. - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\right) \star N^*_\epsilon \phi \\ & \quad + \mathrm{i}a\partial^*_j \left(\mathrm{i}(\delta_\beta \Lambda_\alpha - \delta_\alpha \Lambda_\beta) \right. \\ & \qquad \left. - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\right) \star \epsilon_l M^{lj} \phi. \end{aligned} \tag{91}$$

We use the condition (82) again and obtain the result (90). This demonstrates that (88) is a gauge transformation.

It is also possible to verify the result (90) by a direct calculation. We start with the solution of the Seiberg–Witten map (62)

$$\begin{aligned} \Lambda_\alpha & = \alpha - \frac{1}{4} x^\lambda C_\lambda^{\mu\nu} \{A^0_\mu, \partial_\nu\, \alpha\} + \mathcal{O}(a^2) \\ & =: \alpha + \Lambda^1_\alpha + \mathcal{O}(a^2) \end{aligned} \tag{92}$$

and (65)

$$\begin{aligned}\phi &= \phi^0 - \frac{1}{2}x^\mu C_\mu^{\rho\sigma} A_\rho^0 \partial_\sigma \phi^0 \\ &+ \frac{\mathrm{i}}{8} x^\mu C_\mu^{\rho\sigma}[A_\rho^0, A_\sigma^0]\phi^0 + \mathcal{O}(a^2) \\ &=: \phi^0 + \phi^1 + \mathcal{O}(a^2)\,. \end{aligned} \tag{93}$$

We first apply M^{*in} to (93) and gauge transform the undeformed fields afterwards. This has to be equal to M^{*in} applied to $\delta_\alpha\phi = \mathrm{i}\Lambda_\alpha \star \phi$ up to first order in a. Applying M^{*in} on $\delta_\alpha\phi$, the coproduct (83) has to be taken into account and we obtain

$$\begin{aligned} &M^{*in}(\Lambda_\alpha \star \phi) \\ &= (M^{*in}\Lambda_\alpha)\star\phi + (\mathrm{e}^{\mathrm{i}a\partial_n^*}\Lambda_\alpha)\star(M^{*in}\phi) \\ &\quad + \mathrm{i}a(\partial_j^*\Lambda_\alpha)\star(M^{ij*}\phi). \end{aligned} \tag{94}$$

To write this explicitly to first order we need the operators (21) expanded up to first order in a:

$$\begin{aligned} M^{*in} &= x^i\partial^n - x^n\partial^i + \frac{\mathrm{i}a}{2}x^i\partial_\mu\partial^\mu - \frac{\mathrm{i}a}{2}x^\mu\partial_\mu\partial^i \\ &=: M_0^{*in} + M_1^{*in} \end{aligned} \tag{95}$$

and

$$M^{ij*} = x^i\partial^j - x^j\partial^i =: M_0^{ij*}\,. \tag{96}$$

Now we obtain from (94):

$$\begin{aligned} &\mathrm{i}M^{*in}(\Lambda_\alpha\star\phi)|_{\mathcal{O}(a)} \\ &= \mathrm{i}(M_1^{*in}\alpha)\phi^0 + \mathrm{i}\alpha(M_1^{*in}\phi^0) - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho(M_0^{*in}\alpha)\partial_\sigma\phi^0 \\ &- \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0) + \mathrm{i}(M_0^{*in}\alpha)\phi^1 \\ &+ \mathrm{i}\alpha(M_0^{*in}\phi^1) + \mathrm{i}(M_0^{*in}\Lambda_\alpha^1)\phi^0 + \mathrm{i}\Lambda_\alpha^1(M_0^{*in}\phi^0) \\ &- a\partial_n\alpha(M_0^{*in}\phi^0) - a\partial_j\alpha(M_0^{ij*}\phi^0)\,. \end{aligned} \tag{97}$$

Notice that

$$\begin{aligned} &\delta_\alpha(M_0^{*in}\phi^1) \\ &= \mathrm{i}M_0^{*in}\left(\frac{\mathrm{i}}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0 + \alpha\phi^1 + \Lambda_\alpha^1\phi^0\right), \end{aligned} \tag{98}$$

since ϕ^1 was constructed as solution for the Seiberg–Witten map (65). Besides it can be shown by direct calculation that

$$\begin{aligned} &\mathrm{i}M_1^{*in}(\alpha)\phi^0 + \mathrm{i}\alpha M_1^{*in}(\phi^0) \\ &= \mathrm{i}M_1^{*in}(\alpha\phi^0) - ax^j\partial_\mu\alpha\partial^\mu\phi^0 + \frac{a}{2}x^\mu\partial_\mu\alpha\partial^u\phi^0 \\ &\quad + \frac{a}{2}x^\mu\partial^i\alpha\partial_\mu\phi^0 \end{aligned} \tag{99}$$

as well as that

$$\begin{aligned} &-\frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho(M_0^{*in}\alpha)\partial_\sigma\phi^0 - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0) \\ &= -M_0^{*in}\left(\frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0\right) + \frac{1}{2}M_0^{*in}(x^\mu)C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0 \\ &\quad - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\left(\partial_\rho(M_0^{*in})(\alpha)\partial_\sigma\phi^0 + \partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0)\right), \end{aligned} \tag{100}$$

where $\partial_\rho(M_0^{*in})(\alpha) := \eta_\rho^i\partial^n\alpha - \eta_\rho^n\partial^i\alpha$. Then (98), (99) and (100) yield

$$\begin{aligned} &M^{*in}(\Lambda_\alpha\star\phi)|_{\mathcal{O}(a)} \\ &= \delta_\alpha(M_1^{*in}\phi^0) + \delta_\alpha(M_0^{*in}\phi^1) \\ &\quad -ax^i\partial_\mu\alpha\partial^\mu\phi^0 + \frac{a}{2}x^\mu\partial_\mu\alpha\partial^i\phi^0 \\ &\quad + \frac{a}{2}x^\mu\partial^i\alpha\partial_\mu\phi^0 + \frac{1}{2}M_0^{*in}(x^\mu)C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0 \\ &\quad - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\left(\partial_\rho(M_0^{*in})(\alpha)\partial_\sigma\phi^0 + \partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0)\right). \end{aligned} \tag{101}$$

Calculation shows that the last terms on the right-hand side all cancel and we end up with

$$(M^{*in}\delta_\alpha\phi)|_{\mathcal{O}(a)} = \delta_\alpha(M_1^{*in}\phi^0) + \delta_\alpha(M_0^{*in}\phi^1)\,. \tag{102}$$

Hence we showed explicitly up to first order in a that (90) is true.

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5.4 Deformed Spaces, Symmetries and Gauge Theories

by

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Deformed Spaces, Symmetries and Gauge Theories*

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Abstract

This contribution is based on talks given by Frank Meyer (Section 1) and Marija Dimitrijević (Section 2). In the first section we review the basic concepts of deformed spaces and deformed symmetries. We discuss general features of differential calculi, introduce the star-product and star-product representations of differential operators. As examples we treat the canonically deformed space and the κ-deformed space. In the second section we study gauge theories on deformed spaces. Special attention is given to gauge theory the on κ-deformed space (which was introduced as an example in the first part). Nevertheless, the analysis is done in a rather general way such that it could also be applied to the other deformed spaces.

1. Deformed Spaces and Symmetries

1.1. Deformed Spaces

In gauge theories one usually considers differential space-time manifolds and fibers that admit a representation of a Lie-group. In the noncommutative realm, the notion of a point is no longer well-defined and we have to give up

* The two talks given by the authors are based on common work with Larisa Jonke, Lutz Möller, Efrossini Tsouchnika, Julius Wess and Michael Wohlgenannt [1].

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the concept of differentiable manifolds. However, the space of functions on a manifold is an algebra. A generalization of this algebra can be considered in the noncommutative case. We take the algebra freely generated by the noncommutative coordinates $\hat{x}^\mu$, $\mu = 0 \dots n$, which respect commutation relations of the type

$$[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{1}$$

Mathematically this means that we take the space of formal power series in the coordinates $\hat{x}^\mu$ and divide by the ideal generated by the above relations [2]:

$$\widehat{\mathcal{A}}_{\hat{x}} = \mathbb{C}\langle\langle \hat{x}^0, \dots, \hat{x}^n \rangle\rangle / ([\hat{x}^\mu, \hat{x}^\nu] - C^{\mu\nu}(\hat{x}))\,.$$

This we call a *deformed coordinate space.*

The function $C^{\mu\nu}(\hat{x})$ is unknown. It should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments. Nevertheless, one can consider a power-series expansion

$$C^{\mu\nu}(\hat{x}) = i\,\theta^{\mu\nu} + iC^{\mu\nu}_\rho \hat{x}^\rho + (q\hat{R}^{\mu\nu}_{\rho\sigma} - \delta^\nu_\rho \delta^\mu_\sigma)\,\hat{x}^\rho \hat{x}^\sigma + \dots,$$

where $\theta^{\mu\nu}$, $C^{\mu\nu}_\rho$ and $q\,\hat{R}^{\mu\nu}_{\rho\sigma}$ are constants, and study cases where the commutation relations are constant, linear or quadratic in the coordinates. At very short distances those cases provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$ and lead to the following three structures which are of particular interest since they satisfy the so-called Poincare-Birkhoff-Witt property [1]

1. Canonical structure:
$$[\hat{x}^\mu, \hat{x}^\nu] = i\,\theta^{\mu\nu}. \tag{2}$$
2. Lie algebra structure:
$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\rho \hat{x}^\rho. \tag{3}$$
3. Quantum Space structure:
$$\hat{x}^\mu \hat{x}^\nu = q\,\hat{R}^{\mu\nu}_{\rho\sigma}\,\hat{x}^\rho \hat{x}^\sigma. \tag{4}$$

1.2. Symmetries on Deformed Spaces

In general the commutation relations (1) are not covariant with respect to undeformed symmetries. For example the canonical commutation relations (2) break Lorentz symmetry.

Then the question naturally arises whether we can *deform* the symmetry in such a way that it is consistent with the deformed space and that it

[1] The PBW-property states that the space of polynomials in noncommutative coordinates of a given degree is isomorphic to the space of polynomials in the commutative coordinates.

reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie groups can be deformed in the category of Hopf algebras [2]. The generated objects are called *Quantum Groups.* To make this more explicit we give two examples.

1.2.1. The Canonically Deformed Space

For a long time it was common belief that there does not exist a deformed symmetry for the canonically deformed space. However, recently a quantum group-symmetry was discovered [3] [3]. Let us state the result without deriving it:

$$\begin{aligned} [\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0, \qquad\qquad [\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega_\rho{}^\mu \hat{\partial}_\mu, \\ [\hat{\delta}_\omega, \hat{\delta}'_\omega] &= \hat{\delta}_{\omega\times\omega'}, \qquad (\omega\times\omega)'_\mu{}^\nu = -(\omega_\mu{}^\sigma \omega'_\sigma{}^\nu - \omega'_\mu{}^\sigma \omega_\sigma{}^\nu), \\ \Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes 1 + 1 \otimes \hat{\partial}_\mu, \\ \Delta\hat{\delta}_\omega &= \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega + \frac{i}{2}(\theta^{\mu\nu}\omega_\nu{}^\rho - \theta^{\rho\nu}\omega_\nu{}^\mu)\hat{\partial}_\rho \otimes \hat{\partial}_\mu. \end{aligned} \tag{5}$$

Here the deformed generators of Lorentz-transformations are denoted by $\hat{\delta}_\omega$ with constant transformation parameters ω. Note that the algebra relations are undeformed and the deformation takes place exclusively in the co-sector of the Hopf-algebra. The coproduct $\Delta\hat{\delta}_\omega$ [4] of $\hat{\delta}_\omega$ contains $\theta^{\mu\nu}$-corrections. It is interesting that the coproduct of $\hat{\delta}_\omega$ closes only in the Poincare-algebra and not in the Lorentz-algebra. This may be the reason why this symmetry remained undiscovered for such a long time. The consequences of this new symmetry are part of future investigations by various groups.

1.2.2. κ-deformed Space-time

As an example for the Lie structure we introduce the κ-deformed space-time [5]:

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\rho \hat{x}^\rho, \tag{6}$$

where $C^{\mu\nu}_\lambda = a\,(\eta^\mu_n\,\eta^\nu_\lambda - \eta^\nu_n\,\eta^\mu_\lambda)$ and where we use the signature $\eta^{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$. In the following Latin indices always run from 0 to $n-1$ whereas Greek indices run from 0 to n. The commutation relations (6) are covariant with respect to the κ-deformed Poincare algebra [6]. There is a basis where the Lorentz-algebra remains again undeformed

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}, \tag{7}$$

[2] To be more precise the algebra of functions on a Lie group can be deformed. Since Lie groups themselves form a discrete set, a continuous deformation is not possible.

[3] Actually, a deformed symmetry which is just the dual to the one given here was already introduced some years ago in [4] but was basically unknown to the community of physicists working in that field.

[4] The coproduct is a structure map of a Hopf algebra. It tells us how to act on a product of functions.

[5] The κ-deformed space appears also naturally in the context of Doubly Special Relativity [5].

but the commutators of derivatives with the generators $M^{\mu\nu}$

$$\begin{aligned}
[M^{ij},\hat\partial_\mu] &= \eta^j_\mu\hat\partial^i - \eta^i_\mu\hat\partial^j,\\
[M^{in},\hat\partial_n] &= \hat\partial^i,\\
[M^{in},\hat\partial_j] &= \eta^i_j\,\frac{e^{2ia\hat\partial_n}-1}{2\,i\,a} - \frac{i\,a}{2}\,\eta^i_j\,\hat\partial^l\hat\partial_l + i\,a\,\hat\partial^i\hat\partial_j,\\
[\hat\partial_\mu,\hat\partial_\nu] &= 0
\end{aligned} \tag{8}$$

and the co-algebra sector are deformed

$$\begin{aligned}
\Delta M^{ij} &= M^{ij}\otimes 1 + 1\otimes M^{ij},\\
\Delta M^{in} &= M^{in}\otimes 1 + e^{ia\hat\partial_n}\otimes M^{in} + i\,a\,\hat\partial_k\otimes M^{ik},\\
\Delta\hat\partial_i &= \hat\partial_i\otimes 1 + e^{i\,a\,\hat\partial_n}\otimes\hat\partial_i,\\
\Delta\hat\partial_n &= \hat\partial_n\otimes 1 + 1\otimes\hat\partial_n\,.
\end{aligned} \tag{9}$$

The generators $M^{\mu\nu}$ and $\hat\partial_\mu$ act as follows on the coordinates:

$$\begin{aligned}
[M^{ij},\hat x^\mu] &= \eta^{\mu j}\hat x^i - \eta^{\mu i}\hat x^j,\\
[M^{in},\hat x^\mu] &= \eta^{\mu n}\hat x^i - \eta^{\mu i}\hat x^n + i\,a\,M^{i\mu},\\
[\hat\partial_i,\hat x^\mu] &= \eta^\mu_i - i\,a\,\eta^{\mu n}\hat\partial_i, \qquad [\hat\partial_n,\hat x^\mu] = \eta^\mu_n\,.
\end{aligned} \tag{10}$$

Note that all the commutation relations reduce the classical relations in the limit $a\to 0$.

1.3. Differential Calculus

Derivatives are maps on the deformed coordinate space [7]

$$\hat\partial:\ \widehat{\mathcal{A}}_{\hat x}\to\widehat{\mathcal{A}}_{\hat x}\,.$$

Such a map in particular has to map the ideal generated by the commutation relations (1) into itself. If this is the case we say that the map $\hat\partial$ respects the commutation relations (1) or is compatible with them.
To find a suitable map it is convenient to make a general ansatz for the commutator of a derivative and a coordinate:

$$[\hat\partial_\mu,\hat x^\nu] = \delta^\nu_\mu + \sum_j A^{\nu\rho_1\ldots\rho_j}_\mu\,\hat\partial_{\rho_1}\ldots\hat\partial_{\rho_j}\,. \tag{11}$$

The coefficient functions $A^{\nu\rho_1\ldots\rho_j}_\mu$ are of the order of the deformation parameter and vanish in the commutative limit. Requiring consistency of (11) with the commutation relations of the deformed space leads to conditions on the coefficients $A_\mu^{\nu\rho_1\ldots\rho_j}$. In general a solution for those conditions is not unique.

In the case of a canonically deformed space (1) one immediately verifies that actually the undeformed differential calculus

$$[\hat{\partial}_\mu, \hat{x}^\nu] := \delta_\mu^\nu \tag{12}$$

is compatible with the commutation relations (1).

For the κ-deformed space-time there exist several sets of differential calculi which are all equivalent. The derivatives obtained by requiring that the righthand side of (11) is at most linear in the derivatives are the ones introduced above in Section 1.2. as part of the generators of the κ-deformed Poincare algebra. Of special interest is the following set of derivatives which have a vector-like transformation property with respect to the κ-deformed Poincare symmetry. They will be used later on to establish a gauge theory on the κ-deformed space-time:

$$[M^{\mu\nu}, \hat{D}_\mu] = \eta^\nu_\rho\, \hat{D}^\mu - \eta^\mu_\rho\, \hat{D}^\nu, \tag{13}$$

where

$$\hat{D}_n = \frac{1}{a}\,\sin(a\hat{\partial}_n) - \frac{i\,a}{2}\,\hat{\partial}^l\hat{\partial}_l\, e^{-ia\hat{\partial}_n}, \qquad \hat{D}_i = \hat{\partial}_i e^{-ia\hat{\partial}_n}. \tag{14}$$

1.4. Towards a Physical Theory

So far we described how a deformed symmetry acts on a deformed space $\widehat{\mathcal{A}}_{\hat{x}}$ and how we construct differential calculi. To get a physical theory which makes predictions that can be checked by experiments we will express the noncommutative theory in terms of the known commutative variables. This means that the particle content does not change but the noncommutative theory predicts new interactions [8]. This can be achieved by the following two steps:

1. First we represent the abstract deformed space-time algebra $\widehat{\mathcal{A}}_{\hat{x}}$ on the common algebra of commutative functions $\mathcal{A}_x$ by a new product called *star-product ($\star$-product)* which is a deformation of the commutative product of functions.
2. Then we express all noncommutative fields in terms of their commutative counterparts by the *Seiberg-Witten map* (see Section 2.2.).

Using the results from the second step one can express the action of the noncommutative theory in terms of commutative fields and using the star-product from the first step we can expand this action in terms of the deformation parameter. The zeroth order gives back the commutative theory and one can study corrections of it in higher orders of the deformation parameter. Those two steps will be explained in a bit more detail in the following sections.

1.5. Star Product Approach

1.5.1. The Star Product

If the noncommutative algebra $\widehat{\mathcal{A}}_{\hat{x}}$ satisfies the PBW property (see the beginning of this section), the vector space of noncommutative functions is isomorphic (as a vector space) to the vector space of commutative functions [6]. Let

$$\begin{aligned} \rho: \mathbb{R}[[x^0, \ldots, x^n]] &\rightarrow \widehat{\mathcal{A}}_{\hat{x}} \\ f(x^\mu) &\mapsto \hat{f}(\hat{x}^\mu) \end{aligned}$$

be such an isomorphism of vector spaces.[7]

To render the vector space of commutative functions isomorphic as algebra to $\widehat{\mathcal{A}}_{\hat{x}}$ we just have to equip it with a new, noncommutative product. The isomorphism ρ tells us how to define this new product which we call *star-product* and which we denote with a $\star$:

$$f(x) \star g(x) := \rho^{-1}(\hat{f}(\hat{x}) \cdot \hat{g}(\hat{x})) \,. \tag{15}$$

Again we want to give explicit examples. For the canonically deformed space we have the well-known *Moyal-Weyl product*

$$\begin{aligned} f \star g &= \mu \circ e^{i\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}(f \otimes g) \\ &= fg + \frac{i}{2}\,\theta^{\mu\nu}(\partial_\mu f)(\partial_\nu g) + \ldots \,, \end{aligned} \tag{16}$$

where $\mu(f \otimes g) := fg$ is just the multiplication map. This star-product corresponds to the symmetric ordering prescription.

For the κ-deformed space-time we get the following more complicated expression from the symmetric ordering prescription:

$$\begin{aligned} f \star g(x) &= \lim_{\substack{z\to x \\ y \to x}} \exp\Bigg(x^j \partial_{z^j}\left(\frac{\partial_n}{\partial_{z^n}}\, e^{-ia\partial_{y^n}}\frac{1-e^{-ia\partial_{z^n}}}{1-e^{-ia\partial_n}} - 1\right) \\ &\qquad + x^j \partial_{y^j}\left(\frac{\partial_n}{\partial_{y^n}}\frac{1-e^{-ia\partial_{y^n}}}{1-e^{-ia\partial_n}} - 1\right)\Bigg) f(z) g(y) \\ &= f(x)\, g(x) + \frac{i}{2}\, C^{\mu\nu}_\lambda x^\lambda (\partial_\mu f)(\partial_\nu g) + \ldots \,. \end{aligned} \tag{17}$$

Both star-products start in zeroth order with the usual, commutative product and are deformations of it.

[6] It is obvious that they are not isomorphic as algebras since one is a commutative algebra and the other not.

[7] This isomorphism is not unique and every isomorphism describes an ordering prescription.

1.5.2. The Star-Representation of Differential Operators

An operator $\hat{O}$ acting on $\widehat{\mathcal{A}}_{\hat{x}}$ can be represented by a differential operator O^* acting on commutative functions:

$$\begin{array}{ccc} \hat{f}(\hat{x}) & \xrightarrow{\hat{O}} & \hat{O}(\hat{f}(\hat{x})) \\ \rho^{-1}\downarrow & & \downarrow \rho^{-1} \\ f(x) & \xrightarrow{O^*} & O^*(f(x)) \end{array}$$

The star-representation of the derivatives $\hat{\partial}_\mu$ for the canonically deformed space defined in (12) is quite easy: The differential calculus in this case is undeformed and we get

$$\partial^*_\mu = \partial_\mu. \tag{18}$$

In the case of κ-deformed spaces things are more complicated. For instance, the star-representation of the Dirac-derivatives introduced in (14) and their Leibnitz-rules read:

$$\begin{aligned} D^*_n f(x) &= (\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{i\,a\partial_n^2}\,\partial_j\partial^j)\,f(x)\,, \\ D^*_i f(x) &= \frac{e^{-ia\partial_n}-1}{-i\,a\,\partial_n}\,\partial_i f(x)\,, \end{aligned} \tag{19}$$

$$\begin{aligned} D^*_n(f(x)\star g(x)) &= (D^*_n f(x))\star(e^{-ia\partial_n}g(x)) \\ &\quad + (e^{ia\partial_n}f(x))\star(D^*_n g(x)) \\ &\quad - i\,a\left(D^*_j e^{ia\partial_n}f(x)\right)\star(D^{j\,*}g(x))\,, \end{aligned} \tag{20}$$

$$\begin{aligned} D^*_i(f(x)\star g(x)) &= (D^*_i f(x))\star(e^{-ia\partial_n}g(x)) \\ &\quad + f(x)\star(D^*_i g(x)). \end{aligned} \tag{21}$$

We will see in the next section how the above star-representation of the Dirac-derivatives will be used to establish a gauge theory on κ-deformed space-time.

2. Gauge Theory on Deformed Spaces

Gauge theories are based on a gauge group. This is a compact Lie group with generators T^a

$$[T^a, T^b] = i\,f^{ab}_c\,T^c. \tag{22}$$

Infinitesimal transformation of the matter field ψ^0 is given by

$$\delta_\alpha\psi^0(x) = i\,\alpha(x)\,\psi^0(x), \tag{23}$$

where $\alpha(x) = \alpha^a(x)\,T^a$ is a Lie algebra-valued gauge parameter. Transformations (23) close in the algebra

$$\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \delta_{-i\,[\alpha,\beta]}. \tag{24}$$

In this section we will generalize this concept to deformed spaces as well. We choose to work in the $\star$-product representation and define noncommutative gauge transformations as

$$\delta_\alpha \psi = i\,\Lambda_\alpha \star \psi(x)\,, \tag{25}$$

where Λ_α is the noncommutative gauge parameter and ψ is the noncommutative matter field. Before proceeding to the standard construction of a covariant derivative one should check if this transformations close in the algebra (24). Explicit calculation gives

$$\begin{aligned}(\delta_\alpha\delta_\beta\,-\,\delta_\beta\delta_\alpha)\,\psi(x) &= (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi \\ &= \frac{1}{2}\Big([\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta]\{T^a, T^b\} + \{\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta\}[T^a, T^b]\Big) \star \psi\,. \end{aligned} \tag{26}$$

If we take $\Lambda_\alpha = \Lambda^a_\alpha\, T^a$, that is a Lie algebra-valued gauge parameter, algebra (24) will not close because of the first term in the last line of (26) (anticommutator of two generators is no longer in the Lie algebra of generators). There are two ways of solving this problem. One is to consider only $U(N)$ gauge theories and that one we will not follow here. The other one is to go to the enveloping algebra [9] approach and we continue analysing this one.

2.1. Enveloping Algebra Approach

To start with, we define the basis in the enveloping algebra (we choose symmetric ordering)

$$\begin{aligned} :T^a: \,&=\, T^a\,, \\ :T^a\,T^b: \,&=\, \frac{1}{2}(T^a\,T^b + T^b\,T^a)\,, \\ :T^{a_1}\ldots T^{a_l}: \,&=\, \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\ldots T^{\sigma(a_l)})\,. \end{aligned}$$

Gauge parameter Λ_α is said to be enveloping algebra-valued

$$\begin{aligned}\Lambda_\alpha(x) \,&=\, \sum_{l=1}^{\infty}\sum_{\text{basis}} \alpha_l^{a_1\ldots a_l}(x) : T^{a_1}\ldots T^{a_l} \\ &=\, \alpha^a(x) : T^a : +\;\alpha_2^{a_1 a_2}(x) : T^{a_1}\,T^{a_2} : + \ldots\,. \end{aligned} \tag{27}$$

In this case algebra (24) will close since we work in the enveloping algebra. Now one can proceed and define a covariant derivative $\mathcal{D}_\mu\psi(x) = \partial^*_\mu\psi(x) - iV_\mu \star \psi(x)$ by its transformation law

$$\delta_\alpha(\mathcal{D}_\mu\,\psi(x)) = i\,\Lambda_\alpha \star \mathcal{D}_\mu\psi(x)\,. \tag{28}$$

The choice of ∂^*_μ will depend on the choice of a deformed space on which we want to construct gauge theory. Since we are trying to keep the analysis as general as possible we do not specify (yet) what is ∂^*_μ. The noncommutative gauge field V_μ has to be enveloping algebra-valued as well

$$V_\mu = \sum_{l=1}^{\infty} \sum_{\text{basis}} V^l_{\mu\, a_1 \dots a_l} : T^{a_1} \dots T^{a_l} : .$$

From all this it looks like we have a theory with infinitely many degrees of freedom. This is an unphysical situation and the solution of the problem is given in terms of the Seiberg-Witten map [10].

2.2. Seiberg-Witten Map

The basic idea of this map is to suppose that the noncommutative gauge parameter (field) can be expressed in terms of the commutative gauge parameter and field, for example $\Lambda_\alpha = \Lambda_\alpha(x; \alpha, A^0_\mu)$. Then one uses (24) to calculate explicitly this dependance. Inserting $\Lambda_\alpha = \Lambda_\alpha(x; \alpha, A^0_\mu)$ in (24) gives [8]

$$(\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi + i\,(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) \star \psi = \delta_{-i\,[\alpha,\beta]}\, \psi\, . \qquad (29)$$

What has been said up to now applies for a general deformed space since we have not yet specified the $\star$-product or the derivatives ∂^*_μ. But the equation (29) has to be solved perturbatively, therefore one has to expand the $\star$-product. Since we are mainly interested in the gauge theories on the κ-deformed space-time we use (17) and expand Λ_α as

$$\Lambda_\alpha = \alpha + a\,\Lambda^1_\alpha + \ldots + a^k \Lambda^k_\alpha + \ldots .$$

Up to first order in the deformation parameter a the solution of (29) is

$$\Lambda_\alpha = \alpha - \frac{1}{4}\, x^\lambda C^{\mu\nu}_\lambda \{A^0_\mu, \partial_\nu \alpha\}\, . \qquad (30)$$

This solution is not unique, one can always add to it solutions of the homogeneous equation. Using (25) and solution for gauge parameter (30) one finds solution for the noncommutative matter field as well

$$\psi = \psi^0 - \frac{1}{2}\, x^\lambda C^{\mu\nu}_\lambda A^0_\mu\, \partial_\nu \psi^0 + \frac{i}{8}\, x^\lambda C^{\mu\nu}_\lambda [A^0_\mu, A^0_\nu]\, \psi^0, \qquad (31)$$

where ψ^0 is the commutative matter field, $\delta_\alpha \psi^0 = i\,\alpha\,\psi^0$.

If one compares $\star$-products for the canonically deformed space (16) and for the κ-deformed space-time (17) one sees that up to first order in the

[8] One should notice that now $\delta_\alpha \Lambda_\beta \neq 0$ because Λ_β depends on the commutative gauge field A^0_μ as well and $\delta_\alpha A^0_\mu = \partial_\mu \alpha - i\,[A^0_\mu, \alpha]$.

deformation parameter they are of the same form (just replace $\theta^{\mu\nu}$ with $C_\lambda^{\mu\nu}x^\lambda$). Therefore it is not surprising that the solutions for Λ_α and ψ in the canonically deformed space can be obtained from (30) and (31) by replacing $C_\lambda^{\mu\nu}x^\lambda$ with $\theta^{\mu\nu}$ (and the other way around). However this analogy only applies in first order, in second order new terms will appear in the κ-deformed space-time compared to the canonically deformed space.

In order to solve the Seiberg-Witten map for the gauge field V_μ one first has to choose ∂_μ^* derivatives. In the canonically deformed space $\partial_\mu^* = \partial_\mu$ is the most natural choice. In the κ-deformed space-time there are more possibilities (see Section 1.3.). We choose D_μ^* derivatives because of their vector-like transformation law (13). From $\mathcal{D}_\mu\psi = D_\mu^*\psi - iV_\mu \star \psi$ and

$$\delta_\alpha(\mathcal{D}_\mu\psi) = i\,\Lambda_\alpha \star \mathcal{D}_\mu\psi$$

we get

$$\begin{aligned}(\delta_\alpha V_\mu)\star\psi &= D_\mu^*(\Lambda_\alpha\star\psi) - \Lambda_\alpha\star(D_\mu^*\psi) + i\,[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi \\ &\neq (D_\mu^*\Lambda_\alpha)\star\psi + i\,[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi\,.\end{aligned}$$

The last line follows from the nontrivial Leibnitz rules for D_μ^* derivatives (20,21). In order to continue we split between n and j indices.

First we have a look at the j index.

$$\begin{aligned}(\delta_\alpha V_j)\star\psi &= D_j^*(\Lambda_\alpha\star\psi) - \Lambda_\alpha\star(D_j^*\psi) + i\,[\Lambda_\alpha \stackrel{\star}{,} V_j]\star\psi \\ &= (D_j^*\Lambda_\alpha)\star e^{-ia\partial_n}\psi + i\,[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi\,,\end{aligned} \tag{32}$$

where we have used (21). In order to solve this equation we have to allow for V_j to be derivative-valued, that is we make the following ansatz

$$V_j\star\psi = A_j\star(e^{-ia\partial_n}\psi)$$

and insert it into (32). After using $e^{-ia\partial_n}(f\star g) = (e^{-ia\partial_n}f)\star(e^{-ia\partial_n}g)$ and omitting $e^{-ia\partial_n}\psi$ we have

$$\delta_\alpha A_j = (D_j^*\Lambda_\alpha) + i\,\Lambda_\alpha\star A_j - i\,A_j\star(e^{-ia\partial_n}\Lambda_\alpha)\,. \tag{33}$$

This equation can be solved order by order in the deformation parameter. The solution up to first order in a is

$$\begin{aligned}V_j &= A_j^0 - i\,a\,A_j^0\,\partial_n - \frac{i\,a}{2}\,\partial_n A_j^0 - \frac{a}{4}\,\{A_n^0, A_j^0\} \\ &\quad + \frac{1}{4}\,x^\lambda C_\lambda^{\mu\nu}\Big(\{F_{\mu j}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_j^0\}\Big).\end{aligned} \tag{34}$$

For V_n one follows the same steps, using the Leibnitz rule for the D^*_n derivative (20) this time. The solution up to first order in a is

$$\begin{aligned} V_n \;=\; & A^0_n - i\,a\,A^{0j}\partial_j - \frac{i\,a}{2}\,\partial_j A^{0j} - \frac{a}{2}\,A^0_j\,A^{0j} \\ & + \frac{1}{4}\,x^\lambda C^{\mu\nu}_\lambda \Big(\{F^0_{\mu n}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_n\}\Big)\,. \end{aligned} \tag{35}$$

From (34) and (35) we see that besides being enveloping algebra-valued (consequence of noncommutativity, that is $\star$-product) the gauge field is also derivative-valued. This is the consequence of special properties of κ-deformed space-time, more concretely of nontrivial Leibnitz rules for D^*_μ derivatives.

For completeness we give here also the solution for V_μ in the canonically deformed space

$$V_\rho = A^0_\rho + \frac{1}{4}\,\theta^{\mu\nu}\Big(\{F^0_{\mu\rho}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_\rho\}\Big)\,. \tag{36}$$

This solution is not derivative valued since ∂_μ derivatives have undeformed Leibnitz rule.

Having solutions of the Seiberg-Witten map at hand, one calculates the field-strength tensor defined as

$$\mathcal{F}_{\mu\nu} = i\,[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu]\,. \tag{37}$$

Since the gauge field V_μ is derivative-valued [9] it is not surprising that the field-strength tensor will also be derivative-valued. With a derivative-valued field-strength tensor we do not know how to write down the action for the gauge field. Therefore, we split the tensor $\mathcal{F}_{\mu\nu}$ into "curvature-like" and "torsion-like" terms, like one usually does in gravity theories

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + T^\rho_{\mu\nu}\mathcal{D}_\rho + \ldots + T^{\rho_1\ldots\rho_l}_{\mu\nu} : \mathcal{D}_{\rho_1}\ldots\mathcal{D}_{\rho_l} : + \ldots \tag{38}$$

For the action we will only use the "curvature-like" term $F_{\mu\nu}$ and ignore all "torsion-like" terms. With this we have all the ingredients to write Lagrangian densites up to the first order in a, see [1].

2.3. Integral and the Action

To come from the Lagrangian densities to the action for noncommutative gauge theory we need an integral. It should have the trace property

$$\int f \star g = \int g \star f\,. \tag{39}$$

[9] The following does not apply to the canonically deformed space since V_μ in not derivative valued there.

This is required by gauge invariance of the action for the gauge field and can be used to formulate the variational principle. For the canonically deformed space (39) is automatically fulfilled and the following analysis is not needed there. Unfortunately, for κ-deformed space-time (39) is not fulfilled. The way to repair this is to introduce so-called measure function $\mu(x)$ such that

$$\int \mathrm{d}^{n+1}x\ \mu(x)\ (f \star g) = \int \mathrm{d}^{n+1}x\ \mu(x)\ (g \star f)\,. \tag{40}$$

From this request one gets conditions on $\mu(x)$

$$\partial_n \mu(x) = 0, \qquad x^j \partial_j \mu(x) = -n\,\mu(x)\,. \tag{41}$$

This equation can be solved, however the solution is not unique. But this is not the only problem. It turns out that the solution for $\mu(x)$ is a independent so it does not vanish in the limit $a \to 0$. This means that it will spoil the classical limit of the theory (equations of motion for example). Also, because of its explicit x-dependence [10] it will break the κ-Poincaré invariance of the integral.

On the other hand, one can construct an integral which is κ-Poincaré invariant using quantum trace [11]. The problem with the integral obtained that way is that it does not have the trace property, therefore it is not convenient for analysing gauge theories.

So far there has not been a completely satisfactory answer to the question of proper definition of the integral on κ-deformed space-time. It appears that one has to choose between having a gauge invariant theory or κ-Poincaré invariant theory. In the case of $U(1)$ gauge theory we have been able to write down the action using the first approach [12], but the analysis is still far from being complete.

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5.5 Twisted Gauge Theories

by

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Twisted Gauge Theories

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Abstract

Gauge theories on a space-time that is deformed by the Moyal-Weyl product are constructed by twisting the coproduct for gauge transformations. This way a deformed Leibniz rule is obtained, which is used to construct gauge invariant quantities. The connection will be enveloping algebra valued in a particular representation of the Lie algebra. This gives rise to additional fields, which couple only weakly via the deformation parameter θ and reduce in the commutative limit to free fields. Consistent field equations that lead to conservation laws are derived and some properties of such theories are discussed.

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1 Introduction

The aim of this work is to construct and investigate gauge theories on deformed space-time structures that are defined by an associative but noncommutative product of C^∞ functions. Such products are known as star products; the best known is the Moyal-Weyl product [1, 2]. In this letter we shall deal with this product exclusively.

From previous work [3, 4, 5] we know that the usual algebra of functions and the algebra of vector fields can be represented by differential operators on the deformed manifold. The deformed diffeomorphisms have been used to construct a deformed theory of gravity. Here we shall show that along the same lines a deformed gauge theory can be constructed as well. The algebra, based on a Lie algebra, will not change but the comultiplication rule will. This leads to a deformed Hopf algebra. In turn this gives rise to deformed gauge theories because the construction of a gauge theory involves the Leibniz rule that is based on the comultiplication.

Covariant derivatives can be constructed by a connection. Different to a usual gauge theory the connection cannot be Lie algebra valued. The construction of covariant tensor fields (curvature or field strength) and of an invariant Lagrangian is completely analogue to the undeformed case. Field equations can be derived and it can be shown that they are consistent. This leads to conserved currents. It is for the first time that it is seen that deformed symmetries also lead to conservation laws; note that the Noether theorem is not directly applicable in the noncommutative context.

The deformed gauge theory has interesting new features. We start with a Lie(G)-valued connection and show that twisted gauge transformations close in Lie(G), however consistency of the equation of motion requires the introduction of new vector potentials so that the initial Lie(G)-valued connection becomes an hermitian matrix. The number of these extra vector potentials is representation dependent but remains finite for finite dimensional representations. Concerning the interaction, the Lie algebra valued fields and the new vector fields behave quite differently. The interaction of the Lie algebra valued fields can be seen as a deformation of the usual gauge interactions; for vanishing deformation parameters the interaction will be the interaction of a usual gauge theory. The interactions of the new fields are deformations of a free field theory for vector potentials; for vanishing deformation parameters the fields become free. As the deformation parameters are supposed to be very small we conclude that the new fields are practically dark with respect to the usual gauge interactions.

Finally we discuss the example of a $SU(2)$ gauge group in the two dimensional representation.

The treatment introduced here can be compared with previous ones. In [6] the noncommutative gauge transformations for $U(N)$ have an undeformed comultiplication. The action is the same as in (4.19) if we restrict our discussion, valid for any compact Lie group, to $U(N)$ in the n-dimensional matrix representation. In other terms we show that noncommutative $U(N)$ gauge theories have usual noncommutative gauge transformations and also twisted gauge transformations. In [7, 8, 9, 10, 11] the situation is different because we consider field dependent transformation parameters.

2 Algebraic formulation

A noncommutative coordinate space can be realized with the help of the Moyal-Weyl product [1, 2]. On such a space we are going to construct gauge theories based on a Lie algebra.

We start from the linear space of C^∞ functions on a smooth manifold $\mathcal{M}$, $Fun(\mathcal{M})$. To define an algebra $\mathcal{A}_\theta$ we shall use the associative but noncommutative Moyal-Weyl product. The algebra defined with the usual, commutative point-wise product we refer to as the algebra of C^∞ functions.

The Moyal-Weyl product is defined as follows

$$\begin{aligned} f, g &\in Fun(\mathcal{M}) \\ f \star g &= \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\} \\ \mu\{f \otimes g\} &= f \cdot g, \end{aligned} \tag{2.1}$$

where $\theta^{\rho\sigma} = -\theta^{\sigma\rho}$ is x-independent. The $\star$-product of two functions is a function again

$$\begin{aligned} \mu_\star : \qquad Fun(\mathcal{M}) \otimes Fun(\mathcal{M}) &\to Fun(\mathcal{M}), \\ \mu_\star\{f \otimes g\} &= f \star g. \end{aligned} \tag{2.2}$$

Derivatives are linear maps on $Fun(\mathcal{M})$

$$\begin{aligned} \partial_\rho : \qquad Fun(\mathcal{M}) &\to Fun(\mathcal{M}), \\ f &\mapsto \partial_\rho f. \end{aligned} \tag{2.3}$$

The Leibniz rule extends these maps to the usual algebra of C^∞ functions

$$(\partial_\rho(f \cdot g)) = (\partial_\rho f) \cdot g + f \cdot (\partial_\rho g). \tag{2.4}$$

This concept can be lifted to the algebra $\mathcal{A}_\theta$ [12]

$$\begin{aligned} \partial^\star_\rho : \qquad & f \mapsto \partial^\star_\rho f \equiv \partial_\rho f \\ & \partial^\star_\rho(f \star g) = (\partial^\star_\rho f) \star g + f \star (\partial^\star_\rho g). \end{aligned} \tag{2.5}$$

The last line is true because $\theta^{\mu\nu}$ is x-independent.

Analogously to differential operators acting on the usual algebra of functions we define differential operators on $\mathcal{A}_\theta$

$$\mathcal{D}^\star \star f = \sum_n d^{\rho_1 \cdots \rho_n} \star \partial^\star_{\rho_1} \ldots \partial^\star_{\rho_n} f. \tag{2.6}$$

This is well defined, $\star$ and $\partial^\star_\rho$ always act on functions. The product of such differential operators can be computed with the help of the Leibniz rule.

We can now define the $\star$-product as the action of a bilinear differential operator

$$f \star g = \mu\{\mathcal{F}\, f \otimes g\}, \tag{2.7}$$

with

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}. \tag{2.8}$$

This differential operator can be inverted

$$f \cdot g = \mu_\star\{\mathcal{F}^{-1} f \otimes g\}. \tag{2.9}$$

Equation (2.9) can also be written in the form [3]

$$f \cdot g = \Big(\sum_{n=0}^{\infty}\Big(-\frac{i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n} f\Big)\star\partial^\star_{\sigma_1}\dots\partial^\star_{\sigma_n}\Big)\star g. \tag{2.10}$$

Equation (2.10) shows that the point-wise product $f \cdot g$ can also be interpreted as the $\star$-action of a differential operator $X^\star_f$ on g

$$f \cdot g = X^\star_f \star g = (X^\star_f \star g), \tag{2.11}$$

where

$$X^\star_f = \sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{i}{2}\Big)^n\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}(\partial_{\rho_1}\dots\partial_{\rho_n} f)\star\partial^\star_{\sigma_1}\dots\partial^\star_{\sigma_n}. \tag{2.12}$$

From the associativity of the $\star$-product follows immediately

$$f \cdot g \cdot h = X^\star_{f\cdot g}\star h = X^\star_f \star X^\star_g \star h. \tag{2.13}$$

The differential operators $X^\star_f$ represent the usual algebra of functions

$$X^\star_f \star X^\star_g = X^\star_{f\cdot g}. \tag{2.14}$$

3 Gauge transformations

Ordinary gauge transformations are Lie algebra-valued

$$\alpha(x) = \alpha^a(x)T^a, \quad [T^a, T^b] = if^{abc}T^c. \tag{3.1}$$

The gauge transformation of a field is

$$\delta_\alpha\psi(x) = i\alpha(x)\psi(x) = i\alpha^a(x)T^a\psi(x), \tag{3.2}$$

i.e. $\delta_\alpha\psi = i\alpha\cdot\psi$. This can be viewed as a $\star$-action

$$\hat{\delta}_\alpha\psi = iX^\star_{\alpha^a}\star T^a\psi = iX^\star_\alpha\star\psi = i\alpha\cdot\psi. \tag{3.3}$$

When we deal with a gauge theory in physics we not only use the Lie algebra but also the corresponding Hopf algebra obtained from the comultiplication rule

$$\begin{aligned}\Delta(\delta_\alpha)(\phi\otimes\psi) &= (\delta_\alpha\phi)\otimes\psi + \phi\otimes(\delta_\alpha\psi),\\ \Delta(\delta_\alpha) &= \delta_\alpha\otimes 1 + 1\otimes\delta_\alpha.\end{aligned} \tag{3.4}$$

The transformation of the product of fields is

$$\delta_\alpha(\phi\cdot\psi) = \delta_\alpha\mu\{\phi\otimes\psi\} = \mu\Delta(\delta_\alpha)(\phi\otimes\psi). \tag{3.5}$$

But there are different ways to extend a Lie algebra to a Hopf algebra. A convenient way is by a twist $\mathcal{F}$, that is a bilinear differential operator acting on a tensor product of functions. A well known example is

$$\mathcal{F} = e^{-\frac{1}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}. \tag{3.6}$$

It satisfies all the requirements for a twist [13, 14] and therefore gives rise to a new coproduct (twisted gauge transformations were also introduced in [15])

$$\Delta_\mathcal{F}(\delta_\alpha) = \mathcal{F}\Delta(\delta_\alpha)\mathcal{F}^{-1}. \tag{3.7}$$

This coproduct defines a new Hopf algebra, the Lie algebra is extended by the derivatives, the comultiplication is deformed. This twist can also be used to deform Poincaré transformations [16, 12, 17, 3] respectively diffeomorphisms [3, 4, 5]. In [18] gauge theories consistent with twisted diffeomorphisms where constructed without deforming the coproduct for gauge transformations.

We now look at the transformation law of products of fields based on the deformed coproduct (3.7).

$$\hat\delta_\alpha(\phi\star\psi) = \mu_\star\{\Delta_\mathcal{F}(\hat\delta_\alpha)(\phi\otimes\psi)\}, \tag{3.8}$$

where $\mu_\star$ is defined in (2.2) and $\hat\delta_\alpha$ in (3.3). We obtain

$$\hat\delta_\alpha(\phi\star\psi) = iX^\star_{\alpha^a}\star\Big((T^a\phi)\star\psi + \phi\star(T^a\psi)\Big). \tag{3.9}$$

Note that the operator $X^\star_{\alpha^a}$ is at the left of both terms, this is due to the coproduct $\Delta_\mathcal{F}$. Formula (3.9) is different from

$$\hat\delta_\alpha(\phi\star\psi) = (\hat\delta_\alpha\phi)\star\psi + \phi\star(\hat\delta_\alpha\psi). \tag{3.10}$$

It is exactly the requirement that the $\star$-product of two fields should transform as (3.9) that leads to the twist $\mathcal{F}$. It is by the twisted coproduct that the $\star$-product of fields transforms like (3.3) again. The commutator of two gauge transformation closes in the usual way

$$\hat\delta_\alpha\hat\delta_\beta - \hat\delta_\beta\hat\delta_\alpha = \hat\delta_{-i[\alpha,\beta]}\,. \tag{3.11}$$

To construct an invariant Lagrangian we have to introduce covariant derivatives

$$D_\mu\psi = \partial_\mu\psi - iA_\mu\star\psi. \tag{3.12}$$

From

$$\hat\delta_\alpha\psi = iX^\star_{\alpha^a}\star(T^a\psi)$$

we find

$$\hat\delta_\alpha(D_\mu\psi) = iX^\star_{\alpha^a}\star(T^a(D_\mu\psi)) \tag{3.13}$$

if we use the proper comultiplication for the term $A_\mu \star \psi$ in the covariant derivative and if the vector field transforms as follows

$$\hat{\delta}_\alpha A_\mu = \partial_\mu \alpha + iX^\star_{\alpha^a} \star [T^a, A_\mu]. \tag{3.14}$$

This can also be written in the familiar way:

$$\hat{\delta}_\alpha A_\mu = \partial_\mu \alpha + i[\alpha, A_\mu]. \tag{3.15}$$

The transformation would take Lie algebra-valued objects to Lie algebra-valued objects. For reasons that will become clear in the following we will assume the hermitian field A_μ to be $n \times n$ matrix valued where n is the dimension of the Lie algebra representation. Formula (3.14) will still be true in that case.

The field-strength tensor can be obtained as usual

$$\begin{aligned} F_{\mu\nu} &= i[D_\mu \stackrel{\star}{,} D_\nu], \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu]. \end{aligned} \tag{3.16}$$

Using the deformed coproduct and the gauge variation of the potential we derive the following transformation law,

$$\begin{aligned} \hat{\delta}_\alpha F_{\mu\nu} &= iX^\star_{\alpha^a} \star [T^a, F_{\mu\nu}] \\ &= i[\alpha, F_{\mu\nu}]. \end{aligned} \tag{3.17}$$

4 Field equations

With the tensor $F_{\mu\nu}$ and the covariant derivatives we can construct invariant Lagrangians. Starting from the usual invariant Lagrangians we replace the point-wise product by the $\star$-product and the comultiplication (3.5) with (3.7). We convince ourselves that we can construct an invariant Lagrangian under the deformed Hopf algebra. The expression $F^{\mu\nu} \star F_{\mu\nu}$ transforms as follows

$$\begin{aligned} \hat{\delta}_\alpha (F^{\mu\nu} \star F_{\mu\nu}) &= iX^\star_{\alpha^a} \star [T^a, F^{\mu\nu} \star F_{\mu\nu}] \\ &= i[\alpha, F^{\mu\nu} \star F_{\mu\nu}]. \end{aligned} \tag{4.18}$$

This leads to an invariant and real action

$$S_g = c_1 \int \mathrm{d}^4x \, \mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}). \tag{4.19}$$

The integral introduced in (4.19) has the trace property

$$\int \mathrm{d}^4x \, (f \star g) = \int \mathrm{d}^4x \, (f \cdot g) = \int \mathrm{d}^4x \, (g \star f). \tag{4.20}$$

Therefore we obtain the field equations by writing the varied field to the very left. Varying with respect to the matrix algebra-valued field A_μ leads to the field equations

$$(\partial_\mu F^{\mu\rho})_{AB} - i([A_\mu \stackrel{\star}{,} F^{\mu\rho}])_{AB} = 0. \tag{4.21}$$

Here A and B are matrix indices.

From the field equations and the antisymmetry of $F^{\mu\nu}$ in μ and ν follows the consistency requirement

$$\partial_\rho\Big(i[A_\mu \overset{\star}{,} F^{\mu\rho}]\Big) = 0. \tag{4.22}$$

To show (4.22) we have to use the equation of motion (4.21). We calculate

$$\partial_\rho\Big(i[A_\mu \overset{\star}{,} F^{\mu\rho}]\Big) = i[\partial_\rho A_\mu \overset{\star}{,} F^{\mu\rho}] + i[A_\mu \overset{\star}{,} \partial_\rho F^{\mu\rho}]. \tag{4.23}$$

In the second term we insert the field equation (4.21). In the first term we complete $\partial_\rho A_\mu$ to the tensor $F_{\rho\mu}$ by adding and subtracting the respective terms. We then use

$$[F_{\mu\rho} \overset{\star}{,} F^{\mu\rho}] = 0, \tag{4.24}$$

and obtain

$$+\frac{(i)^2}{2}[[A_\rho \overset{\star}{,} A_\mu] \overset{\star}{,} F^{\mu\rho}] + \frac{(i)^2}{2}[A_\mu \overset{\star}{,} [A_\rho \overset{\star}{,} F^{\mu\rho}]] - \frac{(i)^2}{2}[A_\rho \overset{\star}{,} [A_\mu \overset{\star}{,} F^{\mu\rho}]] = 0$$

for the right hand side of equation (4.23). That it vanishes follows from the Jacobi identity. Thus, we obtained a conservation law

$$\begin{aligned} J^\rho &= i[A_\mu \overset{\star}{,} F^{\mu\rho}], \\ \partial_\rho J^\rho &= 0. \end{aligned} \tag{4.25}$$

From (3.16) follows that $F_{\mu\nu}$ is enveloping algebra valued if A_μ is. From the field equation follows that A_μ and $F_{\mu\nu}$ will remain enveloping algebra valued in the n-dimensional representation of the Lie algebra. Thus, we try to replace matrix algebra valued by enveloping algebra valued for A_μ. As an example we treat the case $SU(2)$ in the two-dimensional representation. In this representation the generators T^a of the Lie algebra satisfy the relations

$$[T^a, T^b] = i\epsilon^{abc}T^c \tag{4.26}$$

and

$$\{T^a, T^b\} = \frac{1}{2}\delta^{ab}\,. \tag{4.27}$$

Note that (4.26) is valid for any representation. The anticommutator is representation dependent. Equation (4.27) is only true in the two dimensional representation. In our example we can write A_μ as follows:

$$A_\mu = B_\mu + A_\mu^d T^d\,.$$

This is consistent with the gauge transformations; the field equations are a consequence of (4.26) and (4.27).

The tensor $F_{\mu\nu}$ is easy to calculate following (3.16):

$$F_{\mu\nu} = G_{\mu\nu} + \tilde{F}^d_{\mu\nu} T^d\,,$$

where

$$
\begin{aligned}
G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu \stackrel{\star}{,} B_\nu] - \frac{i}{4}[A^d_\mu \stackrel{\star}{,} A^d_\nu] \\
\tilde{F}^d_{\mu\nu} &= \partial_\mu A^d_\nu - \partial_\mu A^d_\nu - i[B_\mu \stackrel{\star}{,} A^d_\nu] - i[A^d_\mu \stackrel{\star}{,} B_\nu] + \frac{1}{2}\{A^a_\mu \stackrel{\star}{,} A^b_\nu\}\epsilon^{abd}\,.
\end{aligned}
$$

Varying the Lagrangian (4.19) with respect to B_μ and A^d_μ leads to the field equations

$$
\begin{aligned}
\partial^\mu G_{\mu\nu} - i[B^\mu \stackrel{\star}{,} G_{\mu\nu}] - \frac{i}{4}[A^{\mu a} \stackrel{\star}{,} \tilde{F}^a_{\mu\nu}] &= 0 \\
\partial^\mu \tilde{F}^d_{\mu\nu} - i[A^{\mu d} \stackrel{\star}{,} G_{\mu\nu}] - i[B^\mu \stackrel{\star}{,} \tilde{F}^d_{\mu\nu}] + \frac{1}{2}\epsilon^{abd}\{A^a_\mu \stackrel{\star}{,} \tilde{F}^b_{\mu\nu}\} &= 0\,. \qquad (4.28)
\end{aligned}
$$

These field equations are consistent. They describe a triplet of vector fields A^d_μ as expected and a singlet B_μ. In the limit $\theta \to 0$, B_μ becomes a free field; it interacts only via θ and higher order terms in θ. The triplet A^d_μ satisfies the usual field equations of $SU(2)$ gauge theory in the limit $\theta \to 0$. For $\theta \neq 0$ both the triplet and the singlet couple to conserved currents but the current of B_μ has no θ-independent term.

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5.6 Gauge Theory on Fuzzy $S^2 \times S^2$ and Regularization on Noncommutative $\mathbb{R}^4$

by

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Gauge theory on fuzzy $S^2 \times S^2$ and regularization on noncommutative $\mathbb{R}^4$

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ABSTRACT: We define U(n) gauge theory on fuzzy $S^2_N \times S^2_N$ as a multi-matrix model, which reduces to ordinary Yang-Mills theory on $S^2 \times S^2$ in the commutative limit $N \to \infty$. The model can be used as a regularization of gauge theory on noncommutative $\mathbb{R}^4_\theta$ in a particular scaling limit, which is studied in detail. We also find topologically non-trivial U(1) solutions, which reduce to the known "fluxon" solutions in the limit of $\mathbb{R}^4_\theta$, reproducing their full moduli space. Other solutions which can be interpreted as 2-dimensional branes are also found. The quantization of the model is defined non-perturbatively in terms of a path integral which is finite. A gauge-fixed BRST-invariant action is given as well. Fermions in the fundamental representation of the gauge group are included using a formulation based on SO(6), by defining a fuzzy Dirac operator which reduces to the standard Dirac operator on $S^2 \times S^2$ in the commutative limit. The chirality operator and Weyl spinors are also introduced.

KEYWORDS: Nonperturbative Effects, Solitons Monopoles and Instantons, Non-Commutative Geometry, Matrix Models.

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Contents

1. Introduction

Gauge theories on noncommutative spaces have received much attention in recent years. One of the reasons is the natural realization of such theories in the framework of string theory and D-branes [1], however they deserve interest also in their own right; see [2, 3] for some reviews. One of the most remarkable new features of noncommutative gauge theories is the fact that they can be defined in terms of multi-matrix models, which means that the action involves only products of "covariant coordinates" $X_i = x_i + A_i$, with gauge transformations acting as $X_i \to UX_iU^{-1}$. In particular for certain quantized compact spaces such as fuzzy spheres and tori, these X_i are finite-dimensional hermitean matrices of size N. Nevertheless, the conventional gauge theory is correctly reproduced in the limit $N \to \infty$. This leads to a natural quantization prescription by simply integrating over these matrices. For the much-studied case of the quantum plane $\mathbb{R}^d_\theta$, the matrices X_i are infinite-dimensional, and the precise definition of the models is quite non-trivial. This is particularly obvious by noting that the naive action for gauge theory on $\mathbb{R}^d_\theta$ contains sectors with any rank of the gauge group U(n) [4]. To have a well-defined theory and quantization prescription, a regularization of gauge theory on $\mathbb{R}^d_\theta$ based on the finite compact case is therefore very desirable. Furthermore, the formulation as multi-matrix model leads to the hope that non-trivial results may be obtained using the sophisticated techniques from random matrix theory. We introduce in this paper such a matrix model for fuzzy $S^2 \times S^2$, and study its relationship with $\mathbb{R}^4_\theta$.

In the 2-dimensional case, this matrix-model approach to gauge theory has been studied in considerable detail for the fuzzy sphere S^2_N [5–10] and the noncommutative torus $\mathbb{T}^2_\theta$ [11–14], both on the classical and quantized level. It is well-known that $\mathbb{R}^2_\theta$ can be obtained as scaling limits of these spaces S^2_N and $\mathbb{T}^2_N$ at least locally, which suggests a correspondence also for the gauge theories. This correspondence of gauge theories has been studied in great detail for the case of $\mathbb{T}^2_\theta \to \mathbb{R}^2_\theta$ [12, 15, 16] on the quantized level, exhibiting the role of certain instanton contributions. A matching of gauge theory on the classical level can also be seen for $S^2_N \to \mathbb{R}^2_\theta$ [17, 18], which is implicitly contained in section 7 of the present paper.

In 4 dimensions, the quantization of gauge theory is more difficult, and a regularization using finite-dimensional matrix models is particularly important. The most obvious 4-dimensional spaces suitable for this purpose are $\mathbb{T}^4, S^2 \times S^2$ and $\mathbb{C}P^2$. On fuzzy $\mathbb{C}P^2_N$ [19–21], such a formulation of gauge theory was given in [22]. This can indeed be used to obtain $\mathbb{R}^4_\theta$ for the case of U(2) -invariant θ_{ij}. The case of $\mathbb{R}^2 \times S^2_N$ as regularization of $\mathbb{R}^4_\theta$

with degenerate θ_{ij} was considered in [18, 23], exhibiting a relation with a conventional non-linear sigma model. A formulation of lattice gauge theory for even-dimensional tori has been discussed in [24, 25, 14]. Related "fuzzy" solutions of the string-theoretical matrix models [26] were studied e.g. in [27, 28], see also [29].

In the present paper we give a definition of U(n) gauge theory on fuzzy $S^2_N \times S^2_N$, which can be used to obtain any $\mathbb{R}^4_\theta$ as a scaling limit. The action is a simple generalization of the matrix model approach of [7] for fuzzy S^2_N. It differs from similar string-theoretical matrix models [26] by adding a constraint-term, which ensures that the "vacuum" solution is stable and describes the product of 2 spheres. The fluctuations of the covariant coordinates then correspond as usual to the gauge fields, and the action reduces to ordinary Yang-Mills theory on $S^2 \times S^2$ in the limit $N \to \infty$. The quantization of the model is defined by a finite integral over the matrix degrees of freedom, which is shown to be convergent due to the constraint term. We also give a gauge-fixed action with BRST symmetry.

We then discuss some features of the model, in particular a hidden SO(6) invariance of the action which is broken explicitly by the constraint. This suggests some alternative formulations in terms of "collective matrices", which are assembled from the individual covariant coordinates (matrices). This turns out to be very useful to construct a Dirac operator, and may help to eventually study the quantization of the model. The stability of the model without constraint is also discussed, and we show that the only flat directions of the SO(6) -invariant action are fluctuations of the constant radial modes of the 2 spheres.

As a further test of the proposed gauge theory, we study in section 6 topologically non-trivial solutions (instantons) on $S^2_N \times S^2_N$. We find in particular a simple class of solutions which can be interpreted as U(1) instantons with quantized flux, combined with a singular, localized "flux tube". They are related to the so-called "fluxon" solutions of U(1) gauge theory on $\mathbb{R}^4_\theta$. Solutions which can be interpreted as 2-dimensional spherical branes wrapping one of the two spheres are also found and are matched with similar solutions on $\mathbb{R}^4_\theta$. We then study the relation of the model on $S^2_N \times S^2_N$ with Yang-Mills theory on $\mathbb{R}^4_\theta$, and demonstrate that the usual Yang-Mills action on $\mathbb{R}^4_\theta$ is recovered in the appropriate scaling limit. Some aspects of U(1) instantons ("fluxons") on $\mathbb{R}^4_\theta$ are recalled in section 7.2, and we show in detail how they arise as limits of the above non-trivial solutions on $S^2_N \times S^2_N$. In particular, we are able to match the moduli space of n fluxons, corresponding to their location on $\mathbb{R}^4_\theta$ resp. $S^2_N \times S^2_N$. We find in particular that even though the field strength in the "bulk" vanishes in the limit of $\mathbb{R}^4_\theta$, it does contribute to the action on $S^2_N \times S^2_N$ with equal weight as the localized flux tube. This can be interpreted on $\mathbb{R}^4_\theta$ as a topological or surface term at infinity. Another unexpected feature on $S^2_N \times S^2_N$ is the appearance of certain "superselection rules", restricting the possible instanton numbers. In other words, not all instanton numbers on $\mathbb{R}^4_\theta$ are reproduced for a given matrix size $\mathcal{N}$, however they can be found by considering matrices of different size. This depends on the precise form of the constraint term in the action, which is hence seen to imply also certain topological constraints. To recover the full space of ADHM solutions on $\mathbb{R}^4_\theta$ starting from $S^2_N \times S^2_N$ remains an open challenge, which is non-trivial since the concept of self-duality does not extend naturally to the fuzzy case.

We should mention here that topologically non-trivial configurations have also been discussed more abstractly in terms of projective modules using a somewhat different formulation of gauge theory on fuzzy spaces, see in particular [30, 31].

Finally in section 8 we include charged fermions in the fundamental representation of the gauge group, by giving a Dirac operator $\widehat{D}$ which in the large-N limit reduces to the ordinary gauged Dirac operator on $S^2 \times S^2$. This Dirac operator is covariant under the SO(6) symmetry of the embedding space $S^2 \times S^2 \subset \mathbb{R}^6$, and exactly anti-commutes with a chirality operator. The 4-dimensional physical Dirac spinors are obtained by suitable projections from 8-dimensional SO(6) spinors. This projection however commutes with $\widehat{D}$ only in the large-N limit, and is achieved by giving one of the 2 spinors a large mass. Weyl spinors can then be defined using the exact chirality operator.

2. The fuzzy spaces S^2_N and $S^2_{N_L} \times S^2_{N_R}$

We start by recalling the definition of the fuzzy sphere in order to fix our conventions and notation. The algebra of functions on the fuzzy sphere is the finite algebra S^2_N generated by hermitean operators $x_i = (x_1, x_2, x_3)$ satisfying the defining relations

$$[x_i, x_j] = i\Lambda_N \epsilon_{ijk} x_k \,, \tag{2.1}$$

$$x_1^2 + x_2^2 + x_3^2 = R^2 \,. \tag{2.2}$$

They are obtained from the N-dimensional representation of $su(2)$ with generators λ_i ($i = 1,2,3$) and commutation relations

$$[\lambda_i, \lambda_j] = i\epsilon_{ijk}\lambda_k \,, \qquad \sum_{i=1}^{3} \lambda_i \lambda_i = \frac{N^2-1}{4} \tag{2.3}$$

(see appendix A) by identifying

$$x_i = \Lambda_N \, \lambda_i \,, \qquad \Lambda_N = \frac{2R}{\sqrt{N^2-1}} \,. \tag{2.4}$$

The noncommutativity parameter Λ_N is of dimension length. The algebra of functions S^2_N therefore coincides with the simple matrix algebra $Mat(N, \mathbb{C})$. The normalized integral of a function $f \in S^2_N$ is given by the trace

$$\int_{S^2_N} f = \frac{4\pi R^2}{N} \mathrm{tr}(f) \,. \tag{2.5}$$

The functions on the fuzzy sphere can be mapped to functions on the commutative sphere S^2 using the decomposition into harmonics under the action

$$J_i f = [\lambda_i, f] \tag{2.6}$$

of the rotation group SU(2). One obtains analogs of the spherical harmonics up to a maximal angular momentum $N-1$. Therefore S^2_N is a regularization of S^2 with a UV

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cutoff, and the commutative sphere S^2 is recovered in the limit $N \to \infty$. Note also that for the standard representation (A.2), entries in the upper-left block of the matrices correspond to functions localized at $x_3 = R$. In particular, the fuzzy delta-function at the "north pole" is given by a suitably normalized projector of rank 1,

$$\delta^{(2)}_{NP}(x) = \frac{N}{4\pi R^2}\left|\frac{N-1}{2}\right\rangle\left\langle\frac{N-1}{2}\right| \tag{2.7}$$

where $|\frac{N-1}{2}\rangle$ is the highest weight state with maximal eigenvalue of λ_3. Delta-functions with arbitrary localization are obtained by rotating (2.7).

The simplest 4-dimensional generalization of the above is the product $S^2_{N_L} \times S^2_{N_R}$ of 2 such fuzzy spheres, with generally independent parameters $N_{L,R}$. It is generated by a double set of representations of $su(2)$ commuting with each other, i.e. by λ_i^L, λ_i^R satisfying

$$[\lambda_i^L, \lambda_j^L] = i\epsilon_{ijk}\lambda_k^L\,, \qquad [\lambda_i^R, \lambda_j^R] = i\epsilon_{ijk}\lambda_k^R\,,$$
$$[\lambda_i^L, \lambda_j^R] = 0$$

for $i, j = 1, 2, 3$, and Casimirs

$$\sum_{i=1}^{3} \lambda_i^L \lambda_i^L = \frac{N_L^2 - 1}{4}\,, \qquad \sum_{i=1}^{3} \lambda_i^R \lambda_i^R = \frac{N_R^2 - 1}{4}\,. \tag{2.8}$$

This can be realized as a tensor product of 2 fuzzy sphere algebras

$$\lambda_i^L = \lambda_i \otimes 1_{N_R \times N_R}\,, \tag{2.9}$$
$$\lambda_i^R = 1_{N_L \times N_L} \otimes \lambda_i\,, \tag{2.10}$$

hence as algebra we have $S^2_{N_L} \times S^2_{N_R} \cong \mathrm{Mat}(\mathcal{N}, \mathbb{C})$ where

$$\mathcal{N} = N_L N_R\,. \tag{2.11}$$

The normalized coordinate functions are given by

$$x_i^{L,R} = \frac{2R}{\sqrt{(N^{L,R})^2 - 1}}\,\lambda_i^{L,R}\,, \qquad \sum (x_i^L)^2 = R^2 = \sum (x_i^R)^2\,. \tag{2.12}$$

This space[1] can be viewed as regularization of $S^2 \times S^2 \subset \mathbb{R}^6$, and admits the symmetry group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \subset \mathrm{SO}(6)$. The generators $x_i^{L,R}$ should be viewed as coordinates in an embedding space $\mathbb{R}^6$. The normalized integral of a function $f \in S^2_{N_L} \times S^2_{N_R}$ is now given by

$$\int\limits_{S^2_{N_L} \times S^2_{N_R}} f = \frac{16\pi^2 R^4}{\mathcal{N}} \mathrm{tr}(f) = \frac{V}{\mathcal{N}} \mathrm{tr}(f)\,, \tag{2.13}$$

where we define the volume $V := 16\pi^2 R^4$. We will mainly consider $N_L = N_R$ in the following.

[1] In principle one could also introduce different radii $R^{L,R}$ for the 2 spheres, but for simplicity we will keep only one scale parameter R (and usually we will set $R = 1$).

2.1 The quantum plane limit $\mathbb{R}^4_\theta$

It is well-known [32] that if a fuzzy sphere is blown up near a given point, it can be used to obtain a (compactified) quantum plane: Consider the tangential coordinates $x_{1,2}$ near the "north pole". Setting

$$R^2 = \frac{N\theta}{2}\,, \tag{2.14}$$

they satisfy the commutation relations

$$[x_1, x_2] = i\frac{2R}{N}x_3 = i\frac{2R}{N}\sqrt{R^2 - x_1^2 - x_2^2} = i\theta\left(1 + O\left(\frac{1}{N}\right)\right). \tag{2.15}$$

Therefore in the large-N limit with (2.14) keeping θ fixed, we recover[2] the commutation relation of the quantum plane,

$$[x_1, x_2] = i\theta \tag{2.16}$$

up to corrections of order $1/N$. Similarly, starting with $S^2_{N_L} \times S^2_{N_R}$ and setting

$$R^2 = \frac{N_{L,R}\,\theta_{L,R}}{2}\,, \tag{2.17}$$

we obtain in the large N_L, N_R limit

$$\begin{aligned} [x_i^L, x_j^L] &= i\epsilon_{ij}\theta^L\,, \qquad [x_i^R, x_j^R] = i\epsilon_{ij}\theta^R\,, \\ [x_i^L, x_j^R] &= 0\,. \end{aligned} \tag{2.18}$$

This is the most general form of $\mathbb{R}^4_\theta$ with coordinates $(x_1, \ldots, x_4) \equiv (x_1^L, x_2^L, x_1^R, x_2^R)$ (after a suitable orthogonal transformation). The integral of a function $f(x)$ then becomes

$$\int\limits_{S^2_{N_L}\times S^2_{N_R}} f(x) \to 4\pi^2\theta_L\theta_R \mathrm{tr}(f(x)) =: \int\limits_{\mathbb{R}^4_\theta} f(x)\,, \tag{2.19}$$

which has indeed the standard normalization, giving each "Planck cell" the appropriate volume.

3. Gauge theory on fuzzy $S^2 \times S^2$

We start with the most general case, and construct a matrix model having $S^2_{N_L} \times S^2_{N_R}$ as its ground state. The fluctuations around this ground state will produce a gauge theory. A simplified and more elegant formulation in terms of "collective matrices" similar as in [7] for the fuzzy sphere will be given later in section 4.

In the fuzzy case, it is natural to construct $S^2_L \times S^2_R$ as "submanifold" of $\mathbb{R}^6$. We therefore consider a multi-matrix model with 6 dynamical fields ("covariant coordinates")

[2] One could be more sophisticated and use the stereographic projections as in [32], which leads essentially to the same results.

B_i^L and B_i^R ($i = 1,2,3$), which are $\mathcal{N} \times \mathcal{N}$ hermitean matrices. As action we choose the following generalization of the action in [7, 8],

$$S = \frac{1}{g^2} \int \frac{1}{2} F_{ia\,jb} F_{ia\,jb} + \varphi_L^2 + \varphi_R^2 \tag{3.1}$$

with $a, b = L, R$ and $i, j = 1, 2, 3$; summation over repeated indices is implied. Here $\varphi_{L,R}$ are defined as

$$\varphi_L := \frac{1}{R^2}\left(B_i^L B_i^L - \frac{N_L^2 - 1}{4}\right), \qquad \varphi_R := \frac{1}{R^2}\left(B_i^R B_i^R - \frac{N_R^2 - 1}{4}\right), \tag{3.2}$$

and the terms $\varphi_L^2 + \varphi_R^2$ in the action ensure that the unwanted radial degrees of freedom are suppressed [7, 8]. R denotes the radius of the two spheres, which we keep explicitly to have the correct dimensions. The field strength is defined by

$$\begin{aligned} F_{iL\,jL} &= \frac{1}{R^2}(i[B_i^L, B_j^L] + \epsilon_{ijk} B_k^L)\,, \\ F_{iR\,jR} &= \frac{1}{R^2}(i[B_i^R, B_j^R] + \epsilon_{ijk} B_k^R)\,, \\ F_{iL\,jR} &= \frac{1}{R^2}(i[B_i^L, B_j^R])\,. \end{aligned} \tag{3.3}$$

This model (3.1) is manifestly invariant under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ rotations acting in the obvious way, and $\mathrm{U}(\mathcal{N})$ gauge transformations acting as $B_i^{L,R} \to U B_i^{L,R} U^{-1}$. We will see below that this reduces indeed to the U(1) Yang-Mills action on $S^2 \times S^2$ in the commutative limit. Note that if the action (3.1) is considered as a matrix model, the radius drops out using (2.13). The equations of motion (e.o.m.) for B_i^L are

$$\begin{aligned} &\left\{B_i^L, B_j^L B_j^L - \frac{N_L^2 - 1}{4}\right\} + (B_i^L + i\epsilon_{ijk} B_j^L B_k^L) + \\ &+ i\epsilon_{ijk}[B_j^L, (B_k^L + i\epsilon_{krs} B_r^L B_s^L)] + [B_j^R, [B_j^R, B_i^L]] = 0\,, \end{aligned} \tag{3.4}$$

and those for B_i^R are obtained by exchanging $L \leftrightarrow R$. By construction, the minimum or ground state of the action is given by $F = \varphi = 0$, hence $B_i^{L,R} = \lambda_i^{L,R}$ as in (2.9), (2.10) up to gauge transformations; cp. [22] for a similar approach on $\mathbb{C}P^2$. We can therefore expand the "covariant coordinates" B_i^L and B_i^R around the ground state

$$B_i^a = \lambda_i^a + R A_i^a\,, \tag{3.5}$$

where $a \in \{L, R\}$ and A_i^a is small. Then $A_i^{L,R}$ transforms under gauge transformations as

$$A_i^{L,R} \to A_i'^{L,R} = U A_i^{L,R} U^{-1} + U[\lambda_i^{L,R}, U^{-1}]\,, \tag{3.6}$$

and the field strength takes a more familiar form,[3]

$$\begin{aligned} F_{iL\,jL} &= i\left(\left[\frac{\lambda_i^L}{R}, A_j^L\right] - \left[\frac{\lambda_j^L}{R}, A_i^L\right] + [A_i^L, A_j^L]\right), \\ F_{iR\,jR} &= i\left(\left[\frac{\lambda_i^R}{R}, A_j^R\right] - \left[\frac{\lambda_j^R}{R}, A_i^R\right] + [A_i^R, A_j^R]\right), \\ F_{iL\,jR} &= i\left(\left[\frac{\lambda_i^L}{R}, A_j^R\right] - \left[\frac{\lambda_j^R}{R}, A_i^L\right] + [A_i^L, A_j^R]\right). \end{aligned} \tag{3.7}$$

[3]We do not distinguish between upper and lower indices L, R.

So far, the spheres are described in terms of 3 cartesian covariant coordinates each. In the commutative limit, we can separate the radial and tangential degrees of freedom. There are many ways to do this; perhaps the most elegant for the present purpose is to note that the terms $\int \varphi_L^2 + \varphi_R^2$ in the action imply that $\varphi_{L,R}$ is bounded for configurations with finite action. Using

$$\varphi_L = \frac{\lambda_i^L}{R} A_i^L + A_i^L \frac{\lambda_i^L}{R} + A_i^L A_i^L \,, \tag{3.8}$$

and similarly for φ_R it follows that

$$x_i A_i^a + A_i^a x_i = O(\frac{\varphi}{N}) \tag{3.9}$$

for finite A_i^a. This means that A_i^a is tangential in the (commutative) large-N limit. Alternatively, one could consider $\phi_L = N\varphi_L$, which would acquire a mass of order N and decouple from the other fields.[4] The commutative limit of (3.1) therefore gives the standard action for electrodynamics on $S^2 \times S^2$,

$$S = \frac{1}{2g^2} \int\limits_{S^2 \times S^2} F^t_{ia\,jb} F^t_{ia\,jb}$$

with $a, b = L, R$ and $i, j = 1, 2, 3$. Here $F^t_{iL\,jR}$ denotes the usual tangential field strength. This can be seen most easily noting that e.g. at the north pole $x_3^{L,R} = R$, one can replace

$$i[\frac{\lambda_i^{L,R}}{R}, \cdot] \quad \to \quad -\varepsilon_{ij} \frac{\partial}{\partial x_j^{L,R}} \tag{3.10}$$

in the commutative limit, so that upon identifying the commutative gauge fields $A_i^{(cl)}$ via

$$A_i^{(cl)L,R} = -\varepsilon_{ij} A_i^{L,R} \tag{3.11}$$

the field strength is given by the standard expression $F^t_{iL\,jR} = \partial_i^L A_j^{(cl)R} - \partial_j^R A_i^{(cl)L}$ etc.

U(k) **gauge theory.** The above action generalizes immediately to the nonabelian case, keeping precisely the same action (3.1), (3.2) but replacing the matrices $B_i^{L,R}$ by $k\mathcal{N} \times k\mathcal{N}$ matrices, cp. [7]. Expanding them in terms of (generalized) Gell-Mann matrices, the same action (3.1) is the fuzzy version of nonabelian U(k) Yang-Mills on $S^2 \times S^2$.

4. A formulation based on SO(6)

The above action can be cast into a nicer form by assembling the matrices $B_i^{L,R}$ into bigger "collective matrices", following [7]. Since it is natural from the fuzzy point of view to embed $S^2 \times S^2 \subset R^6$ with corresponding embedding of the symmetry group $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_R \subset \mathrm{SO}(6)$, we consider

$$B_\mu = (B_i^L, B_i^R) \tag{4.1}$$

[4]The constraints $\varphi_L = 0 = \varphi_R$ could also be imposed by hand; however the suppression through the above terms in the action is more flexible, as we will see in section 6.

(Greek indices μ, ν denoting from now on all the six dimensions) to be the 6 -dimensional irrep of $so(6) \cong su(4)$. Since $(4) \otimes (4) = (6) \oplus (10)$, it is natural to introduce the intertwiners

$$\gamma_\mu = (\gamma_i^L, \gamma_i^R) = (\gamma_\mu)^{\alpha,\beta} \tag{4.2}$$

of $(6) \subset (4) \otimes (4)$. where α, β denote indices of (4). We could then assemble our dynamical fields into a single $4\mathcal{N} \times 4\mathcal{N}$ matrix

$$B = B_\mu \gamma_\mu + \text{const} \cdot \mathbb{1} \,. \tag{4.3}$$

Of course the most general such $4\mathcal{N} \times 4\mathcal{N}$ matrix contains far too many degrees of freedom, and we have to constrain these B further. Since SU(4) acts on B as $B \to U^T B U$, the γ_μ can be chosen as totally anti-symmetric matrices, which precisely singles out the $(6) \subset (4) \otimes (4)$. One can moreover impose

$$(\gamma_i^L)^\dagger = \gamma_i^L \,, \qquad (\gamma_i^R)^\dagger = -\gamma_i^R \,, \tag{4.4}$$

and

$$\gamma_i^L \gamma_j^L = \delta_{ij} + i\epsilon_{ijk}\gamma_k^L \,, \tag{4.5}$$

$$\gamma_i^R \gamma_j^R = -\delta_{ij} - \epsilon_{ijk}\gamma_k^R \,, \tag{4.6}$$

$$[\gamma_i^L, \gamma_j^R] = 0 \,, \tag{4.7}$$

which will be assumed from now on; we will give two explicit such representations in (B.5), (D.2). This would suggest to constrain B to be antisymmetric. However, the component fields B_μ are naturally considered as hermitean rather than symmetric matrices. Furthermore, since the $\gamma_\mu = (\gamma_\mu)^{\alpha,\beta}$ have two upper indices, they do not form an algebra. There are now 2 ways to proceed. We can either separate them again by introducing two $4\mathcal{N} \times 4\mathcal{N}$ matrices,

$$B^L = \frac{1}{2} + B_i^L \gamma_i^L \,, \qquad B^R = \frac{i}{2} + B_i^R \gamma_i^R \,, \tag{4.8}$$

breaking SO(6) $\to$ SO(3) $\times$ SO(3). This will be pursued in appendix B. Alternatively, we can use the γ_μ with the above properties to construct the 8×8 Gamma-matrices

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^{\mu\dagger} & 0 \end{pmatrix} , \tag{4.9}$$

which generate the SO(6)-Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = \begin{pmatrix} \gamma^\mu \gamma^{\nu\dagger} + \gamma^\nu \gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu + \gamma^{\nu\dagger}\gamma^\mu \end{pmatrix} = 2\delta^{\mu\nu} \,. \tag{4.10}$$

This suggests to consider the single hermitean $8\mathcal{N} \times 8\mathcal{N}$ matrix

$$C = \Gamma^\mu B_\mu + C_0 = \begin{pmatrix} 0 & B^L \\ B^L & 0 \end{pmatrix} + \begin{pmatrix} 0 & B^R \\ -B^R & 0 \end{pmatrix} =: C^L + C^R \,, \tag{4.11}$$

where $C_0 = C_0^L + C_0^R$ denote the constant 8×8-matrices

$$C_0^L = -\frac{i}{2}\Gamma_1^L\Gamma_2^L\Gamma_3^L = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.12}$$

$$C_0^R = -\frac{i}{2}\Gamma_1^R\Gamma_2^R\Gamma_3^R = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.13}$$

in the above basis. This is very close to the approach in [7], and using the Clifford algebra and the above definitions one finds indeed

$$C^2 = B_\mu B_\mu + \frac{1}{2} + \Sigma_8^{\mu\nu} F_{\mu\nu}\,. \tag{4.14}$$

Here $\Sigma_8^{\mu\nu} = -\frac{i}{4}[\Gamma_\mu, \Gamma_\nu]$, and the field strength $F_{\mu\nu}$ coincides with the definition in (3.3) if written in the $L-R$ notation,

$$F_{ia\,jb} = i[B_{ia}, B_{jb}] + \delta_{ab}\epsilon_{ijk}B_{ka}\,.$$

Therefore the action

$$S_6 = \mathrm{Tr}\left(\left(C^2 - \frac{N^2}{2}\right)^2\right) = 8\mathrm{tr}\left(B_\mu B_\mu - \frac{N^2-1}{2}\right)^2 + 4\mathrm{tr}F_{\mu\nu}F_{\mu\nu} \tag{4.15}$$

is quite close to what we want. The only difference is the term $(B_\mu B_\mu - \frac{N^2-1}{2})^2$ instead of $(B_{iL}B_{iL} - \frac{N_L^2-1}{4})^2 + (B_{iR}B_{iR} - \frac{N_R^2-1}{4})^2$, for $2N^2 = N_L^2 + N_R^2$. This difference is easy to understand: since (4.15) is SO(6)-invariant, the ground state should be some S^5. We therefore have to break this SO(6)- invariance explicitly, which will be done in the next section. However before doing that, let us try to understand action (4.15) better and see whether it leads to a meaningful 4-dimensional field theory. We show in appendix C by carefully integrating out the scalar components of $B_i^{L,R}$ that the SO(6)- invariant constraint term in (4.15) induces the second term in the following effective action

$$S_6^{\mathrm{eff}} \sim 4\mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu} - (F_{iL}x_{iL} - F_{iR}x_{iR})\frac{1}{4(\frac{1}{2} - \partial_\mu\partial_\mu)}(F_{iL}x_{iL} - F_{iR}x_{iR})\right) \tag{4.16}$$

in the commutative limit, where $F_{iL} = \frac{1}{2}\epsilon_{ijk}F_{jL\,kL}$ etc. Comparing the second term with $F_{\mu\nu}F_{\mu\nu}$, we see that the zero mode of the Laplace operator $\partial_\mu\partial_\mu$ can produce a contribution that cancels the corresponding contribution from $F_{\mu\nu}F_{\mu\nu}$, but that all higher modes are smaller by at least a factor of $2(\frac{1}{2} - \partial_\mu\partial_\mu)$. Therefore, the action (4.15) is positive definite except for the obvious zero mode $\delta B_i^L = \epsilon$, $\delta B_i^R = -\epsilon$. This means that the geometry of $S_L^2 \times S_R^2$ is locally stable even with the SO(6)-symmetry unbroken, except for opposite fluctuations of the radii.

4.1 Breaking SO(6) → SO(3) × SO(3)

To obtain the original action (3.1) for $S^2 \times S^2$, we have to break the SO(6)-symmetry down to SO(3) × SO(3). We can do this by using the left and right gauge fields C^L and

C^R introduced in (4.11) separately. Their squares are

$$C_L^2 = B_{iL}B_{iL} + \frac{1}{4} + \begin{pmatrix} \gamma_L^i & 0 \\ 0 & \gamma_L^i \end{pmatrix} (B_{iL} + i\epsilon_{ijk}B_{jL}B_{kL}),$$
$$C_R^2 = B_{iR}B_{iR} + \frac{1}{4} - i \begin{pmatrix} \gamma_R^i & 0 \\ 0 & \gamma_R^i \end{pmatrix} (B_{iR} + i\epsilon_{ijk}B_{jR}B_{kR}).$$

As both γ_L^i, γ_R^i and $\gamma_L^i\gamma_R^j$ are traceless, we have

$$S_{\rm break} := {\rm Tr}\Big(\Big(C_L^2 - \frac{N_L^2}{4}\Big)\Big(C_R^2 - \frac{N_R^2}{4}\Big)\Big) = 8{\rm Tr}\Big(\Big(B_{iL}B_{iL} - \frac{N_L^2-1}{4}\Big)\Big(B_{iR}B_{iR} - \frac{N_R^2-1}{4}\Big)\Big).$$

With these terms we can recover our action as

$$\begin{aligned} S &= S_6 - 2S_{\rm break} \\ &= {\rm Tr}\Big(\Big(C^2 - \frac{N^2}{2}\Big)^2 - \Big\{C_L^2 - \frac{N_L^2}{4}, C_R^2 - \frac{N_R^2}{4}\Big\}\Big) \\ &= 8\,{\rm tr}\Big(\Big(B_{iL}B_{iL} - \frac{N_L^2-1}{4}\Big)^2 + \Big(B_{iR}B_{iR} - \frac{N_R^2-1}{4}\Big)^2 + \frac{1}{2}F_{\mu\nu}F_{\mu\nu}\Big), \end{aligned} \tag{4.17}$$

which is precisely the action (3.1) for gauge theory on $S^2_{N_L} \times S^2_{N_R}$ omitting the overall constants. Hence the action is formulated as 2-matrix model, however with highly constrained matrices C_L, C_R. This formulation using the Gamma-matrices is very natural and useful if one wants to couple the gauge fields to fermions, as discussed in section 8.

For simplicity, we will only consider $N_L = N_R = N$ from now on.

5. Quantization

The quantization of the gauge theory defined by (3.1) or its reformulation (4.17) is straightforward in principle, by a "path integral" over the hermitean matrices

$$Z[J] = \int dB_\mu e^{-S[B_\mu] + {\rm tr} B_\mu J_\mu}. \tag{5.1}$$

Note that there is no need to fix the gauge since the gauge group $U(\mathcal{N})$ is compact. The above path integral is well-defined and finite for any fixed $\mathcal{N}$. To see this, it is enough to show that the integral $\int dB_\mu \exp(-(B_i^L B_i^L - (N^2-1)/4)^2 - (B_i^R B_i^R - (N^2-1)/4)^2)$ converges, since the contributions from the field strength further suppress the integrand. This integral is obviously convergent for any fixed N.

For perturbative computations it is necessary to fix the gauge, and to substitute gauge invariance by BRST-invariance. Such a gauge-fixed action will be presented next.

5.1 BRST symmetry

To construct a gauge-fixed BRST-invariant action, we have to introduce ghost fields c and anti-ghost fields $\bar{c}$. These are fermionic fields, more precisely $\mathcal{N}\times\mathcal{N}-$ matrices with entries which are Grassman variables.

The full gauge-fixed action reads:

$$S_{\text{BRST}} = S + \frac{1}{\mathcal{N}}\text{tr}\Big(\bar{c}[\lambda_\mu, [B_\mu, c]] - \Big(\frac{\alpha}{2}b - [\lambda_\mu, B_\mu]\Big)b\Big)\,,$$

where b is an auxiliary (Nakanishi-Lautrup) field. This action is invariant with respect to the following BRST-transformations:

$$sB_\mu = [B_\mu, c] \qquad sc = cc \tag{5.2}$$

$$s\bar{c} = b \qquad sb = 0 \tag{5.3}$$

(matrix product is understood), where the BRST-differential s acts on a product of fields as follows:

$$s(XY) = X(sY) + (-1)^{\varepsilon_Y}(sX)Y\,.$$

Here ε_Y denotes the Grassman-parity of Y

$$\varepsilon_Y = \begin{cases} 0 & Y\,\text{bosonic} \\ 1 & Y\,\text{fermionic}\,. \end{cases}$$

As usual, it is not difficult to check that these BRST-transformations are indeed nilpotent, i.e.

$$s^2 = 0\,.$$

Integrating out the auxiliary field b leads to the following action

$$S'_{\text{BRST}} = S + \frac{1}{\mathcal{N}}\text{tr}(\bar{c}[\lambda_\mu, [B_\mu, c]] - \frac{1}{2\alpha}[\lambda_\mu, B_\mu][\lambda_\nu, B_\nu])\,.$$

Setting $\alpha = 1$ corresponds to the Feynman gauge. This is indeed what one would obtain by the Faddeev-Popov procedure. The action S' is invariant with respect to the following operations:

$$\begin{aligned} s'B_\mu &= [B_\mu, c] \\ s'c &= cc \\ s'\bar{c} &= [\lambda_\mu, B_\mu]\,. \end{aligned}$$

Since we have used the equations of motion of b, the BRST-differential s' is *not* nilpotent off-shell anymore but still we have

$$s'^2|_{\text{on-shell}} = 0\,.$$

6. Topologically non-trivial solutions on $S^2_N \times S^2_N$

In order to understand better the non-trivial solutions found below, we first note that the classical space $S^2 \times S^2$ is symplectic with symplectic form

$$\omega = \omega^L + \omega^R\,, \tag{6.1}$$

where

$$\omega^L = \frac{1}{4\pi R^3}\epsilon_{ijk}x_i^L dx_j^L dx_k^L \tag{6.2}$$

and similarly ω^R. The normalization is chosen such that

$$\int_{S^2_{L,R}} \omega^{L,R} = 1 = \int_{S^2\times S^2} \omega^L \wedge \omega^R \tag{6.3}$$

so that ω^L, ω^R generate the integer cohomology $H^*(S^2 \times S^2, \mathbb{Z})$. Noting that ω is self-dual while $\tilde{\omega} := \omega^L - \omega^R$ is anti-selfdual, it follows immediately that both $F = 2\pi\omega$ and $F = 2\pi\tilde{\omega}$ are solutions of the abelian field equations. More generally, any

$$F^{(m_L,m_R)} = 2\pi m_L \omega^L + 2\pi m_R \omega^R \tag{6.4}$$

for any integers m_L, m_R is a solution. In bundle language, they correspond to products of 2 monopole bundles with connections and monopole number $m_{L,R}$ over $S^2_{L,R}$. Following the literature we will denote any such non-trivial solution as instanton.

6.1 Instantons and fluxons

We are interested in similar non-trivial solutions of the e.o.m. (3.4) in the fuzzy case. The monopole solutions on the fuzzy sphere S^2_N are given by representations λ_i^{N-m} of $su(2)$ of size $N-m$ [33], which lead to the classical monopole gauge fields in the commutative limit as shown in [7]. It is hence easy to guess that we will obtain solutions on $S^2_N \times S^2_N$ by taking products of these:

$$B_i^L = \alpha^L \lambda_i^{N-m_L} \otimes \mathbb{1}_{N-m_R}\,, \tag{6.5}$$

$$B_i^R = \alpha^R \mathbb{1}_{N-m_L} \otimes \lambda_i^{N-m_R} \tag{6.6}$$

where $\lambda_i^{N-m_{L,R}}$ are the $N-m_{L,R}$ dimensional generators of $su(2)$. It is not difficult to verify that these are solutions of (3.4) with $\alpha^{L,R} = 1 + \frac{m_{L,R}}{N}$ for $m_{L,R} \ll N$, with field strength

$$F_{iLjL} = -\frac{m^L}{2R^3}\epsilon_{ijk}x_k^L\,, \qquad F_{iRjR} = -\frac{m^R}{2R^3}\epsilon_{ijk}x_k^R\,, \qquad F_{iLjR} = 0\,, \tag{6.7}$$

while $B\cdot B - \frac{N^2-1}{4} \to 0$ as $N \to \infty$. This means that $F = -2\pi m^L\omega^L - 2\pi m^R\omega^R$ in the commutative limit, so that indeed

$$\int\limits_{S_2^{L,R}} \frac{F}{2\pi} = -m^{L,R}. \tag{6.8}$$

Notice that the Ansatz (6.6) implies that all matrices have size $\mathcal{N} = (N-m_L)(N-m_R)$, which is inconsistent if we require that $\mathcal{N} = N^2$ in order to have the original $S^2_N \times S^2_N$ vacuum. Therefore it appears that these solutions live in a different configuration space, similar as the commutative monopoles which live on different bundles. However, the situation is in fact more interesting: the above solutions can be embedded in the *same* configuration spaces of $N^2 \times N^2$ matrices as the vacuum solution if we combine them with other solutions, which have finite action in four dimensions.[5] They are in fact crucial to recover some of the known U(1) instantons in the limit $S^2_N \to \mathbb{R}^2_\theta$ resp. $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$, as we will see. Consider the following Ansatz

$$B_i^{L,R} = \mathrm{diag}(d_{i,1}^{L,R}, \ldots, d_{i,n}^{L,R})$$

in terms of diagonal matrices (ignoring the size of the matrices for the moment). These are solutions of (3.4) in two cases,

$$\sum_i d_{i,k}^{L,R} d_{i,k}^{L,R} = \begin{cases} \frac{N^2-3}{4}, & \text{type A} \\ 0, & \text{type B} \end{cases} \tag{6.9}$$

(i.e. $d_{i,k}^{L,R} = 0$ in type B). The associated field strength is

$$F_{iLjL} = \frac{\epsilon_{ijk}}{R^2} \,\mathrm{diag}(d_{k,1}^L, \ldots, d_{k,n}^L)\,, \qquad F_{LR} = 0\,, \tag{6.10}$$

and a similar formula for F_{iRjR}. The constraint term is then $(B \cdot B - \frac{N^2-1}{4}) \to -\frac{1}{2}$ for type A, and $(B \cdot B - \frac{N^2-1}{4}) \to -\frac{N^2-1}{4}$ for type B in the large-N limit. In particular, only the type A solutions will have a finite contribution

$$S_{\text{fluxon}} = \frac{V}{g^2 \mathcal{N}} \left(\frac{n}{4R^4} + \frac{2n}{R^4} \frac{N^2-3}{4} \right) \quad \to \quad \frac{8\pi^2}{g^2}\, n \tag{6.11}$$

to the action,[6] which for $N \to \infty$ is only due to the field strength. We will see below that these type A solutions can be interpreted as a localized flux or vortex, and we will call them “fluxons” since they will reduce in a certain scaling limit to solutions on $\mathbb{R}^4_\theta$ which are sometimes denoted as such [34–36].

One can now combine these “fluxon” solutions with the monopole solutions (6.6) in the form

$$\begin{aligned} B_i^L &= \begin{pmatrix} \alpha^L \lambda_i^{N-m_L} \otimes \mathbb{1}_{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^L, \ldots, d_{i,n}^L) \end{pmatrix}, \\ B_i^R &= \begin{pmatrix} \alpha^R \mathbb{1}_{N-m_L} \otimes \lambda_i^{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^R, \ldots, d_{i,n}^R) \end{pmatrix}. \end{aligned} \tag{6.12}$$

These are now matrices of size $\mathcal{N} = (N - m_L)(N - m_R) + n$, which must agree with $\mathcal{N} = N^2$, say. This is clearly possible for

$$m_L = -m_R = m\,, \qquad n = m^2\,, \tag{6.13}$$

while for $m_L \neq -m_R$ the contribution from the fluxons would be infinite since $n = O(N)$. To understand these solutions, we can compute the gauge field from (3.5),

$$\mathcal{A}_i^L = \frac{1}{R}\left(B_i^L - \lambda_i^N \otimes \mathbb{1}_N\right) = \mathcal{A}_i^L(x^L, x^R)\,. \tag{6.14}$$

[5] as opposed to 2 dimensions, which is the reason why they were not considered in [7].

[6] A finite action can also be obtained for the type B solution using a slightly modified action (6.19), as discussed below.

To evaluate this, we first have to choose a gauge, i.e. a unitary transformation U for (6.12) which allows to express e.g. $\lambda_i^{N-m_L} \otimes 1\!\!1_{N-m_R}$ in terms of $x_i^L \propto \lambda_i^N \otimes 1\!\!1_N$ and $x_i^R \propto 1\!\!1_N \otimes \lambda_i^N$. For example, in the case $m_L = -m_R = m$ this can be done using a unitary map

$$U: \ \mathbb{C}^{N-m} \otimes \mathbb{C}^{N+m} \oplus \mathbb{C}^{m^2} \to \mathbb{C}^N \otimes \mathbb{C}^N, \tag{6.15}$$

mapping a $(N-m) \times (N+m)$ matrix into a $N \times N$ matrix by trivially matching the upper-left corner in the obvious way, and fitting $\mathbb{C}^{m^2}$ into the remaining lower-right corner. With this being understood, one can write

$$\begin{aligned} R\mathcal{A}_i^L(x^L, x^R) &= (\alpha^L \lambda_i^{N-m} - \lambda_i^N) \otimes 1\!\!1_{N+m} + \lambda_i^N \otimes (1\!\!1_{N+m} - 1\!\!1_N) + (d - \text{terms}) \\ &= A_i^{(m)}(x^L) + \text{sing}(x_3^L = -R, x_3^R = -R) \end{aligned} \tag{6.16}$$

where $A_i^{(m)}(x^L)$ is indeed the gauge field of a monopole with charge m on S_L^2 in the large-N limit, as was checked explicitly in [7]. Here $\text{sing}(x_3^L = -R, x_3^R = -R)$ indicates a field localized at the "south pole" of S_L^2 and/or S_R^2 which becomes singular for large N. It originates both from "cutting and pasting" the bottom and right border of the above matrices using U (leading to singular gauge fields but regular field strength at the south poles), as well as the d-block (leading to a singular field strength). To see this recall that in general for the standard representation (A.2) of fuzzy spheres, entries in the lower-right block of the matrices correspond to functions localized at $x_3 = -R$, cp. (2.7). The gauge field near this singularity will be studied in more detail in section 7.3. The field strength is

$$F_{iLjL} = -\frac{m^L}{2R^3}\epsilon_{ijk}x_k^L + \epsilon_{ijk}\frac{1}{R^2}\sum_{i=1}^{n} d_{k,i}^L P_i \tag{6.17}$$

in the commutative limit, where P_i are projectors in the algebra of functions on $S_N^2 \times S_N^2$ of rank 1; recalling (2.7), they should be interpreted as delta-functions $P_i = \frac{V}{N^2}\,\delta^{(4)}(x_3 = -R)$. Similar formulae hold for $\mathcal{A}_i^L(x^L, x^R)$ and F_{iRjR}, while $F_{LR} = 0$.

We assumed above that these delta-functions are localized at the south poles $x_3^L = x_3^R = -R$. However, the location of these delta-functions can be chosen freely using gauge transformations. This can be seen by applying suitable successive gauge transformations using $N-k$-dimensional irreps of SU(2) for $k = 0, 1, \ldots, m-1$, which from the classical point of view all correspond to global rotations, successively moving the individual delta-peaks. Therefore the solution (6.12) should in general be interpreted as monopole on $S^2 \times S^2$ with monopole number $m_L = -m_R = m$, combined with a localized singular field strength characterized by its position and a vector $d_{k,i}^L$. We will see in section 7 that it becomes the "fluxon" solution in the planar limit $\mathbb{R}_\theta^4$; we therefore also call it a "fluxon".

The total action of these solutions (6.12) is the sum of the contributions from the monopole field plus the contribution from the fluxons (6.11), which both give the same contribution

$$S_{(m)} = \frac{4\pi^2}{g^2}\left(2m^2 + 2m^2\right) \tag{6.18}$$

in the large-N limit, using (6.13). The first term is due to the global monopole field (6.7), and the second term is the contribution of the fluxons through the localized field strength.

The interpretation of these solutions depends on the scaling limit $N \to \infty$ which we want to consider. We have seen that in the commutative limit keeping $R = \text{const}$, these solutions become commutative monopoles on $S^2 \times S^2$ with magnetic charges $m_L = -m_R$, plus additional localized "fluxon" degrees of freedom. For large R, the field strength of the monopoles vanishes, leaving only the localized fluxons. In particular, we will see in the following section that in the scaling limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$ only the fluxons survive and become well-known solutions for gauge theory on $\mathbb{R}^4_\theta$. Away from this localized fluxon the gauge field becomes a flat connection, which is however topologically nontrivial. This is very interesting as it shows that one can indeed use these fuzzy spaces as regularization for gauge theory on $\mathbb{R}^{2n}_\theta$.

A final remark is in order: if we fix the size $\mathcal{N}$ of the matrices, only certain fluxon and monopole numbers are allowed, given by (6.13). Otherwise the number n of fluxons and hence the action would diverge with N. This can be seen as an interesting feature of our model: viewed as a regularization of gauge theory on $\mathbb{R}^4_\theta$, this points to possible subtleties of defining the admissible field configurations in infinite-dimensional Hilbert spaces and relations with topological terms in the action. On the other hand, we could accommodate the most general solutions including also type B solutions (6.9) by modifying the action similar as in [7]. For example,

$$S = \frac{1}{g^2}\int \left(\frac{4B^L_iB^L_i}{N^2R^4}\left(B^L_iB^L_i - \frac{N_L^2-1}{4}\right)^2 + \frac{4B^R_iB^R_i}{R^4}\left(B^R_iB^R_i - \frac{N_R^2-1}{4}\right)^2 + \frac{1}{2}F_{ia,jb}F_{ia,jb}\right) \tag{6.19}$$

leads to the same commutative action, but with a vanishing action for the Dirac string in the type B solutions.

6.2 Spherical branes

Consider the following solutions

$$\begin{aligned} B^L_i &= \begin{pmatrix} \alpha^L\,\lambda^{N-m}_i & 0 \\ 0 & \mathrm{diag}(d_{i,1},\ldots,d_{i,m}) \end{pmatrix} \otimes \mathbb{1}_N\,, \\ B^R_i &= \mathbb{1}_N \otimes \lambda^N_i \end{aligned} \tag{6.20}$$

which are matrices of size $\mathcal{N} = N^2$. The corresponding field strength is

$$\begin{aligned} F_{iLjL} &= -\frac{m}{2R^3}\epsilon_{ijk}x^L_k + \epsilon_{ijk}\frac{1}{R^2}\sum_{i=1}^m d_{k,i}P_i \\ F_{RR} &= F_{LR} = 0 \end{aligned} \tag{6.21}$$

where P_i are projectors in the algebra of functions on S^2_L of rank 1 which should be interpreted as delta-functions $P_i = \frac{4\pi R^2}{N}\,\delta^{(2)}(x_3 = -R)$. In particular the gauge field A vanishes on S^2_R, while on S^2_L there is a monopole field together with a singularity at a point. This is similar to the fluxons on the previous section, but now only on S^2_L. This leads to the interpretation as 2-dimensional brane wrapping on S^2_R, located at a point on S^2_L. The action for these solutions is infinite. In the limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$, the flux will be located at a 2-dimensional hyperplane. Such solutions for gauge theory on $\mathbb{R}^4_\theta$ were found in [37, 4], which would be recovered in the scaling limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$ as discussed in section 7. In a similar way, we can interpret solutions with any m_L, m_R as branes wrapping on S^2_L and S^2_R.

7. Gauge theory on $\mathbb{R}^4_\theta$ from $S^2_{N_L} \times S^2_{N_R}$

We saw in section 2.1 that $\mathbb{R}^4_\theta$ can be obtained as a scaling limit of fuzzy $S^2_{N_L} \times S^2_{N_R}$. Here we will extend this scaling also to the covariant coordinates B_μ, thereby relating the gauge theory on $S^2_{N_L} \times S^2_{N_R}$ to that on $\mathbb{R}^4_\theta$ and hence providing a regularization for the latter. We will in particular relate the instanton solutions on these two spaces.

On noncommutative $\mathbb{R}^2_\theta$, all U(1)-instantons were constructed and classified in [4]. They can be interpreted as localized flux solutions, sometimes called fluxons. One can indeed recover these instantons from corresponding solutions on S^2_N, as we will show below. However since we are mainly interested in the 4-dimensional case here, we will only present the corresponding constructions on $S^2_{N_L} \times S^2_{N_R}$ resp. $\mathbb{R}^4_\theta$ here, without discussing the 2-dimensional case separately. It can be recovered in an obvious way from the considerations below.

The situation on $\mathbb{R}^4_\theta$ is more complicated, and there are different types of non-trivial U(1) "instanton" solutions on $\mathbb{R}^4_\theta$. Assuming that $\theta_{\mu\nu}$ is self-dual, there are two types of instantons: first, there exist straightforward generalizations of the localized "fluxon" solutions with self-dual field strength. These will be discussed in detail here, and we will show how these solutions can be recovered as scaling limits of the solutions (6.12) on $S^2_{N_L} \times S^2_{N_R}$. This is one of the main results of the present paper. In particular, the moduli of the fluxon solutions on $\mathbb{R}^4_\theta$ will be related to the free parameters $d_i^{L,R}$ in (6.12). This supports our suggestion to use gauge theory on $S^2_{N_L} \times S^2_{N_R}$ as a regularization for gauge theory on $\mathbb{R}^4_\theta$. However there are other types of U(1) instantons on $\mathbb{R}^4_\theta$ which were found through a noncommutative version of the ADHM equations [38 – 43], in particular anti-selfdual instantons which are much less localized than the fluxon solutions. To find the corresponding solutions on $S^2_{N_L} \times S^2_{N_R}$ is an interesting open challenge.

7.1 The action

The most general noncommutative $\mathbb{R}^4_\theta$ is generated by the coordinates subject to the commutation relations

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}\,, \tag{7.1}$$

where $\mu, \nu \in \{1, \ldots, 4\}$. Using suitable rotations, $\theta_{\mu\nu}$ can always be cast in the following form:

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}.$$

We will assume that $\theta_{12} > 0$ and $\theta_{34} > 0$ for simplicity in this section. Then define

$$X_{1,2} := \sqrt{\frac{2\theta_{12}}{N_L}}\, B^L_{1,2}\,, \tag{7.2}$$

$$X_{3,4} := \sqrt{\frac{2\theta_{34}}{N_R}}\, B^R_{1,2}\,, \tag{7.3}$$

$$\phi^{L,R} := B_3^{L,R} - \frac{N_{L,R}}{2} + \frac{1}{N_{L,R}}((B_1^{L,R})^2 + (B_2^{L,R})^2)\,, \tag{7.4}$$

which should be interpreted as a blow-up near the north pole. In the scaling limit (2.17),

$$R^2 = \frac{1}{2}N_L\theta_{34} = \frac{1}{2}N_R\theta_{12} \quad \to \quad \infty \tag{7.5}$$

the X will become the covariant coordinates on the "tangential" $\mathbb{R}^4_\theta$ as $N_{L,R} \to \infty$, and ϕ remains an auxiliary field. To see this, we compute for the field strength

$$\begin{aligned}
\frac{1}{R^2}([B_1^L, B_1^R]) &= \frac{1}{\theta_{12}\theta_{34}}[X_1, X_3]\,, \qquad \text{etc.,}\\
\frac{1}{R^2}(B_1^L + i[B_2^L, B_3^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\,(X_1 + i[X_2, \phi^L] - \frac{i}{2\theta_{12}}[X_2, (X_1)^2])\\
\frac{1}{R^2}(B_2^L + i[B_3^L, B_1^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\,(X_2 + i[X_1, \phi^L] - \frac{i}{2\theta_{12}}[X_1, (X_2)^2])\\
\frac{1}{R^2}(B_3^L + i[B_1^L, B_2^L]) &= \frac{1}{\theta_{12}\theta_{34}}(\theta_{12} + i[X_1, X_2] + \frac{\theta_{12}\theta_{34}}{R^2}\phi_L - \frac{\theta_{12}\theta_{34}^2}{2R^4}((X_1)^2 + (X_2)^2))\,.
\end{aligned}$$

Analogous expressions hold for B_i^R. For the potential term we get

$$\begin{aligned}
\frac{1}{R^2}\left(B_i^L B_i^L - \frac{N_L^2 - 1}{4}\right) = \frac{1}{\theta_{34}}\phi^L + \frac{2}{R^2}\left((\phi^L)^2 + \frac{1}{4}\right) - \frac{1}{\theta_{12}R^2}\{\phi^L, (X_1)^2 + (X_2)^2\} +\\
+\frac{1}{\theta_{12}^2 R^2}((X_1)^2 + (X_2)^2)^2\,.
\end{aligned}$$

We immediately see that the only terms from action (3.1) involving $\phi^{L,R}$ are

$$\frac{1}{\theta_{34}^2}(\phi^L)^2 + \frac{1}{\theta_{12}^2}(\phi^R)^2 + O\left(\frac{1}{R}\right),$$

and therefore we can integrate them out in the limit $R \to \infty$. In the leading order in R the remaining terms give the standard action

$$S = -\frac{1}{2g^2\theta_{12}^2\theta_{34}^2}\int([X_\mu, X_\nu] - i\theta_{\mu\nu})^2$$

for a gauge theory on $\mathbb{R}^4_\theta$ for general $\theta_{\mu\nu}$. The X_μ are interpreted as "covariant coordinates", which can be written as[7]

$$X_\mu := x_\mu + i\theta_{\mu\nu}A_\nu\,.$$

Hence the gauge fields A_μ describe the fluctuations around the vacuum. In particular, note that our regularization procedure clearly fixes the rank of the gauge group, unlike in the naive definition on $\mathbb{R}^d_\theta$ as discussed in [4]. The generalization to the U(n) case is obvious.

[7] We do not distinguish between upper and lower indices.

7.2 U(1) instantons on $\mathbb{R}^4_\theta$

The construction of instanton solutions for the two-dimensional noncommutative plane given in [4] can be easily generalized to the four-dimensional case. We shall recall and discuss these 4-dimensional "fluxon" solutions in some detail here, in order to understand the relation with the above solutions. To simplify the following formulas, we restrict our discussion from now on to the selfdual case

$$\theta_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma}$$

and denote

$$\theta := \theta_{12} = \theta_{34};$$

the generalizations to the antiselfdual and the general case are obvious. Then the action for U(1) gauge theory on $\mathbb{R}^4_\theta$ reads

$$S = \frac{(2\pi)^2}{2g^2\theta^2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \tag{7.6}$$

where

$$F_{\mu\nu} = i([X_\mu, X_\nu] - i\theta_{\mu\nu}) \tag{7.7}$$

is the field strength. In terms of the complex coordinates

$$x_{\pm L} := x_1 \pm ix_2 \ , \ x_{\pm R} := x_3 \pm ix_4 \, ,$$

the commutation relations (7.1) take the form

$$[x_{+a}, x_{-b}] = 2\theta\delta_{ab}\, , \qquad [x_{+a}, x_{+b}] = [x_{-a}, x_{-b}] = 0\, , \tag{7.8}$$

where $a, b \in \{L, R\}$. The Fock-space representation $\mathcal{H}$ of (7.8) has the standard basis

$$|n_1, n_2\rangle\, , \qquad n_1, n_2 \in \mathbb{N}\, ,$$

with

$$\begin{aligned} x_{-L}|n_1, n_2\rangle &= \sqrt{2\theta}\sqrt{n_1+1}|n_1+1, n_2\rangle\, , \ \ x_{+L}|n_1, n_2\rangle = \sqrt{2\theta}\sqrt{n_1}|n_1-1, n_2\rangle \\ x_{-R}|n_1, n_2\rangle &= \sqrt{2\theta}\sqrt{n_2+1}|n_1, n_2+1\rangle\, , \ x_{+R}|n_1, n_2\rangle = \sqrt{2\theta}\sqrt{n_2}|n_1, n_2-1\rangle\, . \end{aligned}$$

Similarly, using the complex covariant coordinates $X_{\pm a}$

$$X_{\pm L} = X_1 \pm iX_2 \ , \ X_{\pm R} = X_3 \pm iX_4 \tag{7.9}$$

and the corresponding field strength

$$F_{\alpha a,\beta b} = [X_{\alpha a}, X_{\beta b}] - 2\theta\varepsilon_{\alpha\beta}\delta_{ab}$$

with $a, b \in \{L, R\}$ and $\alpha, \beta \in \{+, -\}$, the action (7.6) can be written in the form

$$S = \frac{\pi^2}{g^2\theta^2}\,\mathrm{tr}\Big(\sum_a F_{+a,-a}F_{+a,-a} - \sum_{a,b} F_{+a,+b}F_{-a,-b}\Big)\, .$$

Then the equations of motion are given by:

$$\sum_{a,\alpha}[X_{\alpha a},(F_{\alpha a,\beta b})^{\dagger}]=0\,. \tag{7.10}$$

Let us consider a finite dimensional subvectorspace V_n of $\mathcal{H}$ of dimension n, which we can assume (using a unitary gauge transformation) to be spanned by a finite set of vectors $|n_1,n_2\rangle\in\mathcal{H}$,

$$V_n=\langle\{|i_k,j_k\rangle\,;\;k=1,\dots,n\}\rangle\,. \tag{7.11}$$

Following [4] one finds solutions to the equations of motion given by[8]

$$X^{(n)}_{+L}:=Sx_{+L}S^{\dagger}+\sum_{k=1}^{n}\gamma^{L}_{k}|i_k,j_k\rangle\langle i_k,j_k| \tag{7.12}$$

$$X^{(n)}_{+R}:=Sx_{+R}S^{\dagger}+\sum_{k=1}^{n}\gamma^{R}_{k}|i_k,j_k\rangle\langle i_k,j_k|\,. \tag{7.13}$$

Here $\gamma_k^{L,R}\in\mathbb{C}$ determine the position of the fluxons, and S denotes a partial isometry from $\mathcal{H}$ to $\mathcal{H}\backslash V_n$ with $S^{\dagger}S=\mathbb{1}$, $SS^{\dagger}=\mathbb{1}-P_{V_n}$, where

$$P_{V_n}:=\sum_{k=1}^{n}|i_k,j_k\rangle\langle i_k,j_k|$$

is the projection operator onto the subspace V_n. The field strength $F_{\mu\nu}$ for this solution is

$$F_{\mu\nu}=P_{V_n}\theta_{\mu\nu}\,.$$

In particular, the action corresponding to the instanton solution (7.12), (7.13) is proportional to the dimension of the subspace V_n

$$S[X^{(n)}_{\pm a}]=\frac{8\pi^2}{g^2}\,\mathrm{tr}(P_{V_n})=\frac{8\pi^2}{g^2}\,n\,.$$

We will see in the next section that this class of solutions can be reproduced by instanton solutions (6.12) on $S^2_{N_L}\times S^2_{N_R}$ in a suitable scaling limit. Let us stress again that this is only one class of U(1)-instanton solutions for $\mathbb{R}^4_\theta$ which is called "fluxons", since they can be interpreted as localized flux. The localization can be seen as follows: recall [44] that the above projection operators can be represented on the space of commutative functions (using a normal-ordering prescription) as

$$|k^1,k^2\rangle\langle k^1,k^2|\cong\frac{1}{k^1!k^2!}\left(\frac{x^{-L}}{\sqrt{2\theta}}\right)^{k^1}\left(\frac{x^{+L}}{\sqrt{2\theta}}\right)^{k^1}\left(\frac{x^{-R}}{\sqrt{2\theta}}\right)^{k^2}\left(\frac{x^{+R}}{\sqrt{2\theta}}\right)^{k^2}e^{-\frac{x^{+L}x^{-L}}{2\theta}-\frac{x^{+R}x^{-R}}{2\theta}}\,.$$

Hence the above field strengths $F_{\mu\nu}=P_{V_n}\theta_{\mu\nu}$ are superpositions of Gauss-functions which are localized in a region in space of size $\sqrt{\theta}$.

[8]Note that $[X^{(n)}_{+L},X^{(n)}_{+R}]=[X^{(n)}_{+L},X^{(n)}_{-R}]=[X^{(n)}_{-L},X^{(n)}_{+R}]=[X^{(n)}_{-L},X^{(n)}_{-R}]=0$.

7.3 Instantons on $\mathbb{R}^4_\theta$ from $S^2_N \times S^2_N$

With the scaling limit of section 7.1, the gauge theory on $S^2_N \times S^2_N$ provides us with a regularization for the gauge theory on $\mathbb{R}^4_\theta$. Of course, such a regularization might affect the topological features of the theory, an effect we want to investigate in this section. For this, we will map the topologically nontrivial solutions found in section 6 on $S^2_N \times S^2_N$ to $\mathbb{R}^4_\theta$.

Consider again the solutions (6.12) that combine the fluxon solutions with the monopoles, with the fluxons at the north pole instead of the south pole because we want to study their structure. Their scaling limit as in (7.2) gives

$$X_i = \sqrt{\frac{2\theta}{N}} \begin{pmatrix} \mathrm{diag}(d^L_{i,1}, \ldots, d^L_{i,n}) & 0 \\ 0 & \alpha^L \lambda_i^{N-m} \otimes \mathbb{1} \end{pmatrix}, \tag{7.14}$$

$$X_{i+2} = \sqrt{\frac{2\theta}{N}} \begin{pmatrix} \mathrm{diag}(d^R_{i,1}, \ldots, d^R_{i,n}) & 0 \\ 0 & \alpha^R \mathbb{1} \otimes \lambda_i^{N+m} \end{pmatrix} \tag{7.15}$$

for $i = 1, 2$. Recalling that the rescaled $\lambda_{1,2}$ on $S^2_{N_L} \times S^2_{N_R}$ become the $x_\pm$'s on $\mathbb{R}^4_\theta$ in the scaling limit

$$\sqrt{\frac{2\theta}{N}} (\lambda_1^{L,R} \pm i\lambda_2^{L,R}) \to x_{\pm L,R}\,,$$

we see that (7.14) and (7.15) become the instantons (7.12), (7.13) on $\mathbb{R}^4_\theta$,

$$X_1 + iX_2 \to X^{(n)}_{+L} = Sx_{+L}S^\dagger + \sum_{k=1}^{n} \gamma_k^L |i_k, j_k\rangle\langle i_k, j_k|\,, \tag{7.16}$$

$$X_3 + iX_4 \to X^{(n)}_{+R} = Sx_{+R}S^\dagger + \sum_{k=1}^{n} \gamma_k^R |i_k, j_k\rangle\langle i_k, j_k|\,. \tag{7.17}$$

Here the (d_i)-block acting on a basis $|i_k, j_k\rangle$ of $V_n \subset \mathcal{H} \cong \mathbb{C}^{\mathcal{N}}$ becomes the projector part of (7.16), (7.17) with

$$\sqrt{\frac{2\theta}{N}}\, d^{L,R}_{1,k} \to \mathrm{Re}\gamma_k^{L,R}\,,$$
$$\sqrt{\frac{2\theta}{N}}\, d^{L,R}_{2,k} \to \mathrm{Im}\gamma_k^{L,R}\,, \tag{7.18}$$

and the monopole block becomes $Sx_+S^\dagger$ where S is a partial isometry from $\mathcal{H}$ to $\mathcal{H}\backslash V_n$. Note that we can recover any value for the γ's in this scaling, solving the constraint $d_i d_i = \frac{N^2-3}{4}$ for $d_3 \sim \frac{N}{2}$. Therefore the full moduli space of the fluxon solutions (7.12), (7.13) on $\mathbb{R}^4_\theta$ can be recovered in this way. Furthermore, the meaning of the parameters $\gamma^{L,R}$ is easy to understand in our approach: Note first that using a rotation (which acts also on the indices) followed by a gauge transformation, the d_i can be fixed to be radial at the north pole, $d_i^{L,R} \sim (0, 0, N/2)$. This is a fluxon localized at the north pole. Now apply a "translation" at the north pole, which corresponds to a suitable rotation on the sphere. Rotating the vector $d_i^{L,R}$ in the scaling limit amounts to a translation of the $\gamma_k^{L,R}$ according to (7.18), which therefore parametrize the position of the fluxons.

It has been noted [2] that the $Sx_+S^\dagger$ correspond to a pure (but topologically nontrivial) gauge, which can qualitatively be seen already in two dimensions. There, the isomorphism $S : |k\rangle \to |k+n\rangle$ is basically $(\frac{x_-}{\sqrt{x_-x_+}})^n \sim (\frac{x-iy}{r})^n \sim e^{in\varphi}$ and therefore the gauge field $A_i = S\partial_i S^\dagger$ has a winding number n. The topological nature of the $Sx_+S^\dagger$ is even more evident in our setting, as they are the limit of the monopole solutions (6.5), (6.6) on $S^2_N \times S^2_N$. Moreover, note that their contribution to the action (6.18) survives the scaling: even though the field strength vanishes as $R \to \infty$, the integral gives a finite contribution equal to the contribution of the fluxon part. This topological "surface term" is usually omitted in the literature on $\mathbb{R}^4_\theta$, but becomes apparent in the regularized theory.

So it seems that we recovered all the instantons of section 7.2, but in fact there is an important detail that we haven't discussed jet. It is the embedding of the n-dimensional fluxons and the $(N-m)(N+m)$-dimensional monopole solutions into the N^2-dimensional matrices of the ground state. Such an embedding is clearly only possible for $n = m^2$. This means that the regularized theory has some kind of "superselection rule" for the dimension of the allowed instantons, a rule that did not exist in the unregularized theory.[9]

One way to allow arbitrary instanton numbers is to allow the size $\mathcal{N}$ of the matrices to vary. However, this is less satisfactory as it destroys the unification of topological sectors which is a beautiful feature of noncommutative gauge theory. On the other hand, the type B solutions (6.9) together with the changed action (6.19) might allow the construction of the missing instantons. The idea is to fill up the unnecessary $m^2 - n$ places with $d_i = 0$. The changed action would not suppress such solutions any more, and in fact they would not even contribute to the action. This amounts to adding a discrete sector to the theory which accommodates these type B solutions, but decouples from the rest of the model. Whether or not one wants to do this appears to be a matter of choice. This emphasizes again the importance of a careful regularization of the theory. It would be very interesting to see what happens in other regularizations e.g. using gauge theory on noncommutative tori or fuzzy $\mathbb{C}P^2$.

8. Fermions

8.1 The commutative Dirac operator on $S^2 \times S^2$

To find a form of the commutative Dirac operator on $S^2 \times S^2$ which is suitable for the fuzzy case, one can generalize the approach of [45] for S^2, which is carried out in detail in appendix E.3: One can write the flat SO(6) Dirac operator D_6 in 2 different forms, using spherical coordinates of the spheres and also using the usual flat euclidean coordinates. Then one can relate D_6 with the curved four-dimensional Dirac operator D_4 on $S^2 \times S^2$ in the same spherical coordinates. This leads to an explicit expression for D_4 involving only the angular momentum generators, which is easy to generalize to the fuzzy case. The

[9] Note that this is different in two dimensions. There, a rank n fluxon can be combined with a $(N-n)$-dimensional monopole block and all the instantons on $\mathbb{R}^2_\theta$ can be recovered. Furthermore, the actions for the fluxons and the monopoles scale differently with N. Therefore the action for the monopoles vanishes in the scaling limit that produces a gauge theory on $\mathbb{R}^2_\theta$ with rescaled coupling constant.

result is rather obvious and easy to guess:

$$D_4 = \Gamma^\mu J_\mu + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Gamma^\mu J_\mu + 2C_0 \, , \tag{8.1}$$

which is clearly a SO(3) × SO(3)-covariant hermitean first-oder differential operator. Here Γ^μ generate the SO(6) Clifford algebra (4.10), C_0 is defined in (4.13), and we put $R = 1$ for simplicity here. However this Dirac operator is reducible, acting on 8-dimensional spinors Ψ_8 corresponding to the SO(6) Clifford algebra. Hence Ψ_8 should be a combination of two independent 4-component Dirac spinors on the 4-dimensional space $S^2 \times S^2$. To see this, we will construct explicit projectors projecting onto these 4-dimensional spinors, and identify the appropriate 4-dimensional chirality operators. This will provide us with the desired physical Dirac or Weyl fermions.

8.1.1 Chirality and projections for the spinors

There are 3 obvious operators which anti-commute with D_4. One is the usual 6-dimensional chirality operator

$$\Gamma := i\Gamma_1^L \Gamma_2^L \Gamma_3^L \Gamma_1^R \Gamma_2^R \Gamma_3^R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{8.2}$$

which satisfies

$$\{D_4, \Gamma\} = 0 \, , \qquad \Gamma^\dagger = \Gamma \, , \qquad \Gamma^2 = 1 \, . \tag{8.3}$$

The 8-component spinors Ψ_8 split accordingly into two 4-component spinors $\Psi_8 = (\frac{\psi_\alpha}{\overline{\psi}_\beta})$, which transform as (4) resp. $(\overline{4})$ under $so(6) \cong su(4)$; recall the related discussion in section 4. The other operators of interest are

$$\chi_L = \Gamma^{iL} x_{iL} \qquad \text{and} \chi_R = \Gamma^{iR} x_{iR} \, .$$

They preserve SO(3) × SO(3) ⊂ SO(6), and satisfy

$$\{D_4, \chi_{L,R}\} = 0 = \{\chi_L, \chi_R\}$$

as well as

$$\chi_{L,R}^2 = 1 \, .$$

We will also use

$$\chi = \frac{1}{\sqrt{2}} \Gamma^\mu x_\mu = \frac{1}{\sqrt{2}} (\chi_L + \chi_R) \tag{8.4}$$

which satisfies similar relations. This means that

$$P_\pm = \frac{1}{2}(1 \pm i\chi_L \chi_R) \tag{8.5}$$

with

$$P_\pm^2 = P_\pm \, , \qquad P_+ + P_- = 1 \qquad \text{and} \qquad P_+ P_- = 0 \tag{8.6}$$

are hermitean projectors commuting with the Dirac operator on $S^2 \times S^2$ as well as with Γ,

$$P_\pm^\dagger = P_\pm \qquad \text{and} \qquad [P_\pm, D_4] = [P_\pm, \Gamma] = 0 \, . \tag{8.7}$$

Therefore they project onto subspaces which are preserved by D_4 and Γ, and are invariant under $SO(3) \times SO(3)$. Hence the spinor lagrangian can be written as

$$\Psi_8^\dagger D_4 \Psi_8 = \Psi_+^\dagger D_4 \Psi_+ + \Psi_-^\dagger D_4 \Psi_-$$

involving two Dirac spinors $\Psi_\pm = P_\pm \Psi_8$. In order to get one 4-component Dirac spinor, we can e.g. impose the constraint

$$P_+ \Psi_8 = \Psi_8 \,, \tag{8.8}$$

or equivalently give one of the two components a large mass, by adding a term

$$M_- \Psi_8^\dagger P_- \Psi_8 \tag{8.9}$$

to the action with $M_- \to \infty$. The physical chirality operator is now identified using (8.7) and (8.3) as Γ acting on Ψ_+. It can be used to define 2-component Weyl spinors on $S^2 \times S^2$.

To make the above more explicit, consider the north-pole of the spheres, i.e.

$$x_L = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad x_R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

In the basis (4.9) for the Clifford algebra we then get explicitly

$$P_\pm = \frac{1}{2}\left(1 \pm i \begin{pmatrix} -\gamma_L^1 \gamma_R^1 & 0 \\ 0 & \gamma_L^1 \gamma_R^1 \end{pmatrix}\right) = \frac{1}{2}(1 \pm \sigma_3 \otimes \sigma_3 \otimes \sigma_3) .$$

This means that

$$P_+ = \mathrm{diag}(1,0,0,1,0,1,1,0)$$

projects onto a 4-dimensional subspace exactly as expected.

8.2 Gauged fuzzy Dirac and chirality operators

To find fuzzy analogues of (8.1) and (8.4) coupled to the gauge fields, we recall the connection between the gauge theory on $S^2 \times S^2$ and the SO(6) Gamma matrices established in section 4. In the spirit of that section a natural fuzzy spinor action would involve

$$\Psi^\dagger C \Psi \,, \tag{8.10}$$

where Ψ is now a $8\mathcal{N} \times \mathcal{N}$-matrix (with Grassman entries). Of course, (8.10) does not have the appropriate commutative limit, but we can split C into a fuzzy Dirac operator $\widehat{D}$ and the operator $\widehat{\chi}$ defined by

$$\widehat{\chi}\Psi = \frac{\sqrt{2}}{N}\left(\Gamma^\mu \Psi \lambda_\mu - C_0 \Psi\right) , \tag{8.11}$$

which generalizes (8.4); we used here the definition (4.12),(4.13) of C_0. This operator satisfies

$$\widehat{\chi}^2 = 1 \,,$$

and reduces to (8.4) in the commutative limit. Note also that $\widehat{\chi}$ commutes with gauge transformations, since the coordinates λ_μ are acting from the right in (8.11). Setting

$$\widehat{J}_\mu \Psi = [\lambda_\mu, \Psi]\,,$$

we get for the fuzzy Dirac operator

$$\widehat{D} = C - \frac{N}{\sqrt{2}}\widehat{\chi} = \Gamma^\mu(\widehat{J}_\mu + A_\mu) + 2C_0 = \Gamma^\mu \widehat{\mathcal{D}}_\mu + 2C_0\,. \tag{8.12}$$

Here[10]

$$\widehat{\mathcal{D}}_\mu := \widehat{J}_\mu + A_\mu \tag{8.13}$$

is a covariant derivative operator, i.e. $U\widehat{\mathcal{D}}_\mu\psi = \widehat{\mathcal{D}}'_\mu U\psi$ which is easily verified using (3.6). This $\widehat{D}$ clearly has the correct classical limit (8.1) for vanishing A, and the gauge fields are coupled correctly. In particular, this definition of $\widehat{D}$ applies also to the topologically non-trivial solutions of section 6 without any modifications. Moreover, the chirality operator Γ as defined in (8.2) anti-commutes with $\widehat{D}$ also in the fuzzy case,

$$\{\widehat{D}, \Gamma\} = 0\,. \tag{8.14}$$

In particular there is no need to consider e.g. fuzzy Ginsparg-Wilson operators as in the 2-dimensional case [46 – 48]. However, the anticommutator of $\widehat{D}$ and $\widehat{\chi}$ no longer vanishes. We find

$$\{\widehat{D}, \widehat{\chi}\} = -\frac{\sqrt{2}}{N}\Big(2(\lambda_\mu + A_\mu)\widehat{J}_\mu - 2A_\mu\lambda_\mu + \{\Gamma^\mu, C_0\}\widehat{\mathcal{D}}_\mu + 2\Big) = O\Big(\frac{1}{N}\Big)\,, \tag{8.15}$$

since $x_\mu J_\mu = O(1/N)$ and $x_\mu A_\mu = O(1/N)$ using (3.9). Furthermore, using some identities given at the beginning of section 4 we obtain for $\widehat{D}^2\psi$:

$$\begin{aligned}\widehat{D}^2\psi &= (\Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu + \{\Gamma^\mu, C_0\}\widehat{\mathcal{D}}_\mu + 2)\psi \\ &=: (\Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\Box} + 2)\psi\,,\end{aligned} \tag{8.16}$$

defining the covariant 4-dimensional laplacian $\widehat{\Box}$ acting on the spinors. This corresponds to the usual expression for $\widehat{D}^2$ on curved spaces, and the constant 2 is due to the curvature scalar. Since $\widehat{D}^2$ and $\Sigma^{\mu\nu}F_{\mu\nu}$ are both hermitean and commute with Γ and $\widehat{P}_\pm$ as defined in (8.18) in the large-N limit, it follows that $\widehat{\Box}$ satisfies these properties as well. Note that (8.16) can also be written as

$$(\widehat{D} - C_0)^2 = \Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu + \frac{1}{2}\,, \tag{8.17}$$

which might suggest to interpret $\widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu$ as covariant laplacian; however this is not correct since $\widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu$ does not commute with the projections $\widehat{P}_\pm$ (8.18) even in the commutative limit. The reason for this is our formulation using spinors based on the SO(6) Clifford algebra rather than SO(4) spinors and comoving frames. The corresponding projections to physical Dirac- or Weyl-spinors in the fuzzy case will be discussed next.

[10] We set $R = 1$ in this section for simplicity.

8.2.1 Projections for the fuzzy spinors

For the fuzzy case, we can again consider the following projection operators

$$\begin{aligned}\widehat{\chi}_L\Psi &= \frac{2}{N}(\Gamma^{iL}\Psi\lambda_{iL} + C_0^L\Psi)\,,\\ \widehat{\chi}_R\Psi &= \frac{2}{N}(\Gamma^{iR}\Psi\lambda_{iR} + C_0^R\Psi)\end{aligned}$$

which satisfy

$$\widehat{\chi}_{L,R}^2 = 1\,, \qquad \{\widehat{\chi}_L, \widehat{\chi}_R\} = 0\,.$$

This implies $(\widehat{\chi}_L\widehat{\chi}_R)^2 = -1$, and we can write down the following projection operators

$$\widehat{P}_\pm = \frac{1}{2}(1 \pm i\widehat{\chi}_L\widehat{\chi}_R) \tag{8.18}$$

which have the classical limit (8.5) and the properties (8.6). However, the projector no longer commutes with the fuzzy Dirac operator (8.12):

$$\begin{aligned}[\widehat{D}, \widehat{\chi}_L\widehat{\chi}_R] &= \{\widehat{D}, \widehat{\chi}_L\}\widehat{\chi}_R - \widehat{\chi}_L\{\widehat{D}, \widehat{\chi}_R\}\\ &= -\frac{2}{N}\Big((2(\lambda_{iL} + \mathcal{A}_{iL})\widehat{J}_{iL} - 2A_{iL}\lambda_{iL} + 2C_0^L\,\Gamma^{iL}\widehat{\mathcal{D}}_{iL} + 1)\widehat{\chi}_R -\\ &\qquad - \widehat{\chi}_L(2(\lambda_{iR} + \mathcal{A}_{iR})\widehat{J}_{iR} - 2A_{iR}\lambda_{iR} + 2C_0^R\,\Gamma^{iR}\widehat{\mathcal{D}}_{iR} + 1)\Big)\,,\end{aligned}$$

which only vanishes for $N \to \infty$ and tangential A_μ (3.9). To reduce the degrees of freedom to one Dirac 4-spinor, we should therefore add a mass term

$$M_-\Psi_8^\dagger\widehat{P}_-\Psi_8 \tag{8.19}$$

which for $M_- \to \infty$ suppresses one of the spinors, rather than impose an exact constraint as in (8.8). This is gauge invariant since $\widehat{P}_\pm$ commutes with gauge transformations,

$$\widehat{P}_\pm\psi \to U\widehat{P}_\pm\psi\,.$$

The complete action for a Dirac fermion on fuzzy $S_N^2 \times S_N^2$ is therefore given by

$$S_{\text{Dirac}} = \int \Psi_8^\dagger(\widehat{D} + m)\Psi_8 + M_-\Psi_8^\dagger\widehat{P}_-\Psi_8 \tag{8.20}$$

with $M_- \to \infty$. The physical chirality operator is given by Γ (8.2), which allows to consider Weyl spinors as well.

9. Conclusion and outlook

We have constructed $\mathrm{U}(n)$ gauge theory on fuzzy $S_N^2 \times S_N^2$ as a multi-matrix model. The model is completely finite, and can be considered as a regularization either of Yang-Mills on the commutative $S^2 \times S^2$, or on the noncommutative $\mathbb{R}_\theta^4$ in a suitable scaling limit. The quantization is defined by a finite "path" integral over the matrix degrees of freedom, which is convergent due to the constraint term. A gauge-fixed action with BRST symmetry is also provided. We then discussed some topologically non-trivial solutions in the U(1) case, which reduce to the known "fluxon" solutions on $\mathbb{R}^4_\theta$ in the appropriate scaling limit, reproducing the full moduli space. On $S^2_N \times S^2_N$ they arise as localized flux tubes together with a monopole background field. This provides a very clean non-perturbative definition of noncommutative gauge theory with fixed rank of the gauge group U(n), and a simple description of instantons as solutions of the equation of motion in one single configuration space. Furthermore, we have shown how charged fermions in the fundamental representation can be coupled to the gauge field, by defining a suitable Dirac operator $\widehat{D}$. This is easily extended to Weyl fermions using a chirality operator which exactly anticommutes with $\widehat{D}$. All this supports the programme to formulate and study physically interesting models on noncommutative spaces.

There are many interesting conclusions and applications to be explored. One crucial feature is the fact that the model is completely regularized, i.e. the quantization is well-defined without any divergences for finite N. This should allow to study suitable scaling limits in N in a well-defined framework, and the emergence of an interesting low-energy limit which could be either commutative or noncommutative. Such a matrix regularization is very interesting in view of the UV/IR mixing, which indicates a close relationship between NC field theory and matrix models. For example, one might try to extend the results in [49] in this context. We also explored some alternative formulations using "collective matrices" based on SO(6). Such formulations are possible only in the noncommutative case, and lead to the hope that new non-perturbative techniques in the spirit of random matrix theory may be developed along these lines.

Another important aspect is the coupling to fermions, which could be extended to scalars and allows to study spontaneous symmetry breaking and the possible generation of other gauge groups in the low-energy limit. Finally, a detailed comparison with other finite models of NC gauge theory in 4 dimensions such as [22, 24, 25] would be very desirable, to see which features are generic and which are model-dependent.

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A. The standard representation of the fuzzy sphere

The irreducible N-dimensional representation of the $su(2)$ algebra λ_i (2.3) is given by

$$(\lambda_3)_{kl} = \delta_{kl}\,\frac{N+1-2k}{2}\,, \tag{A.1}$$

$$(\lambda_+)_{kl} = \delta_{k+1,l}\sqrt{(N-k)k}, \tag{A.2}$$

where $k, l = 1, \ldots, N$ and $\lambda_\pm = \lambda_1 \pm i\lambda_2$.

B. Alternative formulation using $4\mathcal{N} \times 4\mathcal{N}$ matrices

Let us rewrite the action (4.17) in terms of the $4\mathcal{N} \times 4\mathcal{N}$ matrices B_L, B_R (4.8). Noting that

$$C_L C_R + C_R C_L = \begin{pmatrix} -[B_L, B_R] & 0 \\ 0 & [B_L, B_R] \end{pmatrix} \tag{B.1}$$

we can rewrite S_6 (4.15) as

$$S_6 = 2\mathrm{Tr}\left(B_L^2 - B_R^2 - \frac{N^2}{2}\right)^2 + 2\mathrm{Tr}\left([B_L, B_R]^2\right), \tag{B.2}$$

where the trace is now over $4\mathcal{N} \times 4\mathcal{N}$ matrices. Similarly

$$S_{\rm break} = -2\mathrm{Tr}\left(B_L^2 - \frac{N^2}{4}\right)\left(-B_R^2 - \frac{N^2}{4}\right) \tag{B.3}$$

and combined we recover (3.1) as

$$S = S_6 - 2S_{\rm break} = 2\mathrm{Tr}\left((B_L^2 - \frac{N^2}{4})^2 + (-B_R^2 - \frac{N^2}{4})^2 + [B_L, B_R]^2\right). \tag{B.4}$$

This looks like a 2-matrix model, however the degrees of freedom B_L, B_R are still very much constrained and span only a small subspace of the $4\mathcal{N} \times 4\mathcal{N}$ matrices. We would like to find an intrinsic characterization without using the γ_μ explicitly. One possibility is to choose the γ_μ to be completely anti-symmetric matrices, see appendix D. However this does not extend to B, since the B_μ should be hermitean and not necessarily symmetric, and moreover the γ_μ are not hermitean (the conjugate being the intertwiner $(6) \subset (\overline{4}) \otimes (\overline{4})$). Another possibility is provided by the following representation of the γ-matrices:

$$\gamma_L^i = \sigma^i \otimes 1\!\!1_{2\times 2}, \qquad \gamma_R^i = 1\!\!1_{2\times 2} \otimes i\sigma^i. \tag{B.5}$$

They satisfy the relations (4.4)–eqg-L-R, but are not antisymmetric. Now note that

$$\gamma_R^i = iP\gamma_L^i P \tag{B.6}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(1 + \sigma^i \otimes \sigma^i) \tag{B.7}$$

permutes the two tensor factors and satisfies

$$P^2 = 1. \tag{B.8}$$

Therefore we can characterize the degrees of freedom in terms of 2 hermitean $2\mathcal{N} \times 2\mathcal{N}$ matrices

$$X_L = B_L^i \sigma_i + \frac{1}{2}, \qquad X_R = B_R^i \sigma_i + \frac{1}{2} \tag{B.9}$$

which are arbitrary up to the constraint that $X^0_{L,R} = \frac{1}{2}$. Then

$$B_L = X_L \otimes 1\!\!1_{2\times 2}\,, \qquad B_R = iP(X_R \otimes 1\!\!1_{2\times 2})P\,; \tag{B.10}$$

they could be extracted from a single complex matrix $\tilde{B} = (X_L + iX_R) \otimes 1\!\!1_{2\times 2}$. Furthermore, matrices of the form $X \otimes 1\!\!1_{2\times 2}$ are characterized through their spectrum, which is doubly degenerate; indeed any such hermitean matrix can be cast into the above form using suitable unitary SU($4\mathcal{N}$) transformations. Similarly, P can also be characterized intrinsically: any matrix P written as

$$P = P_0 \otimes 1\!\!1_{2\times 2} + P_i \otimes \sigma^i \tag{B.11}$$

which satisfies the constraints

$$P_0 = \frac{1}{2}\,, \qquad P^2 = 1\!\!1 \tag{B.12}$$

is given by (B.7) up to an irrelevant unitary transformation $U \otimes 1\!\!1$. We could therefore write down the action (B.4) in terms of three matrices $B_L, -iPB_RP$ and P, all of which are characterized by their spectrum and constraints of the form $(..)_0 = \frac{1}{2}$. The hope is that such a reformulation may allow to apply some of the powerful methods from random matrix theory, in the spirit of [7]. However we will leave this for future investigations.

C. Stability analysis of the SO(6)-invariant action (4.15)

Consider the action (4.15). We will split off the radial degrees of freedom for large-N by setting $R = 1$ and[11]

$$B_{iL} = \lambda_{iL} + A_{iL} = \lambda_{iL} + \mathcal{A}_{iL} + x_{iL}\Phi_L$$

requiring that $\lambda_{iL}\mathcal{A}_{iL} = 0$, and similarly for B_{iR}, The stability of our geometry will depend on the behavior of Φ^L and Φ^R. We calculate that

$$B_\mu B_\mu - \frac{N^2-1}{2} = N(\Phi_L + \Phi_R) + \Phi_L\Phi_L + \Phi_R\Phi_R + \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + O\left(\frac{1}{N}\right),$$

where we used that $\lambda_{ia}\mathcal{A}_{ia} = 0$ and therefore both $\mathcal{A}_{ia}x_{ia} = O(1/N)$ and $\mathcal{A}_{ia}[\lambda_{ia}, \cdot\,] = O(1/N)$ for $a = L, R$. Setting

$$\Phi_L + \Phi_R = \Phi_1\,,$$
$$\Phi_L - \Phi_R = \Phi_2$$

we get

$$B_\mu B_\mu - \frac{N^2-1}{2} = N\Phi_1 + \Phi_1\Phi_1 + \Phi_2\Phi_2 + \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + O\left(\frac{1}{N}\right). \tag{C.1}$$

In the limit $N \to \infty$ we can integrate out Φ_1, as it acquires an infinite mass. Alternatively we can rescale Φ_1 by setting $\phi_1 = \frac{1}{N}\Phi_1$. Then, all the terms involving ϕ_1 but the first one in (C.1) will be of order $\frac{1}{N}$ and we can equally integrate out ϕ_1.

[11]The fact that this leads to non-hermitian fields for finite N is not essential here.

The terms from

$$F_\mu F_\mu - [B_{iL}, B_{iR}]^2$$

involving the remaining Φ_2 will be (in the limit $N \to \infty$)

$$\frac{1}{2}\Phi_2\Phi_2 - J_\mu(\Phi_2)J_\mu(\Phi_2) - F_{iL}x_{iL}\Phi_2 + F_{iR}x_{iR}\Phi_2$$

with the tangential derivatives $J_{ia} = -i\epsilon_{ijk}x_{ja}\partial_{ka}$. Calculating that

$$J_\mu\Phi_2 J_\mu\Phi_2 = -\partial_\mu\Phi_2\partial_\mu\Phi_2 - x_{iL}\partial_{iL}\Phi_2 x_{jL}\partial_{jL}\Phi_2 - x_{iR}\partial_{iR}\Phi_2 x_{jR}\partial_{jR}\Phi_2$$

and using partial integration under the integral this gives

$$\frac{1}{2}\Phi_2\Phi_2 - \Phi_2\partial_\mu\partial_\mu\Phi_2 - x_{iL}\partial_{iL}\Phi_2 x_{jL}\partial_{jL}\Phi_2 - x_{iR}\partial_{iR}\Phi_2 x_{jR}\partial_{jR}\Phi_2 - F_{iL}x_{iL}\Phi_2 + F_{iR}x_{iR}\Phi_2 \,.$$

Expanding both Φ_2 and F in left and right spherical harmonics as

$$\Phi_2 = \sum_{klmn} c_{klmn} Y^L_{km} Y^R_{ln} \quad \text{and} \quad F_{ia}x_{ia} = \sum_{klmn} f^a_{klmn} Y^L_{km} Y^R_{ln}$$

we get for fixed $klmn$, setting $c = c_{klmn}$, $f^a = f^a_{klmn}$ and $p = \frac{1}{2} + l(l+1) + k(k+1)$ the following expression

$$pc^2 - cf^L + cf^R = p(c - \frac{1}{2p}f^L + \frac{1}{2p}f^R)^2 - \frac{1}{4p}(f^L - f^R)^2 \,.$$

Integrating out the c's and putting everything back this leaves us with the additional term

$$-(F_{iL}x_{iL} - F_{iR}x_{iR})\frac{1}{4(\frac{1}{2} - \partial_\mu\partial_\mu)}(F_{iL}x_{iL} - F_{iR}x_{iR})$$

in the action (4.15).

D. Representation of the SO(6)-intertwiners and Clifford algebra

We will use the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k \,. \tag{D.1}$$

With these we define the 4-dimensional antisymmetric matrices

$$\begin{aligned} \gamma^1_L &= \sigma^1 \otimes \sigma^2 \,, & \gamma^2_L &= \sigma^2 \otimes 1 \,, & \gamma^3_L &= \sigma^3 \otimes \sigma^2 \,, \\ \gamma^1_R &= i\,\sigma^2 \otimes \sigma^1 \,, & \gamma^2_R &= i\,1 \otimes \sigma^2 \,, & \gamma^3_R &= i\,\sigma^2 \otimes \sigma^3 \,. \end{aligned} \tag{D.2}$$

They are the intertwiners between SU(4) ⊗ SU(4) and SO(6) and fulfill the following relations:

$$\begin{aligned}(\gamma_L^i)^\dagger &= \gamma_L^i\,,\\ (\gamma_R^i)^\dagger &= -\gamma_R^i\end{aligned}$$

and

$$\begin{aligned}\gamma_L^i\gamma_L^j &= \delta^{ij} + i\epsilon_k^{ij}\gamma_L^k\,,\\ \gamma_R^i\gamma_R^j &= -\delta^{ij} - \epsilon_k^{ij}\gamma_R^k\,,\\ [\gamma_L^i,\gamma_R^j] &= 0\,.\end{aligned}$$

We can now define the 8-dimensional representation of the SO(6)-Clifford algebra as

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^{\mu\dagger} & 0 \end{pmatrix}, \tag{D.3}$$

with the desired anticommutation relations

$$\{\Gamma^\mu,\Gamma^\nu\} = \begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger}+\gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu+\gamma^{\nu\dagger}\gamma^\mu \end{pmatrix} = 2\delta^{\mu\nu}\,.$$

The chirality operator in this basis is

$$\Gamma = i\Gamma_L^1\Gamma_L^2\Gamma_L^3\Gamma_R^1\Gamma_R^2\Gamma_R^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The 8-dimensional SO(6)-rotations are generated by

$$\Sigma_8^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu,\Gamma^\nu] = -\frac{i}{4}\begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger}-\gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu-\gamma^{\nu\dagger}\gamma^\mu \end{pmatrix}.$$

E. The Dirac operator in spherical coordinates

For a general riemannian manifold with metric

$$g = g_{\mu\nu}dx^\mu dx^\nu$$

the Christoffel symbols are given by

$$\Gamma_{\mu\nu}^\sigma = \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})\,. \tag{E.1}$$

We can change to a non-coordinate basis (labeled by latin indices in contrast to the greek indices for the coordinates) by introducing the vielbeins e_a^μ with

$$\begin{aligned}e_\mu^a e_b^\mu &= \delta_b^a\,,\\ g_{\mu\nu} &= e_\mu^a e_\nu^b\delta_{ab}\,, \qquad g^{\mu\nu} = e_a^\mu e_b^\nu\delta^{ab}\,.\end{aligned}$$

With these, the Dirac operator is given by

$$D = -i\gamma^a e_a^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu ab}[\gamma^a, \gamma^b]) ,$$

where the γ^a form a flat Clifford algebra, i. e.

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} , \qquad \gamma^{a\dagger} = \gamma^a$$

and the spin connection ω fulfills

$$\partial_\mu e_\nu^a - \Gamma^\lambda_{\mu\nu} e_\lambda^a + \omega_\mu{}^a{}_b\, e_\nu^b = 0 . \tag{E.2}$$

E.1 The Dirac operator on $\mathbb{R}^6$ in spherical coordinates

We will now write down the flat SO(6) Dirac operator D_6 by splitting $\mathbb{R}^6$ into $\mathbb{R}^3_L \times \mathbb{R}^3_R$ and introducing spherical coordinates on both the left and right hand side. The flat metric becomes

$$\begin{aligned} g_6 = r_L^2\, d\theta_L \otimes d\theta_L + r_L^2 \sin^2\theta_L\, d\phi_L \otimes d\phi_L + dr_L \otimes dr_L + \\ + r_R^2\, d\theta_R \otimes d\theta_R + r_R^2 \sin^2\theta_R\, d\phi_R \otimes d\phi_R + dr_R \otimes dr_R . \end{aligned} \tag{E.3}$$

Looking at the formula for the Christoffel symbols (E.1), we see that all the symbols with both right and left indices vanish. For the symbols with only right or only left indices we get

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta , \tag{E.4}$$

$$\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} = \Gamma^\phi_{\phi\theta} , \tag{E.5}$$

$$\Gamma^r_{\theta\theta} = -r , \tag{E.6}$$

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta , \tag{E.7}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} = \Gamma^\theta_{\theta r} , \tag{E.8}$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r} = \Gamma^\phi_{\phi r} , \tag{E.9}$$

where we have dropped the left or right subscript for simplicity. All other symbols vanish. We want to go to a non-coordinate basis by introducing the vielbeins

$$e^{1_L}_{\theta_L} = r_L ; \qquad e^{2_L}_{\phi_L} = r_L \sin\theta_L ; \qquad e^{3_L}_{r_L} = 1 ; \tag{E.10}$$

$$e^{1_R}_{\theta_R} = r_L ; \qquad e^{2_R}_{\phi_R} = r_R \sin\theta_R ; \qquad e^{3_R}_{r_R} = 1 . \tag{E.11}$$

Calculating the spinor connection by (E.2), we again see that all the terms with both left and right indices vanish. The terms with only left or only right indices are

$$\omega_\phi{}^1{}_2 = -\cos\theta = -\omega_\phi{}^2{}_1 , \tag{E.12}$$

$$\omega_\phi{}^2{}_3 = \sin\theta = -\omega_\phi{}^3{}_2 , \tag{E.13}$$

$$\omega_\theta{}^1{}_3 = 1 = -\omega_\theta{}^3{}_1 , \tag{E.14}$$

where we again dropped the left or right subscripts. Putting all this together we see that D_6 splits up into a left part D_{3L} and a right part D_{3R} as

$$D_6 = D_{3L} + D_{3R} \tag{E.15}$$

with

$$D_{3L} = -i\overline{\Gamma}^1_L \frac{1}{r_L}\left(\partial_{\theta_L} + \frac{\cos\theta_L}{\sin\theta_L}\right) - i\overline{\Gamma}^2_L \frac{1}{r_L\sin\theta_L}\partial_{\phi_L} - i\overline{\Gamma}^3_L\left(\partial_{r_L} + \frac{1}{r_L}\right), \tag{E.16}$$

$$D_{3R} = -i\overline{\Gamma}^1_R \frac{1}{r_R}\left(\partial_{\theta_R} + \frac{\cos\theta_R}{\sin\theta_R}\right) - i\overline{\Gamma}^2_R \frac{1}{r_R\sin\theta_R}\partial_{\phi_R} - i\overline{\Gamma}^3_R\left(\partial_{r_R} + \frac{1}{r_R}\right). \tag{E.17}$$

where the $\overline{\Gamma}$ have to form a SO(6) Clifford algebra.

E.2 The Dirac operator on $S^2 \times S^2$

We now want to calculate the curved Dirac operator D_4 on $S^2 \times S^2$ in the spherical coordinates of the spheres (they are the same spherical coordinates we used before, now restricted to the spheres). The metric on $S^2 \times S^2$ with radii r_L and r_R is

$$\begin{aligned} g_4 = {} & r_L^2\, d\theta_L \otimes d\theta_L + r_L^2 \sin^2\theta_L\, d\phi_L \otimes d\phi_L + \\ & + r_R^2\, d\theta_R \otimes d\theta_R + r_R^2 \sin^2\theta_R\, d\phi_R \otimes d\phi_R\,. \end{aligned}$$

The metric is the same as (E.3) restricted to the spheres, so the Christoffel symbols are the same as (E.4) and (E.5). Again introducing the vielbeins

$$e^{1_L}_{\theta_L} = r_L\,; \qquad e^{2_L}_{\phi_L} = r_L \sin\theta_L\,; \tag{E.18}$$

$$e^{1_R}_{\theta_R} = r_L\,; \qquad e^{2_R}_{\phi_R} = r_R \sin\theta_R\,, \tag{E.19}$$

we see that also the spin connection is the same as (E.12), and therefore we can again split D_4 into a right part D_{2R} and a left part D_{2L} as $D_4 = D_{2L} + D_{2R}$ with

$$D_{2L} = -i\widetilde{\Gamma}^1_L \frac{1}{r_L}\left(\partial_{\theta_L} + \frac{\cos\theta_L}{\sin\theta_L}\right) - i\widetilde{\Gamma}^2_L \frac{1}{r_L\sin\theta_L}\partial_{\phi_L}\,, \tag{E.20}$$

$$D_{2R} = -i\widetilde{\Gamma}^1_R \frac{1}{r_R}\left(\partial_{\theta_R} + \frac{\cos\theta_R}{\sin\theta_R}\right) - i\widetilde{\Gamma}^2_R \frac{1}{r_R\sin\theta_R}\partial_{\phi_R}\,, \tag{E.21}$$

where the $\widetilde{\Gamma}$ form a flat SO(4) Clifford algebra.

E.3 SO(3) × SO(3)-covariant form of the Dirac operator on $S^2 \times S^2$

The flat SO(6) Dirac operator D_6 was split into a left part D_{3L} and a right part D_{3R} using spherical coordinates in (E.15). Of course, D_6 can also be written in the usual euclidean coordinates as

$$D_6 = -i\Gamma^\mu \partial_\mu\,,$$

where again we can split it into a left and a right part as

$$D_6 = D_{3L} + D_{3R}$$

with

$$D_{3L} = -i\Gamma^i_L \partial_i \,, \qquad D_{3R} = -i\Gamma^i_R \partial_i \,, \tag{E.22}$$

$$\{D_{3L}, D_{3R}\} = 0. \tag{E.23}$$

We have left open which representation of the SO(6) Clifford algebra we want to use for the $\overline{\Gamma}$ in (E.16), (E.17), but Γ in (E.22) is really the representation given by (4.9). We will now relate the two expressions for the Clifford algebra and the Dirac operator by first defining

$$J_{iL} = -i\epsilon_{ijk} x_{jL} \partial_{kL} \qquad \text{and} \qquad J_{iR} = -i\epsilon_{ijk} x_{jR} \partial_{kR}$$

and noting that

$$\left(\frac{\Gamma^i_L x_{iL}}{r_L}\right)^2 = \left(\frac{\Gamma^i_R x_{iR}}{r_R}\right)^2 = 1\,. \tag{E.24}$$

We calculate that

$$\left(\frac{\Gamma^j_L x_{jL}}{r_L}\right)^2 \Gamma^i_L \partial_{iL} = \left(\frac{\Gamma^j_L x_{jL}}{r_L}\right) \left(\frac{x_{iL}\partial_{iL}}{r_L} - \frac{1}{r_L} \begin{pmatrix} \gamma^i_L & 0 \\ 0 & \gamma^i_L \end{pmatrix} J_{iL}\right), \tag{E.25}$$

$$\left(\frac{\Gamma^j_R x_{jR}}{r_R}\right)^2 \Gamma^i_R \partial_{iR} = \left(\frac{\Gamma^j_R x_{jR}}{r_R}\right) \left(\frac{x_{iR}\partial_{iR}}{r_R} + \frac{i}{r_R} \begin{pmatrix} \gamma^i_R & 0 \\ 0 & \gamma^i_R \end{pmatrix} J_{iR}\right), \tag{E.26}$$

and therefore

$$D_{3L} = -i\left(\frac{\Gamma^j_L x_{jL}}{r_L}\right) \left(\partial_{r_L} - \frac{1}{r_L} \begin{pmatrix} \gamma^i_L & 0 \\ 0 & \gamma^i_L \end{pmatrix} J_{iL}\right), \tag{E.27}$$

$$D_{3R} = -i\left(\frac{\Gamma^j_R x_{jR}}{r_R}\right) \left(\partial_{r_R} + \frac{i}{r_R} \begin{pmatrix} \gamma^i_R & 0 \\ 0 & \gamma^i_R \end{pmatrix} J_{iR}\right). \tag{E.28}$$

Comparing this with (E.16), (E.17) we see that

$$\overline{\Gamma}^3_L = \left(\frac{\Gamma^i_L x_{iL}}{r_L}\right) \qquad \text{and} \qquad \overline{\Gamma}^3_R = \left(\frac{\Gamma^i_R x_{iR}}{r_R}\right), \tag{E.29}$$

as the J_L and J_R have no radial components. From (E.27), (E.28) we can also deduce that

$$\left[\overline{\Gamma}^i_L, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0 = \left[\overline{\Gamma}^i_R, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \tag{E.30}$$

and

$$\left\{\overline{\Gamma}^i_R, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} = 0 = \left\{\overline{\Gamma}^i_L, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}. \tag{E.31}$$

The curved Dirac operator D_4 on $S^2 \times S^2$ expressed in the spherical coordinates of the spheres also splits up as $D_4 = D_{2L} + D_{2R}$ with right part D_{2R} and left part D_{2L} given in (E.20),(E.21). Comparing this with (E.16), (E.17), we see that the dependence on the tangential coordinates is the same in both expressions. With (E.30), (E.31) we see that the matrices $-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\overline{\Gamma}^3_L\overline{\Gamma}^i_L$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\Gamma}^3_R\overline{\Gamma}^j_R$ for $i, j = 1, 2$ form a SO(4) Clifford algebra and can therefore be used as the $\widetilde{\Gamma}$. Note that this representation is still reducible, a problem we deal with in section 8.1.1. Now we can get a simple relation between the D_3 restricted on the spheres and the D_2

$$-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(i\overline{\Gamma}^3_L D_{3L}|_{res.} - \frac{1}{r_L}\right) = D_{2L}\,,$$

$$-i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(i\overline{\Gamma}^3_R D_{3R}|_{res.} - \frac{1}{r_R}\right) = D_{2R}\,.$$

Inserting (E.27), (E.28) and using (E.29) together with (E.24) we find that

$$D_{2L} = \frac{1}{r_L}\left(\Gamma^i_L J_{iL} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \tag{E.32}$$

$$D_{2R} = \frac{1}{r_R}\left(\Gamma^i_R J_{iR} + i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right). \tag{E.33}$$

Setting $r_L = r_R = 1$ for simplicity, the Dirac operator D_4 on $S^2 \times S^2$ takes the form (8.1).

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Chapter 6

Gravity on Noncommutative Spaces

In this chapter we propose an approach to gravity on noncommutative spaces based on deformed diffeomorphisms. We start with a contribution to the Proceedings of the Modave Summer School 2005, which was held in June 2005 in Modave, Belgium. It is available as a preprint [63]; the proceedings will appear soon. It is a rather detailed introduction to this topic and summarizes a lecture given by the author at this school. Although it is based on the publications [42,43], we present it first since it may serve the reader as an introduction to the two publications to follow, where the formalism is then presented in all detail.

We continue with a publication about gravity on θ-deformed spaces, which was published together with P. Aschieri, C. Blohmann, M. Dimitrijević, P. Schupp and J. Wess in the journal Classical and Quantum Gravity [42]. There deformed infinitesimal diffeomorphisms and gravity covariant with respect to deformed infinitesimal diffeomorphisms are constructed in detail. A deformed Einstein–Hilbert action is obtained, which reduces in the commutative limit to the usual Einstein–Hilbert action and which is invariant with respect to deformed diffeomorphisms.

In the following publication we generalize this construction to noncommutative spaces coming from a generic twist, see Section 4. We construct Einstein's equations for gravity on noncommutative algebras of functions whose product is a $\star$-product defined by a twist. This work was done in collaboration with P. Aschieri, M. Dimitrijević and J. Wess and was also published in the journal Classical and Quantum Gravity [43].

The last section consists of a contribution to the proceedings of the HEP2005 Europhysics Conference in Lisbon, Portugal, which was published in PoS (Proceedings of Science) [66] (see also [67, 68]). It summarizes some main results of this chapter.

6.1 Noncommutative Spaces and Gravity

by

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Noncommutative Spaces and Gravity

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Abstract

We give an introduction to an algebraic construction of a gravity theory on noncommutative spaces which is based on a deformed algebra of (infinitesimal) diffeomorphisms. We start with some fundamental ideas and concepts of noncommutative spaces. Then the θ-deformation of diffeomorphisms is studied and a tensor calculus is defined. A deformed Einstein-Hilbert action invariant with respect to deformed diffeomorphisms is given. Finally, all noncommutative fields are expressed in terms of their commutative counterparts up to second order of the deformation parameter using the $\star$-product. This allows to study explicitly deviations to Einstein's gravity theory in orders of θ. This lecture is based on joined work with P. Aschieri, C. Blohmann, M. Dimitrijević, P. Schupp and J. Wess.

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Contents

1 Noncommutative Spaces

In field theories one usually considers differential space-time manifolds. In the noncommutative realm, the notion of a point is no longer well-defined and we have to give up the concept of differentiable manifolds. However, the space of functions on a manifold is an algebra. A generalization of this algebra can be considered in the noncommutative case. We take the algebra freely generated by the *noncommutative coordinates* $\hat{x}^i$ which respects commutation relations of the type

$$[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{1}$$

Without bothering about convergence, we take the space of formal power series in the coordinates $\hat{x}^i$ and divide by the ideal generated by the above relations

$$\widehat{\mathcal{A}}_{\hat{x}} = \mathbb{C}\langle\langle \hat{x}^0, \ldots, \hat{x}^n \rangle\rangle / ([\hat{x}^\mu, \hat{x}^\nu] - C^{\mu\nu}(\hat{x}))\,.$$

The function $C^{\mu\nu}(\hat{x})$ is unknown. For physical reasons it should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments. Nevertheless, one can consider a power-series expansion

$$C^{\mu\nu}(\hat{x}) = i\theta^{\mu\nu} + iC^{\mu\nu}{}_\rho \hat{x}^\rho + (q\hat{R}^{\mu\nu}{}_{\rho\sigma} - \delta^\nu_\rho \delta^\mu_\sigma)\hat{x}^\rho \hat{x}^\sigma + \ldots,$$

where $\theta^{\mu\nu}$, $C^{\mu\nu}{}_\rho$ and $q\hat{R}^{\mu\nu}{}_{\rho\sigma}$ are constants, and study cases where the commutation relations are constant, linear or quadratic in the coordinates. At very short distances those cases provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$ and lead to the following three structures

1. canonical or θ-deformed case:
$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{2}$$

2. Lie algebra case:
$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}{}_\rho \hat{x}^\rho. \tag{3}$$

3. Quantum Spaces:
$$\hat{x}^\mu \hat{x}^\nu = q\hat{R}^{\mu\nu}{}_{\rho\sigma}\hat{x}^\rho \hat{x}^\sigma. \tag{4}$$

We denote the algebra generated by noncommutative coordinates $\hat{x}^\mu$ which are subject to the relations (2) by $\hat{\mathcal{A}}$. We shall often call it the *algebra of noncommutative functions*. Commutative functions will be denoted by $\mathcal{A}$. In what follows we will exclusively consider the θ-deformed case (2) but we note that the algebraic construction presented here can be generalized to more complicated noncommutative structures of the above type which possess the Poincaré-Birkhoff-Witt (PBW) property. The PBW-property states that the space of polynomials in noncommutative coordinates of a given degree is isomorphic to the space of polynomials in the commutative coordinates. Such an isomorphism between polynomials of a fixed degree is given by an ordering prescription. One example is the *symmetric ordering* (or Weyl-ordering) W which assigns to any monomial the totally symmetric ordered monomial

$$\begin{array}{rcl} W: \mathcal{A} & \rightarrow & \hat{\mathcal{A}} \\ x^\mu & \mapsto & \hat{x}^\mu \\ x^\mu x^\nu & \mapsto & \frac{1}{2}(\hat{x}^\mu\hat{x}^\nu + \hat{x}^\nu\hat{x}^\mu) \\ & \cdots & . \end{array} \tag{5}$$

To study the dynamics of fields we need a differential calculus on the noncommutative algebra $\hat{\mathcal{A}}$. Derivatives are maps on the deformed coordinate space [1]

$$\hat{\partial}_\mu : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}\,.$$

This means that they have to be consistent with the commutation relations of the coordinates. In the θ-constant case a consistent differential calculus can be defined very easily by[1]

$$\begin{aligned} &[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu \qquad (\hat{\partial}_\mu \hat{x}^\nu) = \delta_\mu^\nu \\ &[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \end{aligned} \tag{6}$$

[1]We use brackets to distinguish the action of a differential operator from the multiplication in the algebra of differential operators.

It is the fully undeformed differential calculus. The above definitions yield the usual Leibniz-rule for the derivatives $\hat{\partial}_\mu$

$$(\hat{\partial}_\mu \hat{f}\hat{g}) = (\hat{\partial}_\mu \hat{f})\hat{g} + \hat{f}(\hat{\partial}_\mu \hat{g}). \tag{7}$$

This is a special feature of the fact that $\theta^{\mu\nu}$ are constants. In the more complicated examples of noncommutative structures this undeformed Leibniz-rule usually cannot be preserved but one has to consider deformed Leibniz-rules for the derivatives [2]. Note that (6) also implies that

$$(\hat{\partial}_\mu \hat{f}) = \widehat{(\partial_\mu f)}. \tag{8}$$

The Weyl ordering (5) can be formally implemented by the map

$$f \mapsto W(f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n k \, e^{ik_\mu \hat{x}^\mu} \tilde{f}(k)$$

where $\tilde{f}$ is the Fourier transform of f

$$\tilde{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n x \, e^{-ik_\mu x^\mu} f(x).$$

This is due to the fact that the exponential is a fully symmetric function. Using the Baker-Campbell-Hausdorff formula one finds

$$e^{ik_\mu \hat{x}^\mu} e^{ip_\nu \hat{x}^\nu} = e^{i(k_\mu + p_\mu)\hat{x}^\mu - \frac{i}{2} k_\mu \theta^{\mu\nu} p_\nu}. \tag{9}$$

This immediately leads to the following observation

$$\begin{aligned} \hat{f}\hat{g} &= W(f)W(g) = \frac{1}{(2\pi)^n} \int d^n k d^n p \, e^{ik_\mu \hat{x}^\mu} e^{ip_\nu \hat{x}^\nu} \tilde{f}(k)\tilde{g}(p) \\ &= \frac{1}{(2\pi)^n} \int d^n k d^n p \, e^{i(k_\mu + p_\mu)\hat{x}^\mu} e^{-\frac{i}{2} k_\mu \theta^{\mu\nu} p_\nu} \tilde{f}(k)\tilde{g}(p) \\ &= W(\mu \circ e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g), \end{aligned} \tag{10}$$

where $\mu(f \otimes g) := fg$ is the multiplication map. With (8) we deduce from (10) the equation

$$\mu \circ e^{-\frac{i}{2}\theta^{\mu\nu}\hat{\partial}_\mu \otimes \hat{\partial}_\nu} \hat{f} \otimes \hat{g} = \widehat{fg}. \tag{11}$$

The above formula shows us how the commutative and the noncommutative product are related. It will be important for us later on.

2 Symmetries on Deformed Spaces

In general the commutation relations (1) are not covariant with respect to undeformed symmetries. For example the canonical commutation relations (2) break Lorentz symmetry if we assume that the noncommutativity parameters $\theta^{\mu\nu}$ do not transform.

The question arises whether we can *deform* the symmetry in such a way that it acts consistently on the deformed space (i.e. leaves the deformed space invariant) and such that it reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie algebras can be deformed in the category of Hopf algebras (Hopf algebras coming from a Lie algebra are also called Quantum Groups)[2]. Important examples of such deformations are q-deformations: Drinfeld and Jimbo have shown that there exists a q-deformation of the universal enveloping algebra of an arbitrary semisimple Lie algebra[3]. Module algebras of this q-deformed universal enveloping algebras are noncommutative spaces with commutation relations of type (4). There exists also a so-called κ-deformation of the Poincaré algebra [3, 4] which leads to a noncommutative space of the Lie type (3). A Hopf algebra symmetry acting on the θ-deformed space was for a long time unknown. But recently also a θ-deformation of the Poincaré algebra leading to the algebra (2) was constructed [5–8].

Quantum group symmetries lead to new features of field theories on noncommutative spaces. Because of its simplicity, θ-deformed spaces are very well-suited to study those. First results on the consequences of the θ-deformed Poincaré algebra have already been obtained [6, 8]. However, it remains unknown and subject of future investigations in which precise way this recently discovered quantum group symmetry restricts the degrees of freedom of the noncommutative field theory.

In the following we will construct explicitly a θ-deformed version of diffeomorphisms which consistently act on the noncommutative space (2). Then we present a gravity theory which is invariant with respect to this deformed diffeomorphisms [8,9].

3 Diffeomorphisms

Gravity is a theory invariant with respect to diffeomorphisms. However, to generalize the Einstein formalism to noncommutative spaces in order to establish a gravity

[2]To be more precise the universal enveloping algebra of a Lie algebra can be deformed. The universal enveloping algebra of any Lie algebra is a Hopf algebra and this gives rise to deformations in the category of Hopf algebras.

[3]It is called q-deformation since it is a deformation in terms of a parameter q.

theory, it is important to first understand that diffeomorphisms possess more mathematical structure than the algebraic one: They are naturally equipped with a Hopf algebra structure. In the common formulations of physical theories this additional Hopf structure is hidden and does not play a crucial role. It is our aim to deform the algebra of diffeomorphism in such a way that it acts consistently on a noncommutative space. This can be done by exploiting the full Hopf structure. In this section we first introduce the concept of diffeomorphisms as Hopf algebra in the undeformed setting.

Diffeomorphisms are generated by vector-fields ξ. Acting on functions, vector-fields are represented as linear differential operators $\xi = \xi^\mu \partial_\mu$. Vector-fields form a Lie algebra Ξ over the field $\mathbb{C}$ with the Lie bracket given by

$$[\xi, \eta] = \xi \times \eta$$

where $\xi \times \eta$ is defined by its action on functions

$$(\xi \times \eta)(f) = (\xi^\mu (\partial_\mu \eta^\nu)\partial_\nu - \eta^\mu (\partial_\mu \xi^\nu)\partial_\nu)(f).$$

The Lie algebra of *infinitesimal diffeomorphisms* Ξ can be embedded into its universal enveloping algebra which we want to denote by $\mathcal{U}(\Xi)$. The universal enveloping algebra is an associative algebra and possesses a natural Hopf algebra structure. It is given by the following structure maps[4]:

- An algebra homomorphism called *coproduct* defined by

$$\begin{aligned} \Delta : \mathcal{U}(\Xi) &\rightarrow \mathcal{U}(\Xi) \otimes \mathcal{U}(\Xi) \\ \Xi \ni \xi &\mapsto \Delta(\xi) := \xi \otimes 1 + 1 \otimes \xi. \end{aligned} \tag{12}$$

- An algebra homomorphism called *counit* defined by

$$\begin{aligned} \varepsilon : \mathcal{U}(\Xi) &\rightarrow \mathbb{C} \\ \Xi \ni \xi &\mapsto \varepsilon(\xi) = 0. \end{aligned} \tag{13}$$

- An anti-algebra homomorphism called *antipode* defined by

$$\begin{aligned} S : \mathcal{U}(\Xi) &\rightarrow \mathcal{U}(\Xi) \\ \Xi \ni \xi &\mapsto S(\xi) = -\xi. \end{aligned} \tag{14}$$

[4]The structure maps are defined on the generators $\xi \in \Xi$ and the universal property of the universal enveloping algebra $\mathcal{U}(\Xi)$ assures that they can be uniquely extended as algebra homomorphisms (respectively anti-algebra homomorphism in case of the antipode S) to the whole algebra $\mathcal{U}(\Xi)$.

For a precise definition and more details on Hopf algebras we refer the reader to text books [10–12]. For our purposes it shall be sufficient to note that the coproduct implements how the Hopf algebra acts on a product in a representation algebra (Leibniz-rule). Below we will make this more transparent. It is now possible to study deformations of $\mathcal{U}(\Xi)$ in the category of Hopf algebras. This leads to a deformed version of diffeomorphisms - the fundamental building block of our approach to a gravity theory on noncommutative spaces. Before studying this in detail, let us shortly review the Einstein formalism. This way we first understand better the meaning of the structure maps of a Hopf algebra introduced above.

Scalar fields are defined by their transformation property with respect to infinitesimal coordinate transformations:

$$\delta_\xi \phi = -\xi \phi = -\xi^\mu (\partial_\mu \phi). \tag{15}$$

The product of two scalar fields is transformed using the Leibniz-rule

$$\delta_\xi (\phi\psi) = (\delta_\xi \phi)\psi + \phi(\delta_\xi \psi) = -\xi^\mu (\partial_\mu \phi\psi) \tag{16}$$

such that the product of two scalar fields transforms again as a scalar. The above Leibniz-rule can be understood in mathematical terms as follows: The Hopf algebra $\mathcal{U}(\Xi)$ is represented on the space of scalar fields by infinitesimal coordinate transformations δ_ξ. On scalar fields the action of δ_ξ is explicitly given by the differential operator $-\xi^\mu\partial_\mu$. Of course, the space of scalar fields is not only a vector space - it possesses also an algebra structure - such as $\mathcal{U}(\Xi)$ is not only an algebra but also a Hopf algebra - it possesses in addition the co-structure maps defined above. We say that a Hopf algebra H acts on an algebra A (or more precisely we say that A is a left H-module algebra) if A is a module with respect to the algebra H and if in addition for all $h \in H$ and $a, b \in A$

$$\begin{aligned} h(ab) &= \mu \circ \Delta h(a \otimes b) && (17) \\ h(1) &= \varepsilon(h). && (18) \end{aligned}$$

Here μ is the multiplication map defined by $\mu(a \otimes b) = ab$. In our concrete example where $H = \mathcal{U}(\Xi)$ and A is the algebra of scalar fields we indeed have that the algebra of scalar fields is a $\mathcal{U}(\Xi)-$module algebra. This can be seen easily if we rewrite (16) using (12) for the generators $\xi \in \Xi$ for $\mathcal{U}(\Xi)$:

$$\delta_\xi (\phi\psi) = (\delta_\xi \phi)\psi + \phi(\delta_\xi \psi) = \mu \circ \Delta\xi(\phi \otimes \psi).$$

It is also evident that

$$\delta_\xi 1 = 0 = \varepsilon(\xi)1.$$

Now we are in the right mathematical framework: We study a Lie algebra (here infinitesimal diffeomorphisms Ξ) and embed it in its universal enveloping algebra (here $\mathcal{U}(\Xi)$). This universal enveloping algebra is a Hopf algebra via a natural Hopf structure induced by (12,13,14).

Physical quantities live in representations of this Hopf algebras. For instance, the algebra of scalar fields is a $\mathcal{U}(\Xi)$-module algebra. The action of $\mathcal{U}(\Xi)$ on scalar fields is given in terms of infinitesimal coordinate transformations δ_ξ.

Similarly one studies tensor representations of $\mathcal{U}(\Xi)$. For example vector fields are introduced by the transformation property

$$\begin{aligned}\delta_\xi V_\alpha &= -\xi^\mu(\partial_\mu V_\alpha) - (\partial_\alpha \xi^\mu) V_\mu \\ \delta_\xi V^\alpha &= -\xi^\mu(\partial_\mu V^\alpha) + (\partial_\mu \xi^\alpha) V^\mu.\end{aligned}$$

The generalization to arbitrary tensor fields is straight forward:

$$\begin{aligned}\delta_\xi T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} &= -\xi^\mu(\partial_\mu T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}) + (\partial_\mu \xi^{\mu_1}) T^{\mu\cdots\mu_n}_{\nu_1\cdots\nu_n} + \cdots + (\partial_\mu \xi^{\mu_n}) T^{\mu_1\cdots\mu}_{\nu_1\cdots\nu_n} \\ &\quad -(\partial_{\nu_1}\xi^\nu) T^{\mu_1\cdots\mu_n}_{\nu\cdots\nu_n} - \cdots - (\partial_{\nu_n}\xi^\nu) T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu}.\end{aligned}$$

As for scalar fields, we also find that the product of two tensors transforms like a tensor. Summarizing, we have seen that scalar fields, vector fields and tensor fields are representations of the Hopf algebra $\mathcal{U}(\Xi)$, the universal enveloping algebra of infinitesimal diffeomorphisms. The Hopf algebra $\mathcal{U}(\Xi)$ acts via *infinitesimal coordinate transformations* δ_ξ which are subject to the relations:

$$\begin{aligned}[] [\delta_\xi, \delta_\eta] &= \delta_{\xi\times\eta} \qquad & \varepsilon(\delta_\xi) &= 0 \\ \Delta\delta_\xi &= \delta_\xi \otimes 1 + 1 \otimes \delta_\xi \qquad & S(\delta_\xi) &= -\delta_\xi.\end{aligned} \tag{19}$$

The transformation operator δ_ξ is explicitly given by differential operators which depend on the representation under consideration. In case of scalar fields this differential operator is given by $-\xi^\mu\partial_\mu$.

4 Deformed Diffeomorphisms

The concepts introduced in the previous subsection can be deformed in order to establish a consistent tensor calculus on the noncommutative space-time algebra (2). In this context it is necessary to account the full Hopf algebra structure of the universal enveloping algebra $\mathcal{U}(\Xi)$.

In our setting the algebra $\hat{\mathcal{A}}$ possesses a noncommutative product defined by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{20}$$

We want to deform the structure maps (19) of the Hopf algebra $\mathcal{U}(\Xi)$ in such a way that the resulting deformed Hopf algebra which we denote by $\mathcal{U}(\hat{\Xi})$ consistently acts on $\hat{\mathcal{A}}$. In the language introduced in the previous section this means that we want $\hat{\mathcal{A}}$ to be a $\mathcal{U}(\hat{\Xi})$-module algebra. We claim that the following deformation of $\mathcal{U}(\Xi)$ does the job. Let $\mathcal{U}(\hat{\Xi})$ be generated as algebra by elements $\hat{\delta}_\xi$, $\xi \in \Xi$. We leave the algebra relation undeformed and demand

$$[\hat{\delta}_\xi, \hat{\delta}_\eta] = \hat{\delta}_{\xi\times\eta} \tag{21}$$

but we deform the co-sector

$$\Delta\hat{\delta}_\xi = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{\delta}_\xi\otimes 1 + 1\otimes\hat{\delta}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}, \tag{22}$$

where

$$[\hat{\partial}_\rho, \hat{\delta}_\xi] = \hat{\delta}_{(\partial_\rho\xi)}.$$

The deformed coproduct (22) reduces to the undeformed one (19) in the limit $\theta \to 0$. Antipode and counit remain undeformed

$$S(\hat{\delta}_\xi) = -\hat{\delta}_\xi \qquad \varepsilon(\hat{\delta}_\xi) = 0. \tag{23}$$

We have to check whether the above deformation is a good one in the sense that it leads to a consistent action on $\hat{\mathcal{A}}$. First we need a differential operator acting on fields in $\hat{\mathcal{A}}$ which represents the algebra (21). Let us consider the differential operator

$$\hat{X}_\xi := \sum_{n=0}^{\infty}\frac{1}{n!}(-\frac{i}{2})^n\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}(\hat{\partial}_{\rho_1}\cdots\hat{\partial}_{\rho_n}\hat{\xi}^\mu)\hat{\partial}_\mu\hat{\partial}_{\sigma_1}\cdots\hat{\partial}_{\sigma_n}. \tag{24}$$

This is to be understood like that: A vector-field $\xi = \xi^\mu\partial_\mu$ is determined by its coefficient functions ξ^μ. In Section 1 we saw that there is a vectorspace isomorphism W from the space of commutative to the space of noncommutative functions which is given by the symmetric ordering prescription. The image of a commutative function f under the isomorphism W is denoted by $\hat{f}$

$$W:\ f \mapsto W(f) = \hat{f}.$$

In (24) $\hat{\xi}^\mu$ is therefore to be interpreted as the image of ξ^μ with respect to W. Then indeed we have

$$[\hat{X}_\xi, \hat{X}_\eta] = \hat{X}_{\xi\times\eta}. \tag{25}$$

To see this we use result (11) to rewrite $(\hat{X}_\xi \hat{\phi})$:

$$\begin{aligned}(\hat{X}_\xi \hat{\phi}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}(\hat{\partial}_{\rho_1}\cdots\hat{\partial}_{\rho_n}\hat{\xi}^\mu)(\hat{\partial}_\mu\hat{\partial}_{\sigma_1}\cdots\hat{\partial}_{\sigma_n}\hat{\phi}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}(\hat{\partial}_{\rho_1}\cdots\hat{\partial}_{\rho_n}\hat{\xi}^\mu)(\hat{\partial}_{\sigma_1}\cdots\hat{\partial}_{\sigma_n}\widehat{\partial_\mu\phi}) \\ &= \widehat{\xi^\mu(\partial_\mu\phi)} = \widehat{(\xi\phi)}. \end{aligned} \tag{26}$$

From (26) follows

$$(\hat{X}_\xi(\hat{X}_\eta\hat{\phi})) - (\hat{X}_\eta(\hat{X}_\xi\hat{\phi})) = (\widehat{[\xi,\eta]\phi}) = (\hat{X}_{\xi\times\eta}\hat{\phi}),$$

which amounts to (25) and this is what we wanted to show.

It is therefore reasonable to introduce scalar fields $\hat{\phi} \in \hat{\mathcal{A}}$ by the transformation property

$$\hat{\delta}_\xi\hat{\phi} = -(\hat{X}_\xi\hat{\phi}).$$

The next step is to work out the action of the differential operators $\hat{X}_\xi$ on the product of two fields. A calculation [8] shows that

$$(\hat{X}_\xi(\hat{\phi}\hat{\psi})) = \mu\circ(e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{X}_\xi\otimes 1 + 1\otimes\hat{X}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}\hat{\phi}\otimes\hat{\psi}).$$

This means that the differential operators $\hat{X}_\xi$ act via a *deformed Leibniz rule* on the product of two fields. Comparing with (22) we see that the deformed Leibniz rule of the differential operator $\hat{X}_\xi$ is exactly the one induced by the deformed coproduct (22):

$$\hat{\delta}_\xi(\hat{\phi}\hat{\psi}) = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{\delta}_\xi\otimes 1 + 1\otimes\hat{\delta}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{\phi}\hat{\psi}) = -\hat{X}_\xi\triangleright(\hat{\phi}\hat{\psi}).$$

Hence, the deformed Hopf algebra $\mathcal{U}(\hat{\Xi})$ is indeed represented on scalar fields $\hat{\phi}\in\hat{\mathcal{A}}$ by the differential operator $\hat{X}_\xi$. The scalar fields form a $\mathcal{U}(\hat{\Xi})$-module algebra.

In analogy to the previous section we can introduce vector and tensor fields as representations of the Hopf algebra $\mathcal{U}(\hat{\Xi})$. The transformation property for an arbitrary tensor reads

$$\begin{aligned}\hat{\delta}_\xi\hat{T}^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} &= -(\hat{X}_\xi\hat{T}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}) + (\hat{X}_{(\partial_\mu\xi^{\mu_1})}\hat{T}^{\mu\cdots\mu_n}_{\nu_1\cdots\nu_n}) + \cdots + (\hat{X}_{(\partial_\mu\xi^{\mu_n})}\hat{T}^{\mu_1\cdots\mu}_{\nu_1\cdots\nu_n}) \\ &\quad -(\hat{X}_{(\partial_{\nu_1}\xi^\nu)}\hat{T}^{\mu_1\cdots\mu_n}_{\nu\cdots\nu_n}) - \cdots - (\hat{X}_{(\partial_{\nu_n}\xi^\nu)}\hat{T}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu}).\end{aligned}$$

Up to now we have seen the following:

- Diffeomorphisms are generated by vector-fields $\xi \in \Xi$ and the universal enveloping algebra $\mathcal{U}(\Xi)$ of the Lie algebra Ξ of vector-fields possesses a natural Hopf algebra structure defined by (19).
- The algebra of scalar fields $\phi \in \mathcal{A}$ is a $\mathcal{U}(\Xi)$-module algebra.
- The universal enveloping algebra $\mathcal{U}(\Xi)$ can be deformed to a Hopf algebra $\mathcal{U}(\hat{\Xi})$ defined in (21,22,23).
- $\mathcal{U}(\hat{\Xi})$ consistently acts on the algebra of noncommutative functions $\hat{\mathcal{A}}$, i.e. the algebra of noncommutative functions is a $\mathcal{U}(\hat{\Xi})$-module algebra.
- Regarding $\mathcal{U}(\hat{\Xi})$ as the underlying "symmetry" of the gravity theory to be built on the noncommutative space $\hat{\mathcal{A}}$, we established a full tensor calculus as representations of the Hopf algebra $\mathcal{U}(\hat{\Xi})$.

5 Noncommutative Geometry

The deformed algebra of infinitesimal diffeomorphisms and the tensor calculus covariant with respect to it is the fundamental building-block for the definition of a noncommutative geometry on $\theta-$deformed spaces. In this section we sketch the important steps towards a deformed Einstein-Hilbert action [8]. A first ingredient is the *covariant derivative* $\hat{D}_\mu$. Algebraically, it can be defined by demanding that acting on a vector-field it produces a tensor-field

$$\hat{\delta}_\xi \hat{D}_\mu \hat{V}_\nu \stackrel{!}{=} -(\hat{X}_\xi \hat{D}_\mu \hat{V}_\nu) - (\hat{X}_{(\partial_\mu \xi^\alpha)} \hat{D}_\alpha \hat{V}_\nu) - (\hat{X}_{(\partial_\nu \xi^\alpha)} \hat{D}_\mu \hat{V}_\alpha) \tag{27}$$

The covariant derivative is given by a *connection* $\hat{\Gamma}_{\mu\nu}{}^\rho$

$$\hat{D}_\mu \hat{V}_\nu = \hat{\partial}_\mu \hat{V}_\nu - \hat{\Gamma}_{\mu\nu}{}^\rho \hat{V}_\rho.$$

From (27) it is possible to deduce the transformation property of $\hat{\Gamma}_{\mu\nu}{}^\rho$

$$\hat{\delta}_\xi \hat{\Gamma}_{\mu\nu}{}^\rho = (\hat{X}_\xi \hat{\Gamma}_{\mu\nu}{}^\rho) - (\hat{X}_{(\partial_\mu \xi^\alpha)} \hat{\Gamma}_{\alpha\nu}{}^\rho) - (\hat{X}_{(\partial_\nu \xi^\alpha)} \hat{\Gamma}_{\mu\alpha}{}^\rho) + (\hat{X}_{(\partial_\alpha \xi^\rho)} \hat{\Gamma}_{\mu\nu}{}^\alpha) - (\hat{\partial}_\mu \hat{\partial}_\nu \hat{\xi}^\rho).$$

The *metric* $\hat{G}_{\mu\nu}$ is defined as a symmetric tensor of rank two. It can be obtained for example by a set of vector-fields $\hat{E}_\mu{}^a$, $a = 0, \ldots, 3$, where a is to be understood as a mere label. These vector-fields are called *vierbeins.* Then the symmetrized product of those vector-fields is indeed a symmetric tensor of rank two

$$\hat{G}_{\mu\nu} := \frac{1}{2}(\hat{E}_\mu{}^a \hat{E}_\nu{}^b + \hat{E}_\nu{}^b \hat{E}_\mu{}^a)\eta_{ab}.$$

Here η_{ab} stands for the usual flat Minkowski space metric with signature $(-+++)$. Let us assume that we can choose the vierbeins $\hat{E}_\mu{}^a$ such that they reduce in the commutative limit to the usual vierbeins $e_\mu{}^a$. Then also the metric $\hat{G}_{\mu\nu}$ reduces to the usual, undeformed metric $g_{\mu\nu}$.

The inverse metric tensor we denote by upper indices

$$\hat{G}_{\mu\nu}\hat{G}^{\nu\rho} = \delta^\rho_\mu.$$

We use $\hat{G}_{\mu\nu}$ respectively $\hat{G}^{\mu\nu}$ to raise and lower indices.

The curvature and torsion tensors are obtained by taking the commutator of two covariant derivatives[5]

$$[\hat{D}_\mu, \hat{D}_\nu]\hat{V}_\rho = \hat{R}_{\mu\nu\rho}{}^\alpha \hat{V}_\alpha + \hat{T}_{\mu\nu}{}^\alpha \hat{D}_\alpha \hat{V}_\rho$$

which leads to the expressions

$$\begin{aligned} \hat{R}_{\mu\nu\rho}{}^\sigma &= \hat{\partial}_\nu \hat{\Gamma}_{\mu\rho}{}^\sigma - \hat{\partial}_\mu \hat{\Gamma}_{\nu\rho}{}^\sigma + \hat{\Gamma}_{\nu\rho}{}^\beta \hat{\Gamma}_{\mu\beta}{}^\sigma - \hat{\Gamma}_{\mu\rho}{}^\beta \hat{\Gamma}_{\nu\beta}{}^\sigma \\ \hat{T}_{\mu\nu}{}^\alpha &= \hat{\Gamma}_{\nu\mu}{}^\alpha - \hat{\Gamma}_{\nu\mu}{}^\alpha. \end{aligned}$$

If we assume the *torsion-free* case, i.e.

$$\hat{\Gamma}_{\mu\nu}{}^\sigma = \hat{\Gamma}_{\nu\mu}{}^\sigma,$$

we find an unique expression for the metric connection (Christoffel symbol) defined by

$$\hat{D}_\alpha \hat{G}_{\beta\gamma} \stackrel{!}{=} 0$$

in terms of the metric and its inverse[6]

$$\hat{\Gamma}_{\alpha\beta}{}^\sigma = \frac{1}{2}(\hat{\partial}_\alpha \hat{G}_{\beta\gamma} + \hat{\partial}_\beta \hat{G}_{\alpha\gamma} - \hat{\partial}_\gamma \hat{G}_{\alpha\beta})\hat{G}^{\gamma\sigma}.$$

From the curvature tensor $\hat{R}_{\mu\nu\rho}{}^\sigma$ we get the curvature scalar by contracting the indices

$$\hat{R} := \hat{G}^{\mu\nu}\hat{R}_{\nu\mu\rho}{}^\rho.$$

$\hat{R}$ indeed transforms as a scalar which may be checked explicitly by taking the deformed coproduct (22) into account.

[5]The generalization of covariant derivatives acting on tensors is straight forward [8].

[6]We don't introduce a new symbol for the metric connection.

To obtain an integral which is invariant with respect to the Hopf algebra of deformed infinitesimal diffeomorphisms we need a measure function $\hat{E}$. We demand the transformation property

$$\hat{\delta}_\xi \hat{E} = -\hat{X}_\xi \hat{E} - \hat{X}_{(\partial_\mu \xi^\mu)} \hat{E}. \tag{28}$$

Then it follows with the deformed coproduct (22) that for any scalar field $\hat{S}$

$$\hat{\delta}_\xi \hat{E}\hat{S} = -\hat{\partial}_\mu (\hat{X}_{\xi^\mu}(\hat{E}\hat{S})).$$

Hence, transforming the product of an arbitrary scalar field with a measure function $\hat{E}$ we obtain a total derivative which vanishes under the integral. A suitable measure function with the desired transformation property (28) is for instance given by the determinant of the vierbein $\hat{E}_\mu{}^a$

$$\hat{E} = \det(\hat{E}_\mu{}^a) := \frac{1}{4!}\varepsilon^{\mu_1 \cdots \mu_4}\varepsilon_{a_1 \cdots a_4} \hat{E}_{\mu_1}{}^{a_1} \hat{E}_{\mu_2}{}^{a_2} \hat{E}_{\mu_3}{}^{a_3} \hat{E}_{\mu_4}{}^{a_4}.$$

That $\hat{E}$ transforms correctly can be shown by using that the product of four $\hat{E}_{\mu_i}{}^{a_i}$ transforms as a tensor of fourth rank and some combinatorics.

Now we have all ingredients to write down an Einstein-Hilbert action. Note that having chosen a differential calculus as in (6), the integral is uniquely determined up to a normalization factor by requiring[7] [13]

$$\int \hat{\partial}_\mu \hat{f} = 0$$

for all $\hat{f} \in \hat{\mathcal{A}}$. Then we define the *Einstein-Hilbert action* on $\hat{\mathcal{A}}$ as

$$\hat{S}_{\text{EH}} := \int \det(\hat{E}_\mu{}^a)\hat{R} + \text{complex conj.}.$$

It is by construction invariant with respect to deformed diffeomorphisms meaning that

$$\hat{\delta}_\xi \hat{S}_{\text{EH}} = 0.$$

In this section we have presented the fundamentals of a noncommutative geometry on the algebra $\hat{\mathcal{A}}$ and defined an invariant Einstein-Hilbert action. There is however one important step missing which is subject of the following section: We want to make contact of the noncommutative gravity theory with Einstein's gravity theory. This we achieve by introducing the $\star$-product formalism.

[7] We consider functions that "vanish at infinity".

6 Star Products and Expanded Einstein-Hilbert Action

To express the noncommutative fields in terms of their commutative counterparts we first observe that we can map the whole algebraic construction of the previous sections to the algebra of commutative functions via the vector space isomorphism W introduced in Section 1. By equipping the algebra of commutative functions with a new product denoted by $\star$ be can render W an algebra isomorphism. We define

$$f \star g := W^{-1}(W(f)W(g)) = W^{-1}(\hat{f}\hat{g}) \tag{29}$$

and obtain

$$(\mathcal{A}, \star) \cong \hat{\mathcal{A}}.$$

The $\star$-product corresponding to the symmetric ordering prescription W is then given explicitly by the Moyal-Weyl product[8]

$$f \star g = \mu \circ e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g = fg + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f)(\partial_\nu g) + \mathcal{O}(\theta^2).$$

It is a deformation of the commutative point-wise product to which it reduces in the limit $\theta \to 0$.

In virtue of the isomorphism W we can map all noncommutative fields to commutative functions in $\mathcal{A}$

$$\hat{F} \mapsto W^{-1}(\hat{F}) \equiv F.$$

We then expand the image F in orders of the deformation parameter θ

$$F = F^{(0)} + F^{(1)} + F^{(2)} + \mathcal{O}(\theta^3),$$

where the zeroth order always corresponds to the undeformed quantity. Products of functions in $\hat{\mathcal{A}}$ are simply mapped to $\star$-products of the corresponding functions in $\mathcal{A}$. The same can be done for the action of the derivative $\hat{\partial}_\mu$ and consequently for an arbitrary differential operator acting on $\hat{\mathcal{A}}$ [8].

The fundamental dynamical field of our gravity theory is the vierbein field $\hat{E}_\mu{}^a$. All other quantities such as metric, connection and curvature can be expressed in terms of it. Its image with respect to W^{-1} is denoted by $E_\mu{}^a$. In first approximation we study the case

$$E_\mu{}^a = e_\mu{}^a,$$

[8]This is an immediate consequence of (10).

where $e_\mu{}^a$ is the usual vierbein field. Then for instance the metric is given up to second order in θ by

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}(E_\mu{}^a \star E_\nu{}^b + E_\nu{}^b \star E_\mu{}^a)\eta_{ab} = \frac{1}{2}(e_\mu{}^a \star e_\nu{}^b + e_\nu{}^b \star e_\mu{}^a)\eta_{ab} \\ &= g_{\mu\nu} - \frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2}e_\mu{}^a)(\partial_{\beta_1}\partial_{\beta_2}e_\nu{}^b)\eta_{ab} + \ldots, \end{aligned}$$

where $g_{\mu\nu}$ is the usual, undeformed metric. For the Christoffel symbol one finds up to second order: The zeroths order is the undeformed expression

$$\Gamma^{(0)\rho}_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma} - \partial_\gamma g_{\mu\nu})g^{\gamma\rho}, \tag{30}$$

the first order reads

$$\Gamma^{(1)\rho}_{\mu\nu} = \frac{i}{2}\theta^{\alpha\beta}(\partial_\alpha\Gamma^{(0)\sigma}_{\mu\nu})g_{\sigma\tau}(\partial_\beta g^{\tau\rho}) \tag{31}$$

and the second order

$$\begin{aligned} \Gamma^{(2)\rho}_{\mu\nu} = &= -\frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\Big((\partial_{\alpha_1}\partial_{\alpha_2}\Gamma^{(0)}_{\mu\nu\sigma})(\partial_{\beta_1}\partial_{\beta_2}g^{\sigma\rho}) - 2(\partial_{\alpha_1}\Gamma^{(0)}_{\mu\nu\sigma})\partial_{\beta_1}((\partial_{\alpha_2}g^{\sigma\tau})(\partial_{\beta_2}g_{\tau\xi})g^{\xi\rho}) \\ &- \Gamma^{(0)}_{\mu\nu\sigma}\Big((\partial_{\alpha_1}\partial_{\alpha_2}g^{\sigma\tau})(\partial_{\beta_1}\partial_{\beta_2}g_{\tau\xi}) + g^{\sigma\tau}(\partial_{\alpha_1}\partial_{\alpha_2}e_\tau{}^a)(\partial_{\beta_1}\partial_{\beta_2}e_\xi{}^b)\eta_{ab} \\ &- 2\partial_{\alpha_1}((\partial_{\alpha_2}g^{\sigma\tau})(\partial_{\beta_2}g_{\tau\lambda})g^{\lambda\kappa})(\partial_{\beta_1}g_{\kappa\xi})\Big)g^{\xi\rho} + \frac{1}{2}\Big(\partial_\mu((\partial_{\alpha_1}\partial_{\alpha_2}e_\nu{}^a)(\partial_{\beta_1}\partial_{\beta_2}e_\sigma{}^b)) \\ &+ \partial_\nu((\partial_{\alpha_1}\partial_{\alpha_2}e_\sigma{}^a)(\partial_{\beta_1}\partial_{\beta_2}e_\mu{}^b)) - \partial_\sigma((\partial_{\alpha_1}\partial_{\alpha_2}e_\mu{}^a)(\partial_{\beta_1}\partial_{\beta_2}e_\nu{}^b))\Big)\eta_{ab}g^{\sigma\rho}\Big), \end{aligned} \tag{32}$$

where

$$\Gamma^{(0)}_{\mu\nu\sigma} = \Gamma^{(0)\rho}_{\mu\nu}g_{\rho\sigma}. \tag{33}$$

The expressions for the curvature tensor read

$$\begin{aligned} R^{(1)}_{\mu\nu\rho}{}^\sigma &= -\frac{i}{2}\theta^{\kappa\lambda}\Big((\partial_\kappa R^{(0)}_{\mu\nu\rho}{}^\tau)(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - (\partial_\kappa\Gamma^{(0)\beta}_{\nu\rho})\Big(\Gamma^{(0)\tau}_{\mu\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} \\ &\quad -\Gamma^{(0)\sigma}_{\mu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} + \partial_\mu((\partial_\lambda g_{\beta\gamma})g^{\gamma\sigma}) + (\partial_\lambda\Gamma^{(0)\sigma}_{\mu\beta})\Big) \\ &\quad +(\partial_\kappa\Gamma^{(0)\beta}_{\mu\rho})\Big(\Gamma^{(0)\tau}_{\nu\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - \Gamma^{(0)\sigma}_{\nu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} \\ &\quad +\partial_\nu((\partial_\lambda g_{\beta\gamma})g^{\gamma\sigma}) + (\partial_\lambda\Gamma^{(0)\sigma}_{\nu\beta})\Big)\Big) \end{aligned} \tag{34}$$

$$\begin{aligned} R^{(2)}_{\mu\nu\rho}{}^\sigma &= \partial_\nu\Gamma^{(2)\sigma}_{\mu\rho} + \Gamma^{(2)\gamma}_{\nu\rho}\Gamma^{(0)\sigma}_{\mu\gamma} + \Gamma^{(0)\gamma}_{\nu\rho}\Gamma^{(2)\sigma}_{\mu\gamma} \\ &\quad +\frac{i}{2}\theta^{\alpha\beta}\Big((\partial_\alpha\Gamma^{(1)\gamma}_{\nu\rho})(\partial_\beta\Gamma^{(0)\sigma}_{\mu\gamma}) + (\partial_\alpha\Gamma^{(0)\gamma}_{\nu\rho})(\partial_\beta\Gamma^{(1)\sigma}_{\mu\gamma})\Big) \\ &\quad -\frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2}\Gamma^{(0)\gamma}_{\nu\rho})(\partial_{\beta_1}\partial_{\beta_2}\Gamma^{(0)\sigma}_{\mu\gamma}) \quad - (\mu \leftrightarrow \nu), \end{aligned} \tag{35}$$

where the second order is given implicitly in terms of the Christoffel symbol.

The deformed Einstein-Hilbert action is given by

$$\begin{aligned} S_{\mathrm{EH}} &= \frac{1}{2}\int \mathrm{d}^4x \,\mathrm{det}_\star e_\mu{}^a \star R + \mathrm{c.c.} \\ &= \frac{1}{2}\int \mathrm{d}^4x \,\mathrm{det}_\star e_\mu{}^a \star (R+\bar{R}) \\ &= \frac{1}{2}\int \mathrm{d}^4x \,\mathrm{det}_\star e_\mu{}^a (R+\bar{R}) \\ &= S_{\mathrm{EH}}^{(0)} + \int \mathrm{d}^4x \,(\mathrm{det} e_\mu{}^a) R^{(2)} + (\mathrm{det}_\star e_\mu{}^a)^{(2)} R^{(0)}, \end{aligned} \tag{36}$$

where we used that the integral together with the Moyal-Weyl product has the property[9]

$$\int \mathrm{d}^4x\, f\star g = \int \mathrm{d}^4x\, fg = \int \mathrm{d}^4x\, g\star f.$$

In (36) $\mathrm{det}_\star e_\mu{}^a$ is the $\star$-determinant

$$\begin{aligned} \mathrm{det}_\star e_\mu{}^a &= \frac{1}{4!}\varepsilon^{\mu_1\cdots\mu_4}\varepsilon_{a_1\cdots a_4} e_{\mu_1}{}^{a_1}\star e_{\mu_2}{}^{a_2}\star e_{\mu_3}{}^{a_3}\star e_{\mu_4}{}^{a_4} \\ &= \mathrm{det} e_\mu{}^a + (\mathrm{det}_\star e_\mu{}^a)^{(2)} + \ldots, \end{aligned}$$

where

$$\begin{aligned} (\mathrm{det}_\star)^{(2)} &= -\frac{1}{8}\frac{1}{4!}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\varepsilon^{\mu_1\ldots\mu_4}\varepsilon_{a_1\ldots a_4} \\ &\quad \Big((\partial_{\alpha_1}\partial_{\alpha_2}e_{\mu_1}{}^{a_1})(\partial_{\beta_1}\partial_{\beta_2}e_{\mu_2}{}^{a_2})e_{\mu_3}{}^{a_3}e_{\mu_4}{}^{a_4} \\ &\quad + \partial_{\alpha_1}\partial_{\alpha_2}(e_{\mu_1}{}^{a_1}e_{\mu_2}{}^{a_2})(\partial_{\beta_1}\partial_{\beta_2}e_{\mu_3}{}^{a_3})e_{\mu_4}{}^{a_4} \\ &\quad + \partial_{\alpha_1}\partial_{\alpha_2}(e_{\mu_1}{}^{a_1}e_{\mu_2}{}^{a_2}e_{\mu_3}{}^{a_3})(\partial_{\beta_1}\partial_{\beta_2}e_{\mu_4}{}^{a_4})\Big). \end{aligned} \tag{37}$$

The odd orders of θ vanish in (36) but the even orders of θ give nontrivial contributions.

Equation (36) shows explicitly the corrections to Einsteins gravity predicted by the noncommutative theory.

[9]This follows by partial integration.

Remarks

For an introduction to field theories on noncommutative spaces, we recommend the review articles [13, 14]. To learn more about related approaches to noncommutative geometry the reader is referred to [15, 16]. More about Hopf algebras and Quantum Groups can be found in [10–12]. A good pedagogical introduction to $\star$-products can be found in [17]. The construction of a gravity theory presented in this lecture is based on [8, 9].

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6.2 A Gravity Theory on Noncommutative Spaces

by

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A gravity theory on noncommutative spaces

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Abstract

A deformation of the algebra of diffeomorphisms is constructed for canonically deformed spaces with constant deformation parameter θ. The algebraic relations remain the same, whereas the comultiplication rule (Leibniz rule) is different from the undeformed one. Based on this deformed algebra, a covariant tensor calculus is constructed and all the concepts such as metric, covariant derivatives, curvature and torsion can be defined on the deformed space as well. The construction of these geometric quantities is presented in detail. This leads to an action invariant under the deformed diffeomorphism algebra and can be interpreted as a θ-deformed Einstein–Hilbert action. The metric or the vierbein field will be the dynamical variable as they are in the undeformed theory. The action and all relevant quantities are expanded up to second order in θ.

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1. Introduction

Several arguments are presently used to motivate a deviation from the flat-space concept at very short distances [1, 2]. Among the new concepts are quantum spaces [3–6]. They have the advantage that their mathematical structure is well defined and that, based on this structure, questions on the physical behaviour of these systems can be asked. One of the questions is if physics on quantum spaces can be formulated by field equations, how they deviate from the usual field equations and to what changes they lead in their physical interpretation.

Quantum spaces depend on parameters such that for a particular value of these parameters they become the usual flat space. Thus, we call them deformed spaces. In the same sense, we expect a deformation of the field equations and finally a deformation of their physical predictions [7–11].

Several of these deformations have been studied [12, 13]. They are all based on nontrivial commutation relations of the coordinates. This algebraic deformation leads to a star-product formulation as it is used for deformation quantization [14–16]. In this paper, we start from the star-product deformation and consider the algebraic relations as consequences. We might not have started from the most general realization of the deformed algebra, but certainly from the one that is very useful for physical interpretation. This way deformed gauge theories have been constructed by the use of the Seiberg–Witten map [17–24]. Their field content is the same as in the undeformed theory, the deformation parameters enter the deformed field equations as coupling constants.

The question was still open if gravity theories can be treated in the same way and has been investigated by several authors [25–39]. We present here a positive answer to this question based on a deformed algebra of diffeomorphisms and this way avoiding the concept of general coordinate transformations. In this presentation, we restrict ourselves to the discussion of the canonical quantum space with $\theta^{\mu\nu}$ constant. The construction is now not based on Seiberg–Witten maps. In a forthcoming paper, we shall show how this can be generalized to x-dependent $\theta^{\mu\nu}$.

By outlining the content of the individual sections, we will show the strategy by which a deformed gravity theory can be constructed.

In section 2, we give a short introduction to the θ-deformed quantum algebra defined by the Moyal–Weyl product. Emphasis is on those concepts that shall be used in the rest of the paper. More detailed features of this algebra can be found in the literature and we give some relevant references.

In section 3, the concept of derivatives is introduced. It turns out that there is a natural way to define a derivative on the quantum algebra. We investigate these derivatives as elements of a Hopf algebra and find that the usual derivatives and the derivatives on the quantum space represent the same Hopf algebra.

We also generalize the derivatives to higher order differential operators and define algebras of higher order differential operators both acting on differential manifolds and acting on the deformed space. A map from the algebra of functions on the differential manifold to the algebra of functions on the deformed space is constructed. This map will be the basis for the representation of the diffeomorphism algebra by an algebra of higher order differential operators acting on the deformed space.

In section 4, we study the algebra generated by vector fields and exhibit its Hopf algebra structure. It is the algebra of diffeomorphisms derived from general coordinate transformations. Scalar, vector and tensor fields are representations of this algebra.

In section 5, we construct a morphism between the classical algebra of diffeomorphisms and an algebra acting on the deformed space. At first, this is an algebra morphism but not a Hopf algebra morphism. To find a comultiplication rule, we derive the Leibniz rule for the deformed algebra and show that it can be obtained from an abstract comultiplication which we construct explicitly to all orders in θ. Thus, we have constructed a new Hopf algebra of diffeomorphisms as a deformation of the classical one. A deformed gravity theory will now be investigated as a theory covariant under this deformed Hopf algebra.

In section 6, we restrict the formalism developed so far to vector fields linear in the coordinates. They form a subalgebra. The Lorentz algebra can be obtained in that way and we find a representation of the Lorentz algebra by differential operators that act on the deformed space. The comultiplication rule follows from the general formalism and shows that the derivatives have to be part of the algebra. This way we have found a representation of the Poincaré algebra with nontrivial comultiplication rule. A tensor calculus of fields is developed for this algebra and invariant actions are constructed. All the operations in the definition of the Lagrangian—derivatives and multiplication—are in the deformed algebra. Field equations can be obtained that are Lorentz covariant. This by itself is an interesting result but it also serves as a guideline for the construction of a general theory on the deformed space.

In section 7, we show that all the concepts of differential geometry such as tensor fields, covariant derivatives, connection and curvature can be obtained by a map from the usual commutative space to the deformed space. The relevant formulae are calculated explicitly.

In sections 8 and 9, we turn to a metric space. We define the metric as a symmetric and real tensor that coincides with $g_{\mu\nu}$ in the limit $\theta \to 0$. All other geometrical quantities are constructed in terms of this metric. Finally, we use the curvature scalar expressed in terms of $g_{\mu\nu}$ to construct a Lagrangian for a deformed gravity theory.

In section 10, we expand all these quantities up to second order in θ. The action obtained this way can be used to calculate some effects of the deformation. The deformation parameter θ enters as a coupling constant as it is familiar from gauge theory.

This way it is possible to study deviations from the undeformed classical gravity due to spacetime noncommutativity. The strategy developed here can be generalized to other $\star$-products which then lead to other algebraic structures of spacetime.

2. θ-deformed coordinate algebra

A simple example of a noncommutative coordinate algebra is the θ-deformed or canonical quantum algebra $\mathcal{A}_\theta$. It is based on relations [40, 41]

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{2.1}$$

with $\theta^{\mu\nu}$ constant and real.

This algebra can be realized on the linear space $\mathcal{F}$ of complex functions $f(x)$ of commuting variables. The elements of the algebra $\mathcal{A}_\theta$ are represented by functions of the commuting variables $f(x)$ and their product by the Moyal–Weyl star-product ($\star$-product) [14, 42]

$$f \star g(x) = \exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial x^\rho}\theta^{\rho\sigma}\frac{\partial}{\partial y^\sigma}\right) f(x)g(y)|_{y\to x}. \tag{2.2}$$

This $\star$-product of two functions is again a function. The $\star$-product defines the associative but noncommutative algebra $\mathcal{A}_\theta$. By taking the usual pointwise product of two functions, we obtain the usual algebra of functions. This algebra is associative and commutative. We shall call it $\mathcal{A}_f$. Note that we write $f(x)$ for elements of $\mathcal{A}_f$ as well as for elements of $\mathcal{A}_\theta$.

As far as complex conjugation is concerned, we observe that the $\star$-product of two real functions is not real. Denoting the complex conjugate of f by $\bar{f}$, we find from definition (2.2)

$$\overline{f \star g} = \bar{g} \star \bar{f}. \tag{2.3}$$

From definition (2.2), it also follows that

$$x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu}. \tag{2.4}$$

These are the defining relations for the generators of the algebra $\mathcal{A}_\theta$. Any element of the space of ordinary functions represents an element of $\mathcal{A}_\theta$; there is an invertible map ϕ [43]

$$\phi : \mathcal{F} \to \mathcal{A}_\theta. \tag{2.5}$$

If we know the elements that are represented by the functions f and g, we can ask how the pointwise product of two functions $f \cdot g$ is represented in $\mathcal{A}_\theta$ by $\star$-products of f and g and their derivatives. First, an example

$$x^\mu \cdot x^\nu = x^\mu \star x^\nu - \frac{\mathrm{i}}{2}\theta^{\mu\nu}. \tag{2.6}$$

This follows from (2.2). The pointwise product $x^\mu \cdot x^\nu$ as an element of $\mathcal{A}_\theta$ represents the sum of two elements of $\mathcal{A}_\theta$ modulo relation (2.4). In general, $f \cdot g$ will represent a sum of $\star$-products of f, g and their derivatives

$$f \cdot g = \sum_{n=0}^{\infty} \left(-\frac{\mathrm{i}}{2}\right)^n \frac{1}{n!} \theta^{\rho_1\sigma_1} \cdots \theta^{\rho_n\sigma_n} \left(\partial_{\rho_1} \cdots \partial_{\rho_n} f\right) \star \left(\partial_{\sigma_1} \cdots \partial_{\sigma_n} g\right). \tag{2.7}$$

This is a well-defined formula because the derivatives of functions are functions again and we know how to $\star$-multiply them. Applied to $x^\mu x^\nu$, equation (2.7) reproduces (2.6). The operations on the right-hand side of (2.7) are all in $\mathcal{A}_\theta$. To prove (2.7), we use the $\star$-product in the form that makes use of the tensor product of the vector spaces $\mathcal{F}$

$$f \star g = \mu \left\{ \exp\left(\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma\right) f \otimes g \right\}. \tag{2.8}$$

The bilinear map μ maps the tensor product to the space of functions

$$\mu : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \qquad \mu\{f \otimes g\} \mapsto f \cdot g. \tag{2.9}$$

We now use the obvious equation

$$f \cdot g = \mu \left\{ \exp\left(\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma\right) \exp\left(-\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma\right) f \otimes g \right\}. \tag{2.10}$$

The first exponent will produce $\star$-products and the second one the sum of terms in (2.7). On the other hand, equation (2.2) expresses the $\star$-product $f \star g$ in terms of pointwise products of f and g and their derivatives. All these operations are in $\mathcal{A}_f$.

3. Derivatives on $\mathcal{A}_\theta$

Derivatives on quantum spaces were constructed in [44, 45]. There is, however, a natural way to introduce derivatives on $\mathcal{A}_\theta$ based on the $\star$-product formulation. We know that the derivative of a function $f \in \mathcal{F}$ is again a function. This function can be mapped to $\mathcal{A}_\theta$, the image we call the $\star$-derivative of $f \in \mathcal{A}_\theta$

$$\begin{array}{ccc} f \in \mathcal{F} & \stackrel{\phi}{\longmapsto} & f \in \mathcal{A}_\theta \\ \downarrow{\scriptstyle \partial_\mu} & & \downarrow{\scriptstyle \partial^\star_\mu} \\ (\partial_\mu f) \in \mathcal{F} & \stackrel{\phi}{\longmapsto} & (\partial^\star_\mu \triangleright f) \in \mathcal{A}_\theta \end{array} . \tag{3.1}$$

This defines $\partial^\star_\mu$ acting on $f \in \mathcal{A}_\theta$

$$\partial^\star_\mu \triangleright f := (\partial_\mu f). \tag{3.2}$$

Now we discuss a few properties of the $\star$-derivatives. From definition (3.2) follows

$$\partial_\mu^\star \triangleright x^\rho = \delta_\mu^\rho, \tag{3.3}$$

a property that we demand for a reasonable definition of a derivative. As the $\star$-product of two functions is again a function, we can use definition (3.2) to differentiate $f \star g$

$$\partial_\mu^\star \triangleright (f \star g) = (\partial_\mu (f \star g)). \tag{3.4}$$

For the $\star$-product with x-independent θ, it follows from (2.2) that

$$(\partial_\mu (f \star g)) = (\partial_\mu f) \star g + f \star (\partial_\mu g). \tag{3.5}$$

Using (3.2), we obtain

$$\partial_\mu^\star \triangleright (f \star g) = (\partial_\mu^\star \triangleright f) \star g + f \star (\partial_\mu^\star \triangleright g). \tag{3.6}$$

In this equation, all operations, derivative and product are within $\mathcal{A}_\theta$. We have expressed the $\star$-derivative acting on a $\star$-product by the $\star$-product of $\star$-derivatives.

Applying this rule to (2.4), we find

$$\partial_\mu^\star \triangleright ([x^\rho \overset{\star}{,} x^\sigma] - \mathrm{i}\theta^{\rho\sigma}) = 0. \tag{3.7}$$

This confirms that the derivative (3.2) is a well-defined map on $\mathcal{A}_\theta$. Moreover, from (3.2) follows

$$\partial_\mu^\star \triangleright (\partial_\nu^\star \triangleright f) = (\partial_\mu \partial_\nu f) \tag{3.8}$$

and therefore

$$\partial_\mu^\star \triangleright (\partial_\nu^\star \triangleright f) = \partial_\nu^\star \triangleright (\partial_\mu^\star \triangleright f). \tag{3.9}$$

The action of $\star$-derivatives on a function is commutative.

Derivatives were defined by their action on functions but they can be seen as differential operators as well because equation (3.6) holds for any function g. Thus, it gives rise to the operator equation

$$\partial_\mu^\star \star f = (\partial_\mu^\star \triangleright f) + f \star \partial_\mu^\star. \tag{3.10}$$

We use the $\star$ when the derivative is meant to be an operator. As for ordinary derivatives, we can also use the bracket notation if the derivatives act on a function. To emphasize that the action is meant, we also use the triangle notation

$$(\partial_\mu^\star \star f) \equiv \partial_\mu^\star \triangleright f. \tag{3.11}$$

Taking for f the coordinate x^ρ, we obtain from (3.10)

$$[\partial_\mu^\star \overset{\star}{,} x^\rho] = \delta_\mu^\rho. \tag{3.12}$$

Analogously to equation (3.7), we get

$$\partial_\mu^\star \star ([x^\rho \overset{\star}{,} x^\sigma] - \mathrm{i}\theta^{\rho\sigma}) = ([x^\rho \overset{\star}{,} x^\sigma] - \mathrm{i}\theta^{\rho\sigma}) \star \partial_\mu^\star. \tag{3.13}$$

Equation (3.9), valid for any function g, leads to the commutativity of $\star$-derivative operators

$$[\partial_\mu^\star \overset{\star}{,} \partial_\nu^\star] = 0. \tag{3.14}$$

The derivatives, as maps on the algebra $\mathcal{A}_\theta$, have a Hopf algebra structure [46–48]. This implies the following properties: the derivatives generate an algebra with defining relation (3.14). The coproduct is defined as follows:

$$\Delta(\partial_\mu^\star) = \partial_\mu^\star \otimes 1 + 1 \otimes \partial_\mu^\star. \tag{3.15}$$

It is compatible with the algebra

$$[\Delta(\partial^\star_\mu) \stackrel{\star}{,} \Delta(\partial^\star_\nu)] = 0. \tag{3.16}$$

The coassociativity

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{3.17}$$

can be verified explicitly. When we apply (3.17) to $\partial^\star_\mu$, we obtain

$$\begin{aligned}((\Delta \otimes \mathrm{id}) \circ \Delta)(\partial^\star_\mu) &= (\Delta \otimes \mathrm{id})(\partial^\star_\mu \otimes 1 + 1 \otimes \partial^\star_\mu) \\ &= (\partial^\star_\mu \otimes 1 + 1 \otimes \partial^\star_\mu) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \partial^\star_\mu.\end{aligned} \tag{3.18}$$

That $(\mathrm{id} \otimes \Delta) \circ \Delta$ gives the same result can easily be seen. To define a Hopf algebra, we still need a counit and an antipode. They are given by

$$\varepsilon(\partial^\star_\mu) = 0, \qquad S(\partial^\star_\mu) = -\partial^\star_\mu. \tag{3.19}$$

The Leibniz rule (3.6) can be obtained by applying the bilinear map $\mu_\star\{f \otimes g\} = f \star g$ to the coproduct

$$\begin{aligned}\mu_\star\{\Delta(\partial^\star_\mu) \triangleright f \otimes g\} &= \mu_\star\{(\partial^\star_\mu \star f) \otimes g + f \otimes (\partial^\star_\mu \star g)\} \\ &= (\partial^\star_\mu \triangleright f) \star g + f \star (\partial^\star_\mu \triangleright g).\end{aligned} \tag{3.20}$$

The usual derivatives ∂_μ and $\star$-derivatives $\partial^\star_\mu$ are representations of the same Hopf algebra.

We are going to discuss the algebra of higher order differential operators. Acting on $\mathcal{A}_f$, elements of this algebra are

$$D = \sum_r d^{\mu_1\cdots\mu_r}(x)\partial_{\mu_1}\cdots\partial_{\mu_r}. \tag{3.21}$$

Acting on $\mathcal{A}_\theta$, the elements are

$$D^\star = \sum_r d^{\mu_1\cdots\mu_r}(x)\partial^\star_{\mu_1}\cdots\partial^\star_{\mu_r}, \tag{3.22}$$

where the coefficient function $d^{\mu_1\cdots\mu_r}(x)$ has to be considered as an element of $\mathcal{A}_\theta$. The multiplication of the operators D is standard. The multiplication of $\star$-operators can be defined if we consider the algebra $\mathcal{A}_\theta$ extended by the derivatives. It is always possible to write such a product in the form as in (3.22) with all derivatives on the right by using the operator equation following from the Leibniz rule. This multiplication can essentially be obtained by replacing the product of the coefficient functions by the $\star$-product.

The operator D can be mapped to operators acting on $\mathcal{A}_\theta$. To construct such a map, we re-examine the pointwise product of two functions (2.7) in the light of higher order differential operators

$$f \cdot g = X^\star_f \triangleright g = (X^\star_f \star g), \tag{3.23}$$

where

$$X^\star_f = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n \theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\left(\partial_{\rho_1}\cdots\partial_{\rho_n} f\right) \star \partial^\star_{\sigma_1}\cdots\partial^\star_{\sigma_n}. \tag{3.24}$$

This can easily be generalized to the action of differential operators on g

$$(Dg) = X^\star_D \triangleright g, \tag{3.25}$$

where

$$X^\star_D = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n \theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\left(\partial_{\rho_1}\cdots\partial_{\rho_n} d^{\mu_1\cdots\mu_r}(x)\right) \star \partial^\star_{\sigma_1}\cdots\partial^\star_{\sigma_n}\partial^\star_{\mu_1}\cdots\partial^\star_{\mu_r}. \tag{3.26}$$

Now we define a map

$$f \mapsto X^\star_f, \qquad D \mapsto X^\star_D.$$

This map is actually an algebra map if the multiplications of differential operators are defined as above after (3.22). From (3.25) follows

$$(D \cdot D'g) = (X^\star_D \star X^\star_{D'}) \triangleright g. \tag{3.28}$$

This is true for any function g and thus the map (3.27) can be interpreted as a morphism of algebras.

4. Diffeomorphisms

We will develop a formalism by which the algebra of diffeomorphisms acting on $\mathcal{A}_f$ can be mapped to an algebra of $\star$-differential operators acting on $\mathcal{A}_\theta$.

Let us first recall the concept of diffeomorphisms as a Hopf algebra. They are generated by vector fields acting on a differential manifold. The vector fields are defined as follows:

$$\xi = \xi^\mu(x)\frac{\partial}{\partial x^\mu}. \tag{4.1}$$

The commutator of two vector fields ξ, η is again a vector field:

$$[\xi, \eta] = \xi \times \eta, \tag{4.2}$$

where the vector field $\xi \times \eta$ is given by

$$\xi \times \eta = (\eta^\mu(\partial_\mu \xi^\rho) - \xi^\mu(\partial_\mu \eta^\rho))\frac{\partial}{\partial x^\rho}. \tag{4.3}$$

From the Leibniz rule for derivatives follows the Leibniz rule for vector fields

$$(\xi(f \cdot g)) = (\xi f) \cdot g + f \cdot (\xi g). \tag{4.4}$$

This Leibniz rule follows from an abstract comultiplication rule that defines the action of a vector field on a tensor product

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi. \tag{4.5}$$

It can be verified with (4.2) that the comultiplication (4.5) and the algebraic relation are compatible without making use of ξ represented as a differential operator

$$[\Delta(\xi), \Delta(\eta)] = \Delta(\xi \times \eta). \tag{4.6}$$

This defines a bialgebra. With the counit and the antipode

$$\varepsilon(\xi) = 0, \qquad S(\xi) = -\xi, \tag{4.7}$$

it becomes a Hopf algebra. Here, ξ and η need to be treated as abstract objects. Their product $\xi\eta$ is to be viewed as an abstract product modulo the relation $\xi\eta - \eta\xi = \xi \times \eta$.[7]

Diffeomorphisms are intimately connected with general coordinate transformations defined as follows:

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{4.8}$$

with infinitesimal $\xi^\mu(x)$.

[7] In other words, we are considering the universal enveloping algebra freely generated by elements ξ, η modulo the relation $\xi\eta - \eta\xi = \xi \times \eta$.

A scalar field is defined to transform under general coordinate transformations as follows:

$$\phi'(x') = \phi(x).$$

For infinitesimal transformations (4.8), this becomes

$$\delta_\xi \phi(x) = \phi'(x) - \phi(x) = -\xi^\mu(\partial_\mu \phi(x)) = -(\xi\phi(x)). \tag{4.9}$$

Similarly, we define covariant vector fields

$$\delta_\xi V_\mu = -\xi^\rho(\partial_\rho V_\mu) - (\partial_\mu \xi^\rho) V_\rho \tag{4.10}$$

and contravariant vector fields

$$\delta_\xi V^\mu = -\xi^\rho(\partial_\rho V^\mu) + (\partial_\rho \xi^\mu) V^\rho. \tag{4.11}$$

This can easily be generalized to tensor fields with an arbitrary number of covariant and contravariant indices.

These transformations represent the algebra of diffeomorphisms (4.2)

$$[\delta_\xi, \delta_\eta] = \delta_{\xi\times\eta}, \tag{4.12}$$

with the coproduct

$$\Delta\delta_\xi = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi. \tag{4.13}$$

As a consequence of (4.13), the product of two vector fields transforms like a tensor field of second rank

$$\begin{aligned}\delta_\xi(V_\mu V_\nu) &= \mu\{\Delta(\delta_\xi) V_\mu \otimes V_\nu\} = \mu\{(\delta_\xi V_\mu) \otimes V_\nu + V_\mu \otimes (\delta_\xi V_\nu)\}\\ &= -\xi^\rho \partial_\rho (V_\mu V_\nu) - (\partial_\mu \xi^\rho)(V_\rho V_\nu) - (\partial_\nu \xi^\rho)(V_\mu V_\rho).\end{aligned} \tag{4.14}$$

This can easily be extended to the product of arbitrary tensor fields.

We summarize the Hopf algebra structure of general coordinate transformations

$$\begin{aligned}&[\delta_\xi, \delta_\eta] = \delta_{\xi\times\eta}, \qquad &&\varepsilon(\delta_\xi) = 0, \qquad S(\delta_\xi) = -\delta_\xi,\\ &\Delta\delta_\xi = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi, \qquad &&[\Delta(\delta_\xi), \Delta(\delta_\eta)] = \Delta(\delta_{\xi\times\eta}).\end{aligned} \tag{4.15}$$

This is true for any realization of δ_ξ on arbitrary tensor fields. It is a property of the abstract Hopf algebra and not of a particular representation as differential operator.

5. Diffeomorphism algebra on $\mathcal{A}_\theta$

We know how to map the algebra of higher order classical differential operators acting on $\mathcal{A}_f$ into the corresponding algebra acting on $\mathcal{A}_\theta$. The relevant formulae are (3.24) and (3.26).

In the same way, the action of a vector field

$$\xi = \xi^\mu \partial_\mu \tag{5.1}$$

can be mapped to a higher order differential operator acting on $\mathcal{A}_\theta$

$$(\xi \cdot f) = X^\star_\xi \triangleright f. \tag{5.2}$$

From (3.28) then follows

$$[X^\star_\xi \stackrel{\star}{,} X^\star_\eta] = X^\star_{\xi\times\eta}. \tag{5.3}$$

The operators $X^\star_\xi$ represent the algebra of vector fields. To obtain a Leibniz rule on $\mathcal{A}_\theta$, we apply the operator $X^\star_\xi$ to the $\star$-product of two functions

$$X^\star_\xi \triangleright (f \star g) = (\xi(f \star g)). \tag{5.4}$$

To get a better understanding, we calculate the right-hand side of (5.4) to first order in θ explicitly

$$(\xi(f\star g)) = \left(\xi\left(fg + \frac{\mathrm{i}}{2}\theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g)\right)\right)$$
$$= (\xi f)g + f(\xi g) + \frac{\mathrm{i}}{2}\theta^{\rho\sigma}((\xi\partial_\rho f)(\partial_\sigma g) + (\partial_\rho f)(\xi\partial_\sigma g)) + \cdots. \tag{5.5}$$

We have to express the right-hand side entirely in terms of operations on $\mathcal{A}_\theta$

$$(\xi(f\star g)) = (\xi f)\star g + f\star(\xi g)$$
$$-\frac{\mathrm{i}}{2}\theta^{\rho\sigma}((\partial_\rho(\xi^\mu\partial_\mu f))(\partial_\sigma g) + (\partial_\rho f)(\partial_\sigma(\xi^\mu\partial_\mu g)))$$
$$+\frac{\mathrm{i}}{2}\theta^{\rho\sigma}((\xi^\mu(\partial_\mu\partial_\rho f))(\partial_\sigma g) + (\partial_\rho f)(\xi^\mu(\partial_\mu\partial_\sigma g))) + \cdots. \tag{5.6}$$

Up to first order in θ, this is identical to [49]

$$X^\star_\xi \triangleright (f\star g) = (X^\star_\xi\star f)\star g + f\star(X^\star_\xi\star g)$$
$$-\frac{\mathrm{i}}{2}\theta^{\rho\sigma}([\partial_\rho,\xi^\mu](\partial_\mu f)(\partial_\sigma g) + (\partial_\rho f)[\partial_\sigma,\xi^\mu](\partial_\mu g))$$
$$= (X^\star_\xi\star f)\star g + f\star(X^\star_\xi\star g)$$
$$-\frac{\mathrm{i}}{2}\theta^{\rho\sigma}(([\partial^\star_\rho \stackrel{\star}{,} X^\star_\xi]\star f)\star(\partial^\star_\sigma\star g) + (\partial^\star_\rho\star f)\star([\partial^\star_\sigma \stackrel{\star}{,} X^\star_\xi]\star g)). \tag{5.7}$$

This Leibniz rule follows from an abstract comultiplication rule which reads up to first order in θ

$$\Delta(X^\star_\xi)(f\otimes g) = (X^\star_\xi\star f)\otimes g + f\star(X^\star_\xi\otimes g)$$
$$-\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\left(\left(X^\star_{[\partial^\star_\rho \stackrel{\star}{,}\xi]}\star f\right)\otimes(\partial^\star_\sigma\star g) + (\partial^\star_\rho\star f)\otimes\left(X^\star_{[\partial^\star_\sigma \stackrel{\star}{,}\xi]}\star g\right)\right). \tag{5.8}$$

This comultiplication rule differs from the one we obtained for the classical diffeomorphisms. These two Hopf algebras are different although they are the same on the algebra level.

The Leibniz rule (5.7) can be calculated to all orders in θ, the result is

$$X^\star_\xi \triangleright (f\star g) = \mu_\star\left\{\exp\left(-\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)(X^\star_\xi\otimes 1 + 1\otimes X^\star_\xi)\right.$$
$$\left.\times\exp\left(\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)\triangleright(f\otimes g)\right\}. \tag{5.9}$$

The map $\mu_\star$ was defined in (3.20). Expanding (5.9) to first order in θ, we obtain (5.7). In (5.9) appear $\star$-commutators of $\star$-derivatives and the operator $X^\star_\xi$. A short calculation using the explicit expression for $X^\star_\xi$ given in (3.26) yields the following equation:

$$[\partial^\star_\rho\otimes\partial^\star_\sigma \stackrel{\star}{,} X^\star_\xi\otimes 1] = [\partial^\star_\rho \stackrel{\star}{,} X^\star_\xi]\otimes\partial^\star_\sigma = X^\star_{(\partial_\rho\xi)}\otimes\partial^\star_\sigma. \tag{5.10}$$

Applying this equation inductively, we find an expression where the exponential functions in (5.9) are expanded to all orders

$$X^\star_\xi \triangleright (f\star g) = (X^\star_\xi \triangleright f)\star g + f\star(X^\star_\xi\triangleright g)$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\left(\left(X^\star_{(\partial_{\rho_1}\cdots\partial_{\rho_n}\xi)}\triangleright f\right)\star\left(\partial_{\sigma_1}\cdots\partial_{\sigma_n}g\right)\right.$$
$$\left.+\left(\partial_{\rho_1}\cdots\partial_{\rho_n}f\right)\star\left(X^\star_{(\partial_{\sigma_1}\cdots\partial_{\sigma_n}\xi)}\triangleright g\right)\right). \tag{5.11}$$

Note that $(\partial_\rho\xi)$ and all higher order derivatives of ξ are vector fields again.

We outline the calculation leading to (5.9) and start from (5.4)

$$(\xi(f\star g)) = \xi\mu\left\{\exp\left(\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma\right) f\otimes g\right\}. \tag{5.12}$$

To commute ξ with μ, we use

$$\xi\mu = \mu\{\xi\otimes 1 + 1\otimes\xi\}, \tag{5.13}$$

which can be verified directly by applying it to the tensor product of two functions. We obtain from (5.12)

$$(\xi(f\star g)) = \mu\left\{(\xi\otimes 1 + 1\otimes\xi)\sum_n \frac{1}{n!}\left(-\frac{i}{2}\right)^n \theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\left(\partial_{\rho_1}\cdots\partial_{\rho_n} f\right)\otimes\left(\partial_{\sigma_1}\cdots\partial_{\sigma_n} g\right)\right\}. \tag{5.14}$$

Next, we use the fact that ξ applied to derivatives of a function can be mapped to $\mathcal{A}_\theta$ as in (5.2) because derivatives of functions are functions again. This way we express everything in terms of operators defined on $\mathcal{A}_\theta$. Now we follow the step outlined in (5.7) and obtain the result (5.9).

Let us summarize the Hopf algebra structure of the diffeomorphism algebra on $\mathcal{A}_\theta$. For an element f of $\mathcal{A}_\theta$, we define the transformation

$$\delta_\xi f = -X^\star_\xi \triangleright f \equiv \hat{\delta}_\xi f. \tag{5.15}$$

This can be used to define $\hat{\delta}_\xi$ as an abstract element of an algebra independent of its representation as a differential operator. From (5.3), the defining relation of the algebra follows

$$[\hat{\delta}_\xi, \hat{\delta}_\eta] = \hat{\delta}_{[\xi,\eta]} = \hat{\delta}_{\xi\times\eta}, \tag{5.16}$$

where ξ and η are vector fields and $[\xi,\eta]$ their commutator. The comultiplication from which the Leibniz rule (5.9) follows is[8]

$$\Delta(\hat{\delta}_\xi) = \exp\left(-\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)(y\hat{\delta}_\xi\otimes 1 + 1\otimes\hat{\delta}_\xi)\exp\left(\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right). \tag{5.17}$$

Here, the $\star$-commutator of a $\star$-derivative and $\hat{\delta}_\xi$ is given by

$$[\partial^\star_\rho \stackrel{\star}{,} \hat{\delta}_\xi] = \hat{\delta}_{(\partial_\rho\xi)}. \tag{5.18}$$

This is the abstract version of (5.10). We show that the above comultiplication is compatible with the algebra

$$\begin{aligned}[\Delta(\hat{\delta}_\xi), \Delta(\hat{\delta}_\eta)] &= \Big[\exp\left(-\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)(\hat{\delta}_\xi\otimes 1 + 1\otimes\hat{\delta}_\xi),\\ &\qquad (\hat{\delta}_\eta\otimes 1 + 1\otimes\hat{\delta}_\eta)\exp\left(\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)\Big]\\ &= \exp\left(-\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)(\hat{\delta}_{\xi\times\eta}\otimes 1 + 1\otimes\hat{\delta}_{\xi\times\eta})\exp\left(\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)\\ &= \Delta(\hat{\delta}_{[\xi,\eta]}).\end{aligned} \tag{5.19}$$

Coassociativity can be shown as well, counit and antipode can be defined.

Thus, we have obtained a Hopf algebra with the same algebraic relations as for the ordinary diffeomorphism algebra, but the comultiplication is different.

[8] The derivative $\partial^\star_\rho$ can be considered as a variation $\hat{\delta}_{\partial_\rho} = -\partial^\star_\rho$ in the direction of ∂_ρ.

For later use, we expand the comultiplication (5.17) to first order in θ

$$\Delta(\hat{\delta}_\xi) = \hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi - \frac{i}{2}\theta^{\rho\sigma}([\partial^\star_\rho, \hat{\delta}_\xi] \otimes \partial^\star_\sigma + \partial^\star_\rho \otimes [\partial^\star_\sigma, \hat{\delta}_\xi])$$
$$= \hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi - \frac{i}{2}\theta^{\rho\sigma}\left(\hat{\delta}_{(\partial_\rho\xi)} \otimes \partial^\star_\sigma + \partial^\star_\rho \otimes \hat{\delta}_{(\partial_\sigma\xi)}\right). \tag{5.20}$$

6. Poincaré algebra

The classical vector fields (5.1), when linear in x, form a subalgebra of the algebra of diffeomorphisms

$$\xi_\omega = x^\mu \omega_\mu{}^\nu \partial_\nu, \qquad [\xi_\omega, \xi_{\omega'}] = \xi_{[\omega,\omega']}, \tag{6.1}$$

where $[\omega, \omega']$ is the commutator of the matrices ω. The corresponding operators $X^\star_\omega$ are

$$X^\star_\omega = x^\mu \omega_\mu{}^\nu \partial^\star_\nu - \frac{i}{2}\theta^{\rho\sigma}\omega_\rho{}^\nu \partial^\star_\nu \partial^\star_\sigma. \tag{6.2}$$

Since ξ_ω is linear in x, this is already the exact expression to all orders in θ. The higher order differential operators $X^\star_\omega$ satisfy the same algebra as the vector fields ξ_ω:

$$[X^\star_\omega \stackrel{\star}{,} X^\star_{\omega'}] = X^\star_{[\omega,\omega']}. \tag{6.3}$$

The transformation defined in (5.15) becomes

$$\hat{\delta}_\omega f = -X^\star_\omega \triangleright f = -(\xi_\omega \cdot f). \tag{6.4}$$

These transformations together with the derivatives form a Hopf algebra, the relevant algebraic relations follow from (5.16) and (5.17) and the respective formulae for the derivatives.

Algebra:

$$[\partial^\star_\mu, \partial^\star_\nu] = 0, \qquad [\hat{\delta}_\omega, \hat{\delta}_{\omega'}] = \hat{\delta}_{[\omega,\omega']}, \qquad [\hat{\delta}_\omega, \partial^\star_\rho] = \omega^\mu_\rho \partial^\star_\mu, \tag{6.5}$$

Comultiplication:

$$\Delta\partial^\star_\mu = \partial^\star_\mu \otimes 1 + 1 \otimes \partial^\star_\mu,$$
$$\Delta\hat{\delta}_\omega = \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega - \frac{i}{2}\theta^{\rho\sigma}([\partial^\star_\rho, \hat{\delta}_\omega] \otimes \partial^\star_\sigma + \partial^\star_\rho \otimes [\partial^\star_\sigma, \hat{\delta}_\omega]). \tag{6.6}$$

This comultiplication mixes the $\hat{\delta}_\omega$ transformations and the derivatives. The transformations (6.4) do not form a Hopf algebra by themselves.

We can choose matrices ω that represent the Lorentz algebra

$$[M^{\rho\sigma}, M^{\kappa\lambda}] = \eta^{\rho\lambda}M^{\sigma\kappa} + \eta^{\sigma\kappa}M^{\rho\lambda} - \eta^{\rho\kappa}M^{\sigma\lambda} - \eta^{\sigma\lambda}M^{\rho\kappa}. \tag{6.7}$$

With derivatives representing the translations, we have obtained a Hopf algebra version of the Poincaré algebra [50–55]. The comultiplication is nontrivially deformed.

The algebra (6.5) and (6.6) can also be represented by tensor or spinor fields. Let ψ_A be a representation of the Lorentz algebra

$$\hat{\delta}_\omega \psi_A = \omega_\rho{}^\mu \left(M_\mu{}^\rho\right)_A{}^B \psi_B, \tag{6.8}$$

where $\left(M_\mu{}^\rho\right)_A{}^B$ as a matrix with indices A, B represents the Lorentz algebra (6.7). The transformation $\hat{\delta}_\omega$ can be defined by the 'field transformations'

$$\hat{\delta}_\omega \psi_A = -X^\star_\omega \triangleright \psi_A + \omega_\rho{}^\mu \left(M_\mu{}^\rho\right)_A{}^B \psi_B. \tag{6.9}$$

With (6.9) we have established a Poincaré covariant tensor calculus on $\mathcal{A}_\theta$. The new comultiplication guarantees that the $\star$-product of tensor fields transforms as a tensor.

Now we can construct Poincaré covariant Lagrangians. As an example, we discuss a scalar field. Let ϕ be a classical scalar field

$$\delta_\omega \phi = -(\xi_\omega \phi). \tag{6.10}$$

The transformation law can be mapped to $\mathcal{A}_\theta$ and we can consider ϕ as an element (field) in $\mathcal{A}_\theta$ with the transformation law

$$\hat{\delta}_\omega \phi = -X^\star_\omega \triangleright \phi. \tag{6.11}$$

The $\star$-derivative of a scalar field will transform like a vector field

$$\hat{\delta}_\omega(\partial^\star_\rho \triangleright \phi) = -X^\star_\omega \triangleright (\partial^\star_\rho \triangleright \phi) - X^\star_{(\partial_\rho \xi^\mu)} \triangleright (\partial^\star_\mu \triangleright \phi). \tag{6.12}$$

Thus, the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial^\star_\mu \phi) \star (\partial^{\star\mu} \phi) - \frac{m^2}{2}\phi \star \phi - \lambda \phi \star \phi \star \phi \tag{6.13}$$

is covariant

$$\hat{\delta}_\omega \mathcal{L} = -X^\star_{\xi_\omega} \triangleright \mathcal{L} = -\xi^\lambda_\omega (\partial_\lambda \mathcal{L}). \tag{6.14}$$

It can be expanded in θ and to second order we obtain

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi\phi - \lambda\phi^3 - \frac{1}{16}\theta^{\rho\sigma}\theta^{\alpha\beta}(\partial_\rho\partial_\alpha\partial_\mu\phi)(\partial_\sigma\partial_\beta\partial^\mu\phi) \\ + \frac{m^2}{16}\theta^{\rho\sigma}\theta^{\alpha\beta}(\partial_\rho\partial_\alpha\phi)(\partial_\sigma\partial_\beta\phi) + \frac{3}{8}\lambda\theta^{\rho\sigma}\theta^{\alpha\beta}\phi(\partial_\rho\partial_\alpha\phi)(\partial_\sigma\partial_\beta\phi).\end{aligned} \tag{6.15}$$

Note that a classical transformation of the fields in (6.15) will only reproduce (6.14) if θ is transformed as well. Due to the comultiplication rule (6.6), we do not have to transform θ to obtain an invariant action.

To construct an invariant action, we define the integration on $\mathcal{A}_\theta$ as the usual integration. This integral has the cyclic property

$$\int \mathrm{d}^n x\, \phi \star \chi = \int \mathrm{d}^n x\, \chi \star \phi = \int \mathrm{d}^n x\, \phi\chi, \tag{6.16}$$

which follows by partial integration. The action

$$S = \int \mathrm{d}^n x \mathcal{L}$$

is invariant if $\mathcal{L}$ transforms like (6.14).

To derive the equations of motion, we vary the action with respect to the field ϕ. We use the undeformed Leibniz rule for this functional variation and we can use property (6.16) to cycle the varied field to the very right (or left) of the integrand. For the Lagrangian (6.13), we obtain

$$\begin{aligned}\delta_\phi S &= \delta_\phi \left(\int \mathrm{d}^n x \left(-\frac{1}{2}\phi \star (\partial^{\star\mu}\partial^\star_\mu \phi) - \frac{m^2}{2}\phi \star \phi - \lambda\phi \star \phi \star \phi \right)\right) \\ &= \int \mathrm{d}^n x \delta_\phi \phi(x) \star \left(-2\frac{1}{2}(\partial^{\star\mu}\partial^\star_\mu \phi) - 2\frac{m^2}{2}\phi - 3\lambda\phi \star \phi \right).\end{aligned} \tag{6.17}$$

This leads to the field equations

$$(\partial^{\star\mu}\partial^\star_\mu \phi) + m^2\phi + 3\lambda\phi \star \phi = 0. \tag{6.18}$$

If we expand (6.18) in θ, we obtain to second order in θ the same field equation as from the variation of the action corresponding to the Lagrangian (6.15)

$$S = \int \mathrm{d}^n x \left(\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi\phi - \lambda\phi^3 + \frac{3}{8}\lambda\theta^{\rho\sigma}\theta^{\alpha\beta}\phi(\partial_\rho\partial_\alpha\phi)(\partial_\sigma\partial_\beta\phi) \right). \tag{6.19}$$

Some partial integration is necessary. This example will guide us by the construction of a gravity action.

7. Differential geometry on $\mathcal{A}_\theta$

Gravity theories in general rely on general coordinate transformations which are hard to generalize to noncommutative spaces. The important concept however, on which the gravity theories are based, is the algebra of diffeomorphisms. General relativity can be seen as a theory covariant under diffeomorphisms. We have learned how to deform the diffeomorphism algebra, thus we can construct a deformed gravity theory as a theory covariant under deformed diffeomorphisms.

In section 5, we realized the algebra of vector fields on $\mathcal{A}_f$ on the noncommutative space $\mathcal{A}_\theta$. We now develop a tensor calculus for the deformed algebra in analogy to the tensor calculus of the deformed Poincaré algebra.

We define the transformation law of a scalar field

$$\hat{\delta}_\xi \phi = -X^\star_\xi \triangleright \phi, \tag{7.1}$$

of a covariant vector field

$$\hat{\delta}_\xi V_\mu = -X^\star_\xi \triangleright V_\mu - X^\star_{(\partial_\mu \xi^\rho)} \triangleright V_\rho, \tag{7.2}$$

of a contravariant vector field

$$\hat{\delta}_\xi V^\mu = -X^\star_\xi \triangleright V^\mu + X^\star_{(\partial_\rho \xi^\mu)} \triangleright V^\rho \tag{7.3}$$

and of a general tensor field

$$\begin{aligned}\hat{\delta}_\xi T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_p} = &-X^\star_\xi \triangleright T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_p} - X^\star_{(\partial_{\mu_1}\xi^\rho)} \triangleright T^{\nu_1\cdots\nu_r}_{\rho\cdots\mu_p} - \cdots - X^\star_{(\partial_{\mu_p}\xi^\rho)} \triangleright T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\rho} \\ &+ X^\star_{(\partial_\rho \xi^{\nu_1})} \triangleright T^{\rho\cdots\nu_r}_{\mu_1\cdots\mu_p} + \cdots + X^\star_{(\partial_\rho \xi^{\nu}_r)} \triangleright T^{\nu_1\cdots\rho}_{\mu_1\cdots\mu_p}.\end{aligned} \tag{7.4}$$

The operators $X^\star_\xi$ and $X^\star_{(\partial_\mu \xi^\lambda)}$ follow from (3.26)

$$\begin{aligned}X^\star_\xi &= \sum_{n=0}^\infty \frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\left(\partial_{\rho_1}\cdots\partial_{\rho_n}\xi\right)\star\partial^\star_{\sigma_1}\cdots\partial^\star_{\sigma_n} \\ &= \sum_{n=0}^\infty \frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\left(\partial_{\rho_1}\cdots\partial_{\rho_n}\xi^\lambda\right)\star\partial^\star_{\sigma_1}\cdots\partial^\star_{\sigma_n}\partial^\star_\lambda,\end{aligned} \tag{7.5}$$

$$X^\star_{(\partial_\mu \xi^\lambda)} = \sum_{n=0}^\infty \frac{1}{n!}\left(-\frac{\mathrm{i}}{2}\right)^n \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\left(\partial_{\rho_1}\cdots\partial_{\rho_n}\partial_\mu\xi^\lambda\right)\star\partial^\star_{\sigma_1}\cdots\partial^\star_{\sigma_n}. \tag{7.6}$$

The Leibniz rule that follows from (5.17) can be defined for the action of $\hat{\delta}_\xi$ on the $\star$-product of any of these fields

$$\begin{aligned}\hat{\delta}_\xi\left(T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_p}\star T^{\beta_1\cdots\beta_t}_{\alpha_1\cdots\alpha_s}\right) = \mu_\star\Big\{&\exp\left(-\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)(\hat{\delta}_\xi\otimes 1 + 1\otimes\hat{\delta}_\xi) \\ &\times\exp\left(\frac{\mathrm{i}}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma\right)\triangleright\left(T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_p}\otimes T^{\beta_1\cdots\beta_t}_{\alpha_1\cdots\alpha_s}\right)\Big\}.\end{aligned} \tag{7.7}$$

This definition of the comultiplication ensures that the $\star$-product $T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_p} \star T^{\beta_1\cdots\beta_t}_{\alpha_1\cdots\alpha_s}$ transforms like the tensor field $T^{\nu_1\cdots\nu_r\beta_1\cdots\beta_t}_{\mu_1\cdots\mu_p\alpha_1\cdots\alpha_s}$.

Examples:

The $\star$-product of two scalar fields is a scalar field again

$$\hat{\delta}_\xi(\phi \star \psi) = \mu_\star \left\{\exp\left(-\frac{i}{2}\theta^{\gamma\delta}\partial^\star_\gamma \otimes \partial^\star_\delta\right)(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)\exp\left(\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho \otimes \partial^\star_\sigma\right) \triangleright (\phi \otimes \psi)\right\}$$
$$= -X^\star_\xi \triangleright (\phi \star \psi). \tag{7.8}$$

The proof is the same as for equation (5.9).

Repeating the same calculation, one finds that the $\star$-product of a scalar field and a vector field is a vector field

$$\hat{\delta}_\xi(\phi \star V_\mu) = -X^\star_\xi \triangleright (\phi \star V_\mu) - X^\star_{(\partial_\mu\xi^\rho)} \triangleright (\phi \star V_\rho) \tag{7.9}$$

and the $\star$-product of two vector fields is a tensor field

$$\hat{\delta}_\xi(V_\mu \star V_\nu) = -X^\star_\xi \triangleright (V_\mu \star V_\nu) - X^\star_{(\partial_\mu\xi^\rho)} \triangleright (V_\rho \star V_\nu) - X^\star_{(\partial_\nu\xi^\rho)} \triangleright (V_\mu \star V_\rho). \tag{7.10}$$

The contraction of two indices is respected by the transformation law as well:

$$\hat{\delta}_\xi(V_\mu \star V^\mu) = \mu_\star \left\{\exp\left(-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma\right)(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)\exp\left(\frac{i}{2}\theta^{\gamma\delta}\partial_\gamma \otimes \partial_\delta\right)(V_\mu \otimes V^\mu)\right\}$$
$$= -X^\star_\xi \triangleright (V_\mu \star V^\mu). \tag{7.11}$$

Similar statements are true for the $\star$-product of arbitrary tensors. This is the basic concept of a covariant tensor calculus. Only the derivatives have to be generalized to covariant derivatives by demanding that the covariant derivative itself transforms like a covariant vector

$$\hat{\delta}_\xi D_\mu V_\nu = -X^\star_\xi \triangleright (D_\mu V_\nu) - X^\star_{(\partial_\mu\xi^\rho)} \triangleright (D_\rho V_\nu) - X^\star_{(\partial_\nu\xi^\rho)} \triangleright (D_\mu V_\rho). \tag{7.12}$$

This can be achieved by introducing a connection $\Gamma^\alpha_{\mu\nu}$ and defining the covariant derivative as

$$D_\mu V_\nu := \partial^\star_\mu \triangleright V_\nu - \Gamma^\alpha_{\mu\nu} \star V_\alpha. \tag{7.13}$$

The transformation law of the connection follows from (7.12) if we use the comultiplication (5.17)

$$\hat{\delta}_\xi \Gamma^\alpha_{\mu\nu} = -X^\star_\xi \triangleright \Gamma^\alpha_{\mu\nu} - X^\star_{(\partial_\mu\xi^\rho)} \triangleright \Gamma^\alpha_{\rho\nu} - X^\star_{(\partial_\nu\xi^\rho)} \triangleright \Gamma^\alpha_{\mu\rho} + X^\star_{(\partial_\rho\xi^\alpha)} \triangleright \Gamma^\rho_{\mu\nu} - \partial_\mu\partial_\nu\xi^\alpha. \tag{7.14}$$

The covariant derivative of a tensor field can be obtained by the same procedure as in the undeformed case, too

$$D_\lambda T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_p} = \partial^\star_\lambda \triangleright T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_p} - \Gamma^\alpha_{\lambda\mu_1} \star T^{\nu_1\cdots\nu_r}_{\alpha\cdots\mu_p} - \cdots - \Gamma^\alpha_{\lambda\mu_p} \star T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\alpha}$$
$$+ \Gamma^{\nu_1}_{\lambda\alpha} \star T^{\alpha\cdots\nu_r}_{\mu_1\cdots\mu_p} + \cdots + \Gamma^{\nu_r}_{\lambda\alpha} \star T^{\nu_1\cdots\alpha}_{\mu_1\cdots\mu_p}. \tag{7.15}$$

Curvature and torsion are obtained in complete analogy to the undeformed formalism

$$[D_\mu \stackrel{\star}{,} D_\nu] \star V_\rho = R_{\mu\nu\rho}{}^\sigma \star V_\sigma + T_{\mu\nu}{}^\alpha \star D_\alpha V_\rho. \tag{7.16}$$

Using (7.13), one finds

$$R_{\mu\nu\rho}{}^\sigma = \partial^\star_\nu \triangleright \Gamma^\sigma_{\mu\rho} - \partial^\star_\mu \triangleright \Gamma^\sigma_{\nu\rho} + \Gamma^\beta_{\nu\rho} \star \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} \star \Gamma^\sigma_{\nu\beta}, \tag{7.17}$$

$$T_{\mu\nu}{}^\alpha = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}. \tag{7.18}$$

For a connection symmetric in μ and ν, the torsion vanishes.

8. Metric and Christoffel symbols

Classically, the metric is a symmetric tensor of rank 2

$$\delta_\xi g_{\mu\nu} = -\xi^\rho(\partial_\rho g_{\mu\nu}) - (\partial_\mu \xi^\rho) g_{\rho\nu} - (\partial_\nu \xi^\rho) g_{\mu\rho}. \tag{8.1}$$

This can be mapped to $\mathcal{A}_\theta$ by defining $G_{\mu\nu}$ as a symmetric tensor of rank 2 in $\mathcal{A}_\theta$

$$\hat{\delta}_\xi G_{\mu\nu} = -X^\star_\xi \triangleright G_{\mu\nu} - X^\star_{(\partial_\mu \xi^\rho)} \triangleright G_{\rho\nu} - X^\star_{(\partial_\nu \xi^\rho)} \triangleright G_{\mu\rho}, \tag{8.2}$$

with the condition that

$$G_{\mu\nu}|_{\theta=0} = g_{\mu\nu}. \tag{8.3}$$

A natural choice for $G_{\mu\nu}$ would be $g_{\mu\nu}$ itself. It has the right transformation properties and is θ-independent.

We can also start from four vector fields E^a_μ, where μ is the vector index and a numbers the four vector fields. These vector fields can be chosen to be real. The metric can be defined as follows:

$$G_{\mu\nu} = \tfrac{1}{2}\left(E_\mu{}^a \star E_\nu{}^b + E_\nu{}^a \star E_\mu{}^b\right)\eta_{ab}, \tag{8.4}$$

where η_{ab} is the x-independent symmetric metric of the flat Minkowski space. With the appropriate comultiplication, $G_{\mu\nu}$ is a tensor of second rank. It is symmetric by construction and real since $E_\mu{}^a$ are real vector fields. To meet condition (8.3), we take the classical vierbein $e_\mu{}^a$ for $E_\mu{}^a$. Now $G_{\mu\nu}$ is θ-dependent. The metric $G_{\mu\nu}$ and its inverse can be used to raise and lower indices.

In $\mathcal{A}_\theta$, we have to construct the $\star$-inverse of $G_{\mu\nu}$ which we denote by $G^{\mu\nu\star}$

$$G_{\mu\nu} \star G^{\nu\rho\star} = \delta^\rho_\mu. \tag{8.5}$$

The inverse metric $G^{\mu\nu\star}$ is supposed to be a tensor but not a differential operator. To show how such a tensor can be found, we first invert a function $f \in \mathcal{A}_\theta$. As an element of $\mathcal{A}_f$, f is supposed to have an inverse f^{-1}

$$f \cdot f^{-1} = 1. \tag{8.6}$$

We want to find an inverse of f in $\mathcal{A}_\theta$, we denote it by $f^{-1\star}$

$$f \star f^{-1\star} = 1. \tag{8.7}$$

Obviously, $f^{-1\star}$ will be different from f^{-1}. For its construction, we use the geometric series. We first invert the element

$$f \star f^{-1} = 1 + \mathcal{O}(\theta), \tag{8.8}$$

$$\begin{aligned}(f \star f^{-1})^{-1\star} &= (1 + f \star f^{-1} - 1)^{-1\star} \\ &= \sum_{n=0}^{\infty} (1 - f \star f^{-1})^{n\star}.\end{aligned} \tag{8.9}$$

The $\star$ on the nth power of $1 - f \star f^{-1}$ indicates that all the products are $\star$-products and therefore (8.9) is an expansion in $\mathcal{A}_\theta$. Because of (8.8), it is also an expansion in θ

$$(1 - f \star f^{-1})^n = \mathcal{O}(\theta^n). \tag{8.10}$$

From

$$(f \star f^{-1}) \star (f \star f^{-1})^{-1\star} = 1 \tag{8.11}$$

and the associativity of the $\star$-product follows

$$f^{-1\star} = f^{-1} \star (f \star f^{-1})^{-1\star}. \tag{8.12}$$

The $\star$-inverse of $f \star f^{-1}$ has already been calculated as a power series in (8.9). Expanding the series (8.9), we obtain the following equality which holds up to first order in θ:

$$\begin{aligned} f^{-1\star} &= f^{-1} + f^{-1} \star (1 - f \star f^{-1}) \\ &= 2f^{-1} - f^{-1} \star f \star f^{-1}, \end{aligned} \tag{8.13}$$

respectively,

$$f^{-1\star} = 3f^{-1} - 3f^{-1} \star f \star f^{-1} + f^{-1} \star f \star f^{-1} \star f \star f^{-1}, \tag{8.14}$$

which is valid up to second order in θ. If f transforms classically as a scalar field, f^{-1} will transform as a scalar field as well. With the proper comultiplication, $f^{-1\star}$ will also be a scalar field.

The same method can be used to find $G^{\mu\nu\star}$

$$G_{\mu\nu} \star G^{\nu\rho\star} = \delta_\mu^\rho. \tag{8.15}$$

We first invert the matrix

$$\begin{aligned} G_{\mu\nu} \star G^{\nu\rho} &= (G \star G^{-1})_\mu{}^\rho = \delta_\mu^\rho + \mathcal{O}(\theta), \\ (G \star G^{-1})^{-1\star} &= \sum_{n=0}^{\infty} (1 - G \star G^{-1})^{n\star}. \end{aligned} \tag{8.16}$$

Here, we introduced $G^{\nu\rho}$ as the inverse of $G_{\mu\nu}$ in $\mathcal{A}_f$,

$$G_{\mu\nu} \cdot G^{\nu\rho} = \delta_\mu^\rho.$$

For $G_{\mu\nu} = g_{\mu\nu}, G^{\mu\nu}$ will be $g^{\mu\nu}$. For $G_{\mu\nu}$, θ-dependent $G^{\mu\nu}$ can be computed by a θ-expansion, starting from $g^{\mu\nu}$ as the θ-independent part. In analogy to (8.12), we obtain

$$G^{\mu\nu\star} = G^{\mu\rho} \star (G \star G^{-1})^{-1\star}{}_\rho{}^\nu. \tag{8.17}$$

When we expand the series θ, we get

$$G^{\mu\nu\star} = 2G^{\mu\nu} - G^{\mu\alpha} \star G_{\alpha\beta} \star G^{\beta\nu}, \tag{8.18}$$

which holds up to first order in θ. Note that $G^{\mu\nu\star}$ is not a symmetric tensor.

Using the proper coproduct and the fact that $G^{\mu\nu}$ transforms like a contravariant tensor of second rank, we conclude that $G^{\mu\nu\star}$ is a tensor of second rank as well

$$\hat{\delta}_\xi G^{\mu\nu\star} = -X_\xi^\star \triangleright G^{\mu\nu\star} + X^\star_{(\partial_\rho \xi^\mu)} \triangleright G^{\rho\nu\star} + X^\star_{(\partial_\rho \xi^\nu)} \triangleright G^{\mu\rho\star}. \tag{8.19}$$

If we demand that the covariant derivative of the metric vanishes, we can express the symmetric part of the connection entirely in terms of the metric and its derivatives. This is also true in the θ-deformed case.

We shall now assume that the connection is symmetric

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} \tag{8.20}$$

and when expressed in terms of $G_{\mu\nu}$, we shall call it the Christoffel symbol.

We demand that the covariant derivative of $G_{\mu\nu}$ vanishes

$$D_\alpha G_{\beta\gamma} = \partial^\star_\alpha \triangleright G_{\beta\gamma} - \Gamma^\rho_{\alpha\beta} \star G_{\rho\gamma} - \Gamma^\rho_{\alpha\gamma} \star G_{\beta\rho} = 0. \tag{8.21}$$

From there we proceed as in the classical case. We permute the indices in (8.21) assuming from the very beginning that $G_{\mu\nu}$ is symmetric and add the corresponding equations to obtain

$$2\Gamma^{\rho}_{\alpha\beta} \star G_{\rho\gamma} = \partial^{\star}_{\alpha} \triangleright G_{\beta\gamma} + \partial^{\star}_{\beta} \triangleright G_{\alpha\gamma} - \partial^{\star}_{\gamma} \triangleright G_{\alpha\beta}. \tag{8.22}$$

We can $\star$-invert $G_{\rho\gamma}$ and get the unique result

$$\Gamma^{\sigma}_{\alpha\beta} = \tfrac{1}{2}(\partial^{\star}_{\alpha} \triangleright G_{\beta\gamma} + \partial^{\star}_{\beta} \triangleright G_{\alpha\gamma} - \partial^{\star}_{\gamma} \triangleright G_{\alpha\beta}) \star G^{\gamma\sigma\star}. \tag{8.23}$$

By a direct calculation, we can convince ourselves that $\Gamma^{\sigma}_{\alpha\beta}$ has the right transformation property (7.14) if $G_{\alpha\beta}$ and $G^{\gamma\sigma\star}$ transform like tensors. All we have used is the symmetry of $G_{\mu\nu}$ and its tensor properties.

9. Curvature, Ricci tensor and curvature scalar

To define the curvature tensor, we follow the standard procedure. We compute the commutator of two covariant derivatives acting on a vector field. The covariant derivative of a vector field was defined in (7.13)

$$D_{\mu} V_{\nu} = \partial^{\star}_{\mu} \triangleright V_{\nu} - \Gamma^{\alpha}_{\mu\nu} \star V_{\alpha}. \tag{9.1}$$

In (8.23), we have found a connection $\Gamma^{\alpha}_{\mu\nu}$ symmetric in μ and ν that can be expressed entirely in terms of the metric. From (7.16) follows the curvature tensor, because the torsion vanishes for symmetric $\Gamma^{\alpha}_{\mu\nu}$:

$$[D_{\mu} \stackrel{\star}{,} D_{\nu}] \triangleright V_{\rho} = R_{\mu\nu\rho}{}^{\sigma} \star V_{\sigma}. \tag{9.2}$$

Then, the curvature tensor in terms of the Christoffel symbols is given by (7.17)

$$R_{\mu\nu\rho}{}^{\sigma} = \partial^{\star}_{\nu} \triangleright \Gamma^{\sigma}_{\mu\rho} - \partial^{\star}_{\mu} \triangleright \Gamma^{\sigma}_{\nu\rho} + \Gamma^{\beta}_{\nu\rho} \star \Gamma^{\sigma}_{\mu\beta} - \Gamma^{\beta}_{\mu\rho} \star \Gamma^{\sigma}_{\nu\beta}. \tag{9.3}$$

The curvature tensor is antisymmetric in the indices μ and ν. That the curvature tensor $R_{\mu\nu\rho}{}^{\sigma}$ transforms like a tensor if $\Gamma^{\sigma}_{\mu\rho}$ has the transformation property (7.14) can be checked explicitly. Finally, we express the Christoffel symbols in terms of the metric and obtain the desired form of the curvature tensor in terms of the metric. Its tensor properties then follow from the tensor property of $G_{\mu\nu}$.

From the curvature tensor, we obtain the Ricci tensor

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma}. \tag{9.4}$$

A summation over the third index would not vanish as in the undeformed case, but it would not reproduce the Ricci tensor in the limit $\theta \to 0$ either. The curvature scalar can be defined by contracting the two indices of the Ricci tensor with $G^{\mu\nu\star}$

$$R = G^{\mu\nu\star} \star R_{\nu\mu}. \tag{9.5}$$

By construction, R transforms as a scalar

$$\hat{\delta}_{\xi} R = -X^{\star}_{\xi} \triangleright R = -\xi^{\mu}(\partial_{\mu} R). \tag{9.6}$$

It will however not be real, as can be seen from (2.3). For the Lagrangian to be constructed in the following, we will just add the complex conjugate.

To obtain a covariant action from a scalar that transforms like (9.6), we have to find a measure E that transforms as

$$\delta_{\xi} E = -(\partial_{\mu}(\xi^{\mu} E)) = -(\partial_{\mu}\xi^{\mu})E - \xi^{\mu}(\partial_{\mu} E). \tag{9.7}$$

This has to be mapped to $\mathcal{A}_\theta$

$$\hat{\delta}_\xi E^\star = -X^\star_\xi \triangleright E^\star - X^\star_{(\partial_\mu \xi^\mu)} \triangleright E^\star. \tag{9.8}$$

Such an object we call a scalar $\star$-density.

Using the comultiplication rule (5.17), we can then verify that

$$\begin{aligned}\hat{\delta}_\xi (E^\star \star R) &= -(\partial_\mu \xi^\mu)(E^\star \star R) - \xi^\mu(\partial_\mu (E^\star \star R)) \\ &= -\partial_\mu(\xi^\mu(E^\star \star R))\end{aligned} \tag{9.9}$$

or in the language of $\mathcal{A}_\theta$

$$\hat{\delta}_\xi (E^\star \star R) = -\partial^\star_\mu \triangleright (X^\star_{\xi^\mu} \triangleright (E^\star \star R)). \tag{9.10}$$

The action

$$S = \int \mathrm{d}^n x\, E^\star \star R \tag{9.11}$$

will be invariant

$$\hat{\delta}_\xi \left(\int \mathrm{d}^n x\, E^\star \star R \right) = 0. \tag{9.12}$$

In $\mathcal{A}_f$, the square root of the determinant of the metric will have the transformation properties of a scalar density. It is however complicated to map the concept of a square root to $\mathcal{A}_\theta$. It is easier to express the metric in terms of the vierbein as we have done in (8.4) and then define the $\star$-determinant.

The $\star$-determinant can be defined as

$$E^\star = \det{}_\star E_\mu^{\ a} = \tfrac{1}{4!} \varepsilon^{\mu_1 \cdots \mu_4} \varepsilon_{a_1 \cdots a_4} E_{\mu_1}^{a_1} \star \cdots \star E_{\mu_4}^{a_4}. \tag{9.13}$$

Here, we have assumed that our space is four dimensional. The generalization to n dimensions is obvious.

With this definition, $E^\star$ has the right properties of a scalar $\star$-density. To verify this, we have to use the transformation properties of covariant vector fields and the comultiplication rule (5.17). This reproduces (9.8). From the definition also follows that $E^\star$ is real if the vierbeins are real.

An invariant action on $\mathcal{A}_\theta$ will be

$$S_{\mathrm{EH}} = \frac{1}{2} \int \mathrm{d}^4 x (E^\star \star R + \mathrm{c.c.}). \tag{9.14}$$

Using the reality of $E^\star$ and using property (6.16) of the integral, we obtain for the action (9.14)

$$S_{\mathrm{EH}} = \frac{1}{2} \int \mathrm{d}^4 x\, E^\star \star (R + \bar{R}) = \frac{1}{2} \int \mathrm{d}^4 x\, E^\star (R + \bar{R}). \tag{9.15}$$

This is the Einstein–Hilbert action on the θ-deformed coordinate space. The field equations can be obtained from this action in analogy to (6.17) by moving the field to be varied to the left (or the right) and then varying it, or we could expand the $\star$-products in (9.15) and vary the field $e_\mu{}^a$.

10. Expansion in θ

To get a better insight into the developed formalism, it is useful to study a θ-expansion. Already for the gauge theories, we used such an expansion for the action and considered θ as a coupling constant. Let us therefore list the θ-expansions of all relevant quantities. In zeroth order, we obtain the classical expressions. We denote them with the index (0).

The basic quantity is the vierbein

$$E_\mu{}^a = e_\mu{}^a \tag{10.1}$$

to all orders in θ.

For the metric, we obtain

$$G_{\mu\nu} = \tfrac{1}{2}\big(E_\mu{}^a \star E_\nu{}^b + E_\nu{}^a \star E_\mu{}^b\big)\eta_{ab},$$
$$G^{(0)}_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu} \tag{10.2}$$

and up to second order

$$G_{\mu\nu} = g_{\mu\nu} - \tfrac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\big(\partial_{\alpha_1}\partial_{\alpha_2}e_\mu{}^a\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_\nu{}^b\big)\eta_{ab} + \cdots. \tag{10.3}$$

There is no contribution in the first order of θ. The reason is that θ enters through the $\star$-product only. By definition, $G_{\mu\nu}$ is real but the first order in the $\star$-product of two real functions is purely imaginary. Therefore, the first order has to vanish and the same will be true for all odd orders in θ.

For $G^{\mu\nu\star}$, we obtain

$$\begin{aligned} G^{\mu\nu\star} = g^{\mu\nu} &- \frac{\mathrm{i}}{2}\theta^{\alpha\beta}(\partial_\alpha g^{\mu\gamma})(\partial_\beta g_{\gamma\delta})g^{\delta\nu} \\ &+ \frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\big(\big(\partial_{\alpha_1}\partial_{\alpha_2}g^{\mu\gamma}\big)\big(\partial_{\beta_1}\partial_{\beta_2}g_{\gamma\eta}\big) + g^{\mu\gamma}\big(\partial_{\alpha_1}\partial_{\alpha_2}e_\gamma{}^a\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_\eta{}^b\big)\eta_{ab} \\ &- 2\partial_{\alpha_1}\big(\big(\partial_{\alpha_2}g^{\mu\gamma}\big)\big(\partial_{\beta_2}g_{\gamma\delta}\big)g^{\delta\epsilon}\big)\big(\partial_{\beta_1}g_{\epsilon\eta}\big)\big)g^{\eta\nu}. \end{aligned} \tag{10.4}$$

As constructed, $G^{\mu\nu\star}$ is neither symmetric nor real. There is no reason for the term of first order in θ to drop out. The same is true for the Christoffel symbol and the curvature tensor. For the Christoffel symbol, we get the following expressions up to second order in θ: the zeroth order reads

$$\Gamma^{(0)\rho}_{\mu\nu} = \tfrac{1}{2}(\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma} - \partial_\gamma g_{\mu\nu})g^{\gamma\rho}, \tag{10.5}$$

the first order

$$\Gamma^{(1)\rho}_{\mu\nu} = \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\big(\partial_\alpha\Gamma^{(0)\sigma}_{\mu\nu}\big)g_{\sigma\tau}(\partial_\beta g^{\tau\rho}) \tag{10.6}$$

and the second order

$$\begin{aligned} \Gamma^{(2)\rho}_{\mu\nu} = -\tfrac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}&\big(\big(\partial_{\alpha_1}\partial_{\alpha_2}\Gamma^{(0)}_{\mu\nu\sigma}\big)(\partial_{\beta_1}\partial_{\beta_2}g^{\sigma\rho}) - 2\big(\partial_{\alpha_1}\Gamma^{(0)}_{\mu\nu\sigma}\big)\partial_{\beta_1}\big(\big(\partial_{\alpha_2}g^{\sigma\tau}\big)\big(\partial_{\beta_2}g_{\tau\xi}\big)g^{\xi\rho}\big) \\ &- \Gamma^{(0)}_{\mu\nu\sigma}\big(\big(\partial_{\alpha_1}\partial_{\alpha_2}g^{\sigma\tau}\big)\big(\partial_{\beta_1}\partial_{\beta_2}g_{\tau\xi}\big) + g^{\sigma\tau}\big(\partial_{\alpha_1}\partial_{\alpha_2}e_\tau{}^a\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_\xi{}^b\big)\eta_{ab} \\ &- 2\partial_{\alpha_1}\big(\big(\partial_{\alpha_2}g^{\sigma\tau}\big)\big(\partial_{\beta_2}g_{\tau\lambda}\big)g^{\lambda\kappa}\big)\big(\partial_{\beta_1}g_{\kappa\xi}\big)\big)g^{\xi\rho} + \tfrac{1}{2}\big(\partial_\mu\big(\big(\partial_{\alpha_1}\partial_{\alpha_2}e_\nu{}^a\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_\sigma{}^b\big)\big) \\ &+ \partial_\nu\big(\big(\partial_{\alpha_1}\partial_{\alpha_2}e_\sigma{}^a\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_\mu{}^b\big)\big) - \partial_\sigma\big(\big(\partial_{\alpha_1}\partial_{\alpha_2}e_\mu{}^a\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_\nu{}^b\big)\big)\big)\eta_{ab}g^{\sigma\rho}\big), \end{aligned} \tag{10.7}$$

where

$$\Gamma^{(0)}_{\mu\nu\sigma} = \Gamma^{(0)\rho}_{\mu\nu}g_{\rho\sigma}. \tag{10.8}$$

For the curvature tensor, we also list the first and second orders individually. The zeroth order is just the classical tensor expressed in the metric or the vierbein

$$\begin{aligned} R^{(1)}_{\mu\nu\rho}{}^{\sigma} = -\frac{\mathrm{i}}{2}\theta^{\kappa\lambda}&\big(\big(\partial_\kappa R^{(0)}_{\mu\nu\rho}{}^{\tau}\big)(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - \big(\partial_\kappa\Gamma^{(0)\beta}_{\nu\rho}\big)\big(\Gamma^{(0)\tau}_{\mu\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} \\ &- \Gamma^{(0)\sigma}_{\mu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} + \partial_\mu\big((\partial_\lambda g_{\beta\gamma})g^{\gamma\sigma}\big) + \big(\partial_\lambda\Gamma^{(0)\sigma}_{\mu\beta}\big)\big) \\ &+ \big(\partial_\kappa\Gamma^{(0)\beta}_{\mu\rho}\big)\big(\Gamma^{(0)\tau}_{\nu\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - \Gamma^{(0)\sigma}_{\nu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} \\ &+ \partial_\nu\big((\partial_\lambda g_{\beta\gamma})g^{\gamma\sigma}\big) + \big(\partial_\lambda\Gamma^{(0)\sigma}_{\nu\beta}\big)\big)\big) \end{aligned} \tag{10.9}$$

$$R^{(2)\ \sigma}_{\mu\nu\rho} = \partial_\nu \Gamma^{(2)\sigma}_{\mu\rho} + \Gamma^{(2)\gamma}_{\nu\rho}\Gamma^{(0)\sigma}_{\mu\gamma} + \Gamma^{(0)\gamma}_{\nu\rho}\Gamma^{(2)\sigma}_{\mu\gamma} + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\big(\big(\partial_\alpha \Gamma^{(1)\gamma}_{\nu\rho}\big)\big(\partial_\beta \Gamma^{(0)\sigma}_{\mu\gamma}\big) + \big(\partial_\alpha \Gamma^{(0)\gamma}_{\nu\rho}\big)\big(\partial_\beta \Gamma^{(1)\sigma}_{\mu\gamma}\big)\big) - \frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\big(\partial_{\alpha_1}\partial_{\alpha_2}\Gamma^{(0)\gamma}_{\nu\rho}\big)\big(\partial_{\beta_1}\partial_{\beta_2}\Gamma^{(0)\sigma}_{\mu\gamma}\big) - (\mu \leftrightarrow \nu). \tag{10.10}$$

From the curvature tensor, we obtain the Ricci tensor and the curvature scalar as outlined in the previous section.

The curvature scalar is given by

$$R = R^{(0)} + R^{(1)} + R^{(2)}, \tag{10.11}$$

where $R^{(0)}$ is the classical curvature scalar and

$$R^{(1)} = +\frac{\mathrm{i}}{2}\theta^{\kappa\lambda}\big((\partial_\kappa g^{\mu\nu})\big(\partial_\lambda R^{(0)}_{\mu\nu}\big) - g^{\mu\nu}\big(\big(\partial_\kappa R^{(0)\ \tau}_{\mu\alpha\nu}\big)(\partial_\lambda g_{\tau\gamma})g^{\gamma\alpha} - \big(\partial_\kappa \Gamma^{(0)\beta}_{\alpha\nu}\big)\big(\Gamma^{(0)\tau}_{\mu\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\alpha} - \Gamma^{(0)\alpha}_{\mu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} + \partial_\mu((\partial_\lambda g_{\beta\gamma})g^{\gamma\alpha}) + \big(\partial_\lambda \Gamma^{(0)\sigma}_{\mu\beta}\big)\big) + \big(\partial_\kappa \Gamma^{(0)\beta}_{\mu\nu}\big)\big(\Gamma^{(0)\tau}_{\alpha\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\alpha} - \Gamma^{(0)\alpha}_{\nu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} + \partial_\alpha\big((\partial_\lambda g_{\beta\gamma})g^{\gamma\alpha}\big) + \big(\partial_\lambda \Gamma^{(0)\alpha}_{\nu\beta}\big)\big)\big)\big), \tag{10.12}$$

$$R^{(2)} = G^{(2)\mu\nu\star}R^{(0)}_{\nu\mu} + g^{\mu\nu}R^{(2)}_{\nu\mu} + G^{(1)\mu\nu\star}R^{(1)}_{\nu\mu} + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}(\partial_\alpha g^{\mu\nu})\big(\partial_\beta R^{(1)}_{\mu\nu}\big) - \frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\big(\partial_{\alpha_1}\partial_{\alpha_2}g^{\mu\nu}\big)\big(\partial_{\beta_1}\partial_{\beta_2}R^{(0)}_{\mu\nu}\big). \tag{10.13}$$

For the action, we still need the scalar $\star$-density $E^\star$

$$E^\star = \det\big(e^a_\mu\big) - \frac{1}{8}\frac{1}{4!}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\varepsilon^{\mu_1\cdots\mu_4}\varepsilon_{a_1\cdots a_4}\big(\big(\partial_{\alpha_1}\partial_{\alpha_2}e_{\mu_1}{}^{a_1}\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_{\mu_2}{}^{a_2}\big)e_{\mu_3}{}^{a_3}e_{\mu_4}{}^{a_4} + \partial_{\alpha_1}\partial_{\alpha_2}\big(e_{\mu_1}{}^{a_1}e_{\mu_2}{}^{a_2}\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_{\mu_3}{}^{a_3}\big)e_{\mu_4}{}^{a_4} + \partial_{\alpha_1}\partial_{\alpha_2}\big(e_{\mu_1}{}^{a_1}e_{\mu_2}{}^{a_2}e_{\mu_3}{}^{a_3}\big)\big(\partial_{\beta_1}\partial_{\beta_2}e_{\mu_4}{}^{a_4}\big)\big). \tag{10.14}$$

The Einstein–Hilbert action was defined in (9.14). It is real by definition. Since θ enters only via the $\star$-product, we expect that all terms corresponding to odd orders in θ vanish. Up to second order, we therefore get

$$S_{\mathrm{EH}} = S^{(0)}_{\mathrm{EH}} + \int \mathrm{d}^4x\big(\det\big(e_\mu{}^a\big)R^{(2)} + E^{\star(2)}R^{(0)}\big). \tag{10.15}$$

In this action, the even-order expansion terms in θ do not vanish. Equation (10.14) allows us to study the deviation from gravity theory on a differential manifold.

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6.3 Noncommutative Geometry and Gravity

by

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Noncommutative geometry and gravity

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Abstract
We study a deformation of infinitesimal diffeomorphisms of a smooth manifold. The deformation is based on a general twist. This leads to a differential geometry on a noncommutative algebra of functions whose product is a star product. The class of noncommutative spaces studied is very rich. Non-anticommutative superspaces are also briefly considered. The differential geometry developed is covariant under deformed diffeomorphisms and is coordinate independent. The main target of this work is the construction of Einstein's equations for gravity on noncommutative manifolds.

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1. Introduction

The study of the structure of spacetime at Planck scale, where quantum gravity effects are non-negligible, is one of the main open challenges in fundamental physics. Since the dynamical variable in Einstein general relativity is spacetime itself (with its metric structure), and since in quantum mechanics and in quantum field theory the classical dynamical variables become noncommutative, one is strongly led to conclude that noncommutative spacetime is a feature of Planck-scale physics. This expectation is further supported by Gedanken experiments that aim at probing spacetime structure at very small distances. They show that due to gravitational backreaction one cannot test spacetime at Planck scale[5]. Its description as a (smooth) manifold becomes therefore a mathematical assumption no longer justified by physics. It is then natural to relax this assumption and conceive a more general noncommutative spacetime, where uncertainty relations and discretization naturally arise. In this way, one can argue for the impossibility of an operational definition of continuous Planck-length spacetime (i.e., a definition given by describing the operations to be performed for at least measuring spacetime by a Gedanken experiment). A dynamical feature of spacetime could be incorporated at a deeper kinematical level. As an example, compare Galilean relativity to special relativity. Contraction of distances and time dilatation can be explained in Galilean relativity: they are a consequence of the interaction between ether and the body in motion. In special relativity they have become a kinematical feature.

This line of thought has been pursued in previous works, starting with [1, 2] and more recently in [3–15].

Note that uncertainty relations in position measurements are also in agreement with string theory models [16]. Moreover, non-perturbative attempts to describe string theories have shown that a noncommutative structure of spacetime emerges [17].

A first question to be asked in the context we have outlined is whether one can consistently deform Riemannian geometry into a noncommutative Riemannian geometry. We address this question by considering deformations of the algebra of functions on a manifold obtained via a quite wide class of $\star$-products. In this framework, we successfully construct a noncommutative version of differential and of Riemannian geometry, and we obtain the noncommutative version of the Einstein equations.

Even without physical motivation, the mathematical structure of deformed spaces is a challenging and fruitful research arena. It is very surprising how well $\star$-noncommutative structures can be incorporated in the framework of differential geometry.

The $\star$-products we consider are associated with a deformation by a twist $\mathcal{F}$ of the Lie algebra of infinitesimal diffeomorphisms on a smooth manifold M. Since $\mathcal{F}$ is an arbitrary twist, we can consider it as the dynamical variable that determines the possible noncommutative structures of spacetime.

As argued, $\mathcal{F}$ and its dynamics are relevant at Planck scale, however the physical phenomena they induce can also appear at higher scales. For example, due to inflation, noncommutativity of spacetime at inflation scale (that may be as low as Planck scale) can induce cosmological perturbations; see for example [18]. It is then interesting to apply our present work to study the noncommutative analogue of the Friedmann–Robertson–Walker spacetime, as well as of other classical solutions to the Einstein equations.

In section 2, we construct the universal enveloping algebra $U\Xi$ of the Lie algebra of vectorfields and we give a pedagogical description of its Hopf algebra structure. The twists we consider are elements $\mathcal{F} \in U\Xi \otimes U\Xi$. The notion of twist of a Lie algebra is well known [19, 20]. Multiparametric twists appear in [21]. Other examples of twists (Jordanian deformations) are in [23–25]. In the context of deformed Poincaré group and Minkowski space geometry, twists have been studied in [26, 27] (multiparametric deformations) and in [28–32] (Moyal–Weyl deformations); see also [33].

In the context of Connes noncommutative geometry, the noncommutative torus, the noncommutative spheres [34] and further noncommutative manifolds (so-called isospectral deformations) considered in [34], and in [35, 36], are noncommutative manifolds whose

[5] For example, in relativistic quantum mechanics the position of a particle can be detected with a precision at most of the order of its Compton wavelength $\lambda_C = \lambda/mc$. Probing spacetime at infinitesimal distances implies an extremely heavy particle that in turn curves spacetime itself. When λ_C is of the order of the Planck length, the spacetime curvature radius due to the particle has the same order of magnitude and the attempt to measure spacetime structure beyond Planck scale fails.

deformed algebra of functions is along the lines of Rieffel's twists [38]; see [39] and, for the 4-sphere in [34], see [37, 40].

Our contribution in this section is to consider the notion of twist in the context of an infinite-dimensional Lie algebra, that of vectorfields on M. Several examples of twists and of their corresponding $\star$-noncommutative algebra of functions are then presented. We also extend this notion to the case where M is superspace and describe in a sound mathematical setting a very general class of twists on superspace.

We conclude section 2 by recalling the construction of the Hopf algebra $U\Xi^{\mathcal{F}}$ [20]. This Hopf algebra is closely related to the Hopf algebra of deformed infinitesimal diffeomorphisms.

We begin section 3 by recalling some known facts about Hopf algebra representations and then construct the algebra $U\Xi_\star$ (with product $\star$) as a module algebra on which $U\Xi^{\mathcal{F}}$ acts. The space of vectorfields has a deformed Lie bracket that is realized as a deformed commutator in $U\Xi_\star$. We have constructed the deformed Lie algebra of infinitesimal diffeomorphisms (infinitesimal $\star$-diffeomorphisms). We then construct a natural Hopf algebra structure on $U\Xi_\star$ which proves that vectorfields form a deformed Lie algebra in the sense of [41]; see also [42, 43] and [44] p 41. It can also be proven that $U\Xi_\star$ and $U\Xi^{\mathcal{F}}$ are isomorphic Hopf algebras [45]. In [14, 46, 47] (where $\theta^{\mu\nu}$-constant noncommutativity is considered), the Hopf algebra $U\Xi^{\mathcal{F}}$ rather than $U\Xi_\star$ is used.

In section 4, we study the $\star$-action of the Hopf algebra of infinitesimal $\star$-diffeomorphisms on the algebra of noncommutative functions $A_\star \equiv \mathrm{Fun}_\star(M)$ and on $U\Xi_\star$. In the same way that $A \equiv \mathrm{Fun}(M)$ and $U\Xi$ were deformed in section 3, we here deform the algebra of tensorfields $\mathcal{T}$ into $\mathcal{T}_\star$ and then study the action of $\star$-diffeomorphisms on $\mathcal{T}_\star$. As a further example, we similarly proceed with the algebra of exterior forms.

We then study the pairing between vectorfields and 1-forms, and its $A_\star$-linearity properties. Moving and dual comoving frames (vielbein) are introduced. As in the commutative case, (left) $A_\star$-linear maps $\Xi_\star \to A_\star$ are the same as 1-forms. More generally, tensorfields can be equivalently described as (left) $A_\star$-linear maps.

In section 5, we define the $\star$-covariant derivative in a global coordinate independent way. Locally, the covariant derivative is completely determined by its coefficients $\Gamma_{\mu\nu}^{\sigma}$. Using the deformed Leibniz rule for vectorfields, we extend the covariant derivative to all type of tensorfields.

In section 6, torsion, curvature and the Ricci tensors are defined as (left) $A_\star$-linear maps on vectorfields. The $A_\star$-linearity property is a strong requirement that resolves the ambiguities in the possible definitions of these noncommutative tensorfields.

In section 7, we define the metric as an arbitrary $\star$-symmetric element in the $\star$-tensorproduct of 1-forms $\Omega_\star \otimes_\star \Omega_\star$. Using the pairing between vectorfields and 1-forms, the metric is equivalently described as an $A_\star$-linear map on vectorfields, $(u, v) \mapsto \mathsf{g}(u, v)$. The scalar curvature is then defined and Einstein equations on $\star$-noncommutative space are obtained. Again the requirement of $A_\star$-linearity uniquely fixes the possible ambiguities arising in the noncommutative formulation of Einstein gravity theory.

In section 8, we study reality conditions on noncommutative functions, vectorfields and tensorfields. If the twist $\mathcal{F}$ satisfies a mild natural extra condition then all the geometric constructions achieved in the previous sections admit a real form.

2. Deformation by twists

2.1. Hopf algebras from Lie algebras

Let us first recall that the (infinite-dimensional) linear space Ξ of smooth vectorfields on a smooth manifold M becomes a Lie algebra through the map

$$[\ ]:\quad \Xi \times \Xi \to \Xi$$
$$(u, v) \mapsto [uv]. \tag{2.1}$$

The element $[uv]$ of Ξ is defined by the usual Lie bracket

$$[uv](h) = u(v(h)) - v(u(h)). \tag{2.2}$$

We shall always denote vectorfields by the letters $u, v, z, \ldots,$ and functions on M by $f, g, h, \ldots.$

The Lie algebra of vectorfields (i.e., the algebra of infinitesimal diffeomorphisms) can also be seen as an abstract Lie algebra without referring to the smooth manifold M anymore. This abstract algebra can be extended to a Hopf algebra by first defining the universal enveloping algebra $U\Xi$ that is the tensor algebra (over $\mathbb{C}$) generated by the elements of Ξ and the unit element 1 modulo the left and right ideal generated by all elements $uv - vu - [uv]$. The elements uv and vu are elements in the tensor algebra and $[uv]$ is an element of Ξ. We shall denote elements of the universal enveloping algebra $U\Xi$ by $\xi, \zeta, \eta, \ldots.$

The algebra $U\Xi$ has a natural Hopf algebra structure [48, 49]. On the generators $u \in \Xi$ and the unit element 1, we define

$$\begin{aligned} &\Delta(u) = u \otimes 1 + 1 \otimes u, && \Delta(1) = 1 \otimes 1, \\ &\varepsilon(u) = 0, && \varepsilon(1) = 1, \\ &S(u) = -u, && S(1) = 1. \end{aligned} \tag{2.3}$$

Here, Δ is the coproduct (from which the Leibniz rule for vectorfields follows), S is the antipode (or coinverse) and ε is the counit. The maps Δ, ε and S satisfy the following relations:

$$\begin{aligned} &\Delta(u)\Delta(v) - \Delta(v)\Delta(u) = [uv] \otimes 1 + 1 \otimes [uv] = \Delta([uv]), \\ &\varepsilon(u)\varepsilon(v) - \varepsilon(v)\varepsilon(u) = \varepsilon([uv]), \\ &S(v)S(u) - S(u)S(v) = vu - uv = S([uv]). \end{aligned} \tag{2.4}$$

This allows us to extend Δ and ε as algebra homomorphisms and S as an antialgebra homomorphism to the full enveloping algebra, $\Delta : U\Xi \to U\Xi \otimes U\Xi, \varepsilon : U\Xi \to \mathbb{C}$ and $S : U\Xi \to U\Xi$,

$$\begin{aligned} &\Delta(\xi\zeta) := \Delta(\xi)\Delta(\zeta), \\ &\varepsilon(\xi\zeta) := \varepsilon(\xi)\varepsilon(\zeta), \\ &S(\xi\zeta) := S(\zeta)S(\xi). \end{aligned} \tag{2.5}$$

There are three more propositions that have to be satisfied for a Hopf algebra (we denote the product in the algebra by μ)

$$\begin{aligned} &(\Delta \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \Delta)\Delta(\xi), \\ &(\varepsilon \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \varepsilon)\Delta(\xi) = \xi, \\ &\mu(S \otimes \mathrm{id})\Delta(\xi) = \mu(\mathrm{id} \otimes S)\Delta(\xi) = \varepsilon(\xi)1. \end{aligned} \tag{2.6}$$

It is enough to prove (2.6) on the generators u, 1 of $U\Xi$. We prove the first of them for the coproduct defined in (2.3) using the Sweedler notation $\Delta(u) = u_1 \otimes u_2$ (where a sum over u_1 and u_2 is understood), in this explicit case $\Delta(u) = u_1 \otimes u_2 = u \otimes 1 + 1 \otimes u$,

$$\begin{aligned} (\Delta \otimes \mathrm{id})\Delta(u) &= \Delta(u_1) \otimes u_2 \\ &= u_{1_1} \otimes u_{1_2} \otimes u_2 \\ &= (u \otimes 1 + 1 \otimes u) \otimes 1 + 1 \otimes 1 \otimes u \end{aligned} \tag{2.7}$$

and

$$(\mathrm{id}\otimes\Delta)\Delta(u) = u_1\otimes\Delta(u_2) = u_1\otimes u_{2_1}\otimes u_{2_2} = u\otimes 1\otimes 1 + 1\otimes(u\otimes 1 + 1\otimes u). \tag{2.8}$$

Comparing (2.7) and (2.8), we see that the first condition of (2.6) is satisfied.

After proving the remaining conditions of (2.6) on the generators of $U\Xi$ we have constructed the Hopf algebra $(U\Xi, \cdot, \Delta, S, \varepsilon)$, where $\cdot$ denotes the multiplication map in $U\Xi$; sometimes we denote it by μ and frequently omit any of the symbols $\cdot$ and μ. With abuse of notation we frequently write $U\Xi$ to denote the Hopf algebra $(U\Xi, \cdot, \Delta, S, \varepsilon)$. This Hopf algebra is cocommutative because $\Delta = \Delta^{\mathrm{op}}$ where $\Delta^{\mathrm{op}} = \sigma\circ\Delta$ with σ being the flip map $\sigma(\xi\otimes\zeta) = \zeta\otimes\xi$.

We will extend the notion of enveloping algebra to formal power series in λ and we will correspondingly consider the Hopf algebra $(U\Xi[[\lambda]], \cdot, \Delta, S, \varepsilon)$. In the following for the sake of brevity we will often denote $U\Xi[[\lambda]]$ by $U\Xi$.

2.2. The twist

Definition 1. *A twist $\mathcal{F}$ is an element $\mathcal{F}\in U\Xi[[\lambda]]\otimes U\Xi[[\lambda]]$ that is invertible and that satisfies*

$$\mathcal{F}_{12}(\Delta\otimes\mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id}\otimes\Delta)\mathcal{F}, \tag{2.9}$$

$$(\varepsilon\otimes\mathrm{id})\mathcal{F} = 1 = (\mathrm{id}\otimes\varepsilon)\mathcal{F}, \tag{2.10}$$

where $\mathcal{F}_{12} = \mathcal{F}\otimes 1$ and $\mathcal{F}_{23} = 1\otimes\mathcal{F}$.

In our context, we in addition require[6]

$$\mathcal{F} = 1\otimes 1 + \mathcal{O}(\lambda). \tag{2.11}$$

Property (2.9) states that $\mathcal{F}$ is a two cocycle and it will turn out to be responsible for the associativity of the $\star$-products to be defined. Property (2.10) is just a normalization condition. From (2.11) it follows that $\mathcal{F}$ can be formally inverted as a power series in λ. It also shows that the geometry we are going to construct has the nature of a deformation, i.e. in the 0th order in λ we recover the usual undeformed geometry.

Using the twist $\mathcal{F}$, we now proceed to deform the commutative geometry on M into the twisted noncommutative one. The guiding principle is the observation that every time we have a linear map $X\otimes Y\to Z$, or a linear map $Z\to X\otimes Y$, where X, Y, Z are vectorspaces, and where $U\Xi$ acts on X, Y and Z, we can combine this map with an action of the twist. In this way, we obtain a deformed version of the initial linear map. To preserve algebraic properties of the original maps very particular actions of the twist $\mathcal{F}$ have to be used.

As an example, let $X = Y = Z = A$ where $A \equiv \mathrm{Fun}(M) \equiv C^\infty(M)[[\lambda]]$ is the algebra of smooth functions on M. The elements of $U\Xi$ act on A by the natural extension of the Lie derivative. The Lie derivative on $\mathrm{Fun}(M)$ associated with the vectorfield v is defined as follows:

$$\mathcal{L}_v(h) = v(h) \in A = \mathrm{Fun}(M), \tag{2.12}$$

[6] Actually, it is possible to show that (2.11) is a consequence of (2.9), (2.10) and of $\mathcal{F}$ being at each order in λ a finite sum of finite products of vectorfields.

where $v \in \Xi$ and $h \in \mathrm{Fun}(M)$. From equation (2.12) it follows that the map

$$v \mapsto \mathcal{L}_v, \tag{2.13}$$

satisfies

$$\mathcal{L}_{v'}\mathcal{L}_v(h) = v'(v(h)) \in \mathrm{Fun}(M) \tag{2.14}$$

and therefore it is a Lie algebra homomorphism

$$[\mathcal{L}_{v'}, \mathcal{L}_v](h) = \mathcal{L}_{[v'v]}(h). \tag{2.15}$$

This implies that we can extend the Lie derivative associated with a vectorfield to a Lie derivative associated with elements of $U\Xi$ by[7]

$$\mathcal{L}_{\xi\zeta} = \mathcal{L}_\xi \mathcal{L}_\zeta. \tag{2.16}$$

As in (2.12), we frequently use the notation

$$\xi(h) = \mathcal{L}_\xi(h) \tag{2.17}$$

for the action of $U\Xi$ on $\mathrm{Fun}(M)$. The map we want to deform is the usual pointwise multiplication map between functions

$$\begin{aligned} \mu: \quad & \mathrm{Fun}(M) \otimes \mathrm{Fun}(M) \to \mathrm{Fun}(M) \\ & f \otimes g \mapsto fg. \end{aligned} \tag{2.18}$$

To obtain $\mu_\star$, we first apply $\mathcal{F}^{-1}$ and then μ

$$\begin{aligned} \mu_\star: \quad & \mathrm{Fun}(M) \otimes \mathrm{Fun}(M) \xrightarrow{\mathcal{F}^{-1}} \mathrm{Fun}(M) \otimes \mathrm{Fun}(M) \xrightarrow{\mu} \mathrm{Fun}(M) \\ & f \otimes g \mapsto \mathcal{F}^{-1}(f \otimes g) \mapsto \mu \circ \mathcal{F}^{-1}(f \otimes g). \end{aligned} \tag{2.19}$$

This product is the $\star$-product

$$f \star g \equiv \mu_\star(f, g) := \mu \circ \mathcal{F}^{-1}(f \otimes g). \tag{2.20}$$

We see that $\mu_\star = \mu \circ \mathcal{F}^{-1}$ is a bidifferential operator.

That the $\star$-product is associative follows from (2.9); see the theorem in section 3.1 for the proof. This is only true because we have used $\mathcal{F}^{-1}$ and not $\mathcal{F}$ in (2.19). We also have

$$f \star 1 = f = 1 \star f \tag{2.21}$$

as a consequence of the normalization condition (2.10). From (2.11) it follows that

$$f \star g = fg + \mathcal{O}(\lambda). \tag{2.22}$$

We have thus deformed the commutative algebra of function $A \equiv \mathrm{Fun}(M)$ into the noncommutative one

$$A_\star \equiv \mathrm{Fun}_\star(M). \tag{2.23}$$

We shall frequently use the notation (sum over α understood)

$$\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha, \qquad \mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha, \tag{2.24}$$

so that

$$f \star g := \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(g). \tag{2.25}$$

The elements $\mathrm{f}^\alpha, \mathrm{f}_\alpha, \bar{\mathrm{f}}^\alpha, \bar{\mathrm{f}}_\alpha$ live in $U\Xi$.

[7] Since $\mathcal{L}_\xi$ is a differential operator, we have a map $\mathcal{L} : U\Xi \to \mathrm{Diff}$ where Diff is the algebra of differential operators from A to A. Note that this map is neither surjective nor injective.

In order to get familiar with this notation, we will rewrite equation (2.9) and its inverse,

$$((\Delta \otimes \mathrm{id})\mathcal{F}^{-1})\mathcal{F}_{12}^{-1} = ((\mathrm{id} \otimes \Delta)\mathcal{F}^{-1})\mathcal{F}_{23}^{-1}, \tag{2.26}$$

as well as (2.10) and (2.11) using the notation (2.24), explicitly

$$\mathrm{f}^\beta \mathrm{f}_1^\alpha \otimes \mathrm{f}_\beta \mathrm{f}_2^\alpha \otimes \mathrm{f}_\alpha = \mathrm{f}^\alpha \otimes \mathrm{f}^\beta \mathrm{f}_{\alpha_1} \otimes \mathrm{f}_\beta \mathrm{f}_{\alpha_2}, \tag{2.27}$$

$$\bar{\mathrm{f}}_1^\alpha \bar{\mathrm{f}}^\beta \otimes \bar{\mathrm{f}}_2^\alpha \bar{\mathrm{f}}_\beta \otimes \bar{\mathrm{f}}_\alpha = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_{\alpha_1} \bar{\mathrm{f}}^\beta \otimes \bar{\mathrm{f}}_{\alpha_2} \bar{\mathrm{f}}_\beta, \tag{2.28}$$

$$\varepsilon(\mathrm{f}^\alpha)\mathrm{f}_\alpha = 1 = \mathrm{f}^\alpha \varepsilon(\mathrm{f}_\alpha), \tag{2.29}$$

$$\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha = 1 \otimes 1 + \mathcal{O}(\lambda). \tag{2.30}$$

2.3. *Examples of twists*

(1) Consider the case $M = \mathbb{R}^n$ and the element

$$\mathcal{F} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}} \tag{2.31}$$

where $\theta^{\mu\nu}$ is an antisymmetric matrix of real numbers. The inverse of $\mathcal{F}$ is

$$\mathcal{F}^{-1} = \mathrm{e}^{\frac{\mathrm{i}}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}}.$$

Then we have

$$(\Delta \otimes \mathrm{id})\mathcal{F} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda\theta^{\mu\nu}(\frac{\partial}{\partial x^\mu}\otimes 1 \otimes \frac{\partial}{\partial x^\nu} + 1 \otimes \frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu})}$$

so that property (2.9) follows:

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda\theta^{\mu\nu}(\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}\otimes 1 + \frac{\partial}{\partial x^\mu}\otimes 1 \otimes \frac{\partial}{\partial x^\nu} + 1 \otimes \frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu})} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F}.$$

Property (2.10) trivially holds. The $\star$-product that the twist $\mathcal{F}$ induces on the algebra of functions on $\mathbb{R}^n$ is the usual θ-constant $\star$-product (Moyal–Weyl $\star$-product),

$$(f \star g)(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x}. \tag{2.32}$$

(2) More generally, on a smooth manifold M consider a set of mutually commuting smooth vectorfields $\{X_a\}, a = 1, 2, \dots, s$. These vectorfields are globally defined on the manifold M but can be zero outside a given open of M. Consider then

$$\mathcal{F} = \mathrm{e}^{\lambda\sigma^{ab}X_a\otimes X_b} \tag{2.33}$$

where σ^{ab} are arbitrary constants. The proof that $\mathcal{F}$ is a twist is the same as that of the first example.

In the case that M is a Lie group (and more generally a quantum group) deformations of the form (2.33) appeared in [21]. See also [22] where a few examples that reproduce known q-deformed spaces are explicitly presented.

(2a) A star product that implements the quantum plane commutation relation $xy = qyx$ ($q = \mathrm{e}^{i\lambda}$) can be obtained via the twist

$$\mathcal{F} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda(x\frac{\partial}{\partial x}\otimes y\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}\otimes x\frac{\partial}{\partial x})}. \tag{2.34}$$

Note that the vectorfields $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$ vanish at the origin. In the semiclassical limit, we have a Poisson structure, not a symplectic one.

(2b) Consider the sphere S^2 and the usual polar coordinates $0 \leqslant \varphi < 2\pi, 0 \leqslant \vartheta \leqslant \pi$. Let $f(\varphi)$ and $l(\vartheta)$ be arbitrary smooth functions with support, for example in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{8}, \frac{3\pi}{8}\right)$ respectively. Then,

$$\mathcal{F} = \mathrm{e}^{\lambda f(\varphi)\frac{\partial}{\partial\varphi}\otimes l(\vartheta)\frac{\partial}{\partial\vartheta}} \tag{2.35}$$

gives a well-defined star product on the sphere.

(3) Twists are not necessarily related to commuting vectorfields. For example, consider on a smooth manifold M four vectorfields H, E, A, B that satisfy the Lie algebra relations

$$\begin{aligned} &[H, E] = 2E, \qquad [H, A] = \alpha A, \qquad [H, B] = \beta B, \qquad \alpha + \beta = 2, \\ &[A, B] = E, \qquad [E, A] = 0, \qquad [E, B] = 0. \end{aligned} \tag{2.36}$$

Then the element

$$\mathcal{F} = \mathrm{e}^{\frac{1}{2}H\otimes\ln(1+\lambda E)}\, \mathrm{e}^{\lambda A\otimes B\frac{1}{1+\lambda E}} \tag{2.37}$$

is a twist and gives a well-defined $\star$-product on the algebra of functions on M. These twists are known as extended Jordanian deformations [25]. Jordanian deformations [23, 24] are obtained setting $A = B = 0$ (and keeping the relation $[H, E] = 2E$).

2.3.1. Deformed superspace. Consider the superspace $\mathbb{R}^{m|n}$ with coordinates $(x^\mu, \theta^\alpha) \equiv Z^A$ and partial derivatives $(\partial_\mu, \partial_\alpha) \equiv \partial_A$ that satisfy the following (anti-)commutation relations:

$$[Z^A, Z^B]_\pm = 0, \qquad \partial_A Z^B = \delta_A^B.$$

A generic derivation is of the form $\chi = f^A(Z)\partial_A$, where $f^A(Z)$ are functions on superspace. Consider a set $\{\chi_a, \chi_\varepsilon\} \equiv \{\chi_I\}$ of even derivations χ_a and of odd derivations χ_ε that are mutually (anti-)commuting,

$$[\chi_I, \chi_J]_\pm = 0; \tag{2.38}$$

for instance one can consider the derivations $\{\chi_I\} = \{\partial_\mu, \partial_\alpha\}$ or the derivations $\{\chi_I\} = \left\{\frac{\partial}{\partial x^1}, \theta^1\frac{\partial}{\partial\theta^1}, \theta^2\frac{\partial}{\partial\theta^2}, \frac{\partial}{\partial\theta^3}, \theta^4\frac{\partial}{\partial x^2}\right\}$ (if $m \geqslant 2$ and $n \geqslant 4$).

The universal enveloping superalgebra of the Lie superalgebra (2.38) is as usual the algebra $\mathcal{U}$ over $\mathbb{C}$ generated by the elements χ_I modulo the relations (2.38). The algebra $\mathcal{U}$ becomes a Hopf superalgebra by defining on the generators the following grade preserving coproduct and antipode and the following counit:

$$\Delta(\chi_I) := \chi_I \otimes 1 + 1 \otimes \chi_I, \qquad S(\chi_I) := -\chi_I, \qquad \varepsilon(\chi_I) := 0,$$

where the tensorproduct $\otimes$ is over $\mathbb{C}$. The multiplication in $\mathcal{U} \otimes \mathcal{U}$ is defined as follows for homogeneous elements $\xi, \zeta, \xi', \zeta' \in \mathcal{U}$ (of even or odd degree $|\xi|, |\zeta|, |\xi'|, |\zeta'|$ respectively):

$$(\xi \otimes \zeta)(\xi' \otimes \zeta') = (-1)^{|\zeta||\xi'|}\xi\xi' \otimes \zeta\zeta'. \tag{2.39}$$

The antipode is extended to all elements of $\mathcal{U}$ by requiring it to be linear and graded antimultiplicative, the coproduct is linear and multiplicative (the grading being already present in (2.39)), and the counit is linear and multiplicative:

$$\Delta(\xi\zeta) = \Delta(\xi)\Delta(\eta), \qquad S(\xi\zeta) = (-1)^{|\zeta||\xi|}S(\zeta)S(\xi), \qquad \varepsilon(\xi\zeta) = \varepsilon(\xi)\varepsilon(\zeta). \tag{2.40}$$

We refer to [51] for a concise treatment of Hopf superalgebras.

Consider the even element in $\mathcal{U}[[\lambda]] \otimes \mathcal{U}[[\lambda]]$ given by

$$\mathcal{F} := \mathrm{e}^{\lambda\sigma^{IJ}\chi_I\otimes\chi_J} = \mathrm{e}^{\lambda\sigma^{aa'}\chi_a\otimes\chi_{a'} + \lambda\sigma^{\varepsilon\varepsilon'}\chi_\varepsilon\otimes\chi_{\varepsilon'}}, \tag{2.41}$$

where $\{\sigma_{IJ}\} \equiv \{\sigma_{aa'}, \sigma_{\varepsilon\varepsilon'}\}$ are arbitrary constants ($\mathbb{C}$-numbers). In order to check that $\mathcal{F}$ is a twist as defined in definition 1, we observe that $\mathcal{F}_{12} = \mathrm{e}^{\lambda\sigma^{IJ}\chi_I\otimes\chi_J} \otimes 1 = \mathrm{e}^{\lambda\sigma^{IJ}\chi_I\otimes\chi_J\otimes 1}$ and that

$$(\Delta \otimes \mathrm{id})\mathcal{F} = \mathrm{e}^{\lambda\sigma^{IJ}(\chi_I\otimes 1 + 1\otimes\chi_I)\otimes\chi_J}. \tag{2.42}$$

This last relation holds because $\Delta \otimes \mathrm{id} : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ is multiplicative (the product in $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ is given by $(\xi \otimes \zeta \otimes \eta)(\xi' \otimes \zeta' \otimes \eta') = (-1)^{|\zeta+\eta||\xi'|+|\eta||\zeta'|}\xi\xi' \otimes \zeta\zeta' \otimes \eta\eta'$). Finally,

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathrm{e}^{\lambda\sigma^{IJ}\chi_I \otimes \chi_J \otimes 1 + \lambda\sigma^{IJ}(\chi_I \otimes 1 + 1 \otimes \chi_I)\otimes \chi_J} \tag{2.43}$$

because the arguments of the exponentials are even elements of $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ whose commutator vanishes. One similarly computes $\mathcal{F}_{23}(\mathrm{id} \otimes \mathcal{F})$.

An associative $\star$-product on superspace is then defined by

$$\begin{aligned} g \star h &:= \mu \circ \mathcal{F}^{-1}(g \otimes h) \\ &= (-1)^{|\bar{\mathrm{f}}_\alpha||g|}\bar{\mathrm{f}}^\alpha(g)\bar{\mathrm{f}}_\alpha(h). \end{aligned} \tag{2.44}$$

Associativity depends only on property (2.28) and not on the specific example of twists (2.41). Associativity is explicitly proven in appendix A.3.

As particular cases of this construction we obtain the non-anticommutative superspaces considered in [52]. For twists on superspace, see also [53] and references therein.

2.4. The deformed Hopf algebra $U\Xi^{\mathcal{F}}$

Another deformation via the action of $\mathcal{F}$ leads to a new Hopf algebra

$$(U\Xi^{\mathcal{F}}, \cdot, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon^{\mathcal{F}}) = (U\Xi, \cdot, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon). \tag{2.45}$$

As algebras $U\Xi^{\mathcal{F}} = U\Xi$ and they also have the same counit $\varepsilon^{\mathcal{F}} = \varepsilon$. The new coproduct $\Delta^{\mathcal{F}}$ is given by

$$\begin{aligned} &\Delta^{\mathcal{F}}: \quad U\Xi^{\mathcal{F}} = U\Xi \xrightarrow{\Delta} U\Xi \otimes U\Xi \xrightarrow{Conj_{\mathcal{F}}} U\Xi \otimes U\Xi = U\Xi^{\mathcal{F}} \otimes U\Xi^{\mathcal{F}} \\ &\xi \mapsto \Delta(\xi) \mapsto \Delta^{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}. \end{aligned} \tag{2.46}$$

We deform the antipode, a map from $U\Xi$ to $U\Xi$, using an invertible element χ of $U\Xi$ defined as follows[8]:

$$\chi := \mathrm{f}^\alpha S(\mathrm{f}_\alpha), \qquad \chi^{-1} = S(\bar{\mathrm{f}}^\alpha)\bar{\mathrm{f}}_\alpha. \tag{2.47}$$

The definition of the new antipode is

$$S^{\mathcal{F}}(\xi) = \chi S(\xi)\chi^{-1}. \tag{2.48}$$

We follow the same steps as in subsection (2.1) to show that $U\Xi^{\mathcal{F}} = (U\Xi^{\mathcal{F}}, \cdot, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon)$ is a Hopf algebra.

That $\Delta^{\mathcal{F}}$ and ε are algebra homomorphisms and that $S^{\mathcal{F}}$ is an antialgebra homomorphism follows immediately from the definition

$$\Delta^{\mathcal{F}}(\xi\zeta) = \Delta^{\mathcal{F}}(\xi)\Delta^{\mathcal{F}}(\zeta), \qquad \varepsilon^{\mathcal{F}}(\xi\zeta) = \varepsilon^{\mathcal{F}}(\xi)\varepsilon^{\mathcal{F}}(\zeta), \qquad S^{\mathcal{F}}(\xi\zeta) = S^{\mathcal{F}}(\zeta)S^{\mathcal{F}}(\xi). \tag{2.49}$$

We have now to show that $\Delta^{\mathcal{F}}$ and $S^{\mathcal{F}}$ fulfil the additional conditions (2.6), and therefore that $(U\Xi^{\mathcal{F}}, \cdot, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon)$ is a Hopf algebra. This is done in appendix A.1.

The new Hopf algebra $U\Xi^{\mathcal{F}}$ is triangular, i.e., there exists an invertible element $\mathcal{R} \in U\Xi^{\mathcal{F}} \otimes U\Xi^{\mathcal{F}}$ (called universal $\mathcal{R}$-matrix) such that for all $\xi \in U\Xi$,

$$\Delta^{\mathcal{F}\mathrm{op}}(\xi) = \mathcal{R}\Delta^{\mathcal{F}}(\xi)\mathcal{R}^{-1} \tag{2.50}$$

$$(\Delta^{\mathcal{F}} \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes \Delta^{\mathcal{F}})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.51}$$

$$\mathcal{R}_{21} = \mathcal{R}^{-1}, \tag{2.52}$$

[8] See appendix A.1 for a proof that $\chi\chi^{-1} = \chi^{-1}\chi = 1$.

where $\mathcal{R}_{21} = \sigma(\mathcal{R}) \in U\Xi^{\mathcal{F}} \otimes U\Xi^{\mathcal{F}}$, with σ the flip map, $\sigma(\xi \otimes \zeta) = \zeta \otimes \xi$. The two equations in (2.51) take value in $U\Xi \otimes U\Xi \otimes U\Xi$, and $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, while $\mathcal{R}_{13} \in U\Xi \otimes U\Xi \otimes U\Xi$ has the unit 1 in the middle factor. Defining

$$\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1} \tag{2.53}$$

it can be shown that equations (2.50), (2.51), (2.52) are fulfilled. The cocycle condition of $\mathcal{F}$ was in this context only needed to prove (2.51).[9] In the following, we use the notation

$$\mathcal{R} = R^{\alpha} \otimes R_{\alpha}, \qquad \mathcal{R}^{-1} = \overline{R}^{\alpha} \otimes \overline{R}_{\alpha}. \tag{2.54}$$

Using the notation introduced in (2.24), we obtain

$$\mathcal{R} = R^{\alpha} \otimes R_{\alpha} = \mathrm{f}_{\alpha}\bar{\mathrm{f}}^{\beta} \otimes \mathrm{f}^{\alpha}\bar{\mathrm{f}}_{\beta}, \qquad \mathcal{R}^{-1} = \overline{R}^{\alpha} \otimes \overline{R}_{\alpha} = \mathrm{f}^{\alpha}\bar{\mathrm{f}}_{\beta} \otimes \mathrm{f}_{\alpha}\bar{\mathrm{f}}^{\beta}. \tag{2.55}$$

3. Representations

3.1. Module algebras

Having a Hopf algebra, its modules are certainly of interest in physics and mathematics. They are the representations of the Hopf algebra. Here, we show that to a module algebra $\mathcal{A}$ of the Hopf algebra $U\Xi$ there corresponds a module algebra $\mathcal{A}_{\star}$ of the deformed Hopf algebra $U\Xi^{\mathcal{F}}$.

A module algebra $\mathcal{A}$ is a module $\mathcal{A}$ on which $U\Xi$ acts, which, in addition, has an algebra structure that is compatible with the action of $U\Xi$ for all $\xi \in U\Xi$ and $a, b \in \mathcal{A}$,

$$\xi(ab) = \mu \circ \Delta(\xi)(a \otimes b) = \xi_1(a)\xi_2(b), \qquad \xi(1) = \varepsilon(\xi)1.$$

(where 1 is the unit in $\mathcal{A}$).

We recall a basic theorem concerning representations of twisted Hopf algebras. Given a twist $\mathcal{F} \in U\Xi \otimes U\Xi$, we can construct a deformed algebra $\mathcal{A}_{\star}$. The algebra $\mathcal{A}_{\star}$ has the same vector space structure as $\mathcal{A}$ and the action of $U\Xi^{\mathcal{F}}$ on $\mathcal{A}_{\star}$ is the action of $U\Xi$ on $\mathcal{A}$. The product in $\mathcal{A}_{\star}$ is defined by

$$a \star b = \mu \circ \mathcal{F}^{-1}(a \otimes b) = \bar{\mathrm{f}}^{\alpha}(a)\bar{\mathrm{f}}_{\alpha}(b), \tag{3.1}$$

in accordance with formula (2.20). Compatibility between the action of $U\Xi^{\mathcal{F}}$ and the product in $\mathcal{A}_{\star}$ demands

$$\xi(a \star b) = \xi_{1_{\mathcal{F}}}(a) \star \xi_{2_{\mathcal{F}}}(b), \tag{3.2}$$

where we used the notation $\Delta^{\mathcal{F}}(\xi) = \xi_{1_{\mathcal{F}}} \otimes \xi_{2_{\mathcal{F}}}$.

In order to prove associativity of the new product, we use (2.28) and compute

$$\begin{aligned}(a \star b) \star c &= \bar{\mathrm{f}}^{\alpha}(\bar{\mathrm{f}}^{\beta}(a)\bar{\mathrm{f}}_{\beta}(b))\bar{\mathrm{f}}_{\alpha}(c) = \big(\bar{\mathrm{f}}^{\alpha}_1\bar{\mathrm{f}}^{\beta}\big)(a)\big(\bar{\mathrm{f}}^{\alpha}_2\bar{\mathrm{f}}_{\beta}\big)(b)\bar{\mathrm{f}}_{\alpha}(c)\\ &= \bar{\mathrm{f}}^{\alpha}(a)\big(\bar{\mathrm{f}}_{\alpha_1}\bar{\mathrm{f}}^{\beta}\big)(b)\big(\bar{\mathrm{f}}_{\alpha_2}\bar{\mathrm{f}}_{\beta}\big)(c)\\ &= \bar{\mathrm{f}}^{\alpha}(a)\bar{\mathrm{f}}_{\alpha}(\bar{\mathrm{f}}^{\beta}(b)\bar{\mathrm{f}}_{\beta}(c)) = a \star (b \star c).\end{aligned}$$

We still have to prove (3.2):

$$\begin{aligned}\xi(a \star b) &= \xi(\mu \circ \mathcal{F}^{-1}(a \otimes b)) = \mu \circ \Delta(\xi) \circ \mathcal{F}^{-1}(a \otimes b)\\ &= \mu \circ \mathcal{F}^{-1} \circ \Delta^{\mathcal{F}}(\xi)(a \otimes b) = \xi_{1_{\mathcal{F}}}(a) \star \xi_{2_{\mathcal{F}}}(b).\end{aligned}$$

Also note that if $\mathcal{A}$ has a unit element 1, then $1 \star a = a \star 1$ follows from the normalization condition property (2.10) of the twist $\mathcal{F}$.

[9] We refer to [49] (p 56). See also [50] (p 130) for a proof of (2.51) and for an introduction to twists and their relations to Hopf algebra deformations.

3.2. Examples of module algebras

We now apply this construction to the $U\Xi$-module algebras A and $U\Xi$. In both cases, the action of $U\Xi$ on the corresponding module algebra is given by the Lie derivative.

Algebra of noncommutative functions $A_\star$

We start with the $U\Xi$-module algebra of functions $\mathcal{A} = \mathcal{A} = \mathrm{Fun}(\mathcal{M})$ and we obtain the algebra $A_\star \equiv \mathrm{Fun}_\star(M)$ with the $\star$-product already introduced in (2.20). The algebra $A_\star$, according to section 3.1 is a left $U\Xi^{\mathcal{F}}$-module algebra. In particular, vectorfields $u \in \Xi \subset U\Xi^{\mathcal{F}}$ act according to the deformed Leibniz rule

$$u(h \star g) = u_{1_\mathcal{F}}(h) \star u_{2_\mathcal{F}}(g), \tag{3.3}$$

where

$$\Delta^{\mathcal{F}}(u) = u_{1_\mathcal{F}} \otimes u_{2_\mathcal{F}} = \mathrm{f}^\alpha u \bar{\mathrm{f}}^\beta \otimes \mathrm{f}_\alpha \bar{\mathrm{f}}_\beta + \mathrm{f}^\alpha \bar{\mathrm{f}}^\beta \otimes \mathrm{f}_\alpha u \bar{\mathrm{f}}_\beta. \tag{3.4}$$

The algebra $U\Xi_\star$

We next consider the case $\mathcal{A} = \mathcal{U}\Xi$. This is a module algebra with respect to the Hopf algebra $U\Xi$. The action of $U\Xi$ on $U\Xi$ is given by the extended Lie derivative (adjoint action): the action of $\mathcal{L}_u$ on v is just the Lie bracket $\mathcal{L}_u(v) = [uv]$; the action of $U\Xi$ on Ξ is obtained from the action of vectorfields by defining $\mathcal{L}_{\xi\zeta} = \mathcal{L}_\xi \mathcal{L}_\zeta$ (where composition of the actions $\mathcal{L}_\xi$ and $\mathcal{L}_\zeta$ is understood); finally the action of $U\Xi$ on $U\Xi$ is obtained from the known Leibniz rule $\mathcal{L}_u(vz) = \mathcal{L}_u(v)z + v\mathcal{L}_u(z)$ that implies $\mathcal{L}_\xi(\zeta\eta) = \mathcal{L}_{\xi_1}(\zeta)\mathcal{L}_{\xi_2}(\eta)$.

The deformed algebra $U\Xi_\star$ equals $U\Xi$ as a vectorspace, but it has the deformed product

$$\begin{aligned} &\star : U\Xi \otimes U\Xi \to U\Xi \\ &(\xi, \zeta) \mapsto \xi \star \zeta := \bar{\mathrm{f}}^\alpha(\xi)\bar{\mathrm{f}}_\alpha(\zeta) \end{aligned} \tag{3.5}$$

where $\bar{\mathrm{f}}^\alpha(\xi)$ (and $\bar{\mathrm{f}}_\alpha(\zeta)$) is another notation for the Lie derivative $\mathcal{L}_{\bar{\mathrm{f}}^\alpha}(\xi)$ (and $\mathcal{L}_{\bar{\mathrm{f}}_\alpha}(\zeta)$). The Hopf algebra $U\Xi^{\mathcal{F}}$ acts on $U\Xi_\star$, and compatibility with the $\star$-product of $U\Xi_\star$ is

$$\xi(\zeta \star \eta) = \xi_{1_\mathcal{F}}(\zeta) \star \xi_{2_\mathcal{F}}(\eta). \tag{3.6}$$

This way we have obtained from the theorem in section 3.1 the algebra $U\Xi_\star$. We will show in section 3.3 that it is a Hopf algebra.

In $U\Xi_\star$, we consider the deformed commutator of the vectorfields $u, v \in \Xi$,

$$[u, v]_\star := u \star v - \bar{R}^\alpha(v) \star \bar{R}_\alpha(u). \tag{3.7}$$

This commutator closes in Ξ:

$$\begin{aligned} u \star v - \bar{R}^\alpha(v) \star \bar{R}_\alpha(u) &= \bar{\mathrm{f}}^\gamma(u)\bar{\mathrm{f}}_\gamma(v) - \bar{\mathrm{f}}^\gamma(\bar{R}_\alpha(v))\bar{\mathrm{f}}_\gamma(\bar{R}^\alpha(u)) \\ &= \bar{\mathrm{f}}^\gamma(u)\bar{\mathrm{f}}_\gamma(v) - \bar{\mathrm{f}}^\gamma \mathrm{f}^\alpha \bar{\mathrm{f}}_\beta(v)\bar{\mathrm{f}}_\gamma \mathrm{f}_\alpha \bar{\mathrm{f}}^\beta(u) \\ &= \bar{\mathrm{f}}^\gamma(u)\bar{\mathrm{f}}_\gamma(v) - \bar{\mathrm{f}}_\gamma(v)\bar{\mathrm{f}}^\gamma(u) \\ &= [\bar{\mathrm{f}}^\gamma(u), \bar{\mathrm{f}}_\gamma(v)], \end{aligned}$$

(the first line uses the definition of the $\star$-product, the second line the definition of the $\mathcal{R}$-matrix, $\mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha = \mathrm{f}^\alpha \bar{\mathrm{f}}_\beta \otimes \mathrm{f}_\alpha \bar{\mathrm{f}}^\beta$ as introduced in section 2.4. The third line uses $\mathcal{F}^{-1}\mathcal{F} = 1$). The last term is a sum (over γ) of undeformed commutators between the vectorfields $\bar{\mathrm{f}}^\gamma(u)$ and $\bar{\mathrm{f}}_\gamma(v)$, and therefore $[u, v]_\star \in \Xi$.

We denote by $\Xi_\star$ the linear space of vectorfields Ξ equipped with the multiplication

$$\begin{aligned} &[\,,\,]_\star : \quad \Xi \times \Xi \to \Xi \\ &(u, v) \mapsto [u, v]_\star. \end{aligned} \tag{3.8}$$

This way $\Xi_\star$ becomes a deformed Lie algebra. The elements of $\Xi_\star$ we call $\star$-vectorfields. It is easy to see that the bracket $[\,,\,]_\star$ has the $\star$-antisymmetry property

$$[u, v]_\star = -[\overline{R}^\alpha(v), \overline{R}_\alpha(u)]_\star. \tag{3.9}$$

This can be shown as follows:

$$[u, v]_\star = [\bar{\mathrm{f}}^\alpha(u), \bar{\mathrm{f}}_\alpha(v)] = -[\bar{\mathrm{f}}_\alpha(v), \bar{\mathrm{f}}^\alpha(u)] = -[\overline{R}^\alpha(v), \overline{R}_\alpha(u)]_\star.$$

We recall that $\mathcal{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha = \mathcal{F}\mathcal{F}_{21}^{-1} \in U\Xi \otimes U\Xi$.

A $\star$-Jacobi identity can be proven as well:

$$[u, [v, z]_\star]_\star = [[u, v]_\star, z]_\star + [\overline{R}^\alpha(v), [\overline{R}_\alpha(u), z]_\star]_\star. \tag{3.10}$$

A direct proof of the $\star$-Jacobi identity can be found in appendix A.2.

Finally, we note that any sum of the products of vectorfields in $U\Xi$ can be rewritten as a sum of the $\star$-products of vectorfields via the formula $uv = \mathrm{f}^\alpha(u) \star \mathrm{f}_\alpha(v)$, and therefore the $\star$-vectorfields generate the algebra.

Indeed we have proven (see [45]) that $U\Xi_\star$ is the universal enveloping algebra of $\Xi_\star$.

3.3. $U\Xi_\star$ is a Hopf algebra

We have seen that $U\Xi$ can be equipped with the usual Hopf algebra structure $(U\Xi, \cdot, \Delta, S, \varepsilon)$ or with the twisted Hopf algebra $(U\Xi^{\mathcal{F}}, \cdot, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon)$ or with a new product $U\Xi_\star = (U\Xi, \star)$. It turns out that $U\Xi_\star$ has also a natural Hopf algebra structure,

$$(U\Xi_\star, \star, \Delta_\star, S_\star, \varepsilon_\star). \tag{3.11}$$

We describe it by giving the coproduct, the inverse of the antipode and the counit on the generators u of $U\Xi_\star$:

$$\Delta_\star(u) = u \otimes 1 + X_{\overline{R}^\alpha} \otimes \overline{R}_\alpha(u), \tag{3.12}$$

$$S_\star^{-1}(u) = -\overline{R}^\alpha(u) \star X_{\overline{R}_\alpha}, \tag{3.13}$$

$$\varepsilon_\star(u) = \varepsilon(u) = 0, \tag{3.14}$$

where for all $\xi \in U\Xi$, $X_\xi = \bar{\mathrm{f}}^\alpha \xi \chi S^{-1}(\bar{\mathrm{f}}_\alpha)$. The map $X : U\Xi \to U\Xi$ is invertible and it can be shown [54] that its inverse X^{-1} is

$$X^{-1} = \bar{\mathrm{f}}^\alpha(\xi)\bar{\mathrm{f}}_\alpha =: D(\xi). \tag{3.15}$$

In principle one could directly check that (3.12)–(3.14) define a bona fide Hopf algebra. Another way [45] is to show that the Hopf algebra $U\Xi_\star$ is isomorphic to the Hopf algebra $U\Xi^{\mathcal{F}}$. The isomorphism is given by the map D:

$$D(\xi \star \zeta) = D(\xi)D(\zeta), \tag{3.16}$$

$$\Delta_\star = (D^{-1} \otimes D^{-1}) \circ \Delta^{\mathcal{F}} \circ D, \tag{3.17}$$

$$S_\star = D^{-1} \circ S^{\mathcal{F}} \circ D. \tag{3.18}$$

In particular, since $U\Xi^{\mathcal{F}}$ is a triangular Hopf algebra, $U\Xi_\star$ is also a triangular Hopf algebra. Its R-matrix is

$$\mathcal{R}_\star = (D^{-1} \otimes D^{-1})(\mathcal{R}), \qquad \mathcal{R}_\star = R_\star^\alpha \otimes R_{\star\alpha} = X_{R^\alpha} \otimes X_{R_\alpha}. \tag{3.19}$$

Explicitly, we have

$$\Delta_\star{}^{\mathrm{op}}(\xi) = \mathcal{R}_\star \star \Delta_\star(\xi) \star \mathcal{R}_\star^{-1} \tag{3.20}$$

$$(\Delta_\star \otimes \mathrm{id})\mathcal{R}_\star = \mathcal{R}_{\star 13} \star \mathcal{R}_{\star 23}, \qquad (\mathrm{id} \otimes \Delta_\star)\mathcal{R}_\star = \mathcal{R}_{\star 13} \star \mathcal{R}_{\star 12}, \tag{3.21}$$

$$\mathcal{R}_{\star 21} = \mathcal{R}_\star^{-1}, \tag{3.22}$$

where $\mathcal{R}_\star^{-1}$ is the $\star$-inverse of $\mathcal{R}_\star$, i.e., $\mathcal{R}_\star^{-1} \star \mathcal{R}_\star = \mathcal{R}_\star \star \mathcal{R}_\star^{-1} = 1 \otimes 1$.

Summarizing, we have encountered the Hopf algebras

$$(U\Xi, \cdot, \Delta, S, \varepsilon), \qquad (U\Xi^{\mathcal{F}}, \cdot, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon), \qquad (U\Xi_\star, \star, \Delta_\star, S_\star, \varepsilon).$$

The first is cocommutative, the second is triangular and is obtained by twisting the first, and the third is triangular and isomorphic to the second. The remarkable fact about $U\Xi_\star$ is the Leibniz rule for vectorfields (3.12). We find that $\overline{R}_\alpha(u)$ is again a vectorfield so that

$$\Delta_\star(\Xi_\star) \subset \Xi_\star \otimes 1 + U\Xi_\star \otimes \Xi_\star. \tag{3.23}$$

This is a fundamental property for the construction of a deformed differential calculus in the style of Woronowicz [41]. Note that the coproduct $\Delta^{\mathcal{F}}(u)$ does not have this property, as can be seen explicitly from (3.4). It is interesting to note that a Hopf algebra with comultiplication structure (3.4) is isomorphic to a Hopf algebra with comultiplication structure (3.23). In order to establish a gravity theory which is invariant with respect to deformed infinitesimal diffeomorphisms, we will consider module algebras with respect to $U\Xi_\star$ and not with respect to $U\Xi^{\mathcal{F}}$.

4. Representations of deformed infinitesimal diffeomorphisms

In section 3, we have constructed the Hopf algebra $U\Xi_\star$. Since $U\Xi_\star$ and $U\Xi^{\mathcal{F}}$ are isomorphic as Hopf algebras, any $U\Xi^{\mathcal{F}}$-module has automatically a $U\Xi_\star$-module structure. In particular, $A_\star$ and $U\Xi_\star$ are also $U\Xi_\star$-module algebras.

The action $\mathcal{L}^\star$ of $U\Xi_\star$ on $A_\star$ is given by combining the usual action (Lie derivative $\mathcal{L}$) with the twist $\mathcal{F}$

$$\mathcal{L}^\star_\xi(h) := \mathcal{L}_{\bar{\mathrm{f}}^\alpha(\xi)}\big(\mathcal{L}_{\bar{\mathrm{f}}_\alpha}(h)\big), \tag{4.1}$$

or equivalently, recalling that $D(\xi) = \bar{\mathrm{f}}^\alpha(\xi)\bar{\mathrm{f}}_\alpha$, we see that

$$\mathcal{L}^\star_\xi := \mathcal{L}_{D(\xi)}. \tag{4.2}$$

Similarly for the action of $U\Xi_\star$ on $U\Xi_\star$, that we also denote by $\mathcal{L}^\star$,

$$\mathcal{L}^\star_\xi(\zeta) := \mathcal{L}_{\bar{\mathrm{f}}^\alpha(\xi)}\big(\mathcal{L}_{\bar{\mathrm{f}}_\alpha}(\zeta)\big) = \bar{\mathrm{f}}^\alpha(\xi)(\bar{\mathrm{f}}_\alpha(\zeta)). \tag{4.3}$$

It is easy to see that these actions are well defined: $\mathcal{L}^\star_\xi \circ \mathcal{L}^\star_\zeta = \mathcal{L}^\star_{\xi\star\zeta}$, for example, we find[10]

$$\mathcal{L}^\star_\zeta(\mathcal{L}^\star_\xi(h)) = \mathcal{L}^\star_\zeta((D\xi)(h)) = (D\xi)(D\zeta)(h) = D(\xi \star \zeta)(h) = \mathcal{L}^\star_{\xi\star\zeta}(h) \tag{4.4}$$

where we used (3.16). Compatibility with the $\star$-product in $A_\star$ is also easily proven,

$$\begin{aligned}\mathcal{L}^\star_\xi(h \star g) &= \mathcal{L}_{D\xi}(h \star g) = (D\xi)(h \star g) = (D\xi)_{1_{\mathcal{F}}}(h) \star (D\xi)_{2_{\mathcal{F}}}(g) \\ &= D(\xi_{1_\star})(h) \star D(\xi_{2_\star})(g) = \mathcal{L}^\star_{\xi_{1_\star}}(h) \star \mathcal{L}^\star_{\xi_{2_\star}}(g)\end{aligned} \tag{4.5}$$

where we used (3.17). One proceeds similarly for the action $\mathcal{L}^\star$ of $U\Xi_\star$ on $U\Xi_\star$. The proofs that this action is well defined and that it is compatible with the $\star$-product in $U\Xi_\star$ are exactly the same as in (4.4) and (4.5): just substitute $h, g \in A_\star$ with $\zeta, \eta \in U\Xi_\star$. Here we note in particular that the $\star$-Lie derivative of a vectorfield on a vectorfield gives the $\star$-Lie bracket,

$$\mathcal{L}^\star_u(v) = [u, v]_\star. \tag{4.6}$$

Moreover, it can be shown that the $\star$-Lie derivative of $U\Xi_\star$ on $U\Xi_\star$ equals the $\star$-adjoint action, $\mathcal{L}^\star_\xi(\zeta) = ad^\star_\xi(\zeta) \equiv \xi_{1_\star} \star \zeta \star S_\star(\xi_{2_\star})$. In particular, the $\star$-commutator $[u, v]_\star$ is just the $\star$-adjoint action of u on v.

[10] In [14, 46, 47], we have $\theta^{\mu\nu}$-constant noncommutativity and differential operators $X^\star_u$ that satisfy $X^\star_u \star X^\star_v = X^\star_{uv}$; the relation between $X^\star_u$ and $\mathcal{L}^\star_u$ (for the $\theta^{\mu\nu}$-constant case) is $(X^\star_u \star g) = u(g) = \mathcal{L}^\star_{X_u}(g)$.

4.1. Tensorfields

Our main interest in this subsection is the deformed algebra of tensorfields. We recall that tensorfields on a smooth manifold can be described as elements in[11]

$$\Omega \otimes \Omega \otimes \cdots \Omega \otimes \Xi \otimes \Xi \otimes \cdots \Xi \tag{4.7}$$

where $\otimes$ stands for $\otimes_A$. Functions are in particular type $(0,0)$-tensorfields and the tensorproduct between a function and another tensorfield is as usual not explicitly written. The tensorproduct is an associative product. This in particular implies $\tau \otimes h\tau' = \tau h \otimes \tau'$ and $h(\tau \otimes \tau') = (h\tau) \otimes \tau'$. Tensorfields are a $U\Xi$ module, the action of $U\Xi$ on $\mathcal{T}$ is obtained via the Lie derivative on tensorfields that extends to a map $\mathcal{L} : U\Xi \otimes \mathcal{T} \to \mathcal{T}$. For example, $\mathcal{L}_{uv}(\tau) = \mathcal{L}_u(\mathcal{L}_v(\tau))$.

By using the theorem in section 3.1 and by setting $\mathcal{A} = \mathcal{T}$ where $\mathcal{T}$ is the commutative algebra of tensorfields, we obtain a deformed tensor algebra $\mathcal{T}_\star$ with associative $\star$-tensorproduct

$$\tau \otimes_\star \tau' := \bar{\mathrm{f}}^\alpha(\tau) \otimes \bar{\mathrm{f}}_\alpha(\tau'). \tag{4.8}$$

It follows that in $\mathcal{T}_\star$ we have in particular

$$\tau \otimes_\star h \star \tau' = \tau \star h \otimes_\star \tau', \tag{4.9}$$

$$h \star (\tau \otimes_\star \tau') = (h \star \tau) \otimes_\star \tau'. \tag{4.10}$$

The $\star$-product between a function and a tensor is noncommutative

$$\tau \star h = \mathcal{L}_{\bar{\mathrm{f}}^\alpha}(\tau)\mathcal{L}_{\bar{\mathrm{f}}_\alpha}(h) = \mathcal{L}_{\bar{\mathrm{f}}_\alpha}(h)\mathcal{L}_{\bar{\mathrm{f}}^\alpha}(\tau) = \mathcal{L}_{\bar{R}^\alpha}(h) \star \mathcal{L}_{\bar{R}_\alpha}(\tau) = \bar{R}^\alpha(h) \star \bar{R}_\alpha(\tau). \tag{4.11}$$

We now consider the construction performed at the beginning of this section, but with $\mathcal{T}_\star$ instead of $A_\star$ (or $U\Xi_\star$) and obtain that $\mathcal{T}_\star$ is a $U\Xi_\star$-module algebra. The action of $U\Xi_\star$ on $\mathcal{T}_\star$ is given by the $\star$-Lie derivative

$$\mathcal{L}^\star_\xi(\tau) := \mathcal{L}_{D\xi}(\tau) = \bar{\mathrm{f}}^\alpha(\xi)(\bar{\mathrm{f}}_\alpha(\tau)). \tag{4.12}$$

Compatibility with the $\star$-product in $\mathcal{T}_\star$ is proven as in (4.5)

$$\mathcal{L}^\star_\xi(\tau \star \tau') = \mathcal{L}^\star_{\xi_{1\star}}(\tau) \star \mathcal{L}^\star_{\xi_{2\star}}(\tau').$$

In particular, the $\star$-Lie derivative along vectorfields satisfies the deformed Leibniz rule

$$\mathcal{L}^\star_u(h \star g) = \mathcal{L}^\star_u(h) \star g + \bar{R}^\alpha(h) \star \mathcal{L}^\star_{\bar{R}_\alpha(u)}(g), \tag{4.13}$$

in accordance with the coproduct formula (3.12).

4.1.1. Vectorfields $\Xi_\star$ are an $A_\star$-bimodule. From the definition of the product of tensorfields (4.8), considering functions and vectorfields as particular tensors, we see that we can $\star$-multiply functions with vectorfields from the left and from the right. Because of associativity of the tensorproduct we see that the space of vectorfields $\Xi_\star$ is an $A_\star$-bimodule. In the commutative case, left and right actions of functions on vectorfields coincide, $uh = hu$.[12] In the noncommutative case, the left and right $A_\star$-actions on $\Xi_\star$ are not the same, but are related as in (4.11).

Local coordinate description of vectorfields. In a coordinate neighbourhood U with coordinates x^μ, any vectorfield v can be expressed in the ∂_μ basis as $v = v^\mu \partial_\mu$. We have a similar situation in the noncommutative case.

[11] We assume for simplicity that $\Omega \otimes \cdots \Omega \otimes \Xi \otimes \cdots \Xi \simeq \Gamma(T^*M \otimes \cdots TM \otimes TM \otimes \cdots TM)$. That this is always the case for a smooth manifold M (see for example [57], proposition 2.6) follows from the existence of a finite covering of M that trivializes the tangent bundle TM and the cotangent bundle T^*M; see for example [58], theorem 7.5.16.

[12] Here, uh is just the vectorfield that on a function g gives $(uh)(g) := u(g)h$. This notation should not be confused with the operator notation $u \circ h = u(h) + hu$.

Lemma 1. *In a coordinate neighbourhood U with coordinates x^μ, every vectorfield v can be uniquely written as*

$$v = v^\mu_\star \star \partial_\mu, \tag{4.14}$$

where $v^\mu_\star$ are functions on U.

Proof. We know that v can be uniquely written as $v = v^\mu \partial_\mu$. In order to prove decomposition (4.14), we show that the equation

$$v^\mu_\star \star \partial_\mu = v^\mu \partial_\mu \tag{4.15}$$

uniquely determines order by order in λ the coefficients $v^\mu_\star$ in terms of the v^μ ones. First, we expand v^μ, $v^\mu_\star$ and $\mathcal{F}^{-1}$,

$$v^\mu = v^\mu_0 + \lambda v^\mu_1 + \lambda^2 v^\mu_2 + \cdots, \qquad v^\mu_\star = v^\mu_{\star 0} + \lambda v^\mu_{\star 1} + \lambda^2 v^\mu_{\star 2} + \cdots \tag{4.16}$$

$$\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha = 1 \otimes 1 + \lambda \bar{\mathrm{f}}^{\alpha_1} \otimes \bar{\mathrm{f}}_{\alpha_1} + \lambda^2 \bar{\mathrm{f}}^{\alpha_2} \otimes \bar{\mathrm{f}}_{\alpha_2} + \cdots. \tag{4.17}$$

Then from (4.15) we have

$$v^\mu_{\star 0} = v^\mu_0, \qquad v^\mu_{\star 1} = v^\mu_1 - \bar{\mathrm{f}}^{\alpha_1}(v^\rho) \bar{\mathrm{f}}^\mu_{\alpha_1 \rho} \tag{4.18}$$

where $\bar{\mathrm{f}}^\mu_{\alpha_1 \rho} \partial_\mu = \bar{\mathrm{f}}_{\alpha_1}(\partial_\rho)$. More generally, at order λ^i, we have the equation $v^\mu_{\star i} \partial_\mu + \sum_{j=1}^i \bar{\mathrm{f}}^{\alpha_j} \big(v^\rho_{\star i-j}\big) \bar{\mathrm{f}}^\mu_{\alpha_j \rho} \partial_\rho = v^\mu_i \partial_\mu$ that uniquely determines $v_{\star i}$ in terms of $\mathcal{F}$, v^μ and $v^\mu_{\star j}$ with $j < i$. □

Note that this proof remains true if the local frame $\{\partial_\mu\}$ is replaced by a more general (not necessarily holonomic or λ independent) frame $\{e_a\}$. (Hint: $e_a = e^\mu_a \star \partial_\mu$, $\partial_\mu = e^a_\mu \star e_a$.)

Along these lines one can define a change of reference frame,

$$\partial_\mu \to \partial'_\mu = L^\nu_\mu \partial_\nu = L^\nu_{\star\mu} \star \partial_\nu. \tag{4.19}$$

This is a starting point in order to construct noncommutative transition functions for the tangent bundle TM.

4.1.2. 1-Forms $\Omega_\star$. From the tensorfield product definition (4.8), we see that the space of 1-forms is an $A_\star$-bimodule. The $A_\star$-bimodule structure explicitly reads, $\forall\, h \in A_\star$, $\omega \in \Omega_\star$,

$$\omega \star h = \mathcal{L}^\star_{\overline{R}^\alpha_\star}(h) \star \mathcal{L}^\star_{\overline{R}_{\star\alpha}}(\omega) = \overline{R}^\alpha(h) \star \overline{R}_\alpha(\omega). \tag{4.20}$$

The action of $U\Xi_\star$ on $\Omega_\star$ is given in (4.12).

Local coordinate description of 1-forms and of tensorfields

As in the case of vectorfields, we have that in a coordinate neighbourhood U with coordinates x^μ, every 1-form ω can be uniquely written as

$$\omega = \omega^\star_\mu \star \mathrm{d}x^\mu \tag{4.21}$$

with $\omega^\star_\mu$ functions on U and where $\{\mathrm{d}x^\mu\}$ is the usual dual frame of the vectorfields frame $\{\partial_\mu\}$. We can now show

Lemma 2. *In a coordinate neighbourhood U with coordinates x^μ, every tensorfield $\tau^{p,q}$ can be uniquely written as*

$$\tau^{p,q} = \tau^{\nu_1 \ldots \nu_q}_{\star \mu_1 \ldots \mu_p} \star \mathrm{d}x^{\mu_1} \otimes_\star \cdots \mathrm{d}x^{\mu_p} \otimes_\star \partial_{\nu_1} \otimes_\star \cdots \partial_{\nu_q} \tag{4.22}$$

where $\tau^{\nu_1 \ldots \nu_q}_{\star \mu_1 \ldots \mu_p}$ are functions on U.

Proof. Following the proof of lemma 1 we have that $\tau^{p,q}$ can be uniquely written as $\tau^{p,q} = \tau_\star^{p,q-1\nu} \otimes_\star \partial_\nu$, where $\tau_\star^{p,q-1\nu}$ is a type $(p, q-1)$ tensor. This expression holds for any value of q and therefore (using associativity of the $\otimes_\star$ product) $\tau^{p,q}$ can be uniquely written as $\tau^{p,q} = \tau_\star^{p,0\nu_1\nu_2\ldots\nu_q} \otimes_\star \partial_{\nu_1} \otimes_\star \cdots \partial_{\nu_q}$. Similarly, as in formula (4.21), we find that $\tau^{p,q}$ can be uniquely written as $\tau^{p,q} = \tau_{\star\mu_1}^{p-1,0\nu_1\nu_2\ldots\nu_q} \otimes_\star \mathrm{d}x^{\mu_1} \otimes_\star \partial_{\nu_1} \otimes_\star \cdots \partial_{\nu_q}$. This expression holds for any value of p and q and therefore (using associativity of the $\otimes_\star$ product) we obtain expression (4.22) and its uniqueness. □

4.1.3. Exterior algebra of forms $\Omega_\star^\cdot = \oplus_p \Omega_\star^p$. As another application of the theorem in section 3.1 we consider the algebra of exterior forms $\Omega^\cdot = \oplus_p \Omega^p$, and $\star$-deform the wedge product into the $\star$-wedge product,

$$\vartheta \wedge_\star \vartheta' := \bar{\mathrm{f}}^\alpha(\vartheta) \wedge \bar{\mathrm{f}}_\alpha(\vartheta'). \tag{4.23}$$

We denote by $\Omega_\star^\cdot$ the linear space of forms equipped with the wedge product $\wedge_\star$,

$$\Omega_\star^\cdot := (\Omega^\cdot, \wedge_\star). \tag{4.24}$$

As in the commutative case, it can be shown [45] that the linear space of exterior forms can be seen as the tensor subspace of totally $\star$-antisymmetric (contravariant) tensorfields. The properties of the $\star$-antisymmetrizator imply that there is a top form that has the same degree as in the undeformed case. This is in accordance with (4.23). Explicitly, the $\star$-antisymmetric 2-form $\omega \wedge_\star \omega'$ is defined by (cf (7.1))

$$\omega \wedge_\star \omega' := \omega \otimes_\star \omega' - \mathcal{L}^\star_{\bar{R}^\alpha_\star}(\omega') \otimes_\star \mathcal{L}^\star_{\bar{R}_{\star\alpha}}(\omega). \tag{4.25}$$

It can also be shown [39] that the usual exterior derivative d: $A \to \Omega$ satisfies the Leibniz rule $\mathrm{d}(h \star g) = \mathrm{d}h \star g + h \star \mathrm{d}g$ and is therefore also the $\star$-exterior derivative. This is so because the exterior derivative commutes with the Lie derivative. In the case where A is a Hopf algebra, the fact that the exterior differential on $A_\star$ is not deformed was shown in [55].

4.2. $\star$-Pairing between 1-forms and vectorfields

Following the general prescription outlined in section 2.2, we define the $\star$-pairing between vectorfields and 1-forms as $\langle , \rangle_\star := \langle , \rangle \circ \mathcal{F}^{-1}$. Explicitly, for all $\xi \in \Xi_\star$, $\omega \in \Omega_\star$

$$\langle , \rangle_\star : \Xi_\star \otimes_{\mathbb{C}} \Omega_\star \to A, \tag{4.26}$$

$$(\xi, \omega) \mapsto \langle \xi, \omega \rangle_\star := \langle \bar{\mathrm{f}}^\alpha(\xi), \bar{\mathrm{f}}_\alpha(\omega) \rangle. \tag{4.27}$$

We leave it to the reader to prove the following:

Lemma 3. *The pairing $\langle , \rangle_\star$ is compatible with the $\star$-Lie derivative,*

$$\mathcal{L}^\star_\xi(\langle u, \omega \rangle_\star) = \left\langle \mathcal{L}^\star_{\xi_{1_\star}}(u), \mathcal{L}^\star_{\xi_{2_\star}}(\omega) \right\rangle_\star, \tag{4.28}$$

and satisfies the $A_\star$-linearity properties

$$\langle h \star u, \omega \star k \rangle_\star = h \star \langle u, \omega \rangle_\star \star k, \tag{4.29}$$

$$\langle u, h \star \omega \rangle_\star = \langle u \star h, \omega \rangle_\star = \mathcal{L}^\star_{\bar{R}^\alpha_\star}(h) \star \left\langle \mathcal{L}^\star_{\bar{R}_{\star\alpha}}(u), \omega \right\rangle_\star \tag{4.30}$$

so that $\langle , \rangle_\star : \Xi_\star \otimes_\star \Omega_\star \to A$.

In the commutative case, we can consider locally a moving frame (or vielbein) $\{e_i\}$ and a dual frame of 1-forms ω^j:

$$\langle e_i, \omega^j \rangle = \delta_i^j, \tag{4.31}$$

in particular $\langle \partial_\mu, \mathrm{d}x^\nu \rangle = \delta^\nu_\mu$. In the noncommutative case locally we also have a moving frame $\{\hat{e}_i\}$ and a dual frame of 1-forms ω^j:

$$\langle \hat{e}_i, \omega^j \rangle_\star = \delta^j_i. \tag{4.32}$$

We construct it in the following way: since $\langle e_i, \omega^j \rangle = \delta^j_i$ we have $\langle e_i, \omega^j \rangle_\star = N_i^j$ with N being a $\star$-invertible matrix since $N_i^j = \delta^j_i + O(\lambda)$. We denote by $N^{-1_\star}$ the $\star$-inverse matrix of the matrix N. We have $N^{-1_\star} = 1 + \lambda N_1 + \lambda^2 N_2 + \cdots$ with the generic terms $N_n^{-1_\star}$ recursively given by $N_n^{-1_\star} = -\sum_{l=1}^n N_{n-l}^{-1_\star} \star N_l$; see also [14] for another equivalent explicit expression. Then,

$$\hat{e}_i = N_i^{-1_\star k} \star e_k \tag{4.33}$$

satisfies $\langle \hat{e}_i, \omega^j \rangle_\star = \delta^j_i$ as is easily seen using $A_\star$-linearity of the pairing $\langle , \rangle_\star$. Of course, we also have $\langle e_i, \hat{\omega}^j \rangle_\star = \delta^j_i$ with $\hat{\omega}^j = \omega^k \star N_k^{-1_\star j}$. We denote by $\{\hat{\partial}_\mu\}$ the basis of vectorfields that satisfy

$$\langle \hat{\partial}_\mu, \mathrm{d}x^\nu \rangle_\star = \delta^\nu_\mu, \tag{4.34}$$

and we have $\hat{\partial}_\mu = N_\mu^{-1_\star \nu} \star \partial_\nu$ with $N^\nu_\mu = \langle \partial_\mu, \mathrm{d}x^\nu \rangle_\star$.

Using the pairing $\langle , \rangle_\star$, we associate to any 1-form ω the left $A_\star$-linear map $\langle , \omega \rangle_\star$. It can also be shown [45] that the converse holds: any left $A_\star$-linear map $\Phi : \Xi_\star \to A_\star$ is of the form $\langle , \omega \rangle_\star$ for some ω.

5. Covariant derivative

By now we have acquired enough knowledge of $\star$-noncommutative differential geometry to develop the formalism of covariant derivative, torsion, curvature and Ricci tensors just by following the usual classical formalism.

We define a $\star$-covariant derivative $\nabla^\star_u$ along the vector field $u \in \Xi$ to be a linear map $\nabla^\star_u : \Xi_\star \to \Xi_\star$ such that for all $u, v, z \in \Xi_\star$, $h \in A_\star$

$$\nabla^\star_{u+v} z = \nabla^\star_u z + \nabla^\star_v z, \tag{5.1}$$

$$\nabla^\star_{h\star u} v = h \star \nabla^\star_u v, \tag{5.2}$$

$$\nabla^\star_u (h \star v) = \mathcal{L}^\star_u(h) \star v + \overline{R}^\alpha(h) \star \nabla^\star_{\overline{R}_\alpha(u)} v. \tag{5.3}$$

Note that in the last line we have used the coproduct formula (3.12), $\Delta_\star(u) = u \otimes 1 + \overline{R}^\alpha_\star \otimes \mathcal{L}^\star_{\overline{R}_{\star\alpha}}(u)$. Expression (5.3) is well defined because $\overline{R}_\alpha(u)$ is again a vectorfield.

Local coordinate description

In a coordinate neighbourhood U with coordinates x^μ we have the frame $\{\hat{\partial}_\mu\}$ that is $\star$-dual to the frame $\{\mathrm{d}x^\mu\}$ (cf (4.34)). The (noncommutative) connection coefficients $\Gamma_{\mu\nu}{}^\sigma$ are uniquely determined by

$$\nabla^\star_{\hat{\partial}_\mu} \hat{\partial}_\nu = \Gamma_{\mu\nu}{}^\sigma \star \hat{\partial}_\sigma. \tag{5.4}$$

They uniquely determine the connection; indeed for vectorfields z and u we have

$$\begin{aligned}
\nabla^\star_z u &= \nabla^\star_z (u^\nu_\star \star \hat{\partial}_\nu) \\
&= \mathcal{L}^\star_z(u^\nu_\star) \star \hat{\partial}_\nu + \overline{R}^\alpha(u^\nu_\star) \star \nabla^\star_{\overline{R}_\alpha(z)} \hat{\partial}_\nu \\
&= \mathcal{L}^\star_z(u^\nu_\star) \star \hat{\partial}_\nu + \overline{R}^\alpha(u^\nu_\star) \star \overline{R}_\alpha(z)^\mu \star \nabla^\star_{\hat{\partial}_\mu} \hat{\partial}_\nu \\
&= \mathcal{L}^\star_z(u^\nu_\star) \star \hat{\partial}_\nu + \overline{R}^\alpha(u^\nu_\star) \star \overline{R}_\alpha(z)^\mu \star \Gamma_{\mu\nu}{}^\sigma \star \hat{\partial}_\sigma
\end{aligned} \tag{5.5}$$

where $\overline{R}_\alpha(z)^\mu$ are the coefficients of $\overline{R}_\alpha(z)$, $\overline{R}_\alpha(z) = \overline{R}_\alpha(z)^\mu \star \hat{\partial}_\mu$. With respect to a local frame of vectorfields $\{e_i\}$, we have the connection coefficients

$$\nabla^\star_{e_i} e_j = \Gamma_{ij}{}^k \star e_k. \tag{5.6}$$

Covariant derivative on tensorfields

We define the covariant derivative on bivectorfields extending by linearity the following deformed Leibniz rule for all $u, v, z \in \Xi_\star$:

$$\nabla^\star_u(v \otimes_\star z) := \nabla^\star_u(v) \otimes_\star z + \overline{R}^\alpha(v) \otimes_\star \nabla^\star_{\overline{R}_\alpha(u)} z. \tag{5.7}$$

We now define the covariant derivative on functions to be the $\star$-Lie derivative,

$$\nabla^\star_u(h) = \mathcal{L}^\star_u(h). \tag{5.8}$$

As in the commutative case we also define the covariant derivative on 1-forms $\Omega_\star$ by requiring compatibility with the contraction operator for all $u, v \in \Xi_\star$, $\omega \in \Omega_\star$,

$$\nabla^\star_u \langle v, \omega\rangle_\star = \langle \nabla^\star_u(v), \omega\rangle_\star + \big\langle \overline{R}^\alpha(v), \nabla^\star_{\overline{R}_\alpha(u)} \omega\big\rangle_\star \tag{5.9}$$

so that $\langle v, \nabla^\star_u \omega\rangle_\star = \mathcal{L}^\star_{\overline{R}^\alpha(u)} \langle \overline{R}_\alpha(v), \omega\rangle_\star - \big\langle \nabla^\star_{\overline{R}^\alpha(u)}(\overline{R}_\alpha(v)), \omega\big\rangle_\star$. Finally, we extend the covariant derivative to all tensorfields via the deformed Leibniz rule (5.7) where now $\tau, \tau' \in \mathcal{T}_\star$,

$$\nabla^\star_u(\tau \otimes_\star \tau') := \nabla^\star_u(\tau) \otimes_\star \tau' + \overline{R}^\alpha(\tau) \otimes_\star \nabla^\star_{\overline{R}_\alpha(u)} \tau'. \tag{5.10}$$

6. Torsion and curvature

Definition 2. *The torsion* T *and the curvature* R *associated to a connection* $\nabla^\star$ *are the* $\mathbb{C}$*-linear maps* $\mathsf{T} : \Xi_\star \otimes_{\mathbb{C}} \Xi_\star \to \Xi_\star$ *and* $\mathsf{R}^\star : \Xi_\star \otimes_{\mathbb{C}} \Xi_\star \otimes_{\mathbb{C}} \Xi_\star \to \Xi_\star$ *defined by*

$$\mathsf{T}(u, v) := \nabla^\star_u v - \nabla^\star_{\overline{R}^\alpha(v)} \overline{R}_\alpha(u) - [u, v]_\star, \tag{6.1}$$

$$\mathsf{R}(u, v, z) := \nabla^\star_u \nabla^\star_v z - \nabla^\star_{\overline{R}^\alpha(v)} \nabla^\star_{\overline{R}_\alpha(u)} z - \nabla^\star_{[uv]_\star} z \tag{6.2}$$

for all $u, v, z \in \Xi_\star$.

From the antisymmetry property of the bracket $[\,]_\star$ (see (3.9)) and the triangularity of the $\mathcal{R}$-matrix, it easily follows that the torsion T and the curvature R have the following $\star$-antisymmetry property:

$$\mathsf{T}(u, v) = -\mathsf{T}(\overline{R}^\alpha(v), \overline{R}_\alpha(u)), \tag{6.3}$$

$$\mathsf{R}(u, v, z) = -\mathsf{R}(\overline{R}^\alpha(v), \overline{R}_\alpha(u), z). \tag{6.4}$$

It can be shown [45] that T and R are left $A_\star$-linear maps

$$\begin{aligned} &\mathsf{T} : \Xi_\star \otimes_\star \Xi_\star \to \Xi_\star \\ &\mathsf{R} : \Xi_\star \otimes_\star \Xi_\star \otimes_\star \Xi_\star \to \Xi_\star \end{aligned} \tag{6.5}$$

and therefore that they uniquely define a torsion tensor and a curvature tensor. For the torsion, left $A_\star$-linearity explicitly reads

$$\mathsf{T}(f \star u, v) = f \star \mathsf{T}(u, v), \tag{6.6}$$

$$\mathsf{T}(u, f \star v) = \mathsf{T}(u \star f, v) = \overline{R}^\alpha(f) \star \mathsf{T}(\overline{R}_\alpha(u), v), \tag{6.7}$$

and similarly for the curvature. Instead of entering the technical Hopf algebra aspects of the proof of (6.6) and (6.7), we here present an easy intuitive argument. Recall that

$f \star g = \overline{R}^{\alpha}(g) \star \overline{R}_{\alpha}(f)$. In other terms, the noncommutativity of the $\star$-product is regulated by the $\mathcal{R}$-matrix. Expression $\overline{R}^{\alpha}(g) \star \overline{R}_{\alpha}(f)$ can be read as saying that the initial ordering $f \star g$ has been inverted. Similarly, expression $\overline{R}^{\beta}\overline{R}^{\alpha}(h) \star \overline{R}^{\beta}(f) \star \overline{R}_{\alpha}(g)$ equals $f \star g \star h$ as is easily seen by accounting for the number of elementary transpositions needed to permute (f, g, h) into (h, f, g). In short, $\mathcal{R}^{-1} = \overline{R}^{\alpha} \otimes \overline{R}_{\alpha}$ is a representation of the permutation group on the $\star$-algebra of functions $A_\star$ and similarly on the algebra of vectorfields $U\Xi_\star$. The formula

$$[f \star u, v]_\star = f \star [u, v]_\star - (\mathcal{L}_{\overline{R}^{\beta}(\overline{R}^{\alpha}(v))}\overline{R}_{\beta}(f)) \star \overline{R}_{\alpha}(u) \tag{6.8}$$

can then be intuitively obtained recalling the analogue commutative formula $[fu, v] = f[u, v] - (\mathcal{L}_v f)u$ and keeping track of the transpositions that have occurred. For example, the $\mathcal{R}$-matrices in the last addend agree with the reordering $(f, u, v) \to (v, f, u)$. Recalling again that the initial ordering is (f, u, v), one similarly has

$$\nabla^\star_{\overline{R}^{\alpha}(v)}\overline{R}_{\alpha}(f \star u) = f \star \nabla^\star_{\overline{R}^{\alpha}(v)}\overline{R}_{\alpha}(u) + (\mathcal{L}_{\overline{R}^{\beta}\overline{R}^{\alpha}(v)}\overline{R}_{\beta}(f)) \star \overline{R}_{\alpha}(u). \tag{6.9}$$

The sum of (6.8) and (6.9) gives the left $A_\star$-linearity property (6.6) of the torsion. Formula (6.7) can be similarly obtained. It also follows from the $\star$-antisymmetry property (6.3).

Local coordinates description

We denote by $\{e_i\}$ a local frame of vectorfields (subordinate to an open $U \in M$) and by $\{\theta_j\}$ the dual frame of 1-forms:

$$\langle e_i, \theta^j \rangle_\star = \delta_i^j. \tag{6.10}$$

The coefficients $\mathsf{T}_{ij}{}^l$ and $\mathsf{R}_{ijk}{}^l$ of the torsion and curvature tensors with respect to this local frame are defined by

$$\mathsf{T}_{ij}{}^l = \langle \mathsf{T}(e_i, e_j), \theta^l \rangle_\star, \qquad \mathsf{R}_{ijk}{}^l = \langle \mathsf{R}(e_i, e_j, e_k), \theta^l \rangle_\star.$$

We denote by $\Lambda^\star$ the $\star$-transposition operator; it is the linear operator given by

$$\Lambda^\star(u \otimes_\star v) := \mathcal{L}^\star_{\overline{R}^{\alpha}_\star}(v) \otimes_\star \mathcal{L}^\star_{\overline{R}_{\star\alpha}}(u) = \overline{R}^{\alpha}(v) \otimes_\star \overline{R}_{\alpha}(u). \tag{6.11}$$

It is easily seen to be compatible with the $A_\star$-bimodule and the $U\Xi_\star$-module structure of $\Xi_\star \otimes_\star \Xi_\star$:

$$\Lambda^\star(h \star u \otimes_\star v \star k) = h \star \Lambda^\star(u \otimes_\star v) \star k, \tag{6.12}$$

$$\mathcal{L}^\star_\xi(\Lambda^\star(u \otimes_\star v)) = \Lambda^\star(\mathcal{L}^\star_\xi(u \otimes_\star v)). \tag{6.13}$$

(Hint: use (2.51), (2.52), and (3.20).) Because of the $A_\star$-bilinearity property (6.12), we have that $\Lambda^\star$ is completely determined by its action on a basis of vectorfields. We define the coefficients $\Lambda^{\star kl}_{ij}$ of $\Lambda^\star$ by the expression

$$\Lambda^\star(e_i \otimes_\star e_j) = \Lambda^{\star kl}_{ij} \star e_k \otimes_\star e_l.$$

Recalling the $\star$-antisymmetry property of T and R (see (6.3) and (6.4)), we then immediately have the $\star$-antisymmetry properties of the coefficients $\mathsf{T}_{ij}{}^l$ and $\mathsf{R}_{nij}{}^l$,

$$\mathsf{T}_{ij}{}^l = -\Lambda^{\star km}_{ij} \star \mathsf{T}_{km}{}^l, \qquad \mathsf{R}_{nij}{}^l = -\Lambda^{\star km}_{ij} \star \mathsf{R}_{nkm}{}^l. \tag{6.14}$$

In the commutative case, if the connection is chosen to have vanishing torsion, we have the first Bianchi identities $\mathsf{R}_{ijk}{}^l + \mathsf{R}_{jki}{}^l + \mathsf{R}_{kij}{}^l = 0$, where the lower indices ijk have been cyclically permuted. There is a similar equation in the noncommutative case.

We first define the $\star$-operation of cyclic permutation of three vectors. Recalling the definition of the $\star$-transposition operator we have that $\star$-cyclic permutation of the vectors uvz is given by

$$\mathcal{C}^\star(u \otimes_\star v \otimes_\star z) = u \otimes_\star v \otimes_\star z + \Lambda^\star_{12}\Lambda^\star_{23}(u \otimes_\star v \otimes_\star z) + \Lambda^\star_{23}\Lambda^\star_{12}(u \otimes_\star v \otimes_\star z) \tag{6.15}$$

where $\Lambda^\star_{12} = \Lambda \otimes_\star \mathrm{id}$ and $\Lambda^\star_{23} = \mathrm{id} \otimes_\star \Lambda$. From the $A_\star$-bilinearity property of $\Lambda^\star$, we see that also $\mathcal{C}^\star$ is $A_\star$-bilinear

$$\mathcal{C}^\star(h \star u \otimes_\star v \otimes_\star z \star k) = h \star \mathcal{C}^\star(u \otimes_\star v \otimes_\star z) \star k. \tag{6.16}$$

Since any tensor in $\Xi_\star \otimes_\star \Xi_\star \otimes_\star \Xi_\star$ is of the form $f^{ijk} \star e_i \otimes_\star e_j \otimes_\star e_k$, we have that the $\star$-cyclic permutation operator is completely defined by its action on a basis $\{e_i \otimes_\star e_j \otimes_\star e_k\}$. This action is completely determined by the coefficients $\mathcal{C}^{\star}{}^{lmn}_{ijk}$ of $\mathcal{C}^\star$,

$$\mathcal{C}^\star(e_i \otimes_\star e_j \otimes_\star e_k) = \mathcal{C}^{\star}{}^{lmn}_{ijk} \star e_l \otimes_\star e_m \otimes_\star e_n. \tag{6.17}$$

We can now state the first Bianchi identity in the case of vanishing torsion:

$$\mathcal{C}^\star(\mathsf{R}(u, v, z)) = 0 \tag{6.18}$$

where $\mathcal{C}^\star$ denotes the $\star$-cyclic permutation of u, v and z. In components, the Bianchi identity reads

$$\mathcal{C}^{\star}{}^{lmn}_{ijk} \star \mathsf{R}_{lmn}{}^p = 0. \tag{6.19}$$

The proof of the Bianchi identity follows the classical proof. Since the torsion vanishes we have $\nabla^\star_u \mathsf{T}(v, z) = 0$ and this equation reads

$$\nabla^\star_u \nabla^\star_v(z) - \nabla^\star_u \nabla^\star_{\overline{R}^\alpha(z)} \overline{R}_\alpha(v) - \nabla^\star_{\overline{R}^\alpha([v,z]_\star)} \overline{R}_\alpha(u) - [u, [v, z]_\star]_\star, \tag{6.20}$$

where we have used $\mathsf{T}(u, [v, z]_\star) = 0$. We now add this equation three times, each time $\star$-cyclically permuting the vectors (u, v, z), so that we have the three orderings (u, v, z), $(\overline{R}^\beta \overline{R}^\alpha(z), \overline{R}_\beta(u), \overline{R}_\alpha(v))$ and $(\overline{R}^\delta(v), \overline{R}^\gamma(z), \overline{R}_\gamma \overline{R}_\delta(u))$. The three addends

$$[u, [v, z]_\star]_\star + \ \star\text{-cyclic perm}$$

vanish because of the $\star$-Jacobi identities and the remaining addends give the Bianchi identity. (This can be seen using (2.51), (2.52) and the quantum Yang–Baxter equation $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$, that is a consequence of (2.50), (2.51), (2.52).)

We end this section with the definition of the Ricci tensor. In the commutative case, the Ricci tensor is a contraction of the curvature tensor, $\mathsf{Ric}_{jk} = \mathsf{R}_{ijk}{}^i$. We define the Ricci map to be the following contraction of the curvature:

$$\mathsf{Ric}(u, v) := \langle \theta^i, \mathsf{R}(e_i, u, v)\rangle'_\star, \tag{6.21}$$

where sum over i is understood. The contraction $\langle , \rangle'_\star$ is a contraction between forms on the *left* and vectorfields on the *right*. It is defined through the by now familiar deformation of the commutative pairing,

$$\begin{aligned} \langle \omega, u\rangle'_\star &:= \langle \bar{\mathrm{f}}^\alpha(\omega), \bar{\mathrm{f}}_\alpha(u)\rangle, \\ &= \langle \overline{R}^\alpha(u), \overline{R}_\alpha(\omega)\rangle_\star. \end{aligned} \tag{6.22}$$

The pairing $\langle , \rangle'_\star$ has of course the $A_\star$-linearity properties

$$\langle h \star \omega, u \star k\rangle'_\star = h \star \langle \omega, u\rangle'_\star \star k, \qquad \langle \omega, h \star u\rangle'_\star = \langle \omega \star h, u\rangle'_\star. \tag{6.23}$$

Definition (6.21) is well given because it is independent from the choice of the frame $\{e_i\}$ (and the dual frame $\{\theta^i\}$) and because the Ricci map so defined is an $A_\star$-linear map:

$$\mathsf{Ric}(f \star u, v) = f \star \mathsf{Ric}(u, v), \tag{6.24}$$

$$\mathsf{Ric}(u, f \star v) = \mathsf{Ric}(u \star f, v) = \overline{R}^\alpha(f) \star \mathsf{Ric}(\overline{R}_\alpha(u), v). \tag{6.25}$$

In order to prove this statement, we consider the coefficients $\mathsf{R}^j(e_i, u, v)$ of the vector

$$\mathsf{R}(e_i, u, v) = \mathsf{R}^j(e_i, u, v) \star e_j.$$

The $A_\star$-linearity of R implies the $A_\star$-linearity of the coefficients, $\mathsf{R}^j(h \star e_i, u, v) = h \star \mathsf{R}^j(e_i, u, v)$. This in turn implies (recall the end of section 4) that there exists 1-forms $\omega^j_{\mathsf{R}}(u, v)$ such that

$$\mathsf{R}^j(e_i, u, v) = \langle e_i, \omega^j_{\mathsf{R}}(u, v)\rangle_\star. \tag{6.26}$$

From $\mathsf{R}(e_i, h \star u, v) = \mathsf{R}(e_i \star h, u, v)$, we immediately see that the 1-forms $\omega^j_{\mathsf{R}}(u, v)$ are left linear in u, i.e., $\omega^j_{\mathsf{R}}(h \star u, v) = h \star \omega^j_{\mathsf{R}}(u, v)$. We now have

$$\begin{aligned}\langle \theta^i, \mathsf{R}(e_i, u, v)\rangle'_\star &= \langle \theta^i \star \mathsf{R}^j(e_i, u, v), e_j\rangle'_\star \\ &= \langle \theta^i \star \langle e_i, \omega^j_{\mathsf{R}}(u, v)\rangle_\star, e_j\rangle'_\star \\ &= \langle \omega^j_{\mathsf{R}}(u, v), e_j\rangle'_\star\end{aligned}$$

where in the first line we used (6.23). This formula implies independence from the choice of basis $\{e_i\}$ and left $A_\star$-linearity of Ric.

The coefficients of the Ricci tensor are

$$\mathsf{Ric}_{jk} = \mathsf{Ric}(e_j, e_k). \tag{6.27}$$

7. Metric and Einstein equations

In order to define a $\star$-metric, we need to define $\star$-symmetric elements in $\Omega_\star \otimes_\star \Omega_\star$. In (6.11), we have defined the transposition operator $\Lambda^\star$ on the vectorfields; we can similarly define it on the forms,

$$\Lambda^\star(\omega \otimes_\star \omega') := \mathcal{L}^\star_{\overline{R}^\alpha_\star}(\omega') \otimes_\star \mathcal{L}^\star_{\overline{R}_{\star\alpha}}(\omega) = \overline{R}^\alpha(\omega') \otimes_\star \overline{R}_\alpha(\omega). \tag{7.1}$$

We now recall that $\Omega_\star \otimes_\star \Omega_\star = \Omega \otimes \Omega$ as vectorspaces and we note that the transposition operator $\Lambda^\star : \Omega_\star \otimes_\star \Omega_\star \to \Omega_\star \otimes_\star \Omega_\star$ is just the classical transposition operator $\Lambda : \Omega \otimes \Omega \to \Omega \otimes \Omega$. Indeed, we have

$$\Lambda(\omega \otimes_\star \omega') = \Lambda(\overline{\mathrm{f}}^\alpha(\omega) \otimes \overline{\mathrm{f}}_\alpha(\omega')) = \overline{\mathrm{f}}_\alpha(\omega') \otimes \overline{\mathrm{f}}^\alpha(\omega) = \overline{R}^\alpha(\omega') \otimes_\star \overline{R}_\alpha(\omega) = \Lambda^\star(\omega \otimes_\star \omega'),$$

where in the first equality we have explicitly written the element $\omega \otimes_\star \omega'$ as an element of $\Omega \otimes \Omega$ and then in the second equality we have applied the definition of Λ. This implies that (anti-)symmetric elements in $\Omega \otimes \Omega$ are $\star$-(anti-)symmetric elements in $\Omega_\star \otimes_\star \Omega_\star$.

Since a commutative metric is a nondegenerate symmetric tensor in $\Omega \otimes \Omega$, we conclude that any commutative metric is also a noncommutative metric ($\star$-nondegeneracy of the metric is insured by the fact that at zeroth order in the deformation parameter λ the metric is nondegenerate). Contrary to [8, 56], we see that in our approach, where all (moving) frames are on equal footing, there are infinitely many metrics compatible with a given noncommutative differential geometry; noncommutativity does not single out a preferred metric.

We denote by g the metric tensor. If we write

$$\mathsf{g} = \mathsf{g}^a \otimes_\star \mathsf{g}_a \in \Omega_\star \otimes_\star \Omega_\star \tag{7.2}$$

(for example, locally $\mathsf{g} = \theta^j \otimes_\star \theta^i \star \mathsf{g}_{ij}$), then for every $v \in \Xi_\star$ we can define the 1-form

$$\langle v, \mathsf{g}\rangle_\star := \langle v, \mathsf{g}^a\rangle_\star \star \mathsf{g}_a \tag{7.3}$$

and we can then construct the left $A_\star$-linear map g, corresponding to the metric tensor $\mathsf{g} \in \Omega_\star \otimes_\star \Omega_\star$, as

$$\begin{aligned}&\mathsf{g} : \Xi_\star \otimes_\star \Xi_\star \to A_\star \\ &(u, v) \mapsto \mathsf{g}(u, v) = \langle u \otimes_\star v, \mathsf{g}\rangle_\star := \langle u, \langle v, \mathsf{g}\rangle_\star\rangle_\star.\end{aligned} \tag{7.4}$$

The $\star$-inverse metric $\mathsf{g}^{-1} \in \Xi_\star \otimes_\star \Xi_\star$ is then defined by the following equations, for all $u \in \Xi_\star, \omega \in \Omega_\star$,

$$\langle\langle u, \mathsf{g}\rangle_\star, \mathsf{g}^{-1}\rangle'_\star = u, \tag{7.5}$$

$$\langle\langle \omega, \mathsf{g}^{-1}\rangle'_\star, \mathsf{g}\rangle_\star = \omega, \tag{7.6}$$

where, as in (7.3), we have defined

$$\langle \omega, \mathsf{g}^{-1}\rangle'_\star := \langle \omega, \mathsf{g}^{a-1}\rangle'_\star \star \mathsf{g}_a^{-1}, \tag{7.7}$$

and we have decomposed g^{-1} as

$$\mathsf{g}^{-1} = \mathsf{g}^{a-1} \otimes_\star \mathsf{g}_a^{-1} \in \Xi_\star \otimes_\star \Xi_\star \tag{7.8}$$

(for example, locally $\mathsf{g}^{-1} = \mathsf{g}^{ij\star} \star e_j \otimes_\star e_i$). At zeroth order in the deformation parameter λ, and using local coordinates, we write $\mathsf{g} = \mathsf{g}_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ and the above definition of the inverse metric gives $\mathsf{g}^{-1} = \mathsf{g}^{\mu\nu} \partial_\mu \otimes \partial_\nu$, where $\mathsf{g}^{\mu\nu}$ is the inverse matrix of $\mathsf{g}_{\mu\nu}$, $\mathsf{g}^{\mu\nu}\mathsf{g}_{\nu\rho} = \delta^\mu_\rho$, $\mathsf{g}_{\mu\nu}\mathsf{g}^{\nu\rho} = \delta^\rho_\mu$. For the noncommutative analogue of the relations $\mathsf{g}^{\mu\nu}\mathsf{g}_{\nu\rho} = \delta^\mu_\rho$, $\mathsf{g}_{\mu\nu}\mathsf{g}^{\nu\rho} = \delta^\rho_\mu$, see the end of the following section.

Consider now the connection that has vanishing torsion and that is metric compatible, $\nabla^\star_u \mathsf{g} = 0$. See [45], and see also [14] for the case θ-const. The scalar curvature $\mathfrak{R}$ with respect to this connection is given by

$$\mathfrak{R} := \mathsf{Ric}\big(\mathsf{g}^{a-1}, \mathsf{g}_a^{-1}\big) \tag{7.9}$$

where $\mathsf{g}^{-1} = \mathsf{g}^{a-1} \otimes_\star \mathsf{g}_a^{-1} \in \Xi_\star \otimes_\star \Xi_\star$. Locally, we have $\mathsf{g}^{-1} = \mathsf{g}^{ij\star} \star e_j \otimes_\star e_i$ and

$$\begin{aligned} \mathfrak{R} &= \mathsf{Ric}(\mathsf{g}^{ij\star} \star e_j, e_i) = \mathsf{g}^{ij\star} \star \mathsf{Ric}(e_j, e_i) \\ &= \mathsf{g}^{ij\star} \star \mathsf{Ric}_{ji}. \end{aligned} \tag{7.10}$$

We finally arrive at the noncommutative Einstein equation (in vacuum),

$$\mathsf{Ric} - \tfrac{1}{2}\mathsf{g} \star \mathfrak{R} = 0, \tag{7.11}$$

where the dynamical field is the metric g. This equation is an equality between the left $A_\star$-linear maps Ric and $\mathsf{g} \star \mathfrak{R}$, where

$$(\mathsf{g} \star \mathfrak{R})(u, v) := \langle u \otimes_\star v, \mathsf{g} \star \mathfrak{R}\rangle_\star = \langle u \otimes_\star v, \mathsf{g}\rangle_\star \star \mathfrak{R} = \mathsf{g}(u, v) \star \mathfrak{R}.$$

Because of left $A_\star$-linearity, the curvature scalar must appear on the right of the metric and not on the left in (7.11). Applying (7.11) to the vectors e_i and e_j, we obtain the components' equation

$$\mathsf{Ric}_{ij} - \tfrac{1}{2}\mathsf{g}_{ij} \star \mathfrak{R} = 0, \tag{7.12}$$

where $\mathsf{g}_{ij} = \mathsf{g}(e_i, e_j) = \langle e_i \otimes_\star e_j, \mathsf{g}\rangle_\star$ are the same coefficients appearing in the expression $\mathsf{g} = \theta^j \otimes_\star \theta^i \star \mathsf{g}_{ij}$.

8. Conjugation

In this section, we introduce the notion of complex conjugation on the algebra $A_\star$ and we see that we can impose reality conditions on the $\star$-spaces of functions, vectorfields and tensorfields.

We first briefly recall the commutative $*$-structure. Given a smooth real manifold M, the usual $*$-structure on the complex-valued functions $A = \mathrm{Fun}(M)$ is a map: $A \to A$, where for all $h \in A$ and $m \in M$,

$$h^*(m) = \overline{h(m)}. \tag{8.1}$$

Here the bar denotes complex conjugation. This $*$-structure induces a $*$-structure on the Lie algebra of the vectorfields by defining $*:\Xi\to\Xi$, where for all $u\in\Xi$ and $h\in A$,

$$u^*(h) := (S(u)(h^*))^* = -(u(h^*))^*. \tag{8.2}$$

It is easy to check that the $*$-operation so defined is antimultiplicative with respect to the Lie bracket of Ξ, $[u,v]^* = [v^*,u^*]$. In particular, locally, we can consider the real coordinate functions x^μ, then the partial derivatives ∂_μ are pure imaginary, $\partial_\mu^* = -\partial_\mu$; we also have $u^* = (u^\mu\partial_\mu)^* = -\overline{u^\mu}\partial_\mu$.

The $*$-structure on Ξ is extended to the universal enveloping algebra $U\Xi$ by antilinearity and antimultiplicativity, so that for all $\xi,\zeta\in U\Xi$, $(\xi\zeta)^* = \zeta^*\xi^*$. Applying a vectorfield v to definition (8.2) we obtain $(v^*u^*)(h) = (S(uv)(h^*))^*$, and iterating we obtain that for a generic element of $U\Xi$,

$$\xi^*(h) = (S(\xi)(h^*))^*. \tag{8.3}$$

Similarly, from $u^*(v) = [u^*,v] = [S(u),v^*]^* = (S(u)(v^*))^*$, we have

$$\xi^*(\zeta) = (S(\xi)(\zeta^*))^*. \tag{8.4}$$

Finally, from the local formula $\langle\partial_\mu, \mathrm{d}x^\nu\rangle^* = -\langle\partial_\mu^*, (\mathrm{d}x^\nu)^*\rangle^*$ we have the general formula of compatibility between the $*$-structure and the pairing

$$\langle u,\omega\rangle^* = -\langle u^*,\omega^*\rangle. \tag{8.5}$$

We now study the $*$-operation in the noncommutative context. We define the $*$-structure on $A_\star$ to be the same as that on A. The requirement

$$(h\star g)^* = g^*\star h^* \tag{8.6}$$

is then satisfied if the twist $\mathcal{F}$ satisfies the relation $(S\otimes S)\mathcal{F}_{21} = \mathcal{F}^{*\otimes *}$, i.e.,

$$(S\otimes S)\mathcal{F}_{21}^{-1} = \mathcal{F}^{-1*\otimes *}. \tag{8.7}$$

We similarly define the $*$-structure on $U\Xi$ to be the same as the undeformed one. Using (8.4) it is not difficult to show that the $*$-operation is compatible with the $\star$-product of $U\Xi_\star$ and with the $\star$-Lie bracket of $\Xi_\star$,

$$(\xi\star\zeta)^* = \zeta^*\star\xi^*, \qquad [u,v]_\star{}^* = [v^*,u^*]_\star. \tag{8.8}$$

It can be shown [45] that the $*$-operation is compatible with the triangular Hopf algebra structure of $U\Xi_\star$ (a key point being that on $U\Xi^{\mathcal{F}}$ the $*$-operation reads $\xi^{*_{\mathcal{F}}} := \chi\xi^*\chi^{-1}$). On tensors too the $*$-structure is by definition the undeformed one, and we have for all $\tau,\tau'\in\mathcal{T}_\star$,

$$(\tau\otimes_\star\tau')^* = \overline{R}^\alpha(\tau^*)\otimes_\star\overline{R}_\alpha(\tau'^*). \tag{8.9}$$

Finally, the two pairings $\langle\,,\rangle_\star$ and $\langle\,,\rangle'_\star$ are related by the $*$-operation, for all $u\in\Xi_\star$ and $\omega\in\Omega_\star$,

$$\langle u,\omega\rangle_\star^* = -\langle\omega^*,u^*\rangle'_\star. \tag{8.10}$$

In particular, if locally we consider a basis $\{e_i\}$ and the dual basis $\{\theta^i\}$,

$$\langle e_i,\theta^j\rangle_\star = \delta_i^j,$$

then the $*$-conjugate bases $\{e_i^*\}$ and $\{\theta^{j*}\}$ are (up to a sign) dual with respect to the $\langle\,,\rangle'_\star$ pairing,

$$\langle\theta^{j*},e_i^*\rangle'_\star = -\delta_i^j. \tag{8.11}$$

We can now study, for example, the reality property

$$\mathsf{g}^* = \mathsf{g} \tag{8.12}$$

of the metric tensor $\mathsf{g} \in \Omega_\star \otimes_\star \Omega_\star$. The metric tensor has a convenient expansion in terms of the θ^i and the $\theta^{\bar{j}*}$ 1-forms (here $\bar{j}$ is just an index like i or j). We set

$$\mathsf{g} = \theta^i \otimes_\star \mathsf{g}_{i\bar{j}} \star \theta^{\bar{j}*}. \tag{8.13}$$

In this basis, the reality of the metric, and therefore of the noncommutative Einstein equations, has a very simple explicit expression. Also the explicit expression for the inverse metric is particularly simple in this basis.

We first study the consequences of the reality condition $\mathsf{g} = \mathsf{g}^*$ on the metric coefficients $\mathsf{g}_{i\bar{j}}$. From (8.9), we have

$$\mathsf{g}^* = \overline{R}^\alpha(\theta^{i*}) \otimes_\star \overline{R}_\alpha(\theta^{\bar{j}} \star \mathsf{g}^*_{i\bar{j}}) = \overline{R}^\alpha(\theta^{\bar{j}*}) \otimes_\star \overline{R}_\alpha(\theta^{i} \star \mathsf{g}^*_{\bar{j}i}) \tag{8.14}$$

where in the last equality we have just renamed the indices. In order to compare this expression of g^* with the expression (8.13) of g, we use the $\star$-symmetry property of the metric, $\mathsf{g} = \Lambda^\star \mathsf{g}$, to rewrite the metric as

$$\mathsf{g} = \theta^i \star \mathsf{g}_{i\bar{j}} \otimes_\star \theta^{\bar{j}*} = \overline{R}^\alpha(\theta^{\bar{j}*}) \otimes_\star \overline{R}_\alpha(\theta^{i} \star \mathsf{g}_{i\bar{j}}).$$

Comparison with (8.14) gives $\overline{R}^\alpha(\theta^{\bar{j}*}) \otimes_\star \overline{R}_\alpha(\theta^{i} \star \mathsf{g}^*_{\bar{j}i}) = \overline{R}^\alpha(\theta^{\bar{j}*}) \otimes_\star \overline{R}_\alpha(\theta^{i} \star \mathsf{g}_{i\bar{j}})$ iff $\mathsf{g} = \mathsf{g}^*$. After applying the transposition $\Lambda^\star$ to this equation, we find that the reality of g reads

$$\theta^i \star \mathsf{g}^*_{\bar{j}i} \otimes_\star \theta^{\bar{j}*} = \theta^i \star \mathsf{g}_{i\bar{j}} \otimes_\star \theta^{\bar{j}*},$$

i.e.,

$$\mathsf{g}^*_{\bar{j}i} = \mathsf{g}_{i\bar{j}}. \tag{8.15}$$

Concerning the inverse metric g^{-1}, we find that it is given by the expression

$$\mathsf{g}^{-1} = -e^*_{\bar{i}} \otimes_\star \mathsf{g}^{\bar{i}j\star} \star e_j \tag{8.16}$$

where $\mathsf{g}^{\bar{i}j\star}$ is the $\star$-inverse matrix of $\mathsf{g}_{i\bar{j}}$,

$$\mathsf{g}^{\bar{i}j\star} \star \mathsf{g}_{j\bar{l}} = \delta^{\bar{i}}_{\bar{l}}, \qquad \mathsf{g}_{i\bar{j}} \star \mathsf{g}^{\bar{j}l\star} = \delta^l_i.$$

Indeed, it is not difficult to see that (8.16) satisfies (7.5) and (7.6).

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Appendix

A.1. Proof that $U\Xi^{\mathcal{F}}$ is a Hopf algebra

We start from

$$(\varepsilon \otimes \mathrm{id})\Delta^{\mathcal{F}}(u) = u = (\mathrm{id} \otimes \varepsilon)\Delta^{\mathcal{F}}(u) \tag{A.1}$$

and calculate first the left-hand side

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})\Delta^{\mathcal{F}}(u) &= (\varepsilon \otimes \mathrm{id})(\mathrm{f}^\alpha u_1 \bar{\mathrm{f}}^\beta \otimes \mathrm{f}_\alpha u_2 \bar{\mathrm{f}}_\beta) \\ &= \varepsilon(\mathrm{f}^\alpha u_1 \bar{\mathrm{f}}^\beta)\mathrm{f}_\alpha u_2 \bar{\mathrm{f}}_\beta = \varepsilon(\mathrm{f}^\alpha)\varepsilon(u_1)\varepsilon(\bar{\mathrm{f}}^\beta)\mathrm{f}_\alpha u_2 \bar{\mathrm{f}}_\beta.\end{aligned} \tag{A.2}$$

In the last line we have used that $\varepsilon : U\Xi \to \mathbb{C}$ is an algebra homomorphism. Applying $(\varepsilon \otimes \mathrm{id})$ on the identity

$$\mathcal{F}\mathcal{F}^{-1} = 1 \otimes 1 \tag{A.3}$$

and using (2.29) gives

$$\begin{aligned} 1 &= (\varepsilon \otimes \mathrm{id})\mathcal{F}\mathcal{F}^{-1} = (\varepsilon \otimes \mathrm{id})(\mathrm{f}^{\alpha}\bar{\mathrm{f}}^{\beta} \otimes \mathrm{f}_{\alpha}\bar{\mathrm{f}}_{\beta}) \\ &= \varepsilon(\mathrm{f}^{\alpha})\varepsilon(\bar{\mathrm{f}}^{\beta})\mathrm{f}_{\alpha}\bar{\mathrm{f}}_{\beta} = \varepsilon(\bar{\mathrm{f}}^{\beta})\bar{\mathrm{f}}_{\beta}. \end{aligned} \tag{A.4}$$

Inserting this into (A.2), we finally obtain

$$(\varepsilon \otimes \mathrm{id})\Delta^{\mathcal{F}}(u) = \varepsilon(u_1)u_2 = u. \tag{A.5}$$

In order to calculate the right-hand side of (A.1), one proceeds in an analogous way.

Next, we prove

$$\mu(S^{\mathcal{F}} \otimes \mathrm{id})\Delta^{\mathcal{F}}(u) = \varepsilon(u)1 = \mu(\mathrm{id} \otimes S^{\mathcal{F}})\Delta^{\mathcal{F}}(u). \tag{A.6}$$

To show this, we first have to prove that $\chi^{-1} = S(\bar{\mathrm{f}}^{\alpha})\bar{\mathrm{f}}_{\alpha}$:

$$\begin{aligned} \chi\chi^{-1} &= \mathrm{f}^{\beta}S(\mathrm{f}_{\beta})S(\bar{\mathrm{f}}^{\alpha})\bar{\mathrm{f}}_{\alpha} \\ &= \bar{\mathrm{f}}^{\gamma}\varepsilon(\bar{\mathrm{f}}_{\gamma})\mathrm{f}^{\beta}S(\bar{\mathrm{f}}^{\alpha}\mathrm{f}_{\beta})\bar{\mathrm{f}}_{\alpha} \\ &= \bar{\mathrm{f}}^{\gamma}\mathrm{f}^{\beta}S\big(\bar{\mathrm{f}}_{\gamma_1}\bar{\mathrm{f}}^{\alpha}\mathrm{f}_{\beta}\big)\bar{\mathrm{f}}_{\gamma_2}\bar{\mathrm{f}}_{\alpha} \\ &= \bar{\mathrm{f}}^{\gamma}_1 S\big(\bar{\mathrm{f}}^{\gamma}_2\big)\big)\bar{\mathrm{f}}_{\gamma} \\ &= \varepsilon(\bar{\mathrm{f}}^{\gamma})\bar{\mathrm{f}}_{\gamma} = 1. \end{aligned}$$

In the first line we used the definitions given in (2.47). Next, we inserted $1 = \bar{\mathrm{f}}^{\gamma}\varepsilon(\bar{\mathrm{f}}_{\gamma})$ which we showed in (A.4). The antipode property $S(\xi_1)\xi_2 = \varepsilon(\xi)$ together with the fact that the antipode is an antialgebra homomorphism leads to the next line. Then, we used $\bar{\mathrm{f}}^{\gamma}\mathrm{f}^{\beta} \otimes \bar{\mathrm{f}}_{\gamma_1}\bar{\mathrm{f}}^{\alpha}\mathrm{f}_{\beta} \otimes \bar{\mathrm{f}}_{\gamma_2}\bar{\mathrm{f}}_{\alpha} = \bar{\mathrm{f}}^{\gamma}_1 \otimes \bar{\mathrm{f}}^{\gamma}_2 \otimes \bar{\mathrm{f}}_{\gamma}$ which follows from the cocycle condition (2.28) by multiplying both sides of the equality with $\mathrm{f}^{\beta} \otimes \mathrm{f}_{\beta} \otimes 1$. The next step uses the antipode property $\xi_1 S(\xi_2) = \varepsilon(\xi)$. Finally, we used $\varepsilon(\bar{\mathrm{f}}^{\gamma})\bar{\mathrm{f}}_{\gamma} = 1$. Similarly, one shows that $\chi^{-1}\chi = 1$.

We are now able to prove (A.6). Starting with the left-hand side, we get

$$\begin{aligned} \mu(S^{\mathcal{F}} \otimes \mathrm{id})\Delta^{\mathcal{F}}(u) &= \mu\big(S^{\mathcal{F}}\big(u_{1_{\mathcal{F}}}\big) \otimes u_{2_{\mathcal{F}}}\big) \\ &= \mathrm{f}^{\alpha}S(\mathrm{f}_{\alpha})S(\mathrm{f}^{\gamma}u_1\bar{\mathrm{f}}^{\delta})S(\bar{\mathrm{f}}^{\beta})\bar{\mathrm{f}}_{\beta}\mathrm{f}_{\gamma}u_2\bar{\mathrm{f}}_{\delta} \\ &= \mathrm{f}^{\alpha}S(\mathrm{f}_{\alpha})S(\bar{\mathrm{f}}^{\beta}\mathrm{f}^{\gamma}u_1\bar{\mathrm{f}}^{\delta})\bar{\mathrm{f}}_{\beta}\mathrm{f}_{\gamma}u_2\bar{\mathrm{f}}_{\delta} \\ &= \mathrm{f}^{\alpha}S(\mathrm{f}_{\alpha})S(u_1\bar{\mathrm{f}}^{\delta})u_2\bar{\mathrm{f}}_{\delta} \\ &= \mathrm{f}^{\alpha}S(f_{\alpha})S(\bar{\mathrm{f}}^{\delta})S(u_1)u_2\bar{\mathrm{f}}_{\delta}. \end{aligned} \tag{A.7}$$

Here, we used that S is an antialgebra homomorphism and that $\mathcal{F}\mathcal{F}^{-1} = \bar{\mathrm{f}}^{\beta}\bar{\mathrm{f}}^{\gamma} \otimes \bar{\mathrm{f}}_{\beta}\bar{\mathrm{f}}_{\gamma} = 1 \otimes 1$.

Knowing that Δ is the coproduct in the $U\Xi$ Hopf algebra, we find

$$\mu(S \otimes \mathrm{id})\Delta(u) = S(u_1)u_2 = \varepsilon(u). \tag{A.8}$$

Inserting this relation into (A.7) gives

$$\begin{aligned} \mu(S^{\mathcal{F}} \otimes \mathrm{id})\Delta^{\mathcal{F}}(u) &= \mathrm{f}^{\alpha}S(\mathrm{f}_{\alpha})S(\bar{\mathrm{f}}^{\delta})\varepsilon(u)\bar{\mathrm{f}}_{\delta} \\ &= \chi\chi^{-1}\varepsilon(u) = \varepsilon(u). \end{aligned} \tag{A.9}$$

The right-hand side of (A.6) one proves analogously.

A.2. ⋆-Jacobi identity

In order to prove the ⋆-Jacobi identity, $[u[vz]_\star]_\star = [[uv]_\star z]_\star + [\overline{R}^\alpha(v)[\overline{R}_\alpha(u)z]_\star]_\star$, we use the following:

Lemma.

$$\bar{f}^\alpha \overline{R}^\gamma \otimes \bar{f}_{\alpha_1}\bar{f}^\beta \overline{R}_\gamma \otimes \bar{f}_{\alpha_2}\bar{f}_\beta = \bar{f}^\alpha_{\ 1}\bar{f}_\delta \otimes \bar{f}^\alpha_{\ 2}\bar{f}^\delta \otimes \bar{f}_\alpha \tag{A.10}$$

Proof.

$$\begin{aligned}\bar{f}^\alpha \overline{R}^\gamma \otimes \bar{f}_{\alpha_1}\bar{f}^\beta \overline{R}_\gamma \otimes \bar{f}_{\alpha_2}\bar{f}_\beta &= \bar{f}^\alpha f^\gamma \bar{f}_\delta \otimes \bar{f}_{\alpha_1}\bar{f}^\beta f_\gamma \bar{f}^\delta \otimes \bar{f}_{\alpha_2}\bar{f}_\beta \\ &= \bar{f}^\alpha_{\ 1}\bar{f}^\beta f^\gamma \bar{f}_\delta \otimes \bar{f}^\alpha_{\ 2}\bar{f}_\beta f_\gamma \bar{f}^\delta \otimes \bar{f}_\alpha \\ &= \bar{f}^\alpha_{\ 1}\bar{f}_\delta \otimes \bar{f}^\alpha_{\ 2}\bar{f}^\delta \otimes \bar{f}_\alpha\end{aligned}$$

where in the third line we applied property (2.28), while in the last line we used that $\bar{f}^\beta f^\gamma \otimes \bar{f}_\beta f_\gamma = \mathcal{F}^{-1}\mathcal{F} = 1 \otimes 1$. □

Now we observe that $\forall \xi \in U\Xi$

$$\mathcal{L}_\xi([vz]) = \mathcal{L}_\xi(vz) - \mathcal{L}_\xi(zv) = \mathcal{L}_{\xi_1}(v)\mathcal{L}_{\xi_2}(z) - \mathcal{L}_{\xi_1}(z)\mathcal{L}_{\xi_2}(v) = \left[\mathcal{L}_{\xi_1}(v)\mathcal{L}_{\xi_2}(z)\right], \tag{A.11}$$

where we used $\mathcal{L}_{\xi_1}(z)\mathcal{L}_{\xi_2}(v) = \mathcal{L}_{\xi_2}(z)\mathcal{L}_{\xi_1}(v)$ which holds because the classical coproduct Δ (see (2.3)) is cocommutative. Finally, we have the ⋆-Jacobi identity

$$\begin{aligned}[u[vz]_\star]_\star &= \left[\bar{f}^\alpha(u)\left[\bar{f}_{\alpha_1}\bar{f}^\beta(v)\bar{f}_{\alpha_2}\bar{f}_\beta(z)\right]\right] \\ &= \left[\bar{f}^\alpha_{\ 1}\bar{f}^\beta(u)\left[\bar{f}^\alpha_{\ 2}\bar{f}_\beta(v)\bar{f}_\alpha(z)\right]\right] \\ &= \left[\left[\bar{f}^\alpha_1\bar{f}^\beta(u)\bar{f}^\alpha_2\bar{f}_\beta(v)\right]\bar{f}_\alpha(z)\right] + \left[\bar{f}^\alpha_{\ 2}\bar{f}_\beta(v)\left[\bar{f}^\alpha_{\ 1}\bar{f}^\beta(u)\bar{f}_\alpha(z)\right]\right] \\ &= [[uv]_\star z]_\star + [\overline{R}^\alpha(v)[\overline{R}_\alpha(u)z]_\star]_\star\end{aligned} \tag{A.12}$$

where in the second line we used property (2.28), while in the last line we used the above lemma and the fact that $U\Xi$ is cocommutative.

A.3. Associativity of the ⋆-product on superspace

First, we calculate

$$\begin{aligned}(g \star h) \star k &= \mu \circ \mathcal{F}^{-1}(g \star h \otimes k) \\ &= \mu \circ \mathcal{F}^{-1}((\mu \circ \mathcal{F}^{-1}(g \otimes h)) \otimes k) \\ &= \mu \circ \mathcal{F}^{-1} \circ ((\mu \circ \mathcal{F}^{-1}) \otimes \mathrm{id})(g \otimes h \otimes k) \\ &= \mu \circ \mathcal{F}^{-1} \circ (\mu \otimes \mathrm{id}) \circ (\mathcal{F}^{-1} \otimes \mathrm{id})(g \otimes h \otimes k) \\ &= \mu \circ (\mu \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})\mathcal{F}^{-1} \circ (\mathcal{F}^{-1} \otimes \mathrm{id})(g \otimes h \otimes k) \\ &= \mu \circ (\mu \otimes \mathrm{id}) \circ \left(((\Delta \otimes \mathrm{id})\mathcal{F}^{-1})\mathcal{F}^{-1}_{12}\right)(g \otimes h \otimes k),\end{aligned}$$

where in the last line we used $\mathcal{L}_\xi \circ \mathcal{L}_\zeta = \mathcal{L}_{\xi\zeta}$ (i.e., $\xi \circ \zeta(h) = \xi\zeta(h)$) and in the next to last line we used

$$\begin{aligned}\mathcal{F}^{-1} \circ (\mu \otimes \mathrm{id})(g' \otimes h' \otimes k') &= (-1)^{|\bar{f}_\alpha||g'h'|}\bar{f}^\alpha(g'h') \otimes \bar{f}_\alpha(k') \\ &= (-1)^{|\bar{f}_\alpha||g'h'|+|\bar{f}^\alpha_2||g'|}\bar{f}^\alpha_1(g')\bar{f}^\alpha_2(h') \otimes \bar{f}_\alpha(k') \\ &= (\mu \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})\mathcal{F}^{-1}(g' \otimes h' \otimes k').\end{aligned}$$

Then, we similarly obtain

$$\begin{aligned} g \star (h \star k) &= \mu \circ \mathcal{F}^{-1}(g \otimes (h \star k)) \\ &= \mu \circ \mathcal{F}^{-1}(g \otimes (\mu \circ \mathcal{F}^{-1}(h \otimes k))) \\ &= \mu \circ \mathcal{F}^{-1} \circ (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \mathcal{F}^{-1})(g \otimes h \otimes k) \\ &= \mu \circ (\mathrm{id} \otimes \mu) \circ \left(((\mathrm{id} \otimes \Delta)\mathcal{F}^{-1})\mathcal{F}_{23}^{-1}\right)(g \otimes h \otimes k). \end{aligned}$$

Using (2.26), we finally conclude that $(g \star h) \star k = g \star (h \star k)$ and associativity is proven.

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6.4 A Gravity Theory on Noncommutative Spaces

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A Gravity Theory on Noncommutative Spaces

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The concepts necessary for an algebraic construction of a gravity theory on noncommutative spaces are introduced. The θ-deformed diffeomorphisms are studied and a tensor calculus is defined. This leads to a deformed Einstein-Hilbert action which is invariant with respect to deformed diffeomorphisms.

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1. Noncommutative Spaces

It is expected that in order to obtain a better understanding of physics at short distances and in order to cure the problems occuring when trying to quantize gravity one has to change the nature of space-time in a fundamental way. One way to do so is to implement noncommutativity by taking coordinates which satisfy the commutation relations

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{1.1}$$

The function $C^{\mu\nu}(\hat{x})$ is unknown. For physical reasons it should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments [1]. We denote the algebra generated by noncommutative coordinates $\hat{x}^{\mu}$ which are subject to the relations (1.1) by $\hat{\mathscr{A}}$ (*algebra of noncommutative functions*). In what follows we will exclusively consider the θ-deformed case which may at very short distances provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu} = \text{const.} \tag{1.2}$$

but we note that the algebraic construction presented here can be generalized to more complicated noncommutative structures of the above type which possess the Poincaré-Birkhoff-Witt (PBW) property.

2. Symmetries on Deformed Spaces

In general the commutation relations (1.1) are not covariant with respect to undeformed symmetries. For example the canonical commutation relations (1.2) break Lorentz symmetry if we assume that the noncommutativity parameters $\theta^{\mu\nu}$ do not transform.

The question arises whether we can *deform* the symmetry in such a way that it acts consistently on the deformed space (i.e. leaves the deformed space invariant) and such that it reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie algebras can be deformed in the category of Hopf algebras (Hopf algebras coming from a Lie algebra are also called Quantum Groups)[1]. Quantum group symmetries lead to new features of field theories on noncommutative spaces. Because of its simplicity, θ-deformed spaces are very well-suited to study those.

In the following we will construct explicitly a θ-deformed version of diffeomorphisms which consistently act on the noncommutative space (1.2). It is possible to construct a gravity theory which is invariant with respect to these deformed diffeomorphisms [2, 3, 4].

3. Diffeomorphisms

Diffeomorphisms are generated by vector-fields ξ. Acting on functions, vector-fields are represented as linear differential operators $\xi = \xi^{\mu}\partial_{\mu}$. Vector-fields form a Lie algebra Ξ with the Lie bracket given by

$$[\xi, \eta] = \xi \times \eta$$

[1] To be more precise the universal enveloping algebra of a Lie algebra can be deformed. The universal enveloping algebra of any Lie algebra is a Hopf algebra and this gives rise to deformations in the category of Hopf algebras.

where $\xi \times \eta$ is defined by its action on functions

$$(\xi \times \eta)(f) = (\xi^\mu(\partial_\mu \eta^\nu)\partial_\nu - \eta^\mu(\partial_\mu \xi^\nu)\partial_\nu)(f).$$

The Lie algebra of *infinitesimal diffeomorphisms* Ξ can be embedded into its universal enveloping algebra which we want to denote by $\mathscr{U}(\Xi)$. The universal enveloping algebra is an associative algebra and possesses a natural Hopf algebra structure. The coproduct is defined as follows on the generators[2]:

$$\begin{aligned} \Delta: \mathscr{U}(\Xi) &\to \mathscr{U}(\Xi) \otimes \mathscr{U}(\Xi) \\ \Xi \ni \xi &\mapsto \Delta(\xi) := \xi \otimes 1 + 1 \otimes \xi. \end{aligned} \tag{3.1}$$

For a precise definition and more details on Hopf algebras we refer the reader to text books [5]. For our purposes it shall be sufficient to note that the coproduct implements how the Hopf algebra acts on a product in a representation algebra (Leibniz-rule). Scalar fields are defined by their transformation property with respect to infinitesimal coordinate transformations:

$$\delta_\xi \phi = -\xi \phi = -\xi^\mu(\partial_\mu \phi). \tag{3.2}$$

The product of two scalar fields is transformed using the Leibniz-rule

$$\delta_\xi(\phi\psi) = (\delta_\xi \phi)\psi + \phi(\delta_\xi \psi) = -\xi^\mu(\partial_\mu \phi\psi) \tag{3.3}$$

such that the product of two scalar fields transforms again as a scalar.

Similarly one studies tensor representations of $\mathscr{U}(\Xi)$. For example vector fields are introduced by the transformation property

$$\begin{aligned} \delta_\xi V_\alpha &= -\xi^\mu(\partial_\mu V_\alpha) - (\partial_\alpha \xi^\mu)V_\mu \\ \delta_\xi V^\alpha &= -\xi^\mu(\partial_\mu V^\alpha) + (\partial_\mu \xi^\alpha)V^\mu. \end{aligned}$$

4. Deformed Diffeomorphisms

The concepts introduced in the previous subsection can be deformed in order to establish a consistent tensor calculus on the noncommutative space-time algebra (1.2). In this context it is necessary to account the full Hopf algebra structure of the universal enveloping algebra $\mathscr{U}(\Xi)$.

We want to deform the structure map (3.1) of the Hopf algebra $\mathscr{U}(\Xi)$ in such a way that the resulting deformed Hopf algebra which we denote by $\mathscr{U}(\hat{\Xi})$ consistently acts on $\hat{\mathscr{A}}$. Let $\mathscr{U}(\hat{\Xi})$ be generated as algebra by elements $\hat{\delta}_\xi$, $\xi \in \Xi$. We leave the algebra relation undeformed

$$[\hat{\delta}_\xi, \hat{\delta}_\eta] = \hat{\delta}_{\xi \times \eta} \tag{4.1}$$

but we deform the co-sector

$$\Delta \hat{\delta}_\xi = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma}(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma}, \tag{4.2}$$

[2]The structure maps are defined on the generators $\xi \in \Xi$ and the universal property of the universal enveloping algebra $\mathscr{U}(\Xi)$ assures that they can be uniquely extended as algebra homomorphisms (respectively anti-algebra homomorphism in case of the antipode S) to the whole algebra $\mathscr{U}(\Xi)$.

where $[\hat{\partial}_\rho, \hat{\delta}_\xi] = \hat{\delta}_{(\partial_\rho \xi)}$. The deformed coproduct (4.2) reduces to the undeformed one (3.1) in the limit $\theta \to 0$. We need a differential operator acting on fields in $\hat{\mathscr{A}}$ which represents the algebra (4.1). Let us consider the differential operator

$$\hat{X}_\xi := \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1} \cdots \theta^{\rho_n\sigma_n} (\hat{\partial}_{\rho_1} \cdots \hat{\partial}_{\rho_n} \xi^\mu) \hat{\partial}_\mu \hat{\partial}_{\sigma_1} \cdots \hat{\partial}_{\sigma_n}. \tag{4.3}$$

Then indeed we have

$$[\hat{X}_\xi, \hat{X}_\eta] = \hat{X}_{\xi\times\eta}. \tag{4.4}$$

It is therefore reasonable to introduce scalar fields $\hat{\phi} \in \hat{\mathscr{A}}$ by the transformation property

$$\hat{\delta}_\xi \hat{\phi} = -(\hat{X}_\xi \hat{\phi}).$$

Let us work out the action of the differential operators $\hat{X}_\xi$ on the product of two fields. A calculation [2] shows that

$$(\hat{X}_\xi(\hat{\phi}\hat{\psi})) = \mu \circ (e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{X}_\xi \otimes 1 + 1 \otimes \hat{X}_\xi) e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma} \hat{\phi} \otimes \hat{\psi}).$$

This means that the differential operators $\hat{X}_\xi$ act via a *deformed Leibniz rule* on the product of two fields. Comparing with (4.2) we see that the deformed Leibniz rule of the differential operator $\hat{X}_\xi$ is exactly the one induced by the deformed coproduct (4.2):

$$\hat{\delta}_\xi(\hat{\phi}\hat{\psi}) = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi) e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{\phi}\hat{\psi}) = -\hat{X}_\xi \triangleright (\hat{\phi}\hat{\psi}).$$

Hence, the deformed Hopf algebra $\mathscr{U}(\hat{\Xi})$ is indeed represented on scalar fields $\hat{\phi} \in \hat{\mathscr{A}}$ by the differential operator $\hat{X}_\xi$. The scalar fields form a $\mathscr{U}(\hat{\Xi})$-module algebra.

Up to now we have seen the following:

- Diffeomorphisms are generated by vector-fields $\xi \in \Xi$ and the universal enveloping algebra $\mathscr{U}(\Xi)$ of the Lie algebra Ξ of vector-fields possesses a natural Hopf algebra structure defined by (3.1).
- The algebra of scalar fields $\phi \in \mathscr{A}$ is a $\mathscr{U}(\Xi)$-module algebra.
- The universal enveloping algebra $\mathscr{U}(\Xi)$ can be deformed to a Hopf algebra $\mathscr{U}(\hat{\Xi})$ defined in (4.1,4.2).
- $\mathscr{U}(\hat{\Xi})$ consistently acts on the algebra of noncommutative functions $\hat{\mathscr{A}}$, i.e. the algebra of noncommutative functions is a $\mathscr{U}(\hat{\Xi})$-module algebra.
- Regarding $\mathscr{U}(\hat{\Xi})$ as the underlying "symmetry" of the gravity theory to be built on the noncommutative space $\hat{\mathscr{A}}$, we established a full tensor calculus as representations of the Hopf algebra $\mathscr{U}(\hat{\Xi})$.

5. Noncommutative Geometry

Based on deformed diffeomorphisms it is possible to introduce covariant derivatives, curvature and torsion and to define a metric [2, 3, 4]. This leads to a curvature scalar. Introducing in addition the star-determinant of the vierbein, one can construct a Einstein-Hilbert action which is invariant with respect to deformed diffeomorphisms. It is a deformation of the usual Einstein-Hilbert action. Using the star-product formalism it is possible to map the algebraic quantities to functions depending on commutative variables. Then it is possible to study explicitly deviations of the undeformed theory in orders of a deformation parameter [4, 2]. Very interesting is also to study a generalization of the above concepts to a more general class of noncommutative structures given by a twist [3]. This class contains in particular lattice-like spacetime algebras which may indeed provide a regularization of the field theory under consideration.

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Chapter 7

Summary and Conclusions

In this thesis we studied field theories on noncommutative spaces. We started by introducing and reviewing the concepts underlying the construction of deformed field theories. In Section 2.1 we presented some examples of noncommutative spaces. We learned that noncommutative spaces are in general not covariant with respect to undeformed symmetries and constructed in Section 2.3 deformed symmetries given by Quantum Groups, which act consistently on these spaces. In Chapter 3 we saw how an abstract noncommutative space can be represented on the more familiar, ordinary space of functions on a manifold by introducing a new, noncommutative product, called $\star$-product. In the same way we showed how operators acting on a noncommutative space can be represented as pseudo-differential operators acting on this algebra of functions. In Chapter 4 we introduced deformation by twists. Twists lead to quite a large class of $\star$-products defining a rich class of noncommutative spaces. We have seen that the noncommutative spaces discussed in Section 2.1 belong to this class. Hence, the deformation of the algebra of functions on a manifold by a $\star$-product coming from a twist yields a way to generalize our concept of noncommutative spaces. Furthermore, a deformed symmetry acting on this class of deformed spaces could be introduced: For any twist $\mathcal{F}$, which is an element in $U\Xi \otimes U\Xi$, where $U\Xi$ is the universal enveloping algebra of vector fields, we are able to construct the twisted Hopf algebra $U\Xi^{\mathcal{F}}$. We often call it deformed infinitesimal diffeomorphisms since vector fields generate diffeomorphisms.

In Chapters 5 and 6 we applied the learned tools in order to construct gauge respectively gravity theories on noncommutative spaces. In [35, 36] (Sections 5.1 and 5.2) and [34] (Section5.3) we presented two possibilities to circumvent the problems appearing when trying to construct gauge theories via the Seiberg–

Witten formalism on noncommutative spaces, which have non-trivial derivatives with deformed Leibniz rules. The first possibility is to gauge the commuting frame instead of the partial derivatives [35–37]. This is only possible if the algebra possesses a commuting frame, which in general is not the case, see also [38]. The second possibility proposed in [34, 64] is to consider derivative valued gauge fields. In this approach, a Lagrangian describing the dynamics of the gauge field can be obtained by projecting away torsion-like terms. In Section 5.6, [65], we introduced a model for gauge theories on fuzzy $S^2 \times S^2$. Fuzzy spaces retain the undeformed rotational invariance such that this is an example of a noncommutative space with an undeformed symmetry acting on it. This model reduces in a certain double scaling limit to noncommutative gauge theories on θ-deformed spaces. It therefore serves as a regularization of gauge theories on noncommutative $\mathbb{R}^4$. Nontrivial topological solutions (instantons) become instanton solutions for gauge theories on θ-deformed $\mathbb{R}^4$ in this double scaling limit.

Chapter 6 was devoted to a new approach towards a deformation of Einstein's general relativity. The construction is based on deformed infinitesimal diffeomorphisms. A full tensor calculus could be established, covariant with respect to these deformed coordinate transformations. In the θ-deformed case, this gives rise to a deformed Einstein–Hilbert action [42, 63, 66–68] (Sections 6.1 and 6.2). This action reduces in the commutative limit to the usual, undeformed Einstein–Hilbert action. The first non-trivial contribution in the deformation parameter θ was determined for all relevant quantities, including the action. In [43] (Section 6.3) this model was generalized. We constructed a noncommutative geometry on noncommutative algebras of functions whose product is given by a $\star$-product coming from a twist. This leads to a deformation of Einstein's equations for gravity on this large class of noncommutative spaces. It is still not clear in which precise sense a Quantum Group invariance restricts the degrees of freedom of our theory. The physical meaning of a deformed coproduct and deformed Leibniz rules, which make a derivation of conserved Noether currents in the standard way impossible, is still not well enough understood. In [39] we give first answers to these questions and show how conserved currents can be constructed for twisted gauge theories. We start with an arbitrary Lie algebra and see that consistency of the equations of motion requires us to choose the gauge field in the enveloping algebra. This leads to new fields in addition to the usual gauge field, which couple only weakly via the deformation parameter θ and reduce in the commutative limit to free fields. The new fields depend on the representation and their number is finite if we choose a finite dimensional representation for the enveloping algebra. The results obtained for the construction of twisted gauge theories enable us to introduce matter in the models of noncommutative gravity proposed in [42, 43]. Via the first order formalism for gravity the construction of conserved currents should also lead to Ward identities for such noncommutative gravity theories.

We can make out quite a progress in our understanding of physics on noncommutative spaces: Whereas in first attempts to construct deformed field theories special examples of noncommutative spaces such as the θ-deformed space, the q-deformed Euclidean plane or the κ-deformed space were considered, we are now able to understand noncommutativity on more and more general grounds. An important step in this direction was made by the construction of deformed gravity on the class of noncommutative spaces coming form a twist. A further generalization would be to consider the algebra of functions on arbitrary Poisson manifolds together with a Kontsevich $\star$-product [31]. First results towards this aim have already been obtained [117]. A complete construction of field theories and of deformed gravity in this setting is part of current and future investigations.

The class of $\star$-products obtained by twists is of particular interest since it contains noncommutative spaces with a lattice-like structure. In the case of θ-deformed spaces the eigenvalue spectrum of the coordinates is continuous, as the commutation relations are given by Heisenberg algebras. However, more complicated noncommutative spaces such as q-deformed spaces lead to a discrete eigenvalue spectrum for the coordinates and therefore to a lattice-like space-time [123]. Such spaces are contained in the class of twist-deformed spaces and should indeed provide a regularization of quantum field theories. On the classical level quite some progress has been made here. However, the quantum behaviour of such gravity theories is still completely unknown. Another compelling question, for instance, is the relation of deformed gravity to string theory. In [9] a connection between string theory and noncommutative gauge theories could be established. In current investigations we aim at finding such a relation between string theory and noncommutative gravity [124]. This shall provide a detailed connection between the notion of a fundamental length encompassed in noncommutative theories, with the ultraviolet finiteness incorporated in string theory. We expect new insight about physics at very short distances from a better understanding of gravity on noncommutative spaces, in particular on those with a lattice-like structure, and its interplay with string theory. We hope that the results obtained in the framework of this thesis will prove to be of fundamental importance for the understanding of these issues.

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