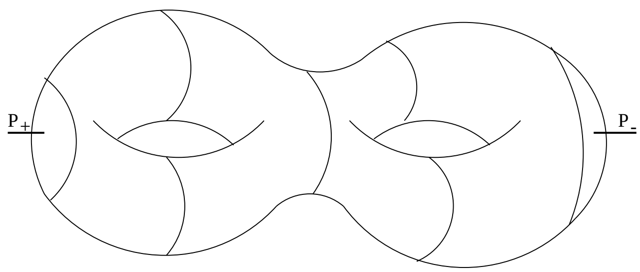
Global Vertex Algebras on Riemann Surfaces



Dissertation
an der Fakultät für Mathematik und Informatik
der Ludwig-Maximilians-Universität München

vorgelegt von
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(München)
2004

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Datum der mündlichen Prüfung: 23. August 2004

Acknowledgements

I wish to express my sincere thanks to Professor Martin Schottenloher for supervising this thesis. I thank him for his patient guidance and also for his recommendations for my stays in the USA and Great Britain.
I would like to express my gratitude to the following people for their support and assistance in writing this thesis:
Dr York Sommerhäuser for organizing a seminar about vertex algebras and for useful discussions concerning this topic.
Roland Friedrich (IAS) for useful discussions.
During my stay in Liverpool from march 2003 to september 2003 I benefitted from discussions with Slava Nikulin and Hugh Morton.
Furthermore I want to express my gratitude to Peter Newstead and Ann Newstead for their hospitality in Liverpool.
I acknowledge the financial support from the EU (Marie Curie Training Site).
Furthermore I acknowledge the financial support from the Ludwig Maximilians Universität (Stipendium nach dem Gesetz zur Förderung des wissenschaftlichen Nachwuches).
Moreover I want to thank the Institute for Pure and Applied Mathematics (IPAM) at the University California of Los Angeles (UCLA) for hospitality in 2002, and I want to thank Professor Giesecker for organizing a semester lasting course in conformal field theory.
I am greatful to Oleg Sheinman for correspondence and Martin Schlichenmaier for sending me his thesis and some reprints.
On a more personal level I would like to thank PD Dr Peter Schuster and Professor Zöschinger for their moral support.
Furthermore I would like to thank Dr Johanna Graf for her friendship and Michael Retter for coffee and talks.
And lastly, which is the most important place, I want to express my gratitude to two very important persons in my life. I want to thank Dr Paul-C. Streidl for his seemingly unlimited belief in me, and especially Kathrin Brunner for her patience and love. Without them this thesis would not have been completed.

Zusammenfassung

Konforme Feldtheorie is eng mit der Theorie der Vertex-Algebren und der Geometrie Riemannscher Flächen verknüpft.
In der vorliegenden Arbeit wird eine neue algebro-geometrische Struktur, genannt Globale Vertex-Algebra, auf Riemannschen Flächen definiert, die als natürliche Verallgemeinerung von Vertex-Algebren verstanden wird.
Dazu wird ein Formaler Kalkül von Feldern auf Riemannschen Flächen entwickelt. Als Beispiel für eine solche Struktur wird die Globale Vertex-Algebra für Bosonen vom Krichever-Novikov-Typ konstruiert.
Zu Beginn der Arbeit wird der Formale Kalkül für die klassischen Vertex-Algebren unter dem Gesichtspunkt von Distributionen in der komplexen Analysis dargestellt.
Darüber hinaus wird ein graphischer Kalkül zur Berechnung von Korrelationsfunktionen von Primärfeldern assoziiert zu affinen Kac-Moody-Algebren vorgestellt.

Abstract

Conformal field theory is intimately connected to the theory of vertex algebras and the geometry of Riemann surfaces.
In this thesis a new algebro-geometric structure called global vertex algebra is defined on Riemann surfaces which is supposed to be a natural generalization of vertex algebras.
In order to define this structure a formal calculus of fields on Riemann surfaces is constructed. The basic objects in vertex algebra theory are fields. They are defined as formal Laurent series with possibly infinite principal part. The coefficients are endomorphisms.
As an example for such a structure the global vertex algebra of bosons of Krichever-Novikov type will be constructed.
At the beginning of this thesis the formal calculus of classical vertex algebras is introduced from the viewpoint of distributions in complex analysis.
Furthermore a graphical calculus for the computation of correlation functions of primary fields associated to affine Kac-Moody algebras is introduced.

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne unerlaubte Beihilfe (im Sinne von Paragraph 5 Absatz 6 der Promotionsordnung der Ludwig-Maximilians-Universität für die Fakultät für Mathematik, Informatik und Statistik vom 15. Januar 2002) angefertigt habe.

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Chapter 0

Introduction

0.0 The Thesis

In this thesis I define the new structure of a **global vertex algebra** on compact Riemann surfaces, and I present a non-trivial example of this structure, namely the Heisenberg algebra of Krichever-Novikov type. I furthermore discuss the correlation functions of WZNW-fields by introducing a pictorial scheme.
When a common structure is generalized there must be a reasonable correspondence to the old structure that has been generalized. Therefore I first present the theory of vertex algebras and its formal calculus upon which it depends.
In order to come to these points I establish the ideas and the notions of vertex algebras, its relevance in conformal field theory and in certain areas of mathematics.
This thesis is in the realm of conformal field theory. The topic of the thesis was motivated by the fact that the suitable framework of conformal field theory as vertex algebras are only defined on the Riemann sphere, but the main goal of conformal field theory is to consider fields and eventually correlation functions on higher genus Riemann surfaces.

0.1 Conformal Field Theory

Since the celebrated paper of Belavin, Polyakov and Zamolodchikov [BPZ] conformal field theory has shown that it sits at the junction of several mathematical and physical roads. From the physical point of view conformal field theories (CFT for short) are the suitable toy models for genuinely interacting quantum field theories, furthermore they describe two-dimensional critical phenomena in statistical physics, and they play a central role in string theory, which is at present supposed to be the most promising candidate for a unifying theory of all forces.
From the mathematical point of view conformal field theory gave and is still giving illuminating inspirations in order to give answers to hard questions in

algebraic geometry, representation theory of infinite dimensional Lie algebras, knot theory, finite group theory, and classical complex analysis.

String Theory

String theory is a quantum field theory in which the basic objects are not point particles but one dimensional manifolds (strings). These strings can either form closed loops or they can have two end-poins. Naively string theory describes the vibration of strings which propagate through the space. The different vibrational modes are interpreted as the particles of the theory.

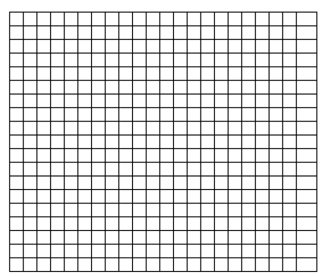
Mathematically we think of a map $f : X \to S$ as describing the propagation of a string through the space S, and X is the world sheet swept out by the string as it propagates. The world sheet X is a Riemann surface.

In other words, allowing Riemann surfaces of arbitrary genus automaticaly includes interactions between strings in the theory. This is a general feature in string theory.

The fields "living" on these Riemann surfaces are operator valued "densities". The operator algebra of these fields reflect the underlying symmetry algebra.

Critical Phenomena

In the description of statistical mechanics we also come across with conformal field theory. This can be illustrated in the example of the **Ising model**. This model consists of a two-dimensional lattice whose lattice sites represent atoms of an two-dimensional crystal.



Each atom is taken to have a spin variable σ_i that can take the values ± 1, and the magnetic energy is given by the sum over the pairs of adjacent atoms

$$E = \sum_{(ij)} \sigma_i \sigma_j .$$

The thermal average behaves exponentially that depends on a parameter ξ, called the correlation length. However this system possesses a critical temperature, at which the correlation length ξ diverges, and the thermal average behaves at the critical temperature in a power law. The continuum theory that describes the correlation functions for distances that are large compared to the lattice spacing is then scale invariant. Due to an important paper of Cardy [C] every scale invariant two-dimensional local quantum field theory is conformally invariant. So roughly speaking the two-dimensional models of statistical mechanics at the critical point can be described by a conformal field theory.

Conformal Field Theory and Mathematics
The study of conformal field theory is intimately connected to infinite dimensional Lie algebras. The most prominent Lie algebra in CFT is the Virasoro algebra. It is the central extension of the Witt algebra, the algebra of polynomial vector fields on the circle.
Furthermore there are the (non-twisted) affine Kac-Moody algebras. They are central extensions of current algebras associated to simple finite dimensional Lie algebras.
From the mathematical point of view it is amazing, how many answers from physicists can be given to rather involved questions in mathematics. There is for instance the **Verlinde formula** which gives the answer to the question about the dimension of the space of section of a certain line bundle on the moduli space of stable vector bundles on Riemann surfaces.
In the paper [B0] Richard Borcherds introduced the notion of a vertex algebra. This was the starting point for Frenkel, Lepowsky and Meurman to construct the **moonshine module** as an orbifold conformal field theory. It turned out that the j-function in the theory of elliptic curves can be considered as the partition function of a conformal field theory.
Furthermore Borcherds [B1] used the **no-ghost theorem** of string theory by proving the so called **moonshine conjectures** of Conway and Thompson. This proof inspired Borcherds to erect a theory of modular forms as infinite products. These products are called **Borcherds products** and they provide new insights and methods in classical complex analysis as well as in algebraic geometry (see chapter 4 for a review).

0.2 Vertex Algebras

Vertex algebras provide the suitable framework in which the considerations of the preceeding section can be formulated rigorously. One can say that a vector space that satisfies the properties of a vertex algebra is a conformal field theory. The fascinating aspect of vertex algebras is their appearance in seemingly different areas of mathematics. The name "vertex algebras" explains its origin from physics. But the first spectacular application of vertex algebras has been made in group theory [FLM].
The central objects of vertex algebras are **fields**. They are endomorphism valued power series $a(z) \in End(V)[[z, z^{-1}]]$ such that $a(z)v \in V[[z]][z^{-1}]$ are formal Laurent series with values in the vector space V.
More precisely: For all $v \in V$ there exists an $n_0 \in \mathbb{Z}$ such that $v_n(v) = 0$ for all $n \geq n_0$. The product of two fields is given as the **operator product expansion** (OPE). The OPE of two fields can be given by the formula

$$a(z)b(w) = \sum_{j<<0} \frac{a(w)_j b(w)}{(z-w)^{j+1}},$$

where $a(w)_j b(w)$ are some fields which may be viewed as bilinear products of fields a and b for all $j \in \mathbb{Z}$.
Roughly speaking a vertex algebra is a vector space V with a distinguished vector v_0, the **vacuum vector**, and an endomorphism $T : V \to V$, the **translation operator**. Furthermore we have a mapping, called the **state field correspondence**, from V to the space of formal power series in z and z^{-1} with $End(V)$-valued coefficients:

$$Y : V \to End(V)[[z, z^{-1}]].$$

The state field correspondence has to satisfy certain conditions. Among them there is the translation axiom

$$\partial_z Y(a, z) = [T, Y(a, z)]$$

and the locality axiom

$$[Y(a, z), Y(b, w)](z - w)^n = 0$$

with $a, b \in V$, and n dependent on a and b.

Formal Calculus
Starting point in the theory of vertex algebras is the calculus of formal distributions in two variables, namely series of the form

$$a(z, w) = \sum_{n \in \mathbb{Z}} a_{n,m} z^{-n-1} w^{-m-1}.$$

I emphasize in this thesis an approach to the theory of formal distributions from the viewpoint of the theory of distributions in complex analysis. In complex analysis distributions can be defined as boundary values of holomorphic functions. They can be interpreted as "jumps" of holomorphic functions expanded in different domains.
There exists an important lemma in the theory of formal distributions: If the distributions are supported on the diagonal, i.e. if $a(z, w) \cdot (z - w)^N = 0$ then they can be expanded by

$$a(z, w) = \sum_{j=0}^{N-1} c^j(w) \partial_w^j \delta(z - w)$$

where $\delta(z - w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$ and the derivatives are formal derivatives.
This lemma is the essential fact on which the important calculations for **operator product expansions** can be built.
The tempting question whether any formal distribution $a(z, w)$ can be expanded in a possibly infinite series in the form of

$$a(z, w) = \sum_{j=0}^{\infty} c^j(w) \partial_w^j \delta(z - w)$$

has a negative answer.
In the first edition of Kac' book **Vertex algebras for beginners** [K] this was implicitly assumed to be true. I give in chapter 1 a counterexample to disprove this assumption.
One important notion in formal calculus is the **normal ordered product**. If we divide the formal distribution $a(z) = \sum_n a_n z^{-n-1}$ into two parts:

$$a_+(z) + a_-(z) = \sum_{n<0} a_n z^{-n-1} + \sum_{n\geq 0} a_n z^{-n-1},$$

then we can define the normal ordered product by

$$: a(z)b(w) : \stackrel{\text{def}}{=} a_+(z)b(w) + b(w)a_-(z).$$

We can iterate this definition

$$: a^1(z)a^2(z)...a^n(z) : \stackrel{\text{def}}{=} : a^1(z)(: a^2(z)...a^n(z) :) : .$$

In section 2 of the first chapter an explicit expression for the iterated normal ordered product is presented:

$$: a^1(z)a^2(z)...a^n(z) := \sum_{p=0}^{n} \sum_{\sigma} a_+^{\sigma(1)}(z)...a_+^{\sigma(p)}(z)a_-^{\sigma(p+1)}(z)...a_-^{\sigma(n)}(z)$$

where σ runs over all permutations of the symmetric group S_n such that

$$\sigma(1) < \sigma(2) < ... < \sigma(p) \text{ and } \sigma(p+1) > \sigma(p+2) > ... > \sigma(n).$$

I call such a permutation a $(p, n-p)$-**anti-shuffle** (where the $(n, 0)$-anti-shuffle is the identity and the $(0, n)$-anti-shuffle is the reversion of the set $\{1, 2, .., n\}$). The notion "shuffle" is well known for instance in the realm of Hopf algebras and quantum groups (see e.g. [Kas]).

Examples
1. The simplest example is the vertex algebra associated to the Fock representation of the Heisenberg algebra. It provides a structure theorem how to built from certain data vertex algebras generally. Therefore the Heisenberg vertex algebra can be considered (in biological terms) as the phylogenetic starting point of vertex algebras. That justifies why I call the corresponding chapter a **trip into the zoo**.
2. Further typical examples of conformal field theories are free fermions (also called b-c systems), and Kac-Moody algebras.
Roughly speaking vertex algebras are built around representations of these algebras in order to define a conformal field theory with the respective underlying symmetry algebra.
The Heisenberg algebra is the central extension of the current algebra of the one-dimensional abelian Lie algebra $\mathbb{C}$. Let $\mathbb{C} \otimes \mathbb{C}[t, t^{-1}]$ denote the **current**

algebra associated to $\mathbb{C}$. The commutator relations of the central extension are given by

$$[a \otimes t^n, a \otimes t^m] = n\delta_{n,-m}K,$$

where K is the central element.
The Fock representation of the Heisenberg algebra is given by the vector space of polynomials in countably many variables over the field $\mathbb{C}$: $V := \mathbb{C}[x_{-1}, x_{-2}, ...]$. The Heisenberg algebra acts on this space as follows

$$\begin{array}{rl} a_n = a \otimes t^n \mapsto \frac{\partial}{\partial x_{-n}} & \text{if } n > 0 \\ a_n = a \otimes t^n \mapsto x_n \cdot & \text{if } n < 0 \\ a_0 = a \otimes t^0 \mapsto 0 \cdot & \end{array}$$

The vector space V is generated by the elements

$$a_{-n_1}...a_{-n_k}1 \text{ for all } n_i > 0.$$

This vector space satisfies the axioms of a vertex algebra by the state field correspondence

$$a_{-n_1}...a_{-n_1}1 \mapsto: \partial^{(n_1-1)}a(z)...\partial^{(n_k-1)}a(z) :$$

where $\partial^{(n)}a(z)$ is the n-th derivative of the field $a(z)$, and $a(z) = \sum a_n z^{-n-1}$ is the "generating" function for the operators a_n.
The next well-studied examples are the lattice vertex algebras. These are related to the Heisenberg algebra, but the abelian Lie algebra $\mathbb{C}$ is replaced by a finite dimensional complex vector space $\mathfrak{h}$ together with a bilinear form $(\cdot, \cdot)$. The resulting vertex algebra is in physicists terms the **toroidal compactification** [Gawedzky].
We give in the table below an overview of some examples:

physics notion	mathematics notion
Free Fermions	Clifford algebra
Free Bosons	Heisenberg algebra
WZNW models	non-twisted affine Kac-Moody algebras

One very prominent example is the **moonshine module** of the monster group. It is essentially a lattice vertex algebra associated to a 24-dimensional complex vector space with the bilinear form induced by the Leech lattice. In the book [FLM] Frenkel, Lepowsky and Meurman proved that the automorphism group of this vertex algebra factorized by an element of order two is the monster group, the largest simple sporadic finite group.

0.3 Krichever-Novikov Algebras

Krichever and Novikov started in a series of three papers [KN1][KN2][KN3] a program to extend the notions of Heisenberg, affine Kac-Moody and Virasoro algebras to higher genus Riemann surfaces. I shall explain their basic philosophy

for affine algebras.
For $\mathfrak{g}$ given a simple finite-dimensional Lie algebra with Killing form $(\cdot,\cdot)$ the central extension $\widehat{\mathfrak{g}}$ of the current algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is obtained by adding a central element K. That means $\widehat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathbb{C}K$ as a vector space, and the structure equation

$$[a \otimes z^n, b \otimes z^m] = [a, b] \otimes z^{n+m}, \qquad a, b \in \mathfrak{g}$$

is replaced by

$$[a \otimes z^n, b \otimes tz^m] = [a, b] \otimes z^{n+m} + n \cdot (a, b)\delta_{n,-m}K, \qquad [a \otimes z^n, K] = 0, \quad a, b \in \mathfrak{g}.$$

The algebra $\widetilde{\mathfrak{g}}$ can be described as the Lie algebra of $\mathfrak{g}$-valued meromorphic functions on the Riemann sphere and holomorphic outside 0 and ∞.
Starting from this description the natural extension to higher genus compact Riemann surfaces X is to replace the algebra $\mathbb{C}[z, z^{-1}]$ by the commutative algebra $\mathcal{A}(X, P_\pm)$ of meromorphic functions on X which are holomorphic outside two (generic) points P_+ and P_-.
That means: For $\mathfrak{g}$ given a simple finite-dimensional Lie algebra with Killing form $(\cdot,\cdot)$ the current algebra $\widetilde{\mathfrak{g}}$ of higher genus is defined as $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A}(X, P_\pm)$.
The algebra $\mathcal{A}(X, P_\pm)$ has a basis $\{A_n\}_{n\in\mathbb{Z}}$ similar to the monomials z^n in the algebra $\mathbb{C}[z, z^{-1}]$. However there is an important difference: The algebra $\mathbb{C}[z, z^{-1}]$ is graded. The grading is given by the degree of the monomials z^n. The algebra $\mathcal{A}(X, P_\pm)$ is only **quasigraded** in the sense that the product of two elements $A_n \cdot A_m$ gives

$$A_n \cdot A_m = \sum_{k=n+m-C}^{n+m+\tilde{C}} \alpha_{n,m}^k A_k.$$

The structure equation of the current algebra of higher genus is given by

$$[a \otimes A_n, a \otimes A_m] = \sum_{k=n+m-c}^{n+m+c} \alpha_{n,m}^k [a, b] \otimes A_k, \qquad a, b \in \mathfrak{g}.$$

There also exists an appropriate notion of central extension of the current algebra $\widetilde{\mathfrak{g}}$. The central extension is $\widehat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathbb{C}K$ as a vector space, and the structure equation is given by

$$[a \otimes t^n, b \otimes t^m] = \sum_{k=n+m-c}^{n+m+c} \alpha_{n,m}^k [a, b] \otimes A_k + \gamma(n, m)K, \qquad a, b \in \mathfrak{g},$$

where $\gamma(n, m)$ is a Lie algebra cocycle.
The original motivation of Krichever and Novikov was to extend especially the Virasoro algebra to higher genus compact Riemann surfaces. The Virasoro algebra can be regarded as the central extension of meromorphic vector fields on

the Riemann sphere which are holomorphic outside the points 0 and ∞. Starting from this description it is natural to consider the algebra $\mathcal{L}$ of meromorphic vector fields holomorphic outside two (generic) points P_+ and P_-. This algebra has a basis $\{e_n\}$. The Lie algebra strucure is given by

$$[e_n, e_m] = \sum C_{nm}^k e_k.$$

The central extension of the algebra $\mathcal{L}$ is $\mathcal{L} \oplus \mathbb{C}K$ with the structure equation

$$[e_n, e_m] = \sum C_{nm}^k e_k + \chi_{nm} K.$$

The most relevant algebra of Krichever-Novikov type for this thesis is the Heisenberg algebra of KN-type.

0.4 Global Vertex Algebras

Theta Functions, Prime Form, and Szegö Kernels
The formal calculus of higher genus relies on existence of certain differentials on Riemann surfaces X and its square $X \times X$. In order to construct them explicitely we have to state some results in the theory of theta functions on Riemann surfaces.
1. Theta functions are convenient building blocks in order to construct forms and functions on Riemann surfaces explicitely.
If X is a compact Riemann surface of genus $g \geq 1$ with normalized period matrix Ω (see chapter 6 for details). Then the series

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{-n^t \Omega n + 2\pi i n^t z}$$

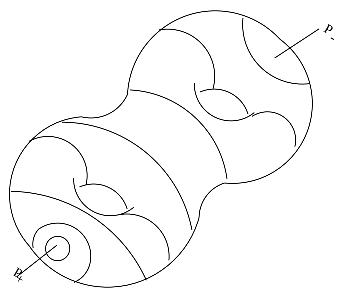
defines a holomorphic function on the space $\mathbb{C} \times \mathbb{H}_g$ where $\mathbb{H}_g$ is the space of complex valued $g \times g$ matrices with positive definite imaginary part (called the Siegel upper half space).
Due to the Abel-Jacobi map $X \to J(X)$ ($J(X)$ is the Jacobi variety) this function can be considered as a section in a line bundle on this Riemann surface.
2. We need for later purposes the notion of the **prime form** $E(P, Q)$ which is the suitable analogue of $(z - w)$ on the Riemann sphere. More precisely the prime form satisfies the properties

1. $E(P, Q) = -E(Q, P)$
2. $E(P, Q) = 0$ only for $P = Q$

It can be defined in terms of theta functions.
3. In the formal calculus for the vertex algebras we came across with the delta distribution which is the formal difference of the expansions of $\frac{1}{z-w}$.
In the higher genus case we need a similar expression. This expression is the Szegö kernel. It looks locally like

$$S(z, w) = \frac{\sqrt{dz}\sqrt{dw}}{z - w} + \text{ higher order terms of } (z - w)$$

The crucial point is that it is possible to expand the Szegö kernel in different domains. These domains are given by the so called **level lines** which are cycles on the Riemann surface. One can define a level on these cycles in order to simulate a kind of norm on the Riemann surface. This means the level of a point P is smaller than the level of a point Q if the levels $\tau(P)$, $\tau(Q)$ of the level lines on which they are lying satisfy $\tau(P) < \tau(Q)$ (see the picture below, and see chapter 7 for more details).



In chapter 7 I shall show that the Szegö kernel associated to certain line bundles can be expanded in the domains $\tau(P) < \tau(Q)$ in terms of the Krichever-Novikov forms

$$S(z, w) = \sum_{n \geq s_\lambda} f_{\lambda,n}(P) f^n_{1-\lambda}(Q).$$

Formal Calculus on Higher Genus Riemann Surfaces

Fields are formal power series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ with coefficients in the algebra of endomorphisms of a vector space: $a_n \in End(V)$ and with the additional property: $a_n v = 0$ for $n >> 0$. We want to introduce an analogous object of higher genus Riemann surfaces. For this purpose we replace the monomials $z^{-n-\lambda}$ by the Krichever-Novikov forms $f^n_\lambda(P)$.

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \rightsquigarrow a(P) = \sum_{n\in\mathbb{Z}'} a_n f^n_\lambda(P).$$

($\mathbb{Z}'$ means $\mathbb{Z}$ if the genus of X is even, and $\mathbb{Z} + \frac{1}{2}$ if the genus is odd)
We are especially interested in the forms $\lambda = 1$. We introduce:

$$\omega^n(P) := f^n_1(P).$$

So the fields of conformal weight $\lambda = 1$ are given by

$$\sum_{n\in\mathbb{Z}'} a_n \omega^n(P).$$

An important notion is the higher genus analogue of the normal ordered product introduced in this thesis. The classical normal ordered product is given by

$$: a(z)b(z) := a_+(z)b(z) + b(z)a_-(z).$$

We mimmick this definition as follows:

$$: a_\lambda(P) b_\mu(P) := a_{\lambda,+}(P) b_\mu(P) + b_\mu(P) a_{\lambda,-}(P).$$

Again I can deduce an explicit expression of the iterated normal ordered product. One important feature is the formal delta distribution. It is understood as the formal sum of the expansion of the Szegö kernel in the two different domains:

$$\Delta_\lambda(P,Q) = i_{P,Q} S_\lambda(P,Q) - i_{Q,P} S_\lambda(P,Q).$$

This is analogous to the genus $g = 0$ case

$$\delta(z,w) = i_{z,w} \frac{1}{z-w} - i_{w,z} \frac{1}{z-w}$$

Global Vertex Algebras as Higher Genus Analogues

The **global vertex algebras** which are introduced in this thesis have an analogous structure as the vertex algebra, but with the important difference that they are defined on compact Riemann surfaces of higher genus than 0.

The idea is the following: If we regard vertex algebras as the generalization of unital associative commutative algebras by adding a parameter z, then the global vertex algebra is the generalization by adding a parameter z which reflects the geometric structure of the Riemann surface.

A global vertex algebra is a graded vector space $V = \bigoplus V_\lambda$ together with mappings

$$\mathcal{Y} : V_\lambda \to End[[f^n_\lambda(P)]].$$

The fields $\mathcal{Y}(v,P)$ have to satisfy certain conditions. One of them is the translation axiom:

$$\nabla \mathcal{Y}(v,P) = [T, \mathcal{Y}(v,P)]$$

(with ∇ is a certain Lie derivative)

and the locality axiom: For $\tau(P) < \tau(Q)$:

$$\mathcal{Y}(v,P)\mathcal{Y}(v,Q)E(P,Q)^N \text{ holomorphic along the diagonal.}$$

Bosons of Krichever-Novikov Type and Global Vertex Algebras

The main theorem of this thesis says that the Fock representation of the Heisenberg algebra of Krichever-Novikov type carries the structure of a global vertex algebra.

More precisely we have a representation of the central extended algebra $\widehat{\mathcal{A}}(X, P_\pm)$ of the algebra $\mathcal{A}(X, P_\pm)$ of meromorphic functions on the Riemann surface that are holomophic outside the generic points $P_\pm$.

The representation is constucted as follows:

The algebra spaces $\mathcal{F}(X, P_\pm)$ are $\mathcal{A}(X, P_\pm)$-modules.

We define the half infinite wedge space H_λ by using the Krichever-Novikov basis $\{f_\lambda^n\}_{n\in\mathbb{Z}'}$. The vectors in H_λ look as

$$f_\lambda^{n_0} \wedge f_\lambda^{n_1} \wedge f_\lambda^{n_2} \wedge ...$$

where $n_i = i$ for $i >> 0$. This procedure of using the half infinite wedge space is well known (see for instance [KR]).
By a suitable regularization procedure we can define a projective representation. This algebra acts on the space of half-infinite wedge forms. So we obtain a space V together with a distinguished vector v_0. The space V is generated by the elements

$$a_{-n_1-\frac{g}{2}}...a_{-n_k-\frac{g}{2}}v_0 \text{ where all } n_i > 0.$$

The state-field correspondence is given by

$$a_{-n_1-\frac{g}{2}}...a_{-n_k-\frac{g}{2}}v_0 \mapsto : \nabla^{n_1-1}a(P)...\nabla^{n_k-1}a(P) : .$$

where $a(P) = \sum a_n\omega^n(P)$ is the "generating function" of the operators a_n of the algebra $\widehat{\mathcal{A}}(X, P_\pm)$.

0.5 Correlation Functions

The physical relevant entities are the correlation functions. Suppose we have

$$\langle \tilde{v}, a^1(z_1)a^2(z_2)...a^n(z_n)v \rangle.$$

Starting from the operator product expansions of the fields it is possible to compute the correlation functions.
I shall provide a pictorial calculus which facilitates the computation of correlation funcions for primary fields associated to the veretx algebra of a non-twisted affine Kac-Moody algebra. It furthermore illustrates the form of the correlation functions in a better way.
The correlation functions for primary fields of non-twisted affine Kac-Moody algebras with v is the vacuum vector and v' the dual vacuum vector (see chapter 5 for details) is given by the formula

$$\langle a^1(z_1)...a^N(z_N) \rangle = \sum_{\rho\in d_n} f_{\sigma_1^\rho} f_{\sigma_2^\rho} ... f_{\sigma_k^\rho}$$

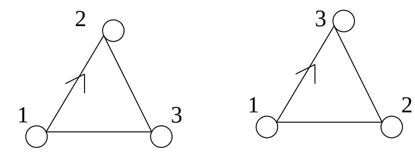
where $\sigma_1^\rho...\sigma_j^\rho = \rho$ are disjoint cycles of the **derangement** ρ. And the factors f_σ are given by

$$f_\sigma = \frac{tr(a^{i_1}...a^{i_m})}{\Delta(z_{i_1}, ..., z_{i_m})},$$

and $\Delta(z_{i_1}, ..., z_{i_m}) = (z_{i_1} - z_{i_2})...(z_{i_m-1} - z_{i_m})(z_{i_m} - z_{i_1})$.
This corresponds pictorially to a graph. We illustrate this for the three-point correlation functions (see chapter 4 for details):

$$\langle a^1(z_1)a^2(z_2)a^3(z_3) \rangle = \frac{tr(a^1a^2a^3)}{(z_1-z_2)(z_2-z_3)(z_3-z_1)} + \frac{tr(a^1a^3a^2)}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}.$$

where $tr(a^{i_1}a^{i_2}...a^{i_k})$ is the trace with respect to a suitable faithful representation of the algebra $\mathfrak{g}$.
The two summands correspond to two graphs:



0.6 Approaches of Conformal Field Theory on Riemann Surfaces

The basic philosophy in CFT over higher genus curves is given by the Segal axioms for conformal field theories [S]. Roughly speaking CFT is in the sense of Segal a functor from the category of parametrized circles (with Riemann surfaces interpolating them as morphisms) to the category of finite-dimensional vector spaces (or at least Hilbert spaces).

Conformal Field Theory on Semi-Stable Curves
The first mathematical rigorous treatment of conformal field theory on the Riemann sphere $\mathbb{P}$ was given by Tsuchiya and Kanie in [TK]. This approach was generalized over any compact Riemann surface by Tsuchiya, Ueno and Yamada [TUY] (see also [U]).
We associate to a compact Riemann surface (with marked points which are labelled by elements of a finite set) a finite dimensional vector space. This space is called the **space of conformal blocks**. The space satisfies furthermore some properties which are called factorisation rules. This approach fits into the above explained philosophy of conformal field theory.

Bundles of Vertex Algebras on Curves and Chiral Algebras
Frenkel and Ben-Zvi [FB-Z] construct bundles of vertex algebras on algebraic curves. Their approach is similar to [TUY]. But it has the disadvantage that they cannot consider curves with nodes and therefore they cannot establish factorisation rules for conformal blocks.
Beilinson and Drinfeld [BD] established the notion of **chiral algebras** which can be considerd as a generalization of vertex algebras.
Frenkel and Ben-Zvi show in their book that the bundles of vertex algebras satisfy the definition of chiral algebras.

0.7 Organization of the Thesis

The Thesis is divided into two main parts. Four additional chapters form the transition from the first part to the second part.

In the first part I recall some notions which are relevant in the theory of vertex algebras.
In the transition chapters I establish a pictorial calculus in order to compute correlation functions for affine Kac-Moody algebras. Furthermore I recall some notions in the theory of theta functions on compact Riemann surfaces, the notions of Krichever-Novikov algebras and their representations.
The second part can be regarded as a mirror of the first part in which I establish the notion of a global vertex algebra. I also construct an example which satisfies the axioms of a global vertex algebra.

The First Part of the Thesis
The first part contains four chapters. In **chapter 1** I establish the formalism of operator product expansion which is important for the theory of vertex algebras.
In **chapter 2** I formulate the notion of current algebras as a preliminary stage for the construction of vertex algebras. In the appendix I present several proves of the Sugawara construction.
In **chapter 3** I give the definition of a vertex algebra, and I give some proves of structure theorems concerning vertex algebras which are sometimes neclected in the literature.
In **chapter 4** I recall examples of vertex algebras. More precisley I show that the Fock representation of the Heisenberg algebra carries the structure of a vertex algebra, and I present the vacuum representation of affine Kac-Moody algebras.

Transition to Higher Genus
In **chapter 5** I discuss the correlation functions for vertex algebras, especially for the free bosons, fermions, and WZW models. For this purpose I develop a pictorial calculus in order to prove the assertions.
In **chapter 6** the relevant parts of the theory of theta functions as bulding blocks for sections of line bundles on abelian varieties are presented. Furthermore I establish notions like the (Schottky-Klein) prime form and the Szegö kernel.
In **chapter 7** I present Krichever-Novikov forms.
In **chapter 8** the relevant notions Krichever-Novikov algebras are introduced. I define the Krichever-Novikov forms and level-lines.
One important feature in this chapter is the result that the Szegö kernels associated to certain bundles can be expanded in terms of KN-forms. I define the Heisenberg algebra of Krichever-Novikov type and discuss the representation of it on the space of half-infinite wedge forms. This is a compulsary technique.

The Second Part of the Thesis
In **chapter 9** I establish some important notions of a formal calculus on higher genus Riemann surfaces. One important notion is the delta distribution (originally introduced by Krichever and Novikov) and my definition of the normal ordered product. I also establish some results concerning the operator product for higher genus.

In **chapter 10** I formulate the notion of a global vertex algebra. It can be seen that this global vertex algebra coincides for $g = 0$ with the "classical" definition of a vertex algebra.
In **chapter 11** I shall prove that a representation of the Heisenberg algebra carries the structure of a global vertex algebra.

The following table illustrates the relationship between the chapters:

Part I			Part II	
Form. Distr.	$\boxed{1}$	$\longrightarrow$	Form. Distr.	$\boxed{9}$
Current alg.	$\boxed{2}$	$\longrightarrow$	Current alg.	$\boxed{9}$
Vertex Alg.	$\boxed{3}$	$\longrightarrow$	Vertex Alg.	$\boxed{10}$
Examples	$\boxed{4}$	$\longrightarrow$	Examples	$\boxed{11}$

Middle Part			
Corr. Funct.	$\boxed{5}$	KN-Forms	$\boxed{7}$
Riem. Surf.	$\boxed{6}$	KN-Alg.	$\boxed{8}$

In the appendix I present two detailed proofs of the Sugawara construction, and I discuss the known approaches of CFT to higher genus.

Chapter 1

Formal Calculus, OPEs and NOPs

In this chapter the foundations of vertex algebras are provided. The key issues in this chapter are

1. Operator Product Expansions (OPE),

2. Normal Ordered Product of Fields (NOP).

Fields are defined as formal power series in w and w^{-1} with values in the algebra of endomorphisms of a complex vector space V. (Throughout the whole thesis the vector spaces are supposed to be defined over $\mathbb{C}$). This means $a(w) \in End(V)[[w, w^{-1}]]$, and with the additional property that $a(w)v \in V[[w]][w^{-1}]$ for any vector $v \in V$.
From the physical point of view the fields should have the property that the product of two fields at nearby points z, w can be expressed in terms of other fields on the small parameter $(z - w)$. Expressed in formulas this means:

$$a(z) \cdot b(w) = \sum_{j=n_0}^{1} \frac{c^{-j}(w)}{(z-w)^j} + \sum_{j=0}^{\infty} c^j(w)(z-w)^j$$

where $c^j(w)$ are fields dependent on $a(z)$ and $b(w)$. This formula is proved in the fourth section of this chapter.
In this chapter we introduce the notion of the delta distribution and formal distributions. The definitions are slightly different from the definition given in the textbooks [Kac] and [FB-Z]. I will define these notions in the spirit of the theory of distributions in the sense of complex analysis. Here distributions are defined as boundary values of holomorphic functions. The distributions are the "jumps" of holomorphic functions expanded in different domains.
Also the normal ordered product can be considered in this sense. In formulas

the normal ordered product of two fields $a(w), b(w)$ is defined by

$$: a(w)b(w) := Res_z \left(a(z)b(w)i_{z,w}\frac{1}{z-w} - b(w)a(z)i_{w,z}\frac{1}{z-w} \right)$$

where $i_{z,w}\frac{1}{z-w}$ (resp. $i_{w,z}\frac{1}{z-w}$) is the expansion of $\frac{1}{z-w}$ in the domain $|z| > |w|$ (resp $|z| < |w|$).
The chapter is organized as follows.
In the first section we discuss the basic properties of formal distributions especially of the delta distribution. In the second section we state the lemma on which the formal operator product expansion theorem (OPE theorem) relies. In the third section we state the OPE theorem. In the fourth section we introduce the normal ordered product and its properties. The iterated normal ordered product is defined recursively by

$$: a^1(w)a^2(w)...a^N(w) :=: a^1(w)(: a^2(w)...a^N(w) :) : .$$

We give an explicit expression of the iterated normal ordered product by considering certain elements of the symmetric group S_n which we shall call "anti-shuffles".

1.1 Formal Distributions

In this section we introduce the notion of formal distributions. We furthermore define the formal delta distribution and prove some of its properties.
Formal distributions are power series in z and z^{-1} with values in a vector space. They are in general not formal Laurent series, because Laurent series have a finite principal part.

1.1.1 Definitions

Roughly speaking a formal distribution is a power series (negative and positive powers) with values in a vector space.

1.1.1 Definition *Let V be a vector space over $\mathbb{C}$.*
A formal expression of the form

$$\sum_{n_1,...,n_N \in \mathbb{Z}} a_{n_1,...,n_N} z_1^{n_1} z_2^{n_2} ... z_N^{n_N}$$

where $a_{n_1,...,n_N} \in V$ is called a ***formal distribution****. The complex vector space*

$$V[[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_N, z_N^{-1}]]$$

is called the space of ***formal distributions****.*

We fix some notations: Throughout the whole thesis we will write the distributions $a(z) \in V[[z, z^{-1}]]$ in the form

$$\boxed{a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}}$$

The n-th coefficient in the series $a(z) = \sum_n a_n z^{-n-1}$ is sometimes written in the form $(a(z))_n$.
Furthermore we will write for formal distributions in two variables:

$$\boxed{a(z,w) = \sum_{n,m\in\mathbb{Z}} a_{n,m} z^{-n-1} w^{-m-1}}$$

We will often drop "$\in \mathbb{Z}$".

1.1.2 Definition (Residue) *Let $a(z,w) = \sum_{n,m\in\mathbb{Z}} a_{n,m} z^{-n-1} w^{-m-1} \in V[[z, z^{-1}, w, w^{-1}]]$ be a formal distribution.*
*The **residue** is defined as the coefficient at the power -1:*

$$Res_z a(z,w) = \sum_m a_{0,m} w^{-m-1}.$$

If we multiply a formal distribution with a Laurent polynomial with coefficients in $\mathbb{C}$ then the product is again a formal distribution. More precisely let $p(z) \in \mathbb{C}[z, z^{-1}]$ be a Laurent polynomial. We write this Laurent polynomial in the form $p(z) = \sum_{k=n_0}^{n_1} p_n z^n$ where $n_0, n_1 \in \mathbb{Z}$. Then we obtain:

$$f(z)p(z) = \left(\sum_{m\in\mathbb{Z}} f_m z^{-m-1}\right)\left(\sum_{n=n_0}^{n_1} p_n z^n\right) = \sum_{m\in\mathbb{Z}}\left(\sum_{n=n_0}^{n_1} f_{m+n} p_n\right) z^{-m-1}. \tag{1.1}$$

1.1.3 Proposition *Let $\mathbb{C}[z, z^{-1}]$ be the algebra of Laurent polynomials in z. Then we have a non-degenerate pairing*

$$V[[z,z^{-1}]] \times \mathbb{C}[z,z^{-1}] \to V \qquad \textit{defined by}$$
$$\langle f, p\rangle = Res_z f(z)p(z).$$

Proof. The product $f(z)p(z)$ is well defined (see equation(1.1)). We obtain immediately:

$$\langle f, p\rangle = Res_z \left(f(z)p(z)\right) =$$
$$= Res_z \left(\sum_m \left(\sum_{n=n_0}^{n_1} f_{m+n} p_n\right) z^{-m-1}\right) = \sum_{n=n_0}^{n_1} f_n p_n$$

Obviously this pairing is non-degenerate. □

1.1.2 Formal Delta Distribution

In this subsection we define the notion of formal delta distributions and prove some of its properties. One of its properties is the formula

$$Res_z\left(a(z)\delta(z-w)\right) = a(w)$$

which justifies the word "delta" distribution.
For further purposes we will show the very important property: *If a formal distribution in two variables is supported on the diagonal, then this formal distribution can be expanded in a finite sum of derivations of the delta distribution.* This very important fact is the starting point of the operator product expansion that will be explained in more detail in the next section.
We recall the definition of the **binomial coefficient**:
Let $\alpha \in \mathbb{R}$ be an arbitrary real number, and $j \in \mathbb{Z}_{>0}$. The binomial coefficient is defined by

$$\binom{\alpha}{j} = \prod_{k=1}^{j} \frac{\alpha - k + 1}{k}. \tag{1.2}$$

We fix furthermore

$$\binom{\alpha}{0} = 1$$

for all $\alpha \in \mathbb{R}$. We have:

$$\binom{\alpha}{0} = 0 \text{ for } \alpha \in \mathbb{Z}, \text{ and } 0 \le \alpha < j. \tag{1.3}$$

We also have the identity ($\alpha \in \mathbb{R}$):

$$\binom{\alpha}{j} + \binom{\alpha}{j+1} = \binom{\alpha+1}{j+1}. \tag{1.4}$$

We are going to define the delta distribution (and its derivatives) as the formal difference of the two expansions of $\frac{1}{z-w}$ (and higher powers) in the domains $|z| < |w|$ and $|z| > |w|$. For this purpose we introduce some notation.

1.1.4 Definition *Let be $R(z,w)$ a rational function in two variables with poles only at $z = 0$, $w = 0$ and $|z| = |w|$.*
Denote by $i_{z,w}R(z,w)$, $(i_{w,z}R(z,w))$ the expansion of $R(z,w)$ in the domain $|z| > |w|$ (resp. $|z| < |w|$).

We expand now the rational function $\frac{1}{z-w}$ in these domains:

$$i_{z,w}\frac{1}{z-w} = \sum_{n=0}^{\infty} z^{-n-1}w^{n} \tag{1.5}$$

$$i_{w,z}\frac{1}{z-w} = -\sum_{n=0}^{\infty} z^{n}w^{-n-1} = -\sum_{n=-\infty}^{-1} z^{-n-1}w^{n} \tag{1.6}$$

More generally we can obtain by deriving j-times ($j \geq 0$) the above equations with respect to w:

$$
\begin{aligned}
i_{z,w}\frac{1}{(z-w)^{j+1}} &= \sum_{n=0}^{\infty}\binom{n}{j}z^{-n-1}w^{n-j} \\
i_{w,z}\frac{1}{(z-w)^{j+1}} &= -\sum_{n=0}^{\infty}\binom{-n-1}{j}z^{n}w^{-n-1-j} = -\sum_{n=-\infty}^{-1}\binom{n}{j}z^{-n-1}w^{n-j}
\end{aligned}
$$

The formal delta distribution is defined as the formal difference of the two expansions of $\frac{1}{z-w}$.

1.1.5 Definition *The **formal delta distribution** $\delta(z-w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ is defined by*

$$\boxed{\delta(z-w) \overset{def}{=} i_{z,w}\frac{1}{(z-w)} - i_{w,z}\frac{1}{(z-w)}}$$

This means we can write the delta distribution as a series as follows

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^{n}.$$

Furthermore we can define the derivatives of the formal delta distribution.

1.1.6 Definition (Derivatives of the Formal Delta Distribution)

$$\partial_w^{(j)}\delta(z-w) := i_{z,w}\frac{1}{(z-w)^{j+1}} - i_{w,z}\frac{1}{(z-w)^{j+1}}. \tag{1.7}$$

1.1.7 Remark *Let $j \in \mathbb{Z}$, $j \geq 0$. The j-th derivative of the delta distribution can be written as*

$$\partial_w^{(j)}\delta(z-w) = \sum_{n\in\mathbb{Z}}\binom{n}{j}z^{-n-1}w^{n-j} \text{ for } j \geq 0. \tag{1.8}$$

Now we turn to the definition of derivatives of formal distributions.

1.1.8 Definition (Derivatives of Formal Distributions) *Let $j \in \mathbb{Z}_{\geq 0}$. The j-th derivative of a formal distribution is defined by*

$$\partial_w^{(j)}a(w) \overset{def}{=} Res_z\left(a(z)i_{z,w}\frac{1}{(z-w)^{j+1}} - a(z)i_{w,z}\frac{1}{(z-w)^{j+1}}\right). \tag{1.9}$$

Denote by $\left(\partial^{(j)}a(w)\right)_+$ (resp. $\left(\partial^{(j)}a(w)\right)_-$) the first (second) summand of the above equation:

$$\left(\partial_w^{(j)}a(w)\right)_+ = Res_z a(z)i_{z,w}\frac{1}{(z-w)^{j+1}} \qquad \left(\partial_w^{(j)}a(w)\right)_- = -Res_z a(z)i_{w,z}\frac{1}{(z-w)^{j+1}}.$$

We obtain immediately:

$$\partial_w^{(j)} a(w) = \left(\partial_w^{(j)} a(w)\right)_+ + \left(\partial_w^{(j)} a(w)\right)_- .$$

More explicitly we have for the coefficients the following remark. This is the formal analogue of the Cauchy formula for holomorphic functions f in the complex plane $\mathbb{C}$

$$\frac{d^n}{dz^n} f(w) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-w)^{n+1}} dz.$$

1.1.9 Remark *We have especially*

$$\left(\partial_w^{(j)} a(w)\right)_+ = \sum_{n<0} (-1)^j \binom{n}{j} a_{n-j} w^{-n-1} \tag{1.10}$$

$$\left(\partial_w^{(j)} a(w)\right)_- = \sum_{n\geq 0} (-1)^j \binom{n}{j} a_{n-j} w^{-n-1} \tag{1.11}$$

Therefore

$$\boxed{\partial_w^{(j)} a(w) = \sum_n (-1)^j \binom{n}{j} a_{n-j} w^{-n-1}} \tag{1.12}$$

Proof.

$$\begin{aligned}
\partial_w^{(j)} a(w)_+ &= Res_z \left(\sum_m a_m z^{-m-1}\right) \left(\sum_{n\geq 0} \binom{n}{j} z^{-n-1} w^{n-j}\right) = \\
&= Res_z \sum_{m,n\geq 0} \binom{n}{j} a_m z^{-(m+n+1)-1} w^{n-j} = \\
&= \sum_{n\geq 0} \binom{n}{j} a_{-n-1} w^{n-j} = \\
&\overset{(*)}{=} \sum_{n<j} \binom{n+j-1}{j} a_{n-j} w^{-n-1} = \\
&\overset{(**)}{=} \sum_{n<0} (-1)^j \binom{n}{j} a_{n-j} w^{-n-1}.
\end{aligned}$$

In (*) we shifted the index $n \mapsto -n+j-1$. In (**) we used the definition of the binomial coefficient, and we used equation (1.3).

Analogously we can compute

$$\begin{aligned}
\partial_w^{(j)} a(w)_- &= Res_z \left(\sum_m a_m z^{-m-1} \right) \left(\sum_{n<0} \binom{n}{j} z^{-n-1} w^{n-j} \right) = \\
&= Res_z \sum_{m,n<0} \binom{n}{j} a_m z^{-(m+n+1)-1} w^{n-j} = \\
&= \sum_{n<0} \binom{n}{j} a_{-n-1} w^{n-j} = \\
&= \sum_{n\geq 0} \binom{n+j-1}{j} a_{n-j} w^{-n-1} = \\
&= \sum_{n\geq 0} (-1)^j \binom{n}{j} a_{n-j} w^{-n-1}. \qquad \square
\end{aligned}$$

Denote especially the $\pm$-part of the 0-th derivative of a formal distribution $a(w)$ by

$$\left(\partial_w^{(0)} a(w)\right)_\pm = a_\pm(w) = \sum_{n \lesseqgtr 0} a_n z^{-n-1}. \tag{1.13}$$

1.1.10 Lemma *The sign "commutes" with the derivative as follows*

$$\partial_w(a_\pm(w)) = (\partial_w a(w))_\pm \,. \tag{1.14}$$

Proof. We use preceding remark:

$$(\partial_w a_+(w))_+ = \sum_{n<0} (-n) a_{n-1} w^{-n-1} \tag{1.15}$$

$$(\partial_w a_+(w))_- = \sum_{n\geq 0} (-n) a_{n-1} w^{-n-1} \tag{1.16}$$

The coefficients for $n \geq 0$ vanish for $a_+(w)$. Thus we obtain the assertion. For the $-$-part it works analogously $\square$

Now we can formulate the main properties of the delta distribution (see also [Kac], Prop. 2.1):

1.1.11 Proposition (Properties of the Delta Distribution) *Let $\delta(z-w)$ be the formal delta distribution. The following identities hold:*

1. $\delta(z-w) = \delta(w-z)$
   $\partial_z \delta(z-w) = -\partial_w \delta(z-w)$
2. $(z-w)\delta(z-w) = 0$
   $(z-w)\partial_w^{(j+1)}\delta(z-w) = \partial_w^{(j)}\delta(z-w)$
   $(z-w)^{j+1}\partial_w^{(j)}\delta(z-w) = 0$ *for* $j \geq 0$
   $(z-w)^N\partial_w^{(j)}\delta(z-w) = 0$ *for* $N > j \geq 0$

3. *Let $a \in U[[z, z^{-1}]]$, then $a(z)\delta(z-w) \in V[[z, z^{-1}, w, w^{-1}]]$ is a well defined distribution in two variables and we have*

$$\begin{aligned} Res_z a(z)\delta(z-w) &= a(w) \\ Res_z a(w)\delta(z-w) &= a(z) \\ Res_z a(z)\partial_w^{(j)}\delta(z-w) &= \partial_w^{(j)} a(w). \end{aligned}$$

Proof.

1. $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n \overset{(m:=-n-1)}{=} \sum_{m\in\mathbb{Z}} z^m w^{-m-1} = \delta(w-z)$.
   $\partial_z\delta(z-w) = \sum_n (-n-1)z^{-n-2}w^n = \sum_n (-n)z^{-n-1}w^{n-1} = -\partial_w\delta(z-w)$.

2. $(z-w)\delta(z-w) = (z-w)\sum_{n\in\mathbb{Z}} z^{-n-1}w^n =$
   $\sum_{n\in\mathbb{Z}} z^{-n}w^n - \sum_{n\in\mathbb{Z}} z^{-(n+1)}w^{n+1} = 0.$

$$\begin{aligned} (z-w)\partial_w^{(j+1)}\delta(z-w) &= (z-w)\sum_{n\in\mathbb{Z}}\binom{n}{j+1} z^{-n-1}w^{n-j-1} = \\ &= \sum_{n\in\mathbb{Z}}\binom{n}{j+1} z^{-n}w^{-n-j-1} - \sum_{n\in\mathbb{Z}}\binom{n}{j+1} z^{-n-1}w^{-n-j} \\ &\overset{(*)}{=} \sum_{n\in\mathbb{Z}}\binom{n+1}{j+1} z^{-n-1}w^{-n-j} - \sum_{n\in\mathbb{Z}}\binom{n}{j+1} z^{-n-1}w^{-n-j} = \\ &= \sum_{n\in\mathbb{Z}}\left(\binom{n+1}{j+1} - \binom{n}{j+1}\right) z^{-n-1}w^{-n-j} = \\ &\overset{(**)}{=} \sum_{n\in\mathbb{Z}}\binom{n}{j} z^{-n-1}w^{-n-j} = \\ = \partial_w^{(j)}\delta(z-w) & \end{aligned}$$

   In $(*)$ we did an index shift in the first sum: $n \mapsto n-1$.
   In $(**)$ we used equation (1.4).

3. These formulas are clear from the definition of the derivatives of formal distributions. □

1.2 Local Distributions

In this section we state the lemma which is the starting point of the operator product expansion. This lemma is proved in [FB-Z] and in [Kac].
I will give a counterexample which shows that not any formal distribution in two variables can be expanded in a series of derivatives of delta distributions.

1.2.1 Definition (Locality) *A formal distribution $a(z,w) \in V[[z,z^{-1},w,w^{-1}]]$ (V a vector space) is called **local** if there exists an integer $N \geq 0$ such that*

$$a(z,w)(z-w)^N = 0.$$

Let V be a vector space. Define the endomorphisms m_N ($N \in \mathbb{N}_0$) on the space of formal distributions $V[[z,z^{-1},w,w^{-1}]]$

$$m_N : V[[z,z^{-1},w,w^{-1}]] \to V[[z,z^{-1},w,w^{-1}]], \quad a(z,w) \mapsto a(z,w) \cdot (z-w)^N.$$

We are interested especially in the kernels of these maps, i.e. the local distributions. Explicitly we have

$$\begin{aligned} a(z,w) \cdot (z-w)^N &= \sum_{n,m} a_{n,m} z^{-n-1} w^{-m-1} \left(\sum_{j=0}^{N} \binom{N}{j} (-1)^j z^{N-j} w^j \right) = \\ &= \sum_{n,m} \sum_{j=0}^{N} \binom{N}{j} (-1)^j a_{n,m} z^{-n+N-j-1} w^{-m+j-1} \end{aligned}$$

By an index shift we obtain eventually:

$$a(z,w) \cdot (z-w)^N = \sum_{n,m} \left(\sum_{j=0}^{N} \binom{N}{j} (-1)^j a_{n+N-j,m+j} \right) z^{-n-1} w^{-m-1}. \quad (1.17)$$

Define now the difference operator

$$\Delta : V^{\mathbb{Z}^2} \to V^{\mathbb{Z}^2}, \quad a_{n,m} \mapsto a_{n+1,m} - a_{n,m+1}.$$

By induction we obtain for the N-times iterated map $\Delta^N = \underbrace{\Delta \circ \ldots \circ \Delta}_{N}$:

$$\Delta^N(a_{n,m}) = \sum_{j=0}^{N} \binom{N}{j} (-1)^j a_{n+N-j,m+j}. \quad (1.18)$$

Therefore we obtain according to the above calculation:

$$m_N(a(z,w)) = \sum_{n,m} (\Delta^N a_{n,m}) z^{-n-1} w^{-m-1}. \quad (1.19)$$

1.2.2 Lemma *Let $a(z) \in V[[z,z^{-1}]]$ be a formal distribution. Then we have the equivalence*

$$a(z,w)(z-w)^N = 0 \Leftrightarrow a(z,w) = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w) \quad (1.20)$$

where $c^j(w) \in V[[w,w^{-1}]]$ for all $1 \leq j \leq N-1$.

Proof of the lemma.
$\Leftarrow$: We start with $a(z,w) = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w)$. From Proposition 1.1.11 we know $\partial_w^{(j)} \delta(z-w)(z-w)^N = 0$ for $j < N$. Hence

$$\sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w)(z-w)^N = 0.$$

$\Rightarrow$: We start with $a(z,w)(z-w)^N = 0$. This means $a(z,w) \in Ker(m_N)$. From equation (1.19) we know that we have to consider the system of difference equations

$$\Delta^N a_{n,m} = 0 \quad n, m \in \mathbb{Z}.$$

Put

$$x_k^{(n+m+N)} := a_{n+m+N-k,k}.$$

From equation (1.18) we can see

$$\Delta^N a_{n,m} = \sum_{j=0}^{N} \binom{N}{j} (-1)^j a_{n+N-j,m+j} = \sum_{j=0}^{N} \binom{N}{j} (-1)^j x_{m+j}^{(n+m+N)}.$$

Thus it remains to consider the difference equation of order N given by the numbers $x_k^{(n+m+N)}$.

$$\sum_{j=0}^{N} \binom{N}{j} (-1)^j x_{m+j}^{(n+m+N)} = 0.$$

The space of solutions of this difference equation is N-dimensional.
We already have N linearly independent solutions of this difference equation. They are given by the coefficients of the N series

$$j! w^{n+m+N+j+1} \partial_w^{(j)} \delta(z-w), \quad 0 \le j \le N-1.$$

This can be seen as follows. Explicitly we have

$$j! w^{n+m+N+j+1} \partial_w^{(j)} \delta(z-w) = \sum_{k \in \mathbb{Z}} \binom{k}{j} j! z^{-k-1} w^{k+n+m+N+1}.$$

The coefficients

$$y_k^{(n+m+N),j} = \binom{k}{j} j!$$

satisfy the equation

$$\sum_{j=0}^{N} \binom{N}{j} (-1)^j y_k^{(j)} = 0$$

Explicitly the $y_k^{(j)}$ are polynomials in k:

$$y_k^{(j)} = k(k-1)...(k-j+1)$$

The polynomials are linearly independent, therefore we get the assertion of the lemma. □
From lemma 1.2.2 and its proof we can determine the $c^j(w)$ more explicitly:

1.2.3 Proposition *Let $a(z,w) \in V[[z,z,^{-1},w,w^{-1}]]$ satisfy $(z-w)^N a(z,w) = 0$. The $c^j(w)$ given by equation (1.20) can be computed by*

$$c^j(w) = Res_z\left(a(z,w)(z-w)^j\right).$$

Proof. We know from proposition 1.1.11:

$$(z-w)^k \partial^{(j)}\delta(z-w) = \begin{cases} \partial_w^{(j-k)}\delta(z-w) & \text{for} \quad k<j \\ \delta(z-w) & \text{for} \quad k=j \\ 0 & \text{for} \quad k>j \end{cases}$$

We know furthermore from proposition 1.1.11: $Res_z a(w)\partial_w^{(j)}\delta(z-w) = 0$ for $(j>0)$. Thus we obtain:

$$Res_z\left(a(z,w)(z-w)^k\right) = Res_z\left(\sum_{j=0}^{N-1} c^j \partial_w^{(j)}\delta(z-w)(z-w)^k\right) = c^k(w)$$

□

1.2.4 Corollary *Let $a(z,w) \in V[[z,z,^{-1},w,w^{-1}]]$ be a local distribution. Then we can write the coefficients in terms of the distributions $c^j(w)$:*

$$a_{n,m} = \sum_{j=0}^{N-1} \binom{n}{j} c^j_{n+m-j}. \tag{1.21}$$

Proof.

$$a(z,w) = \sum_{j=0}^{N-1} c^j(w)\partial^{(j)}\delta(z-w) = \sum_{j=0}^{N-1}\sum_m c^j_m w^{-m-1}\sum_n \binom{n}{j} z^{-n-1}w^{n-j} =$$

$$\stackrel{(m\mapsto m-n+j)}{=} \sum_{m,n}\sum_{j=0}^{N-1} c^j_{m+n-j} w^{-m-n+j-1}\binom{n}{j} z^{-n-1}w^{n-j} =$$

$$= \sum_{m,n}\sum_{j=0}^{N-1}\binom{n}{j} c^j_{m+n-j} z^{-n-1}w^{-m-1} = \sum_{mn} a_{nm} z^{-n-1}w^{-m-1}$$

Comparing the coefficients we get the assertion. □
From equation (1.17) we get:

$$Res_z a(z,w)(z-w)^j = \sum_{m\in\mathbb{Z}}\left(\sum_{k=0}^{j}\binom{j}{k}(-1)^k a_{j-k,m+k}\right) w^{-m-1}. \tag{1.22}$$

We have seen that local distributions can be expanded in the above given way. What about more general formal distributions? The following remark gives a first answer.

1.2.5 Remark *There are formal distributions $a(z,w) \in \mathbb{C}[[z,z^{-1},w,w^{-1}]]$ such that $a(z,w)$ cannot be expanded in the form*

$$a(z,w) = \sum_{j=0}^{\infty} c^j(w)\partial_w^{(j)}\delta(z-w). \tag{1.23}$$

Let $V[[z,z^{-1},w,w^{-1}]]$ be the space of formal distributions. Let the subspaces U_N denote by

$$U_N := Ker(m_N).$$

We have obviously

$$U_1 \subset U_2 \subset U_3 \subset U_4..$$

Furthermore we have the inverse system

$$U_1 \xleftarrow{m_1} U_2 \xleftarrow{m_1} U_3 \xleftarrow{m_1} U_4..$$

We can define the inverse (also called projective) limit

$$U := \varprojlim Ker(m_N)$$

An element $a(z,w) \in U$ is a formal distribution $a(z,w) \in V[[z,z^{-1},w,w^{-1}]]$ which can be given by equation (1.23).
The above remark can be now reformulated as: $U \subsetneqq V[[z,z^{-1},w,w^{-1}]]$.

1.2.6 Remark *Let $a(z,w) \in \mathbb{C}[[z,z^{-1},w,w^{-1}]]$ be a formal distribution over $\mathbb{C}$. Let the coefficients $a_{n,m}$ be defined by $a_{n,p+n} := (-1)^n$, for $p \in \mathbb{Z}$.*
This distribution cannot be expanded in a series of the form of equation (1.23).

To see this we first consider an element $a(z,w) \in U$.
From equation (1.22) we can compute

$$\begin{aligned}
a(z,w) &= \sum_{j=0}^{\infty} c^j(w)\partial_w^{(j)}\delta(z-w) \\
&= \sum_{j=0}^{\infty}\left(\sum_{m\in\mathbb{Z}}\left(\sum_{k=0}^{j}\binom{j}{k}(-1)^k a_{j-k,m+k}\right)w^{-m-1}\right)\left(\sum_{n\in\mathbb{Z}}\binom{n}{j}z^{-n-1}w^{n-j}\right) \\
&= \sum_{n,m\in\mathbb{Z}}\left(\sum_{j=0}^{\infty}\binom{n}{j}\left(\sum_{k=0}^{j}\binom{j}{k}(-1)^k a_{j-k,m+k}\right)\right)z^{-n-1}w^{n-m-j-1} \\
&= \sum_{n,m\in\mathbb{Z}}\left(\sum_{j=0}^{\infty}\binom{n}{j}\left(\sum_{k=0}^{j}\binom{j}{k}(-1)^k a_{j-k,m+n-j+k}\right)\right)z^{-n-1}w^{-m-1}
\end{aligned}$$

From this calculation we obtain:
The coefficients $(a(z,w))_{n,m}$ are well defined for $n \geq 0$, $m \in \mathbb{Z}$ because $\binom{n}{j} = 0$ for $j > n$.
But for $n > 0$ we do not have a well defined expression in general. Consider especially $n = -1$: We have due to the definition of the binomial coefficient

$$\binom{-1}{j} = (-1)^j \text{ for all } j \geq 0.$$

Thus we obtain for the coefficients:

$$\begin{aligned} (a(z,w))_{-1,m} &= \sum_{j=0}^{\infty} \binom{-1}{j} \left(\sum_{k=0}^{j} \binom{j}{k} (-1)^k a_{j-k,m-1-j+k} \right) = \\ &= \sum_{j=0}^{\infty} (-1)^j \left(\sum_{k=0}^{j} \binom{j}{k} (-1)^k (-1)^{j-k} \right) \\ &= \sum_{j=0}^{\infty} (-1)^{2j} \left(\sum_{k=0}^{j} \binom{j}{k} \right) \\ &= \sum_{j=0}^{\infty} 2^j. \end{aligned}$$

This series is obviously not finite. □

1.3 Operator Product Expansion of Formal Distributions

The operator product expansion (OPE) is one of the most essential issues in conformal field theory. Especially in the famous paper [BPZ] (that is regarded as the breaking dawn of conformal field theory) their starting point is the OPE of fields. In order to get a mathematical rigorous notion of OPE we have to provide some notions that are going to be helpful in further contexts.

1.3.1 Definition *Let U be an associative algebra. The product of two formal distributions*

$$a(z) \in V[[z, z^{-1}]], \quad b(w) \in V[[w, w^{-1}]]$$

is defined by

$$a(z)b(w) = \sum_{n,m \in \mathbb{Z}} a_n b_m z^{-n-1} w^{-m-1} \in U[[z, z^{-1}, w, w^{-1}]].$$

This product is well defined because for each n, m we have only one coefficient $a_n b_m$.
Now we come to the definition of mutual local distributions with values in an associative algebra U.

1.3.2 Definition (Locality) *Two formal distributions $a(z), b(z) \in U[[z, z^{-1}]]$ with values in an associative algebra U are called* ***mutually local****, if*

$$\exists N \forall n \geq N : (z-w)^n [a(z), b(w)] = 0.$$

For our purposes it is convenient to define a normal ordered product of two distributions with coefficients in an associative algebra.

1.3.3 Definition (Normal Ordered Product of two Distributions)

$$: a(z)b(w) : \stackrel{def}{=} a_+(z)b(w) + b(w)a_-(z).$$

In the OPE theorem below the product of two formal distributions is the sum of a certain "singular part" and the normal ordered product of the distributions. Especially the equations in 6. can be found in several textbooks about conformal field theory. In physicists language the operator product is sometimes called **radial ordering** (see e.g. [Wal]).

1.3.4 Theorem (OPE Theorem) *Let $a(z), b(z)$ be two formal distributions. Then we have the following equivalent conditions:*

1. *$a(z), b(z)$ are mutually local, i.e. $(z-w)^N [a(z), b(w)] = 0$.*

2. $[a(z), b(w)] = \sum_{j=0}^{N-1} c^j(w) \partial_w^{(j)} \delta(z-w)$, *where* $c^j(w) \in V[[w, w^{-1}]]$

3. $[a_n, b_m] = \sum_{j=0}^{N-1} \binom{n}{j} c^j_{n+m-j}, n, m \in \mathbb{Z}$

4. $[a_n, b(w)] = \sum_{j=0}^{N-1} \binom{n}{j} c^j(w) w^{n-j}, m \in \mathbb{Z}$

5. $[a_-(z), b(w)] = \sum_{j=0}^{N-1} \left(i_{z,w} \frac{1}{(z-w)^{j+1}} \right) c^j(w)$

   $[a_+(z), b(w)] = -\sum_{j=0}^{N-1} \left(i_{w,z} \frac{1}{(z-w)^{j+1}} \right) c^j(w)$

   where $c^j(w) \in \mathfrak{g}[[w, w^{-1}]]$.

6. $a(z)b(w) = \sum_{j=0}^{N-1} \left(i_{z,w} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + : a(z)b(w) :$

   $b(w)a(z) = \sum_{j=0}^{N-1} \left(i_{w,z} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + : a(z)b(w) :$

   where $c(w) \in U[[w, w^{-1}]]$.

Proof. 1. $\Leftrightarrow$ 2.: This is the content of lemma 1.2.2 for $a(z,w) = \sum_{n,m}[a_n, b_m]z^n w^m$.
2. $\Leftrightarrow$ 3.: This is the content of corollary 1.2.4.
3. $\Leftrightarrow$ 4.:

$$[a_n, b(w)] = \sum_m [a_n, b_m] w^{-m-1} \stackrel{(2.)}{=} \sum_m \sum_{j=0}^{N-1} \binom{n}{j} c^j_{n+m-j} w^{-m-1} =$$

$$\stackrel{(m \mapsto n+m-j)}{=} \sum_m \sum_{j=0}^{N-1} \binom{n}{j} c^j_m w^{-m+n-j-1} = \sum_m \sum_{j=0}^{N-1} \binom{n}{j} c^j_m w^{-m-1} w^{n-j} =$$

$$= \sum_m \sum_{j=0}^{N-1} \binom{n}{j} c^j(w) w^{n-j}.$$

2. $\Leftrightarrow$ 5. This follows from the identity $\partial_w^{(j)} \delta(z-w) = i_{z,w} \frac{1}{(z-w)^{j+1}} - i_{w,z} \frac{1}{(z-w)^{j+1}}$.
5 $\Leftrightarrow$ 6.

$$a(z)b(w) = \sum_{nm} a_n b_m z^{-n-1} w^{-m-1} =$$

$$= \sum_{n=0}^{\infty} \sum_m [a_n b_m] z^{-n-1} w^{-m-1} + \sum_{n=-1}^{-\infty} \sum_m a_n b_m z^{-n-1} w^{-m-1} + \sum_{n=0}^{\infty} \sum_m b_m a_n z^{-n-1} w^{-m-1} =$$

$$\sum_{n=0}^{\infty} \sum_m [a_n b_m] z^{-n-1} w^{-m-1} + :a(z)b(w): :=$$

$$\stackrel{(5.)}{=} \sum_{j=0}^{N-1} \left(i_{z,w} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + :a(z)b(w):.$$

□

By abuse of notation we can write with respect of the above theorem and corollary 1.2.4 for the product of two mutually local distributions:

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{Res_z(a(z)b(w)(z-w)^j)}{(z-w)^{j+1}} + :a(z)b(w):, \tag{1.24}$$

(We dropped $i_{z,w}$)
and if we define (provisionally)

$$(a_j b)(w) := Res_z(a(z)b(w)(z-w)^j)$$

then we have the equation

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{(a_j b)(w)}{(z-w)^{j+1}} + :a(z)b(w): \tag{1.25}$$

1.4 Fields and Normal Ordered Products

1.4.1 Normal Ordered Product of Fields

A field is, mathematically speaking, a linear map from some base space with values in the associative algebra of endomorphisms of a vector space.

1.4.1 Definition (Field) *Let V be a vector space, and let $a(z)$ be a formal distribution with coefficients in the associative algebra of endomorphisms of V, i.e. $a(z) \in End(V)[[z, z^{-1}]]$.*
*$a(z)$ is called a **field**, if it satisfies the property:*

$$\forall v \in V \exists n_0 \in \mathbb{Z} : a_n v = 0 \forall n \geq n_0.$$

From the definition we get formal Laurent expansions with values in V by applying $a(z)$ to a vector $v \in V$:

$$a(z)v \in V[[z]][z^{-1}].$$

For $z = w$ it is in general not possible to define the normal ordered product in a straightforward way because it would give rise to divergences in algebraic sense.
Example: $a(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{C}[[z, z^{-1}]]$

$$a(z)a(z) = \sum_{n,m \in \mathbb{Z}} z^n z^m = \sum_{n,m \in \mathbb{Z}} z^{n+m}$$

and e.g. for $n + m = 0$ we have infinitely many pairs $(n, m) \in \mathbb{Z}^2$, such that $n + m = 0$.
The normal ordered product introduced below is to be understood in the spirit of distributions of complex analysis.

1.4.2 Definition (Normal Ordered Product of two Fields) *Let $a(z), b(z)$ be two fields. The **normal ordered product** of two fields is defined by*

$$\boxed{: a(w)b(w) : \stackrel{def}{=} Res_z \left(a(z)b(w) i_{z,w} \frac{1}{z-w} - b(w)a(z) i_{w,z} \frac{1}{z-w} \right).}$$

We also introduce the n-th normal ordered product. i.e. a product as above but with n-th powers of $i_{z,w} \frac{1}{z-w}$ and $i_{w,z} \frac{1}{z-w}$ respectively.

1.4.3 Definition (n-th product) *Let be $n \in \mathbb{Z}$, and let $a(z)$, $b(z)$ be two fields. The **n-th normal ordered Product** (or n-th product for short) of two fields is defined by*

$$\boxed{a(w)_n b(w) \stackrel{def}{=} Res_z \left(a(z)b(w) i_{z,w} (z-w)^n - b(w)a(z) i_{w,z} (z-w)^n \right).}$$

Here we dropped the colons, but instead of them we have the subscript: $a(w)_n b(w)$.
We can thus write for the normal ordered product: $: a(z)b(z) := a(z)_{-1} b(z)$.

1.4.4 Proposition *The n-th product can be written in the form:*

$$a(w)_n b(w) = Res_z[a(z), b(w)](z-w)^n \text{ for } n \in \mathbb{Z}_{\geq 0} \tag{1.26}$$

$$a(w)_n b(w) = :(\partial^{(-n-1)} a(w)) b(w): \text{ for } n \in \mathbb{Z}_{<0} \tag{1.27}$$

The first equation is clear. The second equation follows from the definition 1.1.6 of derivations of formal distributions immediately. □

1.4.5 Proposition *The n-th coefficient* $: a(w)b(w) :_n$ *of the normal ordered product of two fields is given by:*

$$: a(w)b(w) :_n = \sum_{j<0} a_j b_{n-j-1} + \sum_{j \geq 0} b_{n-j-1} a_j \tag{1.28}$$

More generally we can explicitly write down the coefficients of the n-th products:

1.4.6 Theorem *Let* $n \in \mathbb{Z}_+$ *(i.e.* $n \geq 0$*). Then we have for the m-th coefficient*

$$(a(w)_n b(w))_m = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [a_j, b_{m+n-j}] \tag{1.29}$$

$$(a(w)_{-n-1} b(w))_m =$$

$$= \sum_{j<0} (-1)^j \binom{j}{n} a_{j+n+1} b_{m-j-1} + \sum_{j \geq 0} (-1)^j \binom{j}{n} b_{m-j-1} a_{j+n+1} \tag{1.30}$$

Proof of proposition 1.4.5:

$$: a(w)b(w) := Res_z a(z) b(w) i_{z,w} \frac{1}{z-w} - b(w) a(z) i_{w,z} \frac{1}{z-w} =$$

$$= Res_z a(z) b(w) \sum_{k=0}^{\infty} z^{-n-1} w^n + Res_z b(w) a(z) \sum_{k=-1}^{-\infty} z^{-n-1} w^n$$

For the sign change in last line note that $i_{w,z} \frac{1}{z-w} = -\sum_{k=-1}^{-\infty} z^{-n-1} w^n$. The first summand gives

$$Res_z \sum_{n,m} a_n b_m z^{-n-1} w^{-m-1} \sum_{k=0}^{\infty} z^{-k-1} w^k =$$

$$= Res_z \sum_{n,m} \sum_{k=0}^{\infty} a_n b_m z^{-(n+k+1)-1} w^{-m+k-1} =$$

$$= \sum_m \sum_{k=0}^{\infty} a_{-k-1} b_m w^{-m+k-1} = \sum_m \sum_{k=-1}^{-\infty} a_k b_m w^{-m-k}$$

In the last line we substituted $k \mapsto -k-1$. In the line above we substituted $m \mapsto m-k$. $\square$
Proof of the theorem:
Let $n \geq 0$. We obtain from the proposition 1.4.4:

$$a(w)_n b(w) = Res_z[a(z), b(w)](z-w)^n =$$

$$= Res_z \sum_{k,m} [a_k, b_m] z^{-k-1} w^{-m-1} \left(\sum_{j=0}^{n} \binom{n}{j} z^j (-w)^{n-j} \right) =$$

$$= Res_z \sum_{k,m} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [a_k, b_m] z^{-k+j-1} w^{-m+n-j-1} =$$

$$= Res_z \sum_{k,m} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [a_{k+j}, b_{m+n-j}] z^{-k-1} w^{-m-1} =$$

$$= \sum_{m} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [a_j, b_{m+n-j}] w^{-m-1}.$$

Let $m < 0$ then we have according to proposition 1.4.4:

$$a(w)_n b(w) =: (\partial^{(-m-1)} a(w)) b(w) :$$

From proposition 1.4.5 we know

$$: (\partial^{(-m-1)} a(w)) b(w) :_n = \sum_{j<0} (\partial^{(-m-1)} a(w))_j b_{n-j-1} + (-1)^{|a||b|} \sum_{j<0} b_{n-j-1} (\partial^{(-m-1)} a(w))_j$$

For the coefficients of the derivatives we get

$$(\partial^{(-m-1)} a(w))_j = (-1)^j \binom{j}{-m-1} a_{j+m+1}$$

Combining the last two formulas we get the result of the theorem. $\square$
The following corollary will be useful in the third chapter. It is also shown in [Kac] (lemma 3.1.). From the above theorem it is now an easy exercise.

1.4.7 Corollary *Let V be a vector space. Let $a(z), b(z) \in End(V)[[z, z^{-1}]]$ be two fields and let $v_0 \in V$ be a vector such that*

$$a_n v_0 = 0 \text{ and } b_n v_0 = 0 \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

Then

$$a(z)_n b(z) v_0 \in V[[z]]$$

and $(a(z)_n b(z) v_0)_{-1} = a_n b_{-1} v_0$ for all $n \in \mathbb{Z}$.

1.4.8 Corollary *The derivative of the n-th product satisfies the derivation property, i.e.*

$$\partial_w \left(a(w)_n b(w)\right) = \left(\partial_w a(w)\right)_n b(w) + a(w)_n \left(\partial_w b(w)\right) \tag{1.31}$$

where $a(w), b(w)$ are two fields.

Proof. We only have to compare the m-th coefficients of the right hand side and the left hand side of the equation, hence we obtain the assertion. □

1.4.9 Theorem (Normal Ordered Product of Two Fields is a Field) *Let $a(z), b(z) \in End[[z, z^{-1}]]$ be two fields. Then $: a(z)b(z) :\in End[[z, z^{-1}]]$ is a field, too.*

Proof. We have to prove:

$$\forall v \in V \exists n_0, \text{ such that } : a(w)b(w) :_n v = 0$$

for any $n \geq n_0$.
Let be $v \in V$. Because $a(z)$ and $b(z)$ are fields we have the conditions:
$\exists n_0^{a,v}$: $a_n v = 0$ for any $n \geq n_0^{a,v}$, and
$\exists n_0^{b,v}$: $b_n v = 0$ for any $n \geq n_0^{b,v}$.
Choose $n_1 := n_0^{b,v}$. Then we get:

$$\begin{aligned} : a(z)b(z) :_{n_1} v &= \sum_{j=-1}^{-\infty} a_j b_{n_1-j-1} v + \sum_{j=0}^{\infty} b_{n_1-j-1} a_j v = \\ &= 0 + \sum_{j=0}^{\max(0, n_0^{a,v})} b_{n_1-j-1} a_j v \end{aligned}$$

Denote by $v_j := a_j v$ for $0 \leq j \leq n_0^{a,v} - 1$. We have therefore finitely many vectors that are supposed to be killed by a certain b_n. Denote by n_0^{b,v_j} the smallest... then we get by choosing

$$n_0 := \max_{0 \leq j \leq n_0^{a,v}-1} \{(n_0^{b,v_j} + n_0^{a,v}), n_1\}$$

$$\sum_{j=0}^{n_0^{a,v}-1} b_{n_0-j-1} v_j = 0$$

□

1.4.2 Taylor's Formula and Dong's Lemma

1.4.10 Theorem (Taylor) *Let $a(z) \in End(V)[[z, z^{-1}]]$ be a field. Then we have in the domain $|z - w| < |w|$ the expansion:*

$$a(z) = \sum_{n=0}^{\infty} (\partial_w^{(n)} a(w))(z - w)^n \tag{1.32}$$

A proof can be found in [Kac], proposition 2.4. □
The lemma of Dong is very important in order to prove the locality axiom of vertex algebras. It is proved in many references (see e.g. [Kac], [FB-Z]). We prove it very briefly.

1.4.11 Theorem (Dong's Lemma) *Let be $a(z), b(z), c(z)$ three Fields, respectively mutually local.*
Then $[: a(z)b(z) :, c(w)]$ is mutually local.

Proof. Suppose $r \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned} [a(z), b(w)](z-w)^r = 0 \quad &\Leftrightarrow \quad a(z)b(w)(z-w)^r = b(w)a(z)(z-w)^r \\ [b(z), c(w)](z-w)^r = 0 \quad &\Leftrightarrow \quad b(z)c(w)(z-w)^r = c(w)b(z)(z-w)^r \\ [a(z), c(w)](z-w)^r = 0 \quad &\Leftrightarrow \quad a(z)c(w)(z-w)^r = c(w)a(z)(z-w)^r \end{aligned}$$

Define

$$\begin{aligned} A &= a(z)b(w)i_{z,w}\frac{1}{z-w}c(u) - (-1)^{|a||b|}b(z)a(w)i_{w,z}\frac{1}{z-w}c(u) \\ B &= c(u)b(w)a(z)i_{z,w}\frac{1}{z-w} - (-1)^{|a||b|}c(u)a(z)b(w)i_{w,z}\frac{1}{z-w} \end{aligned}$$

We are going to prove the following formula:

$$A(w-u)^{3r} = B(w-u)^{3r} \tag{1.33}$$

In order to prove this equality we use the binomial formula:

$$(w-u)^{3r} = (w-u)^r \underbrace{(w-u)^{2r}}_{(w-z+z-u)^{2r}} = (w-u)^r \sum_{s=0}^{2r} \binom{2r}{s} (w-z)^{2r-s}(z-u)^s$$

For $2r - s \geq r + 1$ the left hand side and the right hand side of equation (1.33) are zero. The reason is the following:
Therefore it remains to consider $(w-u)^r \sum_{s=r}^{3r} \binom{3r}{s} (w-z)^{3r-s}(z-u)^s$. Now the second factor in the sum commutes the fields $a(z), c(u)$ (because $s \geq r$). And the factor $(w-u)^r$ commutes the fields $b(w), c(u)$. hence we get the desired equality. □

1.4.3 Iterated Normal Ordered Products of Fields

In the last subsections we saw that the normal ordered product of two field is again a field. This leads to the definition of an iterated normal ordered product as is given in the present subsection. We define the normal ordered product of n fields and rewrite this expression as an **antishuffle** between the fields of their positive and their negative part. In the appendix we shall see that the normal ordered product does not satisfy, even in the simplest cases, nice properties such as associativity or commutativity.

1.4.12 Definition *Let $a^1(z), ..., a^N(z) \in End(V)[[z, z^{-1}]]$ be N fields. The* ***normal ordered product*** *of these n fields is defined by the recursive formula:*

$$: a^1(z)....a^N(z) : \stackrel{def}{=} : a^1 \left(: a^2......a^N : \right) :$$

with

$$: a(z) := a(z)$$

Example: Let $N = 2$. Then we have

$$: a^1(z)a^2(z) :=: a^1(z)(: a^2(z) :) :$$

This is the usual normal ordered product. Let $N = 3$. Then we get:

$$: a^1(z)a^2(z)a^3(z) :=: a^1(z)(: a^2(z)a^3(z) :) :$$

And so on.
If we write a field $a(z) = a_+(z) + a_-(z)$ as the sum of the negative part and the positive part:

$$a_+(z) = \sum_{n<0} a_n z^{-n-1}, \quad a_-(z) = \sum_{n\geq 0} a_n z^{-n-1},$$

then we can write:

$$: a^1(z)a^2(z) := a^1_+(z)a^2(z) + a^2(z)a^1_-(z)$$

and furthermore

$$: a^1(z)a^2(z) := a^1_+(z)a^2_+(z) + a^1_+(z)a^2_-(z) + a^2_+(z)a^1_-(z) + a^2_-(z)a^1_-(z).$$

Convention: From now on we will drop the argument (z). There should not rise any confusion. We have the equation:

$$: a^1a^2 := a^1_+a^2_+ + a^1_+a^2_- + a^2_+a^1_- + a^2_-a^1_-.$$

For $N = 3$ we get by continuing this procedure:

$$\begin{aligned} : a^1a^2a^3 : \ &= \ a^1_+a^2_+a^3_+ + a^1_+a^2_+a^3_- + \\ &+ \ a^2_+a^3_+a^1_- + a^1_+a^3_+a^2_- + \\ &+ \ a^1_+a^3_-a^2_- + a^2_+a^3_-a^1_- + \\ &+ \ a^3_+a^2_-a^1_- + a^3_-a^2_-a^1_- \end{aligned}$$

Looking carefully at this last equation we conjecture that the normal ordered product of n fields is a sum of 2^N terms. This terms are organized such that the positive parts are in increasing order and the negative parts of the fields are in decreasing order. This is the content of the next theorem.
We first fix some notions:

Let $M, N \subset \mathbb{N}_0$ be two finite subsets of the set of natural numbers. The cardinality is supposed to be $|M| = m$, $|N| = n$.
Denote by $J(M, N)$ the set of strictly increasing mappings from M to N, i.e. the elements $\varphi \in J(M, N)$ satisfy the condition

$$\varphi(i) < \varphi(j) \text{ for } i < j.$$

1.4.13 Remark *For $|M| > |N|$ the set $J(M, N)$ is empty, and for $|M| = |N|$ the set $J(M, N)$ consists of precisely one element (the identity element of S_n). For $|M| < |N|$ the cardinality of the set $J(M, N)$ is given by*

$$|J(M, N)| = \binom{n}{m} \text{ where } |M| = m, |N| = n. \tag{1.34}$$

Reasoning: For $|M| \geq |N|$ the assertion is completely clear because the elements $\varphi \in J(M, N)$ have to be necessarily injective. If we interpret the elements $\varphi \in J(M, N)$ as choosing from n elements m elements by regarding the order, then we obtain the last assertion. $\square$

1.4.14 Definition (Antishuffle) *Let $1 \leq m < n$. Let $\varphi \in J(m, n)$ be a strictly increasing map from $I_m = \{1, ..., m\}$ into $I_n = \{1, 2, ..., n\}$.*
Denote by $i_1, ..., i_{n-m}$ the elements in $I_n \setminus \varphi(I_m)$, with $i_1 < i_2 < ... < i_{n-m}$.
Let the permutation $\rho_\varphi \in S_n$ be defined by

$$\rho_\varphi(i) := \begin{cases} \varphi(i) & i \leq m \\ i_{n-m-j+1} & j \in \{1, ..., n-m\}, i_k \in I_n \setminus \varphi(I_m) \end{cases} \tag{1.35}$$

We call this permutation an ***anti-shuffle****.*
For $1 \leq m = n$ the anti-shuffle is just the identity map.
For $m = 0 < n$ the anti-shuffle is given by the permutation $\begin{pmatrix} 1 & 2 & ... & n-1 & n \\ n & n-1 & ... & 2 & 1 \end{pmatrix}$.

For instance the $\binom{4}{2} = 6$ anti-shuffles with respect to the 6 elements of $J(2, 4)$ are of the form:

$$\left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{array}\right) \quad \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{array}\right) \quad \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{array}\right) \quad \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{array}\right)$$

I use the notion **anti-shuffle** because the usual shuffle is an element σ of the symmetric group S_n such that

$$\sigma(1) < \sigma(2) < ... < \sigma(m) \text{ and } \sigma(m+1) < \sigma(m+2) < ... < \sigma(n).$$

An anti-shuffle is an element $\rho \in S_n$ such that

$$\rho(1) < \rho(2) < ... < \rho(m) \text{ and } \rho(m+1) > \rho(m+2) > ... > \rho(n).$$

The notion "shuffle" can be found in the realm of Hopf algebra theory (see [Kas]).
Now we formulate the theorem which gives an explicit description of the iterated normal ordered product.

1.4.15 Theorem *Let $a^1, a^2, ..., a^N \in End(V)[[z, z^{-1}]]$ be N fields, then there holds the formula:*

$$: a^1 a^2 ... a^N := \sum_{m=0}^{N} \sum_{\varphi \in J(m,N)} (\prod_{i=1}^{m} a_+^{\rho_\varphi(i)})(\prod_{i=m+1}^{N} a_-^{\rho_\varphi(i)}), \tag{1.36}$$

where $J(m,n)$ is the set of strictly increasing mappings from $I_m = \{1, 2, ..., m\}$ to $I_n = \{1, 2, ..., n\}$, and $\rho_\varphi \in S_N$ are antishuffles.
That means, the summands are of the form:

$$\pm a_+^{i_1} a_+^{i_2} ... a_+^{i_m} a_-^{i_{m+1}} a_-^{i_{m+2}} ... a_-^{i_n}$$

where $i_1 < i_2 < ... < i_m$ and $i_{m+1} > i_{m+2} > ... > i_n$ and $i_k \neq i_l$ for $k \neq l$.

Example: Let $N = 2$. Comparing the definition of normal ordered product of two field with the assertion of the theorem we obtain:

$$\sum_{m=0}^{2} \sum_{\varphi \in J(m,2)} (\prod_{i=1}^{m} a_+^{\rho_\varphi(i)})(\prod_{i=m+1}^{n} a_-^{\rho_\varphi(i)}) =$$
$$a_-^2 a_-^1 + a_-^2 a_-^1.$$

Proof of the theorem:
Note that if $\varphi(1) \neq 1$ then $\rho_\varphi^{-1}(1) = N$. This follows form the definition of ρ_φ. Hence the field a_+^1 can only have the left-most position in the sums over $\varphi \in J(m, N)$, and the field a_-^1 can only have the right-most position in the sums over $\varphi \in J(m, N)$ on the right hand side of the equation of the theorem. In formulas:

$$\sum_{m=0}^{N} \sum_{\varphi \in J(m,N)} (\prod_{i=1}^{m} a_+^{\rho_\varphi(i)})(\prod_{i=m+1}^{N} a_-^{\rho_\varphi(i)}) =$$

$$= \sum_{m=0}^{N} \left(\sum_{\varphi \in J(m,N), \varphi(1)=1} a_+^1 (\prod_{i=2}^{m} a_+^{\rho_\varphi(i)})(\prod_{i=m+1}^{N} a_-^{\rho_\varphi(i)}) + \right.$$
$$\left. + \sum_{k=2}^{N} \sum_{\varphi \in J(m,N), \varphi(1)=k} (\prod_{i=1}^{m} a_+^{\rho_\varphi(i)})(\prod_{i=m+1}^{N-1} a_-^{\rho_\varphi(i)}) a_-^1 \right). \tag{1.37}$$

Note furthermore that the case $m = 0$ does not occur in the first sum $\sum_{\varphi \in J(m,N), \varphi(1)=1} \cdots$. Similarly the case $m = N$ does not occur in the sum $\sum_{k=2}^{N} \sum_{\varphi \in J(m,N), \varphi(1)=k} \cdots$. These observations and considerations are the starting point of the induction.
Let $N = 1$: Then $: a(z) := a(z) = a_+(z) + a_-(z)$.
On the other hand because the anti-shuffle for $m = 0 < 1$ and for $m = 1 = N$ is the identity map we obtain

$$(\prod_{i=1}^{0} a_+^{\rho_\varphi(i)})(\prod_{i=1}^{1} a_-^{\rho_\varphi(i)}) + (\prod_{i=1}^{1} a_+^{\rho_\varphi(i)})(\prod_{i=2}^{1} a_-^{\rho_\varphi(i)}) = a_-^1 + a_+^{\rho_\varphi(1)}.$$

We used the convention $\prod_{k=i}^{j} x_k = 1$ for $i > j$.
Let $N - 1 > 1$ and we suppose that the normal ordered product is given as above.
More precisely we study the $N - 1$ fields $a^2, a^3, ..., a^N$. According to our assumption the normal ordered product of these field can be expressed as

$$: a^2 a^3 ... a^N := \sum_{m=1}^{N} \sum_{\varphi \in \hat{J}(m,N)} \prod_{i=2}^{m} a_+^{\hat{\rho}_\varphi(i)} \prod_{i=m+1}^{N} a_-^{\hat{\rho}_\varphi(i)},$$

where $\hat{J}(m, N)$ is the set of strictly increasing maps $\hat{\rho}_\varphi$ from $\hat{I}_m = \{2, 3, ..., m\}$ into $\hat{I}_N = \{2, 3, ..., N\}$. In other word we shifted the set by one: $\hat{I}_n = I_n + 1$.
We consider now the normal ordered product of N fields. From the definition we have

$$: a^1 a^2 ... a^N :=: a^1(: a^2 a^3 ... a^N :) := a_+^1(: a^2 a^3 ... a^N :) + (: a^2 a^3 ... a^N :) a_-^1$$

We insert the assumption:

$$: a_1 a_2 ... a_N := a_+^1 \left(\sum_{m=1}^{N} \sum_{\varphi \in \hat{J}(m,N)} \prod_{i=2}^{m} a_+^{\hat{\rho}_\varphi(i)} \prod_{i=m+1}^{N} a_-^{\hat{\rho}_\varphi(i)} \right) +$$

$$+ \left(\sum_{m=1}^{N} \sum_{\varphi \in \hat{J}(m,N)} \prod_{i=2}^{m} a_+^{\hat{\rho}_\varphi(i)} \prod_{i=m+1}^{N} a_-^{\hat{\rho}_\varphi(i)} \right) a_-^1.$$

The first summand is exactly the desired term for $\varphi \in J(m, N)$ with $\varphi(1) = 1$ (see equation (1.37)).
The second sum is exactly the second sum (for $\varphi(1) \neq 1$) in equation (1.37).
□

1.4.4 Wick Product

Similar to the above theorem we can give the normal ordered product of fields with different arguments explicitly. Let $a(z), b(w)$ be two fields. The normal ordered product is defined by

$$: a(z)b(w) := a_+(z)b(w) + b(w)a_-(z).$$

1.4.16 Definition *Let $a^1(z), ..., a^N(z) \in End(V)[[z, z^{-1}]]$ be N fields, and $b^1(w), ..., b^M(w) \in End(V)[[w, w^{-1}]]$ be M fields. Then the iterated normal ordered product is defined by*

$$: a^1(z)...a^N(z)b^1(w)...b^M(w) :=: a^1(z) : ... : a^N(z)(: b^1(w)...b^M(w) :) : ... : .$$

We obtain immediately from theorem 1.4.15:

$$\begin{aligned} &: a^1(z)...a^N(z)b^1(w)...b^M(w) := \\ &= \sum_{m=0}^{N} \sum_{\varphi \in J(m,N)} (\prod_{i=1}^{m} a_+^{\rho_\varphi(i)}) \left(\sum_{n=0}^{M} \sum_{\psi \in J(n,M)} (\prod_{i=1}^{n} b_+^{\rho_\psi(i)})(\prod_{i=n+1}^{M} b_-^{\rho_\psi(i)}) \right) (\prod_{i=m+1}^{N} a_-^{\rho_\varphi(i)}) \end{aligned} \tag{1.38}$$

I have not found this expression in any article or book. It is far from being obvious.
We state now the Wick theorem (see also [Kac]). It is worth mentioning that in the proof one assumption in [Kac] will not be used.

1.4.17 Theorem *Let $a^1(z), a^2(z), ..., a^N(z)$ and $b^1(z), b^2(z), ..., b^M(z)$ be fields subject to the following conditions:*

1. *$[[a^i(z)_-, b^j(w)], c^k(z)_\pm] = 0$ for all i, j, k, and $c = a$ or b.*
2. *$[a^i(z)_-, b^j(w)_-] = 0$ for all $1 \leq i \leq N$, $1 \leq j \leq M$.*

Then we have

$$\begin{aligned} &\left(: a^1(z)a^2(z)...a^N(z) :\right) \left(: b^1(z)b^2(z)...b^M(z) :\right) = \\ &\qquad = \sum_{s=0}^{\min(M,N)} \sum_{i_1<...<i_s}^{j_1 \neq ... \neq j_s} [a_-^{i_1}, b^{j_1}(w)]...[a_-^{i_s}(z), b^{j_s}(w)] \cdot \\ &\qquad\qquad \cdot : a^1(z)a^2(z)...a^N(z)b^1(w)...b^M(w) :_{(i_1...i_s j_1...j_s)} \end{aligned}$$

where the subscript $(i_1...i_s j_1...j_s)$ means that the fields $a^{i_1}(z), ..., a^{i_s}(z), b^{j_1}(w), ..., b^{j_s}(z)$ are removed.

The important difference to [Kac] is the fact that we dropped the condition $[a^i(z)_+, b^j(w)_+] = 0$.
We give here a proof because we will formulate in later chapter a Wick product for global fields and it will turn out that the proof is the same.
Proof.

From theorem 1.4.15 we know:

$$
\begin{aligned}
: a^1 a^2 ... a^N : &= \sum_{m=0}^{N} \sum_{\varphi \in J(m,N)} (\prod_{i=1}^{m} a_+^{\rho_\varphi(i)})(\prod_{i=m+1}^{N} a_-^{\rho_\varphi(i)}) \\
: b^1 b^2 ... b^M : &= \sum_{n=0}^{M} \sum_{\psi \in J(n,M)} (\prod_{i=1}^{n} a_+^{\rho_\psi(i)})(\prod_{i=n+1}^{N} a_-^{\rho_\psi(i)})
\end{aligned}
$$

Therefore we have to consider the expressions

$$\left(a_+^{\rho_\varphi(1)}(z)...a_+^{\rho_\varphi(m)}(z)a_-^{\rho_\varphi(m+1)}(z)...a_-^{\rho_\varphi(N)}(z)\right)\left(b_+^{\rho_\psi(1)}(w)...b_+^{\rho_\psi(n)}(w)b_-^{\rho_\psi(n+1)}(w)...b_-^{\rho_\psi(M)}(w)\right)$$

We have to "move" the $a_-^{\rho_\varphi(i)}(z)$ across the $b_+^{\rho_\psi(j)}(w)$ in order to bring this product to the normal ordered form

$$a_+^{\rho_\varphi(1)}(z)...a_+^{\rho_\varphi(m)}(z)b_+^{\rho_\psi(1)}(w)...b_+^{\rho_\psi(n)}(w)b_-^{\rho_\psi(n+1)}(w)...b_-^{\rho_\psi(M)}(w)a_-^{\rho_\varphi(m+1)}(z)...a_-^{\rho_\varphi(N)}(z).$$

For this purpose we use the commutation relation

$$[a_-^{\rho_\varphi(i)}(z), b_+^{\rho_\psi(j)}(w)] = a_-^{\rho_\varphi(i)}(z)b_+^{\rho_\psi(j)}(w) - b_+^{\rho_\psi(j)}(w)a_-^{\rho_\varphi(i)}(z)$$

Due to condition two of the theorem one has

$$[a_-^{\rho_\varphi(i)}(z), b_+^{\rho_\psi(j)}(w)] = [a_-^{\rho_\varphi(i)}(z), b^{\rho_\psi(j)}(w)]$$

and due to condition one of the theorem the expressions $[a_-^{\rho_\varphi(i)}(z), b^{\rho_\psi(j)}(w)]$ commute with all fields, hence can be moved to the left. $\square$

Chapter 2

Field Representations

In this chapter we address the representations of certain loop algebras and their central extensions, the current algebras. We also introduce the Virasoro algebra and discuss briefly some aspects of its representations. This chapter can be considered as a "technical" chapter in the sense that the most computations are done in order to explain the examples of vertex algebras without technicalities. Furthermore this chapter has a higher genus "mirror", namely chapter 9.
The chapter is organized as follows: In the first section we start with well known facts about central extensions of Lie algebras. First we give the definition of central extensions which are parametrized by the second cohomology group. The next subsections are devoted to prominent examples of central extensions namely the Virasoro algebra (as central extension of the Witt algebra), the Heisenberg algebra (as the central extension of the loop algebra of a vector space), and the (non-twisted) Kac-Moody algebras (as the central extensions of loop algebras associated to simple finite dimensional Lie algebras).
In the third section we introduce the current algebra associated to a finite dimensional simple Lie algebra and the operator product expansion of their associated distributions.
In the last section we state the Sugawara construction which shows that the Virasoro algebra is contained in the field representations of the Heisenberg algebra and the affine Lie algebras (also called WZNW models).

2.1 Virasoro Algebra and Central Extensions of Loop Algebras

First we recall some facts about central extensions of Lie algebras. Then we introduce the Virasoro algebra, and we introduce the notion of the loop algebra. At the end of this section we introduce central extensions of certain loop algebras namely the Heisenberg algebra, the algebra of free bosons, and non-twisted Kac-Moody algebras.

2.1.1 Central Extensions and Cocycles

Let $\mathfrak{g}$ be any complex Lie algebra. A central extension of a Lie algebra is the middle term of a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

such that the abelian Lie algebra $\mathbb{C}$ is central in $\widehat{\mathfrak{g}}$.
Two central extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ are said to be equivalent if there is a Lie algebra homomorphism $\phi : \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathfrak{g}}_1 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ \| & & \| & & \phi\downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathfrak{g}}_2 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

Central Extensions are classified up to equivalence by the second Lie algebra cohomology group $H^2(\mathfrak{g}, \mathbb{C})$. That means they are classified by 2-cocycles up to coboundaries.
We shall explain this more precisely.
Let $\mathfrak{g}$ be a Lie algebra and let be γ a Lie algebra **two-cocycle** of $\mathfrak{g}$ with values in $\mathbb{C}$, i.e. γ is an anti-symmetric bilinear form of $\mathfrak{g}$ (that means $\gamma(f,g) = -\gamma(g,f)$ $\forall f, g \in \mathfrak{g}$) obeying

$$\gamma([f,g],h) + \gamma([g,h],f) + \gamma([h,f],g) = 0. \ \forall f,g,h \in \mathfrak{g}.$$

A **central extension** of the Lie algebra $\mathfrak{g}$ is the vector space $\mathfrak{g} \oplus \mathbb{C}$ together with the structure equations (where $\hat{f} = (f, 0)$, $\hat{K} = (0, 1)$):

$$[\hat{f}, \hat{g}] = \widehat{[f,g]} + \gamma(f,g)K, \qquad [f, K] = 0, \qquad f, g \in \mathfrak{g}.$$

Two two-cocycles γ_1 and γ_2 of $\mathfrak{g}$ are called **equivalent**, if there exists a linear form $\phi : \mathfrak{g} \to \mathbb{C}$ such that

$$\gamma_1(f,g) - \gamma_2(f,g) = \phi([f,g]).$$

We have as an example of central extensions the following assertion (see [HN] II.5.14.).

2.1.1 Remark (Second Lemma of Whitehead) *The central extensions of simple finite dimensional Lie algebras are trivial.*

2.1.2 Virasoro Algebra

The Virasoro algebra is the most important Lie algebra in conformal field theory. We shall give the basic notions.

2.1.2 Definition (Witt Algebra) *The Witt algebra is defined as the infinite dimensional Lie algebra with basis $\{l_n : n \in \mathbb{Z}\}$ with relations*

$$[l_n, l_m] = (n-m)l_{n+m}.$$

The Witt algebra can be interpreted as the algebra of polynomial vector fields on the unit circle by

$$l_n := -z^{n+1}\frac{d}{dz}$$

We see especially from the definition of the Witt algebra

$$[l_0, l_{-1}] = l_{-1}, \quad [l_0, l_1] = -l_1, \quad [l_{-1}, l_1] = (-2)l_0.$$

Thus the elements l_{-1}, l_0, l_1 generate a Lie subalgebra.
From the mapping

$$\begin{array}{ccc} l_{-1} & \mapsto & \frac{1}{2}\cdot l_{-1} =: f \\ l_0 & \mapsto & (-2)l_0 =: h \\ l_1 & \mapsto & \frac{1}{2}\cdot l_1 =: e \end{array}$$

we can deduce the following.

2.1.3 Remark *The subalgebra generated by the elements l_{-1}, l_0, l_1 is isomorphic to the simple Lie algebra $sl(2,\mathbb{C})$. This is the three-dimensional Lie algebra spanned by vectors e, h, f with relations*

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The following theorem is proved for instance in [Schotten].

2.1.4 Theorem (Virasoro Algebra) *There is up to equivalence only one central extension of the Witt algebra, called the* ***Virasoro algebra****. This means*

$$H^2(Witt, \mathbb{C}) \cong \mathbb{C}.$$

The Virasoro algebra is an infinite dimensional vector space with basis $\{L_n, K\}_{n\in\mathbb{Z}}$ and with the relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n,-m}K \quad [L_n, K] = 0$$

The number $\frac{1}{12}$ is a traditional factor. It comes from the fact

$$\zeta(-1) = -\frac{1}{12}$$

where $\zeta(s)$ is the Riemann zeta function which has for $Re(s) > 1$ the expansion

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

At the end of this chapter we will illustrate where the term $\frac{1}{12}$ comes from.

2.1.3 Loop Algebras and its central extensions

Here we discuss the very important notion of loop algebras and their affine extensions.
Let $\mathfrak{g}$ be a Lie algebra with an invariant symmetric bilinear form $(\cdot|\cdot)$. Invariant means $([a,b]|c) = (a|[b,c])$ (for any $a,b,c \in \mathfrak{g}$), and symmetric means $(a|b) = (b|a)$.
More generally we can consider a Lie **superalgebra** $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with an invariant supersymmetric bilinear form $(\cdot|\cdot)$. Invariant means again

$$([a,b]|c) = (a|[b,c]), \quad a,b,c \in \mathfrak{g},$$

and supersymmetric means

$$(a|b) = (-1)^{|a|}(b|a), \text{ where } |a| \text{ means parity 0 or 1.}$$

By the definition of the supersymmetric bilinear form we have especially

$$(a|b) = (b|a), \quad a,b \in \mathfrak{g}_{\bar{0}}, (a|b) = -(b|a), \quad a,b \in \mathfrak{g}_{\bar{1}}$$

$$\text{and } (a|b) = 0, \quad a \in \mathfrak{g}_{\bar{0}}, b \in \mathfrak{g}_{\bar{1}}, (a|b) = 0, \quad a \in \mathfrak{g}_{\bar{1}}, b \in \mathfrak{g}_{\bar{0}}$$

The first line is clear, and for the second line we get for $a \in \mathfrak{g}_{\bar{0}}, b \in \mathfrak{g}_{\bar{1}}$:

$$(a|b) = (b|a) = -(a|b) \Rightarrow (a|b) = 0 = (b|a).$$

We can now turn to the definition of loop algebras.

2.1.5 Definition (Loop Algebra) *Let $\mathfrak{g}$ be a Lie (super)algebra with an invariant (super)symmetric bilinear form.*
The ***loop algebra*** *is defined by*

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$$

with commutation relations:

$$[a \otimes f(t), b \otimes g(t)] = [a,b] \otimes f(t)g(t).$$

If we write $a_n := a \otimes t^n$, then we can write for the commutation relations:

$$[a_n, b_m] = [a,b]_{n+m}.$$

We consider $\tilde{\mathfrak{g}}$ as the Lie (super)algebra of regular maps from $\mathbb{C}^*$ to $\mathfrak{g}$. This justifies the name **loop algebra**.
A two-cocycle of the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ is given by

$$\gamma(a \otimes f(t), b \otimes g(t)) = (a|b) Res_t g(t) \frac{d}{dt} f(t).$$

This is indeed a two-cocycle. The antisymmetry follows from the property of residues and the product rule for derivations:

$$0 = Res_t \frac{d}{dt}\left(f(t)g(t)\right) = Res_t \frac{d}{dt} f(t)g(t) + Res_t f(t) \frac{d}{dt} g(t).$$

The cocycle-identity follows from the invariance of the bilinear form and a similar equation like the equation above. Therefore the following definition is sensible.

2.1.6 Definition (Affinization, Current Algebra) *Let $\mathfrak{g}$ be a Lie (super)algebra with an invariant (super)symmetric bilinear form $(\cdot|\cdot)$. The* ***affinization*** *of $\mathfrak{g}$ is a central extension of the corresponding loop algebra $\tilde{\mathfrak{g}}$ by a one dimensional center $\mathbb{C}K$:*

$$\widehat{\mathfrak{g}} := \tilde{\mathfrak{g}} \oplus \mathbb{C}K$$

defined by the commutation relations

$$[a \otimes f(t), b \otimes g(t)] = [a, b] \otimes f(t)g(t) + (a|b)Res_t g(t)\frac{d}{dt}f(t)K, \;\; [\widehat{\mathfrak{g}}, K] = 0.$$

The algebra $\widehat{\mathfrak{g}}$ is also called ***current algebra****.*

If we write again $a_n := a \otimes t^n$, then we can write for the commutation relations:

$$[a_n, b_m] = [a, b]_{n+m} + n(a|b)\delta_{n,-m}K.$$

The Kronecker delta comes from the computation of the residue for the monomials t^n, t^m:

$$Res_t t^m \frac{d}{dt}(t^n) = Res_t t^m n t^{n-1} dt = n\delta_{n,-m}.$$

2.1.4 Heisenberg Algebra and Affine Kac-Moody Algebras

In this subsection we fix the notation for the most relevant affinization we are going to deal with in the subsequent chapters.

Heisenberg Algebra

Let $\mathbb{C}$ be the commutative Lie algebra of dimension one over the complex numbers. The bilinear form is just the multiplication of two complex numbers. A basis is given by $\{1\}$. The corresponding loop algebra is

$$\mathbb{C} \otimes \mathbb{C}[z, z^{-1}]$$

with commutator relation:

$$[a \otimes f(t), a \otimes g(t)] = 0 \tag{2.1}$$

If we put especially $a_n := a \otimes t^n$ then we obtain for the structure equations

$$[a_n, a_m] = 0 \qquad n, m \in \mathbb{Z}. \tag{2.2}$$

The affinization is given by

$$[a \otimes f(t), a \otimes g(t)] = Res_t g(t)\frac{d}{dt}f(t) \cdot K \tag{2.3}$$

Especially:

$$[a_n, a_m] = n\delta_{n,-m}K. \tag{2.4}$$

The affinization of the abelian one-dimensional Lie algebra $\mathbb{C}$ is called the **Heisenberg algebra**.

Free Bosons
Let $\mathfrak{h}$ be a complex d-dimensional (super)vector space with invariant bilinear form $(\cdot|\cdot)$. The affinization $\widehat{\mathfrak{h}}$ is given by the relations

$$[a \otimes f(t), b \otimes g(t)] = (a|b) Res_t g(t) \frac{d}{dt} f(t) \cdot K \tag{2.5}$$

Especially

$$[a_n, b_m] = n(a|b)\delta_{n,-m}K. \tag{2.6}$$

The affinization of $\widehat{\mathfrak{h}}$ of $\mathfrak{h}$ is called the algebra of **free bosons**.

Kac-Moody Algebras
The notion affine Lie algebras is the abbreviation for affine Kac-Moody Lie algebras. These algebras are infinite dimensional. They can be constructed as central extensions of loop algebras of simple finite dimensional Lie algebras. Their Cartan matrix is singular, and its proper minors are positive definite. For details we refer the reader to Kac's book [K].
Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Let the loop algebra $\tilde{\mathfrak{g}}$ be defined by

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

Denote by $\hat{\mathfrak{g}}$ be the central extension of the loop algebra $\tilde{\mathfrak{g}}$:

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}} \to 0$$

There is (up to scalar multiplicities) only one central extension.
The commutation relations of $\hat{\mathfrak{g}}$ are

$$[a \otimes f(t), b \otimes g(t)] = [a, b] \otimes f(t)g(t) + (a, b) Res_t g(t) df(t) K, \text{ and } [\hat{\mathfrak{g}}, K] = 0$$

where the pairing $(\cdot|\cdot)$ is the Killing form.
If we set $a_n := a \otimes t^n$ then we have especially:

$$[a_n, b_m] = [a, b]_{n+m} + (a, b)n\delta_{n,-m}.$$

2.2 Operator Product Expansions

2.2.1 Field Representation

The following definition is due to Kac [Kac] (Def. 3.4a)

2.2.1 Definition (Field Representation) *Let $\mathfrak{g}$ be a Lie superalgebra spanned by coefficients of a family of mutually local formal distributions $\left\{a^i(z)\right\}_{i\in I}$ (I an index set).*
*A representation of $\mathfrak{g}$ in a vector space V is called a **field representation** if all the formal distributions $a^i(z)$ are represented by fields, i.e.*

$$\forall v \in V \forall i \in I \exists n_0 \in \mathbb{Z}_+ \forall n \geq n_0 : \quad a^i_n(v) = 0$$

In other words the coefficients of a formal distribution $a(z) = \sum_n a_n z^{-n-1}$ where $a_n \in U$, and U is an associative algebra, are mapped to endomorphisms of a vector space V:

$$a^i_n \mapsto \rho(a^i_n) \in End(V)$$

and $\rho(a(z))(v) \in End(V)[[z]][z^{-1}]$.
We will drop ρ in the next section and will simply write $a(z)$ for $\rho(a(z))$.

2.2.2 OPE of Free Bosons

In this section we address the operator product expansion of the so called free bosons. Free bosons are the affinization of an abelian Lie algebra. They are the most simple example conformal field theory is dealing with.
Let $\mathfrak{h}$ be a finite dimensional (super)vector space of (super)dimension $\text{sdim}(\mathfrak{h}) = d$. Let be $(\cdot|\cdot)$ be an invariant bilinear form.
According to the last section the affinization of $\mathfrak{h}$ is given by the current algebra

$$\hat{\mathfrak{g}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus K$$

with the commutation relations

$$[a_n, b_m] = (a|b) n \delta_{n,-m} K.$$

2.2.2 Proposition *Let $\mathfrak{h}$ be a finite-dimensional abelian Lie (super)algebra, and let $\hat{\mathfrak{g}}$ be the affinization of $\mathfrak{h}$.*
Then the following equations are equivalent:

$$\begin{aligned} [a_n, b_m] &= n(a,b) K \delta_{n,-m} && (2.7) \\ [a(z), b(w)] &= (a,b) K \partial_w \delta(z-w) && (2.8) \\ a(z)b(w) &= i_{z,w} \frac{(a|b)K}{(z-w)^2} + :a(z)b(w): \quad a, b \in \mathfrak{h}. && (2.9) \end{aligned}$$

Proof. This is an immediate consequence of OPE theorem 1.3.4 of the first chapter. More precisely we put

$$[a_n, b_m] = n(a|b) K \delta_{n,-m} = \binom{n}{1} c^1_{n+m-1}$$

where $c^1_{n+m-1} := (a|b) K \delta_{n,-m}$.
Therefore we get $N = 1$, $c^0(w) = 0$, and $c^1(w) = (a|b) K w^0$. $\square$
The operator product expansions of the derivations are given as follows.

2.2.3 Proposition *Let be $\mathfrak{h}$ and $\hat{\mathfrak{g}}$ as in the above proposition. Then we get*

$$[\partial_z^{(n)}a(z), \partial_w^{(m)}b(w)] = (a,b)(-1)^n\partial_w^{(n+m)}\delta(z-w).$$

This corollary is an immediate consequence of the properties of the delta distribution discussed in chapter 1 (see proposition 1.1.11).

2.2.4 Corollary *The distributions $\{\partial^i a(z)\}_{i\in\mathbb{Z}_{\geq 0}}$ are mutually local.*

Because of this corollary we can define a field representation of this family of mutually local distributions.

2.2.3 OPE of WZNW-Fields

In this section we address the operator product expansions for affine Lie algebras.

2.2.5 Proposition *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with the Killing form as invariant bilinear form, and let $\hat{\mathfrak{g}}$ be the associated current algebra of $\mathfrak{g}$. Let a_n denote the elements $a\otimes t^n \in \hat{\mathfrak{g}}$ with $a\in\mathfrak{g}$. Then the following equations are equivalent:*

$$\begin{aligned}
[a_n, b_m] &= [a,b]_{n+m} + (a|b)n\cdot K\delta_{n,-m} &\quad (2.10)\\
[a(z), b(w)] &= [a,b]\delta(z-w) + (a|b)K\partial\delta(z-w) &\quad (2.11)\\
a(z)b(w) &= i_{z,w}\frac{[a,b]}{z-w} + i_{z,w}\frac{(a|b)K}{(z-w)^2} + :a(z)b(w): &\quad (2.12)\\
[a_-(z), b(w)] &= i_{z,w}\frac{[a,b]}{z-w} + i_{z,w}\frac{(a|b)K}{(z-w)^2} &\quad (2.13)
\end{aligned}$$

Proof. The equivalence of the equations is clear from the OPE theorem 1.3.4. We carry out the relation between the first and the second relations. The remaining equivalences are obvious.

$$\begin{aligned}
[a(z), b(w)] &= \sum_{n,m\in\mathbb{Z}}[a_n, b_m]z^{-n-1}w^{-m-1} = \\
= \sum_{n,m\in\mathbb{Z}}[a,b]_{n+m}z^{-n-1}w^{-m-1} &+ \sum_{n,m\in\mathbb{Z}}(a|b)k\delta_{n,-m}z^{-n-1}w^{-m-1} = \\
= \sum_{n,j\in\mathbb{Z}}[a,b]_j z^{-n-1}w^{-(j-n)-1} &+ (a|b)k\sum_{n\in\mathbb{Z}} nz^{-n-1}w^{n-1} = \\
= \sum_{j\in\mathbb{Z}}[a,b]_j w^{-j-1}\sum_{n\in\mathbb{Z}} z^{-n-1}w^n &+ (a|b)k\sum_{n\in\mathbb{Z}} nz^{-n-1}w^{n-1} = \\
= [a,b](w)\delta(z-w) &+ (a|b)k\partial_w\delta(z-w). \quad \square
\end{aligned}$$

2.2.6 Proposition *Let be $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ as above. Then we have for the derivations:*

$$[\partial_z^{(i)}a(z), \partial_w^{(j)}b(w)] = [a,b](-1)^i\partial_w^{(i+j)}\delta(z-w) + (a,b)(-1)^i\partial_w^{(i+j+1)}\delta(z-w).$$

We can follow that the collection of distributions $\{\partial^{(j)}a(z)\}_{a\in\mathfrak{g},j\in\mathbb{Z}_{\geq 0}}$ is mutually local.

2.2.4 Sugawara Construction and OPE of Virasoro Fields

2.2.7 Theorem (Sugawara Construction) *Let $\mathfrak{h}$ be a complex vector (super)space of (super)dimension d.*
Let $\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus K$ be the affinization of $\mathfrak{h}$.
Let be V a vector space such that we have field representation associated to the algebra $\hat{\mathfrak{h}}$, and suppose K acts as $k\cdot id_V$ where k is a non-zero number.
Then the field

$$\begin{aligned} L(z) &\stackrel{\text{def}}{=} \frac{1}{2k}\sum_{i=1}^{d} :a^i(z)a^i(z): &(2.14)\\ &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2} &(2.15)\end{aligned}$$

satisfies the (equivalent) identities

$$\begin{aligned} L(z)L(w) &= \frac{\frac{d}{2}}{(z-w)^4} + \frac{L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)} + :L(z)L(w): \\ [L(z),L(w)] &= \frac{d}{2}\partial_w^{(3)}\delta(z-w) + L(w)\partial_w\delta(z-w) + \partial L(w)\delta(z-w) \\ [L_n,L_m] &= (n-m)L_{n+m} + \frac{1}{12}\cdot\frac{d}{2}\cdot(n^3-n)\delta_{n,-m}. \end{aligned}$$

In the proof of theorem the term $\frac{1}{12}$ occurs naturally (in the appendix the proof is carried out).
In the classification of simple Lie algebras the **dual Coxeter number** plays a crucial role. It is defined as the half of the eigenvalue of the Casimir operator. This is equivalent to the equation:

$$\sum_{i=1}^{d}[a^i,[a^i,x]] = 2h^\vee \qquad \forall x\in\mathfrak{g}.$$

where $\{a^i\}$ is a basis of the Lie algebra $\mathfrak{g}$.
For instance for the simple Lie algebras of type A_n the dual Coxeter number is $h^\vee = n+1$.
Now we can state the Sugawara construction for affine Kac-Moody algebras.

2.2.8 Theorem *Let $\mathfrak{g}$ be a complex simple finite-dimensional Lie algebra.*
Let $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus K$ be the affinization of $\mathfrak{g}$.

Let be V a vector space such that we have field representation associated to the algebra $\hat{\mathfrak{g}}$, and suppose K acts as $k \cdot id_V$ where k is a non-zero number. Then the field

$$\begin{aligned} L(z) &\stackrel{\text{def}}{=} \frac{1}{2(k+h^\vee)} \sum_{i=1}^{d} : a^i(z) a^i(z) : \\ &= \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \end{aligned}$$

satisfies the (equivalent) identities

$$\begin{aligned} L(z)L(w) &= \frac{\frac{d}{2}}{(z-w)^4} + \frac{L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)} + : L(z)L(w) : \\ [L(z), L(w)] &= \frac{d}{2} \partial_w^{(3)} \delta(z-w) + L(w) \partial_w \delta(z-w) + \partial L(w) \delta(z-w) \\ [L_n, L_m] &= (n-m) L_{n+m} + \frac{1}{12} \cdot \frac{d}{2} \cdot (n^3 - n) \delta_{n,-m}. \end{aligned}$$

The equivalence is obvious from the OPE theorem of the first chapter. Therefore it is only necessary to prove one of these equivalent equations. This is done in the appendix.

Chapter 3

Vertex Algebras

Vertex algebras provide the rigorous mathematical framework in which conformal field theory can be studied. Furthermore vertex algebras are important in their own right because some of their applications in mathematics are very spectacular, e.g. Monster Moonshine (see [FLM]) or the vertex algebra of the cohomology ring of Hilbert schemes (see [Nak]).
Vertex algebras can be considered from several points of view.
From an "algebraic" point of view vertex algebras can be considered as a generalization of **unital associative commutative algebras** by inserting an additional parameter. See the discussion in the first section.
In the spirit of this thesis the fields, which are the basic objects of vertex algebras, can be considered geometrically as certain sections in powers of the canonical bundle on the Riemann sphere. This is discussed in the next chapters.
The chapter is organized as follows.
In section 3.1 the definition of a vertex algebra and the notion of a conformal vertex algebra is introduced. We discuss their relation to unital associative commutative algebras (section1). Furthermore some important aspects of the structure theory of vertex algebras is discussed, especially the skewsymmetry and the associativity.
In section 3.2 we present some aspects of the categorical properties of vertex algebras.
In section 3.3 we discuss some aspects of representations of vertex algebras.

3.1 Generalities About Vertex Algebras

3.1.1 Definition

The following definition of a vertex algebra is due to Frenkel and Ben-Zvi [FB-Z].

3.1.1 Definition (Vertex Algebra) *A **vertex algebra** is a vector space V*

with following data:

1. *Space of states: V is a $\mathbb{Z}_+$-graded vector space*
$$V = \bigoplus_{m=0}^{\infty} V_m, \quad \dim V_m < \infty$$
2. *Vacuum vector:* $\mathbf{1} = |0\rangle \in V_0$
3. *Translation operator: a linear operator $T : V \to V$ of degree one.*
4. *Linear operation (Vertex Operators):*
$$Y(\cdot, z) : V \to End(V)[[z, z^{-1}]]$$
with
$$V_m \ni a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$
where $Y(a, z)$ is said to be a field of conformal dimension m.
This means $\deg a_n = -n + m - 1$.

These data are subject to the following axioms

1. *Vacuum axiom: $Y(|0\rangle, z) = id_V$; $\forall a \in V$: $Y(a, z)|0\rangle|_{z=0} = a$*
2. *Translation axiom: $\forall a \in V$: $[T, Y(a, z)] = \partial_z Y(a, z)$ and $T(|0\rangle) = 0$*
3. *Mutual locality: $\forall a, b \in V$ there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$: $[Y(a, z), Y(b, w)](z - w)^n = 0$.*

It is not necessary to consider a graded vector space. But with respect to the higher genus generalization I prefer this definition.
A vertex algebra V given by the above properties with grading deg, vacuum vector v_0, translation operator T and state-field correspondence Y is written for short as $(V, \deg, v_0, T, Y)$.
It is necessary to address the notion of "conformal dimension":
We also could write
$$V_m \ni a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-m}$$
where the operators a_n have degree $-n$, i.e. $a_n V_k \subset V_{k-n}$.
Vertex algebras can be considered as **generalizations of unital associative commutative algebras**:
Let U be an associative commutative algebra with unit $1 \in U$, i.e. for any $a, b, c \in U$ we have the relations
$$a \cdot b = b \cdot a, \quad a \cdot 1 = a, \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

The commutativity corresponds to the mutual locality axiom, the unit corresponds to the vacuum axiom, and the associativity corresponds to the formulas in theorem 3.1.10.
In fact we can realize a unital commutative associative algebra as a vertex algebra (see the discussion in [FB-Z] and [BK]).
We introduce now the notion of a conformal vertex algebra. The examples given in the next section are conformal vertex algebras.

3.1.2 Definition (Conformal Vertex Algebra) *A* ***conformal vertex algebra of central charge c*** *is a vertex algebra $(V, \deg, v_0, T, Y)$ together with a distinguished vector $\omega \in V_2$ such that the coefficients L_n of the corresponding field*

$$Y(\omega, z) = \sum_n L_n z^{-n-2}$$

form a Virasoro algebra where the central element acts as multiplication by the number $c \in \mathbb{C}$.

3.1.2 Consequences

The first trivial observation from the vacuum axiom is

$$a_{-1}\,|0\rangle = a.$$

The next observation is

$$[T, a_n] = (-n)a_{n-1}.$$

3.1.3 Remark

$$T(a) = a_{-2}\,|0\rangle. \tag{3.1}$$

Proof. We use the translation axiom:

$$[T, Y(a,z)]\,|0\rangle|_{z=0} = \partial_z Y(a,z)\,|0\rangle|_{z=0}.$$

On the left hand side we get $[T, Y(a,z)]\,|0\rangle|_{z=0} = TY(a,z)\,|0\rangle|_{z=0} - Y(a,z)T\,|0\rangle|_{z=0}$.
Applying the second and the third property of the vacuum axiom we get for the left hand side $T(a)$.
On the right hand side we have
$\partial_z Y(a,z)\,|0\rangle|_{z=0} = (\sum_n (-n)a_{n-1}\,|0\rangle z^{-n-1})|_{z=0} = a_{-2}\,|0\rangle$.
In the last equation we used the second property of the vacuum axiom. □
The following proposition justifies the notion "translation operator".

3.1.4 Proposition

$$\begin{aligned} e^{zT}(a) &= Y(a,z)\,|0\rangle && (3.2)\\ e^{zT}Y(a,w)e^{-zT} &= Y(a, z+w) \text{ in the domain } |z| < |w| && (3.3)\\ Y(a_n b, z)\,|0\rangle &= Y(a,z)_n Y(b,z)\,|0\rangle && (3.4) \end{aligned}$$

Proof. For the first equation we start with $a = a_{-1}\left|0\right\rangle$, and we know because of equation (3.1): $T(a) = T(a_{-1}\left|0\right\rangle) = a_{-2}\left|0\right\rangle$. Suppose we have $T^n(a) = n!a_{n-1}\left|0\right\rangle$, then we get

$$T^{n+1}(a) = TT^n(a) = T(n!a_{-n-1}\left|0\right\rangle) = n!Ta_{-n-1}\left|0\right\rangle =$$

$$= n![Ta_{-n-1}]\left|0\right\rangle + n!a_{-n-1}T\left|0\right\rangle = n!(n+1)a_{-n-2}\left|0\right\rangle + 0$$

In the last equation we used $[T, a_n] = (-n-1)a_{n-1}$ (translation axiom), and $T\left|0\right\rangle = 0$ (vacuum axiom). Therefore we get the assertion.
For the second equation we write (this equation is discussed in the remark below):

$$e^{zT}Y(a,w)e^{-zT} = \sum_{n=0}^{\infty}\frac{1}{n!}[(zT)^n, Y(a,w)] \tag{3.5}$$

Because of the translation axiom we can write

$$\sum_{n=0}^{\infty}\frac{1}{n!}[(zT)^n, Y(a,w)] = \sum_{n=0}^{\infty}\frac{1}{n!}z^n\partial_w Y(a,w) = e^{z\partial_w}Y(a,w)$$

Now we can apply the Taylor formula: $e^{z\partial_w}Y(a,w) = Y(a, w+z)$.
The left hand side of equation (3.3) satisfies the equation (due to the translation axiom)

$$\partial_z Y(a_n b, z)\left|0\right\rangle = TY(a_n b, z)\left|0\right\rangle.$$

Furthermore the right hand side of equation (3.3) satisfies the equation

$$\partial_z Y(a,z)_n Y(b,z)\left|0\right\rangle = TY(a,z)_n Y(b,z)\left|0\right\rangle$$

Thus the left hand side and the right hand side satisfy the same differential equation of the form

$$\partial f(z) = Tf(z), \quad f(z) \in V[[z]]$$

If the both sides have the same initial value then they coincide. In fact because of corollary 1.4.7 in chapter 1 we have

$$Y(a,z)_n Y(b,z)\left|0\right\rangle|_{z=0} = (Y(a,z)_n Y(b,z))_{-1}\left|0\right\rangle \overset{\text{cor.}}{=} a_n b_{-1}\left|0\right\rangle$$

and because of the vacuum axiom we obtain

$$Y(a_n b, z)\left|0\right\rangle|_{z=0} = a_n b$$

Furthermore we have because of the vacuum axiom $b = b_{-1}\left|0\right\rangle$, hence we have the desired equality. □

3.1.5 Remark *In the proof of the above proposition we used the equation*

$$e^{zT}Y(a,w)e^{-zT} = \sum_{n=0}^{\infty}\frac{1}{n!}[(zT)^n, Y(a,w)].$$

In order to show this equation we differentiate both sides, and we obtain for the right hand side:

$$\begin{aligned}\frac{d}{dz}e^{zT}Y(a,w)e^{-zT} &= Te^{zT}Y(a,w)e^{-zT} - e^{zT}Y(a,w)e^{-zT}T \\ &= [T, e^{zT}Y(a,w)e^{-zT}],\end{aligned}$$

and for the left hand side we get

$$\begin{aligned}\frac{d}{dz}\sum_{n=0}^{\infty}\frac{1}{n!}[(zT)^n, Y(a,w)] &= \sum_{n=1}^{\infty}\frac{1}{n!}nz^{n-1}T^nY(a,w) - \frac{1}{n!}Y(a,w)nz^{n-1}T^n \\ &= [T, \sum_{n=0}^{\infty}\frac{1}{n!}[(zT)^n, Y(a,w)]\end{aligned}$$

We see that the two sides satisfy the same differential equation

$$\frac{d}{dz}f(z) = \mathrm{ad}Tf(z)$$

In order to obtain equality we have to show that both sides satisfy the same initial condition.
In fact for the left hand side we have $e^{zT}Y(a,w)e^{-zT}|_{z=0} = Y(a,w)$, and for the right hand side we obtain $\sum_{n=0}^{\infty}\frac{1}{n!}[(zT)^n, Y(a,w)]|_{z=0} = Y(a,w)$ as well.
The last identity of the proposition is shown similarly as the first identity (see [Kac] proposition 4.1). □
We are going to use the above proposition in order to proof the skewsymmetry and later the associativity. Especially the skew symmetry is important in order to show that we have a reasonable notion of a quotient vertex algebra V/I (where I is an ideal, see next section for these notions).

3.1.6 Theorem (Skewsymmetry) *Let V be a vertex algebra, and let $a, b \in V$ be two elements of V. Then we have*

$$Y(a,z)b = e^{zT}Y(b,-z)a.$$

Proof. This is a consequence of proposition 3.1.4 and the locality axiom. There exists an $N \in \mathbb{N}$ such that

$$(z-w)^N Y(a,z)Y(b,w) = (z-w)^N Y(b,w)Y(a,z)$$

Both sides applied to the vacuum vector v_0 we get

$$(z-w)^N Y(a,z)Y(b,w)v_0 = (z-w)^N Y(b,w)Y(a,z)v_0$$

From equation (3.2) of proposition 3.1.4 we know

$$Y(b,w)v_0 = e^{wT}b, \qquad Y(a,z)v_0 = e^{zT}a.$$

Thus we obtain

$$(z-w)^N Y(a,z)e^{wT}b = (z-w)^N Y(b,w)e^{zT}a.$$

We know from equation (3.3) of proposition 3.1.4

$$e^{-zT}Y(b,w)e^{zT} = Y(b,-z+w) \qquad \text{for } |z|<|w|$$

Hence we get for $|z|<|w|$:

$$(z-w)^N Y(a,z)e^{wT}b = (z-w)^N e^{zT} Y(b,w-z)a.$$

$Y(b,w-z)$ is a field, i.e. $Y(b,w-z)a \in V[[(w-z)]][(w-z)^{-1}]$. If we choose N sufficiently high, then $(z-w)^N Y(b,w-z)a \in V[[(w-z)]]$. For such an N we can put $w=0$ and obtain

$$z^N Y(a,z)b = z^N e^{zT} Y(b,-z)a.$$

Dividing by z^N we obtain the assertion. □

3.1.7 Corollary (Borcherds' nth product) *Let V be a vertex algebra. Then for all $a,b \in V$ we have the identity*

$$a_n b = -\sum_{j=0}^{\infty} \frac{(-1)^{n+j}}{j!} T^j(b_{n+j}a) \quad \forall n \in \mathbb{Z}. \tag{3.6}$$

Proof. The equation is the coefficient version of the skewsymmetry. That means we only have to compare the coefficients of the two sides of the skewsymmetry given in the above theorem. For the left hand side we have

$$Y(a,z)b = \sum_n a_n b z^{-n-1}$$

For the right hand side we have

$$\begin{aligned} e^{zT}Y(b,-z)a &= \left(\sum_{j=0}^{\infty} z^j T^{(j)}\right)\left(\sum_m (-1)^{-m-1} b_m a z^{-m-1}\right) \\ &= \sum_{j=0}^{\infty}\sum_m (-1)^{-m-1} T^{(j)}(b_m a) z^{j-m-1} \\ &= \sum_{j=0}^{\infty}\sum_n (-1)^{-n-j-1} T^{(j)}(b_{n+j})a z^{-n-1}. \end{aligned}$$

Because of $(-1)^{-1} = -1$ we obtain the assertion. □

We can, in some sense, invert the assertion of proposition 3.1.4 as the following theorem shows.

3.1.8 Theorem (Goddards Uniqueness Theorem) *Let V be a vertex algebra and let $B(z)$ a field which is mutually local with all the fields $Y(a,z)$. Suppose for some $b \in V$:*

$$B(z)\,|0\rangle = e^{zT} b.$$

Then $B(z) = Y(b,z)$.

The proof is essentially an application of equation (3.2) of proposition 3.1.4. It can be found in [K] theorem 4.4. $\square$
From the first and last identity of proposition 3.1.4 we have the identity:

$$Y(a_n b, z)\,|0\rangle = Y(a,z)_n Y(b,z)\,|0\rangle = e^{zT}(a_n b).$$

We can now apply the above theorem by defining $B(z) := Y(a_n b, z)$, and we obtain because of the mutual locality of this field the following assertion.

3.1.9 Proposition *For any two elements $a, b \in V$ of a vertex algebra V one has*

$$Y(a,w)_j Y(b,w) = Y(a_j b, w). \tag{3.7}$$

Now we can state the general OPE formula for vertex operators.

3.1.10 Theorem (Associativity) *Let $a, b \in V$. Then we have the following equivalent formulas*

$$\begin{aligned}
[Y(a,z), Y(b,w)] &= \sum_{j=0}^{\infty} Y(a_j b, w)\partial_w^{(j)}\delta(z-w) \\
Y(a,z)\cdot Y(b,w) &= \sum_{j=0}^{\infty} \frac{Y(a_j b, w)}{(z-w)^{n+1}} + :Y(a,z)Y(b,w): \\
[a_n, b_m] &= \sum_{j=0}^{\infty} \binom{n}{j} (a_j b)_{n+m-j} \\
[a_n, Y(b,z)] &= \sum_{j=0}^{\infty} \binom{n}{j} Y(a_j b, z) z^{m-j}
\end{aligned}$$

Proof. From the OPE theorem of the first chapter we know

$$[Y(a,z), Y(b,w)] = \sum_{j=0}^{\infty} \left(Y(a,w)_j Y(b,w)\right) \partial_w^{(j)}\delta(z-w)$$

From the above proposition we know

$$Y(a,w)_j Y(b,w) = Y(a_j b, w).$$

Thus we obtain the first line. The equivalence is obvious from the OPE theorem of the first chapter. $\square$

3.2 Categorical Properties

3.2.1 Definition *Let V be a vertex algebra with vacuum vector v_0. A* ***subalgebra*** *of V is a subspace $U \subset V$ of V with $v_0 = |0\rangle \in U$ such that*

$$a_n b \in U \text{ for all } a, b \in U.$$

This subalgebra is again a vertex algebra. This can be seen as follows:
Let be U a subalgebra of the vertex algebra V. Denote $U_n := U \cap V_n$. The space U is therefore graded with

$$U := \bigoplus U_n$$

and we restrict the endomorphisms to the subspace U:

$$Y(a, z) = \sum_n a_n|_U z^{-n-1}$$

As well for $T_U := T_V|_U$. □
Now we address the definition of homomorphisms between vertex algebras.

3.2.2 Definition (Homomorphism) *Let $(V, \deg, v_0, T, Y)$ and $(V' \deg', v_0', T', Y')$ be two vertex algebras.*
A homomorphism $\varphi : V \to V'$ is a linear map subject to the following properties:

1. *Gradation is preserved, i.e $\varphi(V_n) \in V_n'$.*
2. $$\varphi(Y(a, z)b) = Y(\varphi(a), z)\varphi(b) \text{ for all } a, b \in V, n \in \mathbb{Z}.$$

We can also say: $\varphi(a_n b) = \varphi(a)_n \varphi(b)$.
An **automorphism** $\sigma : V \to V$ is a linear invertible endomorphism of V.

3.2.3 Definition (Ideal)

A vertex algebra ideal $I \subset V$ is a subspace with $T(I) \subset I$ and $|0\rangle \notin I$ such that

$$a_n I \subset I \text{ for all } a \in V.$$

3.2.4 Remark *Due to the skewsymmetry (corollary 3.1.6) we have*

$$a_n V \subset I \text{ for all } a \in I$$

if I is a vertex algebra ideal.

Proof. Let $b \in I$, I an ideal of a vertex algebra. From the definition of I and the field property of the vertex operator we obtain

$$Y(a, z)b \in J[[z]][z^{-1}] \quad \forall a \in V.$$

From the skewsymmetry 3.1.6 we have

$$Y(a, z)b = e^{zT} Y(b, -z)a.$$

Therefore $b_n a \in J$ for all $b \in I$. □

3.2.5 Corollary *Let V be a vertex algebra, and let $I \subset V$ be a vertex algebra ideal. Then the space V/I has a canonical vertex algebra structure, and the homomorphism $\phi : V \to V/I$ given by $v \mapsto v + I$ is a canonical vertex algebra homomorphism.*

Proof. The map ϕ preserves the grading.
Let $a, b \in V$, then $\phi(a) = a' + \tilde{a}$, $\phi(b) = b' + \tilde{b}$ with $a', b' \in V$, $\tilde{a}, \tilde{b} \in I$.
We have

$$(a' + \tilde{a})_n(b' + \tilde{b}) = a'_n b' + a'_n \tilde{b} + \tilde{a}_n b' + \tilde{a}_n \tilde{b}.$$

From the above remark we obtain that the last three summands are elements of I.
Furthermore $\phi(a_n b) = (a_n b)' + \widetilde{a_n b}$.
□

3.3 Representations

3.3.1 Definition (Representation of Vertex Algebras) *Let V be a vertex algebra.*
A vector space M is called a V-module if it is equipped with the following data:

1. *a grading $M = \bigoplus_{n\in\mathbb{Z}} M_n$ such that $M_n = 0$ for small enough n.*
2. *an operation $Y_M : V \to End(M)[[z, z^{-1}]]$ with*

$$a \mapsto a(z) = \sum_n a_n z^{-n-1}$$

where the $a_n \in End(M)$ and $a(z)$ is a field, i.e. for all $v \in M$ $\exists n_0 \in \mathbb{Z}$ such that $a_n(v) = 0$ $\forall n > n_0$.

These data are subject to the following axioms:

1. *$Y_M(v_0, z) = id_M$*
2. *$Y_M(a, z)Y_M(b, w) = Y_M(Y(a, w)b, z - w)$.*

This definition implies that the vertex algebra V is a module over itself.

3.3.2 Proposition 1. *$Y_M(Ta, z) = \partial_z Y_M(a, z)$ for all $a \in V$.*

2. *The fields $Y_M(a, z)$ are mutually local.*

This proposition is proved e.g. in [FB-Z] proposition 4.1.2.
In 1. we apply the second axiom of the definition of a vertex algebra module and use the Taylor expansion.
In 2. we use essentially the skewsymmetry. □
The following definition is due to [FB-Z] (Def. 4.5.1)

3.3.3 Definition (Rational Vertex Algebras) *A conformal vertex algebra V is called **rational** if every V-module is completely reducible.*

A rational conformal vertex algebra has the following properties:

1. V has finitely many simple modules.
2. The graded components of the simple V-modules are finite dimensional: $\dim M_n < \infty$
3. The gradation operator on the simple V-modules coincide with the operator L_0^M (i.e. the 0-th coefficient of the Virasoro field) up to a shift.

3.3.4 Definition (Selfdual Vertex Algebra) *A rational vertex algebra V is called **selfdual** (or "holomorphic") if V has only one simple module namely itself.*

In the literature both notions, self-dual and holomorphic, exist. The word self-dual comes from lattice vertex algebras and the fact that lattice vertex algebras with selfdual lattices have only one simple module. The word "holomorphic" is used in [FB-Z] (page 84).

3.3.5 Definition (character of a rational vertex algebra) *The **character** of a rational vertex algebra is defined by*

$$chM = Tr_M q^{L_0^M - \frac{c}{24}} = \sum_\alpha \dim M_\alpha q^{\alpha - \frac{c}{24}}$$

where M_α is the subspace of M on which L_0^M acts by multiplication by α and c is the central charge of V.

The correcting term $\frac{c}{24}$ comes from the fact that the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

(where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{C}$ with imaginary part > 0) corresponds to the graded dimension of the symmetric algebra of a one-dimensional vector space V given by

$$\dim_* S(V) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)}.$$

By correcting the graded dimension of the symmetric algebra by the factor $q^{-\frac{1}{24}}$ it even becomes a meromorphic function on the upper half plane, namely the Dedekind eta function. See [FLM] for further discussions in this direction.

Chapter 4

Trip to the Zoo: Examples

In this chapter we shall present the most common examples of vertex algebras. Starting from the point of view that vertex algebras are the algebraic counterparts of conformal field theory some of these examples reflect the most common examples of conformal field theories.
Vertex Algebras (or Vertex Operator Algebras) have a wide range of applications in mathematics and mathematical physics. Their use is not only restricted to conformal field theory but can also be used for dealing with questions in group theory and cohomology of Hilbert schemes.
One of the most spectacular examples is the Monster Moonshine vertex algebra constructed by Frenkel, Lepowsky and Meurman [FLM]. They showed that the Monster group, the largest of the 26 simple sporadic groups, acts on an orbifold vertex algebra associated to the 24-dimensional Leech lattice. This construction was the foundation of Borcherds to prove the Conway-Norton conjectures.
The most common examples of vertex algebras are the free fermions and free bosons. Furthermore the representations of Kac-Moody algebras are also well known.
We already gave in chapter 2 the most relevant calculations so that this chapter is rather short.
This chapter is organized as follows:
In section 4.1 we present the well studied vertex algebra associated to the Heisenberg algebra. In order to show the importance of the reconstruction theorem we discuss this vertex algebra in detail. This construction is also relevant for the second part of this thesis because I shall prove a similar result for higher genus Riemann surfaces.
In section 4.2 we present the reconstruction theorem. In section 4.3 we discuss the affine vertex algebras.

4.1 Heisenberg Vertex Algebra

We know from chapter 2 that the Heisenberg algebra is the affinization of the abelian 1-dimensional Lie algebra $\mathbb{C}$. We recall the definition of the Heisenberg algebra.

4.1.1 Definition *The* ***Heisenberg algebra*** *is an infinite dimensional algebra H generated by $\{a_n : n \in \mathbb{Z}\}$ and a central element K.*
These elements satisfy the canonical relations:

$$[a_n, a_m] = n\delta_{n,-m}, \text{ and } [a_n, K] = 0, \qquad n \in \mathbb{Z}.$$

4.1.2 Definition (Induced Representation) *Let $\mathfrak{g}$ be a Lie algebra, and let π be a representation of the subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ in a vector space V. The* ***induced*** *$\mathfrak{g}$-module is the vector space*

$$Ind_{\mathfrak{p}}^{\mathfrak{g}}\pi \quad := \quad U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$$

on which the elements $g \in \mathfrak{g}$ act by left multiplication on the first factor.

Note: $U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.
We define now a representation of the Heisenberg algebra induced by a one-dimensional representation of one "half" of the Heisenberg algebra. More precisely let H_+ be generated by the elements a_n ($n \in \mathbb{Z}_{\geq 0}$). It is a commutative subalgebra, and hence has a trivial one-dimensional representation in the space $\mathbb{C}$. Thus we can define the following representation induced by the one-dimensional representation of the subalgebra H_+.

4.1.3 Definition *The* ***Fock representation*** *of the Heisenberg algebra ist defined by*

$$\pi = Ind_{H_+}^{H}\mathbb{C} = U(H) \otimes_{U(H_+)} \mathbb{C}.$$

We have due to the Poincare-Birkhoff-Witt theorem $U(H) = U(H_+) \otimes U(H_-)$ where H_- is the subalgebra of H generated by the elements a_n ($n < 0$). Because of the definition of induced representations we can deduce:

$$\mathrm{Ind}_{H_+}^{H}\mathbb{C} = U(H) \otimes_{U(H_+)} \mathbb{C} \cong$$

$$\cong U(H_-) \otimes U(H_+) \otimes_{U(H_+)} \mathbb{C} \cong U(H_-) \otimes \mathbb{C} \cong S(H_-)$$

We used the fact that H_- is commutative, hence its tensor algebra is the symmetric algebra.
We obtain the following assertion:

4.1.4 Proposition *The Fock representation can be written as the polynomial ring in countably infinitely many variables:*

$$\pi \cong \mathbb{C}[a_{-1}, a_{-2}, ...]$$

The Heisenberg algebra acts as follows on this space:

$$\begin{array}{lcll} a_n & \mapsto & na_n\cdot & \text{for } n<0 \\ a_n & \mapsto & \frac{\partial}{\partial a_{-n}} & \text{for } n>0 \\ a_0 & \mapsto & 0\cdot & \end{array}$$

4.1.5 Theorem (π is a Vertex Algebra) *The Fock representation π carries the structure of a vertex algebra with the data*

1. *Grading:* $\deg(a_{j_1}...a_{j_k}) = -\sum_{i=1}^{k} j_i$.
2. *Vacuum:* $v_0 = 1$.
3. *Translation operator:* $[T, a_n] = (-n)a_{n-1}$.
4. *Vertex Operator:*

$$Y(a_{n_1}...a_{n_k}1, z) =: \partial^{(-n_1-1)}a(z)...\partial^{(-n_k-1)}a(z) :$$

where $a(z) = \sum_n a_n z^{-n-1}$ *is the "generating series" of the endomorphisms* a_n *in* π.

Before we prove this theorem, we recall some facts from the first chapter of this thesis: The k-th derivative of the formal distribution $a(z)$ is given by

$$\partial_z^{(k)}a(z) = \sum_n (-1)^k \binom{n}{k} a_{n-k}z^{-n-1}.$$

We have due to the definition of the binomial coefficients:

$$k > 0, n < k \Rightarrow \binom{n}{k} = 0. \tag{4.1}$$

We define the coefficients of the k-th derivative by

$$a_n^{(k)} := (-1)^k \binom{n}{k} a_{n-k}.$$

We obtain immediately: $a_n^{(k)} = 0$ for $0 \leq n < k$.

4.1.6 Lemma *For the series $a(z)$ one has*

$$\partial_z^{(k)}a(z)1|_{z=0} = a_{-k-1}1, \quad k \in \mathbb{Z}_{\geq 0}. \tag{4.2}$$

Proof. Form the definition of π we have $a_n 1 = 0$ for all $n \geq 0$. Thus we can compute

$$\begin{aligned} \partial_z^{(k)} a(z)1 &= \sum_n a_n^{(k)} 1 z^{-n-1} = \sum_n (-1)^k \binom{n}{k} a_{n-k} 1 z^{-n-1} \\ &= \sum_{n<k} (-1)^k \binom{n}{k} a_{n-k} 1 z^{-n-1} \\ &= \sum_{n<0} (-1)^k \binom{n}{k} a_{n-k} 1 z^{-n-1} \\ &= (-1)^k \binom{-1}{k} a_{-1-k} 1 z^{1-1} + (-1)^k \binom{-2}{k} a_{-2-k} 1 z^{2-1} + ... \end{aligned}$$

Because of $(-1)^k \binom{-1}{k} = (-1)^k \cdot \frac{(-1)(-2)...(-1-k+1)}{k!} = 1$ we get for $z = 0$:

$$\partial_z^{(k)} a(z)1 = a_{-1-k} 1 \cdot 0^0 + 0$$

Thus we obtain the assertion of the lemma. $\square$
Proof of the theorem:

1. Vacuum axiom: $Y(1, z) = id_V$.
   In general we have to show:

   $$Y(A, z)1|_{z=0} = A$$

   where $A = a_{n_1} ... a_{n_k} 1$. We shall prove the vacuum property by induction.
   Let $k = 1$. $Y(a_{-1}1, z)1 =: a(z) : 1 = \sum_n a_n z^{-n-1} 1 = \sum_{n<0} a_n z^{-n-1} 1$
   For $z \to 0$ it follows: $Y(a, z)1 = a_{-1}$.
   From the lemma above we know $Y(a_{-n}1, z)|_{z=0} = \partial^{-n-1} a(z)1|_{z=0} = a_{-n}1$.
   Let $k > 1$. Suppose $Y(A, z)1|_{z=0} = A$ where $A = a_{n_1} ... a_{n_k} 1$.
   This means more explicitely that for the series $Y(A, z) = \sum_n A_n z^{-n-1}$ we have especially $A_n 1 = 0$ for $n \geq 0$ and $A_{-1} 1 = a_{n_1} ... a_{n_k} 1$.
   We obtain from the defintion of the vertex operator Y for the element $a_{-m} A = a_{-m} a_{n_1} ... a_{n_k} 1$:

   $$Y(a_{-m-1} A, z) =: \partial^{(-m-1)} a(z) Y(a, z) :$$

   We consider this normal ordered product more carefully:

   $$Y(a_{-k} A, z) = \sum_n : \partial^{(k-1)} a(z) Y(a, z) :_n z^{-n-1}.$$

   The n-th coefficient is given by (see chapter 1)

   $$: \partial^{(-m-1)} a(z) Y(a, z) :_n = \sum_{j<0} a_j^{(-m-1)} A_{n-j-1} + \sum_{j \geq 0} A_{n-j-1} a_j^{(-m-1)}$$

Let $n \geq 0$ the we obatin immediately

$$: \partial^{(-m-1)} a(z) Y(a,z) :_n 1 = \sum_{j<0} a_j^{(-m-1)} \underbrace{A_{n-j-1} 1}_{=0} + \sum_{j \geq 0} A_{n-j-1} \underbrace{a_j^{(-m-1)} 1}_{=0}.$$

Therefore it remains to consider the coefficient for $n = -1$.

$$: \partial^{(-m-1)} a(z) Y(a,z) :_{-1} 1 = \sum_{j<0} a_j^{(-m-1)} A_{-1-j-1} 1 + \sum_{j \geq 0} A_{-1-j-1} a_j^{(-m-1)} 1$$

From equation (4.1) we have especially

$$a_j^{(-m-1)} = (-1)^{-m-1} \binom{j}{-m-1} a_{j+m+1} = 0 \text{ for } 0 < j < -m-1.$$

Therefore we obtain due to the vacuum property for the coefficients A_n:

$$: \partial^{(-m-1)} a(z) Y(a,z) :_{-1} 1 = a_{-1}^{(-m-1)} A_{-1} 1 + \sum_{j \geq -m-1} A_{-1-j-1} a_j^{(-m-1)} 1$$

The second sum vanishes due to the definition of $a_j^{(-m-1)}$ and the property $a_n 1 = 0$ for $n \geq 0$, hence we obtain the assertion of the vacuum axiom.

2. Translations axiom: We will prove it by induction as well.
We have to show $[T, Y(A,z)] = \partial Y(A,z)$.
We have $\partial_z Y(a_{-1} z) = \partial_z a(z) = \sum_n (-n) a_n z^{-n-1}$, and $[T, a_n] = (-n) a_{n-1}$, hence

$$[T, Y(a_{-1}, z)] = [T, a(z)] = \partial a(z).$$

For the binomial coefficients we have the identities

$$n \binom{n-1}{m} = \binom{n}{m+1} = \binom{n}{m} (n-m).$$

We compute

$$\begin{aligned}
\partial_z (\partial_z^{(m)} a(z)) &= \partial_z \sum_n (-1)^m \binom{n}{m} a_{n-m} z^{-n-1} \\
&= \sum_n (-1)^m \binom{n}{m} (-n-1) a_{n-m} z^{-n-2} \\
&\overset{\text{index-shift}}{=} \sum_n (-1)^m \binom{n-1}{m} (-n) a_{n-m-1} z^{-n-1} \\
&= \sum_n (-1)^{m+1} \binom{n}{m} a_{n-m-1} z^{-n-1},
\end{aligned}$$

and

$$\begin{aligned}
[T, \partial_z^{(m)} a(z)] &= \sum_n (-1)^m \binom{n}{m} [T, a_{n-m}] z^{-n-1} \\
&= \sum_n (-1)^m \binom{n}{m} (m-n) a_{n-m-1} z^{-n-1} \\
&= \sum_n (-1)^{m+1} \binom{n}{m+1} (m-n) a_{n-m-1} z^{-n-1},
\end{aligned}$$

hence $\partial_z^{(m+1)} a(z) = [T, \partial_z^{(m)} a(z)]$.
Let $k > 1$. Let $A = a_{n_1} ... a_{n_k}$. Suppose, $[T, Y(A, z)] = \partial Y(a, z)$.

$$\begin{aligned}
[T, Y(a_m A, z)] &= \sum_{j<0} [T, a_j^{(-m-1)} A_{n-j-1}] + \sum_{j\geq 0} [T, A_{n-j-1} a_j^{(-m-1)}] \\
&= \sum_{j<0} [T, a_j^{(-m-1)}] A_{n-j-1} + a_j^{(-m-1)} [T, A_{n-j-1}] + \\
&+ \sum_{j\geq 0} [T, A_{n-j-1}] a_j^{(-m-1)}] + A_{n-j-1} [T, a_j^{(-m-1)}] \\
&=: (\partial_z \partial_z^{(-m-1)} a(z)) Y(A, z) : + : \partial_z^{(-m-1)} a(z) \partial_z Y(A, z) : \\
&= \partial_z : Y(a_m A, z) : .
\end{aligned}$$

The last line follows from the fact that the derivation of a normal ordered product satisfies the derivation property (see chapter 1).

3. Locality: We prove it again by induction.
We start with $[a(z), a(w)] = \partial_w \delta(z-w)$. Therefore: $[a(z), a(w)](z-w)^2 = 0$.
We proceed by applying derivatives:

$$\partial_z^{(n)} \partial_w^{(m)} ([a(z), a(w)]) = \partial_z^{(n)} \partial_w^{(m)} \partial_w \delta(z-w)$$

From proposition 1.1.11 (properties of the delta distribution) we get

$$\partial_z^{(n)} \partial_w^{(m)} \partial_w \delta(z-w) = (-1)^n \partial_w^{(m+n+1)} \delta(z-w)$$

and again from this proposition we get:

$$\partial_w^{(m+n+1)} \delta(z-w)(z-w)^{n+m+1} = 0$$

Now we apply Dong's lemma and get the assertion of the theorem. More precisely, let $A = a_{n_1} ... a_{n_s} 1$, and $B = a_{m_1} ... a_{m_t} 1$ where $s, t \geq 1$. Suppose the associated fields are mutually local, i.e.

$$[Y(A, z), Y(B, w)](z-w)^N = 0,$$

and suppose $\partial_z^{(-l-1)} a(z)$ and the fields $Y(A, z), Y(B, z)$ are mutually local. Then due to Dong's lemma there exists an $\tilde{N} \in \mathbb{N}$ such that

$$[Y(a_l A, z), Y(B, w)](z-w)^{\tilde{N}} = [: \partial_z^{(-l-1)} a(z) Y(A, z) : Y(A, z), Y(B, w)](z-w)^{\tilde{N}} = 0.$$

□

4.2 The Reconstruction Theorem

The following theorem is due to [Kac], it is denoted as the "strong reconstruction theorem" in [FB-Z].

4.2.1 Theorem *Let V be a $\mathbb{Z}_+$-graded vector space, $|0\rangle \in V_0$ a non-zero vector, and T a degree 1 endomorphism of V.*
Let $\{a^\alpha(z)\}_{\alpha\in A}$ (A an index set) be a collection of fields
(i.e. $a^\alpha(z) \in End(V)[[z, z^{-n-1}]]$ with $a^\alpha(z)v \in V[[z]][z^{-1}]$)
such that

1. $[T, a^\alpha(z)] = \partial a^\alpha(z)$,
2. $T\,|0\rangle = 0$, $a^\alpha(z)\,|0\rangle|_{z=0} = a^\alpha$ *for all* $\alpha \in A$,
3. *The fields* $a^\alpha(z)$ *are mutually local,*
4. *The vectors* $a^{\alpha_1}_{j_1} a^{\alpha_2}_{j_2} ... a^{\alpha_n}_{j_1}\,|0\rangle$ *with* $j_s < 0$, $\alpha_s \in A$ *span the vector space* V.

Then these structures together with the vertex operation given by

$$Y(a^{\alpha_1}_{j_1} a^{\alpha_2}_{j_2} ... a^{\alpha_n}_{j_n}\,|0\rangle, z) =: \partial_z^{(-j_1-1)} a^{\alpha_1}(z) \partial_z^{(-j_2-1)} a^{\alpha_2}(z) ... \partial_z^{(-j_n-1)} a^{\alpha_n}(z) :$$

defines a unique structure of a vertex algebra on V such that $|0\rangle$ is the vacuum vector, T is the translation operator and

$$Y(a^\alpha, z) = a^\alpha(z), \qquad \alpha \in A.$$

If we apply this theorem to the vector space $\mathbb{C}[]$ with the field $a(z)$ (that means our index set is $A = \{1\}$) then we obtain the vertex algebra of the Heisenberg algebra.
The fourth condition can be relaxed by

4. The vectors $a^{\alpha_1}_{j_1} a^{\alpha_2}_{j_2} ... a^{\alpha_n}_{j_1}\,|0\rangle$ with $j_s \in \mathbb{Z}$, $\alpha_s \in A$ span the vector space V.

Then the vertex operator is given by

$$Y(a^{\alpha_1}_{j_1} a^{\alpha_2}_{j_2} ... a^{\alpha_n}_{j_n}\,|0\rangle, z) \quad = \quad a^{\alpha_1}(z)_{j_1}\,(a^{\alpha_2}(z)_{j_2}(...(a^{\alpha_n}(z)_{j_n} Id_V)) \tag{4.3}$$

4.3 Affine Kac-Moody Algebras and Vertex Algebras

In this section we address to the representation theory of non-twisted affine Kac-Moody algebras. We discuss the fact that the vacuum representation carries the structure of a vertex algebra. This is an easy consequence of the considerations of chapter 2.
We first encounter some basic facts about representations

4.3.1 Definition (Vacuum Representation) *The vacuum representation of weight k of $\widehat{\mathfrak{g}}$ is defined as the induced representation of the one-dimensional representation of $\mathfrak{g}[[t]] \oplus \mathbb{C}K$, given by:*
$\mathfrak{g}[[t]]$ operates by 0, and K by multiplication with a scalar $k \in \mathbb{C}$:

$$V_k(\mathfrak{g}) = Ind^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[[t]]\oplus\mathbb{C}K} = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]]\oplus\mathbb{C}K)} \mathbb{C}_k$$

4.3.2 Proposition

$$\tilde{V}^k = S(\hat{\mathfrak{h}}^<)$$

Sketch of the proof. We start from the definition of the induced module.

$$\begin{aligned}
\tilde{V}^k = \tilde{V}(\pi^k) &= \mathrm{Ind}^{\hat{\mathfrak{h}}'}_{\hat{\mathfrak{h}}_\geq}\pi = \\
\mathfrak{U}(\hat{\mathfrak{h}}') \otimes_{\mathfrak{U}(\hat{\mathfrak{h}}^\geq)} V &= \mathfrak{U}(\hat{\mathfrak{h}}^<) \otimes_{\mathbb{C}} \mathfrak{U}(\hat{\mathfrak{h}}^\geq) \otimes_{\mathfrak{U}(\hat{\mathfrak{h}}^\geq)} V \\
\mathfrak{U}(\hat{\mathfrak{h}}^<) \otimes_{\mathbb{C}} V = \mathfrak{U}(\hat{\mathfrak{h}}^<) &= S(\hat{\mathfrak{h}}^<)
\end{aligned}$$

First line is definition, in the second line we used that V is one-dimensional, in the third line we used $\hat{\mathfrak{h}}^<$ commutative. □

4.3.3 Theorem *$V_k(\mathfrak{g})$ is a vertex algebra with the data:*

1. *vacuum vector: v_k.*
2. *grading:*
$$\deg J^{a_1}_{n_1} ... J^{a_m}_{n_m} v_k = -\sum_{i=1}^m n_i$$
3. *translation: $Tv_k = 0$, $[T, J^a_n] = -nJ^a_{n-1}$.*
4. *Vertex Operator: $Y(v_k, z) = id$,*
$Y(J^a_{-1}v_k, z) = J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$
$Y(J^{a_1}_{n_1} ... J^{a_m}_{n_m} v_k, z) =: \partial^{(-n_1-1)} J^{a_1}(z) ... \partial^{(-n_m-1)} J^{a_m}(z) :$.

Proof. We can use the reconstruction theorem. There is hardly anything to prove because the situation is very similar to the Heisenberg algebra case. □
From chapter 2 we obtain the following result:

4.3.4 Theorem *The vertex algebra associated to an affine Kac-Moody algebra by the vacuum representation $V_k(\mathfrak{g})$ is a conformal algebra of central charge*

$$c = \frac{k \cdot \dim(\mathfrak{g})}{h^\vee + k}.$$

Chapter 5

Correlation Functions

The aim of this chapter is to work out concrete examples of correlation functions of vertex algebras.
From the physicists point of view the observable entities are the expectation values of the operators, or in other words the correlation functions of the conformal field theory.
In this chapter we introduce a pictorial calculus in order to calculate the correlation functions for operator products of primary fields for an affine Kac-Moody vertex algebra. We show that the explicit expressions of the correlation functions can be interpreted as a summation over diagrams which represent "derangements".

5.1 Correlation Functions

We quote a result due to Zhu [Zhu].

5.1.1 Theorem (Zhu) *Let V be a conformal vertex algebra, $M = \bigoplus_{k=0}^{\infty} M_k$ a representation of V, and $M' = \bigoplus_{k=0}^{\infty} M'_k$ (where M'_k are the duals of M_k). Then we have for every $v' \in M'$, $v \in M$ and every $a^1, ..., a^n \in V$:*
The correlation function

$$f_{a^1,..a^n}(z_1, ..., z_n) = \langle v', Y(a^1, z_1)...Y(a^n, z_n)v\rangle$$

is holomorphic on the space $\{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1| > |z_2| > ... > |z_n|\}$.

From the proof of this theorem we shall see that the correlation functions are essentially sums of expressions of the form

$$\frac{c}{(z_{i_1} - z_{i_2})^{n_{12}}...(z_{i_{k-1}} - z_{i_k})^{n_{k-1,k}} z_{i_l}^{n_l}...z_{i_m}^{n_m}}$$

with $c \in \mathbb{C}$.
Proof of the theorem. We have a recursive formula thus we can prove the

assertion of the theorem by induction.
$n = 1$: Let be $v \in M$, $v' \in M'$, and let be $a \in V$. We have due to the field property of $Y(a, z)$:

$$\exists n_0 \in \mathbb{Z}_{\geq 0} \text{ such that } a_n v = 0$$
$$\exists n_1 \in \mathbb{Z}_{<0} \text{ such that } v' a_n = 0$$

Thus we obtain

$$< v', Y(a, z)v >=< v', Y_-(a, z)v >= \sum_{n=0}^{n_0} < v', a_n v > z^{-n-1}$$

The resulting function is a Laurent polynomial. Therefore the 1-correlation function is a holomorphic function in $\mathbb{C} \setminus \{0\}$.
$n - 1 \to n$: Let be given the $(n - 1)$-correlation function

$$< v', Y(a^1, z_1)Y(a^2, z_2)...Y(a^{k-1}, z_{k-1})Y(a^{k+1}, z_{k+1})...Y(a^n, z_n)v >$$

with the property as in the assertion of the theorem.
Consider now the correlation function

$$< v', Y(a^1, z_1)Y(a^2, z_2)...Y(a^{k-1}, z_{k-1})Y(a^k, z_k)Y(a^{k+1}, z_{k+1})...Y(a^n, z_n)v >$$

with $|z_1| > |z_2| > .. > |z_{k-1}| > |z_k| > |z_{k+1}| > ... > |z_n|$.
We will use the operator product expansions of the fields $Y(a^i, z_i)$ for $i \in \{1, ..., n\}$. Explicitly we have for $k, j \in \{1, ..., n\}$ with $k < j$

$$[Y_-(a^k, z_k), Y(a^j, z_j)] = \sum_{s=0}^{\infty} \left(i_{z_k, z_j} \frac{1}{(z_k - z_j)^{s+1}} \right) Y(a^k_s a^j, z_j),$$

and

$$[Y(a^k, z_k), Y_+(a^j, z_j)] = \sum_{s=0}^{\infty} \left(i_{z_k, z_j} \frac{1}{(z_k - z_j)^{s+1}} \right) Y(a^k_s a_j, z_j).$$

We split the field $Y(a^k, z_k) = Y_+(a^k, z_k) + Y_-(a^k, z_k)$ and obtain for the correlation function:

$$\langle v', Y(a^1, z_1)Y(a^2, z_2)...Y(a^k, z_k)...Y(a^n, z_n)v \rangle = \tag{5.1}$$

$$= \langle v', Y(a^1, z_1)Y(a^2, z_2)...Y_+(a^k, z_k)...Y(a^n, z_n)v \rangle + \tag{5.2}$$

$$+ \langle v', Y(a^1, z_1)Y(a^2, z_2)...Y_-(a^k, z_k)...Y(a^n, z_n)v \rangle. \tag{5.3}$$

We "move" now the field $Y_+(a_k, z_k)$ to the left by using the relation

$$[Y(a^j, z_j), Y_+(a^k, z_k)] = Y(a^j, z_j)Y_+(a^k, z_k) - Y_+(a^k, z_k)Y(a^j, z_j).$$

And we move the term $Y_-(a^k, z_k)$ to the right by

$$[Y(a^j, z_j), Y_-(a^k, z_k)] = Y(a^j, z_j)Y_-(a^k, z_k) - Y_-(a^k, z_k)Y(a^j, z_j).$$

We turn to the first summand (5.2):

$$\langle v', Y(a^1, z_1)Y(a^2, z_2)...Y_+(a^k, z_k)...Y(a^n, z_n)v\rangle =$$

$$=< v', Y_+(a^k, z_k)Y(a^1, z_1)Y(a^2, z_2)...\widehat{Y(a^k, z_k)}...Y(a^n, z_n)v > +$$

$$+\sum_{j\le k-1}\sum_{s=0}^{\infty}\left(i_{z_j,z_k}\frac{1}{(z_j-z_k)^{s+1}}\right) < v', Y(a^1, z_1)Y(a^2, z_2)...Y(a^k_s a^j, z_j)...Y(a^n, z_n)v > .$$

For the sum we can apply the assumption. For the first term we can write

$$< v', Y_+(a^k, z_k)Y(a^1, z_1)Y(a^2, z_2)...\widehat{Y(a^k, z_k)}...Y(a^n, z_n v >=$$

$$= \sum_{n=n_1}^{-1} < v', a^k_n Y(a^1, z_1)Y(a^2, z_2)...\widehat{Y(a^k, z_k)}...Y(a^n, z_n v > z_k^{-n-1}.$$

$$= \sum_{n=n_1}^{-1} < a^k_{-n}v', Y(a^1, z_1)Y(a^2, z_2)...\widehat{Y(a^k, z_k)}...Y(a^n, z_n v > z_k^{-n-1}.$$

For this term we can apply our assumption as well.
For the other term (5.2) we calculate in a similar way:

$$\langle v', Y(a^1, z_1)Y(a^2, z_2)...Y_-(a^k, z_k)...Y(a^n, z_n)v\rangle =$$

$$= \langle v', Y(a^1, z_1)Y(a^2, z_2)...\widehat{Y_-(a^k, z_k)}...Y(a^n, z_n)Y_-(a^k, z_k)v\rangle +$$

$$+\sum_{j\ge k+1}\sum_{s=0}^{\infty}\left(i_{z_k,z_j}\frac{1}{(z_k-z_j)^{s+1}}\right) < v', Y(a^1, z_1)Y(a^2, z_2)...Y(a^j_s a^k, z_j)...Y(a^n, z_n)v > .$$

For the first summand we can again apply the assumption.
The sums $\sum_{s=0}^{\infty}$ ... are in fact finite, because the fields are supposed to be mutually local. $\Box$.

5.2 Correlation Functions for WZNW

In this section we give a characterization of the correlation functions of primary fields of affine Kac-Moody algebras. We show that the correlation functions correspond to certain cycles of the symmetric group. Some part of this description was already given in [FreZhu].

5.2.1 Digression: Derangements

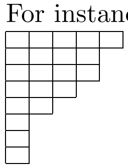
We want to recall some facts about the elements of the symmetric group. We have the well known fact (see e.g. [M]).

5.2.1 Remark *Let be S_n the symmetric group.*

1. *Two elements $\sigma, \sigma' \in S_n$ are in the same conjugacy class if and only if its cycle types are the same, i.e. if σ is of type $(\lambda_1^{a_1}, ..., \lambda_k^{a_k})$, and σ' is of type $(\mu_1^{b_1}, ..., \mu_k^{b_l})$, then $\sigma \sim \sigma' \Leftrightarrow k = l$ and $\lambda_j = \mu_j$, $a_j = b_j$ for all $1 \leq j \leq k$.*

2. *The number of elements in a conjugacy class in the symmetric group of type $(\lambda_1^{a_1}, ..., \lambda_k^{a_k})$ is*
$$\frac{n!}{a_1!a_2!...a_k! \prod_{j=1}^{k} \lambda_j^{a_j}}.$$

Young tableaus are convenient objects in order to visualize partitions. From the preceding remark we know the relationship between partitions and conjugacy classes of elements of the symmetric group.
For instance the partition $(5, 4^2, 3, 2, 1^3)$ can be visualized by the diagram:



We are now especially interested in elements $\sigma \in S_n$ which do not leave any element on its natural place.

5.2.2 Definition (Derangements) *A permutation $\sigma \in S_n$ which has no fixed points is called a **derangement** of S_n.*

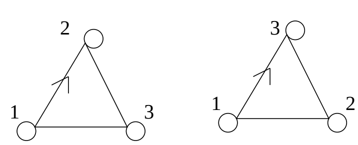
In the sense of cycle types derangements correspond to partitions with $\lambda_j \geq 2$. It is well known (see for instance Jacobs [Jac]) that the cardinality d_n of derangements in S_n is given by

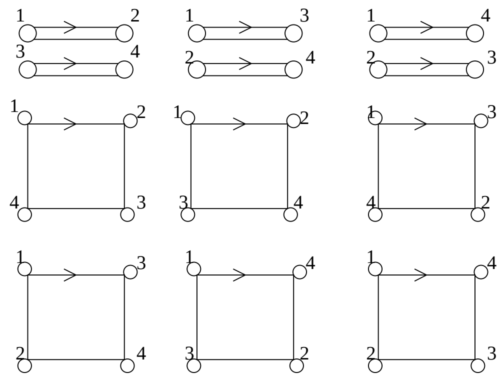
$$d_n = n! \cdot \sum_{n=0}^{n} \frac{(-1)^n}{n!}. \tag{5.4}$$

In some textbooks (see e.g. [Jac]) the cardinalities d_n are called **recontre numbers**.
There exist recurrence relations for the numbers d_n:

$$d_{n+1} = n\left(d_n + d_{n-1}\right), \tag{5.5}$$
$$d_{n+1} = (n+1)d_n + (-1)^{n+1}. \tag{5.6}$$

We can visualize the derangements by graphs as well. For instance for $n = 3$ we have two derangements (123), and (132). Graphically we can illustrate these derangements by two directed graphs:





For $n = 4$ we have the 9 derangements

$$\begin{array}{ccc} (12)(34) & (13)(24) & (14)(23) \\ (1234) & (1243) & (1324) \\ (1342) & (1423) & (1432) \end{array}$$

They correspond to the graphs given above.

5.2.2 Correlation Functions for Primary Fields

We are only considering correlation functions of primary fields.

5.2.3 Definition *Let V be a conformal vertex algebra. A vector $a \in V$ is called **primary of conformal dimension m** if*

$$L_n a = 0, \quad n > 0, \qquad L_0 a = m \cdot a.$$

*The field $Y(a, z)$ associated to a primary vector is called **primary field**.*

According to the OPE theorem the operator product expansion of a primary field with the associated field of the Virasoro element ω is given by

$$\begin{aligned} Y(\omega, z)Y(a, w) &= i_{z,w}\frac{(L_0 a)(w)}{(z-w)^2} + i_{z,w}\frac{(L_{-1}a)(w)}{(z-w)} + :Y(\omega, z)Y(a, w): \\ &= i_{z,w}\frac{m \cdot a}{(z-w)^2} + i_{z,w}\frac{\partial_w a(w)}{(z-w)^2} + :Y(\omega, z)Y(a, w): \end{aligned}$$

We consider the conformal vertex algebra associated to the vacuum representation $V_k(\mathfrak{g})$ of an affine Kac-Moody algebra $\hat{\mathfrak{g}}$ (see chapter 4).
The primary fields of conformal weight 1 are the "generating functions" $a(z) = Y(a_{-1}, z) = \sum_n a_n z^{-n-1}$.
Let $V_k'(\mathfrak{g})$ be the dual of $V_k(\mathfrak{g})$ with the properties

$a_n v' = 0$ for $n \leq 0$
We shall consider the correlation functions

$$\langle v', a^1(z_1)...a^N(z_N)v\rangle$$

which are yet to be defined.
But first we need some facts about the trace for finite matrices.

5.2.4 Remark *Let $a^{i_1}, ..., a^{i_m} \in \mathbb{C}^{k\times k}$ be matrices of size $k \times k$ with values in $\mathbb{C}$. The trace of a $k \times k$-matrix a is defined by $tr(a) = \sum_{i=1}^k a_{i,i}$.*
The trace is linear and it is invariant under cyclic reordering, i.e. it satisfies the following identities:

$$\begin{aligned} tr(a^{i_1}...a^{i_m}) &= tr(a^{i_m}a^{i_1}...a^{i_{m-1}}), & (5.7)\\ tr(a^{i_1}...[a^0, a^{i_j}]...a^{i_m}) &= tr(a^{i_1}...a^0a^{i_j}...a^{i_m}) - tr(a^{i_1}...a^{i_j}a^0...a^{i_m}) & (5.8)\end{aligned}$$

where the Lie bracket $[\cdot,\cdot]$ is defined by

$$[a,b] = ab - ba \quad \forall a, b \in \mathbb{C}^{k\times k}.$$

We fix now the notation

$$\boxed{\Delta(z_{i_1}, ..., z_{i_m}) := (z_{i_1} - z_{i_2})...(z_{i_{m-1}} - z_{i_m})(z_{i_m} - z_{i_1})} \qquad (5.9)$$

It is clear that $\Delta(z_{i_1}, ..., z_{i_m})$ is invariant under cyclic permutation, i.e.

$$\Delta(z_{i_1}, ..., z_{i_m}) = \Delta(z_{i_m}, z_{i_1}, ..., z_{i_{m-1}}). \qquad (5.10)$$

The following lemma is the key lemma for proving the next theorem. It has a pictorial counterpart which will be discussed in the next subsection.

5.2.5 Lemma *Let $a^{i_0}, a^{i_1}, ..., a^{i_m} \in \mathbb{C}^{k\times k}$ be quadratic matrices. Then we have on the domain $\{(z_0, z_{i_1}, ..., z_{i_m}) \in \mathbb{C}^{m+1} : z_j \neq z_k \text{ for } j \neq k\}$ the identity*

$$\frac{1}{\Delta(z_{i_1}, ..., z_{i_m})} \sum_{j=1}^m \frac{tr(a^{i_1}...a^{i_{j-1}}[a^{i_0}, a^{i_j}]a^{i_{j+1}}...a^{i_m})}{z_0 - z_{i_j}} =$$

$$= \sum_{j=1}^m \frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m})}{\Delta(z_{i_1}, ..., z_{i_{j-1}}, z_{i_0}, z_{i_j}, ..., z_{i_m})}.$$

Proof. For the sum on the left hand side we can compute due to the properties of the trace:

$$\sum_{j=1}^m \frac{tr(a^{i_1}...a^{i_{j-1}}[a^{i_0}, a^{i_j}]a^{i_{j+1}}...a^{i_m})}{z_0 - z_{i_j}} =$$

$$= \sum_{j=1}^m \left(\frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}a^{i_{j+1}}...a^{i_m})}{z_0 - z_{i_j}} - \frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_j}a^{i_0}a^{i_{j+1}}...a^{i_m})}{z_0 - z_{i_j}} \right) =$$

$$= \frac{tr(a^{i_0}a^{i_1}...a^{i_m})}{z_0 - z_{i_1}} +$$

$$+ \sum_{j=2}^{m} \left(\frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m})}{z_0 - z_{i_j}} - \frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m})}{z_0 - z_{i_{j-1}}} \right) -$$

$$- \frac{tr(a^{i_1}...a^{i_m}a^{i_0})}{z_0 - z_{i_m}}.$$

Consider the terms in the sum more carefully:

$$\frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m})}{z_0 - z_{i_j}} - \frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m})}{z_0 - z_{i_{j-1}}} =$$

$$= \frac{z_{i_j} - z_{i_{j-1}}}{(z_0 - z_{i_j})(z_0 - z_{i_{j-1}})} tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m}) =$$

$$= \frac{z_{i_{j-1}} - z_{i_j}}{(z_{i_{j-1}} - z_0)(z_0 - z_{i_j})} tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m}).$$

We multiply the last line by $\frac{1}{\Delta(z_{i_1},...,z_{i_m})}$ and cancel the corresponding terms in the numerator and the denominator:

$$\frac{1}{\Delta(z_{i_1},...,z_{i_m})} \cdot \frac{z_{i_{j-1}} - z_{i_j}}{(z_{i_{j-1}} - z_0)(z_0 - z_{i_j})} tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m}) =$$

$$= \frac{1}{(z_{i_1} - z_{i_2})..(z_{i_{j-1}} - z_{i_j})..(z_{i_m} - z_{i_1})} \cdot \frac{z_{i_{j-1}} - z_{i_j}}{(z_{i_{j-1}} - z_0)(z_0 - z_{i_j})} tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m}) =$$

$$= \frac{tr(a^{i_1}...a^{i_{j-1}}a^{i_0}a^{i_j}...a^{i_m})}{\Delta(z_{i_1},...z_{i_{j-1}}, z_0, z_{i_j}, ..., z_{i_m})}.$$

By summing over the terms we obtain the right hand side. □

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. The space of non-degenerate invariant bilinear forms on $\mathfrak{g}$ is one-dimensional (see e.g. [Hum]). Therefore the invariant bilinear form $(\cdot|\cdot)$ of the central extension $\hat{\mathfrak{g}}$ can be given by the trace $tr : End(W) \to \mathbb{C}$ where W is a finite-dimensional representation space of a faithful representation of $\mathfrak{g}$.

5.2.6 Theorem *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let be $\hat{\mathfrak{g}}$ the associated (non-twisted) affine Kac-Moody algebra.*
Let $V_k(\mathfrak{g})$ be the vacuum representation, and let $V_k'(\mathfrak{g})$ be the dual of $V_k(\mathfrak{g})$.
Let $v \in V_k(\mathfrak{g})$ denote the vacuum vector, i.e. $a_n v = 0$ for $n \geq 0$, and let $v' \in V_k'(\mathfrak{g})$ be the dual vector such that $\langle v', v \rangle = 1$.
Then for the primary fields we obtain

$$\langle v', a^1(z_1)...a^N(z_N)v \rangle = \sum_{\rho} f_{\sigma_1^\rho} f_{\sigma_2^\rho} ... f_{\sigma_k^\rho}$$

where the sum is over all derangements $\rho \in S_n$.
Let $\rho = \sigma_1^\rho ... \sigma_k^\rho$ be a derangement where $\sigma_1^\rho ... \sigma_k^\rho$ are the disjoint cycles of the derangement ρ.
Let the cycle σ be a cycle of length m with $\sigma = (i_1 ... i_m)$. The associated factor f_σ is given by

$$f_\sigma = \frac{tr(a^{i_1} ... a^{i_m})}{\Delta(z_{i_1}, ..., z_{i_m})}.$$

The theorem was stated in a different way in [FZ].
Examples: $N = 1$

$$\langle\, a(z)\,\rangle = \langle\, v', a(z)v\,\rangle = \langle\, v', a(z)_+ v\,\rangle + \langle\, v', a(z)_- v\,\rangle = 0 + 0 = 0.$$

$N = 2$:

$$\langle\, a^1(z_1)a^2(z_2)\,\rangle = \langle\, a^1_-(z_1)a^2(z_2)\,\rangle =$$
$$= \langle\, \frac{(a^1|a^2)}{(z_1 - z_2)^2} + \frac{[a^1, a^2](z_2)}{(z_1 - z_2)}\,\rangle =$$
$$= \frac{(a^1|a^2)}{(z_1 - z_2)^2}.$$

Proof of the theorem.
Let be n given and suppose the validity of the theorem for n fields. Consider now the correlation function:

$$\langle\, a^0(z_0)a^1(z_1)...a^n(z_n)\,\rangle = \langle\, a^0_-(z_0)a^1(z_1)...a^n(z_n)\,\rangle =$$
$$= \langle\, [a^0_-(z_0), a^1(z_1)]...a^n(z_n)\,\rangle + \langle\, a^1(z_1)a^0_-(z_0)...a^n(z_n)\,\rangle =$$
$$= \sum_{i=1}^n \frac{tr(a^0 a^i)}{(z_0 - z_i)^2} \langle\, a^1(z_1)...\widehat{a^i(z_i)}...a^n(z_n)\,\rangle + \frac{1}{z_0 - z_i} \langle\, a^1(z_1)...[a^0, a^i](z_i)...a^n(z_n)\,\rangle.$$

Now for the first term we can insert our assumption for $n-1$ variables.
It remains to study the second term:

$$\sum_{i=1}^n \frac{1}{z_0 - z_i} \langle\, a^1(z_1)...[a^0, a^i](z_i)...a^n(z_n)\,\rangle.$$

Now we can compute:

$$\sum_{i=1}^n \frac{1}{z_0 - z_i} \sum_\rho f_{\sigma_1^\rho} .. f_{\sigma_k^\rho} =$$

$$= \frac{1}{z_0 - z_1} \left(... + \frac{tr(a^{i_1} ... a^{i_{m_1}})}{\Delta(z_{i_1}, ..., z_{i_{m_1}})} ... \frac{tr(a^{i_{x-1}+1} .. [a^0, a^1] .. a^{i_{m_x}})}{\Delta(z_{i_1}, ..., z_{i_{k_1}})} ... \frac{tr(a^{i_{m-1}+1} ... a^{i_{m_n}})}{\Delta(z_{i_{m-1}+1}, ..., z_{i_{m_1}})} + ... \right) + ...$$

$$... + \frac{1}{z_0 - z_n} \left(... + \frac{tr(a^{i_1} ... a^{i_{m_1}})}{\Delta(z_{i_1}, ..., z_{i_{m_1}})} ... \frac{tr(a^{i_{y-1}+1} .. [a^0, a^n] .. a^{i_{m_y}})}{\Delta(z_{i_1}, ..., z_{i_{k_1}})} ... \frac{tr(a^{i_{m-1}+1} ... a^{i_{m_k}})}{\Delta(z_{i_{m-1}+1}, ..., z_{i_{m_k}})} + ... \right).$$

We collect now those $\frac{1}{z_0-z_i}$-terms which belong to the same cycle and obtain:

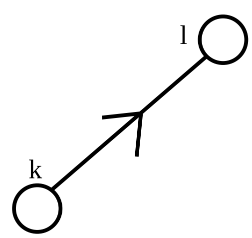
$$\sum_{i=1}^{n} \frac{1}{z_0 - z_i} \sum_{\rho} f_{\sigma_1^\rho}..f_{\sigma_k^\rho} =$$

$$= \sum_{\rho} \sum_{l(\rho)=k} \sum_{x=1}^{k} f_{\sigma_1^\rho}..\widehat{f_{\sigma_x}^{\rho}}...f_{\sigma_k^\rho} \left(\frac{1}{\Delta(z_{i_{m_{x-1}}+1},...,z_{i_{m_x}})} \sum_{j=m_{x-1}}^{m_x} \frac{tr(a^{i_{m_{x-1}}+1}...[a^0,a^j]...a^{m_x})}{z_0 - z_j} \right).$$

We can now apply the preceding lemma and obtain the assertion of the theorem.
□

5.3 Pictorial Calculus

Theorem 5.2.6 can be illustrated as summation over certain graphs which represent derangements. This will be explained in this section.
We consider the following graphs. Each vertex is labeled by a number, and from each vertex goes an arrow to another vertex, and to each vertex there is an arrow pointed.
We introduce the following entities:
The basic arrow:
The expression $\frac{1}{z_k-z_l}$ is illustrated by an arrow which starts from the circle k and spots to the circle l:



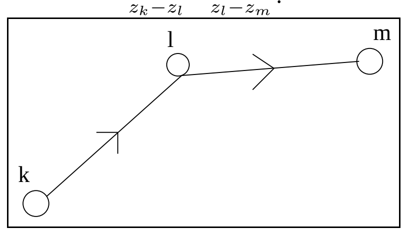
Multiplication:
The multiplication of two expressions $\frac{1}{z_k-z_l}$, $\frac{1}{z_l-z_m}$ is illustrated by concatenation of the arrows (see picture (i) on next page).
The multiplication of two expressions $\frac{1}{z_k-z_l}$, $\frac{1}{z_m-z_n}$ with k, l, m, n distinct is illustrated by picture (ii) on the next page.
From the multiplication rule we get especially for the expression

$$\frac{1}{\Delta(z_{i_1},...,z_{i_m})} = \frac{1}{(z_{i_1} - z_{i_2})...(z_{i_{m-1}} - z_{i_m})(z_{i_m} - z_{i_1})}$$

the corresponding picture (iii) is given on the next page.
Addition:
The addition of two expressions $\frac{1}{z_k-z_l}$, $\frac{1}{z_m-z_n}$ is illustrated by picture (iv)

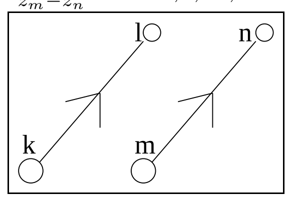
Picture (i)

$\frac{1}{z_k - z_l} \cdot \frac{1}{z_l - z_m}$:



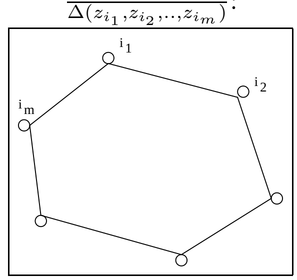
Picture (ii)

$\frac{1}{z_k - z_l} \cdot \frac{1}{z_m - z_n}$ with k, l, m, n distinct:



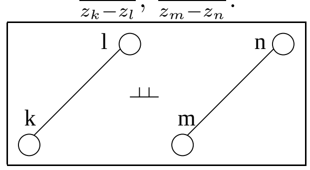
Picture (iii)

$\frac{1}{\Delta(z_{i_1}, z_{i_2}, .., z_{i_m})}$:



Picture (iv)

$\frac{1}{z_k - z_l}$, $\frac{1}{z_m - z_n}$:



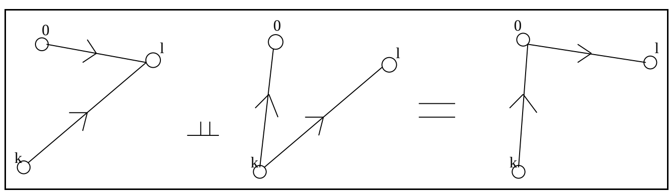
Gluing arrows:

We shall illustrate the identity

$$\frac{1}{z_k - z_l}\left(\frac{1}{z_0 - z_l} - \frac{1}{z_0 - z_k}\right) = \frac{1}{z_k - z_l}\cdot\frac{1}{z_0 - z_l} + \frac{1}{z_k - z_l}\cdot\frac{1}{z_k - z_0} \tag{5.11}$$

$$= \frac{1}{(z_k - z_0)(z_0 - z_l)} \tag{5.12}$$

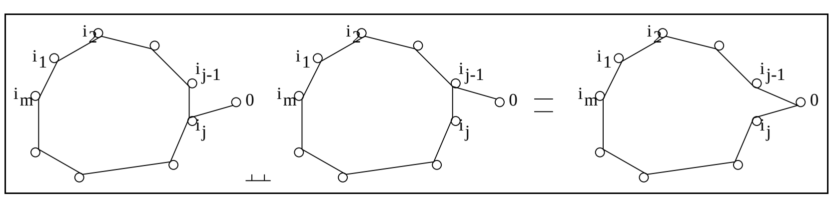
by the following graphical rule:



Especially for the expressions $\Delta(z_{i_1}, ..., z_{i_m})$ we obtain

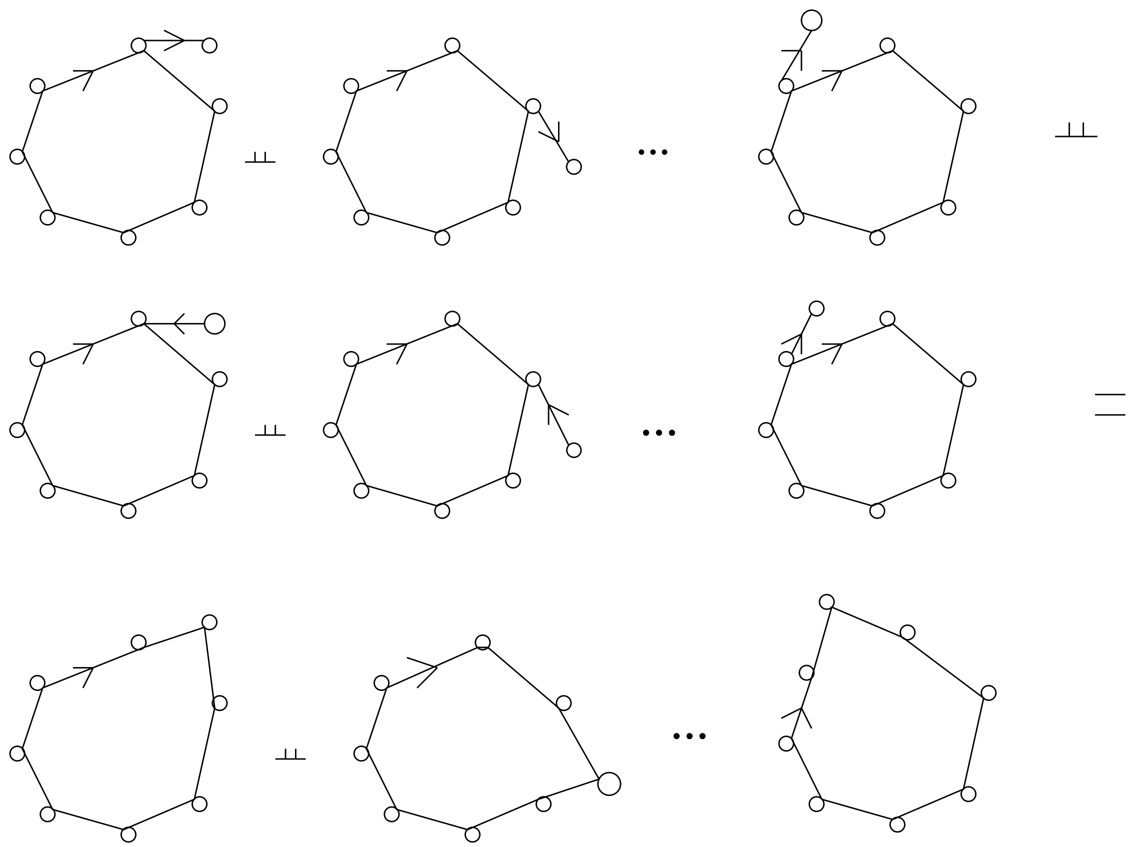
$$\frac{1}{\Delta(z_{i_1}, ..., z_{i_m})}\frac{1}{z_0 - z_{i_j}} - \frac{1}{\Delta(z_{i_1}, ..., z_{i_m})}\frac{1}{z_0 - z_{i_{j-1}}} = \frac{1}{\Delta(z_{i_1}, ..., z_{i_{j-1}}, z_0, z_{i_j}, ..., z_{i_m})}.$$

The illustration is



The set of the above defined graphs is a module over $\mathbb{C}$.

We address now the **pictorial illustration of lemma 5.2.5.**
The proof of lemma 5.2.5 can be illustrated as follows:



Rules of adding a vertex

This diagrammatic point of view suggests now a procedure how to obtain from the pictorial description of a correlation functions for n fields a correlation function for $n+1$ fields. Especially the recursive structure of the correlation functions (which we were using in the proof of theorem 5.2.6) allows one to give rules of constructing the next higher corresponding graphs.
Let be given a labeled m-gon, i.e. a polygon with m vertices.

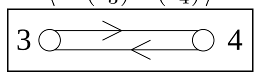
1. A vertex can only be attached to two vertices which are connected by (at least) one line.
2. The new arrow to the the new vertex starts from that vertex whose former arrow started from.
3. The arrow from the new vertex goes to the second vertex where the arrow spotted to.

The pictures illustrate as well the denominators of the correlation function as the numerator.
For the numerator it illustrates the expressions $tr(a^{i_1}...a^{i_k})$.

We now illustrate the correlation function for the first 3 cases.

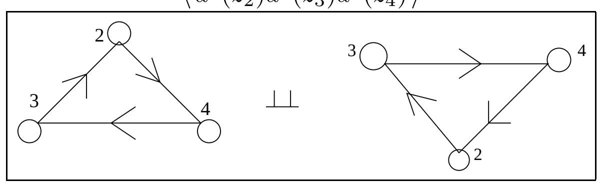
Two-Point Correlation Function:

$\langle a^3(z_3)a^4(z_4)\rangle$



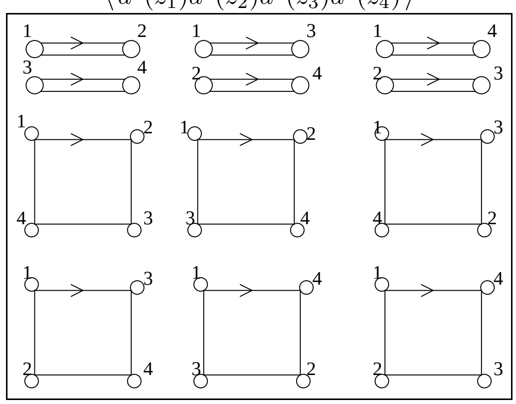
Three-Point Correlation Function:

$\langle a^2(z_2)a^3(z_3)a^4(z_4)\rangle$



Four-Point Correlation Function:

$\langle a^1(z_1)a^2(z_2)a^3(z_3)a^4(z_4)\rangle$



Chapter 6

Theta Functions and Differentials on Riemann Surfaces

Theta functions are building blocks in order to construct explicitely functions or sections in line bundles over compact Riemann surfaces.
In this chapter some basic properties of theta functions, the prime form and Szegö kernels are collected.
The chapter is organized as follows:
In the first section we are devoted to the definition and basic properties of theta functions. Theta functions $\Theta(z, \Omega)$ are defined as certain functions with periodicity properties on the space $\mathbb{C}^g \times \mathbb{H}_g$ where $\mathbb{H}_g$ is the Siegel upper half space of complex symmetric g by g matrices whose imaginary part is positive definite.
The Siegel upper half space is the higher dimensional analogue of the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\} \subset \mathbb{C}$.
In the second section we connect the theta functions with the geometry on Riemann surfaces. To this end we state the Riemann vanishing theorem. We also state the Riemann singularities theorem. It will be relevant for studying divisors of degree $g-1$ more carefully.
In the third section we establish the notion of the prime form. It is the higher genus counterpart of the function $f(z,w) = (z-w)$ in $\mathbb{C} \times \mathbb{C}$. The prime form is a multivalued form on $X \times X$ of weight $-\frac{1}{2}$ in each argument.
In the fourth section we introduce certain differentials on Riemann surfaces that will be relevant in the next chapters. Among them is the sigma differential, an important building block in constructing Krichever-Novikov forms explicitely. It is a multi-valued form of weight $\frac{g}{2}$. It has no zeros on X. We furthermore introduce the notion of affine and projective connections on Riemann surfaces. These connections will play a crucial role in considering central extensions of Krichever-Novikov algebras (see chapter 9).

In the last section we introduce the notion of a Szegö kernel. It can be defined as a section in a suitable vector bundle on $X \times X$. It will play a central role in the next chapters.
First of all we shall fix some notations for later purposes:
Throughout this chapter and later chapters X will denote a compact Riemann surface of genus $g \geq 0$.

6.1 Theta Functions and Properties

In this section we address basic properties of theta functions.
We denote by $\mathbb{H}_g$ the Siegel upper half space, i.e.

$$\mathbb{H}_g = \{\Omega \in \mathbb{C}^{g\times g} : Im(\Omega) \text{ positive definite}\}.$$

We write for short $Im(\Omega) > 0$ for $Im(\Omega)$ positive definite.

6.1.1 Definition (Theta Series) *Let $\Omega \in \mathbb{H}_g$.*
The ***theta series*** *for $z \in \mathbb{C}^g$ is defined by*

$$\Theta(z, \Omega) = \sum_{n\in\mathbb{Z}^g} e^{\pi i n^t \Omega n + 2\pi i n^t z}.$$

Sometimes we will drop the argument Ω if it is clear which matrix Ω is used.
The following assertions can be found e.g. in Mumford's Tata lectures [Mum1].

6.1.2 Remark *The series $\Theta(z, \Omega)$ converges absolutely and uniformly in z and Ω in each set*

$$\max_i |Im(z_i)| < \frac{c_1}{2\pi} \text{ and } Im\Omega \geq c_2 E_g$$

where E_g is the $g \times g$ unity matrix, and $c_1, c_2 \in \mathbb{R}_{>0}$.
Hence it defines a holomorphic function on the space $\mathbb{C}^g \times \mathbb{H}_g$.

The basic property of $\Theta(z)$ is the **quasi-periodicity** with respect to the lattice $L = \mathbb{Z}^g + \Omega\mathbb{Z}^g$. Quasi-periodic means periodic up to a simple multiplicative factor. More precisely: Let $m \in \mathbb{Z}^g$, then we have

$$\Theta(z + m, \Omega) = \Theta(z, \Omega) \tag{6.1}$$
$$\Theta(z + \Omega m, \Omega) = e^{-\pi i m^t \Omega m - 2\pi i m^t z}\Theta(z, \Omega). \tag{6.2}$$

We now discuss the properties of theta fuctions with characteristics.
Consider the real homomorphism $\mathbb{R}^{2g} \to \mathbb{C}^g$ defined by

$$\begin{bmatrix} \epsilon \\ \delta \end{bmatrix} \mapsto e = \delta + \Omega\epsilon.$$

The inverse of this isomorphism is given by

$$e \mapsto \begin{bmatrix} (Im\Omega)^{-1} Im(e) \\ Re(e) - Re\Omega(Im\Omega)^{-1} Im(e) \end{bmatrix}.$$

6.1.3 Definition (Theta Functions with Characteristics) *Let $\epsilon, \delta \in \mathbb{R}^g$ be two real vectors.*
*A theta function with **characteristics** $\begin{pmatrix} \epsilon \\ \delta \end{pmatrix}$ is defined by*

$$\Theta[e](z) \overset{def}{=} \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (z) \overset{def}{=} \sum_{n \in \mathbb{Z}^g} e^{\pi i (n+\epsilon)^t \Omega (n+\epsilon) + 2\pi i (n+\epsilon)^t (z+\delta)}$$

where $\mathbb{C}^g \ni e = \delta + \Omega\epsilon$.

The quasi-periodic property of theta functions with characteristics is given by the formulas:

$$\Theta[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}](z) = e^{\pi i \epsilon^t \Omega \epsilon + 2\pi i \epsilon^t (z+\delta)} \Theta(z + \Omega\epsilon + \delta) \tag{6.3}$$

$$\Theta[\begin{smallmatrix} \epsilon + n \\ \delta + m \end{smallmatrix}](z) = e^{2\pi i \epsilon^t m} \Theta[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}](z) \tag{6.4}$$

$$\Theta[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}](z + m) = e^{2\pi i \epsilon^t m} \Theta[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}](z) \tag{6.5}$$

$$\Theta[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}](z + \Omega m) = e^{-2\pi i \delta^t m} e^{-\pi m^t \Omega m - 2\pi i m^t z} \Theta[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}](z) \tag{6.6}$$

6.1.4 Definition *Let $\epsilon, \delta \in \{0, \frac{1}{2}\}^g$.*
A characteristic $[e] = \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$ with entries from the set $\{0, \frac{1}{2}\}$ is called even (resp. odd), if $4\epsilon \cdot \delta$ is even (resp. odd).

6.1.5 Remark *There are 2^{2g} characteristics with entries from the set $\{0, \frac{1}{2}\}$. $2^{g-1}(2^g - 1)$ are even and $2^{g-1}(2^g + 1)$ are odd.*

6.2 Riemann's Theorem

We now introduce some notions and facts of function theory on a compact Riemann surfaces.
We fix now some notations: Let X be a compact Riemann surface of genus $g \geq 1$. As customary let K denote the canonical bundle on X.
We refer to Forster's book [Fo] for standard facts concerning Riemann surfaces and sheaf cohomology techniques. All line bundles appearing in this and the subsequent chapters are holomorphic line bundles. We use the same symbol to denote a holomorphic line bundle and is associated sheaf of germs of holomorphic sections. If D is a divisor in a compact connected complex manifold (we essentially consider Riemann surfaces and its Jacobians), we denote the associated line bundle as well as its sheaf of germs of holomorphic sections by $\mathcal{O}(D)$. The set of holomorphic line bundles forms a group, called the **Picard group**.

The abelian multiplication is the tensor product $\otimes$, the inverse of a holomorphic line bundle β is the dual bundle, which we denote by β^{-1}, and the identity of this group is the trivial line bundle.
Let X be a compact Riemann surface of genus $g \geq 1$. We denote by $Pic(X)$ its Picard group. $Pic^d(X)$ denotes the subset of $Pic(X)$ of holomorphic line bundles of degree $d \in \mathbb{Z}$.
$Pic^0(X)$ is a g-dimensional complex torus denoted as the **Picard variety**. Choosing a fixed element $\beta \in Pic^d(X)$ we obtain a bijective correspondence between $Pic^0(X)$ and $Pic^d(X)$ (by $Pic^0(X) \ni \alpha \mapsto \alpha \otimes \beta \in Pic^d(X)$.). Thus it is g-dimensional complex manifold as well.
We choose on X a (symplectic) basis of g α-cycles and g β-cycles of $H_1(X, \mathbb{Z})$ and a canonical basis of g holomorphic differentials $\omega_1, ..., \omega_g$ spanning $H^0(X, K)$.
Then the Riemann period matrix is in canonical form

$$(\int_{\alpha_i} \omega_j | \int_{\beta_i} \omega_j) = (E|\Omega),$$

where $E \in \mathbb{C}^{g \times g}$ is the unity matrix and $\Omega \in \mathbb{H}_g$.
More precisely:

$$\int_{\alpha_i} \omega_j = \delta_{ij}, \quad 1 \leq i, j \leq g$$
$$\int_{\beta_i} \omega_j = \Omega_{ij}, \quad 1 \leq i, j \leq g.$$

6.2.1 Proposition *Let X be a compact Riemann surface of genus $g \geq 1$, $\omega_1, ..., \omega_g$ a basis of $H^0(X, K)$. The Riemann surface X can be embedded via the Jacobi map into its Jacobian. Let Q be a base point $Q \in X$ and put*

$$X \to J(X)$$
$$P \mapsto J(P) := \left(\int_Q^P \omega_1, \int_Q^P \omega_2, ..., \int_Q^P \omega_g \right) \mod \mathbb{Z}^g + \Omega\mathbb{Z}^g,$$

where $\int_Q^P$ is an arbitrary path from Q to P, and Ω is the non-trivial part of the period matrix $(E|\Omega)$.

6.2.2 Theorem (see e.g. [Fo] 21.7) *Let X be a Riemann surface of genus $g \geq 1$. Let $Q_1 + Q_2 + ... + Q_n - (P_1 + P_2 + ... + P_n)$ be an arbitrary divisor on X of degree zero. The map*

$$j : Pic^0 \to J(X)$$

given by

$$\mathcal{O}(Q_1+Q_2+...+Q_n-(P_1+P_2+...+P_n)) \mapsto \left(\int_{P_1+P_2+...+P_n}^{Q_1+Q_2+...+Q_n} \omega_1, ..., \int_{P_1+P_2+...+P_n}^{Q_1+Q_2+...+Q_n} \omega_g \right)$$

is an isomorphism.

Injectivity is the content of the theorem of Abel, surjectivity is the content of the Jacobi inversion theorem.

6.2.3 Theorem (Riemann Vanishing Theorem) *There is a vector $\Delta \in \mathbb{C}^g$ so that for all $z \in \mathbb{C}^g$*
either $\Theta(J(P)+z) = 0$ for all $P \in X$.
or $\Theta(J(P)+z)$ has exactly (non necessarily different) g zeros $Q_1, ..., Q_g$ with $\sum_{j=1}^{g} J(Q_j) = -z + \Delta \mod (\mathbb{Z} + \Omega\mathbb{Z})$.

6.2.4 Remark *The Riemann vector Δ has the explicit expression in terms of the period matrix Ω:*

$$J(\Delta)_k := i\pi - i\pi\Omega_{kk} + \sum_{l,k(l\neq k)}^{g} \oint_{\alpha_i} \omega_i(P) \int_{P_0}^{P} \omega_k.$$

From the Riemann vanishing theorem we get the following immediate corollary (see [Schl]):

6.2.5 Corollary *Let X be a compact Riemann surface of genus $g \geq 1$, then the function*

$$\Theta(J(P) - gJ(P_0) + \Delta)$$

has exactly a zero in P_0 with muliplicity g.

Proof. We set $Q_i =: P_0$ as in the theorem above and obtain the assertion with $gJ(P_0) = -z + \Delta \mod (\mathbb{Z} + \Omega\mathbb{Z})$. $\square$
If we choose the points $P_1, ..., P_g$ in the above theorem generic, then the first case will never occur (in Farkas-Kra [FK], VI.3.3 this is discussed in more detail.)
One can built functions on Riemann surfaces in terms of theta functions.
There are three ways to get meromorphic functions on a Riemann surface X from $\Theta(z, \Omega)$:

1. As products
$$\frac{\prod_{k=1}^{n} \Theta(z + a_k)}{\prod_{k=1}^{n} \Theta(z + b_k)}$$
if $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{C}^g$ $2n$ vectors such that $\sum_{i=1}^{n} a_i - b_i = 0 \mod \mathbb{Z}^g$

2. As differences of logarithmic deriviatives
$$d \log \frac{\Theta(z + a)}{\Theta(z)}$$

3. As second logarithmic derivatives
$$\frac{\partial^2}{\partial z_i \partial z_j} \log \Theta(z)$$

The third method gives for instance for genus $g = 1$:

$$\frac{d^2}{dz^2} \log \Theta(z, \tau) = \wp(z),$$

where $\theta(z, \tau)$ is the theta function on $\mathbb{C} \times \mathbb{H}$ (where $\mathbb{H}$ is the upper half space in $\mathbb{C}$), and $\wp(z)$ is the Weierstrass functions with respect to τ.
Let $\mathcal{E}$ denote the subset of $Pic^{g-1}(X)$ consisting of holomoprhic line bundles of degree $g - 1$ which have a nonzero holomorphic section, i.e. $H^0(X, \xi) \neq 0$ ($\xi \in Pic^{g-1}(X)$).
Let denote by $X^{(d)}$ the d-fold symmetric product. $\mathcal{E}$ is the image of $X^{(g-1)}$ under the Jacobi map , which is birational on its image.
Hence $\mathcal{E}$ is a divisor on $Pic^{g-1}(X)$ and defines a line bundle $\mathcal{O}(\mathcal{E})$ on $Pic^{g-1}(X)$.
The Riemann theta function defines its divisor of zeros θ in the Jacobi variety $J(X)$. $\Theta(z)$ is the unique (up to multiplicatives) holomorphic section of the corresponding line bundle $\mathcal{O}(\theta)$ on $J(X)$.
A theta function with characteristics is a holomorphic section of the line bundle associated with a translate of θ by some element of $J(X)$.
We now introduce the notion of a theta characteristic.

6.2.6 Definition *A* ***theta characteristic*** $\alpha \in Pic^{g-1}$ *is a holomorphic line bundle of degree* $g - 1$ *such that* $\alpha^2 = K$ *where* K *is the canonical bundle.*

Riemann's vanishing theorem 6.2.3 states essentially that there exists a theta characteristic $\kappa \in Pic^{g-1}(X)$ such that

$$\theta = \kappa^{-1} \otimes \mathcal{E}.$$

6.2.7 Definition *Let* $\xi \in \mathcal{E}$. *Let denote by* $\Theta[\xi](z)$ *the theta function with characteristics* $\kappa^{-1} \otimes \xi$, *which is a holomorphic section of the line bundle* $\mathcal{O}((\kappa \otimes \xi) \otimes \theta)$ *on* $J(X)$.

6.3 The Prime Form

Recall: $\mathcal{E}$ denotes the subset of $Pic^{g-1}(X)$ consisting of holomorphic line bundles of degree $g - 1$ which have a nonzero holomorphic section. This subset is a subvariety. We now state the Riemann's singularity theorem in order to define the unique sections h_α of odd theta characteristics. These sections are involved in the definition of the prime form.

6.3.1 Theorem (Riemann's Singularities Theorem) *Let* $\mathcal{E}$ *be the subvariety as defined above. Then the multiplicity of a point* ξ *of* $\mathcal{E}$ *is given by the identity*

$$\dim H^0(X, \xi) = mult\,_{\xi}\mathcal{E}.$$

According to the definition of the theta function this means

$$\theta[\xi](0) = 0 \Leftrightarrow \xi \in \mathcal{E}$$

If a theta characteristic $\alpha \in Pic^{g-1}(X)$ is a non-singular point on $\mathcal{E}$, then $\dim H^0(X, \alpha) = 1$,

$$\theta[\alpha](0) = 0, \quad d\theta[\alpha](0) \neq 0. \tag{6.7}$$

Let α be an odd theta characteristic satisfying (6.7). We denote by h_α its unique holomorphic section.
The following assertion can be found e.g. in Fay's book [F1] or in Mumford's Tata lectures [Mum2].

6.3.2 Proposition *Let α be an odd nonsingular theta characteristic satisfying equation (6.7). Let h_α be its unique holomorphic section. Then h_α can be expressed by theta functions with characteristics as follows:*

$$h_\alpha(z)^2 = \sum_{j=1}^{g} \frac{\partial \Theta[\alpha]}{\partial z_j}(0)\omega_j(z)$$

We present a lemma which is proved for instance in the Tata lectures of Mumford [Mum2]. The proof is repeated in Raina's paper [R1]. Therefore we sketch the proof briefly.

6.3.3 Lemma *Let be β an odd theta characteristic with $\beta = \mathcal{O}(P_1 + ... + P_{g-1})$ and $h^0(X, \beta) = 1$ (the last condition means: β is non-singular).*
Then $\theta[\beta](Q - P) = 0 \Leftrightarrow$
(i) $Q = P$
(ii) $P = P_i$
(iii) $Q = P_i$

Proof. We know from the definition of β:

$$\theta[\beta](Q - P) = 0 \Leftrightarrow \mathcal{O}(Q - P) \otimes \beta \in \mathcal{E} \Leftrightarrow h^0(X, \mathcal{O}(Q - P) \otimes \beta) \neq 0$$

$$\overset{(*)}{\Leftrightarrow} h^0(X, \mathcal{O}(Q) \otimes \beta \otimes \mathcal{O}(-P)) \neq 0$$

Riemann Roch:

$$\begin{aligned} h^0(X, \mathcal{O}(Q) \otimes \beta) - h^1(X, \mathcal{O}(Q) \otimes \beta) &= 1 - g + g \\ h^0(X, \mathcal{O}(Q) \otimes \beta) - h^0(X, K \otimes \mathcal{O}(-Q) \otimes \beta^{-1}) &= 1 \\ h^0(X, \mathcal{O}(Q) \otimes \beta) - h^0(X, \mathcal{O}(-Q) \otimes \beta) &= 1 \end{aligned}$$

Suppose $h^0(X, \mathcal{O}(-Q) \otimes \beta) = 0$ then we have a unique (up to a constant) section with zeros in $P_1, ..., P_{g-1}, Q$. The condition $\theta[\beta](Q - P) = 0$ is due to (*) equivalent to the condition $h^0(X, \mathcal{O}(Q) \otimes \beta \otimes \mathcal{O}(-P)) \neq 0$, so P must be Q or some P_i. This proves (i) and (ii).
Suppose $h^0(X, \mathcal{O}(-Q) \otimes \beta) \neq 0$. We know that $H^0(X, \beta) = 1$, so Q must be some P_i. This proves (iii). $\square$

6.3.4 Definition (Prime Form) *Let $e \in Pic^{g-1}$ be a non-singular odd theta characteristic, i.e. $mult_e \Theta = 1$.*
*The **prime form** $E(z,w)$ is defined by*

$$E(P,Q) = \frac{\theta[e](J(P)-J(Q))}{h_e(P)h_e(Q)}. \tag{6.8}$$

From the definition of the prime form we see that the zeros of the theta function off the diagonal are canceled by the h_e. So the main property is immediately clear: The prime form has a zero of order one along the diagonal.
The prime form also satisfies a certain quasi-periodicity:

6.3.5 Remark (Periods of the Prime Form) *We fix*

$$\chi := e^{2\pi i(\epsilon^t n - \delta^t m)} \qquad \epsilon, \delta \in \{0, \frac{1}{2}\}^g; n, m \in \mathbb{Z}^g. \tag{6.9}$$

We obtain $\epsilon^t n - \delta^t m \in \frac{1}{2}\mathbb{Z}$. Thus we have $\chi \in \{-1, +1\}$.
The periods of the prime form are given by

$$\begin{aligned} E(P + \sum_{j=1}^{g} n_j\alpha_j + \sum_{j=1}^{g} m_j\beta_j, Q) &= \chi \cdot e^{-\pi i m^t \Omega m - 2\pi i m^t(J(P)-J(Q))} E(P,Q) \\ E(P, Q + \sum_{j=1}^{g} n_j\alpha_j + \sum_{j=1}^{g} m_j\beta_j) &= \chi \cdot e^{-\pi i m^t \Omega m + 2\pi i m^t(J(P)-J(Q))} E(P,Q) \end{aligned}$$

This is a consequence of the periodicities of the theta function with characteristics. $\Box$
The following theorem can be found for instance in the book of Mumford [Mum2]. It illustrates that the prime form plays the role of $(z-w)$ in the higher genus case.

6.3.6 Theorem (Properties of the Prime Form) *Let X be a compact Riemann surface of genus $g \geq 1$. The prime form satisfies the following two properties:*

1. *$E(P,Q) = 0 \Leftrightarrow P = Q$.*
2. *$E(P,Q) = -E(Q,P)$*
3. *Let be $f \in \mathcal{M}(X)$ a meromorphic function on the Riemann surface X. Let be $Q_1, ..., Q_d$ the zero set of f, and let be $P_1, .., P_d$ the set of poles of f. (where $Q_i \neq p_j$ $\forall i \neq j$). Then f is given by*

   $$f(z) = C \cdot \frac{\prod_{j=1}^{d} E(z, Q_j)}{\prod_{j=1}^{d} E(z, P_j)}$$

   with $C \in \mathbb{C}$.

6.4 Differentials

6.4.1 Differentials of the Second Kind

We cite now a result which can be found e.g. in [Mum2].

6.4.1 Proposition *The differential given by*

$$\omega_{ab} = \partial_z \log \frac{E(z,a)}{E(z,b)}$$

is a 1-form on X with the properties:

1. *ω_{ab} has zero-periods with respect to α_j*

2. *ω_{ab} has two simple poles in a and b with residues $+1$ and -1.*

Starting from the differential ω_{ab} we can add a suitable linear combination of the holomorphic differentials $\omega_1, ..., \omega_g$ in order to achieve that the resulting differential has only purely imaginary periods. The following corollary gives the linear combination explicitely. See also [Schlbook] for this result.

6.4.2 Corollary (Differential with Purely Imaginary Periods) *Let ω_{ab} be the differential given as in the above proposition. The differential*

$$\rho = \omega_{ab} + i \sum_{k,j=1}^{g} \left(Re \int_{\beta_j} \omega_{ab} \right) (Im\Omega)^{-1}_{kj}\, \omega_k$$

has purely imaginary periods.

Proof.

$$\int_{\alpha_l} \rho = \int_{\alpha_l} \omega_{ab} + i \sum_{k,j=1}^{g} \left(Re \int_{\beta_j} \omega_{ab} \right) (Im\Omega)^{-1}_{kj}\, \omega_k =$$

$$= 0 + i \sum_{k,j=1}^{g} \left(Re \int_{\beta_j} \omega_{ab} \right) (Im\Omega)^{-1}_{kj} \left(\int_{\alpha_l} \omega_k \right)$$

$$= +i \sum_{k,j=1}^{g} \left(Re \int_{\beta_j} \omega_{ab} \right) (Im\Omega)^{-1}_{kj}\, \delta_{lk} =$$

$$= +i \sum_{j=1}^{g} \left(Re \int_{\beta_j} \omega_{ab} \right) (Im\Omega)^{-1}_{lj} \in i\mathbb{R}.$$

And

$$\int_{\beta_l} \rho = \int_{\beta_l} \omega_{ab} + i \sum_{k,j=1}^{g} \left(Re \int_{\beta_j} \omega_{ab} \right) (Im\Omega)^{-1}_{kj}\, \omega_k =$$

$$= Re\int_{\beta_l}\omega_{ab} + iIm\int_{\beta_l}\omega_{ab} + i\sum_{k,j=1}^{g}\left(Re\int_{\beta_j}\omega_{ab}\right)(Im\Omega)^{-1}_{kj}\left(\int_{\beta_l}\omega_k\right) =$$

$$= Re\int_{\beta_l}\omega_{ab} + iIm\int_{\beta_l}\omega_{ab} + i\sum_{k,j=1}^{g}\left(Re\int_{\beta_j}\right)(Im\Omega)^{-1}_{kj}\underbrace{(\Omega)_{kl}}_{=(Re\Omega+iIm\Omega)_{kl}} =$$

$$= Re\int_{\beta_l}\omega_{ab} + iIm\int_{\beta_l}\omega_{ab} +$$

$$+ i\sum_{k,j=1}^{g}\left(Re\int_{\beta_j}\omega_{ab}\right)(Im\Omega)^{-1}_{kj}(Re\Omega)_{kl} - \sum_{k,j=1}^{g}\left(Re\int_{\beta_j}\right)(Im\Omega)^{-1}_{kj}(Im\Omega)_{kl} =$$

$$= Re\int_{\beta_l}\omega_{ab} + iIm\int_{\beta_l}\omega_{ab} - \sum_{j=1}^{g}\left(Re\int_{\beta_j}\omega_{ab}\right)\delta_{jl} + i\sum_{k,j=1}^{g}\left(Re\int_{\beta_j}\omega_{ab}\right)(Im\Omega)^{-1}_{kj}(Re\Omega)_{kl} =$$

$$= Re\int_{\beta_l}\omega_{ab} - \left(Re\int_{\beta_l}\omega_{ab}\right) + iIm\int_{\beta_l}\omega_{ab} + i\sum_{k,j=1}^{g}\left(Re\int_{\beta_j}\omega_{ab}\right)(Im\Omega)^{-1}_{kj}(Re\Omega)_{kl} \in i\mathbb{R}$$

$\square$

6.4.3 Remark (The Case $g = 0$) *Let X be the Riemann sphere $\mathbb{P}$. Choose $a = 0$, and $b = \infty$. The analogous differential is*

$$\rho = \frac{1}{z}dz.$$

6.4.2 The Sigma-Differential

The following differential of weight $\frac{g}{2}$ is essential in order to construct explicit sections.
The sigma differential goes probably back to F. Klein. Fay uses this differential in [F2] (chapter 1, prop. 1.2).

6.4.4 Definition (Sigma Differential) *For a given Riemann surface the function*

$$\sigma(P) = \exp\left(-\sum_{i=1}^{g}\omega_i(Q)\log E(P,Q)\right)$$

defines a nowhere vanishing differential of weight $\frac{g}{2}$.

John Fay gives in chapter 1 of [F2] the following expression of sigma differentials in terms of theta functions and prime forms:

6.4.5 Proposition *Let X be a compact Riemann surface of genus $g \geq 1$. For all $P, P_0, P_1, ..., P_g$ we have*

$$\frac{\sigma(P)}{\sigma(P_0)} = \frac{\theta(\sum_{i=1}^{g}J(P_i) - J(P) - \Delta)}{\theta(\sum_{i=1}^{g}J(P_i) - J(P_0) - \Delta)}\prod_{i=1}^{g}\frac{E(P_i,P_0)}{E(P_i,P)}. \qquad (6.10)$$

6.4.6 Corollary *The sigma differential can also be expressed as*

$$\sigma(P) = \frac{\theta(J(P) - gJ(P_0) + \Delta)}{(E(P,P_0))^g}. \tag{6.11}$$

A proof can be found e.g. in [Schl]. □

6.4.7 Remark (Periods of the Sigma Differential) *The periods of $\sigma(P)$ are given by*

$$\sigma(P + na + mb) = \chi^{-g} \cdot e^{-\pi i m^t \Omega m(g-1) - 2\pi i m^t(\Delta-(g-1)P)} \sigma(P). \tag{6.12}$$

where χ is the factor ± 1 from equation (6.9).

6.5 Bidifferentials, Projective Connections

Let X be a Riemann surface of genus $g \geq 1$. Let L and L' be line bundles over X. Denote by $L \boxtimes L'$ the line bundle

$$L \boxtimes L' := \pi_1^* L \otimes \pi_2^* L'$$

on $X \times X$ where the π_i are defined as the projections $\pi_i : X \times X$ on the i-th factor.
Let denote

$$L \boxtimes L'(m\Delta) := \pi_1^* L \otimes \pi_2^* L' \otimes \mathcal{O}(m\Delta), \quad m \in \mathbb{Z},$$

where Δ is the diagonal divisor on $X \times X$.

6.5.1 Definition ([T] Def.1.3.2) *Let X be a Riemann surface of genus $g \geq 1$. Let K be the canonical bundle on X. A* ***symmetric bidifferential*** *$\omega(z,w)$ of the second kind is a section in $K \boxtimes K(2\Delta)$, i.e.*
$\omega(z,w)$ is defined on $X \times X$ and for $z_0 \in X$:

1. *$\omega(z, z_0)$ has only a pole of order 2 in z_0*
2. *$\omega(z,w) = \omega(w,z)$.*

Locally we have

$$\omega(z,w) = \frac{\alpha \cdot dzdw}{(z-w)^2} + H(z,w)dzdw$$

with $H(z,w)dzdw$ a holomorphic bidifferential. The complex number α does not depend on the choice of the parameters.

6.5.2 Definition (Biresiduum [T] Def.1.3.3) *The number α as above is called* ***Biresiduum*** *of $\omega(z,w)$, short: $Bires\omega = \alpha$.*

6.5.3 Proposition 1. *The map $Bires : H^0(X \times X, K \boxtimes K(2\Delta)) \to \mathbb{C}$ is a homomorphism of vector spaces.*

2. *$KerBires = H^0(X \times X, K \boxtimes K)$ is the space of holomorphic symmetric differentials.*

6.5.4 Lemma ([U] 1.4.5) *Let $\omega(P,Q)$, and $\widetilde{\omega}(P,Q)$ be symmetric bilinear differentials on $X \times X$ holomorphic outside the diagonal with poles of order 2 at the diagonal and with*

$$Bires(\omega) = Bires(\widetilde{\omega}).$$

Then

$$\Omega(P,Q) = \omega(P,Q) - \widetilde{\omega}(P,Q)$$

is a holomorphic symmetric bidifferential on $X \times X$.

Define now

$$T_\omega(z)dz^2 = 6 \lim_{x,y \to z} \left(\omega(x,y)dxdy - \frac{dxdy}{(x-y)^2} \right)$$

6.5.5 Definition (Schwarzian Derivative) *Let f be a holomorphic function on a domain $U \subset \mathbb{C}$. The **Schwarzian derivative** is defined by*

$$\{f,z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian derivative has the following important property:
$\{f,z\} = 0$, if f is a Möbius transformation, i.e. $f(z) = \frac{az+b}{cz+d}$ $(a,b,c,d \in \mathbb{C}, ad-bc \neq 0)$.

6.5.6 Theorem

$$T_\omega(z)dz^2 = T_\omega(w)dz^2 - \{g,w\}dw^2.$$

The proof goes back to Tyurin [T]. He again cites Wirtinger [Wirt].

6.5.7 Definition (Projective Connection) *Let X be a Riemann surface of genus $g \in \mathbb{N}_0$. Let $(U_\alpha, z_\alpha)_{\alpha \in I}$ a covering of X by holomorphic coordinates, with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$. A **(holomorphic, meromorphic) projective connection** is a system of local (holomorphic, meromorphic) functions $R_\alpha(z_\alpha)$ if it transforms as*

$$R_\beta(z_\beta)\left(f'_{\beta\alpha}\right)^2 = R_\alpha(z_\alpha) + \{f_{\beta\alpha}, z_\alpha\}. \tag{6.13}$$

From the above theorem we obtain

6.5.8 Corollary *The symmetric bilinear differentials of the second kind define projective connections.*

We have the following assertion which can be found e.g. in Gunning [G].

6.5.9 Theorem *Let X be a Riemann surface of genus $g \geq 0$. Then there exists a projective connection.*

6.6 Digression to Higher Rank Bundles

6.6.1 Generalities

In this section we recall the notions of stability and semi-stability of vector bundles and some of the well-known results in this theory. Let E be a holomorphic vector bundle of rank r. This means the fibers of the vector bundle are r-dimensional complex vector spaces. The **degree** d of E is given by the degree (i.e. the first Chern class) of the associated determinant line bundle $\det E$. (i.e. $d = c(\det E)$).

6.6.1 Definition (Simple Bundles) *Let be E a holomorphic vector bundle on a Riemann surface X. E is called **simple** if*

$$\dim H^0(X, End(E)) = 1.$$

For example line bundles are simple because $H^0(X, End(L)) = H^0(X, L^* \otimes L) = H^0(X, \mathcal{O}_X) = 1$.

6.6.2 Definition *Let E be a holomorphic vector bundle of rank r and degree d. Denote by μ the ratio of the degree and the rank of E:*

$$\mu(E) := \frac{d}{r}$$

*This ratio is called the **slope** of E.*

6.6.3 Definition

1. *A holomorphic vector bundle is **semi-stable** if for any proper nonzero subbundle $F \subset E$ we have*

$$\mu(F) \leq \mu(E)$$

2. *A holomorphic vector bundle is **stable** if for any proper nonzero subbundle $F \subset E$ we have*

$$\mu(F) < \mu(E)$$

The proof of the following proposition can be found e.g. in [Newstead].

6.6.4 Proposition

1. *Line bundles are stable*

2. *If F is a stable bundle, and L is a line bundle then $F \otimes L$ is stable.*

3. *Let E and F be two stable bundles and $\mu(E) = \mu(F)$. Then all (non-trivial) homomorphisms $E \to F$ are isomorphisms.*

6.6.5 Corollary *Every stable bundle ist simple.*

6.6.2 (Semi-)Stable Vector Bundles on $\mathbb{P}^1$

The classification of vector bundles for $g = 0$ is easy because of the following Grothendieck theorem.

6.6.6 Theorem (Grothendieck) *Consider the Riemann surface $X = \mathbb{P}$. Let E be a holomorphic vector bundle of rank n and degree d on X.*
Then we have

$$E \cong \mathcal{O}(d_1) \oplus \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n)$$

where $\sum_{i=1}^n d_i = d$, $d_i \in \mathbb{Z}$ $(1 \leq i \leq n)$.

The Theorem of Grothendieck can be considered as an application (or reformulation) of the Birkhoff decomposition (see [PS] Theorem 8.1.2.). It asserts that any element of the loop group $\gamma \in LGL(n, \mathbb{C})$ can be factorized in the form

$$\gamma = \gamma_- \Lambda \gamma_+$$

where $\gamma_\pm \in LGL^\pm(n, \mathbb{C})$, and the groups $LGL^\pm(n, \mathbb{C})$ consist of loops which are the boundary values of holomorphic maps form inside the unity circle or outside the unity circle. This gives rise to the proof the theorem.
We know from above that every stable bundle is simple, but in general the other direction is not true. Though in the special case of the Riemann sphere we have the following assertions.

6.6.7 Proposition *Let X be the Riemann sphere $\mathbb{P}$. Then we have the following assertions:*

1. *Every simple bundle is stable.*
2. *If E is stable then E is a line bundle.*
3. *If E is semistable of rank n then $E \cong \bigoplus_{i=1}^n \mathcal{O}(d), \quad d \in \mathbb{Z}$.*

The proof can be found e.g. in [Schork] proposition VII.4. □

6.7 Szegö-Kernels

Suppose E a holomorphic vector bundle, then by $E^\vee = E^{-1} \otimes K$ we denote the Serre dual of the vector bundle.
We quote the following assertion (see for instance [R1]):

6.7.1 Theorem *Let X be a compact Riemann surface. Let E be a holomorphic vector bundle over X with $rk(E) = n$, $\deg(E) = n(g-1)$ with the properties*
(i) $\dim H^0(X, E) = 0$
(ii) $\dim H^0(X, End(E)) = 1$
Then $h^0(X \times X, E \boxtimes E^\vee(\Delta)) = 1$.

6.7.2 Definition *Let X and E be as in the theorem above. The elements in*

$$H^0(X \times X, E \boxtimes E^\vee(\Delta))$$

are called ***Szegö kernels****.*
They are due to the above theorem unique up to a scalar multiple.

From this theorem we can see that stable vector bundles of degree $n(g-1)$ are good candidates for Szegö kernels.
Due to the preceeding proposition there does not exist a Szegö kernel of higher rank than 1 for $X = \mathbb{P}$.
There is a detailed discussion in [Schork] from a physical viewpoint concerning Szegö kernels of higher rank. From a mathematical point of view see [F2].

6.7.3 Remark (The "classical" Szegö Kernel) *Let X be a compact Riemann surface of genus $g \geq 1$. Let $\alpha \in Pic^{g-1}(X)$ be a line bundle of degree $g-1$. Let be $H^0(X, E) = 0$ (i.e. $\theta(\alpha) \neq 0$). The Szegö kernel can be expressed in terms of theta functions by*

$$S_\alpha(x, y) = \frac{\theta(y - x + \alpha)}{\theta(\alpha) E(x, y)}. \tag{6.14}$$

Locally (i.e. in a suitable neighborhood of the diagonal) the Szegö kernels can be expanded by

$$\frac{S_\alpha(x, y)}{\sqrt{dx}\sqrt{dy}} = \frac{1}{x - y} + s_0 + s_1(x - y) + s_2(x - y)^2 + ...$$

where $s_i \in \mathbb{C}$.
More generaly for higher rank bundles we can expand locally

$$\frac{S(x, y)}{\sqrt{dx}\sqrt{dy}} = \frac{E_n}{x - y} + S_0 + S_1(x - y) + S_2(x - y)^2 + ...$$

where E_n is n by n unity matrix, and the S_i are n by n matrices.

Chapter 7

Krichever Novikov Forms and Szegö Kernels

In this chapter we address the higher genus analogues of the monomials z^n ($n \in \mathbb{Z}$) on the Riemann sphere.
Krichever-Novikov forms are in principle meromorphic sections in certain line bundles which are holomorphic outside a given set of points.
The idea is the following: Consider the monomials z^{-n-k}. One can regard these monomials as sections in the k-times power of the canonical bundle on the Riemann sphere with possible poles in 0 and ∞.
Now it becomes a question of algebraic geometry if it is possible to find sections in the canonical bundle holomorphic outside two given points P_+ and P_-. More generally one can ask if there exist sections in powers of the canonical bundle holomorphic outside a generic set of finitely many points.
Krichever and Novikov gave the answer for 2 points, Schlichenmaier [Schl] gave the answer for more than two points.
The chapter is organized as follows.
In the first section we briefly show under which conditions the space of sections in the kth power of the canonical bundle with certain allowed poles is exactly one-dimensional. In the second section we introduce the Krichever-Novikov pairing and the Krichever-Novikov forms for two points. In the third section we study Szegö kernels with respect to certain bundles of degree $g-1$. In the fourth section we show that appropriate Szegö kernels can be expanded in terms of Krichever-Novikov forms.

7.1 Preparations

7.1.1 Generalized Weierstrass Points and Fundamental Lemma

For the following considerations we need well-known facts about divisors (or holomorphic line bundles) on compact Riemann surfaces X.

Let K be the canonical line bundle of X. Its associated sheaf is the sheaf of holomorphic differentials. I will follow the common practice not to distinguish between the line bundle and its associated invertible sheaf of sections.
By $H^i(X, D)$ we denote the i-th cohomology group with respect to the divisor D. The dimension of $H^i(X, D)$ is denoted by $\dim H^i(X, D) = h^i(X, D)$.

7.1.1 Remark (Ingredients) *Let X be a compact Riemann surface with genus $g \geq 0$, let K be the canonical divisor, and D a divisor of X.*

$$\begin{aligned} \textit{Riemann-Roch: } h^0(X,D) - h^1(X,D) &= 1 - g + \deg(D); && (7.1)\\ \textit{Serre-Duality: } h^1(X,D) &= h^0(X, K-D); && (7.2)\\ \textit{negative degree: } \deg(D) < 0 &\Rightarrow h^0(X,D) = 0. && (7.3) \end{aligned}$$

We know furthermore

$$\deg(K) = 2g - 2. \tag{7.4}$$

For $g \geq 2$, $\lambda \geq 2$ ($\lambda \in \mathbb{Z}$) the divisor λK is a non-special divisor (i.e. $h^1(X, \lambda K) = 0$) and we have

$$h^0(X, \lambda K) = (2\lambda - 1)(g - 1). \tag{7.5}$$

For $g = 1$ we have $h^0(X, \lambda K) = 1$ for all $\lambda \in \mathbb{Z}$.
We are considering now points $P_1, ..., P_N \in X$ on a Riemann surface such that

$$h^0(X, \lambda K - \sum_{i=1}^{N} n_i P_i) = 1$$

For this purpose we have to adjust the n_i appropriately, and we have to give some conditions on the points P_i. The right hand side of the Riemann-Roch theorem (7.1) is supposed to be 1, that means in our situation:

$$(2\lambda - 1)(g - 1) - \sum_{i=1}^{N} n_i \stackrel{!}{=} 1.$$

We have therefore to study under which conditions the divisor $\lambda K - \sum_{i=1}^{N} n_i P_i$ is not special.

7.1.2 Definition (Wronskian) *Suppose $f_1, ..., f_l$ are holomorphic functions on a domain $U \subset \mathbb{C}$. By the* ***Wronskian determinant*** *one means the determinant of the matrix of derivatives $f_k^{(j)}$, where $0 \leq j \leq l-1$, $1 \leq k \leq l$, i.e.*

$$W(f_1, ..., f_l) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_l \\ f_1' & f_2' & \cdots & f_l' \\ & & & \\ f_1^{(l-1)} & f_2^{(l-1)} & \cdots & f_l^{(l-1)} \end{pmatrix}.$$

If the functions $f_1, ..., f_l$ are linearly independent, then the Wronskian doesn't vanish identically.

7.1.3 Definition (L-Weierstrass Points) *Let X be a compact Riemann surface. Let L be a line bundle with $\dim H^0(X,L) = h^0(X,L) = l \geq 0$.*
By a ***Weierstrass point with respect to*** *L (for short: an L-Weierstrass point) we mean a point $P \in X$ such that the Wronskian does vanish.*

$$W(f_1, ..., f_l)(P) = 0,$$

where $f_1, ..., f_l$ are functions representing the l linearly independent sections of $H^0(X,L)$ on an appropriate neighborhood of P.

7.1.4 Proposition *Let X be a compact Riemann surface. Let L be a line bundle with $\dim H^0(X,L) = h^0(X,L) = l \geq 0$.*
Then we have the equivalence

$$h^0(X, L - lP) \neq 0 \Leftrightarrow P \text{ is a L-Weierstrass point on } X \Leftrightarrow W(f_1, ..., f_l)(P) = 0.$$

Furthermore there are only finitely many Weierstrass points.

Proof. We only have to prove the first equivalence.
Let $f_1, ..., f_l$ be a basis of $H^0(X,L)$. Let P be an L-Weierstrass point on X, i.e. $W(f_1, ..., f_l)(P) = 0$. This implies:
There exists a vector $(c_1, ...c_l) \neq (0, ..., 0)$ such that it is a solution of the equation

$$\begin{pmatrix} f_1 & f_2 & ... & f_l \\ f_1' & f_2' & ... & f_l' \\ & & & \\ f_1^{(l-1)} & f_2^{(l-1)} & ... & f_l^{(l-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \end{pmatrix}.$$

This means

$$\sum_i^l c_i f_i^{(k)} = 0 \text{ for } 0 \leq k \leq l-1.$$

If we define

$$f := \sum_i^l c_i f_i$$

then we have a non-zero section in the space $H^0(X, L - lP)$.
□

7.1.5 Lemma (Schlichenmaier) *Let X be a compact Riemann surface, and let L be a line bundle on X with*

$$h^0(X,L) = l.$$

Let $P \in X$ be a non-Weierstrass point on X. Then we have for $n \in \mathbb{Z}_+$:

$$h^0(X, L - nL_P) = \max\{l - n, 0\}.$$

Proof. We claim for a general point $P \in X$ we have the inequalities

$$l-1 \leq h^0(X, L-P) \leq l$$

More generally

$$l-k \leq h^0(X, L-kP) \leq l$$

If P is now a non-Weierstrass point with respect to L then we have $h^0(X, L-lP) = 0$.
This means that for $0 \leq k \leq l$ the numbers $h^0(X, L-kP)$ **must** decrease by 1. This implies the assertion of the lemma. □
The original notion of Weierstrass points comes from the question for which divisors nP we have global functions f such that $(f)_\infty = nP$.

7.1.6 Remark (Canonical Bundle) *If L is the canonical bundle K, then we get the usual definition of Weierstrass points.*
If L is a λ-differential λK (where $\lambda \geq 1$) then we have the definition like in [FK] III.5.9.

Now we address the question of the dimension of a λ-differential with given poles and zeros.

7.1.7 Theorem *Let X be a compact Riemann surface of genus g.*

1. *Let $g \geq 2$, and let $\lambda \in \mathbb{Z}$, but $\lambda \neq 0, 1$. Let $n_1, ...n_N \in \mathbb{Z}$ be arbitrary integers. Then we have*

$$h^0(X, \lambda K - \sum_{i=1}^{N} n_i P_i) = \max\{(2\lambda - 1)(g-1) - \sum_{i=1}^{N} n_i, 0\}.$$

   *where the points P_i are **generic**.*

2. *Let $g \geq 1$. Then we have*

   (a)

$$h^0(X, K - \sum_{i=1}^{N} n_i P_i) = \begin{cases} \max\{g-1-\sum_{i=1}^N n_i, 0\}, & \textit{at least one } n_i < 0 \\ \max\{g-\sum_{i=1}^N n_i, 0\}, & \textit{all } n_i \geq 0 \end{cases}$$

   (b)

$$h^0(X, -\sum_{i=1}^{N} n_i P_i) = \begin{cases} \max\{-g+1-\sum_{i=1}^N n_i, 0\}, & \textit{at least one } n_i > 0 \\ \max\{-g-\sum_{i=1}^N n_i, 0\}, & \textit{all } n_i \leq 0 \end{cases}$$

3. *Let $g = 1$, and let $\lambda \in \mathbb{Z}$. We have*

$$h^0(X, \lambda K - \sum_{i=1}^{N} n_i P_i) = \begin{cases} \max\{g-1-\sum_{i=1}^N n_i, 0\}, & \textit{at least one } n_i < 0 \\ \max\{g-\sum_{i=1}^N n_i, 0\}, & \textit{all } n_i \geq 0 \end{cases}$$

4. *Let $g = 0$ Then*

$$h^0(X, \lambda K - \sum_{i=1}^{N} n_i P_i) = \max\{1 - 2\lambda - \sum_{i=1}^{N} n_i, 0\}.$$

Where the points P_i are arbitrary.

The proof will show what "generic" means more precisely. In principle it means that we have to exclude finitely many Weierstrass points with respect to certain line bundles corresponding to certain divisors.
Proof.

1. Consider first $\lambda \geq 2$. Let (after a possible reordering) $n_1, ..., n_{N'}$ be negative (where N' can range between 0 and N). The divisor $\lambda K - \sum_{i=1}^{N'} n_i P_i$ has degree $2\lambda(g-1) - \sum_{i=1}^{N'} n_i$, for its dual divisor we obtain

$$\deg\left((1-\lambda)K + \sum_{i=1}^{N'} n_i P_i\right) = \underbrace{(1-\lambda)}_{<0} \cdot 2 \underbrace{(g-1)}_{>0} + \underbrace{\sum_{i=1}^{N'} n_i}_{<0} < 0.$$

Thus we get from Riemann-Roch:

$$h^0(X, K - \sum_{i=1}^{N'} n_i P_i) = (2\lambda - 1)(g-1) - \sum_{i=1}^{N'} n_i.$$

For the (possibly) remaining $n_{N'+1}, ..., n_N$ we can apply the lemma 7.1.5 and obtain the assertion. Here it turns out what "generic" means: We have to avoid the Weierstrass points of the divisors $D_j = (K - \sum_{i=1}^{N'} n_i P_i) - \sum_{i=N'+1}^{j} n_i P_i$.
We address now to the case $\lambda \leq -1$: We start now from the divisor $(1-\lambda)K + \sum_{i=1}^{N} n_i P_i$. Let be (after a possible reordering) $n_1, ..., n_{N'}$ positive (where N' can range between 0 and N). We get by a similar consideration like in the $\lambda \geq 2$ case:

$$h^0(X, (1-\lambda)K + \sum_{i=1}^{N} n_i P_i) = \max\{2(\lambda-1)(g-1) + \sum_{i=1}^{N} n_i, 0).$$

Riemann-Roch gives

$$h^0(X, \lambda K - \sum_{i=1}^{N} n_i P_i) - h^0(X, (1-\lambda)K + \sum_{i=1}^{N} n_i P_i) = 2(\lambda-1)(g-1) - \sum_{i=1}^{N} n_i$$

And we obtain

$$h^0(X, \lambda K - \sum_{i=1}^N n_i P_i) = 2(\lambda-1)(g-1) - \sum_{i=1}^N n_i +$$
$$+ \begin{cases} 0 \\ (1-2\lambda)(g-1) + \sum_{i=1}^N n_i \end{cases} =$$
$$= \begin{cases} 2(\lambda-1)(g-1) - \sum_{i=1}^N n_i \\ 0 \end{cases} .$$

2. (a) If all $n_i \geq 0$ then we apply lemma 7.1.5 and we obtain because of $h^0(K) = g$ the assertion for this case.
   Let now (without restriction of the generality) $n_1, .., n_{N'} < 0$ negative. Therefore the divisor $K - \sum_{i=1}^{N'} n_i P_i$ has degree $2g-2-\sum_{i=1}^{N'} n_i$. Thus $\deg(K - K + \sum_{i=1}^{N'} n_i P_i) = \sum_{i=1}^{N'} n_i P_i < 0$. In other words the divisor $K - \sum_{i=1}^{N'} n_i P_i$ is not special and we know

   $$h^0(K - \sum_{i=1}^{N'} n_i P_i) = g - 1 + K - \sum_{i=1}^{N'} n_i$$

   Now we apply once again lemma 7.1.5 and obtain the assertion.

   (b) We use the above consideration for the Serre-dual divisor: $K + \sum_{i=1}^N n_i P_i$. From the above we know

   $$h^0(X, K + \sum_{i=1}^N n_i P_i) = \begin{cases} \max\{g-1-\sum_{i=1}^N n_i, 0\}, & \text{at least one } n_i > 0 \\ \max\{g - \sum_{i=1}^N n_i, 0\}, & \text{all } n_i \leq 0 \end{cases}$$

   We use Riemann Roch

   $$h^0(X, -\sum_{i=1}^N n_i P_i) = 1 - g - \sum_{i=1}^N n_i P_i + h^0(X, K + \sum_{i=1}^N n_i P_i)$$

   and obtain the assertion.

3. $g = 1$. In genus $g = 1$ the canonical bundle is trivial: $K \cong \mathcal{O}$. Therefore the assertion follows from the above consideration.

4. $g = 0$. We calculate in the same way like in the first case:
   Let be $\lambda \leq 1$. Let be (after a possible reordering) $n_1, ..., n_{N'}$ negative (where N' can range between 0 and N). Consider the divisor $(1-\lambda)K + \sum_{i=1}^{N'} n_i P_i$. We have (due to $\deg(K) = -2$ for $g = 0$)

   $$\deg((1-\lambda)K + \sum_{i=1}^{N'} n_i P_i) = -2\underbrace{(1-\lambda)}_{\geq 0} + \sum_{i=1}^{N'} \underbrace{n_i}_{<0} < 0.$$

Thus we obtain by the compulsory arguments the assertion.
For $\lambda > 1$ we consider the divisor $\lambda K - \sum_{i=1}^{N'} n_i P_i$ with $n_i > 0$ and obtain again the assertion.

□

7.1.2 Restriction to the Desired Bundles

We want the right hand side of the Riemann-Roch theorem (7.1) to be 1. That means

$$1 \overset{!}{=} 1 - g + 2\lambda(g-1) - \sum_{i=1}^{N} n_i.$$

7.1.8 Theorem *Let X be compact Riemann surface of genus g. Let $P_1, ..., P_N$ generic points on X.*

1. *Let $\lambda \in \mathbb{Z}$, but $\lambda \neq 0, 1$. Let $n_1, ...n_N \in \mathbb{Z}$ integers with $\sum_{i=1}^{N} n_i = (2\lambda - 1)g - 2\lambda$. Then we get*

$$h^0(X, \lambda K - \sum_{i=1}^{N} n_i P_i) = 1.$$

2. (a) *Let $n_1, ...n_N \in \mathbb{Z}$ integers with $\sum_{i=1}^{N} n_i = g - 2$. Then we get*

$$h^0(X, K - \sum_{i=1}^{N} n_i P_i) = 1.$$

   (b) *Let $n_1, ...n_N \in \mathbb{Z}$ integers with $\sum_{i=1}^{N} n_i = -g$. Then we get*

$$h^0(X, - \sum_{i=1}^{N} n_i P_i) = 1.$$

Proof. From the above proposition we know

$$h^0(X, K - \sum_{i=1}^{N} n_i P_i) = \max\{(2\lambda - 1)(g-1) - \sum_{i=1}^{N} n_i, 0\}$$

Because $\sum_{i=1}^{N} n_i = (2\lambda - 1)g - 2\lambda$ we obtain:

$$h^0(X, K - \sum_{i=1}^{N} n_i P_i) = \max\{(2\lambda - 1)(g-1) - (2\lambda - 1)g + 2\lambda, 0\} = 1.$$

□

We are especially interested in the two-point case: $\lambda K - n_+ P_+ - n_- P_-$. Define

$$\boxed{s_\lambda := \frac{(1-2\lambda)g}{2} + \lambda.} \tag{7.6}$$

For the special case of two points we obtain the following corollary:

7.1.9 Corollary *Let X be a compact Riemann surface of genus $g \geq 1$.*

1. *For $g \geq 2$ and $\lambda \in \mathbb{Z} \setminus \{0, 1\}$ we have*

$$h^0(X, \lambda K + (-n + s_\lambda)P_+ + (n + s_\lambda)P_-) = 1.$$

   where $n \in \mathbb{Z}$ for g even, and $n \in \mathbb{Z} + \frac{1}{2}$ for g odd.

2. (a) i. *For $n > \frac{g}{2}$ or $n \leq -\frac{g}{2}$ we have*

$$h^0(X, K - (n + \frac{g}{2} - 1)P_+ - (-n + \frac{g}{2} - 1)P_-) = 1$$

   ii. *For $-\frac{g}{2} < n \leq \frac{g}{2}$ we have*

$$h^0(X, K - (n + \frac{g}{2} - 1)P_+ - (-n + \frac{g}{2})P_-) = 1$$

   (b) *Let $g \geq 1$. We obtain*

   i. *For $|n| > \frac{g}{2}$ we have*

$$h^0(X, -(n - \frac{g}{2})P_+ - (-n - \frac{g}{2})P_-) = 1$$

   ii. *For $|n| \leq \frac{g}{2}$ we have*

$$h^0(X, -(n - \frac{g}{2})P_+ - (-n - \frac{g}{2} - 1)P_-) = 1$$

Proof.

1. The first part is trivial because of the first part of theorem 7.1.7.

2. We can apply theorem 7.1.7 for the according cases: For $n > \frac{g}{2}$ or $n \leq -\frac{g}{2}$ we have equivalently $n + \frac{g}{2} - 1 > g - 1$, $-n + \frac{g}{2} - 1 < -1$ or $n + \frac{g}{2} - 1 \leq -1$, $-n + \frac{g}{2} - 1 \geq g - 1$. Thus we have the case of one negative multiplicity. For $-\frac{g}{2} < n \leq \frac{g}{2}$ we have equivalently $-1 < n + \frac{g}{2} - 1 \leq g - 1$, $g > -n + \frac{g}{2} \geq 0$. Thus we can apply the case for only positive multiplicities.

3. For $|n| > \frac{g}{2}$ we can apply the case for at least one positive multiplicity. For $|n| \leq \frac{g}{2}$ we can apply the other case.

□

7.1.10 Definition (L-Weierstrass Gaps) *A* ***L-Weierstrass gap*** *is a number $m \geq 0$ such that $h^0(X, L - mP) \neq h^0(X, L - (m-1)P)$.*

We see from this definition that the gap sequence for a non-Weierstrass point with respect to L has the gap sequence $1, 2, ..., l$.

7.1.11 Proposition *Let L be line bundle of positive degree $\deg(L) =: d \geq 0$ and with $h^0(X, L) = l$.*
Then m is a gap number if and only if $h^0(X, K - L + mp) = h^0(X, K - L + (m-1)p)$.

Proof. We apply Riemann Roch (7.1) an Serre duality (7.2):

$$\begin{aligned} h^0(X, K - L + mp) - h^0(X, L) &= 1 - g + 2g - 2 - d + m && (7.7) \\ h^0(X, K - L + (m-1)p) - h^0(X, L) &= 1 - g + 2g - 2 - d + m - 1 && (7.8) \end{aligned}$$

Subtracting these equation we obtain:

$$h^0(X, K - L + mp) - h^0(X, K - L + (m-1)p) = 1 + h^0(X, L) - h^0(X, L).$$

□

7.2 Krichever-Novikov-Forms

In this section we address the differentials (or "weights") that are the objects we have to deal with in later chapters.
We consider meromorphic λ-forms which are holomorphic outside two given points P_+ and P_-.

7.2.1 Level Lines

We introduce the level lines which are important in order to give a kind of norm on Riemann surfaces. We can also say we introduce a time on the Riemann surface by defining the level lines.

7.2.1 Definition (Level Lines) *Let be P_+ and P_- be two given points on a compact Riemann surface.*
Let be ρ the unique differential of the third kind with purely imaginary periods, i.e. ρ has poles in the points $P_\pm$ with the residues: $Res_{P_+}\rho = 1$, and $Res_{P_-}\rho = -1$.
Furthermore we have

$$\oint_{\alpha_i} \rho \text{ and } \oint_{\beta_i} \rho \in i \cdot \mathbb{R}$$

Let be P_0 a reference point different from $P_\pm$.
The ***level line*** *for τ be defined by*

$$C_\tau := \{P \in X : Re \int_{P_0}^{P} \rho = \tau\}$$

In fact such a differential ρ exists. The next proposition gives a more explicit description of this differential.

7.2.2 Remark *We recall some facts from the last chapter (see proposition 6.4.1 and the discussion below).*
Let $\omega_\pm(z) = d_z \log \frac{E(z,P_+)}{E(z,P_-)}$ be the unique 1 differential with vanishing α-periods and residues ± 1 at $P_\pm$.
Then the unique differential ρ of the third kind with purely imaginary periods is given by

$$\rho(z) = \omega_\pm(z) - \vec{v}(z)(Im\Omega)^{-1} Re(\int_{\vec{\beta}} \omega_\pm).$$

The formula can also be written as

$$\rho(z) = \omega_\pm(z) - i \sum_{i,j=1}^{g} v_i(z) \, (Im\Omega)^{-1}_{ij} \left(Re(\int_{\beta_j} \omega_\pm) \right).$$

This differential and its explicit form can be found e.g. in Fay's first book [F1] and in his second book [F2].
The level lines for $g = 0$ are defined by

$$Re \int_1^P \frac{1}{z} dz$$

Thus the level lines are circles around the origin in the complex plane.

7.2.2 Definition and Explicit Presentation of Krichever-Novikov Forms

In this section we address the explicit presentation of the sections by using theta functions. For the multipoint case including the two-point case this has been already done by Schlichenmaier in his thesis [Schl]. So I restrict myself only to the presentation and a sketch of the proof.
Let denote by $\mathcal{F}^\lambda$ the (infinite dimensional) vector space of meromorphic sections on K^λ which are holomorphic outside the points P_+, P_-.

7.2.3 Definition (Krichever-Novikov Pairing) *The **Krichever-Novikov pairing** (KN pairing) is the pairing between $\mathcal{F}^\lambda$ and $\mathcal{F}^{1-\lambda}$ given by*

$$\mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \to \mathbb{C},$$

$$\langle f, g \rangle := \frac{1}{2\pi i} \int_C f \otimes g = Res_{P_+} (f \otimes g) = -Res_{P_-} (f \otimes g),$$

where C is any cycle separating the points $P_\pm$.

The last equality follows from the residue theorem. The integral does not depend on the separating cycle chosen.

7.2.4 Definition *1. $g \geq 2$, and $\lambda \in \mathbb{Z} \setminus \{0,1\}$:*

$$f_{n,\lambda} \in H^0\left(X, K^{\otimes\lambda} \otimes L_{P_+}^{\otimes -n+s_\lambda} \otimes L_{P_-}^{\otimes +n+s_\lambda}\right)$$

such that

$$\oint_{C_\tau} f_{n,\lambda}(z) f_\lambda^m(z) = \delta_{n,m},$$

where

$$\boxed{f_\lambda^n := f_{n,-\lambda}} \tag{7.9}$$

2. Let $\lambda = 0$:

$$f_{n,0} = A_n \in \begin{cases} H^0\left(X, L_{P_+}^{\otimes -n+\frac{g}{2}} \otimes L_{P_-}^{\otimes +n+\frac{g}{2}}\right) & \text{for } |n| > \frac{g}{2} \\ H^0\left(X, L_{P_+}^{\otimes -n+\frac{g}{2}+1} \otimes L_{P_-}^{\otimes +n+\frac{g}{2}}\right) & \text{for } -\frac{g}{2} \leq n < \frac{g}{2} \end{cases}$$

$$f_{\frac{g}{2},0} = A_{\frac{g}{2}} = 1$$

3. Let $\lambda = 1$:

$$f_1^n = \omega^n \in \begin{cases} H^0\left(X, K \otimes L_{P_+}^{\otimes n-\frac{g}{2}} \otimes L_{P_-}^{\otimes -n+\frac{g}{2}}\right) & \text{for } |n| > \frac{g}{2} \\ H^0\left(X, K \otimes L_{P_+}^{\otimes -n+\frac{g}{2}+1} \otimes L_{P_-}^{\otimes +n+\frac{g}{2}}\right) & \text{for } -\frac{g}{2} \leq n < \frac{g}{2} \end{cases}$$

$$f_1^{\frac{g}{2}} = \omega^{\frac{g}{2}} = \rho$$

such that

$$\oint_{C_\tau} A_n \omega^m = \delta_{n,m}$$

7.2.5 Theorem ([Bo]) *Let X be a compact Riemann surface of genus $g \geq 2$.*

1. Let $\lambda \neq 0, 1$. The sections can be presented as follows:

$$f_{n,\lambda}(P) = N_n^\lambda \cdot \frac{E(P,P_+)^{n-s_\lambda}}{E(P,P_-)^{n+s_\lambda}} \cdot \sigma^{2\lambda-1}(P) \cdot \frac{\Theta(P+u)}{\Theta(u)} \tag{7.10}$$

where

$$u = u(\lambda, n) = (n-s)P_+ - (n+s_\lambda)P_- + (1-2\lambda)\Delta.$$

2. Let $\lambda = 0$. For $|n| > \frac{g}{2}$ we have

$$A_n(P) = N_n^0 \cdot \frac{E(P,P_+)^{n-\frac{g}{2}}}{E(P,P_-)^{n+\frac{g}{2}}} \left(\sigma(P)\right)^{-1} \frac{\Theta(P+u)}{\Theta(u)}. \tag{7.11}$$

For $-\frac{g}{2} \leq n \leq \frac{g}{2} - 1$ we have

$$A_n = N_n^0 \cdot \frac{E(P,P_+)^{n-\frac{g}{2}}}{E(P,P_-)^{n+\frac{g}{2}+1}} \cdot E(P, P_{g+1}) \left(\sigma(P)\right)^{-1} \Theta(P + \hat{a}_n), \tag{7.12}$$

where $\hat{a}_n$ is a vector dependent on $P_\pm$ and an additional point P_{g+1} different from $P_\pm$.
For $n = \frac{g}{2}$ we have $a_{\frac{g}{2}} = 1$.

3. Let $\lambda = 1$. For $|n| > \frac{g}{2}$ we have

$$\omega^n(P) = N^1_{-n} \cdot \frac{E(P,P_-)^{n-\frac{g}{2}+1}}{E(P,P_+)^{n+\frac{g}{2}-1}} \sigma(P) \frac{\Theta(P+u)}{\Theta(u)}. \tag{7.13}$$

For $-\frac{g}{2} \leq n \leq \frac{g}{2} - 1$ we have

$$\omega^n(P) = N^1_{-n} \cdot \frac{E(P,P_-)^{n+\frac{g}{2}} \cdot \sigma(P)}{E(P,P_+)^{n-\frac{g}{2}+1} E(P,P_{\frac{g}{2}})} \cdot \Theta(P+e_n), \tag{7.14}$$

where e_n depends on $P_\pm$ and an additional point P_{g+1} different from $P_\pm$. For $n = \frac{g}{2}$ we have

$$\omega^{\frac{g}{2}}(P) = \rho(P) = d\log \frac{E(P,P_+)}{E(P,P_-)} - i \sum_{i,j=1}^{g} v_i(z)\, (Im\Omega)^{-1}_{ij} \left(Re(\int_{\beta_j} \omega_\pm) \right) \tag{7.15}$$

The N^λ_n are some constants such that the condition

$$\oint_{C_\tau} f_{n,\lambda}(z) f^m_\lambda(z) = \delta_{n,m}$$

is satisfied (for all $\lambda \in \mathbb{Z}$.)
For the case $g = 1$ we have similar expressions.
Sketch of the proof. We only consider the first item of the theorem. The first term of the right hand side is a normalization, the next term gives the multiplicities of the poles and zeros, the third term is the differential, and the theta function corrects the multivaluedness of the other differentials.
First we can count the weights of the several terms in order to obtain the weight of f. This is an easy exercise:

$$-\frac{1}{2}(n - s_\lambda) + \frac{1}{2}(n + s_\lambda) + g(2\lambda - 1) = (1 - 2\lambda)g + \lambda + g(2\lambda - 1) = \lambda.$$

We have to furthermore to show:

$$f^\lambda_n(P + n^t\alpha + m^t\beta) = f^\lambda_n(P),$$

where $n, m \in \mathbb{Z}^g$ and α, β is the homology basis with respect to the Riemann surface X.
In order to prove this equation we consider the factors of automorphy of the several terms in equation (7.10) and multiply the factors (keeping in mind the definition of s_λ).

$$e^{-\pi i m^t \Omega m (n - s_\lambda)} \cdot e^{\pi i m^t \Omega m (n + s_\lambda)} \cdot e^{\pi i (g-1) m^t \Omega m (2\lambda - 1)} \cdot e^{-\pi i m^t \Omega m} = 1$$

and

$$\begin{aligned} & e^{2\pi i m^t (J(P_+)-J(P))(n-s_\lambda)} \cdot e^{-2\pi i m^t (J(P_-)-J(P))(n+s_\lambda)} \\ & \qquad \cdot e^{-2\pi i m^t (\Delta-(g-1)J(P))(2\lambda-1)} \cdot e^{2\pi i m^t (J(P)+u)} = \\ & \qquad\qquad = e^{(n-s_\lambda)J(P_+)-(n-s_\lambda)J(P_-)-\Delta(2\lambda-1)-u} \end{aligned} \tag{7.16}$$

The exponential has to be 1. This shows the assertion.
□

7.2.6 Remark (Local Behavior) *The elements above have the following local behavior for $\lambda \in \mathbb{Z} \setminus \{0,1\}$:*

$$f_{n,\lambda}(z_\pm) = \alpha_n^{\lambda,\pm} z_\pm^{\pm n - s_\lambda} \left(1 + O(z_\pm)\right) (dz_\pm)^\lambda,$$

where $z_\pm(P_\pm)$ are local coordinates at $P_\pm$.
For $\lambda = 0$ we have for $n \geq \frac{g}{2} + 1$

$$A_n(z_\pm) = \alpha_n^{0,\pm} z_\pm^{\pm n - \frac{g}{2}} \left(1 + O(z_\pm)\right),$$

and for $n = \frac{g}{2}$ we put $A_{\frac{g}{2}} = 1$.
For $-\frac{g}{2} \leq n < \frac{g}{2}$ we have

$$A_n(z_\pm) = \alpha_n^{0,\pm} z_\pm^{\pm n - \frac{g}{2} + 0/-1} \left(1 + O(z_\pm)\right).$$

For $\lambda = 0$ we have for $n \geq \frac{g}{2} + 1$

$$\omega^n(z_\pm) = \alpha_n^{1,\pm} z_\pm^{\pm n + \frac{g}{2} - 1} \left(1 + O(z_\pm)\right),$$

and for $-\frac{g}{2} \leq n < \frac{g}{2}$ we have

$$\omega^n(z_\pm) = \alpha_n^{1,\pm} z_\pm^{\pm n + \frac{g}{2} - 1/+0} \left(1 + O(z_\pm)\right).$$

7.2.3 Multipoint-Case

Denote by $f_\lambda(n_1, ..., n_N)(P)$ the (up to a scalar multiple) unique element in the space $H^0(X, \lambda K - \sum_{i=1}^N n_i P_i)$ with $\sum_{i=1}^N n_i = (2\lambda - 1)g - 2\lambda$ (see theorem 7.1.8) For more than two points we get the following generalization which is derived in [Schl].

7.2.7 Theorem *Let be $g \geq 2$, and $\lambda \neq 0, 1$. Let be given N integers $n_1, ..., n_N \in \mathbb{Z}$ such that*

$$\sum_{i=1}^N n_i = (2\lambda - 1)(g - 1) - 1,$$

then there is a constant $C \in \mathbb{C}^*$ so that

$$f_\lambda(n_1, ..., n_N)(P) = C \cdot \prod_{i=1}^{N} E(P, P_i)^{n_i} \cdot \sigma(P)^{(2\lambda-1)} \cdot$$
$$\cdot \theta(J(P) + \sum_{i=1}^{N} n_i J(P_i) - (2\lambda - 1)\Delta). \quad (7.17)$$

For the remaining cases $g \geq 2$, $\lambda = 0$ or $\lambda = 1$ we have suitable expressions as well. This can be found in [Schl].

7.2.3.1 The $g = 0$ Case

For the case of the Riemann sphere ($g = 0$) it is very easy to give the multipoint version of KN-forms and the generalized 2-point version (generalized in the sense, that the usual 2-point forms are $z^{n-\lambda}(dz)^\lambda$).

7.2.8 Theorem

$$f_{\lambda,n}(z)_{(p_1,..p_N)} = C \prod_{i=1}^{N} (z - p_i)^{(n_i - s^i_\lambda)} (dz)^\lambda$$

This is an expression in the quasi-global coordinate z.

We have to check that in ∞ there is no pole.
For this purpose we consider the local coordinate around ∞: $w = \frac{1}{z}$. We get for the differential: $dz = -\frac{1}{w^2}dw$, therefore $(dz)^\lambda = (-1)^\lambda w^{-2\lambda}(dw)^\lambda$. In this coordinate we have:

$$\begin{aligned}
f_{\lambda,n}(w)_{(p_1,..p_N)} &= C \prod_{i=1}^{N} (\frac{1}{w} - p_i)^{(n_i - s^i_\lambda)} (-1)^\lambda w^{-2\lambda} (dw)^\lambda \\
&= C \prod_{i=1}^{N} (\frac{1}{w})^{(n_i - s^i_\lambda)} (1 - p_i w)^{(n_i - s^i_\lambda)} (-1)^\lambda w^{-2\lambda} (dw)^\lambda \\
&= C w^{-2\lambda} \prod_{i=1}^{N} (\frac{1}{w})^{(n_i - s^i_\lambda)} \prod_{i=1}^{N} (1 - p_i w)^{(n_i - s^i_\lambda)} (-1)^\lambda (dw)^\lambda \\
&= (-1)^\lambda \prod_{i=1}^{N} (1 - p_i w)^{(n_i - s^i_\lambda)} (dw)^\lambda
\end{aligned}$$

In the last line we used the condition: $\sum_i n_i - s^i_\lambda = -2\lambda$.
Altogether: $ord_\infty(f_\lambda(z)) = 0$.

7.2.4 The Bilinear Form γ_{nm}

For later purposes we introduce the following pairing between two forms of weight zero.

7.2.9 Definition *Define for all $n, m \in \mathbb{Z}$ or $\in \mathbb{Z} + \frac{1}{2}$ (dependent on the genus):*

$$\gamma_{nm} := \frac{1}{2\pi i} \oint_C A_n(P) dA_m(P). \tag{7.18}$$

We have the obvious fact (due to the product rule of derivation):

$$\gamma_{nm} + \gamma_{mn} = 0$$

Thus γ_{nm} is antisymmetric.

7.2.10 Proposition

$$\begin{aligned} \gamma_{nm} \neq 0 \quad &\Rightarrow \quad |n+m| \leq g, |n|, |m| > \frac{g}{2} \\ \gamma_{nm} \neq 0 \quad &\Rightarrow \quad |n+m| \leq g+1, |n| or |m| \leq \frac{g}{2}. \end{aligned}$$

Proof. This is already discussed in [KN2].
We consider the local behavior around the (possible) poles of $P_{\pm}$.
γ_{nm} can only be non-zero if one of the two following inequalities is satisfied:

$$\begin{aligned} n - \frac{g}{2} + m - \frac{g}{2} - 1 \quad &< \quad 0 \\ -n - \frac{g}{2} - m - \frac{g}{2} - 1 \quad &< \quad 0 \end{aligned}$$

This is the generic case. For the other cases we have to modify the second inequality in the right way and obtain the assertion. □
Note that for $n = \frac{g}{2}$ we obtain $\gamma_{\frac{g}{2},m} = 0$.

7.3 Szegö Kernels of Certain Bundles

7.3.1 Certain Bundles for Two and More Points

In this section we define the bundles which are relevant in order to define the associated Szegö kernel. This bundle has degree $g-1$ and it has no non-trivial sections.
We use again the the number s_λ:

$$s_\lambda = \frac{(1-2\lambda)g}{2} + \lambda$$

and we have $s_\lambda + s_{1-\lambda} = 1$ and $s_\lambda - s_{1-\lambda} = (1-2\lambda)(g-1)$.

7.3.1 Definition *Let $\mathcal{L}_{\lambda,N}$ denote the line bundle*

$$\mathcal{L}_{\lambda,N} := L_{P_+}^{\otimes(-N-s_{1-\lambda})} \otimes L_{P_-}^{\otimes(N+s_\lambda)} \otimes K^{\otimes\lambda}.$$

The degree of this line bundle is
$\deg \mathcal{L}_{\lambda,N} = N - s_{1-\lambda} - N + s_\lambda + (2g-2)\lambda = (1-2\lambda)(g-1) + (2g-2)\lambda = g-1$.
For generic points $P_\pm$ we have

$$\dim H^0(X, \mathcal{L}_{\lambda,N}) = 0.$$

In the next section it will turn out what generic means in this situation.
We can generalize this bundle for more than two points.
Let be $\vec{n} \in \mathbb{Z}^N$ defined by $\vec{n} = (n_1, n_2, .., n_N)$ where $\sum_{i=1}^N n_i = (1-2\lambda)(g-1)$.
Denote by $\mathcal{L}_{\lambda,n_1,...,n_N} = \mathcal{L}_{\lambda,\vec{n}}$ the bundle

$$\mathcal{L}_{\lambda,n_1,...,n_N} = L_{P_1}^{\otimes n_1} \otimes L_{P_2}^{\otimes n_2} \otimes ... \otimes L_{P_N}^{\otimes n_N} \otimes K^\lambda$$

Again the degree is $\deg \mathcal{L}_{\lambda,\vec{n}} = g-1$. And again for generic points $P_1, ..., P_N$ we have

$$\dim H^0(X, \mathcal{L}_{\lambda,\vec{n}}) = 0.$$

7.3.2 Explicit Presentation of Szegö Kernels

In this section we give formulas for the Szegö Kernel in terms of theta functions. More precisely we give the formulas in terms of the prime form, the sigma differential and theta functions.

7.3.2.1 Szegö Kernel - Two Points

Starting from the bundle of degree $g-1$ above we know from chapter 6 that there exists a Szegö kernel, i.e. a non-trivial section $s_\mathcal{L}(z,w) \in H^0(X \times X, \mathcal{L} \boxtimes \mathcal{L}^\vee(\Delta))$. In the proof of the following theorem we will use the periodicity properties of the prime from, the sigma differential and the theta function, respectively.
If we replace P by P' in the following way

$$\begin{aligned} P \mapsto P' &= P + \sum_{i=1}^g n_i \alpha_i + \sum_{i=1}^g m_i \beta_i \\ &= P + n^t \alpha + m^t \beta \end{aligned}$$

then we get the quasi-periodicities $((\alpha_i, \beta_j)_{i,j}$ form a homology basis of X):

$$\begin{aligned} E(P',Q) &= \chi \cdot e^{-\pi i m^t \Omega m + 2\pi i m^t (J(Q)-J(P))} E(P,Q) \\ E(P,Q') &= \chi \cdot e^{-\pi i m^t \Omega m - 2\pi i m^t (J(Q)-J(P))} E(P,Q) \\ \sigma(P') &= \chi^g \cdot e^{\pi i (g-1) m^t \Omega m - 2\pi i m^t (\Delta - (g-1)J(P))} \sigma(P) \\ \theta(P') &= \chi \cdot e^{-\pi i m^t \Omega m - 2\pi i m^t J(P)} \theta(P), \end{aligned} \tag{7.19}$$

χ is the sign dependent on the characteristic as it was discussed in chapter 6.

7.3.2 Theorem *A section in the bundle $\mathcal{L}_{\lambda,N} \boxtimes L_{1-\lambda,N}^\vee(\Delta)$ is given by*

$$S_{\lambda,N}(P,Q) =$$

$$= \frac{1}{E(P,Q)} \frac{E(P,P_+)^{N+s_{1-\lambda}}}{E(P,P_-)^{N+s_\lambda}} \frac{E(P_-,Q)^{N+s_\lambda}}{E(P_+,Q)^{N+s_{1-\lambda}}} \left(\frac{\sigma(P)}{\sigma(Q)}\right)^{2\lambda-1} \frac{\Theta(J(P)-J(Q)+u)}{\Theta(u)}. \tag{7.20}$$

where

$$u = J(P_+)(N+s_{1-\lambda}) - J(P_-)(N+s_\lambda) - \Delta(2\lambda-1).$$

and $\Theta(u) \neq 0$ *for generic* $P_\pm$.

Proof.
The last condition is satisfied because of the property of the theta divisor.
We are going to use the following identity several times: $s_\lambda - s_{1-\lambda} = (1-2\lambda)(g-1)$.
We have to show, (i) $S_{N,\lambda}(P,Q)$ is a $(\lambda, 1-\lambda)$ differential, and (ii) $S_{N,\lambda}(P,Q)$ is a proper section in a bundle, i.e. it is invariant under periods.
(i) Counting weights
with respect to P:

$$\overbrace{\frac{1}{2}}^{\frac{1}{E(P,Q)}} - \overbrace{(N+s_{1-\lambda})\frac{1}{2}}^{E(P,P_+)^{N+s_{1-\lambda}}} + \overbrace{(N+s_\lambda)\frac{1}{2}}^{\frac{1}{E(P,P_-)^{N+s_\lambda}}} + \overbrace{\frac{g}{2}(2\lambda-1)}^{\sigma(P)^{(2\lambda-1)}} + \overbrace{0}^{\theta} =$$

$$= \frac{1}{2} + \frac{1}{2}(s_\lambda - s_{1-\lambda}) + \frac{g}{2}(2\lambda-1) = \frac{1}{2} + \frac{(1-2\lambda)(g-1)}{2} + \frac{g}{2}(2\lambda-1) = \lambda.$$

with respect to Q we are doing the corresponding considerations and obtain:

$$\frac{1}{2} - (N+s_\lambda)\frac{1}{2} + (N+s_{1-\lambda})\frac{1}{2} + \frac{g}{2}(1-2\lambda) = 1-\lambda.$$

(ii) Periods: Recall the periods of the prime form, the sigma differential and the theta function (7.19).
From these periods we can follow for $S_{\lambda,N}(P,Q)$ with respect to P (note that $\chi^2 = 1$):

$$S_{\lambda,N}(P',Q) = \chi \cdot e^{\pi i m^t \Omega m (p_1+p_2+p_3+p_4+p_5)} e^{2\pi i m^t (q_1+q_2+q_3+q_4+q_5)} S_{\lambda,N}(P,Q)$$

where

$$\begin{array}{rcc|rcc} p_1 & = & 1 & p_2 & = & (-1)(N+s_{1-\lambda}) \\ p_3 & = & (+1)(N+s_\lambda) & p_4 & = & (+1)(g-1)(2\lambda-1) \\ p_5 & = & -1 & & & \end{array}$$

And it follows immediately for the addition of our p_i:

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0$$

For the q_i we obtain the following result:

$$\begin{array}{rcc|rcc} q_1 & = & J(P)-J(Q) & q_2 & = & (J(P_+)-J(P))(N+s_{1-\lambda}) \\ q_3 & = & (-1)(J(P_-)-J(P))(N+s_\lambda)) & q_4 & = & (-1)(\Delta-(g-1)J(P))(2\lambda-1) \\ q_5 & = & (-1)(J(P)-J(Q)+u) & & & \end{array}$$

Summation of the q_i gives

$$\overbrace{J(P)-J(Q)}^{q_1}+\overbrace{(J(P_+)-J(P))(N+s_{1-\lambda})}^{q_2}+\overbrace{(-1)(J(P_-)-J(P))(N+s_\lambda))}^{q_3}+$$

$$+\underbrace{(-1)(\Delta-(g-1)J(P))(2\lambda-1)}_{q_4}+\underbrace{(-1)(J(P)-J(Q)+u)}_{q_5}=$$

$$=J(P_+)(N+s_{1-\lambda})-J(P_-)(N+s_\lambda)-\Delta(2\lambda-1)-u.$$

The sum is supposed to be zero. So if we choose u as in the assertion of the theorem we get indeed the desired result.
Now we turn to Q:

$$S_{\lambda,N}(P,Q')=\chi\cdot e^{\pi i m^t\Omega m(f_1+f_2+f_3+f_4+f_5)}e^{2\pi i m^t(h_1+h_2+h_3+h_4+h_5)}S_{\lambda,N}(P,Q)$$

where

$$\begin{array}{rcc|rcc} f_1 & = & 1 & f_2 & = & (-1)(N+s_{1-\lambda}) \\ f_3 & = & (+1)(N+s_\lambda) & f_4 & = & (+1)(g-1)(2\lambda-1) \\ f_5 & = & -1 & & & \end{array}$$

and we get immediately

$$f_1+f_2+f_3+f_4+f_5=0$$

For the h_i we obtain the following result:

$$\begin{array}{rcc|rcc} h_1 & = & J(Q)-J(P) & h_2 & = & (J(Q)-J(P_+))(N+s_{1-\lambda}) \\ h_3 & = & (-1)(J(Q)-J(P_-))(N+s_\lambda)) & h_4 & = & (\Delta-(g-1)J(Q))(2\lambda-1) \\ h_5 & = & J(P)-J(Q)+u & & & \end{array}$$

Summation of the h_i gives

$$\overbrace{J(Q)-J(P)}^{h_1}+\overbrace{(J(Q)-J(P_+))(N+s_{1-\lambda})}^{h_2}+\overbrace{(-1)(J(Q)-J(P_-))(N+s_\lambda))}^{h_3}+$$

$$+\underbrace{(\Delta-(g-1)J(P))(2\lambda-1)}_{h_4}+\underbrace{(J(P)-J(Q)+u)}_{h_5}=$$

$$=-J(P_+)(N+s_{1-\lambda})+J(P_-)(N+s_\lambda)+\Delta(2\lambda-1)+u.$$

□

7.3.2.2 Szegö Kernel - More Points

We give now the explicit formula for the multipoint version. The proof of the theorem is similar to that of the preceding subsection.

7.3.3 Theorem *We have as a section in* $H^0(X \times X, \mathcal{L}_{\lambda,\vec{n}} \boxtimes K \otimes \mathcal{L}^{-1}_{\lambda,\vec{n}}(\Delta))$ *with* $\sum_{i=1}^N n_i = (1-2\lambda)(g-1)$*:*

$$S_{\lambda,\vec{n}} = \frac{1}{E(P,Q)} \prod_{i=1}^N E(P,P_i)^{n_i} E(P_i,Q)^{-n_i} \left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)} \frac{\Theta(J(P)-J(Q)+u)}{\Theta(u)}. \tag{7.21}$$

where

$$u = \sum_{i=1}^N n_i J(P_i) + (2\lambda - 1)\Delta. \tag{7.22}$$

and $\Theta(u) \neq 0$ *for generic* P_i.

Proof.
First we have to count the weights with respect to P and Q.
(i) P: (the minus sign in the second term comes from the fact that the prime form is a $(-\frac{1}{2}, -\frac{1}{2})$-form.)

$$\overbrace{\frac{1}{2}}^{\frac{1}{E(P,Q)}} - \overbrace{\frac{1}{2}\sum_{i=1}^N n_i}^{\prod_{i=1}^N E(P,P_i)^{n_i}} + \overbrace{\frac{g}{2}(2\lambda-1)}^{\sigma(P)^{(2\lambda-1)}} = \frac{1}{2} - \frac{(1-2\lambda)(g-1)}{2} + \frac{g}{2}(2\lambda-1) = \lambda.$$

Accordingly for Q (by taking in account that we have now minus powers of the prime form):

$$\frac{1}{2} + \frac{1}{2}\sum_{i=1}^N n_i - \frac{g}{2}(2\lambda-1) = \frac{1}{2} + \frac{(1-2\lambda)(g-1)}{2} - \frac{g}{2}(2\lambda-1) = 1-\lambda.$$

(ii) Now we consider the periods:
For P we get:

$$S_{\lambda,N}(P',Q) = \chi \cdot e^{\pi i m^t \Omega m (p_0 + \sum_{i=1}^N p_i + p_{N+1} + p_{N+2})} e^{2\pi i m^t (q_0 + \sum_{i=1}^N q_i + q_{N+1} + q_{N+2})} S_{\lambda,N}(P,Q)$$

where

$$\begin{array}{rcc|rcc} p_0 & = & 1 & p_i & = & (-1)n_i \quad (1 \le i \le N) \\ p_{N+1} & = & (+1)(g-1)(2\lambda-1) & p_{N+2} & = & -1 \end{array}$$

Thus we obtain:

$$p_0 + \sum_{i=1}^N p_i + p_{N+1} + p_{N+2} = 1 - (1-2\lambda)(g-1) + (g-1)(2\lambda-1) - 1 = 0.$$

We address now to the q's:

$$\begin{array}{rcc|rcc} q_0 & = & J(P) - J(Q) & q_i & = & (J(P_i) - J(P))n_i \quad (1 \le i \le N) \\ q_{N+1} & = & (-1)(\Delta - (g-1)J(P))(2\lambda-1) & q_{N+2} & = & (-1)(J(P) - J(Q) + u) \end{array}$$

Thus we obtain

$$q_i + \sum_{i=1}^{N} q_i + q_{N+1} + q_{N+2} = J(P) - J(Q) + \sum_{i=1}^{N} J(P_i)n_i + J(P)(2\lambda - 1)(g-1) -$$

$$-(\Delta - (g-1)J(P))(2\lambda - 1) - (J(P) - J(Q) + u) = \sum_{i=1}^{N} J(P_i)n_i - \Delta(2\lambda - 1) - u.$$

The last line has to be zero, so we obtain the assertion.
We apply the same procedure to the argument Q and get the similar result.
□

7.3.4 Remark *If $N = 2$ in the above theorem we obtain the theorem for two points as follows:*
Put $n_1 := N + s_{1-\lambda}$, and put $n_2 := -N - s_\lambda$. We get immediately the assertion of theorem 7.3.2.

7.4 Expansions of Szegö Kernels

Note that the Szegö kernels given in the last section are unique up to a scalar multiple.
In this section we give series expansions of the Szegö kernel. This is going to be relevant to introduce the normal ordered product for higher genus fields.

7.4.1 Theorem *Let X be a Riemann surface. Then a Szegö kernel can be expanded as follows*

$$\tau(P) < \tau(Q) : S_{\frac{g}{2}-1,0}(P,Q) = \sum_{n=\frac{g}{2}}^{\infty} A_n(P)\omega^n(Q) \tag{7.23}$$

$$\tau(P) > \tau(Q) : S_{\frac{g}{2}-1,0}(P,Q) = -\sum_{n=-\infty}^{\frac{g}{2}-1} A_n(P)\omega^n(Q). \tag{7.24}$$

Note that this Szegö kernel is now unique.
Proof. The convergence of the right hand side follows from the inequalities (yet to be explained):

$$|f_{\lambda,n}(P)| \left| f^n_{1-\lambda}(Q) \right| \leq C \cdot e^{n\tau(P)} \cdot e^{(-n)\tau(Q)}$$

where C is a suitable constant.
Thus the series $\left|\sum_{n=N+1}^{\infty} f_{\lambda,n}(P) f^n_{1-\lambda}(Q)\right|$ is bounded by a geometric series $C \cdot \sum e^{n(\tau(P)-\tau(Q))}$ and for $\tau(P) < \tau(Q)$ this series is convergent. The same consideration applies to the second line.
The asymptotic behavior of the elements on the right hand side of the equations coincides with the asymptotic behavior of the Szegö kernel.

It is necessary to explain the inequalities: Let $P \in X$, $\tau := \tau(P)$. Let $f_{\lambda,n}$ be a KN-form.

$$|f_{\lambda,n}(P)| = \left|\frac{E(P,P_+)}{E(P,P_-)}\right|^n \cdot \frac{1}{|E(P,P_+)|^{s_\lambda}|E(P,P_-)|^{s_\lambda}} \cdot |\sigma(P)|^{2\lambda-1} |\theta(J(P)-u)|.$$

Because $P \neq P_\pm$ we can find a constants C_1, C_2 such that $C_1 > |E(P,P_\pm)| > C_2$. The theta function is also bounded. The value $\tau(P)$ is given by

$$\tau(P) = Re\int_{P_0}^{P} d\log\frac{E(z,P_+)}{E(z,P_-)} - \left(i\sum_{i,j=1}^{g}\left(Re\int_{P_0}^{P} v_i(z)\right)(Im\Omega)^{-1}_{ij}\left(Re(\int_{\beta_j} d\log\frac{E(z,P_+)}{E(z,P_-)})\right)\right) =$$

$$= \log\left|\frac{\frac{E(P,P_+)}{E(P,P_-)}}{\frac{E(P_0,P_+)}{E(P_0,P_-)}}\right| + const.$$

Therefore

$$|f_{\lambda,n}(z)| \leq D \cdot \left|\frac{E(P,P_+)}{E(P,P_-)}\right|^n \leq Me^{n\tau(P)}, \quad D, M \in \mathbb{C}$$

□

Similar considerations lead to the following theorem.

7.4.2 Theorem *Let $\lambda \in \mathbb{Z} \setminus \{0,1\}$.*

$$\tau(P) < \tau(Q) : S_{N,\lambda}(P,Q) = \sum_{n=N+1}^{\infty} f_{\lambda,n}(P) f^n_{1-\lambda}(Q) \tag{7.25}$$

$$\tau(P) > \tau(Q) : S_{N,\lambda}(P,Q) = \sum_{n=-\infty}^{N} f_{\lambda,n}(P) f^n_{1-\lambda}(Q). \tag{7.26}$$

The following proposition can be found in a similar fashion in [KN3].

7.4.3 Proposition (Bidifferential $\omega(P,Q)$) *Let be $P, Q \in X$ two points with $\tau(P) < \tau(Q)$ and $P, Q \neq P_\pm$. The differential given by the expansion*

$$\omega_+(P,Q) = \sum_{n \geq \frac{g}{2}, m} \gamma_{nm}\omega^n(P)\omega^m(Q)$$

defines an even bidifferential with pole order two.

This means that we can expand $\omega_+(P,Q)$ in a suitable neighborhood of the diagonal by

$$\omega_+(z,w) = \frac{dzdw}{(z-w)^2} + ... + O(z-w).$$

Proof. From the definition of the numbers γ_{nm} we know: $\gamma_{nm} = 0$ for $|n+m| > g$. Therefore the right hand side is

$$\sum_{n \geq \frac{g}{2}, m < \frac{g}{2}} \gamma_{nm} \omega^n(P) \omega^m(Q)$$

The convergence follows from the lemma above.
By the definition of γ_{nm} we obtain (with $\tau(R) < \tau(P)$, $\tau(R) < \tau(Q)$):

$$\begin{aligned}
\sum_{n \geq \frac{g}{2}, m < \frac{g}{2}} \gamma_{nm} \omega^n(P) \omega^m(Q) = & \\
&= \sum_{n \geq \frac{g}{2}, m < \frac{g}{2}} \int A_n(R) dA_m(R) \omega^n(P) \omega^m(Q) = \\
&= \int \sum_{n \geq \frac{g}{2}, m < \frac{g}{2}} A_n(R) dA_m(R) \omega^n(P) \omega^m(Q) = \\
&\qquad\qquad = \int S_{\frac{g}{2}-1}(R, P) d_R S_{\frac{g}{2}-1}(R, Q)
\end{aligned}$$

The analytic property of the Szegö kernel provides the analytic property of $\gamma_+(P, Q)$. $\square$
For chapter 9 we need a special Szegö kernel. Therefore we introduce the following notation.

7.4.4 Definition *The Szegö kernel for* $N = -s_\lambda = s_{1-\lambda} - 1$ *is defined by*

$$S_\lambda(P, Q) := S_{-s_\lambda, 1-\lambda}(P, Q) \tag{7.27}$$

So the expansion is given in the according way.

Chapter 8

Algebras of KN-type and Representations

In this chapter we are going to define (in the sense of Krichever and Novikov) the basic objects of two-dimensional conformal field theory as global meromorphic objects on Riemann surfaces.
More precisely: Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra. The current algebra on higher genus is defined by $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ where $\mathcal{A}$ is the algebra of meromorphic functions on X holomorphic outside two generic points P_+ and P_-. This is the higher genus analogue of the current algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$.
Central extensions are given by certain cocycles.
In this chapter we introduce the Virasoro algebra of Krichever-Novikov type, the Heisenberg algebra and the affine algebras of KN-type.
We study the representations of the algebra $\mathcal{A}$ in more detail. To this end we use the regularization procedure of the algebra $\mathfrak{a}_\infty$ of infinite matrices with "finitely many diagonals" (as is explained e.g. in [KR]).
This chapter is organized as follows.
In the first section we introduce the notion of quasi-graded (Lie-)algebras and quasi-graded modules of algebras. We discuss the quasi-gradedness the algebras $\mathcal{A}$ (the algebra of meromorphic functions holomorphic outside two given points P_+, P_-), $\mathcal{L}$ (the Lie algebra of meromorphic vector fields holomorphic outside two given points P_+, P_-). We show that the spaces $\mathcal{F}^\lambda$ of meromorphic λ-forms holomorphic outside two given points P_+, P_- form quasi-graded modules of theses algebras.
In the second section we discuss central extensions of the algebras $\mathcal{A}, \mathcal{L}, \mathcal{D}^1$.
In the third section we introduce the algebra $\overline{\mathfrak{a}}_\infty$ in which the representation of the algebra $\mathcal{A}$ can be embedded.
In the fourth section we study the central extension of the algebra $\mathcal{A}$ more carefully.

8.1 Introduction

Let X be a compact Riemann surface of genus $g \geq 1$ and $P_\pm$ two distinguished points in general position. Denote by K the canonical line bundle on X.
Let $\lambda \in \mathbb{Z}$. Denote by $\mathcal{F}^\lambda$ the infinite dimensional vector space of global meromorphic sections of $K^\lambda := K^{\otimes\lambda}$ which are holomorphic outside $P_\pm$.
For $\lambda = 0$, i.e. for the space of meromorphic functions holomorphic outside $P_\pm$ we will write $\mathcal{A} := \mathcal{A}(X, P_\pm) := \mathcal{F}^0$. By multiplying sections in $\mathcal{F}^\lambda$ by functions we again obtain sections in $\mathcal{F}^\lambda$. Let be e.g. $f, g \in \mathcal{A}(X, P_\pm)$, then we obtain

$$f \cdot g \in \mathcal{A}(X, P_\pm).$$

In this way the space $\mathcal{A}$ becomes an associative algebra.
Let $f \in \mathcal{A}(X, P_\pm)$ and let $g \in \mathcal{F}^\lambda$, then

$$f \cdot g \in \mathcal{F}^\lambda,$$

and the $\mathcal{F}^\lambda$ become modules over $\mathcal{A}$.
The spaces $\mathcal{F}^\lambda$ have bases, these bases are called Krichever-Novikov bases.
With respect to these bases the space $\mathcal{A}(X, P_\pm)$ becomes a quasi-graded associative algebra. That means the graded-ness condition is a bit relaxed (see next section for a precise definition).
As well for the $\mathcal{A}(X, P_\pm)$-modules $\mathcal{F}^\lambda$. They become quasi graded modules.
The algebra $\mathcal{A}$ of meromorphic functions holomorphic outside two given points is a quasigraded algebra. The modules of this algebra turn out to be quasigraded as well. As we have already seen the Virasoro algebra is the central extension of the Witt algebra, the algebra of vector fields on the circle.
Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra.
Then

$$\bar{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$$

is called the **Krichever-Novikov current algebra**. The Lie bracket is given by the relations

$$[x \otimes A, y \otimes B] = [x, y] \otimes AB.$$

8.2 Quasigraded Algebras $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{D}^1$

8.2.1 Quasigraded Algebras

In this section we introduce the basic notions of quasigradings.

8.2.1 Definition (Quasigraded Algebras and Modules) *Let $Z = \mathbb{Z}$ or $Z = \mathbb{Z} + \frac{1}{2}$.*

1. *Let $\mathcal{L}$ be a Lie algebra or an associative algebra that admits a decomposition into a direct sum of finite dimensional subspaces $\mathcal{L}_n$:*

$$\mathcal{L} = \bigoplus_{n \in Z} \mathcal{L}_n, \dim L_n < \infty.$$

The algebra $\mathcal{L}$ is said to be **quasigraded** *if there exist constants R and S such that*

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-R}^{n+m+S} \mathcal{L}_h, \forall n,m \in Z$$

(here $\cdot$ means the Lie bracket, if $\mathcal{L}$ is a Lie algebra, or the multiplication, if $\mathcal{L}$ is an associative algebra).
The elements of the subspaces $\mathcal{L}_n$ are called homogeneous elements of degree n.

2. *Let $\mathcal{L}$ be a quasigraded Lie algebra or a quasigraded associative algebra, and let $\mathcal{M}$ be an $\mathcal{L}$-module that admits a decomposition into a direct sum of subspaces $\mathcal{M}_n$:*

$$\mathcal{M} = \bigoplus_{n \in Z} \mathcal{M}_n$$

The module $\mathcal{M}$ is said to be **quasigraded** *if $\dim \mathcal{M}_n < \infty$ and there exist constants R' and S' such that*

$$\mathcal{L}_n \cdot \mathcal{M}_m \subseteq \bigoplus_{h=n+m-R'}^{n+m+S'} \mathcal{M}_h, \forall n,m \in Z$$

The elements of the subspaces $\mathcal{M}_n$ are called homogeneous elements of degree n.

8.2.2 The Algebras $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{D}^1$

Convention: Let $\mathbb{Z}'$ denote either $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$. If X is a Riemann surface of odd genus g, then $\mathbb{Z}'$ is supposed to be $\mathbb{Z}$. If the genus is even, then $\mathbb{Z}' = \mathbb{Z}$.

8.2.2 Definition *Denote by $\mathcal{F}^\lambda$ the space of global meromorphic sections of weight λ which are holomorphic outside two given points $P_\pm$.*
Especially $\mathcal{A} := \mathcal{A}(X, P_\pm) := \mathcal{F}^0$, and
$\mathcal{L} := \mathcal{L}(X, P_\pm) := \mathcal{F}^{-1}$

We exhibit a special basis, the Krichever-Novikov basis, of the space $\mathcal{F}^\lambda$ for any $\lambda \in \mathbb{Z}$.
For every $n \in \mathbb{Z}'$ certain elements $f_{\lambda,n} \in \mathcal{F}^\lambda$ are exhibited. The $f_{\lambda,n}$ form a basis of a certain subspace $\mathcal{F}_n^\lambda$ and it is shown in chapter 8 that

$$\mathcal{F}^\lambda = \bigoplus_{n \in \mathbb{Z}'} \mathcal{F}_n^\lambda.$$

The subspace $\mathcal{F}_n^\lambda$ is called the **homogeneous subspace of degree n**.
We recall the Krichever-Novikov pairing between $\mathcal{F}^\lambda$ and $\mathcal{F}^{1-\lambda}$ given by

$$\begin{aligned} \mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} &\rightarrow \mathbb{C} \\ \langle f, g \rangle := \frac{1}{2\pi i} \int_C f \otimes g &= Res_{P_+}(f \otimes g) = -Res_{P_-}(f \otimes g) \end{aligned}$$

where C is an arbitrary non-singular level line. The second equality follows from the residue theorem. The integral does not depend on the level line chosen. We will call such a level line or any cycle cohomologous to such a level line a **separating cycle**.
We choose the basis elements such that the basis elements fulfill the duality relation with respect to the Krichever-Novikov pairing

$$\langle f_{\lambda,n}, f_{1-\lambda,m} \rangle = \frac{1}{2\pi i} \int_C f \cdot g = \delta_{n,-m}. \tag{8.1}$$

8.2.3 Lemma *The algebra $\mathcal{A}$ acts on the bases of $\mathcal{F}^\lambda$ as follows:*

$$A_n(P) f_\lambda^m(P) = \sum_k \beta_{nk}^{\lambda,m} f_\lambda^k(P) \tag{8.2}$$

where

$$\beta_{nk}^{\lambda,m} = \frac{1}{2\pi i} \int_C (A_n(P) f_\lambda^m(P) f_{1-\lambda,k}(P))$$

and $\beta_{nk}^{\lambda,m} \neq 0$ only for finitely many k. More precisely there exist constants c_1, c_2 such that

$$\beta_{nk}^{\lambda,m} \neq 0 \Rightarrow n - m - c_1 \leq k \leq n - m + c_2.$$

Proof. Due to the duality relation we get

$$\beta_{nk}^{\lambda,m} \delta_{k,j} = \langle \sum_k \beta_{nk}^{\lambda,m} f_\lambda^k(P), f_{j,1-\lambda}(P) \rangle =$$

$$= \langle A_n(P) f_\lambda^m(P), f_{j,1-\lambda}(P) \rangle = \frac{1}{2\pi i} \int_C (A_n(P) f_\lambda^m(P) f_{j,1-\lambda}(P)).$$

We first consider the asymptotic behavior of $A_n(P) f_\lambda^m(P) f_{k,1-\lambda}(P)$ around the points $P_\pm$:

$$P_+ : \quad n - \frac{g}{2} - m - s_\lambda + k - s_{1-\lambda}$$

$$P_- : \quad -n - \frac{g}{2} - x + m - s_\lambda - y - k - s_{1-\lambda} - z$$

The numbers x, y, z are zero in the "generic case", i.e. for $|n| > \frac{g}{2}$, $\lambda \neq 0, 1$. For the remaining cases we have slightly changes. The worst values the numbers x, y, z can take are ± 1.
□
We obtain the observation:

8.2.4 Corollary *The βs are related in the following way:*

$$\beta_{nk}^{\lambda,m} = \beta_{n,-m}^{1-\lambda,-k}. \tag{8.3}$$

For $\lambda = 0$ with $\alpha_{nk}^m := \beta_{nk}^{0,m}$ we have:

$$A_n(P)\omega^m(P) = \sum_k \alpha_{nk}^m \omega^k(P) \tag{8.4}$$

$$A_n(P)A_k(P) = \sum_m \alpha_{nk}^m A_m(P). \tag{8.5}$$

8.2.5 Definition (The Algebra $\mathcal{L}$) *The elements in $\mathcal{L}$ which are vector fields operate on the spaces $\mathcal{F}^\lambda$ by taking the Lie derivative. In local coordinates*

$$\nabla_e(g)| := \left(e(z)\frac{d}{dz}\right).\left(g(z)(dz)^\lambda\right) = \left(e(z)\frac{dg}{dz}(z) + \lambda g(z)\frac{de}{dz}(z)\right)(dz)^\lambda. \tag{8.6}$$

Here $e \in \mathcal{F}^{-1}$ and $g \in \mathcal{F}^\lambda$.

The space $\mathcal{L}$ becomes a Lie algebra with respect to the above equation, and the spaces $\mathcal{F}^\lambda$ become Lie modules [SchSh1].

8.2.6 Definition (The Algebra $\mathcal{D}^1$) *The differential algebra $\mathcal{D}^1$ is the semidirect product of $\mathcal{A}$ and $\mathcal{L}$, i.e. $\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{L}$ as a vector space, and the Lie structure is defined by*

$$[(g, e), (h, f)] = (e.h - f.g, [e, f]) \qquad e, f \in \mathcal{L}; g, h \in \mathcal{A}.$$

Now we address certain modules of these algebras. It turns out that the spaces $\mathcal{F}^\lambda$ are modules over the above algebras. We obtain the following result.

8.2.7 Theorem

The algebras $\mathcal{A}, \mathcal{L}$ are almost graded with respect to the above defined degree. The spaces $\mathcal{F}^\lambda$ are almost graded modules over them.

8.3 Central Extensions and Local Cocycles of $\mathcal{A}$

In this section we briefly recall the definition of central extensions of Lie algebras. Then we define the notion of local cocycles for quasi-graded Lie algebras. Roughly speaking local cocycles are the cocycles such that the central extended quasi-graded Lie algebra is again quasi-graded.
After introducing this notion we consider local cocycles for the abelian Lie algebra $\mathcal{A}$ and for the Lie algebra of vector fields $\mathcal{L}$. We quote the result due to Schlichenmaier, which asserts that we have essentially a unique central extension of these algebras.
We recall the definition for Lie algebra cocycles:

8.3.1 Definition (Cocycles) *Let $\mathfrak{g}$ be a Lie algebra. A* ***Lie algebra two-cocycle*** *γ of $\mathfrak{g}$ with values in $\mathbb{C}$ is an anti-symmetric bilinear form of $\mathfrak{g}$ (that means $\gamma(f, g) = -\gamma(g, f)$ $\forall f, g \in \mathfrak{g}$) obeying*

$$\gamma([f, g], h) + \gamma([g, h], f) + \gamma([h, f], g) = 0. \ \forall f, g, h \in \mathfrak{g}.$$

Two two-cocycles γ_1 and γ_2 of $\mathfrak{g}$ are called **equivalent**, if there exists a linear form $\phi : \mathfrak{g} \to \mathbb{C}$ such that

$$\gamma_1(f,g) - \gamma_2(f,g) = \phi([f,g]).$$

Central extensions of Lie algebras are given by two-cocycles.

8.3.2 Definition (Local Cocycle for Quasi-Graded Lie Algebras) *Let $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ be a quasi-graded Lie algebra. A cocycle γ for $\mathfrak{g}$ is called **local** with respect to the quasi-grading, if there exist $M_1, M_2 \in \mathbb{Z}$ with*

$$\gamma(\mathfrak{g}_n, \mathfrak{g}_m) \neq 0 \Rightarrow M_1 \leq n+m \leq M_2 \ \forall n,m \in \mathbb{Z}.$$

By defining for the central element K: $\deg(K) := 0$ the central extension $\hat{\mathfrak{g}}$ is almost graded if it is given by a local cocycle γ.
We are going to consider now two-cocycles of certain Lie algebras and we quote the result of Schlichenmaier [Schl3].
Let $\mathcal{A}_{P_\pm}$ be the algebra of meromorphic functions on a Riemann surface with poles only in the points $P_\pm$. This algebra is a commutative Lie algebra. Define now

$$\gamma_C(g,h) := \frac{1}{2\pi i}\int_C g dh,$$

where C is a separating cycle (i.e. it separates $P_\pm$).
γ_C is obviously a two-cocycle of the algebra $\mathcal{A}_{P_\pm}$. Furthermore it is local due to section 7 where we introduced γ_{nm}.
The following notions are due to Schlichenmaier [Schl3] Def. 3.3.

8.3.3 Definition 1. *A cocyle γ for $\mathcal{A}_{P_\pm}$ is **multiplicative** if it fulfills a cocycle condition for the associative algebra $\mathcal{A}_{P_\pm}$, i.e.*

$$\gamma(f\cdot g,h) + \gamma(g\cdot h,f) + \gamma(h\cdot f,g) = 0, \quad \forall f,g,h \in \mathcal{A}_{P_\pm}.$$

2. *A cocycle γ for $\mathcal{A}$ is $\mathcal{L}$-invariant if*

$$\gamma(e.f,g) = \gamma(e.g,f) \quad \forall e \in \mathcal{L}, \forall f,g \in \mathcal{A}.$$

8.3.4 Proposition *The cocycle $\gamma_C(g,h)$ defined above is multiplicative and $\mathcal{L}$-invariant.*

Proof. The multiplicativity follows from the product rule of derivations (Leibniz rule).
The $\mathcal{L}$-invariance follows from $e.dh = d(e.h)$ and $e.\omega = d(\omega\cdot e)$ for $\omega \in \mathcal{F}^1$. (see [Schl3]) □
The cocycle $\gamma_C(g,h)$ defined above is up to scalar multiples the only one which satisfies the conditions of multiplicativity and $\mathcal{L}$-invariance (see [Schl3] Theorem 4.3(a)):

8.3.5 Theorem (Schlichenmaier) *A cocyle γ for $\mathcal{A}_{P_\pm}$ which is either multiplicative or $\mathcal{L}$-invariant is local if and only if it is a multiple of the cocycle $\gamma_C(g,h)$.*

8.4 Representations

In this section we are going to construct projective representations of the function algebra $\mathcal{A}$.
Following the lines of the book of Kac and Raina [KR] we consider the algebra $\overline{\mathfrak{a}}_\infty$. The central extended algebra $\overline{\mathfrak{a}}$ goes back to Kac and Peterson, and Date, Jimbo, Kashiwara and Miwa [DJKM].
By the action of $\mathcal{A}$ on the spaces $\mathcal{F}^\lambda$ we can embed the algebra $\mathcal{A}$ in the space $\overline{\overline{\mathfrak{a}}_\infty}$.

8.4.1 Definition (Projective Representation) *Let $\mathfrak{g}$ be an arbitrary complex Lie algebra, V a vector space, and $\pi : \mathfrak{g} \to End(V)$ a linear map. The space V is called a* ***projective*** *$\mathfrak{g}$-module if for all pairs $a, b \in \mathfrak{g}$ there exists $\gamma(a,b) \in \mathbb{C}$ such that*

$$\pi([a,b]) = [\pi(a), \pi(b)] + \gamma(a,b) \cdot id_V.$$

If V is a projective $\mathfrak{g}$-module then the above γ is necessarily a Lie algebra 2-cocycle.
With respect to our further considerations we introduce the notion of an admissible representation.

8.4.2 Definition (Admissible Representation) *Let $\mathfrak{g}$ be an almost graded Lie algebra. A projective $\mathfrak{g}$-module V is called* ***admissible*** *if*

$$\forall v \in V \exists n = n(v) \in \mathbb{N} \textit{ such that } \forall g \in \mathfrak{g} \textit{ with } \deg(g) \geq n : gv = 0.$$

8.4.1 The Matrix Algebra $\overline{\mathfrak{a}}_\infty$

We first consider the algebra of infinite matrices with finitely many non-zero diagonals. The definition is due to [KR]

8.4.3 Definition *Define by $\overline{\mathfrak{a}}_\infty$ the space of infinite matrices with finitely many non-zero diagonals:*

$$\overline{\mathfrak{a}}_\infty := \{(a_{ij})_{ij\in\mathbb{Z}} : a_{ij} = 0 \textit{ for } |i-j| >> 0\}.$$

Let $E_{i,j} \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ denote the matrix with entry 1 at the position (i,j), and zero otherwise.
Let $a_k(\mu)$ with $\mu \in \mathbb{C}^\mathbb{Z}$ be defined by

$$a_k(\mu) = \sum_{i\in\mathbb{Z}} \mu_i E_{i,i+k}, \qquad \mu_i \in \mathbb{C}. \tag{8.7}$$

Obviously $a_k(\mu) \in \overline{\mathfrak{a}}_\infty$. The space $\overline{\mathfrak{a}}_\infty$ is generated by the $a_k(\mu)$ $(k \in \mathbb{Z})$, i.e. a typical element $a \in \overline{\mathfrak{a}}_\infty$ is given by

$$a = \sum_{k=R}^{L} a_k(\mu^k). \tag{8.8}$$

We define the matrix product by:

$$(a \cdot b)_{ij} = (\sum_k a_{ik} b_{kj})_{ij}. \tag{8.9}$$

Define furthermore the Lie bracket by $[a, b] = a \cdot b - b \cdot a$. This means especially for the matrices E_{ij}, E_{kl}:

$$\begin{aligned} E_{ij} \cdot E_{kl} &= \delta_{jk} E_{il} && (8.10) \\ [E_{ij}, E_{kl}] &= \delta_{jk} E_{il} - \delta_{il} E_{kj}. && (8.11) \end{aligned}$$

From this we can see easily:
The product of two elements of $\overline{\mathfrak{a}}_\infty$ is in $\overline{\mathfrak{a}}_\infty$, $\overline{\mathfrak{a}}_\infty$ is an associative algebra, it becomes by the Lie bracket a Lie algebra.

8.4.4 Proposition *For two matrices $a_n(\mu)$ and $a_m(\nu)$ the Lie bracket is given by*

$$[a_n(\mu), a_m(\nu)] = a_{n+m}(\widetilde{\mu}) \tag{8.12}$$

where

$$\widetilde{\mu} = \det \begin{pmatrix} \mu_i & \mu_{i+m} \\ \nu_i & \nu_{i+n} \end{pmatrix}. \tag{8.13}$$

Proof. We use equation (8.11) and the linearity of the Lie bracket.

$$\begin{aligned} [a_n(\mu), a_m(\nu)] &= \sum_{i,j} \mu_i \nu_j [E_{i,i+n}, E_{j,j+m}] \\ &= \sum_{i,j} \mu_i \nu_j (\delta_{i+n,j} E_{i,j+m} - \delta_{j+m,i} E_{j,i+n}) \\ &= \sum_i \mu_i \nu_{i+n} E_{i,i+n+m} - \sum_j \mu_{j+m} \nu_j E_{j,j+n+m} \\ &= \sum_i (\mu_i \nu_{i+n} - \mu_{i+m} \nu_i) E_{i,i+n+m} \end{aligned}$$

This proves the assertion. □
An immediate consequence is given for matrices $a_n(\mu)$ $(n \in \mathbb{Z})$ where $\mu_i = 1$ $\forall i \in \mathbb{Z}$:

8.4.5 Corollary *If we define $a_n := a_n(\mu)$ where $\mu_i = 1$ for all $i \in \mathbb{Z}$, then the set of matrices $\{a_n : n \in \mathbb{Z}\}$ forms a commutative sub-algebra of $\overline{\mathfrak{a}}_\infty$, i.e.*

$$[a_n, a_m] = 0.$$

We introduce now the action of the matrices discussed above on the space $\mathbb{C}^\infty$. We follow [KR].
Let V be a complex vector space with $V = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} f_n$, where $\{f_n : n \in \mathbb{Z}\}$ is a fixed basis of V. We shall identify the vectors f_n with the column vector with

1 as the n-th entry and 0 elsewhere. Any vector in V has only a finite, but arbitrary, number of nonzero coordinates. This identifies V with $\mathbb{C}^{\mathbb{Z}}$, the space of such column vectors.
The following space is contained in the space $\overline{\mathfrak{a}}_\infty$.

8.4.6 Definition

$$gl_\infty = \{(a_{ij}) \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}} : \textit{only finitely many } a_{ij} \neq 0\}.$$

Let $a \in gl_\infty$. Then a acts on the space V discussed above in a natural way by multiplication $a \cdot v$ $(v \in V)$. We have especially for the matrices $E_{i,j} \in gl_\infty$:

$$E_{i,j} \cdot f_n = \delta_{j,n} f_i$$

The matrices $a \in \overline{\mathfrak{a}}_\infty$ act on the space as well.

8.4.7 Remark *Consider the matrices $a_k = a_k(\mu) \in \overline{\mathfrak{a}}_\infty$ with $\mu_i = 1$ for all $i \in \mathbb{Z}$. These matrices act on the basis vectors by*

$$a_k \cdot f_n = f_{n-k}. \tag{8.14}$$

This follows from

$$a_k \cdot f_n = \sum_i E_{i,i+k} \cdot f_n = \sum_i \delta_{i+k,n} f_i = f_{n-k}. \qquad \square$$

Kac and Raina call the matrices in the above remark "shift operators" (see [KR] eq. (4.7)).

8.4.2 The Embedding of $\mathcal{A}$ in $\overline{\mathfrak{a}}_\infty$

We are now considering the space of representations of the associative, commutative algebra $\mathcal{A}(X, P_\pm)$. The KN-forms form a quasi-graded module over this algebra:

$$A_n f_{\lambda,m} = \sum_k \beta_{nm}^{(\lambda),k} f_{\lambda,k}$$

where $\beta_{nm}^{(\lambda),k} = \frac{1}{2\pi i} \oint A_n f_{\lambda,m} f_{1-\lambda}^k$.

8.4.8 Proposition *Let $\lambda \in \mathbb{Z} \setminus \{0,1\}$. Then we get*

$$\beta_{\frac{g}{2},m}^k = \delta_{m,k}.$$

For $-\frac{g}{2} \leq n < \frac{g}{2}$ we have

$$\beta_{nm}^k \neq 0 \Rightarrow n + m - \frac{g}{2} \leq k \leq n + m + \frac{g}{2} + 1 \textit{ for all } \lambda \in \mathbb{Z}. \tag{8.15}$$

For $|n| > \frac{g}{2}$ we have

$$\beta_{nm}^k \neq 0 \Rightarrow n + m - \frac{g}{2} \leq k \leq n + m + \frac{g}{2} \textit{ for all } \lambda \in \mathbb{Z}. \tag{8.16}$$

Proof. The last identity is straightforward:

$$\beta_{nm}^{(\lambda),k} = \frac{1}{2\pi i}\oint A_n f_{\lambda,m} f_{1-\lambda}^k = \frac{1}{2\pi i}\oint A_n f_\lambda^{-m} f_{1-\lambda,-k} = \beta_{n,-k}^{(1-\lambda),-m}.$$

For the inequality we consider the orders of the KN-forms in the points $P_\pm$. Let be $|n| > \frac{g}{2}$. Then we have

$$P_+ : n - \frac{g}{2} + m - s_\lambda - k - s_{1-\lambda}$$

and

$$P_- : -n - \frac{g}{2} - m - s_\lambda + k - s_\lambda$$

And we obtain immediately the inequality.
□
Starting from this proposition we obtain the following observations:

8.4.9 Corollary *From the above proposition we obtain for the action of A_n on elements of $\mathcal{F}^\lambda$ $(\lambda \neq 0, 1)$:*

1. *For $n > \frac{g}{2}$ we get $A_n f_m \in span\{f_k : k > m\}$.*
2. *For $n < -\frac{g}{2}$ we get $A_n f_m \in span\{f_k : k < m\}$*
3. *For $n = \frac{g}{2}$ we have due to the definition of $A_{\frac{g}{2}} = 1$: $A_{\frac{g}{2}} f_m = f_m$.*

We denote the range $-\frac{g}{2} \leq n < \frac{g}{2}$ as the **critical strip**.

8.4.10 Proposition *The map $\psi : \mathcal{A} \to \overline{\mathfrak{a}}_\infty$ given by*

$$A_n \mapsto \psi(A_n) = (a_{ij}) = (\beta_{n,j}^i)$$

is an embedding of the algebra $\mathcal{A}$.

$$A_n \mapsto \sum_{k=-\frac{g}{2}}^{\frac{g}{2}+1} a_{-n-k}(\beta_n) \tag{8.17}$$

and

$$a_{-n-k}(\beta_n) = \sum_{i\in\mathbb{Z}} \beta_{n,i-n-k}^i E_{i,i-n-k}. \tag{8.18}$$

Proof. We know from the above proposition

$$A_n f_m = \sum_k \beta_{nm}^k f_k$$

$$a_{-n-k}(\beta_n) f_m = \sum_{i\in\mathbb{Z}} \beta_{n,i-n-k}^i E_{i,i-n-k} f_m =$$

$$= \beta_{n,m}^{m+n+k} f_{m+n+k}.$$

We used the fact $E_{i,j} f_l = \delta_{j,l} f_i$. □

8.4.3 The Half-Infinite Wedge-Space

8.4.11 Definition *The vector space of half infinite forms $\mathcal{H}_{\geq m}$ is defined defined by formal elements*

$$f_{i_m} \wedge f_{i_{m+1}} ... \wedge f_{i_{m+k}} \wedge ...$$

where $i_{m+k} = m + k$ for $k >> 0$.
The reference vector

$$f_m \wedge f_{m+1} \wedge f_{m+2} \wedge ...$$

*is called **vacuum vector of charge m**.*

8.4.12 Definition (Degree) *Let be $F \in \mathcal{H}_{\geq m}$. The number*

$$\deg F = \sum_{s=0}^{\infty} (s - m - i_{s-m}) =$$

$$= \sum_{s} (i_s < m \text{ which occur}) - \sum (i_s > m \text{ which do not occur}).$$

*is called **degree** of the vector F.*

The first sum in the definition is well defined. We have only finite sums due to the definition of these forms (because $i_{m+s} = m + s$ for $s >> 0$). The equality in the definition is clear.
We can represent the algebra gl_∞ in the space $H_{\geq m}$ as follows:

$$r : gl_\infty \to H_{\geq m}$$

$$r(A)\,(f_m \wedge f_{m+1} \wedge f_{m+2} \wedge f_{m+3} \wedge ...) =$$

$$= r(A) f_m \wedge f_{m+1} \wedge f_{m+2} \wedge f_{m+3} \wedge ... +$$

$$+ f_m \wedge r(A) f_{m+1} \wedge f_{m+2} \wedge f_{m+3} \wedge ... + ...$$

If want to represent the algebra $\overline{\mathfrak{a}}_\infty$ in the space $H_{\geq m}$ we obtain some divergences. For instance for the element $a_0 = \sum_i E_{i,i}$ we obtain:

$$a_0 \Phi_m = (1 + 1 + 1 + ...)\Phi_m.$$

In order to remove this problem we are going to apply the techniques contained in [KR]. We define a projective representation of the algebra $\overline{\mathfrak{a}}_\infty$.
We define the following:

$$\begin{array}{lll} \hat{r}_m(E_{ij}) = r(E_{ij}) & \text{if } i \neq j & \text{or } i = j < M \\ \hat{r}_m(E_{ii}) = r(E_{ii}) - id & \text{if } i \geq 0 & \end{array} \tag{8.19}$$

It can be seen immediately:

8.4.13 Lemma *The commutator relations are given by*

$$\begin{aligned} [\hat{r}_m(E_{ij}),\hat{r}_m(E_{kl})] &= 0 \qquad \text{for } j \neq k, l \neq i && (8.20)\\ [\hat{r}_m(E_{ij}),\hat{r}_m(E_{jl})] &= \hat{r}(E_{il}) \qquad \text{for } l \neq i && (8.21)\\ [\hat{r}_m(E_{ij}),\hat{r}_m(E_{ki})] &= -\hat{r}(E_{kj}) \qquad \text{for } j \neq k && (8.22) \end{aligned}$$

and

$$[\hat{r}_m(E_{ij}),\hat{r}(E_{ji})] = \hat{r}_m(E_{ii}) - \hat{r}_m(E_{jj}) + \begin{cases} I & \text{for} \quad i \leq 0, j > 0 \\ -I & \text{for} \quad i > 0, j \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8.23}$$

We can also write

$$[\hat{r}_m(E_{ij}),\hat{r}_m(E_{kl})] = \hat{r}_m([E_{ij},E_{ji}]) + \alpha(E_{ij},E_{kl}) \tag{8.24}$$

where

$$\alpha(E_{ij},E_{kl}) = \begin{cases} E & \text{for } j = k, i = l, i \leq 0, j > 0 \\ -E & \text{for } j = k, i = l, i > 0, j \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8.25}$$

Convention: From now on we drop the index m in $\hat{r}_m$.

8.4.14 Proposition

$$[\hat{r}(a_n(\mu)),\hat{r}(a_m(\nu))] = \hat{r}(a_{n+m}(\widetilde{\mu})) + \delta_{n,-m} \cdot \begin{cases} \displaystyle\sum_{i=1-n}^{0} \mu_i \nu_{i+n} E & \text{for } n > 0 \\ 0 & \text{for } n = 0 \\ \displaystyle -\sum_{i=1}^{-n} \mu_i \nu_{i+n} E & \text{for } n < 0 \end{cases} \tag{8.26}$$

where $\widetilde{\mu}_i = \det \begin{pmatrix} \mu_i & \mu_{i+m} \\ \nu_i & \nu_{i+n} \end{pmatrix}$.

Proof.

$$[\hat{r}(a_n(\mu)),\hat{r}(a_m(\nu))] = [\sum_i \mu_i \hat{r}(E_{i,i+n}), \sum_j \nu_j \hat{r}(E_{j,j+m})] =$$

$$= \sum_{i,j} \mu_i \nu_j [\hat{r}(E_{i,i+n}), \hat{r}(E_{j,j+m})] =$$

$$= \sum_{i,j} \mu_i \nu_j \left(\hat{r}([E_{i,i+n}, E_{j,j+m}]) + \alpha(E_{i,i+n}, E_{j,j+m})\right).$$

For the first term we use proposition 8.4.4.
We study the α-part more carefully: We know from equation (8.25):

$$\alpha(E_{i,i+n},E_{j,j+m}) = \begin{cases} E & \text{for } i+n = j, i = j+m, i \leq 0, i+n > 0 \\ -E & \text{for } i+n = j, i = j+m, i > 0, i+n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus we obtain

$$\alpha(E_{i,i+n}, E_{j,j+m}) = \delta_{n,-m} \cdot \begin{cases} E & \text{for } i+n=j, -n < i \leq 0 \\ -E & \text{for } i+n=j, -n \geq i > 0 \end{cases}$$

□

We give as an example the representation of a commutative algebra:

8.4.15 Corollary (see also [KR] formula (4.9)) *The representation $\hat{r}$ gives*

$$[\hat{r}(a_n), \hat{r}(a_m)] = n\delta_{n,-m}E.$$

First proof. We apply the above proposition. Because $\mu_i = \nu_i = 1$ for all $i \in \mathbb{Z}$ we obtain immediately the assertion.
Second proof. Let be $n \neq -m$.

$$[\hat{r}(a_n), \hat{r}(a_m)] = [\sum_i \hat{r}(E_{i,i+n}), \sum_j \hat{r}(E_{j,j+m})] = \sum_{ij} [\hat{r}(E_{i,i+n}), \hat{r}(E_{j,j+m})] =$$

$$= \sum_{ij} (\delta_{i+n,j}\hat{r}(E_{i,j+m}) - \delta_{j+m,i}\hat{r}(E_{j,i+n})) = \sum_i r(E_{i,i+n+m}) - \sum_j r(E_{j,j+n+m}) = 0.$$

Let be now $n = -m$:

$$\begin{aligned}
[\hat{r}(a_n), \hat{r}(a_{-n})] &= \sum_{i,j\in\mathbb{Z}} [\hat{r}(E_{i,i+n}), \hat{r}(E_{j,j-n})] \\
&= \sum_{ij} (\delta_{i+n,j}\hat{r}(E_{i,j-n}) - \delta_{j-n,i}\hat{r}(E_{j,i+n})) \\
&= \sum_j \hat{r}(E_{j-n,j-n}) - \sum_j \hat{r}(E_{j,j}) \\
&= \sum_{j\leq n} (r(E_{j,j}) - I) + \sum_{j>n} r(E_{j,j}) - \sum_{j\leq 0} (r(E_{j,j}) - I) - \sum_{j>0} r(E_{j,j}) \\
&= nI
\end{aligned}$$

In the last line we studied the case $n > 0$: $\sum_{j\leq n}(r(E_{j,j}) - I) - \sum_{j\leq 0}(r(E_{j,j}) - I) = \sum_{j=1}^{n}(r(E_{j,j}) - I)$. The $r(E_{ij})$-terms are canceled, therefore it remains nI. The analogous consideration gives the case $n \leq 0$. □

8.4.4 Representation of $\mathcal{A}$ in the Wedge Space

Denote by Φ_m the **vacuum vector** of $\mathcal{H}^{\lambda}_{\geq m}$, i.e.

$$f_{\lambda,m} \wedge f_{\lambda,m+1} \wedge f_{\lambda,m+2} \wedge \ldots$$

The functions A_n act on the space of half infinite wedge forms by

$$\begin{aligned}
A_n\left(f_m \wedge f_{m+1} \wedge f_{m+2} \wedge \ldots\right) &= (A_n f_m) \wedge f_{m+1} \wedge f_{m+2} \wedge \ldots \\
&+ f_m \wedge (A_n f_{m+1}) \wedge f_{m+2} \wedge \ldots \\
&+ f_m \wedge f_{m+1} \wedge (A_n f_{m+2}) \wedge \ldots
\end{aligned}$$

For $n > \frac{g}{2}$ we have

$$A_n \left(f_m \wedge f_{m+1} \wedge f_{m+2} \wedge ...\right) = 0$$

For $n < -\frac{g}{2}$ we have

$$A_n \left(f_m \wedge f_{m+1} \wedge f_{m+2} \wedge ...\right) = \text{finitely many terms}$$

It remains a range: $-\frac{g}{2} \leq n \leq \frac{g}{2}$ for which we can have infinitely many summands.
This problem is similar to the problem discussed in in the last subsection. We have by the embedding of the algebra $\mathcal{A}$ in the algebra $\overline{\mathfrak{a}}_\infty$:
The matrices $\hat{r}(A_n)$ can be represented by

$$\hat{r}(A_n) = \sum_{k=-\frac{g}{2}}^{\frac{g}{2}} a_{-n-k}(\beta_n).$$

$$a_k(\beta_n) = \sum_i \beta^i_{n,i-n-k} E_{i,i-n-k}.$$

Convention: We write $\hat{A}_n$ for $\hat{r}(A_n)$.
We can now state the cocycle relation in for the algebra $\mathcal{A}$.

8.4.16 Theorem *The commutation relations for $\hat{A}_n$ are given by*

$$[\hat{A}_n, \hat{A}_m] = \left(\sum_{k \leq 0, j > 0} \beta^{j+k}_{n,j} \beta^{j+k}_{m,j} - \sum_{k > 0, j \leq 0} \beta^{j+k}_{n,j} \beta^{j+k}_{m,j} \right) E. \tag{8.27}$$

where the sums are finite.

Proof.

$$[\hat{A}_n, \hat{A}_m] = [\sum_{k=-\frac{g}{2}}^{\frac{g}{2}} a_{-n-k}(\beta_n), \sum_{l=-\frac{g}{2}}^{\frac{g}{2}} a_{-m-l}(\beta_m)] =$$

$$= \sum_{k,l=-\frac{g}{2}}^{\frac{g}{2}} [a_{-n-k}(\beta_n), a_{-m-l}(\beta_m)]$$

We obtain

$$[a_{-n-k}(\beta_n), a_{-m-l}(\beta_m)] = \begin{cases} \displaystyle\sum_{i=1+n+k}^{0} \beta^i_{n,i-n-k} \beta^{i-k-n}_{m,i} E & \text{for } -n-k > 0 \\ 0 & \text{for } k = 0 \\ \displaystyle -\sum_{i=1}^{n+k} \beta^{i+k}_{n,i} \beta^{i-k}_{m,i} E & \text{for } -n-k < 0 \end{cases}$$

□

Chapter 9

Higher Genus Formal Calculus

In this chapter we address the formal calculus of higher genus Riemann surfaces. This chapter is supposed to be a "mirror" of the first and the second chapter of this thesis. The first notion in the first chapter were *formal distributions*. We extend this notions to higher genus Riemann surfaces. The monomials $z^{-n-\lambda}$ are replaced by Krichever-Novikov forms on a compact Riemann surface of genus g:

$$\sum_{n\in\mathbb{Z}} a_n z^{-n-\lambda} \rightsquigarrow \sum_{n\in\mathbb{Z}'} a_n f_\lambda^n \quad \text{where } \mathbb{Z}' = \mathbb{Z} + \begin{cases} \frac{1}{2} & g \text{ odd} \\ 0 & g \text{ even} \end{cases}$$

The crucial role in the first two chapters of this thesis was played by the *delta distribution*:

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1} w^n = i_{z,w}\frac{1}{z-w} - i_{w,z}\frac{1}{z-w}.$$

Krichever and Novikov [KN2] introduced an analogous object for the higher genus case:

$$\Delta(P,Q) = \sum_{n\in\mathbb{Z}'} A_n(P)\omega^n(Q) = i_{P,Q}S(P,Q) - i_{Q,P}S(P,Q),$$

where $S(P,Q)$ is the Szegö kernel given in chapter 7. And $i_{P,Q}S(P,Q)$ is the expansion of the Szegö kernel in the domain $\tau(P) > \tau(Q)$.
The next important notion in the first chapter was the *normal ordered product*. I introduce a normal ordered product for the higher genus case which is from my point of view a feasible generalization of the "classical" case. In the classical case we had the equalities

$$\begin{aligned} :a(w)b(w): &= a_+(w)b(w) + b(w)a_-(w) \\ &= Res_{z=0}\left(a(z)b(w)i_{z,w}\frac{1}{z-w} - b(w)a(z)i_{w,z}\frac{1}{z-w}\right). \end{aligned}$$

The higher genus normal ordered product is given by

$$\begin{aligned} :a(Q)b(Q): &= a_+(Q)b(Q) + b(Q)a_-(Q) \\ &= Res_{P=P_+}\left(a(P)b(Q)i_{P,Q}S(P,Q) - b(Q)a(P)i_{Q,P}S(P,Q)\right). \end{aligned}$$

We introduce a special Lie derivative ∇ which will play the role of the derivative of fields in the "classical" case.
The chapter is organized as follows:
Section 1 is devoted to the study of the higher genus version of formal distributions, fields and the delta distribution.
In the second section I introduce the notion of a normal ordered product of fields on higher genus Riemann surfaces. This definition is new, it was earlier mentioned in my preprint [L].
In section 3 the Lie derivatives are studied in more detail. I show that the special choice of the Lie derivative ∇ leads to analogous properties of the derived fields like in the "classical" case.
In the fourth section the notion of current algebras are introduced and I compute the OPE of certain field algebras. At the end I present the higher genus Wick product formula which is quite similar to the classical case.

9.1 Fields and Formal Delta Distributions

In the "classical" case we considered formal power series of endomorphisms with a formal variable z. We replace this formal variable now by the Krichever-Novikov forms in order to reflect the geometric properties of fields on a Riemann surface. As in the "classical" case we can introduce a formal delta distribution. However we obtain several formal delta distributions dependent on λ. We also study the derivative of one special case of the delta distributions.
From now on we fix X to be a compact Riemann surface of genus $g \geq 1$.

9.1.1 Formal Distributions and Fields

Recall the definition of a field in the "classical" case: Let V be a complex vector space. A field is a formal series in z and z^{-1}: $a(z) \in End_{\mathbb{C}}(V)[[z, z^{-1}]]$ with the condition: $a(z)v \in V[[z]][z^{-1}]$, i.e. $a(z)v$ is a formal Laurent series with values in V.
We replace the monomials z^m by KN-forms. More precisely we write instead of

$$a(z) = \sum_n a_n z^{-n-\lambda}$$

for the monomials $z^{-n-\lambda} \rightsquigarrow f_\lambda^n(P)$. Thus we obtain

$$a(P) = \sum_{n \in \mathbb{Z}'} a_n f_\lambda^n(P),$$

and we define for these series the suitable notions for a higher genus formal calculus.

Let $\mathcal{F}^\lambda$ denote the space of meromorphic λ-forms holomorphic outside two generic points P_+, P_-.
From now on we fix the Krichever-Novikov basis for any $\lambda \in \mathbb{Z}$: $\{f_\lambda^n\}_{n\in\mathbb{Z}'}$.
For $\lambda = 1$ we have especially $\{\omega^n\}_{n\in\mathbb{Z}'}$.
For $\lambda = 0$ we have especially $\{A_n\}_{n\in\mathbb{Z}'}$.

9.1.1 Definition (Formal Distribution) *Let X be a compact Riemann surface of genus $g \geq 1$. Let V be a vector space.*
*A **formal distribution** on a higher genus Riemann surface is a formal series*

$$a_{\lambda,\mu}(P,Q) = \sum_{n,m\in\mathbb{Z}'} a_{n,m} f_\lambda^n(P) f_\mu^m(Q)$$

where $a_{n,m} \in V$, $f_\lambda^n(P) \in \mathcal{F}_n^\lambda$, and $f_\mu^m(Q) \in \mathcal{F}_m^\mu$.
This means a formal distribution is an element of

$$V[[f_\lambda^n(P), f_\mu^m(Q)]]_{n,m\in\mathbb{Z}'}$$

*For one variable a formal distribution of **conformal weight** λ is defined by*

$$a(P) = \sum_{n\in\mathbb{Z}'} a_n f_\lambda^n(P) \in V[[f_\lambda^n(P)]].$$

As in the "classical" case the next step is the definition of a field. A field is a formal distribution with an additional property.

9.1.2 Definition (Field) *Let V be a vector space. Denote by $End(V)$ the endomorphism ring of the vector space V.*
*A **field of conformal weight** λ is an element of $End(V)[[f_\lambda^n]]]_{n\in\mathbb{Z}'}$ with the condition:*

$$\forall v \in V \exists n_0 \in \mathbb{Z}'_{>0} : a_n v = 0 \quad \textit{for all } n \geq n_0$$

That means we have for a field $a(P)$:

$$a(P)v \in V[f_\lambda^n]_{n\in\mathbb{Z}'_{>s_\lambda}}[[f_\lambda^n]]_{n\in\mathbb{Z}'_{\leq s_\lambda}}.$$

9.1.2 Formal Delta Distributions

In this section we introduce the higher genus analogue of the formal delta distributions.
In the "classical" case the delta distribution was defined by

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1} w^n.$$

This distribution was interpreted as the formal difference of the two expansions of the function $\frac{1}{z-w}$ (considered as a function in the variable w) in the domains $|w| < |z|$ and $|w| > |z|$. More precisely:

$$i_{z,w}\frac{1}{z-w} - i_{w,z}\frac{1}{z-w} = \delta(z-w).$$

We give now the higher genus analogue of the delta distribution.

9.1.3 Definition (Delta Distribution of Higher Genus) *The delta distribution* $\Delta_\lambda(P,Q) \in \mathbb{C}[[f_{\lambda,n}(P), f_{1-\lambda,m}(Q)]]_{n,m\in\mathbb{Z}'}$ *is defined by*

$$\Delta_\lambda(P,Q) = \sum_{n\in\mathbb{Z}'} f_{\lambda,n}(P) f^n_{1-\lambda}(Q). \tag{9.1}$$

We define especially for $\lambda = 0$*:*

$$\Delta_0(z,w) =: \Delta(P,Q) = \sum_{n\in\mathbb{Z}'} A_n(P)\omega^n(Q). \tag{9.2}$$

Recall now the expansion of the Szegö kernel in chapter 7. As in the "classical" case we interpret the delta distribution as the formal difference of the two expansions of the Szegö kernel in the two areas:

$$\Delta_\lambda(P,Q) = i_{Q,P}S_\lambda(P,Q) - i_{P,Q}S_\lambda(P,Q) \tag{9.3}$$

where $S_\lambda(P,Q)$ is the Szegö kernel with respect to λ, and $i_{Q,P}S_\lambda(P,Q)$ $(i_{Q,P}S_\lambda(P,Q))$ denotes the expansion in the area $\tau(P) < \tau(Q)$ $(\tau(P) > \tau(Q))$.
$S_\lambda(P,Q)$ was defined in Definition 7.4.4. $S_\lambda(P,Q)$ has the expansions:

$$\tau(P) < \tau(Q) : S_\lambda(P,Q) = \sum_{n\geq s_{1-\lambda}} f_{1-\lambda,n}(P) f^n_\lambda(Q) \tag{9.4}$$

$$\tau(P) > \tau(Q) : S_\lambda(P,Q) = \sum_{n< s_{1-\lambda}} f_{1-\lambda,n}(P) f^n_\lambda(Q). \tag{9.5}$$

9.1.4 Proposition (Properties of $\Delta_\lambda(z,w)$) *Let* f_λ *and* $g_{1-\lambda}$ *be two fields of conformal weight* λ *and* $1-\lambda$ *respectively, then we have the identity:*

$$\frac{1}{2\pi i}\oint_{C_\tau} g_{1-\lambda}(P)\Delta_\lambda(P,Q) = g_{1-\lambda}(Q),$$

furthermore

$$\frac{1}{2\pi i}\oint_{C_\tau} f_\lambda(Q)\Delta_\lambda(P,Q) = f_\lambda(P).$$

Proof. We use the duality:

$$\frac{1}{2\pi i}\oint_{C_\tau} f_{\lambda,n} f^m_{1-\lambda} = \delta_{n,m}$$

for the Krichever-Novikov basis.
The field $g_{1-\lambda}(P)$ has the series expansion:

$$g_{1-\lambda}(P) = \sum_m a_m f^m_{1-\lambda}(P).$$

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{C_\tau} g_{1-\lambda}(P)\Delta_\lambda(P,Q) &= \frac{1}{2\pi i}\oint_{C_\tau}\sum_m a_n f_\lambda^n(P)\Delta_\lambda(P,Q) \\
&= \frac{1}{2\pi i}\oint_{C_\tau}\sum_m a_m f_{1-\lambda}^m(P)\sum_n f_{\lambda,n}(P) f_{1-\lambda}^n(Q) \\
&= \frac{1}{2\pi i}\sum_{m,n} a_m \oint_{C_\tau} f_{1-\lambda}^m(P) f_{\lambda,n}(P) f_{1-\lambda}^n(Q) \\
&= \sum_{m,n} a_m \delta_{m,n} f_{1-\lambda}^n(Q) = \sum_n a_n f_{1-\lambda}^n(Q).
\end{aligned}
$$

Analogously it works for f_λ. □

9.1.3 Derivative of a Formal Delta Distribution

In the "classical" case we had the formal derivation of the delta distribution:

$$\partial_w \delta(z-w) = \sum_n n z^{-n-1} w^{n-1}$$

We are going to imitate this for the special case $\Delta_0(P,Q) = \Delta(P,Q)$. But first we study the derivative of the KN-forms of weight $\lambda = 0$:

$$dA_n = \sum_m \eta_{n,m}\omega^m \tag{9.6}$$

with

$$\eta_{n,m} = \frac{1}{2\pi i}\oint A_m dA_n. \tag{9.7}$$

This can be seen by applying the duality relation $\frac{1}{2\pi i}\oint A_m \omega^n = \delta_{n,m}$.
We have from the previous chapter the cocycle

$$\gamma_{nm} = \frac{1}{2\pi i}\oint A_n dA_m$$

Thus we obtain

$$\eta_{nm} = -\gamma_{nm} = \gamma_{mn}$$

and therefore

$$dA_n = \sum_m \gamma_{mn}\omega^m. \tag{9.8}$$

Now we consider the derivation of the delta distribution $\Delta(P,Q)$ with respect to P:

$$d_P\Delta(P,Q) = \sum_n d_P A_n(P)\omega^n(Q).$$

We obtain from the above considerations:

$$d_P\Delta(P,Q) = \sum_{n,m} \gamma_{mn}\omega^m(P)\omega^n(Q).$$

The derivation of the expansion of the Szegö kernel in the domain $\tau(P) < \tau(Q)$ gives

$$d_P i_{Q,P} S(P,Q) = \sum_{n \geq \frac{g}{2}, m} \gamma_{mn} \omega^m(P) \omega^n(Q)$$

In the same way we obtain for the domain $\tau(P) > \tau(Q)$:

$$d_P i_{P,Q} S(P,Q) = -\sum_{n < \frac{g}{2}, m} \gamma_{mn} \omega^m(P) \omega^n(Q)$$

Thus we obtain

$$d_P \Delta(P,Q) = d_P i_{Q,P} S(P,Q) - d_P i_{P,Q} S(P,Q). \tag{9.9}$$

9.2 Normal Ordered Products

In this section I define a normal ordered product that is supposed to be the suitable analogue of the "classical" case.

9.2.1 Higher Genus Normal Ordered Product

Let $a_\lambda(P)$ be a formal distribution of conformal weight λ with values in an associative algebra $\mathfrak{h}$, i.e. $a(P) \in \mathfrak{h}[[f^n_\lambda]]_{n \in \mathbb{Z}'}$
We split the formal distribution into two parts:

$$a_{\lambda,+}(P) := \sum_{n < s_{1-\lambda}} a_n f^n_\lambda \tag{9.10}$$

$$a_{\lambda,-}(P) := \sum_{n \geq s_{1-\lambda}} a_n f^n_\lambda \tag{9.11}$$

This is justified by the fact that

$$\mathrm{ord}_{P_+} f^n_\lambda = -n - s_\lambda$$

This implies: For $n \leq -s_\lambda$ we have $\mathrm{ord}_{P_+} f^n_\lambda \geq 0$. Because of $s_\lambda + s_{1-\lambda} = 1$ the condition $n \leq -s_\lambda$ is equivalent to $n < s_{1-\lambda}$.

9.2.1 Definition (Normal Ordered Product of Distributions) *Let $\mathfrak{h}$ be an associative algebra. Let $a_\lambda(P) \in \mathfrak{h}[[f^n_\lambda]]_n$ and $b_\mu(P) \in \mathfrak{h}[[f^n_\mu]]_n$ be two formal distributions of weight λ and μ respectively. The* ***normal ordered product*** *is defined as*

$$\boxed{: a_\lambda(P) b_\mu(Q) := a_{\lambda,+}(P) b_\mu(Q) + b_\mu(Q) a_{\lambda,-}(P)} \tag{9.12}$$

Recall the normal ordered product for formal distributions in the "classical" case:

Let be $a(z), b(z) \in \mathfrak{h}[[z, z^{-1}]]$ two formal distributions in a formal distribution algebra, i.e. the coefficients a_n and b_n corresponding to the series $a(z) = \sum_n a_n z^{-n-1}$ and $b(z) = \sum_n b_n z^{-n-1}$ are elements in an associative algebra $\mathfrak{h}$. The normal ordered product for two formal distributions was given by:

$$: a(z)b(w) := a(z)_+ b(w) + b(w)a_-(z) \tag{9.13}$$

That means that the powers of z in $a(z)_+$ are only positive, and the powers of z in $a(z)_-$ are only negative. More precisely:

$$: a(z)b(w) := \sum_{n<0, m\in\mathbb{Z}} a_n b_m z^{-n-1} w^{-m-1} + \sum_{n\geq 0, m\in\mathbb{Z}} b_m a_n z^{-n-1} w^{-m-1} \tag{9.14}$$

More generally: If we write $a(z) = \sum_n a_n z^{-n-\lambda}$ and $b(z) = \sum_n b_n z^{-n-\mu}$ for certain $\lambda, \mu \in \mathbb{Z}$, then it means

$$: a(z)b(w) := \sum_{n<\lambda-1, m\in\mathbb{Z}} a_n b_m z^{-n-\lambda} w^{-m-\mu} + \sum_{n\geq\lambda-1, m\in\mathbb{Z}} b_m a_n z^{-n-\lambda} w^{-m-\mu}. \tag{9.15}$$

Now we recall the definition of the normal ordered product for **fields**:
Let $a(z) = \sum_n a_n z^{-n-\lambda}$ and $b(z) = \sum_n b_n z^{-n-\mu}$ be two fields.
The normal ordered product for two fields was given by:

$$: a(w)b(w) := Res_z \left(a(z)b(w) i_{z,w} \frac{1}{z-w} - b(w)a(z) i_{w,z} \frac{1}{z-w} \right) \tag{9.16}$$

For the coefficients we obtained:

$$: a(w)b(w) :_n = \sum_{j<0} a_j b_{n-j-1} + \sum_{j\geq 0} b_{n-j-1} a_j \tag{9.17}$$

9.2.2 Definition (Normal Ordered Product of Fields) *Let $a_\lambda(P)$ and $b_\mu(P)$ be two fields. The normal ordered product is defined by*

$$\boxed{: a_\lambda(Q) b_\mu(Q) := a_{\lambda,+}(Q) b_\mu(Q) + b_\mu(Q) a_{\lambda,-}(Q).} \tag{9.18}$$

For our further considerations we study the product of two KN-forms more carefully. That means we study the products $f_\lambda^n \cdot f_{\mu,m}$ given by the pairing

$$\mathcal{F}^\lambda \times \mathcal{F}^\mu \to \mathcal{F}^{\lambda+\mu}, \quad (g, h) \mapsto g \otimes h.$$

That means: The pairing of a λ and μ form gives a $\lambda + \mu$-form and we can express this product as linear combination of the $\lambda + \mu$-KN-basis.
For this purpose we introduce the following expressions:

9.2.3 Definition (The coefficients $l^{km}_{(\lambda\mu)n}$)

$$l^{km}_{(\lambda\mu)n} \stackrel{def}{=} \frac{1}{2\pi i}\oint_C f^m_\mu(Q) f^k_\lambda(Q) f_{1-(\mu+\lambda),n}(Q)$$

where C is a separating cycle with respect to the points P_+, P_-.

9.2.4 Lemma *We have for the numbers $l^{km}_{(\lambda\mu)n}$:*

1. *Symmetry: $l^{km}_{(\lambda\mu)n} = l^{mk}_{(\mu\lambda)n}$*
2. *(Generic case) For $\lambda \neq 0,1$ and $\mu \neq 0,1$ or $\lambda \in \{0,1\}$ but $|k| > \frac{g}{2}$ ($\mu \neq 0,1$) or $\mu \in \{0,1\}$ but $|m| > \frac{g}{2}$ ($\lambda \neq 0,1$) we have*
$$l^{km}_{(\lambda\mu)n} \neq 0 \Rightarrow \left(k+m-\frac{g}{2} \leq n \leq k+m+\frac{g}{2}\right) \tag{9.19}$$
3. *For $\lambda = 0$:*
   (a) *$k = -\frac{g}{2}$: $l^{-\frac{g}{2},m}_{(0,\mu)n} = \delta_{m,n}$*
   (b) *$-\frac{g}{2} < k \leq \frac{g}{2}$, $\mu \neq 0,1$*
$$l^{km}_{(0\mu)n} \neq 0 \Rightarrow \left(k+m-\frac{g}{2}-1 \leq n \leq k+m+\frac{g}{2}\right) \tag{9.20}$$
   (c) *$\mu = 0$*
      i. *$-\frac{g}{2} < k \leq \frac{g}{2}$, and $|m| > \frac{g}{2}$:*
$$l^{km}_{(00)n} \neq 0 \Rightarrow \left(k+m-\frac{g}{2}-1 \leq n \leq k+m+\frac{g}{2}\right) \tag{9.21}$$
      ii. *$-\frac{g}{2} < k \leq \frac{g}{2}$, and $-\frac{g}{2} < m \leq \frac{g}{2}$:*
$$l^{km}_{(00)n} \neq 0 \Rightarrow \left(k+m-\frac{g}{2}-1 \leq n \leq k+m+\frac{g}{2}\right) \tag{9.22}$$
4. *For $\lambda = 1$:*
   (a) *$-\frac{g}{2} \leq |k| < \frac{g}{2}$,and $\mu \neq 0,1$:*
$$l^{km}_{(1\mu)n} \neq 0 \Rightarrow \left(k+m-\frac{g}{2}-1 \leq n \leq k+m+\frac{g}{2}\right) \tag{9.23}$$
   (b) *$\mu = 1$:*
      i. *$-\frac{g}{2} \leq |k| < \frac{g}{2}$, and $|m| > \frac{g}{2}$:*
$$l^{km}_{(11)n} \neq 0 \Rightarrow \left(k+m-\frac{g}{2}+1 \leq n \leq k+m+\frac{g}{2}\right) \tag{9.24}$$

ii. $-\frac{g}{2} \leq |k|, |m| < \frac{g}{2}$:

$$l^{km}_{(11)n} \neq 0 \Rightarrow \left(k + m - \frac{g}{2} + 2 \leq n \leq k + m + \frac{g}{2}\right) \tag{9.25}$$

iii. $k, m = \frac{g}{2}$:

$$l^{\frac{g}{2},\frac{g}{2}}_{(1,1)n} \neq 0 \Rightarrow \left(-\frac{3g}{2} \leq n \leq \frac{3g}{2}\right) \tag{9.26}$$

Proof.
The symmetry is obvious. We address to the proof of the remaining assertions. For the "generic" case, i.e. for the first assertion we obtain from the definition of s_λ:

$$s_\lambda + s_\mu + s_{1-(\lambda+\mu)} = \frac{g}{2} + 1.$$

We consider now the local behavior around P_+. The pole orders are given by

$$-m - s_\mu - k - s_\lambda + n - s_{1-(\mu+\lambda)} \tag{9.27}$$

and around P_- we have:

$$m - s_\mu + k - s_\lambda - n - s_{1-(\mu+\lambda)} \tag{9.28}$$

If these lines are supposed to be less than zero we obtain:

$$m + k - \frac{g}{2} - 1 < n < m + k + \frac{g}{2} + 1$$

For the remaining cases we compute in the same way. We only have to take into account the modified pole orders.
□

9.2.5 Proposition *For the coefficients of the normal ordered product there holds the identity:*

$$: a_\lambda(P) b_\mu(P) :_n = \sum_m \sum_{k < s_{1-\lambda}} a_k b_m l^{km}_{(\lambda\mu)n} + \sum_m \sum_{k \geq s_{1-\lambda}} b_m a_k l^{km}_{(\lambda\mu)n}$$

The sum over m in the equation of the proposition is finite (due to the lemma above).

Proof.

$$
\begin{aligned}
:a_\lambda(P)b_\mu(P): &= a_{\lambda,+}(P)b_\mu(P)+b_\mu(P)a_{\lambda,-}(P)\\
&= \left(\sum_{k<s_{1-\lambda}} a_n f_\lambda^k(P)\right)\left(\sum_m b_m f_\mu^m(P)\right)+\\
&+ \left(\sum_m b_m f_\mu^m(P)\right)\left(\sum_{k\geq s_{1-\lambda}} a_n f_\lambda^k(P)\right)\\
&= \sum_{k<s_{1-\lambda},m} a_n b_m \underbrace{f_\lambda^k(P) f_\mu^m(P)}+\sum_{k>s_{1-\lambda},m} b_m a_n \underbrace{f_\mu^m(P) f_\lambda^k(P)}\\
&= \sum_{k<s_{1-\lambda},m} a_n b_m \sum_n l_{(\lambda\mu)n}^{km} f_{\lambda+\mu}^n(P)+\\
&+ \sum_{k\geq s_{1-\lambda},m} b_m a_n \sum_n l_{(\lambda\mu)n}^{km} f_{\lambda+\mu}^n(P)
\end{aligned}
$$

Now we sort the coefficients according to the KN-basis $f_{\lambda+\mu}^n(P)$ and obtain

$$:a_\lambda(P)b_\mu(P):=$$

$$=\sum_n\left(\sum_{k<s_{1-\lambda},m} b_m a_n l_{(\lambda\mu)n}^{km}+\sum_{k\geq s_{1-\lambda},m} b_m a_n l_{(\lambda\mu)n}^{km}\right) f_{\lambda+\mu}^n(P).$$

This was the assertion of the proposition. □

9.2.6 Theorem *The normal ordered product can be expressed by*

$$
\begin{aligned}
:a_\lambda(Q)b_\mu(Q): &= \frac{1}{2\pi i}\int_{C_\tau} a_\lambda(P)b_\mu(Q) i_{Q,P} S_{1-\lambda}(P,Q)+\\
&+\frac{1}{2\pi i}\int_{C_\tau} b_\mu(Q)a_\lambda(P) i_{P,Q} S_{1-\lambda}(P,Q)
\end{aligned}
$$

Proof.
If we use the expansion of the Szegö kernel then we can write the above expression as a series as follows:

$$\frac{1}{2\pi i}\int_{C_\tau}\left(\sum_n a_n f_\lambda^n(P)\right)\left(\sum_m b_m f_\mu^m(Q)\right)\left(\sum_{k<s_{1-\lambda}} f_{1-\lambda,k}(P) f_\lambda^k(Q)\right)+$$

$$+\left(\sum_m b_m f_\mu^m(Q)\right)\left(\sum_n a_n f_\lambda^n(P)\right)\left(\sum_{k\geq s_{1-\lambda}} f_{1-\lambda,k}(P) f_\lambda^k(Q)\right)$$

We obtain:

$$\frac{1}{2\pi i}\int_{C_\tau}\sum_{n,m,k<s_{1-\lambda}} a_n b_m f_\lambda^n(P) f_\mu^m(Q) f_{1-\lambda,k}(P) f_\lambda^k(Q)+$$

$$+ \sum_{n,m,k \geq s_{1-\lambda}} b_m a_n f_\lambda^n(P) f_\mu^m(Q) f_{1-\lambda,k}(P) f_\lambda^k(Q)$$

We use the the duality:

$$\frac{1}{2\pi i} \oint f_\lambda^n(P) f_{1-\lambda,k}(P) = \delta_{n,k}.$$

Thus we obtain

$$\sum_{m,n<s_{1-\lambda}} a_n b_m f_\mu^m(Q) f_\lambda^n(Q) + \sum_{m,n \geq s_{1-\lambda}} b_m a_n f_\mu^m(Q) f_\lambda^n(Q)$$

This is exactly the normal ordered product (see the proof of proposition above).
□

9.2.2 Field Property and Iterated Normal Ordered Product

The normal ordered product of two fields is supposed to be a field again. This property of the normal ordered product is the assertion of the next theorem. Because of this property we have a multiplication on the space of fields. But this multiplication in general neither commutative nor associative.

9.2.7 Theorem (Field Property of the Normal Ordered Product) *Let be $a_\lambda(P) \in End(V)[[f_\lambda^n]]_{n \in \mathbb{Z}'}$ and $b(P) \in End(V)[[f_\mu^n]]_{n \in \mathbb{Z}'}$ two fields.*
Then the normal ordered product

$$: a_\lambda(P) b_\mu(P) :$$

is a field, i.e.
for all $v \in V$ $\exists n_0 \in \mathbb{Z}$: $: a(P)b(P) :_n v = 0$ for all $n \geq n_0$.

Proof. $a_\lambda(P)$ and $b_\mu(P)$ are fields, i.e. for all $v \in V$ there exists $n_1 \in \mathbb{Z}$ such that $a_n v = 0$ for all $n \geq n_1$, and
for all $v \in V$ there exists $n_2 \in \mathbb{Z}$ such that $b_n v = 0$ for all $n \geq n_2$.
From the above proposition we have for the coefficients of the normal ordered product:

$$: a_\lambda(Q) b_\mu(Q) :_n = \sum_{j<s_{1-\lambda},m} a_j b_m l_{(\lambda\mu)n}^{jm} + \sum_{j \geq s_{1-\lambda},m} b_m a_j l_{(\lambda\mu)n}^{jm}$$

We study the first term:

$$\sum_{j<s_{1-\lambda}} \sum_m a_j b_m l_{(\lambda\mu)n}^{jm} v$$

we know $l_{(\lambda\mu)n}^{jm} \neq 0 \Rightarrow n - j - \frac{g}{2} \leq m \leq n - j + \frac{g}{2}$.
Let be $m_0 \in \mathbb{Z}$ such that $b_{m_0} v = 0$. This means that for $n > m_0 - \frac{g}{2}$ we have

$\sum_{j<s_{1-\lambda}} \sum_m a_j b_m v l^{jm}_{(\lambda\mu)n} = 0$.
Now we consider the second term: Because of the field property of $a_\lambda(Q)$ we have the finite sum

$$\sum_{j\geq s_{1-\lambda},m}^{n_1-1} b_m a_j v l^{jm}_{(\lambda\mu)n}$$

We have to show that for sufficiently high n this sum vanishes.
Denote by $v_j := a_j v$. Let be N be the maximum of the ns such that $b_n v_j = 0$:

$$N := \max_{s_{1-\lambda}<j<n_1} \{n : b_n v_j = 0\}$$

Set now $\tilde{n} := N + j + \frac{g}{2}$ the we get:

$$\sum_{j\geq s_{1-\lambda},m}^{n_1-1} b_m a_j v l^{jm}_{(\lambda\mu)\tilde{n}} = 0$$

$\square$

9.2.8 Definition (Iterated Normal Ordered Product) *For N fields $a^1(P), a^2(P), ..., a^N(P)$ of conformal weight 1 the* ***iterated normal ordered product*** *is defined by*

$$: a^1(P)a^2(P)..a^N(P) :=: a^1(P)\left(: a^2(P)...a^N(P) :\right) : \tag{9.29}$$

From the above theorem we know that the iterated normal ordered product of fields is again a field.

9.2.9 Remark *As in the "classical" case (chapter 1) we can give an explicit expression of the normal ordered product.*

$$: a^1(P)a^2(P)..a^N(P) := \sum_{m=0}^{N} \sum_{\varphi\in J(m,n)} (\prod_{i=1}^{m} a_+^{\rho_\varphi(i)})(\prod_{i=m+1}^{n} a_-^{\rho_\varphi(i)}) \tag{9.30}$$

where $\rho_\varphi \in S_N$ is an element of the symmetric group S_N which I call an ***anti-shuffle****.*

9.3 Derivatives

Recall

The formal derivatives of formal distributions were given by

$$\partial_z a(z) = \partial_z \sum_n a_n z^{-n-1} = \sum_n (-n) a_{n-1} z^{-n-1}.$$

We recall some facts from formal calculus in the "classical" case.
We first recapitulate the computations for formal distributions:

$$a(z) = \sum_n a_n z^{-n-1}$$

We can write this sum as two separate expressions:

$$a(z) = a(z)_+ + a(z)_-$$

with

$$a(z)_+ = \sum_{n<0} a_n z^{-n-1} \qquad a(z)_- = \sum_{n\geq 0} a_n z^{-n-1} \tag{9.31}$$

We have

$$(\partial a_\pm(z)) = \partial(a_\pm(z)) \tag{9.32}$$

Furthermore we can state:

$$\partial_z^{(j)} a(z) = \sum_n (-1)^j \binom{n}{j} a_{n-j} z^{-n-1}$$

If we write

$$(-1)^j \binom{n}{j} a_{n-j} =: a_n^{(j)}$$

Then we can say because of the properties of the binomial coefficients:

$$a_n^{(j)} = 0 \text{ for } j > n \geq 0 \tag{9.33}$$

And in general we have

$$(\partial^{(j)} a_\pm(z)) = \partial^{(j)}(a_\pm(z)). \tag{9.34}$$

Higher Genus

Now we address to derivatives of higher genus.
The Lie derivation with respect to the vector fields $e_k \in \mathcal{F}^{-1}$ was given by

$$\nabla_{e_k} \omega^n = \sum_v \zeta_{k,v}^n \omega^v$$

where

$$\zeta_{k,v}^n = \oint_C e_k \omega^n dA_v$$

From the right hand side of the last equation we can show that the sum is finite.

9.3.1 Proposition *The value of $\zeta_{k,v}^n$ is non-zero only in a critical strip. More precisely:*

$$\zeta_{k,v}^n \neq 0 \Rightarrow \left(n - k - \frac{3g}{2} + 1 < v < n - k + \frac{3g}{2} + 1\right). \tag{9.35}$$

Proof.
Due to the definition of $\zeta^n_{k,v}$ we have to consider the pole behavior around the two points P_+ and P_-. For P_+ we have the multiplicities:

$$k-(\frac{3g}{2}-1)-n+\frac{g}{2}-1+v-\frac{g}{2}-1$$

and around P_- we have

$$-k-(\frac{3g}{2}-1)+n+\frac{g}{2}-1-v-\frac{g}{2}-1$$

These lines are concerned with the "generic case" $|n|>\frac{g}{2}$. For the case $|n|\leq\frac{g}{2}$ only the last line is modified. For the generic case the lines are negative only in a critical strip:

$$n-k-\frac{3g}{2}+1<v<n-k+\frac{3g}{2}+1$$

For the other case we have to modify the right hand side of the inequalities. □
With these preparations we fix now the Lie derivative with respect to the field $e_{\frac{3g}{2}-1}$.

9.3.2 Definition

$$\boxed{\nabla:=\nabla_{e_{\frac{3g}{2}-1}}}$$

According to this definition we have:

$$\nabla a(P)=\sum_{n,u}\zeta^n_u a_n\omega^u(P)$$

where the coefficients ζ^n_u are given by

$$\zeta^n_u=\oint\left(\omega^n(Q)e_{\frac{3g}{2}-1}(Q)dA_u(Q)\right). \tag{9.36}$$

From the above proposition we obtain immediately the following

9.3.3 Lemma *For the numbers ζ^n_u we have the property:*

$$\zeta^n_{\frac{g}{2}}=0\forall n. \tag{9.37}$$

Furthermore: There exists $C>0$ such that

$$\zeta^m_n\neq 0\Rightarrow n-1\leq m\leq C+n. \tag{9.38}$$

We apply the definition of the derivation ∇ to formal distributions:

$$\begin{aligned}\nabla a(P) &= \nabla\sum_n a_n\omega^n=\sum_n a_n\nabla\omega^n\\ &= \sum_n a_n\sum_v\zeta^n_v\omega^v=\sum_{n,v}\zeta^n_v a_n\omega^v\end{aligned}$$

we put for the coefficient of the derived distribution:

$$a_u^{(1)} := \sum_n \zeta_u^n a_n \tag{9.39}$$

Thus we get

$$\nabla a(P) = \sum_n a_n^{(1)} \omega^n = \sum_{n,m} \zeta_n^m a_m \omega^n$$

9.3.4 Definition *The iterated derivation is defined by*

$$\nabla^k a(P) \stackrel{def}{=} \nabla \left(\nabla^{k-1} a(P)\right). \tag{9.40}$$

Especially

$$\begin{aligned}\nabla^2 a(P) = \nabla^1 \sum_n a_n^{(1)} \omega^n(P) &= \sum_u \left(\sum_n \zeta_u^n a_n^{(1)} \right) \omega^u \\ &= \sum_n \left(\sum_{u_2,u_1} \zeta_n^{u_2} \zeta_{u_2}^{u_2} a_{u_2} \right) \omega^n(P).\end{aligned}$$

Iterated derivation gives:

$$\nabla^k a(P) = \sum_n \left(\sum_{u_k,\dots,u_1} \zeta_n^{u_k} \dots \zeta_{u_2}^{u_1} a_{u_1} \right) \omega^u(P) = \sum_u a_u^{(k)} \omega^u(P)$$

9.3.5 Definition *The coefficient of the k-times iterated derivation of $a(P)$ by ∇ with respect to the KN-basis is defined by*

$$a_n^{(k)} \stackrel{def}{=} \sum_{u_k,u_{k-1},\dots,u_1} \zeta_n^{u_k} \dots \zeta_{u_2}^{u_1} a_{u_1}. \tag{9.41}$$

These sums are finite due to eq. (9.38).

9.3.6 Remark *If we write $a_n^{(k)}$ in the form*

$$a_n^{(k)} = \sum_j q_n^{(k),j} a_j. \tag{9.42}$$

then the sum is bounded by $n-k$ from below, more precisely:

$$q_n^{(k),j} = 0 \text{ for } j < n-k$$

Proof. We use eq. (9.38).
From $a_n^{(k)} \stackrel{def}{=} \sum_{u_k,u_{k-1},...,u_1} \zeta_n^{u_k}...\zeta_{u_2}^{u_1} a_{u_1}$ and due to eq. (9.38) we obtain

$$n-1 \leq u_k, \quad u_k - 1 \leq u_{k-1}, \quad ..., u_2 - 1 \leq u_1$$

therefore

$$n-k \leq u_1.$$

□
We can now state the analogous result of equation (9.34) in the higher genus case.

9.3.7 Theorem *The derivation ∇ respects the partition of distribution in positive and negative part. More precisely:*

1.
$$\nabla(a_\pm(P)) = (\nabla a)_\pm(P) \tag{9.43}$$

2.
$$\nabla^k(a_\pm(P)) = (\nabla^k a)_\pm(P) \tag{9.44}$$

Proof. The second equation is a consequence of the first equation. We turn to the left hand side of the first equation:

$$\nabla(a_+(P)) = \nabla(\sum_{n<\frac{g}{2}} a_n \omega^n(P)) = \sum_n (\sum_{u<\frac{g}{2}} \zeta_n^u a_u)\omega^n(P)$$

For $u = \frac{g}{2} - 1$ we have

$$\sum_n \zeta_n^{\frac{g}{2}-1} a_{\frac{g}{2}-1} = \sum_{n=\frac{g}{2}-1} \zeta_n^{\frac{g}{2}-1} a_{\frac{g}{2}-1}$$

because of equations (9.37) and (9.38).
For $u = \frac{g}{2} - 1$ we have due to eq (9.38) only $n < \frac{g}{2} - 1$. This proves the assertion.
□
We get a similar result as eq. (9.33) given in the following corollary.

9.3.8 Corollary *The coefficients of equation (9.42) obey the inequalities:*

$$q_n^{(k),j} = 0 \text{ for } j < \frac{g}{2}, \frac{g}{2} \leq n \leq \frac{g}{2} + k - 1 \tag{9.45}$$

Consider now the derivatives with respect to bidifferentials:

$$\nabla_P^k \nabla_Q^h \omega_-(P,Q). \tag{9.46}$$

More explicitly,

$$\nabla_P^k \nabla_Q^h \omega_-(P,Q) = \nabla_P^k \nabla_Q^h \sum_{nm} \gamma_{nm} \omega^n(P)\omega^m(Q) =$$

$$= \sum_{n \geq \frac{g}{2}, m} \sum_{v_1...v_k} \gamma_{n,m} \zeta_{v_1}^n \zeta_{v_2}^{v_1} ... \zeta_{v_{k-1}}^{v_k} \omega^{v_k}(P) \zeta_{u_1}^m \zeta_{u_2}^{u_1} ... \zeta_{u_{h-1}}^{u_h} \omega^{u_h}(Q). \tag{9.47}$$

9.3.9 Remark *The derivatives of the bidifferential given above define bidifferentials of possibly higher order, i.e. there exist integers $N_0 = N_0(k,h) > 0$ such that*

$$\left(\nabla_P^k \nabla_Q^h \omega_-(P,Q)\right) E^N(P,Q)$$

is regular on the diagonal ($\forall N \geq N_0$).

9.4 Operator Product Expansions and Wick Product

We give now the operator products of fields for affine Krichever-Novikov algebras. To this end we define first the higher genus analogues of field algebras.

9.4.1 Theorem (OPE) *Let $\mathfrak{g}$ be a semi-simple or abelian Lie algebra, let $\hat{\mathfrak{g}}$ be its current algebra. Then we have the following equivalent formulas:*

$$[a_n, b_m] = \sum_k [a,b]\alpha_{nm}^k A_k + (a,b)\gamma_{nm} K \tag{9.48}$$

$$[a(P), b(Q)] = [a,b](P)\Delta(P,Q) + (a,b)K d_P \Delta(P,Q) \tag{9.49}$$

For $\tau(P) < \tau(Q)$:

$$[a_-(P), b(Q)] = [a,b](Q) i_{P,Q} S(Q,P) + (a,b)K\omega_+(P,Q) \tag{9.50}$$

For $\tau(P) < \tau(Q)$:

$$a(P)b(Q) = [a,b](P) i_{P,Q} S(P,Q) + (a,b)K\omega_-(P,Q) + :a(P)b(Q): \tag{9.51}$$

We recall the operator product expansions for the classical current algebras. They are essentially the same as in the above theorem but with the crucial difference that $i_{z,w}\frac{1}{z-w}$ and $i_{z,w}\frac{1}{(z-w)^2}$ are replaced by the Szegö kernel and the bidifferential $\omega(P,Q)$.
More precisely the correspondence of equation (9.48) to the classical case is illustrated as follows

$$\begin{array}{ccccc} [a_n, b_m] & = & \sum_k [a,b]\alpha_{nm}^k A_k & + & (a,b)\gamma_{nm} K \\ & & \uparrow & & \uparrow \\ [a_n, b_m] & = & [a,b]_{n+m} & + & (a,b) n \delta_{n,-m} K. \end{array}$$

That means for $\alpha_{nm}^k = \delta_{n+m,k}$ we obtain the classical commutator relations. Equation (9.49) corresponds to the classical case by

$$\begin{array}{ccccc} [a(P), b(Q)] & = & [a,b](P)\Delta(P,Q) & + & (a,b) d_P \Delta(P,Q) \\ [a(z), b(w)] & = & [a,b](w)\delta(z,w) & + & (a,b)\partial_w \delta(z,w) \end{array}$$

Proof of the theorem. We start with the observation:

$$A_m(P)\cdot\omega^n(P)=\sum_k \beta^n_{m,k}\omega^k(P)$$

with

$$\beta^n_{m,k}=\frac{1}{2\pi i}\int_C A_m(P)\omega^n(P)A_k(P)$$

We see: $\beta^n_{m,k}=\alpha^n_{mk}$.
We address now to equation (9.49):

$$\begin{aligned}
[a(P),b(Q)] &= \sum_{nm}[a_n,b_m]\omega^n(P)\omega^m(Q)\\
&= \sum_{nm}\left(\sum_k \alpha^k_{nm}[a,b]_k+(a,b)\gamma_{nm}K\right)\omega^n(P)\omega^m(Q)\\
&= \sum_{nm}\sum_k \alpha^k_{nm}[a,b]_k\omega^n(P)\omega^m(Q)+\sum_{nm}(a,b)\gamma_{nm}K\omega^n(P)\omega^m(Q)\\
&= \sum_{nm}\sum_k \alpha^k_{nm}[a,b]_k\omega^n(P)\omega^m(Q)+\\
&+ (a,b)K\sum_{nm}\gamma_{nm}\omega^n(P)\omega^m(Q)
\end{aligned}$$

From the second summand we know:

$$(a,b)K\sum_{nm}\gamma_{nm}\omega^n(P)\omega^m(Q)=(a,b)Kd_P\Delta(P,Q),$$

and:

$$\begin{aligned}
\sum_{nm}\sum_k \alpha^k_{nm}[a,b]_k\omega^n(P)\omega^m(Q) &= \sum_m\sum_k[a,b]_k(A_m(P)\omega^k(P))\omega^m(Q)\\
&= \sum_k[a,b]_k\omega^k(P)\underbrace{\sum_m A_m(P)\omega^m(Q)}_{=\Delta(P,Q)}
\end{aligned}$$

We prove equation (9.50): This is essentially the same calculation like the above.

$$\begin{aligned}
[a_-(P),b(Q)] &= \sum_{n\geq\frac{g}{2},m}[a_n,b_m]\omega^n(P)\omega^m(Q)\\
&= \sum_{n\geq\frac{g}{2},m}\left(\sum_k \alpha^k_{nm}[a,b]_k+(a,b)\gamma_{nm}K\right)\omega^n(P)\omega^m(Q)\\
&= \sum_{n\geq\frac{g}{2},m}\sum_k \alpha^k_{nm}[a,b]_k\omega^n(P)\omega^m(Q)+\\
&+ (a,b)K\sum_{n\geq\frac{g}{2},m}\gamma_{nm}\omega^n(P)\omega^m(Q)
\end{aligned}$$

From the second summand we know:

$$(a,b)K\sum_{n\geq\frac{g}{2},m}\gamma_{nm}\omega^n(P)\omega^m(Q)=(a,b)K\omega_-(P,Q),$$

and

$$\begin{aligned}\sum_{n\geq\frac{g}{2},m}\sum_k\alpha^k_{nm}[a,b]_k\omega^n(P)\omega^m(Q) &\overset{(*)}{=} \sum_{n\geq\frac{g}{2}}\sum_k[a,b]_k\omega^n(P)(A_n(Q)\omega^k(Q))\\ &= \sum_k[a,b]_k\omega^k(Q)\underbrace{\sum_{n\geq\frac{g}{2}}A_n(Q)\omega^m(P)}_{=i_{P,Q}S(Q,P)}\end{aligned}$$

The equation (*) is a consequence of the equation $A_n(Q)\omega^k(Q)=\sum_m\alpha^k_{nm}\omega^k(Q)$.
We prove equation (9.51):

$$a(P)b(Q)=[a_-(P),b(Q)]+\underbrace{a_+(P)b(Q)+b(Q)a_-}_{=:a(P)b(Q):}.$$

We can apply equation (9.50) and obtain the assertion. □
We address now to the Wick product formula for commuting fields.

9.4.2 Theorem (Wick Product Formula) *Let $a^1(P),...,a^N(P),b^1(P),...,b^M(P)$ a collection of fields with the properties*

1. $[a^i_-(P),b^j_-(Q)]=0$ *for all* i,j
2. $[[a^i_-(P),b^j(Q)]c(Q)]=0$

Then we obtain as the product

$$\begin{aligned}&\left(:a^1(z)a^2(z)...a^N(z):\right)\left(:b^1(z)b^2(z)...b^M(z):\right)=\\ &\quad=\sum_{s=0}^{\min(M,N)}\sum_{i_1<...<i_s}^{j_1\neq...\neq j_s}[a^{i_1}_-,b^{j_1}(w)]...[a^{i_s}_-(z),b^{j_s}(w)]\cdot\\ &\qquad\cdot:a^1(z)a^2(z)...a^N(z)b^1(w)...b^M(w):_{(i_1...i_sj_1...j_s)}\end{aligned}$$

where the subscript $(i_1...i_sj_1...j_s)$ means that the fields $a^{i_1}(z),...,a^{i_s}(z),b^{j_1}(w),...,b^{j_s}(z)$ are removed.

Proof. The theorem is the complete analogue of the classical case. □

Chapter 10

Global Vertex Algebra

In this chapter I am going to define the notion of a **Global Vertex Algebra**. This notion is new and I haven't found a similar notion in the literature. It is supposed to be a suitable generalization of the notion of a vertex algebra.
In order to generalize a well known mathematical structure there must be a non-trivial example that satisfies the new conditions. This example is given in the next chapter. Furthermore there must be a reasonable correspondence to the old structure that has been generalized. The correspondence is given in this chapter.
As the "classical" vertex algebra a global vertex algebra is a state field correspondence in the sense that elements of a vector space are mapped to fields.

10.1 Recap and Preliminaries

For a vertex algebra we had the state-field correspondence

$$A \mapsto Y(A,z) = \sum_n A_n z^{-n-1}$$

where $A_n \in End(V)$ are endomorphisms and we had the field property that $A_n v = 0$ for any $v \in V$ and $n \gg 0$.
Furthermore we had the vacuum property which asserts that the field $Y(A,z)$ applied to the vacuum vector $|0\rangle$ gives a power series:

$$Y(A,z)\,|0\rangle = \sum_{n<0} A_n z^{-n-1}$$

And we had the additional property that $Y(A,z)v|_{z=0} = A$.
According to the considerations of the last chapters if we replace the monomials z^{-n-1} be the sections $\omega^n(z)$ then we have to replace the vacuum property by

$$Y(A,P)\,|0\rangle = \sum_{n<\frac{g}{2}} A_n \omega(P)$$

which means that we have a series expansion with sections that have only zeros (and no poles) in the point P_+.
The locality axiom was for the vertex algebra $[Y(A,z), Y(B,w)](z-w)^n = 0$ for some $n \in \mathbb{Z}_{>0}$. As it was discussed in earlier chapters this means that the field $[Y(A,z), Y(B,w)]$ is supported on the diagonal.

10.2 Definition

10.2.1 Definition *Let be X a compact Riemann surface, P_+ and P_- two distinguished points on X in general position.*
A ***global vertex algebra*** *is a collection of data:*
- A $\mathbb{Z}_{\geq 0}$ graded vector space V,
- (vacuum vector) a vector $v_0 \in V$,
- (vertex operator) a linear map from the homogeneous elements to fields

$$\begin{array}{rcl} V_\lambda & \to & End(V)[[f_\lambda^n]]_{n \in \mathbb{Z}'} \\ A & \mapsto & \mathcal{Y}(A,P) = \sum_n A_n f_\lambda^n(P). \end{array}$$

These data are subject to the following axioms:

1. *(vacuum) $\mathcal{Y}(v_0, P) = id_V$.*
   For any $A \in V_\lambda$ of conformal dimension λ we have

   $$\mathcal{Y}(A,P)v_0 = \sum_{n < -s_\lambda} (A_n v_0) f_\lambda^n(P).$$

   and

   $$\mathcal{Y}(A,P)v_0|_{P=P_0} = A + ...$$

   where the dots ... denote vectors of lower degree.
   In other words, $A_n v_0 = 0$, $n > -s_\lambda$, and $A_{-s_\lambda} v_0 = A$.

2. *(locality) There exists an $N \in \mathbb{N}$ such that*

   $$[\mathcal{Y}_-(A,P), \mathcal{Y}(B,Q)](E(P,Q))^N \text{ regular on the diagonal } \Delta \subset X \times X.$$

   where

   $$\mathcal{Y}_-(A,P) = \sum_{n=n_0}^{\infty} A_n f_\lambda^n \text{ for a suitable } n_0 \in \mathbb{Z}.$$

Note that a translation axiom is missing.

10.2.2 Remark *Let X be now the Riemann sphere $X = \mathbb{P}$. Let be $P_+ = 0$ and $P_- = \infty$. The Krichever-Novikov basis can be written in quasi-global coordinates by*

$$f_\lambda^n = z^{-n-\lambda}.$$

By these considerations one can see that the global vertex algebra coincides with the classical vertex algebra for a graded vector space (up to the translation axiom).
The prime form in the locality axiom is replaced by the function $(z - w)$.

There is a relationship to the moduli space of Riemann surfaces:
By the explicit representation of the Krichever-Novikov forms one sees that they vary analytically when the complex structure of the Riemann surface is deformed. Therefore it should be possible to define a global vertex algebra on the moduli space of Riemann surfaces.

Chapter 11

Heisenberg Global Vertex Algebra

In this chapter I will show that the Fock representation V of the Heisenberg algebra of Krichever-Novikov type carries the structure of a global vertex algebra.

11.1 The Fock Representation of the Heisenberg Algebra of KN Type

Let X be a compact Riemann surface of genus $g \geq 2$ with two generic points P_+ and P_-. Let $\mathcal{F}^\lambda$ ($\lambda \in \mathbb{Z}$) denote the space of meromorphic sections of the line bundle $K^{\otimes \lambda}$ holomorphic outside P_+ and P_-.
Let $\mathcal{A} := \mathcal{F}^0$ be the space of meromorphic functions holomorphic outside $P_\pm$.
Recall the representation of $\mathcal{A}$ in the space of half-infinite wedge forms (see 8.4 and especially 8.4.2).
Let V be the space H_λ.
Put

$$v_0 := f_\lambda^{s_\lambda} \wedge f_\lambda^{s_\lambda+1} \wedge f_\lambda^{s_\lambda+2} \wedge .... \tag{11.1}$$

We have

$$a_{\frac{g}{2}+h} v_0 = 0 \text{ for } h \geq 0. \tag{11.2}$$

V is spanned by the vectors

$$a_{-n_1+\frac{g}{2}-1} a_{-n_2+\frac{g}{2}-1} .... a_{-n_M+\frac{g}{2}-1} v_0 \tag{11.3}$$

where $n_1 \geq ... \geq n_M \geq 0$. We can define a gradation:
Let $v \in V$ be a vector as in eq. (11.3). The degree of v is defined by

$$\deg(v) = \sum_{j=1}^{M} n_j.$$

Define a linear map (the state-field correspondence)

$$\mathcal{Y} : V \to End(V)[[f_\lambda^n]]_{n\in\mathbb{Z}',\lambda\in\mathbb{N}}$$

by

$$\begin{aligned} a_{-n_1+\frac{g}{2}-1}...a_{-n_M+\frac{g}{2}-1}v_0 &\mapsto \mathcal{Y}(a_{-n_1+\frac{g}{2}-1}...a_{-n_M+\frac{g}{2}-1}v_0, P) \\ &\stackrel{def}{=} \; : \nabla^{n_1}a(P)...\nabla^{n_M}a(P) : \end{aligned}$$

11.1.1 Theorem *This state field correspondence above defines the structure of a Global Vertex Algebra.*
That means

1. $Y(v,P)v_0 \in V[[f_\lambda^n]]_{n\le s_\lambda}$
   and
   $Y(v,P)v_0|_{P=P_+} = c\cdot v +$ *terms of lower degree.*
2. $[Y_-(v,P), Y(w,Q)]E(P,Q)$ *is holomorphic on the diagonal* $\Delta \subset X\times X$.

In the next subsections we are going to prove this theorem.

11.2 Vacuum Property

In this section we prove the vacuum property of the state field correspondence. First we prove the vacuum property for the fields of conformal dimension 1.

11.2.1 Fields of Conformal Dimension 1 and 2

We start with the field $a(P) = \sum a_n\omega^n(P)$.

11.2.1 Proposition *For the field* $a(P) := \mathcal{Y}(a_{\frac{g}{2}-1}v_0, P) = \sum_n a_n\omega^n(P)$ *we have*

1. $a(P)v_0 \in V[[\omega^n]]_{n\le\frac{g}{2}-1}$
2. $a(P)v_0|_{P=P_+} = a_{\frac{g}{2}-1}v_0$.

Proof. We have due to the construction of the representation π: $a_nv_0 = 0$ for all $n \ge \frac{g}{2}$. Thus we obtain

$$a(P)v_0 = \sum_n a_nv_0\omega^n(P) = \sum_{n<\frac{g}{2}} a_nv_0\omega^n(P)$$

The forms $\omega^n(P)$ look locally around P_+:

$$z_+^{-n+\frac{g}{2}-1}(1+O(z_+))dz_+.$$

For $n \leq \frac{g}{2}-1$ we get non-negative powers and therefore

$$\omega^n(P_+) = \begin{cases} 1 & \text{for } n = \frac{g}{2}-1 \\ 0 & \text{for } n < \frac{g}{2}-1 \end{cases} \tag{11.4}$$

The two assertions of the theorem follow immediately. □

11.2.2 Lemma *Denote by $a_n^{(k)}$ the coefficients of the series*

$$\nabla^k a(P) = \sum_n a_n^{(k)} \omega^n(P).$$

Then we have

$$a_n^{(k)} v_0 = 0 \text{ for all } n \geq \frac{g}{2}. \tag{11.5}$$

Proof. We obtain from remark 9.3.6 of section three of chapter 9:

$$a_n^{(k)} = \sum_{n-k}^{\infty} q_n^{(k),m} a_m.$$

This sum is properly finite.
We use the fact $a_n v_0 = 0$ for all $n \geq \frac{g}{2}$. Thus we have

$$a_n^{(k)} v_0 = \sum_{m=n-k}^{\frac{g}{2}-1} q_n^{(k),m} a_m v_0.$$

We obtain that $a_n^{(k)} v_0 = 0$ for $n-k \geq \frac{g}{2} \Leftrightarrow n \geq \frac{g}{2}+k$.
In order to prove the assertion of the lemma we only have to consider the strip

$$\frac{g}{2} \leq n \leq \frac{g}{2}+k-1.$$

We use now the important fact (due to remark 9.3.6):

$$q_n^{(k),m} = 0 \text{ for } m < \frac{g}{2} \text{ and } \frac{g}{2} \leq n \leq \frac{g}{2}+k-1.$$

Therefore we have for $\frac{g}{2} \leq n \leq \frac{g}{2}+k-1 \Leftrightarrow \frac{g}{2}-k+1 \leq n-k+1 \leq \frac{g}{2}$:

$$a_n^{(k)} v_0 = \sum_{m=n-k}^{\frac{g}{2}-1} q_n^{(k),m} a_m v_0 = \sum_{m=n-k}^{\frac{g}{2}-1} 0 \cdot a_m v_0 = 0.$$

□

This crucial lemma facilitates the next calculation considerably. For the derivations of the field $a(P)$ we get the following result:

11.2.3 Corollary *For the fields $\nabla^k a(P)$ we have*

1. $\nabla^k a(P)v_0 \in V[[\omega^n]]_{n\le \frac{g}{2}-1}$
2. $\nabla^k a(P)v_0|_{P=P_+} = \eta \cdot a_{-k+\frac{g}{2}-1}v_0 + ...$
   where $\eta \in \mathbb{C}$*, and where ... means summands of lower degree* ($\eta \in \mathbb{C}$).

Proof. This is a consequence of the preceding lemma.
Starting from

$$\nabla^k a(P) = \sum_n a_n^{(k)} \omega^n(P)$$

we have due to the above lemma: $a_n^{(k)} v_0 = 0$ for $n \ge \frac{g}{2}$.
So we get

$$\begin{aligned}
\nabla^k a(P)v_0 &= \sum_{n\le \frac{g}{2}-1} a_n^{(k)} v_0 \omega^n(P) &(11.6)\\
&= a^{(k)}_{\frac{g}{2}-1} v_0 \omega^{\frac{g}{2}-1}(P) + a^{(k)}_{\frac{g}{2}-2} v_0 \omega^{\frac{g}{2}-2}(P) + ... &(11.7)
\end{aligned}$$

For $P \to P_+$ we obtain:

$$\begin{aligned}
\nabla^k a(P)v_0|_{P=P_+} &= \left(q^{(k),\frac{g}{2}-k-1}_{\frac{g}{2}-1} a_{\frac{g}{2}-k-1} + q^{(k),\frac{g}{2}-k}_{\frac{g}{2}-1} a_{\frac{g}{2}-k} + ... \right) v_0 \\
&= \left(q^{(k),\frac{g}{2}-k-1}_{\frac{g}{2}-1} a_{\frac{g}{2}-k-1} v_0 + q^{(k),\frac{g}{2}-k}_{\frac{g}{2}-1} a_{\frac{g}{2}-k} v_0 + ... \right)
\end{aligned}$$

The first vector in this sum has the highest degree k. $\square$

11.2.2 General Fields

We address now the general state-field correspondence. It is quite interesting to see that up to some technicalities the proof is quite similar to the proof for the Fock representation of the "classical" Heisenberg algebra.

11.2.4 Theorem *The Fields*

$$: \nabla^{n_1-1} a(P) \nabla^{n_2-1} a(P) .... \nabla^{n_M-1} a(P) :$$

satisfy the properties:

1.
$$: \nabla^{n_1-1} a(P) \nabla^{n_2-1} a(P) .... \nabla^{n_M-1} a(P) :_n v_0 = 0 \text{ for } n > -s_M \tag{11.8}$$

   and

2.
$$\begin{aligned}
: \nabla^{n_1-1} a(P) .... \nabla^{n_M-1} a(P) : v_0|_{P=P_+} &= \; : \nabla^{n_1-1} a(P) .... \nabla^{n_M-1} a(P) :_{-s_M} v_0 \\
&= C \cdot \left(a_{-n_1+\frac{g}{2}} .... a_{-n_M+\frac{g}{2}} \right) v_0 + ...
\end{aligned}$$

   where ... means lower degree vectors. C *is a scalar.*

Proof. We prove this by induction for $M \in \mathbb{Z}_{>0}$:
Consider first the case if we have only one field and no derivation, i.e. let be $M = 1$ and $n_M = n_1 = 1$. We thus consider the field

$$a(P) = \sum a_n \omega^n(P)$$

We know from the preceding proposition:

$$a(P)v_0|_{P=P_+} = a_{\frac{g}{2}-1}v_0$$

We have $s_1 = \frac{(1-2\cdot1)g}{2} + 1 = -\frac{g}{2} + 1$. This is the assertion of the theorem for that case.
For $n_M = n_1 > 1$ we have due to the preceding corollary:

$$\nabla^k a(P)v_0|_{P=P_+} = \eta \cdot a_{-k+\frac{g}{2}-1}v_0 + ...$$

and we obtain the assertion of the theorem because $s_1 = -\frac{g}{2} + 1$.
Let be $M = 2$ and $n_1 = n_2 = 1$. That means we consider

$$: a(P)a(P) := \sum_n : a(P)a(P) :_n \Omega^n(P)$$

We have to show that

$$: a(P)a(P) :_n v_0 = 0 \text{ for } n \geq -s_2 = \frac{3g}{2} - 2.$$

and

$$: a(P)a(P) :_n v_0|_{P=P_+} = c \cdot a_{\frac{g}{2}-1}a_{\frac{g}{2}-1}v_0 + ...$$

$$: a(P)a(P) :_n v_0 = \sum_{j<\frac{g}{2},m} a_j a_m v_0 l_n^{jm} + \sum_{j\geq\frac{g}{2},m} a_m a_j v_0 l_n^{jm}$$

The second term is zero. We only have to address the first summand.
The second summand reduces to

$$\sum_{j<\frac{g}{2},m<\frac{g}{2}} a_j a_m v_0 l_n^{jm},$$

and we have especially $l_n^{jm} \neq 0 \Rightarrow n \leq j + m + \frac{g}{2} - 1 < \frac{3g}{2} - 1$.
Let be now $M \geq 2$ and suppose that

$$Y(A,P)v_0|_{P=P_+} = A_{-s_M}v_0 = A$$

where $A_{-s_M} = a_{-n_1+\frac{g}{2}}....a_{-n_M+\frac{g}{2}}v_0$.
Consider now

$$: \nabla^k a(P)Y(A,P) := \sum_n : \nabla^k a(P)Y(A,P) :_n f^n_{M+1}(P)$$

We have to show that

$$:\nabla^k a(P)Y(A,P):_n v_0 = 0 \text{ for } n > -s_{M+1}$$

in order to get

$$:\nabla^k a(P)Y(A,P): v_0|_{P=P_+} =: \nabla^k a(P)Y(A,P):_{-s_{M+1}} v_0.$$

We have by the definition of the normal ordered product:

$$:\nabla^k a(P)Y(Av_0,P) := \sum_n \left(\underbrace{\sum_{j<\frac{g}{2},m} a_j^{(k)} A_m l_n^{jm}}_{(I)} + \underbrace{\sum_{j\geq\frac{g}{2},m} A_m a_j^{(k)} l_n^{jm}}_{(II)} \right) f_{M+1}^n(P)$$

where $l_n^{jm} = l_{(1,M)n}^{jm}$ (see chapter 10 for a detailed discussion).
We investigate the sums (I) and (II) more carefully. The next lemma asserts that sum (II) is trivial.

11.2.5 Lemma *For all n we have*

$$\sum_{j\geq\frac{g}{2},m} A_m a_j^{(k)} v_0 l_n^{jm} = 0$$

This is a consequence of eq. (9.45) and the condition $a_{\frac{g}{2}+x} v_0 = 0$ for $x = 0,1,2,...$ □

11.2.6 Lemma *If $A_m v_0 = 0$ for $m > -s_M$ then for $n > -s_{M+1}$ we have*

$$\sum_{j<\frac{g}{2},m} a_j^{(k)} A_m v_0 l_n^{jm} = 0$$

Proof. We have to consider $n = -s_{M+1} + x$ where $x = 1,2,3,..$ and $j = \frac{g}{2} - y$ where $y = 1,2,3,....$ Then we get because of lemma 9.2.4:

$$l_{-s_{M+1}+x}^{\frac{g}{2}-y,m} \neq 0 \Rightarrow -s_{M+1} - g + x + y \leq m \leq -s_{M+1} + x + y$$

We have the identity

$$s_{M+1} = \frac{(-1-2M)g}{2} + M + 1 = \frac{(1-2M)g}{2} - g + M + 1 = s_M - g + 1$$

thus we get:

$$l_{-s_{M+1}+x}^{\frac{g}{2}-y,m} \neq 0 \Rightarrow -s_M + x + y - 1 \leq m \leq -s_M + g + x + y - 1.$$

It follows that for $x, y = 1,2,3,...$ we have $m > -s_M$ and because of $A_m v_0 = 0$ for $m > -s_M$ we get for the first sum: $\sum_{j<\frac{g}{2}} a_j^{(k)} A_m v_0 l_n^{j,m} = 0$. □

It remains to show the second assertion of the theorem:
Because of $A_m v_0 = 0$ for $m > -s_M$ we get

$$\sum_{j<\frac{g}{2},m} a_j^{(k)} A_m v_0 l^{jm}_{-s_{M+1}} = (l^{\frac{g}{2}-1,-s_M}_{-s_{M+1}}) a_{-k+\frac{g}{2}-1} A_{-s_M} v_0 + ...$$

□

11.3 Locality

For the proof of the locality condition it is sufficient to consider the operator products $\mathcal{Y}(A,P) \cdot \mathcal{Y}(B,Q)$.
From chapter 9 (theorem 9.4.1) we know

$$a(P) \cdot a(Q) = \omega_-(P,Q) + :a(P)b(Q):.$$

11.3.1 Proposition

$$[\nabla^k a(P)_-, \nabla^h a(Q)_-] = 0 \tag{11.9}$$

and

$$[[\nabla^k a(P)_-, \nabla^h a(Q)], \nabla^f a(Q)] = 0 \textit{ for all } k,h,f \tag{11.10}$$

Proof. We use the fact

$$\gamma_{nm} \neq 0 \Rightarrow |n+m| > g+1$$

$$[\nabla^k a(P)_-, \nabla^h a(Q)_-] = \sum_{n \leq \frac{g}{2}, m \leq \frac{g}{2}} [a_n^{(k)}, a_m^{(h)}] \omega^n(P) \omega^m(Q)$$

We focus our attention to the coefficients of this series. The coefficients are given by

$$a_n^{(k)} = \sum_{n,v_1,\dots,v_k} \zeta_n^{v_1} ... \zeta_{v_{k-1}}^{v_k} a_{v_k}.$$

We know furthermore $[a_n, a_m] = \gamma_{nm}$. For $n,m \geq \frac{g}{2}$ we have $[a_n, a_m] = 0$.
□
The proposition above shows that the fields $\nabla^k a(P)$ obey the conditions for the Wick theorem. Therefore we can state the following theorem.

11.3.2 Theorem *The operator product of is given by*

$$(:\nabla^{n_1} a(P) ... \nabla^{n_M} a(P):)(:\nabla^{m_1} a(Q) ... \nabla^{m_N} a(Q):) =$$

$$= \sum_{s=0}^{\min(M,N)} \sum_{i_1 < \dots < i_s}^{j_1 \neq \dots \neq j_s} [\nabla^{i_1} a_-(P), \nabla^{j_1} a(Q)] ... [\nabla^{i_s} a_-(P), \nabla^{j_s} a(Q)] \cdot$$

$$\cdot : \nabla^k a(P) ... \nabla^k a(P) \nabla^k a(Q) ... \nabla^k a(Q) :_{(i_1 ... i_s, j_1 ... j_s)}$$

where the subscript $(i_1...i_s, j_1...j_s)$ *means that the fields* $\nabla^{i_1} a(z)...\nabla^{i_s} a(z)$*, and* $\nabla^{j_1} a(z)...\nabla^{j_s} a(z)$ *are removed.*

Proof. We have only to show that the conditions of the Wick product is satisfied by the fields $\nabla^i a(P)$. But this is clear because of equation (9.43).
□

11.3.3 Theorem *There exists an* $N_0 \in \mathbb{N}$ *such that for all* $N \geq N_0$

$$[\mathcal{Y}(A, P), \mathcal{Y}(B, Q)] E^N(P, Q)$$

is regular along the diagonal.

Proof. We know

$$[a_-(P), a(Q)] = \omega_-(P, Q).$$

Furthermore: $\omega_-(P, Q) E^N(P, Q)$ is regular along the diagonal.
For the derivatives we have

$$[\nabla^k a_-(P), \nabla^h a(Q)] = \nabla^k_P \nabla^h_Q \omega_-(P, Q).$$

According to remark 9.3.9 we obtain the assertion.
□

Appendix A

Distributions in Complex Analysis

A distribution in the sense of functional analysis is a linear continuous functional on a space of functions. The most common example is the space of Schwartz distributions on the space of $\mathcal{C}^\infty$ functions with compact support.
Schwartz distributions can also be represented by "jumps" of analytic functions.
Let (Φ) be a complex linear space of complex valued functions in $n \geq 1$ real variables: $\phi(t_1, ..t_n)$. Let a notion of convergence be defined on (Φ) with the following property:
Each sequence $\{\phi_j\}$ of functions in (Φ) is classified either as convergent or divergent. With each convergent sequence $\{\phi_j\}$ is associated an nonempty set of limit functions. If a sequence $\{\phi_j\}$ converges to ϕ_0, and a sequence $\{\psi_j\}$ converges to ψ_0 then the sequences $\{(a\phi + b\psi)_j\}$ converge to $\{(a\phi + b\psi)_0\}$ for a, b arbitrary complex numbers.
A **distribution** T is a linear continuous functional on the space (Φ), where continuous means

$$\lim_{j\to\infty} \langle T, \phi_j \rangle = \langle T, \lim_{j\to\infty} \phi_j \rangle$$

and $\langle\ < T, \phi \rangle >$ means the complex value of T when applied to the function $\phi \in (\Phi)$, and the limit on the right hand side of the equation is convergence in the sense of the space Φ. The limit on the left hand side means convergence of complex numbers.
We restrict ourselves to the space of $(\mathcal{D})$, i.e. to space of all $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$ with compact support. The space of Schwartz distributions $(\mathcal{D}')$ is the space of all linear functionals on $(\mathcal{D})$ that are continuous in the following sense: A functional $T \in (\mathcal{D}')$ is called continuous in the sense of $(\mathcal{D})$ if and only if its restriction to any space $(\mathcal{D}_K)$ is continuous in $(\mathcal{D}_K)$ where $(\mathcal{D}_K)$ is the space of all $\mathcal{C}^\infty$-functions whose support is contained in some compact set in $\mathbb{R}^n$.
The notion of convergence here we are dealing with is uniform convergence.
We have the theorem [Brem]:

A.0.4 Theorem *Let $T \in (\mathcal{D}')$ be a Schwartz distribution. Then there exists a function $F(z)$, analytic everywhere except (possibly) on the real axis, such that*

$$\lim_{\epsilon \to 0+} \int_{-\infty}^{\infty} [F(t+i\epsilon) - F(t-i\epsilon)]\, \phi(t) dt = \langle T, \phi \rangle$$

for any test function of class $(\mathcal{D})$. $F(z)$ is called the analytic representation of T.

The derivative of a distribution is defined by

$$\langle T', \phi \rangle = \langle T, \phi' \rangle$$

and more generally

$$\langle T^{(n)}, \phi \rangle = \langle T, \phi^{(n)} \rangle$$

Define [Brem] p. 57:

$$\hat{S}_n(z) = \frac{n!}{2\pi i} \langle T, (t-z)^{-n-1} \rangle.$$

$\hat{S}$ represents the n-th derivative of T in the following way:

$$\lim_{\epsilon \to 0+} \int_{-\infty}^{\infty} \left[\hat{S}_n(x+i\epsilon) - \hat{S}_n(x-i\epsilon)\right] \phi(x) dx = \langle T^{(k)}, \phi \rangle.$$

Roughly speaking this theorem asserts that distributions are the "jump" values of two holomorphic functions with respect to the different domains $Im(z) > 0$ and $Im(z) < 0$.
We compare the above considerations with our case of **formal distributions**.
We mimic this "jump" behavior by the following equation:
We can represent our formal distribution $a(z) = \sum_n a_n z^{-n-1}$ by:

$$\langle a, \phi \rangle = Res_z \left((a(z) i_{z,w} \frac{1}{z-w} - a(z) i_{w,z} \frac{1}{z-w}) \phi(z) \right).$$

With the above representation of distributions by analytic functions in mind we can define the n-th derivation by:

$$\langle a^{(n)}, \phi \rangle = Res_z \left((a(z) i_{z,w} \frac{1}{(z-w)^{n+1}} - a(z) i_{w,z} \frac{1}{(z-w)^{n+1}}) \phi(z) \right).$$

This illustrates the similarity between the theory of distributions in complex analysis and the theory of formal distributions in the first chapter.

Appendix B

Sugawara Construction

B.1 Free Bosons

We present two proofs of the Sugawara construction for the most simple case of free bosons. This shows that we have also a representation of the Virasoro algebra in the field representation of free bosons. These two proofs are already suggested in [Kac].

Convention: From now on consider a field representation of the algebra of distributions $\{\partial^i a(z)\}_{i\in\mathbb{Z}_{\geq 0}}$, where the central element K acts on a vector space V by multiplication with a scalar k.

In other words we consider now field algebras. For fields we can define the normal ordered product.

B.1.1 Proposition *Let be a_i, a^i for $i = 1....d$ a dual basis with respect to the bilinear form $(\cdot|\cdot)$. Define by $L(z)$ the field*

$$L(z) := \frac{1}{2k}\sum_{i=1}^{d} : a_i(z)a^i(z) := \sum_n L_n z^{-n-2}$$

Then we have the equivalent relations:

$$[L_n, a_m] = (-m)a_{n+m} \qquad \text{(B.1)}$$

$$[L(z)a(w)] = a(w)\partial_w\delta(z-w) \qquad \text{(B.2)}$$

$$L(z)a(w) = \frac{a(w)}{(z-w)^2} + : L(z)a(w) : \qquad \text{(B.3)}$$

The equivalence of these relations is clear from the OPE theorem of chapter 1. We only have to take into account the fact that the coefficients L_n correspond to the field $L(z) = \sum L_n z^{-n-2}$, $L(z)$ is a field of conformal weight 2. But we can calculate directly from $[L_n, a_m] = (-m)a_{n+m}$:

$$[L(z)a(w)] = \sum_{n,m}[L_n, a_m]z^{-n-2}w^{-m-1} = \sum_{n,m}(-m)a_{n+m}z^{-n-2}w^{-m-1} =$$

$$= \sum_{k,m}(-m)a_k z^{-k-2+m}w^{-m-1} = \sum_{k,m}(-m)a_k z^{-k-1}z^{m-1}w^{-m-1} =$$

$$= (\sum_k a_k z^{-k-1})\sum_m(-m)z^{m-1}w^{-m-1} = a(z)\sum_m m z^{-m-1}w^{m-1}.$$

The proof of this proposition will be given after the following theorem.

B.1.2 Theorem *Let be a_i, a^i for $i = 1....d$ a dual basis with respect to the bilinear form $(\cdot|\cdot)$. Define by $L(z)$ the field*

$$L(z) := \frac{1}{2}\sum_{i=1}^{d} : a_i(z)a^i(z) := \sum_n L_n z^{-n-2}$$

Then we have the relations:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{d}{12}(n^3-n)\delta_{n,-m} && \text{(B.4)}\\
[L(z), L(w)] &= \frac{d}{2}\partial^{(3)}\delta(z-w) + 2L(w)\partial\delta(z-w) + \partial L(w)\delta(z-w) && \text{(B.5)}\\
L(z)L(w) &= \frac{\frac{d}{2}}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w} && \text{(B.6)}
\end{aligned}$$

Proof of the proposition:
We proof the first relation:

$$\begin{aligned}
[L_n, a_m] &= [\frac{1}{2}\sum_{i=1}^{d} : a_i(z)a^i(z) :_n, a_m] = \\
&= \frac{1}{2}\sum_{i=1}^{d}[: a_i(z)a^i(z) :_n, a_m] = \\
&= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<0}[a_{i,j}a^i_{n-j}, a_m] + \sum_{j\geq 0}[a^i_{n-j}a_{i,j}, a_m]
\end{aligned}$$

We use the relation $[ab, c] = a[b, c] + [a, c]b$:

$$\begin{aligned}
[L_n, a_m] &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<0} a_{i,j}[a^i_{n-j}, a_m] + [a_{i,j}, a_m]a^i_{n-j} + \\
&+ \sum_{j\geq 0} a^i_{n-j}[a_{i,j}, a_m] + [a^i_{n-j}, a_m]a_{i,j}
\end{aligned}$$

We use now the relation

$$[a_{i,n}, a^j_m] = (a_i|a^j)n\delta_{n,-m}. \tag{B.7}$$

$$\begin{aligned}
[L_n, a_m] &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<0} a_{i,j}(a^i|a)(n-j)\delta_{n-j,m} + (a_i|a)j\delta_{j,m}a^i_{n-j} + \\
&+ \sum_{j\geq 0} a^i_{n-j}(a_i|a)j\delta_{j,m} + (a^i|a)(n-j)\delta_{n-j,m}a_{i,j} = \\
&= \frac{1}{2}\sum_{i=1}^{d}\sum_{j}(a^i|a)(n-j)\delta_{n-j,m}a_{i,j} + (a_i|a)j\delta_{j,m}a^i_{n-j} = \\
&= \frac{1}{2}\sum_{i=1}^{d}(a^i|a)ma_{i,m-n} + (a_i|a)ma^i_{n-m} = \\
&= ma_{m-n}
\end{aligned}$$

□

B.1.1 First Proof of the Theorem

$$\begin{aligned}
[L_n, L_m] &= [L_n, \frac{1}{2}\sum_{i=1}^{d} : a_i(z)a^i(z) :_m] = \\
&= \frac{1}{2}\sum_{i=1}^{d}[L_n, : a_i(z)a^i(z) :_m] = \\
&= \frac{1}{2}\sum_{i=1}^{d}[L_n, a_{i,j}a^i_{m-j}] + [L_n, a^i_{m-j}a_{i,j}]
\end{aligned}$$

We apply ones again the relation $[a, bc] = [a, b]c + b[a, c]$ to the terms. Therefore we get

$$\begin{aligned}
[L_n, a_{i,j}a^i_{m-j}] &= [L_n, a_{i,j}]a^i_{m-j} + a_{i,j}[L_n, a^i_{m-j}] \\
[L_n, a^i_{m-j}a_{i,j}] &= [L_n, a^i_{m-j}]a_{i,j} + a^i_{m-j}[L_n, a_{i,j}]
\end{aligned}$$

Now applying the relation $[L_n, a_m] = (-m)a_{n+m}$ we obtain

$$\begin{aligned}
[L_n, a_{i,j}a^i_{m-j}] &= (-j)a_{i,n+j}a^i_{m-j} + (j-m)a_{i,j}a^i_{m+n-j} \\
[L_n, a^i_{m-j}a_{i,j}] &= (j-m)a^i_{m+n-j}a_{i,j} + (-j)a^i_{m-j}a_{i,n+j}
\end{aligned}$$

Plugging in this result in the original ansatz we get

$$\begin{aligned}
[L_n, a_m] &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<0}(-j)a_{i,n+j}a^i_{m-j} + (j-m)a_{i,j}a^i_{m+n-j} + \\
&+ \sum_{j\geq 0}(j-m)a^i_{m+n-j}a_{i,j} + (-j)a^i_{m-j}a_{i,n+j}
\end{aligned}$$

We are now doing an index shift:

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<n}(n-j)a_{i,j}a^i_{m+n-j} + \sum_{j<0}(j-m)a_{i,j}a^i_{m+n-j} + \\
&+ \sum_{j\geq 0}(j-m)a^i_{m+n-j}a_{i,j} + \sum_{j\geq n}(n-j)a^i_{m+n-j}a_{i,j}
\end{aligned}$$

For $n > 0$ we get:

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<0}(n-j)a_{i,j}a^i_{m+n-j} + \sum_{j<0}(j-m)a_{i,j}a^i_{m+n-j} + \\
&+ \sum_{j\geq 0}(j-m)a^i_{m+n-j}a_{i,j} + \sum_{j\geq 0}(n-j)a^i_{m+n-j}a_{i,j} + \\
&+ \sum_{j=0}^{n}(n-j)a_{i,j}a^i_{m+n-j} - (n-j)a^i_{m+n-j}a_{i,j}
\end{aligned}$$

By arranging the terms we can write:

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<0}(n-j)a_{i,j}a^i_{m+n-j} + \sum_{j\geq 0}(n-j)a^i_{m+n-j}a_{i,j} + \\
&+ \sum_{j<0}(j-m)a_{i,j}a^i_{m+n-j} + \sum_{j\geq 0}(j-m)a^i_{m+n-j}a_{i,j} + \\
&+ \sum_{j=0}^{n}(n-j)a_{i,j}a^i_{m+n-j} - (n-j)a^i_{m+n-j}a_{i,j} = \\
&= \frac{1}{2}\sum_{i=1}^{d}(n-j) : a_i(z)a^i(z) :_{n+m} + (j-m) : a_i(z)a^i(z) :_{n+m} \\
&+ \sum_{j=0}^{n}(n-j)a_{i,j}a^i_{m+n-j} - (n-j)a^i_{m+n-j}a_{i,j} \\
&= \frac{1}{2}\sum_{i=1}^{d}(n-m) : a_i(z)a^i(z) :_{n+m} \\
&+ \sum_{j=0}^{n}(n-j)[a_{i,j}, a^i_{m+n-j}]
\end{aligned}$$

We know that

$$(n-j)[a_{i,j}, a^i_{m+n-j}] = (n-j)(a_i|a^i)j\delta_{j,m+n-j} = (n-j)j(a_i|a^i)\delta_{m,-n} \quad \text{(B.8)}$$

Therefore

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{2}\sum_{i=1}^{d}(n-m) : a_i(z)a^i(z) :_{n+m} + \\
&+ \frac{1}{2}\sum_{i=1}^{d}\delta_{m,-n}\sum_{j=0}^{n}(n-j)j \\
&= (n-m)L_{n+m} + \frac{1}{2}\sum_{i=1}^{d}(a_i|a^i)\delta_{m,-n}\underbrace{\sum_{j=0}^{n}(n-j)j} \\
&= (n-m)L_{n+m} + \frac{1}{2}\sum_{i=1}^{d}(a_i|a^i)\delta_{m,-n}\cdot\frac{n^3-n}{6} \\
&= (n-m)L_{n+m} + \frac{d(n^3-n)}{12}\delta_{m,-n}
\end{aligned}$$

We only considered the case $n > 0$, now we turn briefly to the case $n < 0$ which is in principle analogous:

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j<0}(n-j)a_{i,j}a^i_{m+n-j} + \sum_{j\geq 0}(n-j)a^i_{m+n-j}a_{i,j} + \\
&+ \sum_{j<0}(j-m)a_{i,j}a^i_{m+n-j} + \sum_{j\geq 0}(j-m)a^i_{m+n-j}a_{i,j} + \\
&+ \sum_{j=0}^{n}(n-j)a^i_{m+n-j}a_{i,j} - (n-j)a_{i,j}a^i_{m+n-j}
\end{aligned}$$

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=-1}^{-n}(n-j)[a^i_{m+n-j}, a_{i,j}] \\
&= (n-m)L_{n+m} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=-1}^{-n}(n-j)(a^i|a_i)(m+n-j)\delta_{m+n-j,j} \\
&= (n-m)L_{n+m} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=0}^{n}(n-j)j(a^i|a_i)\delta_{m,-n} \\
&= (n-m)L_{n+m} + \frac{1}{2}\sum_{i=1}^{d}(a^i|a_i)\delta_{m,-n}\sum_{j=0}^{n}(n-j)j
\end{aligned}$$

□

B.1.2 Second Proof (by Wick's Theorem)

In this proof we we will use the following fact:
Wick's theorem for commuting fields, especially:

$$: a(z)b(z) :: c(w)d(w) :=$$

$$= [a(z)_-, c(w)][b(z)_-, d(w)] + [a(z)_-, c(w)][b(z)_-, d(w)] + [a(z)_-, c(w)] : b(z)d(w) : +$$

$$+ [a(z)_-, d(w)] : b(z)c(w) : + [b(z)_-, c(w)] : a(z)d(w) : + [b(z)_-, d(w)] : a(z)c(w) : +$$

$$+ : a(z)b(z)c(w)d(w) :$$

Now we turn to the computation:

$$\left(\frac{1}{2k}\sum_{i=1}^{d} : a^i(z)a^i(z) :\right)\left(\frac{1}{2k}\sum_{j=1}^{d} : a^j(z)a^j(z) :\right) =$$

$$= \frac{1}{4k^2}\sum_{i,j=1}^{d} \big(: a^i(z)a^i(z) :\big)\big(: a^i(z)a^i(z) :\big) =$$

$$= \frac{1}{4k^2}\sum_{i,j=1}^{d} \big([a^i(z)_-, a^j(w)][a^i(z)_-, a^j(w)] + [a^i(z)_-, a^j(w)][a^i(z)_-, a^j(w)] +$$

$$+ [a^i(z)_-, a^j(w)] : a^i(z)a^j(w) : + [a^i(z)_-, a^j(w)] : a^i(z)a^j(w) : +$$

$$+ [a^i(z)_-, a^j(w)] : a^i(z)a^j(w) : + [a^i(z)_-, a^j(w)] : a^i(z)a^j(w) : +$$

$$+ : a^i(z)a^i(z)a^j(w)a^j(w) :\big) =$$

$$\frac{1}{4k^2}\sum_{i,j=1}^{d}\left(\frac{2k^2\delta_{i,j}}{(z-w)^4} + \frac{4k\delta_{i,j}}{(z-w)^2} : a^i(z)a^j(w) : + : a^i(z)a^i(z)a^j(w)a^j(w) :\right) =$$

$$= \frac{\frac{d}{2}}{(z-w)^4} + \sum_{i}^{d} \frac{1}{k} : a^i(z)a^i(w) : + : a^i(z)a^i(z)a^j(w)a^j(w) :=$$

$$= \frac{\frac{d}{2}}{(z-w)^4} + \sum_{i}^{d} \frac{1}{k(z-w)^2} : a^i(w)a^i(w) : + \sum_{i}^{d} \frac{1}{k(z-w)^2} : \partial_w a^i(w)a^i(w) : (z-w) + \ldots$$

$$+ : a^i(z)a^i(z)a^j(w)a^j(w) :=$$

$$=$$

$$\frac{\frac{d}{2}}{(z-w)^4} + \frac{1}{2k(z-w)^2}L(w) + \frac{1}{k(z-w)^2}\partial_w L(w) + \ldots$$

□

B.2 Affine Kac-Moody Algebras

B.2.1 Definition (Casimir Operator) *The Casimir operator is defined as an element of $U(\mathfrak{g})$ by*

$$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} u_i u^i.$$

The Casimir operator is independent from the choice of the basis.

B.2.2 Proposition *The dual Coxeter number is the half eigenvalue of the Casimir operator:*

$$\sum_{i=1}^{\dim \mathfrak{g}} [u_i[u^i, x]] = 2h^\vee x.$$

B.2.3 Proposition *Let $\mathfrak{g}$ a finite dimensional simple Lie algebra, and let be $h^\vee$ its dual Coxeter number. Let be $\{u_i, u^i\}_{i=1,..,d}$ a dual basis of $\mathfrak{g}$ with respect to the Killing form. Let be*

$$L(z) = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{d} : u_i(z) u^i(z) :$$

Then we have the equivalent relations:

$$\begin{aligned} [L_n, a_m] &= (-m) a_{n+m} && \text{(B.9)} \\ [L(z)a(w)] &= a(w)\partial_w \delta(z-w) && \text{(B.10)} \\ L(z)a(w) &= \frac{a(w)}{(z-w)^2} + : L(z)a(w) : && \text{(B.11)} \end{aligned}$$

The equivalence is clear (see also the discussion in the previous subsection). The proof is almost the same as in the abelian case.

B.2.4 Theorem *Let $\mathfrak{g}$ a finite dimensional simple Lie algebra, and let be $h^\vee$, u_i, u^i as in the above proposition. Let be*

$$L(z) = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{d} : u_i(z) u^i(z) :$$

Then we have the equivalent relations:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{d}{2(k+h^\vee)} \frac{1}{12}(n^3 - n)\delta_{n,-m}, \\ [L(z), L(w)] &= \frac{d}{2(k+h^\vee)} \partial^{(3)}\delta(z-w) + 2L(w)\partial\delta(z-w) + \partial L(w)\delta(z-w), \\ L(z)L(w) &= \frac{\frac{d}{2(k+h^\vee)}}{(z-w)^4} + \frac{L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w} + : L(z)L(w) : . \end{aligned}$$

Proof of the proposition: The proof is analogous to the abelian case. The difference begins with equation (B.7). We start with the identity

$$\begin{aligned}
[L_n, a_m] &= [\frac{1}{2(k+h^\vee)}\sum_{i=1}^{d} :u_i(z)u^i(z):_n, a_m] = \\
&= \frac{1}{2(k+h^\vee)}\sum_{i=1}^{d}\sum_{j<0} u_{i,j}[u^i_{n-j}, a_m] + [u_{i,j}, a_m]u^i_{n-j} + \\
&+ \sum_{j\geq 0} u^i_{n-j}[u_{i,j}, a_m] + [u^i_{n-j}, a_m]u_{i,j}.
\end{aligned}$$

We have the relations:

$$[u_{i,n}, a_m] = [u_i, a]_{n+m} + (u_i|a)nk\delta_{n,-m},$$
$$[u^i_n, a_m] = [u^i, a]_{n+m} + (u^i|a)nk\delta_{n,-m}.$$

Therefore we obtain:

$$\begin{aligned}
[L_n, a_m] &= \frac{1}{2(k+h^\vee)}\sum_{i=1}^{d}\sum_{j<0} u_{i,j}\left([u^i, a]_{n-j+m} + (u^i|a)(n-j)k\delta_{n-j,-m}\right) + \\
&+ \left([u_i, a]_{j+m} + (u_i|a)jk\delta_{j,-m}\right)u^i_{n-j} + \\
&+ \sum_{j\geq 0} u^i n - j\,() + \\
&+ ()
\end{aligned}$$

Proof of the theorem

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{2(k+h^\vee)}\sum_{i=1}^{d}\sum_{j<n}(n-j)u_{i,j}u^i_{m+n-j} + \sum_{j<0}(j-m)u_{i,j}u^i_{m+n-j} + \\
&+ \sum_{j\geq 0}(j-m)u^i_{m+n-j}u_{i,j} + \sum_{j\geq n}(n-j)u^i_{m+n-j}u_{i,j}.
\end{aligned}$$

Let be $n > 0$, then we obtain

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{2(k+h^\vee)}\sum_{i=1}^{d}(n-m):u_i(z)u^i(z):_{m+n} + \\
&+ \sum_{j=0}^{n}(n-j)[u^i_{m+n-j}, a_{i,j}] = \\
&= (n-m)L_{n+m} + \sum_{i=1}^{d}\sum_{j=0}^{n}(n-j)[u^i, u_i]_{m+n} + (n-j)jk(u^i, u_i)\delta_{n,m} = \\
&= (n-m)L_{n+m} + 2h^\vee\sum_{j=0}^{n}(n-j) + (n-j)jk(u^i, u_i)\delta_{n,m}
\end{aligned}$$

□

Appendix C

Higher Genus Generalizations

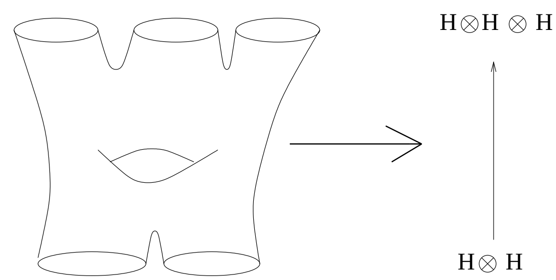
Conformal field theory is conceptually a functor from the category of parametrized circles which are interpolated by Riemann surfaces (the morphisms of this category) to the category of vector spaces. This is essentially the content of the Segal-Kontsevich axioms for conformal field theory.
Under conformal field theory we can understand the association of a finite dimensional vector space to a pointed nodal projective curve over $\mathbb{C}$ whose marked points are labeled by elements of a certain finite set. The associated finite dimensional vector space is called the **space of conformal blocks** or **space of covacua**.
In the groundbreaking paper [TUY] **Conformal field theory on universal family of stable curves with gauge symmetries** Tsuchiya, Ueno and Yamada (for short: TUY) started in principle from this "naive" ansatz and developed a theory that gives rise to a proof of the celebrated Verlinde formula (see below). On each of these points there is a representation of an affine (non-twisted) Kac-Moody algebra attached.
The normalization of one of the nodal curves on one of the nodes enables them to find factorization rules that give rise to the famous Verlinde formula.

Frenkel and Ben-Zvi consider in their book "Vertex Algebras and Algebraic Curves" sheaves of vertex algebras on algebraic curves. They follow essentially the ideas of TUY.
Unfortunately they cannot give factorization rules because their algebraic curves have no nodes. Roughly speaking they are not able to go the boundary of the moduli space of curves.

In a very influential paper **Geometric realization of conformal field theory on Riemann surfaces** Kawamoto, Namikawa, Tsuchiya and Yamada (for short: KNTY) considered the Universal Grassmannian due to Sato in order to embed all moduli spaces of curves in this Grassmannian. This approach is important from the point of view of KP hierarchy.

C.1 Basic Philosophy

The basic philosophy consists in the following consideration: Start naively from a category whose objects are parametrized circles, and whose morphisms are Riemann surfaces which interpolate the circles. A conformal field theory is now a functor from this category to the category of infinite dimensional Hilbert spaces. The circles are mapped to vector spaces, and the Riemann surfaces are mapped to homomorphisms.



This is essentially the content of the Segal axioms [S]. Recently Hu and Kriz [HK] rigorized this ansatz.
From the Segal Axioms we can see that conformal field theory is a topological quantum field theory in the sense of Atiyah [At] but with additional properties which reflect the conformal structure of the Riemann surfaces. Hence we have to take into account the different conformal structures on a Riemann surface.

C.2 TUY and Frenkel-Ben Zvi

Tsuchiya, Ueno and Yamada associate in [TUY] to a pointed nodal curve a finite dimensional vector space. This space is called the space of conformal blocks. They proved the Verlinde formula by using so called factorization rules. Faltings writes in [Fal] that he had been skeptical about their proof of factorization, but "it is entirely correct".
The Verlinde formula is a formula which gives the dimension of the space of sections in certain line bundles (namely powers of the generalized theta divisor) on the moduli space of Riemann surfaces.
Frenkel and Ben Zvi [FB-Z] define bundles of (conformal) vertex algebras. They also construct a space of conformal blocks. Their idea is quite similar to the work of Tsuchiya, Ueno and Yamada.
Recently Schlichenmaier and Sheinman constructed in [SchSh1] and [SchSh2] spaces of conformal blocks by using a global operator approach.

C.3 Chiral Algebras

First we introduce some notation.
Let X be a smooth curve. Let M, N denote sheaves on X.
Let $M \boxtimes N(\infty\Delta)$ denote the sheaf on $X \times X$ whose sections are sections of $M \boxtimes N$ with arbitrary poles on the diagonal:

$$M \boxtimes N(\infty\Delta) = \lim_{\rightarrow} M \boxtimes N(n\Delta).$$

Define

$$\Delta_! M = \frac{K \boxtimes M(\infty\Delta)}{K \boxtimes M}.$$

A Lie algebra is a K-vector space $\mathfrak{g}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions:

1. (skew-symmetry) $[a, b] = -[b, a]$
2. (Jacobi identity) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

With this definition in mind we introduce the notion of a chiral algebra.

C.3.1 Definition *Let X be a Riemann surface.*
*A **chiral algebra** on X is a right $\mathcal{D}$ module $\mathcal{A}$, equipped with a $\mathcal{D}$-module homomorphism*

$$\mu : \mathcal{A} \boxtimes \mathcal{A}(\infty\Delta) \to \Delta_! \mathcal{A}$$

on X^2, satisfying the following conditions:

- *(skew-symmetry) $\mu = -\sigma_{12} \circ \mu \circ \sigma_{12}$.*
- *(Jacobi identity) $\mu_{1\{23\}} = \mu_{\{12\}3} + \mu_{2\{13\}}$.*
- *(unit) We are given an embedding (unit map) $K \hookrightarrow \mathcal{A}$ (K is the canonical sheaf of X) compatible with the homomorphism $\mu_K : K \boxtimes K \to \Delta_! K$. That is, the following diagram is commutative*

$$\begin{array}{ccc} K \boxtimes \mathcal{A}(\infty\Delta) & \to & \mathcal{A} \boxtimes \mathcal{A}(\infty\Delta) \\ \downarrow & & \downarrow \\ \Delta_! \mathcal{A} & \to & \Delta_! \mathcal{A} \end{array}$$

C.3.2 Lemma *Let K be the canonical bundle on a compact Riemann surface X. Denote by μ_K the composition of the maps*

$$K \boxtimes \Omega(\infty\Delta) \to \omega_{X^2}(\infty\Delta) \to \Delta_! \Omega$$

The first map is an isomorphism. The map μ_Ω is a morphism of right $\mathcal{D}$-modules. Ω equipped with μ_Ω is a chiral algebra.

Frenkel and Ben Zvi showed in [FB-Z] (Theorem 18.3.3) that their bundles of vertex algebras carry the structure of a chiral algebra.

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