Field Theoretical Models on Non-Commutative Spaces

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HELP ME WITH THIS HOMEWORK, OK? WHAT'S 6+3?
6+3, EH? WELL, THIS ONE IS A BIT TRICKY.
FIRST WE CALL THE ANSWER "Y," AS IN "Y DO WE CARE?" NOW Y MAY BE A SQUARE NUMBER, SO WE'LL DRAW A SQUARE AND MAKE THIS SIDE 6 AND THAT SIDE 3. THEN WE'LL MEASURE THE DIAGONAL.
I DON'T REMEMBER THE TEACHER EXPLAINING IT LIKE THIS.
SHE PROBABLY DOESN'T KNOW HIGHER MATH. WHEN YOU DEAL WITH HIGH NUMBERS, YOU NEED HIGHER MATH.
BUT THIS DIAGONAL IS JUST A LITTLE UNDER TWO.
OK, HERE, I'LL DRAW A BIGGER SQUARE.

B

Zusammenfassung

Aus vielen voneinander unabhängigen Überlegungen wird klar, daß die Raum-Zeit im Kleinen, oder mit sehr großen Energien betrachtet, in irgendeiner Form nichtkommutativ oder quantisiert sein muss. Diese Arbeit beschäftigt sich mit zwei verschiedenen Arten der Nichtkommutativität der Raum-Zeit und mit eichtheoretischen Modellen auf solchen Räumen. Wir werden die nichtkommutativen Konzepte via eines Isomorphismus auf kommutative Räume übertragen. Die Information über die nichtkommutative Struktur versteckt sich in einem neuen nichtabelschen Produkt, dem sogenannten $*$-Produkt. Das Sternprodukt ist gegeben durch eine störungstheoretische Formel. Daher ist der kommutative Limes, in dem die Nichtkommutativität verschwindet, und die gewohnten Strukturen zurückkehren, sehr gut erkennbar.

Wir betrachten also die Konstruktion des Sternproduktes als ersten Schritt in Richtung feldtheoretischer Modelle auf einem nichtkommutativen Raum. So werden im ersten Teil die Sternprodukte für den 4-dimensionalen q-deformierten Euklidischen und Minkowski Raum in Normalordnung berechnet. Hierfür können wir geschlossene Ausdrücke angeben. Allerdings werden q-deformierte Räume in dieser Arbeit nicht weiter verfolgt. Stattdessen werden wir uns mit kanonisch deformierten und κ-deformierten Räumen befassen. Kanonisch deformierte Räume haben den Nachteil, dass die klassischen Symmetrien gebrochen sind. Dagegen erlauben sowohl q- als auch κ-deformierte Räume verallgemeinerte Symmetriestrukturen. Die Symmetrien werden durch Quantengruppen beschrieben.

Rechnerisch sind kanonische Strukturen leichter handzuhaben. Wir werden das Standardmodell der Elemetarteilchenphysik auf kanonischer Raum-Zeit formulieren. Dabei legen wir großen Wert darauf, zu zeigen, dass sowohl der Higgs Mechanismus als auch der Yukawa Sektor im nichtkommutativen Modell implementiert werden können. Wir entwickeln die Wirkung störungstheoretisch bis zur ersten Ordnung in der Nichtkommutativität. Die zusätzlichen Terme in erster Ordnung entsprechen neuen Wechselwirkungen. Diese neuen Wechselwirkungen haben weitreichende phänomenologische Bedeutung und erlauben eine experimentelle Suche nach Anzeichen, die auf die Nichtkommutativität der Raum-Zeit hindeuten.

Darüber hinaus sind wir bemüht, auch Modelle auf κ-deformierten Räumen zu betrachten, die sowohl eine verallgemeinerte Poincaré Symmetry besitzen, als auch symmetrisch unter einer beliebigen Eichgruppe sind. Dabei legen wir der Eichtheorie die gleichen Konzepte zugrunde wie im Falle der kanonischen Raum-Zeit. Da die Strukturen vielfältiger sind, werden wir auf interessante Unterschiede stoßen. So ist das Eichfeld nicht nur ein Element der einhüllenden Algebra der Eichgruppe, sondern auch der Poincaré Gruppe. Für die Formulierung von Lagrange-Modellen fehlt allerdings im Moment noch ein invariantes Integral. Feldgleichungen können allerdings hergeleitet werden. Wir werden, auf eindeutige Weise, eine κ-Poincaré kovariante Klein-Gordon und Dirac Gleichung aufstellen. Weiters werden wir alle Ergebnisse in den $*$-Formalismus und auf kommutative Räume übersetzen.

Acknowledgement

First of all, I want to thank my "Doktorvater" Julius Wess for opening up in front of me a very interesting and rich field of research, for many fruitful blackboard discussions, for his steady support and for sharing some of his immense physical insight and intuition.

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Contents

Chapter 1

Introduction

Non-commutative spaces have a long history. Even in the early days of quantum mechanics and quantum field theory, continuous space-time and Lorentz symmetry was considered inappropriate to describe the small scale structure of the universe [1]. It was also argued that one should introduce a fundamental length scale limiting the precision of position measurements. In [2, 3] the introduction of a fundamental length is suggested to cure the ultraviolet divergencies occuring in quantum field theory. H. Snyder was the first to formulate these ideas mathematically [4]. He introduced non-commutative coordinates. Therefore a position uncertainty arises naturally. The success of the renormalisation programme made people forget about these ideas for some time. But when the quantisation of gravity was considered thoroughly, it became clear that the usual concepts of space-time are inadequate and that space-time has to be quantised or non-commutative, in some way.

In order to combine quantum theory and gravitation (geometry), one has to describe both in the same language, this is the language of algebras [5]. Geometry can be formulated algebraically in terms of abelian $\mathbf{C}^*$ algebras and can be generalised to non-abelian $\mathbf{C}^*$ algebras (non-commutative geometry). Quantised gravity may even act as a regulator of quantum field theories. This is encouraged by the fact that non-commutative geometry introduces a lower limit for the precision of position measurements. There is also a very nice argument showing that, on a classical level, the selfenergy of a point particle is regularised by the presence of gravity [6]. Let us consider an electron and a shell of radius ϵ around the electron. The selfenergy of the electron is the selfenergy of the shell $m(\epsilon)$, in the limit $\epsilon \to 0$. $m(\epsilon)$ is given by

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon},$$

where m_0 is the restmass and e the charge of the electron. In the limit $\epsilon \longrightarrow 0$, $m(\epsilon)$ will diverge. Including Newtonian gravity we have to modify this equation,

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{Gm_0^2}{\epsilon},$$

G denotes Newton's gravitational constant. $m(\epsilon)$ will still diverge for $\epsilon \longrightarrow 0$, unless the mass and the charge are fine tuned. Considering general relativity, we know that energy, therefore also the energy of the electron's electric field, is the source of a gravitational field. Again, we have to modify the above equation,

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{Gm(\epsilon)^2}{\epsilon}.$$

The solution of this quadratic equation is straight forward,

$$m(\epsilon) = -\frac{\epsilon}{2G} \pm \frac{\epsilon}{2G}\sqrt{1 + \frac{4G}{\epsilon}(m_0 + \frac{e^2}{\epsilon})}.$$

We are interested in the positive root. Miraculously, the limit $\epsilon \longrightarrow 0$ is finite,

$$m(\epsilon \longrightarrow 0) = \frac{e}{\sqrt{G}}.$$

This is a non-perturbative result, since $m(\epsilon \longrightarrow 0)$ cannot be expanded around $G = 0$. $m(\epsilon \longrightarrow 0)$ does not depend on m_0, therefore there is no fine tuning present. Classical gravity regularises the selfenergy of the electron, on a classical level. However, this does not make the quantisation of space-time unnecessary, since quantum corrections to the above picture will again introduce divergencies. But it provides an example for the regularisation of physical quantities by introducing gravity. So hope is raised that the introduction of gravity formulated in terms of non-commutative geometry will regularise physical quantities even on the quantum level.

The world of our daily perceptions is continuous. The idea behind non-commutative space-time is that at some critical energy (or distance) there is a phase transition from a continuous to a non-commutative space-time. At which energies this phase transition might take place, or at which energies non-commutative effects occur is a point much debated on. From various theories generalised to non-commutative coordinates, limits on the non-commutative scale have been derived. These generalisations have mainly considered the socalled canonical non-commutativity,

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta^{ij},$$

$\theta^{ij} = -\theta^{ji} \in \mathbb{C}$. For more details on this and other kinds of non-commutative coordinates, see Chapter 2. Let us name a few estimates of the non-commutativity scale. A very weak limit on the non-commutative scale Λ_{NC} is obtained from an additional energy loss in stars due to a coupling of the neutral neutrinos to the photon, $\gamma \to \nu\bar{\nu}$ [7]. They get

$$\Lambda_{NC} > 81\, GeV.$$

The estimate is based on the argument that any new energy loss mechanism must not exceed the standard neutrino losses from the Standard Model by much. A similar limit is obtained in [8] from the calculation of the energy levels of the hydrogen atom and the Lamb shift within non-commutative quantum electrodynamics ,

$$\Lambda_{NC} \gtrsim 10^4\, GeV.$$

If $\Lambda_{NC} = \mathcal{O}(TeV)$, measurable effects may occur for the anomalous magnetic moment of the muon which may account for the reported discrepancy between the Standard Model prediction and the measured value [9]. Also in cosmology and astrophysics non-commutative effects might be observable. One suggestion is that the modification of the dispersion relation due to ($\kappa-$)non-commutativity may explain the time delay of high energy γ rays, e.g., from the active galaxy Makarian 142 [10, 11].

As we already mentioned, we will provide an introduction to non-commutative geometry in Chapter 2. We will mainly develop the quantum group point of view. This approach has the advantage to deform not only the space, but also its symmetry structure. The canonical deformation breaks Lorentz symmetry, only a translational symmetry is left unbroken. In the quantum group case, the symmetry is deformed to a quantum group, but still present. Quantum groups depend on the deformation parameter q. In the limit $q \to 1$, we have to regain all the concepts of the undeformed world. We will use the term "classical theory" to denote ordinary theories on commutative space-time, and the term "classical fields" to denote fields (abelian or non-abelian) on commutative space-time.

In Chapter 3, we will discuss a powerful perturbative approach to non-commutativity, $*$-products. The algebra of non-commutative functions $\hat{\mathcal{A}}$,

$$\hat{\mathcal{A}} = \frac{\langle\langle \hat{x}^1, \ldots, \hat{x}^n \rangle\rangle}{\mathcal{I}},$$

the free algebra generated by non-commutative coordinates divided by an Ideal generated by the relations

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta^{ij}(\hat{x}),$$

is mapped to the commutative algebra of functions $\mathcal{A}$. The non-commutativity is now hidden in a new non-abelian multiplication

$$f * g = \sum_{n=0}^{\infty} h^n B_n(f,g),$$

where h is a formal deformation parameter ($e^h = q$, for quantum groups). $B_0(f,g) = fg$ is the commutative limit, $h \to 0$. The $*$-products for the 4-dimensional quantum Euclidean and Minkowski space will be presented. These results were obtained together with Hartmut Wachter [12].

In Chapter 4 and 5, we will concern ourselves with the canonical space-time structure. The construction of gauge theories on non-commutative spaces [13,14] will be revised in Chapter 4. These ideas will be crucial in Chapter 5 where we will formulate the Standard Model of theoretical particle physics on canonical space-time. Some phenomenological and experimental implications will also be discussed. These results were obtained in collaboration with Xavier Calmet, Branislav Jurčo, Peter Schupp and Julius Wess [15].

A special quantum deformation will be under scrutiny in Chapter 6, κ-deformation [16, 17]. First of all, we will study the algebra of κ-Euclidean space and κ-deformed rotation algebra very carefully, construct invariants and wave equations. The most important result is the generalisation of the ideas developed in Chapter 4 to spaces symmetric under a quantum group. So we will develop a model symmetric under both, κ-Poincaré (rotation) symmetry and (arbitrary) gauge symmetry. So far, only scalar field theory has been considered, and no gauge theory has been tackled on κ-deformed spaces. The work is still in progress and is conducted in collaboration with Marija Dimitrijević, Lutz Möller, Efrossini Tschouchnika and Julius Wess [18].

Essentially, this work consists of two parts, The first part is concerned with the calculation of $*$-products of the $q-$deformed Euclidean and Minkowski space. This is considered as a first step towards field theoretical models on q-deformed spaces, formulated within the $*$-product approach. Further steps have been conducted in [19, 20]. Gauge theoretical models are at the heart of the second part. Models symmetric under gauge transformations are constructed on canonical and κ-deformed space-time.
It is the aim to understand non-commutative spaces, especially spaces that allow for a deformed symmetry, such that we are enabled to construct realistic particle models and to predict some observable effects.

Chapter 2

Non-Commutative Geometry

Let us try to present some handwaving idea what picture we have in mind when we talk about non-commutative geometry. Some examples will show, where non-commutativity has already shown up in physics, and how these ideas might be useful. After all these motivations, we will formulate some aspects of non-commutative geometry mathematically. We will mainly be concerned with quantum groups and quantum spaces.

Non-commutative geometry is based on non-commuting coordinates

$$[\hat{x}^i, \hat{x}^j] \neq 0, \tag{2.1}$$

i.e., coordinates are non-commutative operators and we have to think in quantum mechanical terms. The $\hat{x}^i$'s cannot all be diagonalised simultaneously. Space-time is the collection of the eigenvalues (spectrum) of the operators $\hat{x}^i$. If the spectrum is discrete, space-time will be discrete. Commutative coordinates induce a continuous spectrum. Therefore, also space-time will be continuous.
The theory of non-commutative geometry is based on the simple idea of replacing ordinary coordinates with non-commuting operators. We will see how this idea can be formulated mathematically.

2.1 Physical Motivation for Non-Commuting Coordinates

But before we do so, let us consider some examples.

2.1.1 Divergencies in QFT

In quantum field theories, loop contributions to the transition amplitudes diverge. Consider a real scalar particle ϕ which is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{2.2}$$

The contribution of the diagram shown in Fig. 2.1 reads

$$\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m^2}. \tag{2.3}$$

The result is divergent. There may be other divergencies as well, and the renormalisation procedure may remove some of these infinities. The theory is called renormalisable, if all

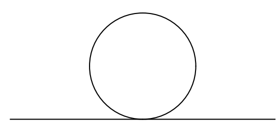

Figure 2.1: Loop contribution

divergencies can be removed with a finite number of counter terms. The theory defined by (2.2) is renormalisable, in four dimensions. However, no renormalisable quantum field theory of gravity is known so far. Discretising space-time may introduce a momentum cut-off in a canonical way and render a theory finite or at least renormalisable. The hope is that this can be accomplished by making space-time non-commutative.

2.1.2 Quantum Gravity

All kind of models for and approaches to quantum gravity seem to lead to a fundamental length scale, i.e., to a lower bound to any position measurement [21]. This seems to be a model independent feature. The uncertainty in space-time measurements can be explained by replacing coordinates by non-commutative operators.

2.1.3 String Theory

In open string theory with a constant background B-field, the endpoints of the strings are confined to submanifolds (D-branes) and become non-commutative [22]. This is true even on an operator level,

$$[X^i, X^j] = i\theta^{ij}, \tag{2.4}$$

where $\theta^{ij} = -\theta^{ji} \in \mathbb{R}$, and X^i are the coordinates of the 2-dimensional world sheet embedded in the target space (e.g., $\mathbb{R}^{10}$), i.e., operator valued bosonic fields. Therefore, we also have for the propagator

$$\langle\, [X^i, X^j]\, \rangle = i\theta^{ij}. \tag{2.5}$$

2.1.4 Classical Non-Commuting Coordinates

Consider a particle with charge e moving in a homogeneous and constant magnetic field. The action is given by

$$S = \int dt \left(\frac{1}{2} m \dot{x}_\mu \dot{x}^\mu - \frac{e}{c} B_{\mu\nu} x^\mu \dot{x}^\nu \right), \tag{2.6}$$

where $B_{\mu\nu}$ is an antisymmetric tensor defining the vector potential A_μ, $B_{\mu\nu} = -B_{\nu\mu}$ and $A_\nu = B_{\mu\nu} x^\mu$. The classical commutation relations are

$$\{\pi_\mu, x^\nu\} = \delta^\nu_\mu, \tag{2.7}$$

where $\{\, ,\, \}$ is the classical Poisson structure. Writing it out explicitly, we get

$$\{\dot{x}_\mu, x^\nu\} + \frac{eB_{\mu\sigma}}{c\, m} \{x^\sigma, x^\nu\} = \frac{1}{m} \delta^\mu_\nu, \tag{2.8}$$

where $\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m\dot{x}_\mu + \frac{e}{c} B_{\mu\nu} x^\nu$. Let us assume strong magnetic field B and small mass m - i.e., we restrict the particle to the lowest Landau level [23]. In this approximation, eqn. (2.8) simplifies, and we get [24]

$$\{x^\sigma, x^\nu\} = \frac{c\,(B^{-1})^{\sigma\nu}}{e}. \tag{2.9}$$

The coordinates perpendicular to the magnetic field do not commute, on a classical level.

2.2 Systematic Approach

Let us examine the classical situation depicted in Fig. 2.2. We start with a smooth and compact manifold $\mathcal{M}$. The topology of $\mathcal{M}$ is uniquely determined by the algebra of continuous complex (real) valued functions on $\mathcal{M}$, $C(\mathcal{M})$ with the usual involution

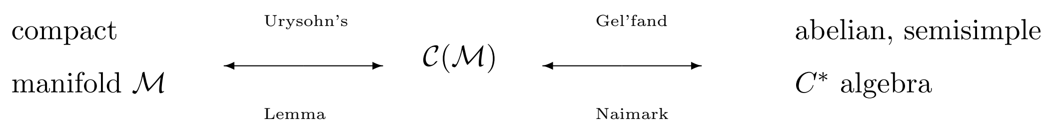

Figure 2.2: Classical algebraic geometry

(Urysohn's Lemma [25]). The Gel'fand-Naimark theorem [26] relates the function algebra to an abelian C^*-algebra. The algebra of continuous functions over a compact manifold $\mathcal{M}$ is isomorphic to an abelian unital C^*-algebra. The algebra of continuous functions vanishing at infinity over a locally compact Hausdorff space $C^0(\mathcal{M})$ is isomorphic to an abelian C^*-algebra (not necessarily unital).
Coordinates on the manifold are replaced by coordinate functions in $C(\mathcal{M})$, vector fields by derivations of the algebra. Points are replaced by maximal ideals, cf. Fig. 2.3.

points	$\rightarrow$	maximal ideals
$x \in \mathcal{M}$	$\mapsto$	ideal $J = \{f \vert f(x) = 0\} \subset C(\mathcal{M})$
coordinates	$\rightarrow$	coordinate functions
vector field	$\rightarrow$	derivations of the algebra
geometry	$\rightarrow$	algebraic geometry

Figure 2.3: Algebraic geometry

The trick in non-commutative geometry is to replace the abelian C^* algebra by a non-abelian one and to reformulate as much of the concepts of algebraic geometry as possible in terms of non-abelian C^* algebras [27, 28].
In the following, this non-commutative algebra $\mathcal{A}$ will be given by the algebra of formal power series generated by the non-commutative coordinate functions $\hat{x}^i$, divided by an ideal $\mathcal{I}$ of relations generated by the commutator of the coordinate functions,

$$\mathcal{A} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ...\hat{x}^n \rangle\rangle}{\mathcal{I}}, \tag{2.10}$$

where $[\hat{x}^i, \hat{x}^j] \neq 0$. Most commonly, the commutation relations are chosen to be either constant or linear or quadratic in the generators. In the canonical case the relations are constant,

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{2.11}$$

where $\theta^{ij} \in \mathbb{C}$ is an antisymmetric matrix, $\theta^{ij} = -\theta^{ji}$. The linear or Lie algebra case

$$[\hat{x}^i, \hat{x}^j] = i\lambda^{ij}_k \hat{x}^k, \tag{2.12}$$

where $\lambda^{ij}_k \in \mathbb{C}$ are the structure constants, basically has been discussed in two different approaches, fuzzy spheres [29] and κ-deformation [16,17,18]. Last but not least, we have quadratic commutation relations

$$[\hat{x}^i, \hat{x}^j] = (\frac{1}{q}\widehat{R}^{ij}_{kl} - \delta^i_l \delta^j_k)\hat{x}^k \hat{x}^l, \tag{2.13}$$

where $\widehat{R}^{ij}_{kl} \in \mathbb{C}$ is the so-called $\widehat{R}$-matrix which will be discussed in some detail in Subsection 2.2.1, corresponding to quantum groups. For a reference, see e.g., [30, 31].

In Chapter 5, we will deal with the Standard Model on canonical space-time, and Chapter 6 will discuss wave equations and gauge theory in the framework of κ-deformation. We will consider canonical spaces as an approximation in some sense, to quantum spaces with a quantum group as its underlying symmetry. The advantage of quantum spaces is that the concept of symmetry can be generalised to quantum groups. Whereas canonical space-time does not allow for a generalised Lorentz symmetry. Let us now discuss one specific approach to non-commutative geometry in some more detail, namely quantum groups.

2.2.1 q-Deformed Case

Classically, symmetries are described by Lie algebras or Lie groups. Physical spaces are representation spaces of its symmetry algebra - or respectively co-representations of the function algebra over its symmetry group. Therefore, we will introduce non-commutative spaces as representation spaces of some quantum algebra. The interpretation has already been discussed in Chapter 1: Space-time is a continuum in the low energy domain, at high energies - Planck energy or below - space-time undergoes a phase transition and becomes a "fuzzy" non-commutative space. Therefore the symmetries are not broken, but deformed to a quantum group.

What is a quantum group? Let us start with the function algebra over a classical Lie group $\mathcal{F}(\mathcal{G})$. $\mathcal{F}(\mathcal{G})$ is a Hopf algebra whose structure will be defined later in this Section. Then there is a well defined transition from the classical function algebra to the respective quantum group, $\mathcal{F}(\mathcal{G}) \to \mathcal{F}(\mathcal{G})_q$, introducing the non-commutativity parameter $q \in \mathbb{C}$. In the classical limit, $q \to 1$, we have to regain the classical situation. This is the basic property of a deformation.

As we mentioned before, the enveloping algebra of a Lie algebra and the function algebra over a Lie group are in a natural way Hopf algebras. Most importantly, q-deformation does not lead out of the category of Hopf algebras.

Hopf algebra

A Hopfalgebra A (see e.g., [32]) consists of an algebra and a co-algebra structure which are compatible with each other. Additionally, there is a map called antipode, which corresponds to the inverse of a group. A is an algebra, i.e., there is a multiplication m and a unit element η,

$$\begin{aligned} m : A \otimes A &\to A, \\ a \otimes b &\mapsto ab, \\ \eta : \mathbb{C} &\to A, \\ c &\mapsto c\,\mathbf{1}_A, \end{aligned}$$

such that the multiplication satisfies the associativity axiom (Fig. 2.4) and η satisfies

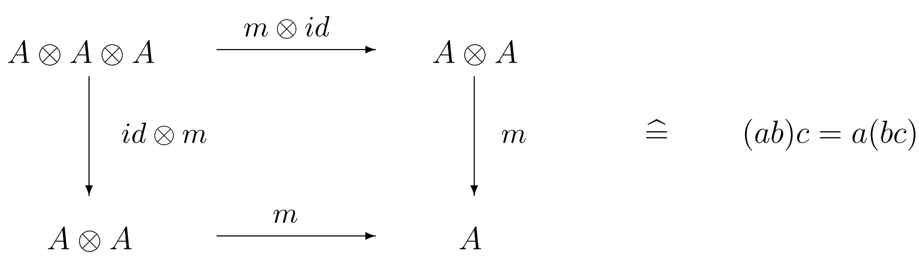

Figure 2.4: Associativity

the axiom depicted in Fig. 2.5.
Reversing all the arrows in Figs. 2.4 and 2.5 and replacing m by the co-product Δ and η by the co-unit ϵ gives us the axioms for the structure maps of a co-algebra. The co-product and the co-unit,

$$\begin{aligned} \Delta : A &\to A \otimes A, \\ \epsilon : A &\to \mathbb{C} \end{aligned}$$

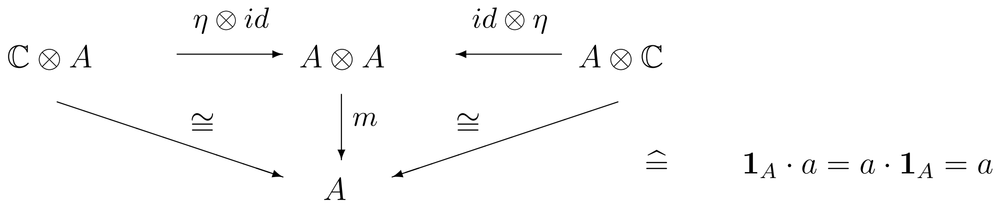

Figure 2.5: Unity axiom

are the dual to m and η, respectively. Compatibility between algebra and co-algebra structure means that the co-product Δ and the co-unit ϵ are algebra homomorphisms,

$$\Delta(ab) = \Delta(a)\Delta(b), \tag{2.14}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \tag{2.15}$$

where $a, b \in A$. The antipode $S : A \rightarrow A$ satisfies the axiom shown in Fig. 2.6 below. It is an anti-algebra homomorphism.

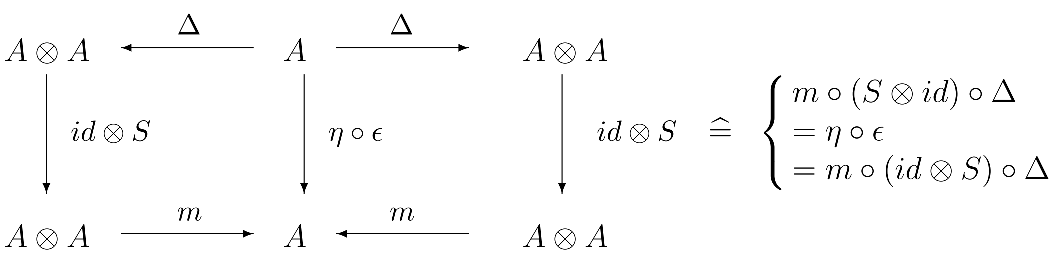

Figure 2.6: Antipode axiom

If A is the algebra of functions over some matrix group, the antipode S is the inverse,

$$S(t^i_j) = (t^{-1})^i_j, \tag{2.16}$$

where t^i_j are the coordinate functions and generate the algebra of functions.

Let me quote the structure maps for the function algebra and its dual. Let $\mathcal{G}$ be an arbitrary, (for simplicity) finite group and $\mathcal{F}(\mathcal{G})$ the Hopf algebra of all complex-valued functions on $\mathcal{G}$. Then the algebra structure of $\mathcal{F}(\mathcal{G})$ is given by

$$\begin{aligned} m : \mathcal{F}(\mathcal{G}) \otimes \mathcal{F}(\mathcal{G}) &\rightarrow \mathcal{F}(\mathcal{G}), \\ m(f_1 \otimes f_2)(g) &= f_1(g) f_2(g), \end{aligned} \tag{2.17}$$

$$\eta : \mathbb{C} \to \mathcal{F}(\mathcal{G}), \qquad \eta(k) = k\,\mathbf{1}_{\mathcal{F}(\mathcal{G})}. \tag{2.18}$$

And we have the following co-algebra structure

$$\Delta : \mathcal{F}(\mathcal{G}) \to \mathcal{F}(\mathcal{G}) \otimes \mathcal{F}(\mathcal{G}), \qquad \Delta(f)(g_1 \otimes g_2) = f(g_1 g_2), \tag{2.19}$$

$$\epsilon : \mathcal{F}(\mathcal{G}) \to \mathbb{C}, \qquad \epsilon(f) = f(e), \tag{2.20}$$

where e is the unit element of $\mathcal{G}$. Eventually, the antipode is given by

$$(S(f))(g) = f(g^{-1}). \tag{2.21}$$

$\mathcal{F}(\mathcal{G})$ is a commutative Hopf algebra.

Let us consider its dual. Let $\mathbf{g}$ be a Lie algebra. The universal enveloping algebra is defined as

$$\mathcal{U}(\mathbf{g}) = \frac{T(\mathbf{g})}{x \otimes y - y \otimes x - [x, y]}, \tag{2.22}$$

where $T(g)$ is the universal tensor algebra. Its algebra structure is given by the commutator and its unit element by the unit in $T(\mathbf{g})$. The other structure maps are consistently defined as

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x, \tag{2.23}$$

$$\epsilon(x) = 0, \tag{2.24}$$

$$S(x) = -x. \tag{2.25}$$

$\mathcal{U}(g)$ is a co-commutative Hopf algebra, i.e., the co-product is symmetric.

Quantum group

A quantum group is a Hopf algebra with one additional structure. Let us concentrate on the function algebra $\mathcal{F}(\mathcal{G})$ over some Lie group $\mathcal{G}$ rather than on its dual, the universal enveloping algebra of the Lie algebra. The additional structure is the R-form,

$$R : A \otimes A \to \mathbb{C}.$$

$\mathcal{F}(\mathcal{G})$ is a commutative algebra, the R-form describes the almost commutativity of the product in the deformed algebra. Let us denote this quantum group by $\mathcal{F}(\mathcal{G})_q$, since

R depends on the non-commutativity parameter q. Let t^i_j be the coordinate functions generating $\mathcal{F}(\mathcal{G})_q$. The generators satisfy the so-called RTT-relations expressing the almost commutativity

$$R^{ij}_{kl}\, t^k_m\, t^l_n = t^j_l\, t^i_k\, R^{kl}_{mn}, \tag{2.26}$$

where $R(t^i_k \otimes t^j_l) = R^{ij}_{kl}$. In the commutative limit, $R \to \mathbf{1}$, we have commutative generators,

$$t^k_m\, t^l_n = t^l_n\, t^k_m. \tag{2.27}$$

R is a solution of the Quantum-Yang-Baxter-Equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.28}$$

where $(R_{13})^{ijk}_{lmn} = \delta^j_m R^{ik}_{ln}$, R_{12} and R_{23} are defined accordingly.

Quantum Spaces, $\mathcal{M}_q$

A quantum space for a quantum group $\mathcal{F}(\mathcal{G})_q$ has two basic properties. $\mathcal{M}_q$ is a $\mathcal{F}(\mathcal{G})_q-$co-module algebra and in the commutative limit, $q \to 1$, $\mathcal{M}$ is the proper $\mathcal{F}(\mathcal{G})-$co-module space.

$$\mathcal{M}_q \equiv \mathbb{C}\langle\langle \hat{x}^1, \ldots, \hat{x}^n \rangle\rangle / \mathcal{I}, \tag{2.29}$$

where $\mathcal{I}$ is the ideal generated by the commutation relations of the generators $\hat{x}^i$. But how can the commutation relations be chosen consistently? The product in $\mathcal{M}_q$ has to be compatible with the co-action of $\mathcal{F}(\mathcal{G})_q$.

First let us introduce $\widehat{R} \equiv R \circ \tau$. In the classical limit, $\widehat{R}$ is just the permutation τ, $\tau(a \otimes b) = b \otimes a$. $\widehat{R}$ can be decomposed into projectors,

$$\widehat{R} = \lambda_1 \widehat{P}_S + \lambda_2 \widehat{P}_A, \tag{2.30}$$

where $\widehat{P}_A$ is the q-deformed generalisation of the antisymmetriser, $\widehat{P}_S$ of the symmetriser, respectively. The relations

$$\widehat{P}_A{}^{mn}_{ij}\, \hat{x}^i \hat{x}^j = 0 \tag{2.31}$$

on $\mathcal{M}_q$ satisfy both of the above requirements. In the commutative limit, (2.31) means that the commutator of two coordinates vanishes. It is also covariant under the co-action ρ of the quantum group

$$\rho : \mathcal{M}_q \to \mathcal{F}(\mathcal{G})_q \otimes \mathcal{M}_q, \tag{2.32}$$

$$\rho(\hat{x}^i) \;=\; t^i_j \otimes \hat{x}^j, \tag{2.33}$$

since

$$\widehat{P}_A{}^{mn}_{ij} (t^i_k \otimes \hat{x}^k)(t^j_l \otimes \hat{x}^l) = t^m_i\, t^n_j \otimes \widehat{P}_A{}^{ij}_{kl}\, \hat{x}^k \hat{x}^l = 0. \tag{2.34}$$

$\widehat{P}_A$ is a polynomial in $\widehat{R}$, and the $\widehat{R}TT$ relations (2.26) can be applied ($R^{ij}_{kl} = \widehat{R}^{ji}_{kl}$).

Differentials, $\hat{\partial}_A$

$\hat{\partial}_A$ satisfy the same commutation relations as the coordinates [33],

$$\widehat{P}_{A}{}^{ij}{}_{kl}\,\hat{\partial}_i\hat{\partial}_j = 0. \tag{2.35}$$

This follows from the assumptions on the exterior derivative d. The exterior derivative $d = \xi^A\hat{\partial}_A$ shall have the same properties as in the classical case,

$$d^2 = 0, \tag{2.36}$$
$$d\hat{x}^A = \xi^A + \hat{x}^A d, \tag{2.37}$$

where the coordinate differentials ξ^A are supposed to anticommute, i.e.,

$$\widehat{P}_{S}{}^{AB}{}_{CD}\,\xi^C\xi^D = 0. \tag{2.38}$$

Consequently, the differentials satisfy a modified Leibniz rule

$$\hat{\partial}_A(\hat{f}\hat{g}) = (\hat{\partial}_A\hat{f})\hat{g} + \mathcal{O}_A{}^B(\hat{f})\,\hat{\partial}_B\hat{g}, \tag{2.39}$$

where the operator $\mathcal{O}_A{}^B$ is a homomorphism $\mathcal{O}_A{}^B(\hat{f}\hat{g}) = \mathcal{O}_A{}^C(\hat{f})\,\mathcal{O}_C{}^B(\hat{g})$.

We finish this Section on quantum groups and quantum spaces with a popular two dimensional example, the Manin plane.

Example: Manin Plane

See e.g., [34]. The Manin plane is generated by the two coordinates $\hat{x}, \hat{y}$. The underlying symmetry is given by the quantum algebra $\mathcal{U}_q(sl_2)$.

- The coordinates satisfy
$$\hat{x}\hat{y} = q\hat{y}\hat{x}. \tag{2.40}$$

- The differentials fulfill the same relation, except for some scaling factor,
$$\hat{\partial}_x\hat{\partial}_y = \frac{1}{q}\hat{\partial}_y\hat{\partial}_x. \tag{2.41}$$

- The crossrelations compatible with the above structures are given by

$$\hat{\partial}_x \hat{x} = 1 + q^2 \hat{x} \hat{\partial}_x + q\lambda \hat{y} \hat{\partial}_y, \tag{2.42}$$
$$\hat{\partial}_x \hat{y} = q \hat{y} \hat{\partial}_x, \tag{2.43}$$

$$\hat{\partial}_y \hat{x} = q \hat{x} \hat{\partial}_y, \tag{2.44}$$
$$\hat{\partial}_y \hat{y} = 1 + q^2 \hat{y} \hat{\partial}_y, \tag{2.45}$$

where $\lambda = q - \frac{1}{q}$.

- The symmetry algebra $\mathcal{U}_q(sl_2)$ is generated by T^+, T^-, T^3, which satisfy the following defining relations

$$\begin{aligned} \frac{1}{q} T^+ T^- - q T^- T^+ &= T^3, \\ q^2 T^3 T^+ - \frac{1}{q^2} T^+ T^3 &= (q + \frac{1}{q}) T^+, \\ q^2 T^- T^3 - \frac{1}{q^2} T^3 T^- &= (q + \frac{1}{q}) T^-. \end{aligned} \tag{2.46}$$

- The action of the generators on coordinates is given by

$$\begin{aligned} T^3 \hat{x} &= q^2\, \hat{x} T^3 - q \hat{x}, \\ T^3 \hat{y} &= \frac{1}{q^2} \hat{y} T^3 + \frac{1}{q} \hat{y}, \\ \\ T^+ \hat{x} &= q \hat{x} T^+ + \frac{1}{q} \hat{y}, \\ T^+ \hat{y} &= \frac{1}{q} \hat{y} T^+, \\ \\ T^- \hat{x} &= q \hat{x} T^-, \\ T^- \hat{y} &= \frac{1}{q} \hat{y} T^- + q \hat{x}. \end{aligned} \tag{2.47}$$

In the classical limit, the symmetry algebra fulfills the usual sl_2 relations

$$\begin{aligned} [T^+, T^-] &= T^3, \\ [T^3, T^+] &= 2\, T^+, \\ [T^-, T^3] &= 2\, T^-. \end{aligned} \tag{2.48}$$

Eqns. (2.47) reduce to the usual action of the generators of angular momentum on a spin-1/2 state.

2.2.2 Lie Algebra Case

A special example of Lie algebra valued coordinates,

$$[\hat{x}^n, \hat{x}^i] = ia\hat{x}^i, \tag{2.49}$$

where $a \in \mathbb{R}$, $i = 1, \ldots, n-1$, will be discussed in detail in Chapter 6. This structure also allows for a quantum group as symmetry group. This case is called κ-deformation in the literature, cf. [16, 17, 18] and references given in Chapter 6.

2.2.3 Canonical Case

As discussed before, Minkowski space with canonical commutation relations does not allow for a Lorentz symmetry. Only a translational symmetry is present. Compared to the quantum group case or other more sophisticated examples, calculations can be done more easily and more interesting models can be studied. The commutator of two coordinates is a constant

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{2.50}$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{C}$. The derivatives act on coordinates as in the classical case,

$$[\hat{\partial}_\nu, \hat{x}^\mu] = \delta^\mu_\nu. \tag{2.51}$$

However, there are two consistent ways to define commutation relations of derivatives. By observing that

$$\hat{x}^\mu - i\theta^{\mu\nu}\hat{\partial}_\nu \tag{2.52}$$

commutes with all coordinates $\hat{x}^\nu$ and all derivatives $\hat{\partial}_\nu$ one may assume that this expression equals some constant, 0 say. Thus, we can define a derivative in terms of the coordinates (for invertible θ),

$$\hat{\partial}_\mu = -i\theta^{-1}_{\mu\nu}\hat{x}^\nu. \tag{2.53}$$

The commutator of derivatives is given by

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = i(\theta^{-1})_{\mu\nu}. \tag{2.54}$$

The other possibility compatible with the coordinate algebra relations and with (2.51) is

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \tag{2.55}$$

The integral is given by the usual four dimensional integral over commutative functions

$$\int \hat{f}\hat{g} := \int d^4x\, f * g\,(x) = \int d^4x\, f(x)\, g(x). \tag{2.56}$$

In the next Chapter, we will discuss what we mean by the map $f \rightarrow \hat{f}$, mapping functions f depending on commutative coordinates x^μ onto non-commutative functions $\hat{f}$, and by the product $*$.

All the necessary prerequisites for field theory with action integral are met. But before we are going to turn to physics, to gauge theory, we will talk about the $*$-product approach. In this approach the commutative limit is very transparent.

Chapter 3

Star Products

Let us consider the non-commutative algebra of functions $\widehat{\mathcal{A}}$ on a non-commutative space

$$\hat{\mathcal{A}} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ..., \hat{x}^n \rangle\rangle}{\mathcal{I}}, \tag{3.1}$$

where $\mathcal{I}$ is the ideal generated by the commutation relations of the coordinate functions, and the commutative algebra of functions

$$\mathcal{A} = \frac{\mathbb{C}\langle\langle x^1, ..., x^n \rangle\rangle}{[x^i, x^j]} \equiv \mathbb{C}[[x^1, ..., x^n]], \tag{3.2}$$

i.e., $[x^i, x^j] = 0$. Our aim in this Section is to relate these algebras by an isomorphism. Let us first consider the vector space structure of the algebras, only. In order to construct a vector space isomorphism, we have to choose a basis (ordering) in $\hat{\mathcal{A}}$ - satisfying the Poincaré-Birkhoff-Witt property - e.g., the basis of symmetrically ordered polynomials,

$$1, \quad \hat{x}^i, \quad \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i), \quad \ldots. \tag{3.3}$$

Now we map the basis monomials in $\mathcal{A}$ onto the according symmetrically ordered basis elements of $\hat{\mathcal{A}}$

$$\begin{aligned} W : \mathcal{A} &\to \hat{\mathcal{A}}, \\ x^i &\mapsto \hat{x}^i, \\ x^i x^j &\mapsto \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i) \equiv \,:\hat{x}^i\hat{x}^j:\,. \end{aligned} \tag{3.4}$$

The ordering is indicated by $::$. W is an isomorphism of vector spaces. In order to extend W to an algebra isomorphism, we have to introduce a new non-commutative multiplication $*$ in $\mathcal{A}$. This $*$-product is defined by

$$W(f * g) := W(f) \cdot W(g) = \hat{f} \cdot \hat{g}, \tag{3.5}$$

where $f, g \in \mathcal{A}$, $\hat{f}, \hat{g} \in \hat{\mathcal{A}}$.

$$(\mathcal{A}, *) \cong (\hat{\mathcal{A}}, \cdot), \tag{3.6}$$

i.e., W is an algebra isomorphism. The information on the non-commutativity of $\hat{\mathcal{A}}$ is encoded in the $*$-product.

3.1 Construction of a $*$-Product of Functions

Let us choose symmetrically ordered monomials as basis in $\hat{\mathcal{A}}$. The commutation relations of the coordinates are

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x}), \tag{3.7}$$

where $\theta(\hat{x})$ is an arbitrary expression in the coordinates $\hat{x}$, for now. In just a moment we will discuss the special cases (2.11 - 2.13). The Weyl quantisation procedure [35, 36] is given by the Fourier transformation,

$$\hat{f} = W(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k \, e^{ik_j \hat{x}^j} \tilde{f}(k), \tag{3.8}$$

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x \, e^{-ik_j x^j} f(x), \tag{3.9}$$

where we have replaced the commutative coordinates by non-commutative ones ($\hat{x}^i$) in the inverse Fourier transformation (3.8). The exponential takes care of the symmetrical ordering. Using eqn. (3.5), we get

$$W(f * g) = \frac{1}{(2\pi)^n} \int d^n k \, d^n p \, e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j} \tilde{f}(k) \tilde{g}(p). \tag{3.10}$$

Because of the non-commutativity of the coordinates $\hat{x}^i$, we need the Campbell-Baker-Hausdorff (CBH) formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]-\frac{1}{12}[[A,B],A]+\dots}. \tag{3.11}$$

Clearly, we need to specify $\theta^{ij}(\hat{x})$ in order to evaluate the CBH formula.

Canonical Case

Due to the constant commutation relations, the CBH formula will terminate, terms with more than one commutator will vanish,

$$\exp(ik_i\hat{x}^i)\exp(ip_j\hat{x}^j) = \exp\left(i(k_i+p_i)\hat{x}^i - \frac{i}{2}k_i\theta^{ij}p_j\right). \tag{3.12}$$

Eqn. (3.10) now reads

$$f * g\,(x) = \frac{1}{(2\pi)^n}\int d^nkd^np\, e^{i(k_i+p_i)x^i - \frac{i}{2}k_i\theta^{ij}p_j}\tilde{f}(k)\tilde{g}(p) \tag{3.13}$$

and we get for the $*$-product the Moyal-Weyl product [37]

$$f * g\,(x) = \exp(\,\frac{i}{2}\frac{\partial}{\partial x^i}\,\theta^{ij}\,\frac{\partial}{\partial y^j})\,f(x)g(y)\Big|_{y\to x}. \tag{3.14}$$

Lie Algebra Case

The coordinates build a Lie algebra

$$[\hat{x}^i,\hat{x}^j] = i\lambda^{ij}_k\hat{x}^k, \tag{3.15}$$

with structure constants λ^{ij}_k. In this case the CBH sum will not terminate and we get

$$\exp(ik_i\hat{x}^i)\exp(ip_j\hat{x}^j) = \exp\left(i(k_i+p_i)\hat{x}^i + \frac{i}{2}g_i(k,p)\hat{x}^i\right), \tag{3.16}$$

where all the terms containing more than one commutator are collected in $g_i(k,p)$. (3.10) becomes

$$f * g\,(x) = \frac{1}{(2\pi)^n}\int d^nkd^np\, e^{i(k_i+p_i)x^i + \frac{i}{2}g_i(k,p)x^i}\tilde{f}(k)\tilde{g}(p). \tag{3.17}$$

The symmetrically ordered $*$-product takes the form

$$f * g\,(x) = e^{\frac{i}{2}x^i\,g_i(-i\frac{\partial}{\partial y},-i\frac{\partial}{\partial z})}\,f(y)g(z)\Big|_{\substack{y\to x\\ z\to x}}. \tag{3.18}$$

In general, it will not be possible to write down a closed expression for the $*$-product, since the CBH formula can be summed up only for very few examples.

q-Deformed Case

The CBH formula cannot be used explicitly, we have to use eqns. (3.4), instead. Let us first write functions as a power series in x^i,

$$f(x) = \sum_J c_J \ (x^1)^{j_1} \cdot \ldots (x^n)^{j_n}, \tag{3.19}$$

where $J = (j_1, \ldots, j_n)$ is a multi-index. In the same way, non-commutative functions are given by power series in ordered monomials

$$\hat{f}(\hat{x}) = \sum_J c_J \ : (\hat{x}^1)^{j_1} \cdot \ldots (\hat{x}^n)^{j_n} : . \tag{3.20}$$

In a next step, we have to express the product of two ordered monomials in the non-commutative coordinates again in terms of ordered monomials, i.e., we have to find coefficients a_K such that

$$: (\hat{x}^1)^{i_1} \ldots (\hat{x}^n)^{i_n} : : (\hat{x}^1)^{j_1} \ldots (\hat{x}^n)^{j_n} : = \sum_K a_K : (\hat{x}^1)^{k_1} \ldots (\hat{x}^n)^{k_n} : . \tag{3.21}$$

Knowing the a_K, we know the $*$-product for monimials. It is simply given by

$$(\hat{x}^1)^{i_1} \ldots (\hat{x}^n)^{i_n} \ * \ (\hat{x}^1)^{j_1} \ldots (\hat{x}^n)^{j_n} = \sum_K a_K (\hat{x}^1)^{k_1} \ldots (\hat{x}^n)^{k_n}, \tag{3.22}$$

using the same coefficients a_K as in (3.21). The whole procedure makes use of the isomorphism W defined in eqns. (3.4) and (3.5). In a last step we have to generalise the above expression to functions f and g, and express the $*$-product in terms of ordinary derivatives on the functions f and g, respectively. This merely amounts to replacing q^{i_k} - where i_k refers to the power ot the k^{th} coordinate in (3.22) - by the differential operator $q^{x^k \partial_k}$, where no summation over k is implied. We will work out in detail the $*$-product for physically relevant quantum spaces in the next Sections. But for a better illustration, let us consider some examples first.

Examples

- In Quantum mechanics, cf. [38], we have Heisenberg commutation relations between momenta and position operators,

$$[Q^i, P_j] = i\hbar \delta^i_j, \tag{3.23}$$

$i, j = 1, \ldots, n$. In normal ordering, where all the momenta are on the right and all coordinates on the left, $Q^l P^k$, we get for the $*$-product

$$f *_N g\,(Q, P) = m \circ \exp(-i\hbar\partial_{P_i} \otimes \partial_{Q^i}) f(Q, P) \otimes g(Q, P), \qquad (3.24)$$

where m is the multiplication map, $m(a \otimes b) = ab$. For symmetrical ordering the $*$-product reads

$$f *_S g\,(P, Q) = m \circ \exp\left(\frac{i\hbar}{2}\left(\partial_{Q^i} \otimes \partial_{P_i} - \partial_{P_i} \otimes \partial_{Q^i}\right)\right) f(Q, P) \otimes g(Q, P). \quad (3.25)$$

- The Manin plane is always an eligible candidate,

  $$\hat{x}\,\hat{y} = q\,\hat{y}\,\hat{x}.$$

  First, we consider normal ordering, i.e., a normal ordered monomial has the form

  $$:\hat{y}^3\,\hat{x}^2\,\hat{y} := \hat{x}^2\,\hat{y}^4.$$

  Following the above prescription, we end up with the following $*$-product [13]

  $$f *_N g\,(x, y) = m \circ q^{-y\frac{\partial}{\partial y} \otimes x\frac{\partial}{\partial x}} f(x, y) \otimes g(x, y). \qquad (3.26)$$

  Let us now consider a *symmetric ordering*, where the factor $k!$ in the denominator is replaced by $[k]_{q^{1/2}}!$, e.g.,

  $$W[xy] = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{[2]_{q^{1/2}}!}, \qquad (3.27)$$

  where $[a]_{q^b} \equiv \frac{q^{ab} - q^{-ab}}{q^b - q^{-b}}$. The only difference to symmetrically ordered polynomials is the normalisation.

  $$f *_S g\,(x, y) = m \circ q^{-\frac{1}{2}\left(y\frac{\partial}{\partial y} \otimes x\frac{\partial}{\partial x} - x\frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y}\right)} f(x, y) \otimes g(x, y). \qquad (3.28)$$

- The $GL(n)_q$ quantum plane is a generalisation of the Manin plane. We have n generators, $\hat{x}^1, \ldots, \hat{x}^n$ satisfying the relations

  $$\hat{x}^i\hat{x}^j = q\hat{x}^j\hat{x}^i, \qquad i < j. \qquad (3.29)$$

  In the same symmetric ordering as above, the $*$-product is given by [34]

  $$f *_S g\,(x) = m \circ \exp\left(\frac{-h}{2}\sum_{i<j}\left(x^j\partial_j \otimes x^i\partial_i - x^i\partial_i \otimes x^j\partial_j\right)\right) f(x) \otimes g(x). \quad (3.30)$$

- In the case of the q-deformed 3 dimensional Euclidean space, the algebra of functions is generated by the coordinates $\hat{x}^+, \hat{x}^3, \hat{x}^-$. Again, we consider normal ordering,
$$: (\hat{x}^3)^{i_3}\, (\hat{x}^+)^{i_+}\, (\hat{x}^-)^{i_-} \; := (\hat{x}^+)^{i_+}\, (\hat{x}^3)^{i_3}\, (\hat{x}^-)^{i_-}.$$
For the $*$-product we obtain [12]
$$f * g = \sum_{i=0}^{\infty} \lambda^i \frac{x_3^{2i}}{[[i]]_{q^4}!}\; q^{2(\hat{\sigma}_3\hat{\sigma}'_+ + \hat{\sigma}_-\hat{\sigma}'_3)} \left(D_{q^4}^-\right)^i f(x) \cdot \left(D_{q^4}^+\right)^i g(x')\Big|_{x'\to x}, \tag{3.31}$$
where $D_q^A f(x) = \frac{f(x_A)-f(qx_A)}{x_A - qx_A}$ is the discrete Jackson derivative, $[[i]]_{q^i} = \frac{1-q^{in}}{1-q^i}$ and $q^{\hat{\sigma}_A} = q^{x_A \frac{\partial}{\partial x_A}}$, where no summation is implied.

In Section 3.5, we will examine the connection between $*$-products corresponding to different oderings in more detail. Deformations using the CBH formula, such as (3.14) or (3.18), are sometimes called CBH-quantisations.

3.2 q-Deformed 4-Dimensional Euclidean Space

The procedure to get the $*$-product for the 4-dimensional Euclidean space is very much the same as described in the previous Section. The quantum space algebra is freely generated by the coordinates $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$ and $\hat{X}_4$, divided by the ideal generated by the following relations [30, 39]
$$\begin{aligned} \hat{X}_1\hat{X}_2 = q\hat{X}_2\hat{X}_1,\ \hat{X}_1\hat{X}_3 = q\hat{X}_3\hat{X}_1, \\ \hat{X}_3\hat{X}_4 = q\hat{X}_4\hat{X}_3,\ \hat{X}_2\hat{X}_4 = q\hat{X}_4\hat{X}_2, \\ \hat{X}_2\hat{X}_3 = \hat{X}_3\hat{X}_2,\ \hat{X}_4\hat{X}_1 - \hat{X}_1\hat{X}_4 = \lambda\hat{X}_2\hat{X}_3. \end{aligned} \tag{3.32}$$
As a basis we use the ordered monomials $\hat{X}_1^{i_1}\hat{X}_2^{i_2}\hat{X}_3^{i_3}\hat{X}_4^{i_4}$, and
$$W(x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}) = \hat{X}_1^{i_1}\hat{X}_2^{i_2}\hat{X}_3^{i_3}\hat{X}_4^{i_4}. \tag{3.33}$$
The $*$-product on monomials is then defined by the condition
$$\begin{aligned} W((x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}) \,*\, (x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4})) = \\ = W(x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4})W(x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}). \end{aligned} \tag{3.34}$$
The right hand side of (3.34) has to be rewritten in normal ordering, using the relations (3.32). For this aim, we need to calculate the commutation relations for $\hat{X}_2^{n_2}\hat{X}_1^{m_1}$,

$\hat{X}_1^{n_1}\hat{X}_3^{m_3}$, $\hat{X}_4^{n_4}\hat{X}_3^{m_3}$, $\hat{X}_4^{n_4}\hat{X}_2^{m_2}$, $\hat{X}_3^{n_3}\hat{X}_2^{m_2}$ and $\hat{X}_4^{n_4}\hat{X}_1^{m_1}$.
These commutation relations read

$$\begin{aligned}\hat{X}_2^{n_2}\hat{X}_1^{m_1} = q^{-n_2m_1}\hat{X}_1^{m_1}\hat{X}_2^{n_2},\ \ \hat{X}_3^{n_3}\hat{X}_1^{m_1} = q^{-n_3m_1}\hat{X}_1^{m_1}\hat{X}_3^{n_3},\\ \hat{X}_4^{n_4}\hat{X}_3^{m_3} = q^{-n_4m_3}\hat{X}_3^{m_3}\hat{X}_4^{n_4},\ \ \hat{X}_4^{n_4}\hat{X}_2^{m_2} = q^{-n_4m_2}\hat{X}_2^{m_2}\hat{X}_4^{n_4},\\ \hat{X}_3^{m_3}\hat{X}_2^{n_2} = \hat{X}_2^{n_2}\hat{X}_3^{m_3},\\ \hat{X}_4^{n_4}\hat{X}_1^{m_1} = \sum_{i=0}^{\min\{n_4,m_1\}}\lambda^i B_i^{n_4,m_1}\hat{X}_1^{m_1-i}\hat{X}_2^i\hat{X}_3^i\hat{X}_4^{n_4-i},\end{aligned} \tag{3.35}$$

where

$$B_i^{n_4,m_1} = \frac{1}{[[i]]_{q^{-2}}!}\frac{[[n_4]]_{q^{-2}}!\ [[m_1]]_{q^{-2}}!}{[[n_4-i]]_{q^{-2}}!\ [[m_1-i]]_{q^{-2}}!}. \tag{3.36}$$

Therefore the $*$-product of two monomials has the form

$$\begin{aligned}&(x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}) * (x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}) =\\ &= \sum_{i=0}^{\min\{n_4,m_1\}}\lambda^i q^{-(n_2+n_3)(m_1-i)-(m_2+m_3)(n_4-i)}B_i^{n_4,m_1}\ \times\\ &\quad\times\ x_1^{n_1+m_1-i}x_2^{n_2+m_2+i}x_3^{n_3+m_3+i}x_4^{n_4+m_4-i}.\end{aligned} \tag{3.37}$$

Using the substitution

$$q^{n_A} \to q^{\hat{\sigma}_A} = q^{x_A\frac{\partial}{\partial x_A}}, \tag{3.38}$$

we obtain for $f, g \in \mathcal{A}$

$$f * g = \sum_{i=0}^{\infty}\lambda^i\frac{(x_2x_3)^i}{[[i]]_{q^{-2}}!}\ q^{-(\hat{\sigma}_2+\hat{\sigma}_3)\hat{\sigma}_1'-(\hat{\sigma}_2'+\hat{\sigma}_3')\hat{\sigma}_4}\left(D_{q^{-2}}^4\right)^i f(x)\cdot\left(D_{q^{-2}}^1\right)^i g(x')\Big|_{x'\to x}, \tag{3.39}$$

where $D_{q^i}^A f(x) = \frac{f(x_A)-f(q^ix_A)}{x_A-q^ix_A}$ are Jackson derivatives, and $[[n]]_{q^i} = \frac{1-q^{in}}{1-q^i}$. Let us expand expressions (3.37) and (3.39) in terms of $h = \ln q$. We find

$$\begin{aligned}(x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4})\ *\ (x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}) &= x_1^{n_1+m_1}x_2^{n_2+m_2}x_3^{n_3+m_3}x_4^{n_4+m_4}\ +\\ &+ h\Big(a_0^{(1)}\ x_1^{n_1+m_1}x_2^{n_2+m_2}x_3^{n_3+m_3}x_4^{n_4+m_4}\ +\\ &\quad+\theta(n_4)\theta(m_1)\ a_1^{(1)}\ x_1^{n_1+m_1-1}x_2^{n_2+m_2+1}x_3^{n_3+m_3+1}x_4^{n_4+m_4-1}\Big)\\ &+ h^2\Big(a_0^{(2)}\ x_1^{n_1+m_1}x_2^{n_2+m_2}x_3^{n_3+m_3}x_4^{n_4+m_4}\\ &\quad+\theta(n_4)\theta(m_1)\ a_1^{(2)}\ x_1^{n_1+m_1-1}x_2^{n_2+m_2+1}x_3^{n_3+m_3+1}x_4^{n_4+m_4-1}\\ &\quad+\theta(n_4-1)\theta(m_1-1)\ a_2^{(2)}\ x_1^{n_1+m_1-2}x_2^{n_2+m_2+2}x_3^{n_3+m_3+2}x_4^{n_4+m_4-2}\Big)\\ &+ \mathcal{O}(h^3),\end{aligned} \tag{3.40}$$

where $a_i^{(j)} = a_i^{(j)}(\underline{n}, \underline{m})$,

$$
\begin{aligned}
a_0^{(1)}(\underline{n}, \underline{m}) &= -(n_2 + n_3)m_1 - (m_2 + m_3)n_4, \\
a_1^{(1)}(\underline{n}, \underline{m}) &= 2n_4 m_1, \\
a_0^{(2)}(\underline{n}, \underline{m}) &= \frac{1}{2}((n_2 + n_3)m_1 + (m_2 + m_3)n_4)^2, \\
a_1^{(2)}(\underline{n}, \underline{m}) &= -2n_4 m_1(((n_2 + n_3) + 1)(m_1 - 1) + ((m_2 + m_3) + 1)(n_4 - 1)), \\
a_2^{(2)}(\underline{n}, \underline{m}) &= 2n_4(n_4 - 1)m_1(m_1 - 1).
\end{aligned}
\tag{3.41}
$$

And in terms of derivatives we find

$$
\begin{aligned}
f * g &= f(x)g(x) \\
&+ h\Big(- (\hat{\sigma}_2 + \hat{\sigma}_3)\hat{\sigma}'_1 - (\hat{\sigma}'_2 + \hat{\sigma}'_3)\hat{\sigma}_4 + 2\frac{x_2 x_3}{x_1 x_4}\hat{\sigma}_4\hat{\sigma}'_1\Big) f(x)g(x')\Big|_{x' \to x} \\
&+ h^2\Big(\frac{1}{2}((\hat{\sigma}_2 + \hat{\sigma}_3)\hat{\sigma}'_1 + (\hat{\sigma}'_2 + \hat{\sigma}'_3)\hat{\sigma}_4)^2 + 2\left(\frac{x_2 x_3}{x_1 x_4}\right)^2 \hat{\sigma}_4(\hat{\sigma}_4 - 1)\hat{\sigma'}_1(\hat{\sigma}'_1 - 1) \\
&- 2\frac{x_2 x_3}{x_1 x_4}\hat{\sigma}_4\hat{\sigma}'_1\left(((\hat{\sigma}_2 + \hat{\sigma}_3) + 1)(\hat{\sigma}'_1 - 1) + ((\hat{\sigma}'_2 + \hat{\sigma}'_3) + 1)(\hat{\sigma}_4 - 1)\right)\Big) f(x)g(x')\Big|_{x' \to x} \\
&+ \mathcal{O}(h^3).
\end{aligned}
\tag{3.42}
$$

The symmetry in all these expressions between x_1 and x_4, respectively n_4 and m_1 is remarkable. In eqn. (3.40) the exponents of the variables x_1 and x_4 are always diminished by the same number. These powers are distributed symmetrically among the coordinates x_2 and x_3. This stems from the fact that $\mathcal{U}_q(so(4))$ can be decomposed into 2 independent copies of $\mathcal{U}_q(su(2))$, as in the classical case. In case of the Lorentz group its decomposition also leads to the tensor product of 2 copies of $\mathcal{U}_q(sl_2(\mathbb{C}))$, which are related to each other via complex conjugation. Thus we will not be able to observe this symmetry between the corresponding Minkowski coordinates, x_0 and x_3. Additional terms in (3.40) will occur where the powers taken away from x_- and x_+ are not symmetrically distributed among x_0 and x_3. But still some remnants of the symmetry are present, cf. (3.51).

3.3 q-Deformed Minkowski Space

The maybe most important case we want to discuss here is a q-deformed version of the Minkowski space, the co-module algebra of the q-deformed Lorentz group [31, 33, 40, 41, 42]. q-Minkowski space is generated by the four coordinates $\hat{X}_0$, $\hat{X}_+$, $\hat{X}_3$ and $\hat{X}_-$, they satisfy the following relations

$$
\begin{gathered}
\hat{X}_-\hat{X}_0 = \hat{X}_0\hat{X}_-, \quad \hat{X}_+\hat{X}_0 = \hat{X}_0\hat{X}_+, \quad \hat{X}_3\hat{X}_0 = \hat{X}_0\hat{X}_3, \\
\hat{X}_-\hat{X}_3 - q^2\hat{X}_3\hat{X}_- = (1-q^2)\hat{X}_0\hat{X}_-, \quad \hat{X}_3\hat{X}_+ - q^2\hat{X}_+\hat{X}_3 = (1-q^2)\hat{X}_0\hat{X}_+, \\
\hat{X}_-\hat{X}_+ - \hat{X}_+\hat{X}_- = \lambda\left(\hat{X}_3\hat{X}_3 - \hat{X}_0\hat{X}_3\right).
\end{gathered} \tag{3.43}
$$

In order to make the calculations easier, we introduce a new set of coordinates $\hat{X}_0$, $\hat{X}_+$, $\hat{\tilde{X}}_3$, $\hat{X}_-$, where

$$
\hat{\tilde{X}}_3 \equiv \hat{X}_3 - \hat{X}_0. \tag{3.44}
$$

Thus, the relevant relations of (3.43) become

$$
\begin{aligned}
&\hat{X}_-\hat{\tilde{X}}_3 = q^2\hat{\tilde{X}}_3\hat{X}_-, \quad \hat{\tilde{X}}_3\hat{X}_+ = q^2\hat{X}_+\hat{\tilde{X}}_3, \\
&\hat{X}_-\hat{X}_+ - \hat{X}_+\hat{X}_- = \lambda\left(\hat{\tilde{X}}_3\hat{\tilde{X}}_3 + \hat{X}_0\hat{\tilde{X}}_3\right).
\end{aligned} \tag{3.45}
$$

We again introduce the isomorphism W from the commutative coordinate algebra into the q-deformed Minkowski space algebra

$$
W(x_0^{n_0}x_+^{n_+}\tilde{x}_3^{n_3}x_-^{n_-}) = \hat{X}_0^{n_0}\hat{X}_+^{n_+}\hat{\tilde{X}}_3^{n_3}\hat{X}_-^{n_-}, \tag{3.46}
$$

the right hand side is defined as our normal ordering. Using relations (3.45), we get

$$
\begin{aligned}
\hat{\tilde{X}}_3^{n_3}\hat{X}_+^{m_+} &= q^{2n_3m_+}\hat{X}_+^{m_+}\hat{\tilde{X}}_3^{n_3}, \\
\hat{X}_-^{n_-}\hat{\tilde{X}}_3^{m_3} &= q^{2n_-m_3}\hat{\tilde{X}}_3^{m_3}\hat{X}_-^{n_-}, \\
\hat{X}_-^{n_-}\hat{X}_+^{m_+} &= \sum_{i=0}^{\min\{n_-,m_+\}} \lambda^i\hat{X}_+^{m_+-i}F_i^{n_-,m_+}(\hat{X}_0,\hat{\tilde{X}}_3)\,\hat{X}_-^{n_--i},
\end{aligned} \tag{3.47}
$$

where the coefficients $F_i^{n,m}(\hat{X}_0,\hat{\tilde{X}}_3)$ satisfy the recursion relation

$$
\begin{aligned}
F_i^{n,m}(\hat{X}_0,\hat{\tilde{X}}_3) &= F_i^{n,m-1}(\hat{X}_0,\hat{\tilde{X}}_3) + F_{i-1}^{n,m-1}(\hat{X}_0,\hat{\tilde{X}}_3) \times \\
&\quad \times \left(q^{4(m-i)}[[n-(i-1)]]_{q^4}\hat{\tilde{X}}_3^{2} + q^{2(m-i)}[[n-(i-1)]]_{q^2}\hat{X}_0\hat{\tilde{X}}_3\right), \\
F_0^{n,m}(\hat{X}_0,\hat{\tilde{X}}_3) &= 1.
\end{aligned} \tag{3.48}
$$

We could not deduce a closed expression for $F_i^{n,m}(\hat{X}_0, \hat{\tilde{X}}_3)$ solving the recursion relations (3.48). However, we can write down what we have so far for the $*$-product of ordered monomials,

$$
\begin{aligned}
(x_0^{n_0} x_+^{n_+} \tilde{x}_3^{n_3} x_-^{n_-}) \; * \; (x_0^{m_0} x_+^{m_+} \tilde{x}_3^{m_3} x_-^{m_-}) = \\
= \sum_{i=0}^{\min\{n_-, m_+\}} \lambda^i q^{2(n_3(m_+ - i) + m_3(n_- - i))} \times \\
\times \quad F_i^{n_-, m_+}(x_0, \tilde{x}_3)\, x_0^{n_0+m_0} x_+^{n_+ + m_+ - i} \tilde{x}_3^{n_3+m_3} x_-^{n_- + m_- - i}.
\end{aligned}
\tag{3.49}
$$

We can rewrite the recursion formula for $F_i^{n_-, m_+}(x_0, \tilde{x}_3)$

$$
\begin{aligned}
F_j^{n,m} &= \sum_{i=0}^{m-j} \left(q^{4i} [[n-(j-1)]]_{q^4} \tilde{x}_3^2 + q^{2i} [[n-(j-1)]]_{q^2} x_0 \tilde{x}_3 \right) F_{j-1}^{n, i+(j-1)} \\
&= \sum_{i_0=0}^{m-j} \sum_{i_1=0}^{i_0} \cdots \sum_{i_{j-1}=0}^{i_{j-2}} \prod_{k=0}^{j-1} \sum_{l=0}^{n-(j-k)} \left(q^{4(l+i_k)} \tilde{x}_3^2 + q^{2(l+i_k)} x_0 \tilde{x}_3 \right)
\end{aligned}
\tag{3.50}
$$

and expand this expression in powers of $h = \ln q$. The expansion of $F_i^{n_-, m_+}$ enables us to write down the $*$-product up to order h^2. In order to deduce a closed expression we will use the identification of the generators of q-deformed Minkowski space with combinations of the generators of the Drinfeld-Jimbo algebra $\mathcal{U}_q(sl_2)$ [43,44,32]. Expanding expression (3.49) in powers of h reads

$$
\begin{aligned}
(x_0^{n_0} x_+^{n_+} \tilde{x}_3^{n_3} x_-^{n_-}) \; * \; (x_0^{m_0} x_+^{m_+} \tilde{x}_3^{m_3} x_-^{m_-}) = x_0^{n_0+m_0} x_+^{n_+ + m_+} \tilde{x}_3^{n_3+m_3} x_-^{n_- + m_-} \\
+ \; h \Big(a_{0,0}^{(1)}(\underline{n}, \underline{m}) \; x_0^{n_0+m_0} x_+^{n_+ + m_+} \tilde{x}_3^{n_3+m_3} x_-^{n_- + m_-} \\
+ \theta(n_-)\theta(m_+) \sum_{i=0,1} a_{1-i,1+i}^{(1)}(\underline{n}, \underline{m}) \times \\
\times \quad x_0^{n_0+m_0+(1-i)} x_+^{n_+ + m_+ - 1} \tilde{x}_3^{n_3+m_3+(1+i)} x_-^{n_- + m_- - 1} \Big) \\
+ \; h^2 \Big(a_{0,0}^{(2)}(\underline{n}, \underline{m}) \; x_0^{n_0+m_0} x_+^{n_+ + m_+} \tilde{x}_3^{n_3+m_3} x_-^{n_- + m_-} \\
+ \theta(n_-)\theta(m_+) \sum_{i=0,1} a_{1-i,1+i}^{(2)}(\underline{n}, \underline{m}) \times \\
\times \quad x_0^{n_0+m_0+(1-i)} x_+^{n_+ + m_+ - 1} \tilde{x}_3^{n_3+m_3+(1+i)} x_-^{n_- + m_- - 1} \\
+ \theta(n_- - 1)\theta(m_+ - 1) \sum_{i=0}^{2} a_{2-i,2+i}^{(2)}(\underline{n}, \underline{m}) \times
\end{aligned}
\tag{3.51}
$$

$$\times \quad x_0^{n_0+m_0+(2-i)} x_+^{n_++m_+-2} \tilde{x}_3^{n_3+m_3+(2+i)} x_-^{n_-+m_--2}\Big) + \mathcal{O}(h^3),$$

where

$$\begin{aligned}
a^{(1)}_{0,0}(\underline{n},\underline{m}) &= 2(n_3m_+ + m_3n_-),\\
a^{(1)}_{1,1}(\underline{n},\underline{m}) &= a^{(1)}_{0,2}(\underline{n},\underline{m}) = 2n_-m_+,\\
a^{(2)}_{0,0}(\underline{n},\underline{m}) &= 2(n_3m_+ + m_3n_-)^2,\\
a^{(2)}_{1,1}(\underline{n},\underline{m}) &= 2n_-m_+((2n_3+1)(m_+-1) + (2m_3+1)(n_--1)),\\
a^{(2)}_{0,2}(\underline{n},\underline{m}) &= 4n_-m_+((n_3+1)(m_+-1) + (m_3+1)(n_--1)),\\
a^{(2)}_{2,2}(\underline{n},\underline{m}) &= \frac{1}{2}a^{(2)}_{1,3}(\underline{n},\underline{m}) = a^{(2)}_{0,4}(\underline{n},\underline{m}) = 2n_-(n_--1)m_+(m_+-1).
\end{aligned} \tag{3.52}$$

And in terms of derivatives we find

$$\begin{aligned}
f * g = {} & f(x)g(x)\\
&+ h\Big(2(\hat{\sigma}_3\hat{\sigma}'_+ + \hat{\sigma}'_3\hat{\sigma}_-) + 2\frac{\tilde{x}_3^2 + x_0\tilde{x}_3}{x_+x_-}\hat{\sigma}_-\hat{\sigma}'_+\Big) f(x)g(x')\Big|_{x'\to x}\\
&+ h^2\Big(2(\hat{\sigma}_3\hat{\sigma}'_+ + \hat{\sigma}'_3\hat{\sigma}_-)^2 + 4\frac{\tilde{x}_3^2}{x_+x_-}\hat{\sigma}_-\hat{\sigma}'_+((\hat{\sigma}_3+1)(\hat{\sigma}'_+-1) + (\hat{\sigma}'_3+1)(\hat{\sigma}_--1))\\
&\quad + 2\frac{x_0\tilde{x}_3}{x_+x_-}\hat{\sigma}_-\hat{\sigma}'_+((2\hat{\sigma}_3+1)(\hat{\sigma}'_+-1) + (2\hat{\sigma}'_3+1)(\hat{\sigma}_--1))\\
&\quad + 2\left(\frac{\tilde{x}_3^2 + x_0\tilde{x}_3}{x_+x_-}\right)^2 \hat{\sigma}_-(\hat{\sigma}_--1)\hat{\sigma}'_+(\hat{\sigma}'_+-1)\Big) f(x)g(x')\Big|_{x'\to x}\\
&+ \mathcal{O}(h^3).
\end{aligned} \tag{3.53}$$

Finally, we want to deduce a closed expression for the $*$-product (3.49). To this aim we have a look at the algebra $\mathcal{U}_q(sl_2)$ [32]. The algebra is generated by the four generators E, F, K, K^{-1}, satisfying the relations

$$\begin{aligned}
KE = q^2EK, \quad KF = q^{-2}FK, \quad KK^{-1} = K^{-1}K = 1\ ,\\
EF - FE = \tfrac{K-K^{-1}}{q-q^{-1}}.
\end{aligned} \tag{3.54}$$

Further we have [32]

$$\begin{aligned}
F^nE^m = {} & E^mF^n \\
&+ \sum_{i=1}^{\min\{n,m\}} (-\lambda)^{-i}\frac{[n]!\ [m]!}{[i]!\ [n-i]!\ [m-i]!}\left(\prod_{j=0}^{i-1} Kq^{n-m+j} - K^{-1}q^{-n+m-j}\right) E^{m-i}F^{n-i},
\end{aligned} \tag{3.55}$$

where $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$. The operators L_A, W defined in eqn. (3.56)) can be interpreted as $q-$angular momentum operators [31]. They span a proper subalgebra of $\mathcal{U}_q(su_2)$

$$
\begin{aligned}
L_+ &\equiv q^{-3}[2]^{-1/2}E, \\
L_- &\equiv -q^{-2}[2]^{-1/2}KF, \\
L_3 &\equiv q^{-3}[2]^{-1}(qFE - q^{-1}EF), \\
W &\equiv K + q^3\lambda L_3.
\end{aligned} \tag{3.56}
$$

Because of (3.54), these generators satisfy the following relations

$$
\begin{aligned}
L_3L_+ - q^2L_+L_3 &= -\frac{W}{q^2}L_+, \\
L_-L_3 - q^2L_3L_- &= -\frac{W}{q^2}L_-, \\
L_-L_+ - L_+L_- &= -\frac{W}{q^3}L_3 + \lambda L_3L_3, \\
1 = W^2 - q^6\lambda^2(L_3L_3 &- qL_+L_- - q^{-1}L_-L_+).
\end{aligned} \tag{3.57}
$$

With the substitution $W \to q^3 l\lambda\hat{X}_0$, $L_A \to l\hat{X}_A$, $A \in \{+, 3, -\}$, $\mathbf{1} \to q^6l^2\lambda^2\hat{r}^2$ we regain the relations of q-Minkowski coordinates (3.43) [45]. l has the dimension of an inverse length and is introduced only for dimensional reasons. Now we return to the third equation of (3.47). Using eqn. (3.55), one gets

$$
\begin{aligned}
\hat{X}_-^n\hat{X}_+^m = q^{2nm}\hat{X}_+^m\hat{X}_-^n + &\sum_{i=1}^{\min\{n,m\}} \frac{[n]!\,[m]!}{[i]!\,[n-i]!\,[m-i]!}\left(\frac{\lambda_-}{\lambda_+}\right)^i q^{2nm+i^2-2im} \\
&\times \left(\prod_{k=0}^{i-1} q^{n-m+k}\hat{\tilde{X}}_3^2 - q^{-n+m-k}\hat{r}^2\right)\hat{X}_+^{m-i}\hat{X}_-^{n-i},
\end{aligned} \tag{3.58}
$$

where $\hat{r}^2 = -q^{-2}\hat{\tilde{X}}_3^2 - (1+q^{-2})\hat{X}_0\hat{\tilde{X}}_3 + (q+q^{-1})\hat{X}_+\hat{X}_-$, and $\lambda_\pm = q \pm q^{-1}$. The right hand side of eqn. (3.58) still has to be ordered according to the normal ordering. Note that $\hat{\tilde{X}}_3^2$ and $\hat{r}^2$ commute, therefore we find

$$
\begin{aligned}
q^{i^2-2im}\prod_{k=0}^{i-1}\left(q^{n-m+k}\hat{\tilde{X}}_3^2 - q^{-n+m-k}\,\hat{r}^2\right)\hat{X}_+^{m-i}\hat{X}_-^{n-i} = \\
= q^{-i^2}\sum_{k=0}^{i}(-1)^k q^{(1/2i-k)(i-1)}\begin{bmatrix} i \\ k \end{bmatrix}_q \left(q^{i-2k}\hat{X}_+\right)^{m-i}\hat{\tilde{X}}_3^{2(i-k)}\hat{r}^{2k}\left(q^{i-2k}\hat{X}_-\right)^{n-i},
\end{aligned} \tag{3.59}
$$

where $\begin{bmatrix} i \\ k \end{bmatrix}_q = \frac{[i]!}{[k]!\ [k-i]!}$. One can also calculate $W^{-1}(\hat{\tilde{X}}_3^{2(i-k)} \hat{r}^{2k})$ the last missing link to write down the $*$-product for q-deformed Minkowski space, and after a lengthy calculation one gets

$$W^{-1}(\hat{\tilde{X}}_3^{2(i-k)} \hat{r}^{2k}) = \tilde{x}_3^{2(i-k)} \sum_{p=0}^{k} (q^{4(i-k)}\, \lambda_+ x_+ x_-)^p S_{k,p}(x_0, \tilde{x}_3), \qquad (3.60)$$

where

$$S_{k,p}(x_0, \tilde{x}_3) = \begin{cases} 1, & \text{if } p = k \\ \sum\limits_{j_1=0}^{p} \sum\limits_{j_2=0}^{j_1} \cdots \sum\limits_{j_{k-p}=0}^{j_{k-p-1}} \prod_{l=1}^{k-p} a(x_0, q^{2j_l}\, \tilde{x}_3), & \text{if } 0 \le p < k \end{cases},$$

$$a(x_0, \tilde{x}_3) = -q^{-2} \tilde{x}_3^2 - (1 + q^{-2}) x_0 \tilde{x}_3. \qquad (3.61)$$

Eqns. (3.58), (3.60) and (3.61) enable us to order any two monomials in the q-Minkowski generators and to write down the $*$-product for q-deformed Minkowski space in a closed expression,

$$f * g = \sum_{i=0}^{\infty} \left(\frac{\lambda_-}{\lambda_+}\right)^i \sum_{k+j=i} \frac{R_{k,j}\,(\underline{x})}{[[k]]_{q^2}!\ [[j]]_{q^2}!}\ q^{(2\hat{\sigma}_3 + \hat{\sigma}_- + i)\hat{\sigma}'_+ + (2\hat{\sigma}'_3 + \hat{\sigma}'_+ + i)\hat{\sigma}_-} \times \qquad (3.62)$$
$$\times \left[(D^-_{q^2})^i f\right](x_0, x_+, \tilde{x}_3, q^{j-k} x_-) \cdot \left[(D^+_{q^2})^i g\right](x'_0, q^{j-k} x'_+, \tilde{x}'_3, x'_-)\Big|_{x' \to x},$$

where $\underline{x} = (x_0, x_+, x_3, x_-)$ and with the polynomials

$$R_{k,j}(x_0, x_+, \tilde{x}_3, x_-) = (-q)^k (q^j \tilde{x}_3^2)^j \sum_{p=0}^{k} S_{k,p}(x_0, \tilde{x}_3)\, \lambda_+^p (q^{4j} x_+ x_-)^p =$$
$$= W^{-1}\left((q^j \hat{\tilde{X}}_3^{\,2})^j (-q\, \hat{r}^2)^k\right). \qquad (3.63)$$

So finally, we have found both, the expansion of the $*$-product in powers of h (3.53) and a closed expression (3.62).

3.4 Remarks

Let us end these Sections with a few comments on eqns. (3.31), (3.39) and (3.62). First of all, we can see that the formulas for the $*$-product have a similiar structure in all three cases. The commutative product is modified by an infinite sum of corrections,

$$f * g = fg + \sum_{i=1}^{\infty} h^i C_i(f, g),$$

where C_i are local differential operators, see Section 3.5. The i^{th} term is of order $\mathcal{O}(\lambda^i) = \mathcal{O}(h^i)$.

Additionally, there are some kind of mixed scaling operators of the form $q^{a\hat{\sigma}'\hat{\sigma}}$, which lead to a displacement effect. The derivatives in the exponent will shift the argument of the function, such that the value of the $*$-product at a given point will also depend on the values of the functions at neighbouring points. The displacement effect is present in all dimensions and shows that non-commutativity induced by $q-$deformation implies some kind of non-locality. Especially in Minkowski space, one is forced to reinterpret the concept of causality, as the $*-$product, which can be considered as some kind of interaction (cf. Chapter 5), does not only depend on the nearby past but also on the nearby future.

The remaining operators and factors are responsible for an effect we have already mentioned at the end of Section 3.2. This substitution effect is absent in less than 3 dimensions. It transforms the (plane) coordinates X_+ and X_- (X_1 and X_4, respectively) into the transverse coordinate X_3 and the time coordinate X_0 (X_2 and X_3, respectively). It also shows that physical quantities like charge densities initially restricted to a plane may disperse in transverse directions or undergo an evolution in time.

3.5 Mathematical Approach to $*$-Products

Definition 1 *(Poisson Bracket)*
Let $\mathcal{M}$ be a smooth manifold, a Poisson bracket is a bi-linear map $\{,\} : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ satisfying

$f, g, h \in \mathcal{C}^\infty(\mathcal{M})$

(i) $\{f, g\} = -\{g, f\}$, *antisymmetry*

(ii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, *Jacobi identity*

(iii) $\{f, gh\} = \{f, g\}h + g\{f, h\}$, *Leibniz rule*

Locally, we can always write the Poisson bracket with the help of an antisymmetric tensor

$$\{f, g\} = \theta^{ij}\, \partial_i f \partial_j g, \tag{3.64}$$

where $\theta^{ij} = -\theta^{ji}$. Because of the Jacobi identity θ^{ij} has to satisfy

$$\theta^{ij}\partial_j\theta^{kl} + \theta^{kj}\partial_j\theta^{li} + \theta^{lj}\partial_j\theta^{ik} = 0. \tag{3.65}$$

Definition 2 *($*$-Product)*
Let $f, g \in \mathcal{C}^\infty(\mathcal{M})$ and $C_i : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ be local bi-differential operators. Then we define the $*$-product $* : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})[[h]]$, by

$$f * g = \sum_{n=0}^{\infty} h^n C_n(f, g), \tag{3.66}$$

such that the following axioms are satisfied:

(i) $*$ *is an associative product.*

(ii) $C_0(f, g) = f\,g$, *classical limit.*

(iii) $\frac{1}{h}[f \overset{*}{,} g] = -i\{f, g\}$, *in the limit $h \to 0$, semiclassical limit.*

The rhs. of definition (3.66) is an element of $\mathcal{C}^\infty(\mathcal{M})[[h]]$, the algebra of formal power series in the formal parameter h with coefficients in $\mathcal{C}^\infty(\mathcal{M})$. Therefore we can generalise the given definition of the $*$-product to a $\mathbb{C}[[h]]$-linear product in $\mathcal{B} = \mathcal{C}^\infty(\mathcal{M})[[h]]$ by

$$\begin{aligned}(\sum_n f_n h^n) * (\sum_m g_m h^m) &= \sum_{k,l} f_k g_l h^{k+l} \\ &+ \sum_{k,l\geq 0, m\geq 1} C_m(f_k, g_l) h^{k+l+m}.\end{aligned} \tag{3.67}$$

Theorem 3 *(Theorem by M. Kontsevich [46])*
$*$-products exist for any given Poisson bi-vector field α in a domain of $\mathbb{R}^n$. It is given by the formula

$$f *_K g = \sum_{n=0}^{\infty} h^n \sum_{\Gamma \in G_n} \omega_\Gamma B_{\Gamma,\alpha}(f, g). \tag{3.68}$$

Let us explain the symbols occuring in (3.68) only very briefly. In order to compute the Kontsevich $*$-product, one has to consider the upper half complex plain and to draw n points in that plane, corresponding to the h^n term. Every point is the starting point of two arrows pointing to another point or to points 0 and 1. Every such graph is associated a multi-vectorfield, B in (3.68) assigns a differential operator to these multi-vectorfields.

The arrows correspond to the derivatives, the points to the Poisson structure. If an arrow ends on another point in the plane, the derivative acts on the Poisson structure of that point. If it ends on 0 or 1, respectively the derivative acts on the function f or g, respectively. Γ_n denotes the set of admissable graphs, which is a proper subset of the set of all graphs. The weight ω_Γ of a graph Γ is given by a complicated complex integration. For details see [46], for the explicit calculations of some weights, see e.g., [47, 48].
Changing the ordering in the non-commutative algebra leads to *gauge equivalent* $*$-products. The $*$-products are related by a transformation $\mathcal{D}$,

$$\mathcal{D}f * \mathcal{D}g = \mathcal{D}(f *' g), \tag{3.69}$$

where

$$\mathcal{D}f = f + \sum_{n\geq 1} (ih)^n \, \mathcal{D}_n(f). \tag{3.70}$$

$\mathcal{D}_n$ is a differential operator of order n.

Let us reconsider the examples at the end of Section 3.1. There $*_N$ and $*_S$ are gauge equivalent $*$-products. For simplicity, let us consider 1-dimensional Quantum Mechanics. The generalisation to n dimensions is straight forward. In this case, the $*$-products (3.24) and (3.25) read

$$f *_N g\,(q,p) = m \circ \exp\left(-i\hbar\partial_p \otimes \partial_q\right)\, f(q,p) \otimes g(q,p), \tag{3.71}$$

$$f *_S g\,(q,p) = m \circ \exp\left(\frac{i\hbar}{2}(\partial_q \otimes \partial_p - \partial_p \otimes \partial_q)\right) f(q,p) \otimes g(q,p). \tag{3.72}$$

Using matrices in the exponent, these formulae can be written very succinctly as

$$f *_N g\,(q,p) = m \circ exp\left(\frac{i\hbar}{2}\tilde{\alpha}^{ij}\partial_i \otimes \partial_j\right)\, f(q,p) \otimes g(q,p),$$

$$f *_S g\,(q,p) = m \circ \exp\left(\frac{i\hbar}{2}\alpha^{ij}\partial_i \otimes \partial_j\right)\, f(q,p) \otimes g(q,p),$$

where

$$\left(\tilde{\alpha}^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \qquad \left(\alpha^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.73}$$

(α) is the antisymmetric part of $(\tilde{\alpha})$. These $*$-products are connected by the transformation $\mathcal{D}$,

$$\mathcal{D} = \exp\left(\frac{i\hbar}{4}\theta_S^{ij}\partial_i\partial_j\right) = \exp\left(-\frac{i\hbar}{2}\partial_q\partial_p\right), \tag{3.74}$$

where (θ_S) is the symmetric part of $(\tilde{\alpha})$, i.e., $\tilde{\alpha}^{ij} = \alpha^{ij} + \theta_S^{ij}$. We have

$$f *_S g = \mathcal{D}^{-1} \left(\mathcal{D} f *_N \mathcal{D} g \right). \tag{3.75}$$

The same is true in case of the Manin plane. There we have

$$\left(\tilde{\alpha}^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & 0 \\ -2y \otimes x & 0 \end{pmatrix}, \quad \left(\alpha^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & x \otimes y \\ -y \otimes x & 0 \end{pmatrix},$$
$$\left(\theta_S^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & -x \otimes y \\ -y \otimes x & 0 \end{pmatrix}. \tag{3.76}$$

The transformation $\mathcal{D} = \exp\left(\frac{ih}{4} \theta_S^{ij} \partial_i \partial_j\right)$ connects the normal ordered $*$-product (3.26) and the symmetrical ordered $*$-product (3.28) (with q-numbers as normalisation factors). Again, we have $f *_S g = \mathcal{D}^{-1} \left(\mathcal{D} f *_N \mathcal{D} g \right)$.

In Chapter 6, we will consider a quantum space covariant under κ-deformed Poincaré symmetry. Therefore, it is very important to briefly consider the covariance properties of $*$-products, see [49] and references therein. Covariance of a space $\mathcal{M}$ under the symmetry algebra $\mathcal{U}_\hbar(\mathbf{g})$ means that $\mathcal{M}$ is a $\mathcal{U}_\hbar(\mathbf{g})-$module,

$$g \triangleright m(x \otimes y) = m(\Delta_\hbar(g) \triangleright [x \otimes y]) = m(g_{(1)} \triangleright x \otimes g_{(2)} \triangleright y), \tag{3.77}$$

where $g \in \mathcal{U}_\hbar(\mathbf{g})$, $x, y \in \mathcal{M}$. Drinfel'd's theorem [50] establishes an isomorphism $\mathcal{U}_\hbar(\mathbf{g}) \cong (\mathcal{U}(\mathbf{g})[[\hbar]], \Delta_\hbar, \epsilon_\hbar, S_\hbar)$, where

$$\Delta_\hbar(g) = \mathcal{F} \Delta(g) \mathcal{F}^{-1}.$$

Δ is the classical co-product of $\mathcal{U}(\mathbf{g})$. $\mathcal{F}$ is the socalled Drinfel'd twist. Covariant $*$-products are defined in [49] using some appropriate Drinfel'd twist by

$$x * y = m(\mathcal{F}^{-1} \triangleright [x \otimes y]). \tag{3.78}$$

The twist is defined uniquely by the requirement of covariance up to a central 2-coboundary. The covariance ot the $*$-product (6.110) in Section 6.4 has been checked explicitly.

Chapter 4

Gauge Theory on Non-Commutative Space-Time

We will now concentrate on physics. We will discuss the Standard Model on a canonically deformed space-time in Chapter 5. Before we can do so, we have to think about gauge theory on canonically deformed spaces, in general. Let us first briefly recall classical gauge theory. We will discuss in some detail the features that are essential for the non-commutative generalisation.

4.1 Gauge Theory on Classical Space-Time

Internal symmetries are described by Lie groups or Lie algebras, respectively. The elements T^a

$$[T^a, T^b] = f_c^{ab} T^c \tag{4.1}$$

are generators of the Lie algebra, where f_c^{ab} are its structure constants. Fields are given by n-dimensional vectors carrying an irreducible representation of the gauge group. Elements of the symmetry algebra are represented by $n \times n$ matrices. The free action of the field ψ is given by

$$\mathcal{S} = \int d^4x \mathcal{L} = \int d^4x\, \partial_\mu \psi\, \partial^\mu \psi. \tag{4.2}$$

Requiring the gauge invariance of the action $\mathcal{S}$, one has to introduce additional fields, gauge fields and to replace the usual derivatives by covariant derivatives $\mathcal{D}_\mu$.
Let us start with the field ψ building an irreducible representation of the gauge group, i.e.,

$$\delta\psi(x) = i\alpha(x)\psi(x), \tag{4.3}$$

where α is Lie algebra valued,

$$\alpha(x) = \alpha_a(x)T^a.$$

Observe that the derivative of a field ψ does not transform covariantly,

$$\delta\partial_\mu\psi \neq i\alpha(x)\partial_\mu\psi(x). \tag{4.4}$$

Replacing the usual derivatives ∂_μ by covariant derivatives $\mathcal{D}_\mu$ and demanding that $\mathcal{D}_\mu\psi$ transforms covariantly, one has to introduce a gauge potential $A_\mu(x)$,

$$\begin{aligned}
\mathcal{D}_\mu &= \partial_\mu - igA_\mu(x), \\
A_\mu(x) &= A_{\mu a}(x)T^a, \\
\delta A_\mu(x) &= \frac{1}{g}\partial_\mu\alpha(x) + [\alpha(x), A_\mu(x)].
\end{aligned}$$

As it is well known, the interaction fields are a consequence of the gauge invariance of the action. Interactions are gauge interactions. The modified action reads

$$\mathcal{S} = \int d^4x\, \mathcal{D}_\mu\psi\mathcal{D}^\mu\psi, \tag{4.5}$$

including gauge Fields A_μ. Forgetting about mass terms, we still need a kinetic term for the gauge fields in our action. The only requirement is the gauge invariance of the kinetic term, and the theory must be renormalisable at the end. That fixes the kinetic term uniquely. This is a crucial point, and the situation will be different in the case of the Non-Commutative Standard Model. The action reads

$$\mathcal{S} = \int d^4x\, \left(\mathcal{D}_\mu\psi\mathcal{D}^\mu\psi + Tr\, F_{\mu\nu}F^{\mu\nu}\right), \tag{4.6}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ is the field strength. Considering abelian gauge symmetry, commutators in $F_{\mu\nu}$ and in δA_μ will vanish. Let us make one more important remark: there is a sharp distinction between internal and external symmetry transformations. As we will see, that is not true in the case of non-commutative gauge theory.

4.2 Non-Commutative Gauge Theory

Non-commutative gauge theory, as presented in [13, 14], is based on essentially three principles,

- Covariant coordinates,
- Locality and classical limit,
- Gauge equivalence conditions.

Let us first briefly recall our starting point. We have non-commutative coordinates

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},$$
$$\widehat{\mathcal{A}} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ..., \hat{x}^n \rangle\rangle}{\mathcal{I}},$$

the product of function $f, g \in \mathcal{A}$ is given by the Weyl-Moyal product

$$f * g\,(x) = \exp(\frac{i}{2}\frac{\partial}{\partial x^i}\,\theta^{ij}\,\frac{\partial}{\partial y^j})\,f(x)g(y)\Big|_{y\to x}.$$

4.2.1 Covariant Coordinates

Let ψ be a non-commutative field, i.e., $\widehat{\psi} \in \oplus_{i=1}^n \widehat{\mathcal{A}}$,

$$\widehat{\delta}\widehat{\psi}(\widehat{x}) = i\widehat{\alpha}\widehat{\psi}(\widehat{x}) \tag{4.7}$$

or

$$\widehat{\delta}\psi(x) = i\alpha * \psi(x), \tag{4.8}$$

in the $*$ formalism, where $W(\alpha) = \widehat{\alpha}$. Now, a similar situation arises as in eqn. (4.4), only the derivatives are replaced by coordinates. The product of a field and a coordinate does not transform covariantly, since the $*$-product is not commutative,

$$\widehat{\delta}(x * \psi(x)) = i\,x * \alpha(x) * \psi(x) \neq i\,\alpha(x) * x * \psi(x). \tag{4.9}$$

The arguments are the same as before, and we introduce covariant coordinates

$$X^\mu \equiv x^\mu + A^\mu, \tag{4.10}$$

such that

$$\widehat{\delta}(X^\mu * \psi) = i\alpha * (X^\mu * \psi). \tag{4.11}$$

The coovariant coordinates and the gauge potential transform under a non-commutative gauge transformation in the following way

$$\widehat{\delta}X^\mu = i[\alpha \stackrel{*}{,} X^\mu], \tag{4.12}$$
$$\widehat{\delta}A^\mu = i[\alpha \stackrel{*}{,} x^\mu] + i[\alpha \stackrel{*}{,} A^\mu]. \tag{4.13}$$

Other covariant objects can be constructed from covariant coordinates, such as a generalisation of the field strength,

$$F^{\mu\nu} = [X^\mu \overset{*}{,} X^\nu] - i\theta^{\mu\nu}, \qquad \widehat{\delta} F^{\mu\nu} = i[\alpha \overset{*}{,} F^{\mu\nu}]. \tag{4.14}$$

For non degenerate θ, we can define another gauge potential V_μ

$$\widehat{\delta} V_\mu = \partial_\mu \alpha + i[\alpha \overset{*}{,} V_\mu], \tag{4.15}$$
$$F_{\mu\nu} = \partial_\nu V_\mu - \partial_\nu V_\nu - i[V_\mu \overset{*}{,} V_\nu], \tag{4.16}$$
$$\widehat{\delta} F_{\mu\nu} = i[\alpha \overset{*}{,} F_{\mu\nu}], \tag{4.17}$$

using

$$A^\mu = \theta^{\mu\nu} V_\nu, \;\; F^{\mu\nu} = i\theta^{\mu\sigma}\theta^{\nu\tau} F_{\sigma\tau}, \tag{4.18}$$
$$i\theta^{\mu\nu}\partial_\nu f = [x^\mu \overset{*}{,} f].$$

And we get for the covariant derivatives

$$\mathcal{D}_\mu \psi = (\partial_\mu - iV_\mu) * \psi, \tag{4.19}$$
$$\widehat{\delta}(D_\mu * \psi) = i\alpha * \mathcal{D}_\mu \psi.$$

Even for abelian gauge groups, the $*-$commutators in eqns. (4.15) and (4.16) do not vanish, and the theory has similarities to a non-abelian gauge theory on a commutative space-time.

Let us have a closer look at the gauge parameters and the gauge fields. In classical theory, the gauge parameters and the gauge field are Lie algebra valued, as we have mentioned before. Two subsequent gauge transformations are again a gauge transformation,

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{-i[\alpha,\beta]}, \tag{4.20}$$

where $-i[\alpha, \beta] = \alpha^a \beta_b f_c^{ab} T^c$. However, there is a remarkable difference to the non-commutative case. Let M^α be some matrix basis of the enveloping algebra of the internal symmetry algebra. We can expand the gauge parameters in terms of this basis, $\alpha = \alpha_a M^a, \; \beta = \beta_b M^b$. Two subsequent non-commutative gauge transformations again take the form

$$\widehat{\delta}_\alpha \widehat{\delta}_\beta - \widehat{\delta}_\beta \widehat{\delta}_\alpha = \widehat{\delta}_{-i[\alpha,\beta]}, \tag{4.21}$$

but the commutator of two gauge transformations involves the $*$-commutator of the gauge parameters, and

$$[\alpha \overset{*}{,} \beta] = \frac{1}{2}\{\alpha_a \overset{*}{,} \beta_b\}[M^a, M^b] + \frac{1}{2}[\alpha_a \overset{*}{,} \beta_b]\{M^a, M^b\}, \tag{4.22}$$

where $\{M^a, M^b\} = M^a M^b + M^b M^a$ is the anti-commutator. The difference to (4.20) is the anti-commutator $\{M^a, M^b\}$, respectively the $*-$commutator of the gauge parameters, $[\alpha_a \stackrel{*}{,} \beta_b]$. This term causes some problems. Let us assume that M^α are the Lie algebra generators. Does the relation (4.22) close? Or does (4.22) rule out Lie algebra valued gauge parameters? Clearly, the only crucial term is the anti-commutator. The anti-commutator of two hermitian matrices is again hermitian. But the anti-commutator of traceless matrices is in general not traceless. Relation (4.22) will be satisfied for the generators of the fundamental representation of $U(n)$. Therefore it has been argued [51, 52, 53] that $U(n)$ - and with some difficulty SO(n) and Sp(n) [54, 55] - is the only gauge group that can be generalised to non-commutative spaces. But in fact arbitrary gauge groups can be tackled. But the gauge parameters α, β and the gauge fields A_μ have to be enveloping algebra valued [13, 56], in general. Gauge fields and parameters now depend on infinitely many parameters, since the enveloping algebra is infinite dimensional. Luckily, the infinitely many degrees of freedom can be reduced to a finite number, namely the classical parameters, by the so-called Seiberg-Witten maps we will discuss in the next paragraph.

4.2.2 Locality and Classical Limit

The non-commutative $*$-product can be written as an expansion in a formal parameter h,

$$f * g = f \cdot g + \sum_{n=1}^{\infty} h^n C_n(f, g).$$

In the commutative limit $h \to 0$, the $*$-product reduces to the pointwise product of functions. One may ask, if there is a similiar commutative limit for the fields? The solution to this question was given by [22],

$$\widehat{A}_\mu[A] = A_\mu + \frac{1}{2}\theta^{\sigma\tau}\left(A_\tau \partial_\sigma A_\mu + F_{\sigma\mu} A_\tau\right) + \mathcal{O}(\theta^2), \tag{4.23}$$

$$\widehat{\psi}[\psi, A] = \psi + \frac{1}{2}\theta^{\mu\nu} A_\nu \partial_\mu \psi + \mathcal{O}(\theta^2), \tag{4.24}$$

$$\widehat{\alpha} = \alpha + \frac{1}{2}\theta^{\mu\nu} A_\nu \partial_\mu \alpha + \mathcal{O}(\theta^2), \tag{4.25}$$

where A_μ is the commutative gauge field and α the commutative gauge parameter. The gauge field and gauge parameter defined above is not hermitian. The solutions are not unique and we can use this freedom to define a unique hermitian solution, cf. 4.2.3.

First of all, let me introduce an important convention to which we will stick from now on. Quantities with "hat" ($\widehat{\psi}$, $\widehat{A}$, $\widehat{\alpha} \ldots \in (\mathcal{A}, *)$) refer to non-commutative fields and gauge parameters, respectively which can be expanded (cf. above) in terms of the ordinary commutative fields and gauge parameters, respectively (ψ, A, α).

The Seiberg-Witten maps (4.23 - 4.25) reduce the infinitely many parameters of the enveloping algebra to the classical gauge parameters.
The origins of this map are in string theory. It is there that gauge invariance depends on the regularisation scheme applied [22]. Pauli-Villars regularisation provides us with classical gauge invariance

$$\delta a_i = \partial_i \lambda, \tag{4.26}$$

where a_i is the gauge field and λ the gauge paramter, whence point-splitting regularisation comes up with non-commutative gauge invariance

$$\widehat{\delta} \widehat{A}_i = \partial_i \widehat{\Lambda} + i[\widehat{\Lambda} \overset{*}{,} \widehat{A}_i]. \tag{4.27}$$

Seiberg and Witten argued that consequently there must be a local map from ordinary gauge theory to non-commutative gauge theory

$$\widehat{A}[a], \ \widehat{\Lambda}[\lambda, a], \tag{4.28}$$

satisfying

$$\widehat{A}[a + \delta_\lambda a] = \widehat{A}[a] + \widehat{\delta}_\lambda \widehat{A}[a]. \tag{4.29}$$

The Seiberg-Witten maps are solutions of (4.29). By locality we mean that in each order in the non-commutativity parameter there is only a finite number of derivatives.

4.2.3 Gauge Equivalence Conditions

Let us remember that we consider arbitrary gauge groups. Non-commutative gauge fields $\widehat{A}$ and gauge parameters $\widehat{\Lambda}$ are enveloping algebra valued. Let us choose a symmetric basis in the algebra, T^a, $\frac{1}{2}(T^a T^b + T^b T^a)$, ...

$$\widehat{\Lambda}(x) = \widehat{\Lambda}_a(x) T^a + \widehat{\Lambda}^1_{ab}(x) \ : T^a T^b : + \dots , \tag{4.30}$$

$$\widehat{A}_\mu(x) = \widehat{A}_{\mu a}(x) T^a + \widehat{A}_{\mu ab}(x) \ : T^a T^b : + \dots . \tag{4.31}$$

Eqn. (4.29) defines the Seiberg-Witten maps for the gauge field and the gauge parameter. However, it is more practical to find equations for the gauge parameter and the gauge field alone [14]. First we will concentrate on the gauge parameters $\widehat{\Lambda}$. We already encountered the consistency condition

$$\widehat{\delta}_\alpha \widehat{\delta}_\beta - \widehat{\delta}_\beta \widehat{\delta}_\alpha = \widehat{\delta}_{-i[\alpha,\beta]}.$$

More explicitly, it reads

$$i\widehat{\delta}_\alpha\widehat{\beta}[A] - i\widehat{\delta}_\beta\widehat{\alpha}[A] + [\widehat{\alpha}[A] \stackrel{*}{,} \widehat{\beta}[A]] = (\widehat{[\alpha,\beta]})[A]. \tag{4.32}$$

Keeping in mind the results from Subsection 4.2.2, we can expand $\widehat{\alpha}$ in terms of the non-commutativity θ,

$$\widehat{\alpha}[A] = \alpha + \alpha^1[A] + \alpha^2[A] + \ldots, \tag{4.33}$$

where α^n is $\mathcal{O}(\theta^n)$. The consistency relation (4.32) can be solved order by order in θ.

$$0^{\text{th}} \text{ order}: \ \alpha^0 = \alpha, \tag{4.34}$$

$$\begin{aligned} 1^{\text{st}} \text{ order}: \ \alpha^1 &= \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu\alpha, A_\nu\} \\ &= \frac{1}{2}\theta^{\mu\nu}\partial_\mu\alpha_a A_{\mu b} : T^a T^b : . \end{aligned} \tag{4.35}$$

For fields $\widehat{\psi}$, the condition

$$\delta_\alpha\widehat{\psi}[A] = \widehat{\delta}_\alpha\widehat{\psi}[A] = i\widehat{\alpha}[A] * \widehat{\psi}[A] \tag{4.36}$$

has to be satisfied. That means that the ordinary gauge transformation induces a non-commutative gauge transformation. We expand the fields in terms of the non-commutativity

$$\widehat{\psi} = \psi^0 + \psi^1[A] + \psi^2[A] + \ldots \tag{4.37}$$

and solve eqn. (4.36) order by order in θ. In first order, we have to find a solution to

$$\delta_\alpha\psi^1[A] = i\alpha\psi^1[A] + i\alpha^1[A]\,\psi - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu\psi. \tag{4.38}$$

It is given by

$$0^{\text{th}} \text{ order}: \ \psi^0 = \psi, \tag{4.39}$$

$$1^{\text{st}} \text{ order}: \ \psi^1 = -\frac{1}{2}\theta^{\mu\nu}A_\mu\partial_\nu\psi + \frac{i}{4}\theta^{\mu\nu}A_\mu A_\nu\psi. \tag{4.40}$$

The gauge fields $\widehat{A}_\mu$ have to satisfy

$$\delta_\alpha\widehat{A}_\mu[A] = \partial_\mu\widehat{\alpha}[A] + i[\widehat{\alpha}[A] \stackrel{*}{,} \widehat{A}_\mu[A]]. \tag{4.41}$$

Using the expansion

$$\widehat{A}_\mu[A] = A^0_\mu + A^1_\mu[A] + A^2_\mu[A] + \ldots \tag{4.42}$$

and solving (4.41) order by order, we end up with

$$0^{\text{th}} \text{ order}: \; A^0_\mu = A_\mu, \tag{4.43}$$

$$1^{\text{st}} \text{ order}: \; A^1_\mu = -\frac{1}{4}\theta^{\tau\nu}\{A_\tau, \partial_\nu A_\mu + F_{\nu\mu}\}, \tag{4.44}$$

where $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu - i[A_\nu, A_\mu]$. Similarly, we have for the field strength $\widehat{F}_{\mu\nu}$

$$\delta_\alpha \widehat{F}_{\mu\nu} = i[\widehat{\alpha}, \widehat{F}_{\mu\nu}] \text{ and} \tag{4.45}$$

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\theta^{\sigma\tau}\{F_{\mu\sigma}, F_{\nu\tau}\} - \frac{1}{4}\theta^{\sigma\tau}\{A_\sigma, (\partial_\tau + \mathcal{D}_\tau)F_{\mu\nu}\}, \tag{4.46}$$

where $\mathcal{D}_\mu F_{\tau\nu} = \partial_\mu F_{\tau\nu} - i[A_\mu, F_{\tau\nu}]$.

4.2.4 Remarks

Let us conclude this section with some remarks and observations.

- Seiberg-Witten maps provide solutions to the gauge equivalence relations.
- Gauge equivalence relations are not the only possibile approach to Seiberg-Witten maps. Another approach is via non-commutative Wilson lines, see e.g., [57].
- However, a certain ambiguity in the Seiberg-Witten map remains. They are unique modulo classical field redefinition and non-commutative gauge transformation. We used these ambiguities in order to choose $\widehat{\Lambda}$, $\widehat{A}_\mu$ hermitian. The freedom in Seiberg-Witten map may also be essential for renormalisation issues. There, parameters characterising the freedom in the Seiberg-Witten maps become running coupling constants [58]. Discussing tensor products of gauge groups, this freedom will also be of crucial importance, in Section 5.2.
- Gauge groups in non-commutative spaces contain space-time translations. Since

  $$\partial f = -i\theta^{-1}_{ij}[\hat{x}^j, f], \tag{4.47}$$

  we can express the infinitesimal translation of the field A_i as

  $$\delta A_i = v^j \partial_j A_i = i[\epsilon \stackrel{*}{,} A_i], \tag{4.48}$$

  where $\epsilon = -v^j \theta^{-1}_{jk} x^k$. The gauge transformation of A_i with gauge parameter ϵ gives

  $$\widehat{\delta}_\epsilon A_i = i[\epsilon \stackrel{*}{,} A_i] - v^j \theta^{-1}_{ji}.$$

This agrees with (4.48), ignoring the overall constant, which has no physical effect [59].

- Non-commutative gauge theory allows the construction of realistic particle models on a non-commutative space-time with an arbitrary gauge group as internal symmetry group. Non-commutative gauge parameters and gauge fields are enveloping algebra valued, in general (e.g., for $SU(n)$), but via Seiberg-Witten maps the infinite number of degrees of freedom is reduced to the classical gauge parameters. Therefore these models will have the proper number of degrees of freedom.

Chapter 5

The Non-Commutative Standard Model

5.1 Particle Content

The symmetry group of the Standard Model is $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)$. This is also the gauge group we want to generalise to a non-commutative space-time. Let me stress that we will therefore not introduce any new degrees of freedom or any new parameters we would have to get rid off in the end. Naively spoken, the method we use is very simple. We just write down the Standard Model Lagrangian, replace $\cdot$ by $*$ and the fields Ψ, A by the Seiberg-Witten transformed fields $\widehat{\Psi}[\Psi, A]$, $\widehat{A}[A]$. Of course, it is not as easy as that. We have written down the non-commutative Lagrangian in (5.44). In the rest of this Chapter we are going to explain what all these terms mean. Let us remind ourselves of the convention that fields with a hat are non-commutative whereas those without a hat are ordinary fields.
First of all, we have to address some restrictions on non-commutative gauge theories proposed in [51, 52, 53]. In Section 4.2, we have already discussed the problem of generalising other gauge groups than $U(n)$ to non-commutative spaces. Further it is argued in [53] that a non-commutative field may be charged under at most two gauge groups only. This is a problem since we have three gauge groups in the Standard Model. It will be resolved in Section 5.2 combining all the gauge fields of the Standard Model into a single "master gauge field" und applying the Seiberg-Witten map. We have the

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_Q$
e_R	**1**	**1**	-1	-1
$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	**1**	**2**	$-1/2$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
u_R	**3**	**1**	$2/3$	$2/3$
d_R	**3**	**1**	$-1/3$	$-1/3$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	**3**	**2**	$1/6$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$
$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	**1**	**2**	$1/2$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
B^i	**1**	**3**	0	$(\pm 1, 0)$
$\mathcal{A}$	**1**	**1**	0	0
G^a	**8**	**1**	0	0

Table 5.1: The Standard Model fields. The electric charge is given by the Gell-Mann-Nishijima relation $Q = (T_3 + Y)$. The fields B^i with $i \in \{+,-,3\}$ denote the three electroweak gauge bosons. The gluons G^i are in the octet representation of $SU(3)_C$.

following particle spectrum, see Table 5.1,

$$\Psi_L^{(i)} = \begin{pmatrix} L_L^{(i)} \\ Q_L^{(i)} \end{pmatrix}, \qquad \Psi_R^{(i)} = \begin{pmatrix} e_R^{(i)} \\ u_R^{(i)} \\ d_R^{(i)} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{5.1}$$

where Ψ_L denotes the left handed fermions - the leptons L_L and the quarks Q_L - , Ψ_R the right handed fermions. $(i) \in \{1,2,3\}$ is the generation index, and ϕ^+ and ϕ^0 are the complex scalar fields of the scalar Higgs doublet. $g'\mathcal{A}_\mu(x)Y$ is the gauge field of the hypercharge symmetry $U(1)_Y$, $B_\mu(x) = \frac{g}{2}B_{\mu a}(x)\sigma^a$ the field of the weak interaction $SU(2)_L$ and $G_\mu(x) = \frac{g_S}{2}G_{\mu a}(x)\lambda^a$ of the strong interaction $SU(3)_C$, respectively. The coupling constants of the gauge groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ are respectively denoted by g', g and g_S. σ^a are the usual Pauli matrices and λ^a the Gell-Mann matrices. We have to discuss four serious problems, namely

the tensor product of gauge groups,

the so-called charge quantisation problem in non-commutative QED,

the gauge invariance of the Yukawa couplings and

ambiguities in the choice of the kinetic terms for the gauge fields in the Lagrangian (5.44).

5.2 Tensor Product of Gauge Groups

There are several possibilities to deal with the tensor product of gauge groups which correspond to the freedom in the choice of the Seiberg-Witten maps. The most symmetric and natural choice is to take the classical tensor product

$$V_\mu = g' A_\mu(x) Y + \frac{g}{2} \sum_{a=1}^{3} B_{\mu a} \sigma^a + \frac{g_S}{2} \sum_{a=1}^{8} G_{\mu a} \lambda^a, \tag{5.2}$$

defining one overall "master" gauge field V_μ. The corresponding gauge paramter Λ is given by

$$\Lambda = g' \alpha(x) Y + \frac{g}{2} \sum_{a=1}^{3} \alpha_a^L(x) \sigma^a + \frac{g_S}{2} \sum_{b=1}^{8} \alpha_b^S(x) \lambda^b. \tag{5.3}$$

The non-commutative gauge field $\widehat{V}[V]$ and gauge parameter $\widehat{\Lambda}[\Lambda, V]$ are given by the Seiberg-Witten maps (4.44) and (4.35)

$$\widehat{V}_\xi[V] = V_\xi + \frac{1}{4}\theta^{\mu\nu}\{V_\nu, \partial_\mu V_\xi\} + \frac{1}{4}\theta^{\mu\nu}\{F_{\mu\xi}, V_\nu\} + \mathcal{O}(\theta^2), \tag{5.4}$$

$$\widehat{\Lambda} = \Lambda + \frac{1}{4}\theta^{\mu\nu}\{V_\nu, \partial_\mu \Lambda\} + \mathcal{O}(\theta^2), \tag{5.5}$$

with the ordinary field strength $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]$. Note that this is not equal to a naive sum of non-commutative gauge parameters and fields corresponding to the three factors in G_{SM}. This is due to the nonlinearity of the Seiberg-Witten maps and ultimately a consequence of the nonlinear consistency condition (4.32). As a consequence, the gauge groups mix in higher order in θ and cannot be viewed independently.
Let me also say a few words on the general tensor product of two gauge groups [60]. The most general solution of the gauge consistency condition (4.32) - for one gauge group - is given by

$$\widehat{\Lambda}[A] = \Lambda + \frac{1}{2}\theta^{\mu\nu}\{A_\nu, \partial_\mu \Lambda\}_c + \mathcal{O}(\theta^2), \tag{5.6}$$

where

$$\{A, B\}_c := \frac{1}{2}\{A, B\} + (c - 1/2)[A, B], \tag{5.7}$$

c is a complex function of space-time. We have $\{A, B\}_{1/2} = \frac{1}{2}\{A, B\}$. The according gauge field is of the form

$$\widehat{A}_\mu[A] = A_\mu + \frac{1}{2}\theta^{\nu\sigma}\{A_\sigma, \partial_\nu A_\mu\}_c + \frac{1}{2}\theta^{\nu\sigma}\{F_{\nu\mu}, A_\sigma\}_c + \mathcal{O}(\theta^2). \tag{5.8}$$

The gauge parameter $\widehat{\Lambda}_{(\Lambda,\Lambda')}[A,A']$ of the tensor product of two gauge groups G and G' consists of two parts

$$\widehat{\Lambda}_{(\Lambda,\Lambda')}[A,A'] = \widehat{\Lambda}_{\Lambda}[A,A'] + \widehat{\Lambda}'_{\Lambda'}[A,A'], \tag{5.9}$$

because of the linearity in the classical case. $\widehat{\Lambda}_{(\Lambda,\Lambda')}[A,A']$ has to fulfill the consistency relation (4.32). Therefore both, $\widehat{\Lambda}_{\Lambda}[A,A']$ and $\widehat{\Lambda}'_{\Lambda'}[A,A']$ have to satisfy (4.32) separately, and there is an additional cross relation

$$[\widehat{\Lambda}_{\Lambda} \stackrel{*}{,} \widehat{\Lambda}'_{\Lambda'}] + i\delta_{\Lambda}\widehat{\Lambda}'_{\Lambda'} - i\delta_{\Lambda'}\widehat{\Lambda}_{\Lambda} = 0. \tag{5.10}$$

The solution is given by [60]

$$\begin{aligned}\widehat{\Lambda}_{(\Lambda,\Lambda')}[A,A'] = \Lambda + \Lambda' + \frac{1}{2}\theta^{\mu\nu}\Big(\{A_\nu, \partial_\mu \Lambda\}_c + \{A'_\nu, \partial_\mu\Lambda'\}_d\Big) \\ +(1-\frac{\gamma(x)}{2})\,\theta^{\mu\nu}A'_\nu\partial_\mu\Lambda + \frac{\gamma(x)}{2}\theta^{\mu\nu}A_\nu\partial_\mu\Lambda' + \mathcal{O}(\theta^2).\end{aligned} \tag{5.11}$$

$\gamma(x)$ is a real function, $c-1/2$ and $d-1/2$ are purely imaginary. Solving eqns. (4.36) and (4.41) using (5.11) will provide us with the Seiberg-Witten maps for matter and gauge fields. The symmetric choice in (5.5) is recovered by choosing $\gamma = 1$ and $c = d = 1/2$.

5.3 Charge Quantisation in Non-Commutative QED

It seems that in non-commutative QED only charges $\pm q, 0$ can be accounted for, once q is fixed [61, 62]. The minimal coupling is given by

$$\widehat{\mathcal{D}}_\mu\widehat{\psi} = \partial_\mu\widehat{\psi} - iq\widehat{A}_\mu * \widehat{\Psi}. \tag{5.12}$$

The only other couplings of the field $\widehat{A}_\mu$ to a matter field consistent with the non-commutative gauge transformation $\widehat{\delta}_\alpha\widehat{A}_\mu = \partial_\mu\widehat{\alpha} + i[\widehat{\alpha} \stackrel{*}{,} \widehat{A}_\mu]$ are

$$\widehat{\mathcal{D}}^-_\mu\widehat{\psi}^- = \partial_\mu\widehat{\psi}^- + iq\widehat{\psi}^- * \widehat{A}_\mu, \tag{5.13}$$

$$\widehat{\mathcal{D}}^0_\mu\widehat{\psi}^0 = \partial_\mu\widehat{\psi}^0, \tag{5.14}$$

$$\widehat{\mathcal{D}}^{0'}_\mu\widehat{\psi}^{0'} = \partial_\mu\widehat{\psi}^{0'} - iq[\widehat{A}_\mu \stackrel{*}{,} \widehat{\psi}^{0'}], \tag{5.15}$$

corresponding to charge $-q$ and 0, respectively. Other charges $q^{(n)}$ cannot be absorbed into the respective field $\widehat{A}_\mu$, because of the commutator in

$$\widehat{F}_{\mu\nu} = \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu + ieq[\widehat{A}_\mu \stackrel{*}{,} \widehat{A}_\nu], \tag{5.16}$$

$$\widehat{\delta}_\alpha\widehat{A}_\mu = \partial_\mu\widehat{\alpha} + i[\widehat{\alpha} \stackrel{*}{,} ,\widehat{A}_\mu]. \tag{5.17}$$

Classically, we can have two particles ψ and ψ' with charges q and q' coupling to the same gauge field. The gauge transformation of these fields has the form

$$\begin{aligned}\delta\psi &= iqe\lambda\psi,\\ \delta\psi' &= iq'e\lambda'\psi',\end{aligned}$$

with covariant derivatives

$$\begin{aligned}\mathcal{D}_\mu\psi &= \partial_\mu\psi - ieqa_\mu\psi,\\ \mathcal{D}_\mu\psi' &= \partial_\mu\psi' - ieq'a'_\mu\psi',\end{aligned}$$

where

$$\begin{aligned}\delta a_\mu &= \partial_\mu\lambda,\\ \delta a'_\mu &= \partial_\mu\lambda'.\end{aligned}$$

Now, let us assume that $\lambda' = \lambda$. We can consistently define

$$a_\mu = \frac{q'}{q}a'_\mu, \qquad f_{\mu\nu} = \frac{q'}{q}f'_{\mu\nu}.$$

The $*$-commutators spoil this simple picture. The solution to this quantisation problem is again provided by the Seiberg-Witten maps. We have to introduce a different gauge field $\widehat{a}_\mu^{(n)}$ for each distinct charge $q^{(n)}$ that appears in the theory. It seems that we have introduced too many degrees of freedom, but the Seiberg-Witten map for $\widehat{a}_\mu^{(n)}$ is an expansion in the (single) commutative gauge field a_μ and θ only,

$$\widehat{a}_\mu^{(n)} = a_\mu + \frac{eq^{(n)}}{4}\theta^{\sigma\tau}\{\partial_\sigma a_\mu, a_\tau\} + \frac{eq^{(n)}}{4}\theta^{\sigma\tau}\{f_{\sigma\mu}, a_\tau\} + \mathcal{O}(\theta^2). \tag{5.18}$$

The degrees of freedom are reduced to the classical ones.

5.4 Yukawa Couplings

Let us now consider the Yukawa coupling terms in (5.44) and their behaviour under gauge transformation. They involve products of three fields, e.g.,

$$-\sum_{i,j=1}^{3}\Big(W^{ij}\,(\bar{\widehat{L}}_L^{(i)} * \rho_L(\widehat{\Phi})) * \widehat{e}_R^{(j)} + W^{\dagger ij}\,\bar{\widehat{e}}_R^{(i)} * (\rho_L(\widehat{\Phi})^\dagger * \widehat{L}_L^{(j)})\Big). \tag{5.19}$$

Only in the case of commutative space-time, Φ commutes with the generators of $U(1)$ and $SU(3)_C$. Therefore, the Higgs field needs to transform from both sides in order to "cancel charges" from fields on either side (e.g., $\widehat{\bar{L}}_L^{(i)}$ and $\widehat{e}_R^{(j)}$ in (5.19)). The expansion of $\widehat{\Phi}$ transforming on the left and on the right under arbitrary gauge groups is called hybrid Seiberg-Witten map,

$$\begin{aligned}\widehat{\Phi}[\Phi, A, A'] = \phi + \tfrac{1}{2}\theta^{\mu\nu}A_\nu\Big(\partial_\mu\phi - \tfrac{i}{2}(A_\mu\phi + \phi A'_\mu)\Big) \\ -\tfrac{1}{2}\theta^{\mu\nu}\Big(\partial_\mu\phi - \tfrac{i}{2}(A_\mu\phi + \phi A'_\mu)\Big)A'_\nu + \mathcal{O}(\theta^2),\end{aligned} \tag{5.20}$$

with $\widehat{\delta}\widehat{\Phi} = i\widehat{\Lambda} * \widehat{\Phi} - i\widehat{\Phi} * \widehat{\Lambda}'$. In the above Yukawa term (5.19), we have

$$\rho_L(\widehat{\Phi}) = \widehat{\Phi}[\phi, V, V'], \tag{5.21}$$

with

$$\begin{aligned}V_\mu &= -\frac{1}{2}g'\mathcal{A}_\mu + gB_\mu^a T_L^a, \\ V'_\mu &= g'\mathcal{A}_\mu.\end{aligned}$$

We further need a different representation in each of the Yukawa couplings. The missing representations of the gauge potentials appearing in (5.44) are

$$\rho_Q(\widehat{\Phi}) = \widehat{\Phi}[\phi,\, \frac{1}{6}g'\mathcal{A}_\mu + gB_\mu^a T_L^a + g_S G_\mu^a T_S^a,\, \frac{1}{3}g'\mathcal{A}_\nu - g_S G_\nu^a T_S^a], \tag{5.22}$$

$$\rho_{\bar{Q}}(\widehat{\Phi}) = \widehat{\Phi}[\phi,\, \frac{1}{6}g'\mathcal{A}_\mu + gB_\mu^a T_L^a + g_S G_\mu^a T_S^a,\, -\frac{2}{3}g'\mathcal{A}_\nu - g_S G_\nu^a T_S^a]. \tag{5.23}$$

The respective sum of the gauge fields on both sides gives the proper quantum numbers of the Higgs shown in Table 5.1. The representation ρ_0 of these gauge potentials in the kinetic term of the Higgs and in the Higgs potential is the simplest one possible,

$$\rho_0(\widehat{\Phi}[\phi, V_\mu, V'_\nu]) = \widehat{\Phi}[\phi,\, \frac{1}{2}g'\mathcal{A}_\mu + gB_\mu^a T_L^a,\, 0]. \tag{5.24}$$

Of course, there is an ambiguity in the choice of the gauge fields. One can add an abritrary $U(1)_Y$ or $SU(3)_C$ gauge field on the right side, as long as it is subtracted again on the left side. Eventually, physical criteria should single out the right choice. These criteria may include, e.g., renormalisation, CPT invariance, anomaly freedom, or any kind of symmetry one might want to impose on the action.

5.5 Kinetic Terms for the Gauge Bosons

As we have mentioned earlier in Section 4.1, the kinetic terms for the gauge field in the classical theory are determined uniquely by the requirements of gauge invariance and renormalisability. In the non-commutative case, we do not have a principle like renormalisability at hand. Gauge invariance alone does not fix these terms in the Lagrangian. The non-commutative Standard Model as defined here, has rather to be considered as an effective theory, where renormalisability is not applicable. Otherwise, the role of the non-commutativity θ has to be considered very carefully. θ may become a space-time field with a kinetic term of its own. Therefore, the representations used in the trace of the kinetic terms for the gauge bosons are not uniquely determined. We will take the simpliest choice, since we are interested in a version of the Standard Model on non-commutative space-time with minimal modifications. This choice is discussed in Subsection 5.5.1. In Subsection 5.5.2 we will consider a maybe more physical and natural choice of representation. Considering a Standard Model originating from a $SO(10)$ GUT theory [60], these terms have a unique non-commutative generalisation.

5.5.1 Minimal Non-Commutative Standard Model

The simplest choice for the gauge kinetic terms is named Minimal Non-Commutative Standard Model. The gauge kinetic terms have the form displayed in eqn. (5.44),

$$-\int d^4x \frac{1}{2g'}\mathbf{tr_1}\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g}\mathbf{tr_2}\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g_S}\mathbf{tr_3}\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu}.$$

It is the sum of traces over the $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ sectors. $\mathbf{tr_2}$ and $\mathbf{tr_3}$ are the usual $SU(2)_L$ and $SU(3)_C$ traces, respectively. $\mathbf{tr_1}$ is the trace over the $U(1)_Y$ sector with

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as representation of the charge generator.

5.5.2 Non-Minimal Non-Commutative Standard Model

A perhaps more physical version of the Non-Commutative Standard Modell is obtained, if we consider a charge matrix Y containing all the fields of the Standard Model with covariant derivatives acting on them. For the simplicity of presentation we will only consider one family of fermions and quarks. The situation taking account of all the families can be recovered by a redefinition of the coupling constants given in (5.40). Then the charge matrix has the form

$$Y = \begin{pmatrix} -1 & & & & & & & \\ & -1/2 & & & & & & \\ & & -1/2 & & & & & \\ & & & 2/3 & & & & \\ & & & & 2/3 & & & \\ & & & & & 2/3 & & \\ & & & & & & -1/3 & \\ & & & & & & & \ddots \end{pmatrix}, \tag{5.25}$$

according to Table 5.1. It acts on fields given by column vectors containing all the particles of the theory,

$$\Psi = \begin{pmatrix} e_R \\ L_L \\ u_R \\ d_R \\ Q_L \\ \phi \end{pmatrix}. \tag{5.26}$$

The kinetic term for the gauge field is given by

$$S_{\text{gauge}} = -\int d^4x \,\mathrm{Tr}\, \frac{1}{2G^2} \widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu}, \tag{5.27}$$

where $\widehat{F}_{\mu\nu} = \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu - i[\widehat{V}_\mu \stackrel{*}{,} \widehat{V}_\nu]$. The classical gauge field V_μ is the sum

$$V_\mu = g' A_\mu(x) Y + \frac{g}{2} \sum_{a=1}^{3} B_{\mu a} \sigma^a + \frac{g_S}{2} \sum_{a=1}^{8} G_{\mu a} \lambda^a,$$

where Y is defined by (5.25), and $\widehat{V}_\mu = \widehat{V}_\mu[V]$. G takes the role of the coupling constant for the "master" gauge field $\widehat{V}_\mu$. It is an operator commuting with all the generators of (Y, T_L^a, T_c^b). It can be expressed in terms of Y and the six constants $g_1, \ldots, g_2$ referring to the six multiplets (cf. Table 5.1),

$$\begin{aligned} G e_R &= g_1 e_R, \\ G L_L &= g_2 L_L, \\ &\vdots \end{aligned}$$

In the classical limit only three combinations of these six constants are relevant. They correspond to the usual coupling constants, g', g, g_S,

$$\mathbf{Tr}\frac{1}{\mathbf{G}^2}\,Y^2 = \frac{1}{2g'^2}, \tag{5.28}$$

$$\mathbf{Tr}\frac{1}{\mathbf{G}^2}\,T_L^aT_L^b = \delta^{ab}\frac{1}{2g^2}, \tag{5.29}$$

$$\mathbf{Tr}\frac{1}{\mathbf{G}^2}\,T_S^aT_S^b = \delta^{ab}\frac{1}{2g_S^2}. \tag{5.30}$$

The above equations read, taking the charge matrix (5.25) into account

$$\frac{1}{g_1^2} + \frac{1}{2g_2^2} + \frac{4}{3g_3^2} + \frac{1}{3g_4^2} + \frac{1}{6g_5^2} + \frac{1}{2g_6^2} = \frac{1}{2g'^2}, \tag{5.31}$$

$$\frac{1}{g_2^2} + \frac{3}{g_5^2} + \frac{1}{g_6^2} = \frac{1}{g^2}, \tag{5.32}$$

$$\frac{1}{g_3^2} + \frac{1}{g_4^2} + \frac{2}{g_5^2} = \frac{1}{g_S^2}. \tag{5.33}$$

These three equations define for fixed g', g, g_S a three-dimensional simplex in the six-dimensional moduli space spanned by $1/g_1^2$, ..., $1/g_6^2$. The remaining three degrees of freedom become relevant at order θ in the expansion of the non-commutative action. Interesting are in particular the following traces corresponding to triple gauge boson vertices:

$$\mathbf{Tr}\frac{1}{\mathbf{G}^2}Y^3 = -\frac{1}{g_1^2} - \frac{1}{4g_2^2} + \frac{8}{9g_3^2} - \frac{1}{9g_4^2} + \frac{1}{36g_5^2} + \frac{1}{4g_6^2}, \tag{5.34}$$

$$\mathbf{Tr}\frac{1}{\mathbf{G}^2}YT_L^aT_L^b = \frac{1}{2}\delta^{ab}\left(-\frac{1}{2g_2^2} + \frac{1}{2g_5^2} + \frac{1}{2g_6^2}\right), \tag{5.35}$$

$$\mathbf{Tr}\frac{1}{\mathbf{G}^2}YT_S^cT_S^d = \frac{1}{2}\delta^{cd}\left(\frac{2}{3g_3^2} - \frac{1}{3g_4^2} + \frac{1}{3g_5^2}\right). \tag{5.36}$$

We may choose, e.g., to maximise the traces over Y^3 and $YT_L^aT_L^b$. This will give $1/g_1^2 = 1/(2g'^2) - 4/(3g_S^2) - 1/(2g^2)$, $1/g_3^2 = 1/g_S^2$, $1/g_6^2 = 1/g^2$, $1/g_2^2 = 1/g_4^2 = 1/g_5^2 = 0$ and

$$\mathbf{Tr}\tfrac{1}{\mathbf{G}^2}Y^3 = -\tfrac{1}{2g'^2} + \tfrac{3}{4g^2} + \tfrac{20}{9g_S^2}, \tag{5.37}$$

$$\mathbf{Tr}\tfrac{1}{\mathbf{G}^2}YT_L^aT_L^b = \tfrac{1}{4g^2}\delta^{ab}, \tag{5.38}$$

$$\mathbf{Tr}\tfrac{1}{\mathbf{G}^2}YT_S^cT_S^d = \tfrac{2}{6g_S^2}\delta^{cd}. \tag{5.39}$$

In the scheme that we have presented in Section 5.5.1 the three traces above are all zero. One consequence is that while non-commutativity does not *require* a triple photon vertex, such a vertex is nevertheless consistent with non-commutativity. It is important to note that the values of all three traces are bounded for any choice of constants. By rescaling of the coupling constants $g_1, \ldots, g_6$,

$$\begin{aligned} g_1, g_3, g_4, g_6 &\longrightarrow g_1, g_3, g_4, g_6, \\ g_2 &\longrightarrow \frac{g_2}{\sqrt{3}}, \\ g_5 &\longrightarrow \frac{g_5}{\sqrt{3}}, \end{aligned} \tag{5.40}$$

we recover the case where we sum over all three families of fermions.

5.6 The Model

Let us work out the Lagrangian of the Non-Commutative Standard Model expanded in the non-commutativity θ up to first order. The non-commutative fermion fields $\widehat{\Psi}^{(n)}$ corresponding to particles labelled by (n) are

$$\widehat{\Psi}^{(n)} = \Psi^{(n)} + \frac{1}{2}\theta^{\mu\nu}\rho_{(n)}(V_\nu)\partial_\mu\Psi^{(n)} + \frac{i}{8}\theta^{\mu\nu}[\rho_{(n)}(V_\mu), \rho_{(n)}(V_\nu)]\Psi^{(n)} + \mathcal{O}(\theta^2). \tag{5.41}$$

The expansion of the gauge field has been given in eqn. (5.4) and of the gauge parameter in (5.5). The non-commutative field strength is

$$\widehat{F}_{\mu\nu} = \partial_\mu\widehat{V}_\nu - \partial_\nu\widehat{V}_\mu - i[\widehat{V}_\mu \stackrel{*}{,} \widehat{V}_\nu]. \tag{5.42}$$

The non-commutative Higgs field $\widehat{\Phi}$ is given by the hybrid Seiberg-Witten map (5.20),

$$\begin{aligned} \widehat{\Phi} = \phi &+ \frac{1}{2}\theta^{\mu\nu}V_\nu\Big(\partial_\mu\phi - \frac{i}{2}(V_\mu\phi + \phi V'_\mu)\Big) \\ &- \frac{1}{2}\theta^{\mu\nu}\Big(\partial_\mu\phi - \frac{i}{2}(V_\mu\phi + \Phi V'_\mu)\Big)V'_\nu + \mathcal{O}(\theta^2). \end{aligned} \tag{5.43}$$

The non-commutative Standard Model Lagrangian can now be written in a very compact way,

$$S_{\text{NCSM}} = \int d^4x \sum_{i=1}^{3} \overline{\widehat{\Psi}}_L^{(i)} * i\widehat{\not{D}}\widehat{\Psi}_L^{(i)} + \int d^4x \sum_{i=1}^{3} \overline{\widehat{\Psi}}_R^{(i)} * i\widehat{\not{D}}\widehat{\Psi}_R^{(i)}$$

$$
\begin{aligned}
&-\int d^4x \frac{1}{2g'}\mathbf{tr_1}\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g}\mathbf{tr_2}\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} \\
&-\int d^4x \frac{1}{2g_S}\mathbf{tr_3}\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} + \int d^4x \Big(\rho_0(\widehat{D}_\mu\widehat{\Phi})^\dagger * \rho_0(\widehat{D}^\mu\widehat{\Phi}) \\
&-\mu^2\rho_0(\widehat{\Phi})^\dagger * \rho_0(\widehat{\Phi}) - \lambda\rho_0(\widehat{\Phi})^\dagger * \rho_0(\widehat{\Phi}) * \rho_0(\widehat{\Phi})^\dagger * \rho_0(\widehat{\Phi})\Big) \\
&+\int d^4x \Big(-\sum_{i,j=1}^{3}\Big(W^{ij}\,(\bar{\widehat{L}}_L^{(i)} * \rho_L(\widehat{\Phi})) * \widehat{e}_R^{(j)} + W^{\dagger ij}\,\bar{\widehat{e}}_R^{(i)} * (\rho_L(\widehat{\Phi})^\dagger * \widehat{L}_L^{(j)})\Big) \\
&-\sum_{i,j=1}^{3}\Big(G_u^{ij}\,(\bar{\widehat{Q}}_L^{(i)} * \rho_{\bar{Q}}(\widehat{\bar{\Phi}})) * \widehat{u}_R^{(j)} + G_u^{\dagger ij}\,\bar{\widehat{u}}_R^{(i)} * (\rho_{\bar{Q}}(\widehat{\bar{\Phi}})^\dagger * \widehat{Q}_L^{(j)})\Big) \\
&-\sum_{i,j=1}^{3}\Big(G_d^{ij}\,(\bar{\widehat{Q}}_L^{(i)} * \rho_Q(\widehat{\Phi})) * \widehat{d}_R^{(j)} + G_d^{\dagger ij}\,\bar{\widehat{d}}_R^{(i)} * (\rho_Q(\widehat{\Phi})^\dagger * \widehat{Q}_L^{(j)})\Big)\Big),
\end{aligned}
\tag{5.44}
$$

with $\bar{\Phi} = i\tau_2\Phi^*$. The matrices W^{ij}, G_u^{ij} and G_d^{ij} are the Yukawa couplings. The gauge fields in the Seiberg-Witten maps and covariant derivatives of the fermions are summarised in Table 5.2.

$\Psi^{(n)}$	$\rho_{(n)}(V_\nu)$
e_R	$-g'\mathcal{A}_\nu(x)$
$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$-\frac{1}{2}g'\mathcal{A}_\nu(x) + gB_{\nu a}(x)T_L^a$
u_R	$\frac{2}{3}g'\mathcal{A}_\nu(x) + g_S G_{\nu b}(x)T_S^b$
d_R	$-\frac{1}{3}g'\mathcal{A}_\nu(x) + g_S G_{\nu b}(x)T_S^b$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{6}g'\mathcal{A}_\nu(x) + gB_{\nu a}(x)T_L^a + g_S G_{\nu b}(x)T_S^b$

Table 5.2: The gauge fields in the Seiberg-Witten maps of the fermions and in the covariant derivatives of the fermions in the non-commutative Standard Model. (The symbols T_L^a and T_S^b are here the Pauli and Gell-Mann matrices, respectively.)

5.6.1 The Non-Commutative Electroweak Sector

We shall apply the Seiberg-Witten map to the electroweak non-commutative Standard Model. For the lepton field $L_L^{(i)}$ of the i^{th} generation which is in the fundamental representation of $SU(2)_L$ and in the Y representation of $U(1)_Y$, we have the following expansion

$$\widehat{L}_L^{(i)}[\mathcal{A}, B] = L_L^{(i)} + L_L^{(i)1}[\mathcal{A}, B] + \mathcal{O}(\theta^2), \tag{5.45}$$

with

$$\begin{aligned} L_L^{(i)1}[\mathcal{A}, B] = & -\frac{1}{2} g' \theta^{\mu\nu} \mathcal{A}_\mu \partial_\nu L_L^{(i)} - \frac{1}{2} g \theta^{\mu\nu} B_\mu \partial_\nu L_L^{(i)} \\ & + \frac{i}{8} \theta^{\mu\nu} [g' \mathcal{A}_\mu + g B_\mu,\, g' \mathcal{A}_\nu + g B_\nu]\, L_L^{(i)}. \end{aligned} \tag{5.46}$$

For a right handed lepton field of the i^{th} generation,

$$\widehat{e}_R^{(i)}[\mathcal{A}] = e_R^{(i)} + e_R^{(i)1}[\mathcal{A}] + \mathcal{O}(\theta^2), \tag{5.47}$$

we have

$$e_R^{(i)1}[\mathcal{A}] = -\frac{1}{2} g' \theta^{\mu\nu} \mathcal{A}_\mu \partial_\nu e_R^{(i)}. \tag{5.48}$$

The expansion reads

$$\widehat{Q}_L^{(i)}[\mathcal{A}, B, G] = Q_L^{(i)} + Q_L^{(i)1}[\mathcal{A}, B, G] + \mathcal{O}(\theta^2) \tag{5.49}$$

for a left-handed quark doublet $\widehat{Q}_L^{(i)}$ of the i^{th} generation, where

$$\begin{aligned} Q_L^{(i)1}[\mathcal{A}, B, G] = & -\frac{1}{2} g' \theta^{\mu\nu} \mathcal{A}_\mu \partial_\nu Q_L^{(i)} - \frac{1}{2} g \theta^{\mu\nu} B_\mu \partial_\nu Q_L^{(i)} - \frac{1}{2} g_S \theta^{\mu\nu} G_\mu \partial_\nu Q_L^{(i)} \\ & + \frac{i}{8} \theta^{\mu\nu} [g' \mathcal{A}_\mu + g B_\mu + g_S G_\mu,\, g' \mathcal{A}_\nu + g B_\nu + g_S G_\nu]\, Q_L^{(i)}. \end{aligned} \tag{5.50}$$

For a right-handed quark, e.g., $\widehat{u}_R^{(i)}$, we have

$$\widehat{u}_R^{(i)}[\mathcal{A}, G] = u_R^{(i)} + u_R^{(i)1}[\mathcal{A}, G] + \mathcal{O}(\theta^2), \tag{5.51}$$

$$\begin{aligned} u_R^{(i)1}[\mathcal{A}, G] = & -\frac{1}{2} g' \theta^{\mu\nu} \mathcal{A}_\mu \partial_\nu u_R^{(i)} - \frac{1}{2} g_S \theta^{\mu\nu} G_\mu \partial_\nu u_R^{(i)} \\ & + \frac{i}{8} \theta^{\mu\nu} [g' \mathcal{A}_\mu + g_S G_\mu,\, g' \mathcal{A}_\nu + g_S G_\nu]\, u_R^{(i)}. \end{aligned} \tag{5.52}$$

The same expansion is obtained for a right-handed down type quark $d_R^{(i)}$. The field strength $\widehat{F}_{\mu\nu} = \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu - i[\widehat{V}_\mu \stackrel{*}{,} \widehat{V}_\nu]$ has the expansion

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + F^1_{\mu\nu} + \mathcal{O}(\theta^2), \tag{5.53}$$

where

$$F_{\mu\nu} = g' f_{\mu\nu} + g F^L_{\mu\nu} + g_S F^S_{\mu\nu}. \tag{5.54}$$

$f_{\mu\nu}$ is the field strength corresponding to the group $U(1)_Y$, $F^L_{\mu\nu}$ refers to $SU(2)_L$ and $F^S_{\mu\nu}$ to $SU(3)_C$, respectively. The leading order correction in θ is given by

$$F^1_{\mu\nu} = \frac{1}{2}\theta^{\alpha\beta}\{F_{\mu\alpha}, F_{\nu\beta}\} - \frac{1}{4}\theta^{\alpha\beta}\{V_\alpha, (\partial_\beta + D_\beta)F_{\mu\nu}\}, \tag{5.55}$$

with

$$D_\beta F_{\mu\nu} = \partial_\beta F_{\mu\nu} - i[V_\beta, F_{\mu\nu}]. \tag{5.56}$$

The leading order expansion for the mathematical field V is given by

$$\widehat{V}_\mu = V_\mu + i\Gamma_\mu + \mathcal{O}(\theta^2), \tag{5.57}$$

with

$$\begin{aligned}\Gamma_\mu = \frac{i}{4}\theta^{\alpha\beta}\,\{&g'\mathcal{A}_\alpha + gB_\alpha + g_S G_\alpha,\, g'\partial_\beta\mathcal{A}_\mu + g\partial_\beta B_\mu + g_S\partial_\beta G_\mu \\ &+ g' f_{\beta\mu} + g F^L_{\beta\mu} + g_S F^S_{\beta\mu}\}.\end{aligned} \tag{5.58}$$

The action of the non-commutative electroweak Standard Model reads

$$S_{\text{NCSM}} = S_{\text{Leptons}} + S_{\text{Quarks}} + S_{\text{Gauge}} + S_{\text{Higgs}} + S_{\text{Yukawa}}. \tag{5.59}$$

Let us first consider the fermions, i.e., leptons and quarks.

Fermionic Part of the Electroweak Standard Model

The fermionic matter part is

$$\begin{aligned}S_{\text{Fermions}} &= S_{\text{Leptons}} + S_{\text{Quarks}} \\ &= \int d^4x \left(\sum_f \overline{\widehat{\Psi}}^{(f)}_L * i\widehat{D\!\!\!/}\widehat{\Psi}^{(f)}_L + \sum_f \overline{\widehat{\Psi}}^{(f)}_R * i\widehat{D\!\!\!/}\widehat{\Psi}^{(f)}_R \right),\end{aligned} \tag{5.60}$$

where $\widehat{\Psi}^{(f)}_L$ denotes the left-handed $SU(2)$ doublets, $\widehat{\Psi}^{(f)}_R$ the right-handed $SU(2)_L$ singlets and the index f runs over the three flavours. We thus have

$$\Psi^{(1)}_L = \begin{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ \begin{pmatrix} u^r_L \\ d^r_L \end{pmatrix} \\ \begin{pmatrix} u^y_L \\ d^y_L \end{pmatrix} \\ \begin{pmatrix} u^b_L \\ d^b_L \end{pmatrix} \end{pmatrix}, \qquad \Psi^{(1)}_R = \begin{pmatrix} e_R \\ u^r_R \\ d^r_R \\ u^y_R \\ d^y_R \\ u^b_R \\ d^b_R \end{pmatrix}, \tag{5.61}$$

for the first generation. Expanding the fermion part of the action (5.60) yields

$$\begin{aligned}
S_{\text{Leptons}} &= \int d^4x \Big(\sum_i \left(\bar{L}_L^{(i)} + \bar{L}_L^{(i)1}\right) * i \left(\not{D}^{SM} + \not{\Gamma}\right) * \left(L_L^{(i)} + L_L^{(i)1}\right) \\
&\quad + \sum_i \left(\bar{e}_R^{(i)} + \bar{e}_R^{(i)1}\right) * i \left(\not{D}^{SM} + \not{\Gamma}\right) * \left(e_R^{(i)} + e_R^{(i)1}\right) \Big) + \mathcal{O}(\theta^2) \\
&= \int d^4x \sum_i \left(\bar{L}_L^{(i)} i\not{D}^{SM} L_L^{(i)} + \bar{e}_R^{(i)} i\not{D}^{SM} e_R^{(i)}\right) \qquad (5.62) \\
&\quad -\frac{1}{2}\theta^{\mu\nu} \int d^4x \sum_i \Big\{ \bar{L}_L^{(i)} \left(\frac{1}{2} F^W_{\mu\nu} i\not{D}^{SM} + \gamma^\alpha F^W_{\alpha\mu} i D^{SM}_\nu\right) L_L^{(i)} \\
&\qquad + \bar{e}_R^{(i)} \left(\frac{1}{2} g' f_{\mu\nu} i\not{D}^{SM} + g'\gamma^\alpha f_{\alpha\mu} i D^{SM}_\nu\right) e_R^{(i)} \Big\} + \mathcal{O}(\theta^2),
\end{aligned}$$

where we have used the definition

$$F^W_{\mu\nu} \equiv g' f_{\mu\nu} + g F^L_{\mu\nu}.$$

The kinetic terms involving the right and left handed quarks reads

$$\begin{aligned}
S_{\text{Quarks}} &= \int d^4x \Big(\sum_i \left(\bar{Q}_L^{(i)} + \bar{Q}_L^{(i)1}\right) * i \left(\not{D}^{SM} + \not{\Gamma}\right) * \left(Q_L^{(i)} + Q_L^{(i)1}\right) \\
&\quad + \sum_i \left(\bar{u}_R^{(i)} + \bar{u}_R^{(i)1}\right) * i \left(\not{D}^{SM} + \not{\Gamma}\right) * \left(u_R^{(i)} + u_R^{(i)1}\right) \Big) \\
&\quad + \sum_i \left(\bar{d}_R^{(i)} + \bar{d}_R^{(i)1}\right) * i \left(\not{D}^{SM} + \not{\Gamma}\right) * \left(d_R^{(i)} + d_R^{(i)1}\right) + \mathcal{O}(\theta^2) \\
&= \int d^4x \sum_i \left(\bar{Q}_L^{(i)} i\not{D}^{SM} Q_L^{(i)} + \bar{u}_R^{(i)} i\not{D}^{SM} u_R^{(i)} + \bar{d}_R^{(i)} i\not{D}^{SM} d_R^{(i)}\right) \qquad (5.63) \\
&\quad -\frac{1}{2}\theta^{\mu\nu} \int d^4x \sum_i \Big\{ \bar{Q}_L^{(i)} \left(\frac{1}{2} F_{\mu\nu} i\not{D}^{SM} + \gamma^\alpha F_{\alpha\mu} i D^{SM}_\nu\right) Q_L^{(i)} \\
&\qquad + \bar{u}_R^{(i)} \left(\frac{1}{2} F^R_{\mu\nu} i\not{D}^{SM} + \gamma^\alpha F^R_{\alpha\mu} i D^{SM}_\nu\right) u_R^{(i)} \\
&\qquad + \bar{d}_R^{(i)} \left(\frac{1}{2} F^R_{\mu\nu} i\not{D}^{SM} + \gamma^\alpha F^R_{\alpha\mu} i D^{SM}_\nu\right) d_R^{(i)} \Big\} + \mathcal{O}(\theta^2),
\end{aligned}$$

where we have used the substitutions

$$\begin{aligned}
F_{\mu\nu} &\equiv g' f_{\mu\nu} + g F^L_{\mu\nu} + g_S F^S_{\mu\nu}, \\
F^R_{\mu\nu} &\equiv g' f_{\mu\nu} + g_S F^S_{\mu\nu}.
\end{aligned}$$

We recover the commutative Standard Model, but some new interactions appear. The most striking feature are point-like interactions between gluons, electroweak bosons and quarks.

Gauge Kinetic Part of the Electroweak Standard Model

For the gauge part of the action, one finds, cf. [14],

$$\begin{aligned}
S_{\text{Gauge}} &= -\int d^4x \frac{1}{2g'} \mathbf{tr_1} \widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g} \mathbf{tr_2} \widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g_S} \mathbf{tr_3} \widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} \\
&= -\frac{1}{2} \int d^4x \left(\frac{1}{2} f_{\mu\nu} f^{\mu\nu} + \operatorname{Tr} F^L_{\mu\nu} F^{L\mu\nu} + \operatorname{Tr} F^S_{\mu\nu} F^{S\mu\nu} \right) \qquad (5.64) \\
&\quad - \theta^{\mu\nu} \operatorname{Tr} \int d^4x \left(g F^L_{\mu\rho} F^L_{\nu\sigma} F^{L\rho\sigma} + g_S F^S_{\mu\rho} F^S_{\nu\sigma} F^{S\rho\sigma} \right) \\
&\quad + \frac{1}{4} g_S \, \theta^{\mu\nu} \operatorname{Tr} \int d^4x \, F^S_{\mu\nu} F^S_{\rho\sigma} F^{S\rho\sigma} + \mathcal{O}(\theta^2).
\end{aligned}$$

The coefficients of the triple vertex in the $U(1)$ sector are also different from plain non-commutative QED with a single electron. These coefficients depend on the representation we are choosing for Y (cf. Section 5.5) . For the simple choice that we have taken $\mathbf{tr_1} Y^3 = 0$, and this coefficient is zero. So there is no triple vertex for the $U(1)_Y$-photon. Note that the term

$$+\frac{1}{4} g \, \theta^{\mu\nu} \operatorname{Tr} \int d^4x \, F^L_{\mu\nu} F^L_{\rho\sigma} F^{L\rho\sigma} \qquad (5.65)$$

vanishes, the trace over the three Pauli matrices yields $2i\epsilon^{abc}$ and the sum $\epsilon^{abc} F^{bL}_{\rho\sigma} F^{cL\rho\sigma}$ vanishes. The trace over $\tau^3\tau^3\tau^3$ is zero. Therefore, there is also no cubic self-interaction term for the electromagnetic photon coming from the $SU(2)_L$ sector. Limits on non-commutative QED found from triple photon self-interactions do therefore not apply to the Minimal Non-Commutative Standard Model.

Higgs Branch

As in the usual commutative Standard Model, the Higgs mechanism can be applied to break the $SU(2)_L \times U(1)_Y$ gauge symmetry and thus to generate masses for the electroweak gauge bosons. The non-commutative action for a scalar field ϕ in the fundamental representation of $SU(2)_L$ and with the hypercharge $Y = 1/2$ reads

$$S_{\text{Higgs}} = \int d^4x \Big(\rho_0 \left(D_\mu \widehat{\Phi} \right)^\dagger * \rho_0 \left(D^\mu \widehat{\Phi} \right) \qquad (5.66)$$

$$- \mu^2 \rho_0(\widehat{\Phi})^\dagger * \rho_0(\widehat{\Phi}) - \lambda(\rho_0(\widehat{\Phi})^\dagger * \rho_0(\widehat{\Phi})) * (\rho_0(\widehat{\Phi})^\dagger * \rho_0(\widehat{\Phi}))\Big).$$

In the leading order of the expansion in θ, we obtain

$$\begin{aligned} S_{\text{Higgs}} = & \int d^4x \Big((D_\mu^{SM}\phi)^\dagger D^{SM\mu}\phi - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger\phi)(\phi^\dagger\phi) \Big) \\ & + \int d^4x \Big((D_\mu^{SM}\phi)^\dagger \left(D^{SM\mu}\rho_0(\phi^1) + \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha V^\mu \partial_\beta \phi + \Gamma^\mu \phi \right) \\ & + \left(D_\mu^{SM}\rho_0(\phi^1) + \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha V_\mu \partial_\beta\phi + \Gamma_\mu \phi \right)^\dagger D^{SM\mu}\phi \\ & + \frac{1}{4}\mu^2 \theta^{\mu\nu}\phi^\dagger (g' f_{\mu\nu} + g F^L_{\mu\nu})\phi - \lambda i \theta^{\alpha\beta}\phi^\dagger\phi (D_\alpha^{SM}\phi)^\dagger (D_\beta^{SM}\phi) \Big) + \mathcal{O}(\theta^2), \end{aligned} \tag{5.67}$$

where

$$D_\mu^{SM} = \partial_\mu - ig'\mathcal{A}_\mu - igB_\mu, \tag{5.68}$$

$$\Gamma_\mu = -iV_\mu^1 = \frac{i}{4}\theta^{\alpha\beta}\, \{g'\mathcal{A}_\alpha + gB_\alpha,\ g'\partial_\beta \mathcal{A}_\mu + g\partial_\beta B_\mu + g' f_{\beta\mu} + gF^L_{\beta\mu}\}. \tag{5.69}$$

We have also used the representation ρ_0 (5.24),

$$\rho_0(\widehat{\Phi}) = \phi + \rho_0(\phi^1) + \mathcal{O}(\theta^2), \tag{5.70}$$

where

$$\rho_0(\phi^1) = -\frac{1}{2}\theta^{\alpha\beta}(g'\mathcal{A}_\alpha + gB_\alpha)\,\partial_\beta\phi + \frac{i}{8}\theta^{\alpha\beta}\,[g'\mathcal{A}_\alpha + gB_\alpha,\, g'\mathcal{A}_\beta + gB_\beta]\,\phi. \tag{5.71}$$

For $\mu^2 < 0$ the $SU(2)_L \times U(1)_Y$ gauge symmetry is spontaneously broken to $U(1)_Q$, which is the gauge group describing the electromagnetic interaction. We have gauge invariance and may choose the so-called unitarity gauge,

$$\phi = \begin{pmatrix} 0 \\ \eta + v \end{pmatrix} \frac{1}{\sqrt{2}}, \tag{5.72}$$

where v is the vacuum expectation value. Since the zeroth order of the expansion of the non-commutative action corresponds to the Standard Model action, the Higgs mechanism generates the same masses for electroweak gauge bosons as in the commutative Standard Model,

$$M_{W^\pm} = \frac{gv}{2} \quad \text{and} \quad M_Z = \frac{\sqrt{g^2 + g'^2}}{2}v, \tag{5.73}$$

where the physical mass eigenstates $W^{\pm}$, Z and A are as usual defined by

$$W_\mu^{\pm} = \frac{B_\mu^1 \mp i B_\mu^2}{\sqrt{2}}, \quad Z_\mu = \frac{-g'\mathcal{A}_\mu + g B_\mu^3}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad A_\mu = \frac{g\mathcal{A}_\mu + g' B_\mu^3}{\sqrt{g^2 + g'^2}}. \tag{5.74}$$

There are no corrections to the masses of order θ at tree level, since these terms involve derivatives and therefore do not resemble mass terms. The Higgs mass is given by $m_\eta^2 = -2\mu^2$. Rewriting the term Γ_μ in terms of the mass eigenstates, using

$$B_\mu^3 = \frac{g Z_\mu + g' A_\mu}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad \mathcal{A}_\mu = \frac{g A_\mu - g' Z_\mu}{\sqrt{g^2 + g'^2}}, \tag{5.75}$$

one finds that besides the usual Standard Model couplings, numerous new couplings between the Higgs boson and the electroweak gauge bosons appear. We note that the non-commutative version of the Standard Model is also compatible with the alternative to the Higgs mechanism proposed in [63].

Yukawa Branch

The Yukawa terms generate masses for the fermions, they have the form

$$\begin{aligned} S_{\text{Yukawa}} = \int d^4x \Bigg(& - \sum_{i,j=1}^{3} \Big(W^{ij} \, (\bar{\widehat{L}}_L^{(i)} * \rho_L(\widehat{\Phi})) * \widehat{e}_R^{(j)} + W^{\dagger ij} \bar{\widehat{e}}_R^{(i)} * (\rho_L(\widehat{\Phi})^\dagger * \widehat{L}_L^{(j)}) \Big) \\ & - \sum_{i,j=1}^{3} \Big(G_u^{ij} \, (\bar{\widehat{Q}}_L^{(i)} * \rho_{\bar{Q}}(\bar{\widehat{\Phi}})) * \widehat{u}_R^{(j)} + G_u^{\dagger\, ij} \, \bar{\widehat{u}}_R^{(i)} * (\rho_{\bar{Q}}(\bar{\widehat{\Phi}})^\dagger * \widehat{Q}_L^{(j)}) \Big) \\ & - \sum_{i,j=1}^{3} \Big(G_d^{ij} \, (\bar{\widehat{Q}}_L^{(i)} * \rho_Q(\widehat{\Phi})) * \widehat{d}_R^{(j)} + G_d^{\dagger\, ij} \, \bar{\widehat{d}}_R^{(i)} * (\rho_Q(\widehat{\Phi})^\dagger * \widehat{Q}_L^{(j)}) \Big) \Bigg), \end{aligned} \tag{5.76}$$

where $\widehat{\Phi}[\Phi, V, V']$'s are given in (5.21)-(5.23). The sum is over the generation index. The leading order expansion yields

$$\begin{aligned} S_{\text{Yukawa}} &= S_{\text{Yukawa}}^{SM} \\ &- \int d^4x \Bigg(\sum_{i,j=1}^{3} \Big\{ W^{ij} \Big((\bar{L}_L^{(i)} \phi) \, e_R^{(j)1} + (\bar{L}_L^{(i)} \rho_L(\phi^1)) \, e_R^{(j)} + (\bar{L}_L^{(i)1} \phi) \, e_R^{(j)} \\ &+ \frac{i}{2} \theta^{\alpha\beta} (\partial_\alpha \bar{L}_L^{(i)} \partial_\beta \phi) \, e_R^{(j)} \Big) + W^{\dagger ij} \Big(\bar{e}_R^{(i)} (\phi^\dagger L_L^{(j)1}) + \bar{e}_R^{(i)} (\rho_L(\phi^1)^\dagger L_L^{(j)}) \end{aligned}$$

$$
\begin{aligned}
&+\bar{e}_R^{(i)1}(\phi^\dagger L_L^{(j)}) + \frac{i}{2}\theta^{\alpha\beta}\bar{e}_R^{(i)}(\partial_\alpha\phi^\dagger\partial_\beta L_L^{(j)})\Big)\Big\} \qquad (5.77)\\
-&\sum_{i,j=1}^{3}\Big\{G_u^{ij}\Big((\bar{Q}_L^{(i)}\bar{\phi})\,u_R^{(j)1} + (\bar{Q}_L^{(i)}\rho_{\bar{Q}}(\bar{\phi}^1))\,u_R^{(j)} + (\bar{Q}_L^{(i)1}\bar{\phi})\,u_R^{(j)}\\
&+\frac{i}{2}\theta^{\alpha\beta}(\partial_\alpha\bar{Q}_L^{(i)}\partial_\beta\bar{\phi})\,u_R^{(j)}\Big) + {G_u^\dagger}^{ij}\Big(\bar{u}_R^{(i)}(\bar{\phi}^\dagger Q_L^{(j)1}) + \bar{u}_R^{(i)}(\rho_{\bar{Q}}(\bar{\phi}^1)^\dagger Q_L^{(j)})\\
&+\bar{u}_R^{(i)1}(\bar{\phi}^\dagger Q_L^{(j)}) + \frac{i}{2}\theta^{\alpha\beta}\bar{u}_R^{(i)}(\partial_\alpha\bar{\phi}^\dagger\partial_\beta Q_L^{(j)})\Big)\Big\}\\
-&\sum_{i,j=1}^{3}\Big\{G_d^{ij}\Big((\bar{Q}_L^{(i)}\phi)\,d_R^{(j)1} + (\bar{Q}_L^{(i)}\rho_Q(\phi^1))\,d_R^{(j)} + (\bar{Q}_L^{(i)1}\phi)\,d_R^{(j)}\\
&+\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\bar{Q}_L^{(i)}\partial_\beta\phi\,d_R^{(j)}\Big) + {G_d^\dagger}^{ij}\Big(\bar{d}_R^{(i)}(\phi^\dagger Q_L^{(j)1}) + \bar{d}_R^{(i)}(\rho_Q(\phi^1)^\dagger Q_L^{(j)})\\
&+\bar{d}_R^{(i)1}(\phi^\dagger Q_L^{(j)}) + \frac{i}{2}\theta^{\alpha\beta}\bar{d}_R^{(i)}(\partial_\alpha\phi^\dagger\partial_\beta Q_L^{(j)})\Big)\Big\}\Big) + \mathcal{O}(\theta^2),
\end{aligned}
$$

where $L_L^{(i)}$ stands for a left-handed leptonic doublet of the i^{th} generation, $e_R^{(i)}$ for a leptonic singlet of the i^{th} generation, $Q_L^{(i)}$ for a left-handed quark doublet of the i^{th} generation, $u_R^{(i)}$ for a right-handed up-type quark singlet of the i^{th} and $d_R^{(i)}$ stands for a right-handed down-type quark singlet of the i^{th} generation. We used

$$
\rho(\Phi) = \phi + \rho(\phi^1) + \mathcal{O}(\theta^2), \qquad (5.78)
$$

where ρ stands for ρ_L, ρ_Q and $\rho_{\bar{Q}}$, respectively. $\rho(\phi^1)$ is given by (5.20),

$$
\begin{aligned}
\rho(\phi^1) = &\frac{1}{2}\theta^{\mu\nu}\rho(V_\nu)\Big(\partial_\mu\phi - \frac{i}{2}\rho(V_\mu)\,\phi - \frac{i}{2}\phi\,\rho(V'_\mu)\Big) \qquad (5.79)\\
&-\frac{1}{2}\theta^{\mu\nu}\Big(\partial_\mu\phi - \frac{i}{2}\rho(V_\mu)\,\phi - \frac{i}{2}\phi\,\rho(V'_\mu)\Big)\rho(V'_\nu).
\end{aligned}
$$

Once again we recover the Standard Model, but some new interactions arise. The Yukawa coupling matrices can be diagonalised using bi-unitary transformations. We thus obtain a Cabibo-Kobayashi-Maskawa matrix in the charged currents, as in the Standard Model and as long as right-handed neutrinos are absent, we do not predict lepton flavour changing currents. We will present the Lagrangian for the charged and neutral currents in Section 5.7. We will also extract the currents which have their origin in (5.77).

5.7 Currents in the Non-Commutaive Standard Model

5.7.1 Currents in the Commutative Standard Model

Let us calculate charged and neutral currents of the quarks in the Standard Model on commutative space-time. The kinetic term for the left handed quark fields is given by

$$S_{\text{Quarks}} = \int d^4x \sum_{i=1}^{3} \bar{Q}_L^{(i)} i \not{D}^{SM} Q_L^{(i)}. \tag{5.80}$$

But let us also forget about the three different families of the quarks, let us consider only one of them. Explicitly, we obtain

$$\bar{Q}_L i \not{D}^{SM} Q_L = \bar{Q}_L \gamma^\mu \left(i\partial_\mu - ig'Y\mathcal{A}_\mu - \frac{ig}{2} B_{\mu a}\sigma^a \right) Q_L. \tag{5.81}$$

We only consider electroweak currents, therefore the strong interaction does not contribute. We get

$$\left(\bar{u}\ \bar{d} \right) i\not{\partial} \begin{pmatrix} u \\ d \end{pmatrix} + \left(\bar{u}\ \bar{d} \right) \begin{pmatrix} g'Y\not{\mathcal{A}} + \frac{g}{2}\not{B}_3 & \frac{g}{\sqrt{2}}\not{W}^+ \\ \frac{g}{\sqrt{2}}\not{W}^- & g'Y\not{\mathcal{A}} - \frac{g}{2}\not{B}_3 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}. \tag{5.82}$$

The currents are defined by the matrix in (5.82),

$$S_{\text{Quarks}} = \int d^4x \left(\bar{Q}_L i\not{\partial} Q_L + \frac{g}{\sqrt{2}} \bar{u}_L J_1 d_L + \frac{g}{\sqrt{2}} \bar{d}_L J_2 u_L + \bar{u}_L \not{N}_1 u_L + \bar{d}_L \not{N}_2 d_L \right), \tag{5.83}$$

where J_1 and J_2 are the charged currents, $\not{N}_1$ and $\not{N}_2$ the neutral currents, respectively. They are given by

$$J_1 = \not{W}^+, \quad J_2 = \not{W}^-, \tag{5.84}$$

$$\begin{aligned} \not{N}_1 &= g\sin\theta_W (Y + 1/2)\not{A} + (g/2\cos\theta_W - g'Y\sin\theta_W)\not{Z}, \\ \not{N}_2 &= g\sin\theta_W (Y - 1/2)\not{A} - (g/2\cos\theta_W + g'Y\sin\theta_W)\not{Z}. \end{aligned} \tag{5.85}$$

We may now identify $g\sin\theta_W$ in (5.85) and (5.84) with the electromagnetic coupling (electron charge) e, and $Y \pm 1/2$ with the electromagnetic charge Q of the quark. The second identification is just the Gell-Mann - Nishijima relation,

$$Q = T_3 + Y.$$

5.7.2 Charged Currents

In this Section, we give the explicit formulas for the electroweak charged currents in the non-commutative Standard Model up to leading order of the expansion in θ.

$$\mathcal{L} = (\,\bar{u} \quad \bar{c} \quad \bar{t}\,)_L \, V_{CKM} J_1 \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L + (\,\bar{d} \quad \bar{s} \quad \bar{b}\,)_L \, V_{CKM}^{\dagger} J_2 \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L . \tag{5.86}$$

For the current J_1 we obtain

$$\begin{aligned} J_1 = \; & \frac{1}{\sqrt{2}} g \not{W}^+ + (\frac{1}{2}\theta^{\mu\nu}\gamma^\alpha + \theta^{\nu\alpha}\gamma^\mu)\Bigg(\Big(-\frac{\sqrt{2}}{4} Y g' g (\cos\theta_W A_{\mu\nu} - \sin\theta_W Z_{\mu\nu}) W_\alpha^+\Big) \\ & + g\frac{\sqrt{2}}{8}\Big(W_{\mu\nu}^+ - ig\left((\cos\theta_W Z_\mu + \sin\theta_W A_\mu)W_\nu^+ - W_\mu^+(\cos\theta_W Z_\nu + \sin\theta_W A_\nu)\right)\Big) \\ & \times(-2i\partial_\alpha + (2Yg'\sin\theta_W + g\cos\theta_W)Z_\alpha + g'\cos\theta_W(1-2Y)A_\alpha) \\ & - \frac{\sqrt{2}}{8} g^2 \Big(\cos\theta_W Z_{\mu\nu} + \sin\theta_W A_{\mu\nu} - igW_{\mu\nu}^{\pm}\Big) W_\alpha^+ \Bigg) + \mathcal{O}(\theta^2), \end{aligned} \tag{5.87}$$

where

$$\begin{aligned} W_{\mu\nu}^{\pm} &= W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-, \\ A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ Z_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu, \\ W_{\mu\nu}^+ &= \partial_\mu W_\nu^+ - \partial_\nu W_\mu^+. \end{aligned}$$

$W_{\mu\nu}^-$ is defined accordingly. For the current J_2 we obtain

$$\begin{aligned} J_2 = \; & \frac{1}{\sqrt{2}} g \not{W}^- + (\frac{1}{2}\theta^{\mu\nu}\gamma^\alpha + \theta^{\nu\alpha}\gamma^\mu)\Bigg(\Big(-\frac{\sqrt{2}}{4} Y g' g (\cos\theta_W A_{\mu\nu} - \sin\theta_W Z_{\mu\nu}) W_\alpha^-\Big) \\ & + g\frac{\sqrt{2}}{8}\Big(W_{\mu\nu}^- - ig\left(W_\mu^-(\cos\theta_W Z_\nu + \sin\theta_W A_\nu) - (\cos\theta_W Z_\mu + \sin\theta_W A_\mu)W_\nu^-\right)\Big) \\ & \times(-2i\partial_\alpha + (2Yg'\sin\theta_W - g\cos\theta_W)Z_\alpha - g'\cos\theta_W(2Y+1)A_\alpha) \\ & + \frac{\sqrt{2}}{8} g^2 \Big(\cos\theta_W Z_{\mu\nu} + \sin\theta_W A_{\mu\nu} - igW_{\mu\nu}^{\pm}\Big) W_\alpha^- \Bigg) + \mathcal{O}(\theta^2). \end{aligned} \tag{5.88}$$

Note that as previously we have not included the interactions with the gluons in the "electroweak" charged currents. We have used the identity $g'\cos\theta_W = g\sin\theta_W$ in order to simplify above formulae.

5.7.3 Neutral Currents

The explicit formula for the neutral current in the leading order of the expansion in θ is derived similarly. Let $\not{N}_1$ be the current connecting left handed up-type quarks, $\bar{u}_L \not{N}_1 u_L$. $\not{N}_2$ connects left handed down-type quarks, $\bar{d}_L \not{N}_2 d_L$, and the right handed quarks are connected by $\not{N}_3$ and $\not{N}_4$, respectively, i.e., $\bar{u}_R \not{N}_3 u_R$, $\bar{d}_R \not{N}_4 d_R$. As a result we obtain

$$\begin{aligned}
\not{N}_1 &= g\sin\theta_W(Y+1/2)\not{A} + (g/2\cos\theta_W - Yg'\sin\theta_W)\not{Z} \\
&+ \left(\frac{1}{2}\theta^{\mu\nu}\gamma^\alpha + \theta^{\nu\alpha}\gamma^\mu\right)\left(\frac{g'Y}{4} f_{\mu\nu}\left(-2i\partial_\alpha - 2g'Y\mathcal{A}_\alpha - gB^3_\alpha\right)\right. \\
&+ \frac{g}{8}(B^3_{\mu\nu} - igW^{\pm}_{\mu\nu})\left(-2i\partial_\alpha - 2g'Y\mathcal{A}_\alpha - gB^3_\alpha\right) \\
&\left. - \frac{g^2}{4}\left(W^+_{\mu\nu} - ig(B^3_\mu W^+_\nu - W^+_\mu B^3_\nu)\right)W^-_\alpha\right),
\end{aligned} \tag{5.89}$$

$$\begin{aligned}
\not{N}_2 &= g\sin\theta_W(Y-1/2)\not{A} - (g/2\cos\theta_W + Yg'\sin\theta_W)\not{Z} \\
&+ \left(\frac{1}{2}\theta^{\mu\nu}\gamma^\alpha + \theta^{\nu\alpha}\gamma^\mu\right)\left(\frac{g'Y}{4} f_{\mu\nu}\left(-2i\partial_\alpha - 2g'Y\mathcal{A}_\alpha + gB^3_\alpha\right)\right. \\
&- \frac{g}{8}(B^3_{\mu\nu} - igW^{\pm}_{\mu\nu})\left(-2i\partial_\alpha - 2g'Y\mathcal{A}_\alpha + gB^3_\alpha\right) \\
&\left. - \frac{g^2}{4}\left(W^-_{\mu\nu} - ig(W^-_\mu B^3_\nu - B^3_\mu W^-_\nu)\right)W^+_\alpha\right),
\end{aligned} \tag{5.90}$$

$$\begin{aligned}
\not{N}_3 &= g'Y(\cos\theta_W\not{A} - \sin\theta_W\not{Z}) \\
&+ \left(\frac{1}{2}\theta^{\mu\nu}\gamma^\alpha + \theta^{\nu\alpha}\gamma^\mu\right)\left(\frac{g'Y}{2} f_{\mu\nu}\left(-i\partial_\alpha - g'Y\mathcal{A}_\alpha\right)\right),
\end{aligned} \tag{5.91}$$

$$\begin{aligned}
\not{N}_4 &= g'Y(\cos\theta_W\not{A} - \sin\theta_W\not{Z}) \\
&+ \left(\frac{1}{2}\theta^{\mu\nu}\gamma^\alpha + \theta^{\nu\alpha}\gamma^\mu\right)\left(\frac{g'Y}{2} f_{\mu\nu}\left(-i\partial_\alpha - g'Y\mathcal{A}_\alpha\right)\right),
\end{aligned} \tag{5.92}$$

where $f_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ is the field strength of the $U(1)_Y$ photon, $B^3_{\mu\nu} = \partial_\mu B^3_\nu - \partial_\nu B^3_\mu$. For simplicity or presentation, we have used the unphysical $U(1)_Y$ photon $\mathcal{A}_\mu$ and $SU(2)$ boson B_μ in the above formulae. They have to be replaced by the physical eigenstates, the electromagnetic photon A_μ and the neutral Z boson,

$$\begin{aligned}
\mathcal{A}_\mu &= \cos\theta_W A_\mu - \sin\theta_W Z_\mu, \\
B^3_\mu &= \cos\theta_W Z_\mu + \sin\theta_W A_\mu \text{ and} \\
f_{\mu\nu} &= \cos\theta_W A_{\mu\nu} - \sin\theta_W Z_{\mu\nu}, \\
B^3_{\mu\nu} &= \cos\theta_W Z_{\mu\nu} + \sin\theta_W A_{\mu\nu}.
\end{aligned}$$

Note that we have not included the interactions with the gluons in the "electroweak" neutral currents.

5.7.4 Currents from the Yukawa Terms

Let us expand the Yukawa terms (5.77) to first order in θ. We will include strong interaction. In terms of currents the Yukawa terms read

$$\begin{aligned} S_{\text{Yukawa}} &= S^{SM}_{\text{Yukawa}} \\ &+ \sum_{i,j=1}^{3} \Bigg(W^{ij}\, \bar{L}^{(i)}_L \frac{1}{2}\theta^{\mu\nu}\mathbf{C}^1_{\mu\nu} e^{(j)}_R + W^{\dagger\, ij}\, \bar{e}^{(i)}_R \frac{1}{2}\theta^{\mu\nu}(\mathbf{C}^1)^\dagger_{\mu\nu} L^{(j)}_L \\ &+ G^{ij}_u\, \bar{Q}^{(i)}_L \frac{1}{2}\theta^{\mu\nu}\mathbf{C}^2_{\mu\nu} u^{(j)}_R + G^{\dagger\, ij}_u\, \bar{u}^{(i)}_R \frac{1}{2}\theta^{\mu\nu}(\mathbf{C}^2)^\dagger_{\mu\nu} Q^{(j)}_L \\ &+ G^{ij}_d\, \bar{Q}^{(i)}_L \frac{1}{2}\theta^{\mu\nu}\mathbf{C}^3_{\mu\nu} d^{(j)}_R + G^{\dagger\, ij}_d\, \bar{d}^{(i)}_R \frac{1}{2}\theta^{\mu\nu}(\mathbf{C}^3)^\dagger_{\mu\nu} Q^{(j)}_L \Bigg) + \mathcal{O}(\theta^2), \end{aligned} \tag{5.93}$$

where

$$\begin{aligned} \mathbf{C}^1_{\mu\nu} &= g'\mathcal{A}_\mu\phi\,\partial_\nu + \left(-\frac{3}{2}g'\mathcal{A}_\nu + gB_\nu\right)\left(\partial_\mu - \frac{i}{2}(\frac{1}{2}g'\mathcal{A}_\mu + gB_\mu)\right)\phi \\ &\quad - \overleftarrow{\partial_\nu}\left(-\frac{1}{2}g'\mathcal{A}_\mu + gB_\mu + i\partial_\mu\right)\phi + \frac{i}{4}[-\frac{g'}{2}\mathcal{A}_\mu + gB_\mu,\, -\frac{g'}{2}\mathcal{A}_\nu + gB_\nu]\,\phi, \end{aligned} \tag{5.94}$$

$$\begin{aligned} \mathbf{C}^2_{\mu\nu} &= -\frac{2}{3}g'\mathcal{A}_\mu\bar{\phi}\,\partial_\nu - g_S G_\mu\,\bar{\phi}\,\partial_\nu - \overleftarrow{\partial_\nu}\,(V_\mu\bar{\phi} + i\partial_\mu\bar{\phi}) \\ &+ \frac{i}{4}[\frac{2}{3}g'\mathcal{A}_\mu + g_S G_\mu,\, \frac{2}{3}g'\mathcal{A}_\nu + g_S G_\nu]\,\bar{\phi} + \frac{i}{4}[V_\mu,\, V_\nu]\bar{\phi} \\ &+ \left(\frac{5}{6}g'\mathcal{A}_\nu + gB_\nu + 2g_S G_\nu\right)\left(\partial_\mu + \frac{i}{4}g'\mathcal{A}_\mu - \frac{i}{2}gB_\mu\right)\bar{\phi}, \end{aligned} \tag{5.95}$$

$$\begin{aligned} \mathbf{C}^3_{\mu\nu} &= \left(\frac{1}{3}g'\mathcal{A}_\mu - g_S G_\mu\right)\phi\,\partial_\nu - \overleftarrow{\partial_\nu}\,(V_\mu\phi + i\partial_\mu\phi) \\ &+ \frac{i}{4}[-\frac{1}{3}g'\mathcal{A}_\mu + g_S G_\mu,\, -\frac{1}{3}g'\mathcal{A}_\nu + g_S G_\nu]\,\phi + \frac{i}{4}[V_\mu,\, V_\nu]\phi \\ &+ \left(\frac{5}{6}g'\mathcal{A}_\nu + gB_\nu + 2g_S G_\nu\right)\left(\partial_\mu\phi - \frac{i}{2}(\frac{1}{2}g'\mathcal{A}_\mu + gB_\mu)\right), \end{aligned} \tag{5.96}$$

(5.97)

We have used the definition

$$V_\mu = \frac{1}{6}g'\mathcal{A}_\mu + gB_\mu + g_S G_\mu. \tag{5.98}$$

$\mathcal{A}_\mu$ and B^3_μ still have to be replaced by the physical photon A_μ and the neutral Z-boson Z_μ, which we have not done for the sake of brevity.

5.8 Discussion and Remarks

We have shown in Section 5.6 that the commutative Standard Model is the zeroth order approximation in an expansion in θ of the action of the Non-Commuative Standard Model. Although we have considered a minimal version of the Non-Commutative Standard Model (Section 5.5.1), there is a basic difference between the commutative and the non-commutative version. In the non-commutative model, the different interactions cannot be considered separately as the master field V_μ, which is a superposition of the different gauge fields has to be introduced. In zeroth order of the expansion in θ, the gauge bosons of the different gauge groups decouple, but mix in higher orders due to the Seiberg-Witten map. Therefore, some new vertices appear where the gauge bosons of different gauge groups are connected to the quarks. A kind of mixing or unification between all the interactions appears as we have vertices where e.g., $SU(3)_C$ gauge bosons couple to the $U(1)_Y$ gauge boson and to quarks. This type of unification implies that parity is broken in non-commutative QCD.

Up to the first order, we do not find couplings of neutral particles like the Higgs boson to the electromagnetic photon in the minimal version of the Non-Commutative Standard Model. We find new vertices in the pure gauge sector. One might expect to find a self-interacting vertex of the $U(1)_Y$ gauge boson which is not the case in the minimal version.But one does find vertices with five and six gauge bosons for the gauge group $SU(3)_C$ and $SU(2)_L$. Neutral decays of heavy particle, e.g., of the b and t-quarks might also reveal the non-commutative nature of space-time. New vertices appear in QCD. We find a point-like interaction between two quarks a gluon and a photon, thus opening new decay modes for hadrons.

A main result is that all the important features of the ordinary Standard Model can be implemented in this non-commutative version, in particular the Higgs mechanism and the Yukawa sector. Bi-unitary transformations can be applied to diagonalise the matrices of Yukawa couplings.

Recently a model based on the gauge group $U(3) \times U(2) \times U(1)$ was proposed [64]. This model involves a clever extra Higgs mechanism to deal with the problems of charge quantisation and tensor products, but it contains two gauge bosons which are not present in the usual Standard Model. What we are doing is fundamentally different as we are considering the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$, directly. We thus have proposed a *minimal* non-commutative extension of the Standard Model.

We have presented the first order expansion in $\theta^{\mu\nu}$ of the non-commutative Standard Model, which only represents a low energy effective theory. The limits that can be found in the literature on the combination $\Lambda\theta$ are based on the assumption that $\theta^{\mu\nu}$ is constant [65, 66], clearly the limits are much weaker if the assumption is relaxed. The effects are expected to be small for light particles. But, they could be sizable for heavy particles. In particular it is conceivable that a phase transition occurs at high energy, nature could be non-commutative above that scale but commutative under the scale of this phase transition, as discussed in Chapter 1.

The Non-Commutative Standard Model predicts a lot of new physics beyond the Standard Model. In particular as we have seen, we expect the charged and neutral currents to be considerably affected by non-commutative physics. The extraction of the CKM matrix elements and in particular of the phase at the origin of CP-violation would be strongly influenced by that type of new physics. One expects that the effects should become larger with the mass of the decaying particle, especially if a phase transition exists. The expansion of the Higgs part of the action still has to be calculated explicitly, and the currents have to be extracted, as done in Sections 5.7.2 - 5.7.4. One-loop corrections to the Non-Commutative Standard Model may also reveal additional substantial insight to the understanding of this model.

There are attempts to search for processes that are sensitive to the non-commutativity of space-time in both, the minimal and non-minimal Non-Commutative Standard Model [67,68]. For this aim, the expansion of the non-commutative Lagrangian in terms of currents is crucial.

One is tempted to think that our model is renormalisable to all orders in the coupling constants and in θ. A study in the framework of non-commutative quantum electrodynamics [58] has shown that the photon self-energy is renormalizable to all order. But, it occured that non-commutative QED might not be power-counting renormalisable in perturbation theory [69]. Additional symmetries may still render non-commutative QED and the Standard Model renormalisable. A proof of the renormalisability of our model is also still to be furnished. This is may be related to the still open question of anomaly freedom of the model [70]. Arguing as above that this model represents a low energy effective theory, ideas of renormalisation are not applicable. The problem of ultra-violet and infra-red mixing which plagues non-commutative quantum field theories [71], should be reconsidered in the framework of the Seiberg-Witten expansion used in our approach [72].

Chapter 6

Towards Gauge Theory on $\kappa-$Deformed Space-Time

6.1 Introduction

$\kappa-$deformed Euclidean space and a gauge theoretical model on this deformed space will be of scrutiny in this Chapter. So far, we have discussed models on canonical space-time. In [16] the $\kappa-$deformed Poincaré algebra has been constructed via a contraction of the anti-de-Sitter algebra $\mathcal{U}_q(o(3,2))$. Analogously, the contraction of $o(3,2)$ leads to the classical Poincaré algebra. The $\kappa-$deformed Poincaré algebra $\mathcal{P}_\kappa$ is a quantum group. In [17] a $\kappa-$deformed Minkowski space $\mathcal{M}_\kappa$ has been introduced. $\mathcal{M}_\kappa$ is a $\mathcal{P}_\kappa-$module algebra. This enables us to construct fields and knowing an invariant integral Lagrangian models.

The algebra relations of the Lorentz generators are undeformed. The difference lies in the co-product structure of the boost generators (6.18). I.e., the rotational sector is completely undeformed. The derivatives ∂_μ do not only generate a sub-algebra of $\mathcal{P}_\kappa$, but also a sub-Hopf algebra. This is the key fact for the construction of the $\kappa-$Poincaré algebra as a bicrossed product algebra [17]. The advantage of this deformation is its mildness. In Section 6.9 we will comment on different choices of generators of $\mathcal{P}_\kappa$.

The algebra relations of the generators of $\mathcal{M}_\kappa$ are [17]

$$[\hat{x}^0, \hat{x}^i] = ia\hat{x}^i, \tag{6.1}$$

where $i = 1, 2, 3$ and $a \in \mathbb{R}$. Clearly, the non-commutativity parameter a has the dimension of length. In order to compare with [17], we have to make the substitution

$$a \equiv -\frac{1}{i\kappa}. \tag{6.2}$$

The most general linear quantum space structure compatible with a deformed version of Poincaré symmetry is given by [73]

$$[\hat{x}^\mu, \hat{x}^\nu] = i\,(a^\mu \delta^\nu_\sigma - a^\nu \delta^\mu_\sigma)\,\hat{x}^\sigma, \tag{6.3}$$

where a^μ is a constant $4-$vector "pointing into the direction of non-commutativity". Its components also play the role of Lie algebra structure constants. Choosing

$$a^\mu = a\delta^{\mu 0} \tag{6.4}$$

we get relation (6.1) and $\mathcal{P}_\kappa$ as symmetry algebra. In order to be able to consider any $4-$vector a^μ or the non-commutativity pointing in any direction, we will formulate the Euclidean version of the $\kappa-$deformed Poincaré algebra (algebra of rotations) and the corresponding $\kappa-$deformed Euclidean space. In the Euclidean case, all directions are equivalent. We will construct wave equations and will also formulate all results using the $*$-product formalism. Last but not least, a gauge theoretical model on $\kappa-$deformed Euclidean space will be discussed. These results will be translated to $\kappa-$Minkowski space and the $\kappa-$deformed Poincaré algebra in Section 6.8.

Some field theoretical aspects on $\kappa-$Minkowski space have already been discussed in [74, 75, 76, 77, 78, 79]. Nevertheless, gauge theory has not been tackled. The gauge theory discussed here will be based on Seiberg-Witten maps introduced in Chapter 4. These methods will have new implications on the nature of the gauge fields.

6.2 Quantum Space and Symmetry Algebra - The Setting

Let us consider a n dimensional Euclidean space with coordinates $x^1, \ldots, x^n$. In the following, Latin indices range from 1 to $n-1$, greek indices from 1 to n. In Euclidean spaces all directions are equivalent. For convenience, the non-commutativity will point into the $n-$direction, i.e.,

$$a^\mu = a\,\delta^{n\mu}. \tag{6.5}$$

The n dimensional $\kappa-$Euclidean space algebra $\mathcal{E}_\kappa$ is generated by the coordinates $\hat{x}^1, \ldots, \hat{x}^n$. They satisfy the relations

$$[\hat{x}^n, \hat{x}^i] = ia\hat{x}^i, \tag{6.6}$$

all other commutators vanish,

$$[\hat{x}^i, \hat{x}^j] = 0. \tag{6.7}$$

The symmetry group of the κ-Euclidean space is a deformed version of the n-dimensional rotation group. It is generated by the rotations $M^{\mu\nu}$. Since the n-direction is special, we will denote the generators M^{nl} by N^l and will call them boosts, in analogy to the Lorentz algebra. The relations between symmetry generators and coordinates are have to be compatible with the algebra structure on $\mathcal{E}_\kappa$ and are supposed to be linear. We get as a result

$$\begin{aligned} M^{rs}\hat{x}^k &= \delta^{rk}\hat{x}^s - \delta^{sk}\hat{x}^r + \hat{x}^k M^{rs}, && (6.8)\\ M^{rs}\hat{x}^n &= \hat{x}^n M^{rs}, && (6.9)\\ N^l\hat{x}^i &= -\delta^{li}\hat{x}^n + \hat{x}^i N^l - iaM^{li}, && (6.10)\\ N^l\hat{x}^n &= \hat{x}^l + (\hat{x}^n + ia)\, N^l. && (6.11) \end{aligned}$$

In the commutative limit, $a \to 0$, the relations are the usual relations for a 4-dimensional Euclidean space. The consistent choice of algebra relations is given by

$$\begin{aligned} [N^l, N^k] &= M^{lk}, && (6.12)\\ [M^{rs}, N^l] &= \delta^{rl}N^s - \delta^{sl}N^l, && (6.13)\\ [M^{rs}, M^{kl}] &= \delta^{rl}M^{ks} - \delta^{sl}M^{kr} - \delta^{rk}M^{ls} + \delta^{sk}M^{lr}. && (6.14) \end{aligned}$$

These are just the undeformed algebra relations. The difference arises in the co-algebra structure. The commutation relations (6.8) - (6.11) can be generalised to non-commutative functions, which are given as power series in the non-commutative coordinates. For these relations we obtain

$$\begin{aligned} N^l\hat{f}(\hat{x}) &= N^l \triangleright \hat{f}(\hat{x}) + \hat{f}(\hat{x}^i, \hat{x}^n + ia)N^l - ia(\hat{\partial}_b \triangleright \hat{f}(\hat{x}))\, M^{lb}, && (6.15)\\ M^{rs}\hat{f}(\hat{x}) &= M^{rs} \triangleright \hat{f}(\hat{x}) + \hat{f}(\hat{x})\, M^{rs}. && (6.16) \end{aligned}$$

We can read off the co-product structure of the rotation generators from the above formulae, using the crossed product

$$T\hat{x}^\nu = (T_{(1)} \triangleright \hat{x}^\nu)T_{(2)}. \qquad (6.17)$$

We obtain

$$\begin{aligned} \Delta N^l &= N^l \otimes \mathbf{1} + \exp(ia\hat{\partial}_n) \otimes N^l - ia\hat{\partial}_b \otimes M^{lb}, && (6.18)\\ \Delta M^{rs} &= M^{rs} \otimes \mathbf{1} + \mathbf{1} \otimes M^{rs}. && (6.19) \end{aligned}$$

Now let us define derivatives on the this κ-Euclidean space. We introduce them by finding a deformed Leibniz rule compatible with the algebra relations (6.6). Since the coordinate algebra is the freely generated algebra divided by the ideal generated by relations (6.6) and (6.7), the derivatives have to map cosets on cosets. Consistent Leibniz rules are given by

$$\begin{aligned} \hat\partial_n \hat x^i &= \hat x^i \hat\partial_n, \\ \hat\partial_n \hat x^n &= 1 + \hat x^n \hat\partial_n, \\ \hat\partial_i \hat x^j &= \delta_i^j + \hat x^j \hat\partial_i, \\ \hat\partial_i \hat x^n &= (\hat x^n + ia)\,\hat\partial_i. \end{aligned} \tag{6.20}$$

However, these relations are not unique. Additionally, the derivatives have to form a module algebra of the deformed rotation algebra, i.e., they have to constitute a vector under rotations. This singles out the above choice of Leibniz rules. For the action of the symmetry generator on the derivatives we obtain

$$[M^{rs}, \hat\partial_i] = \delta^{ri}\hat\partial_s - \delta^{si}\hat\partial_r, \tag{6.21}$$

$$[M^{rs}, \hat\partial_n] = 0, \tag{6.22}$$

$$[N^l, \hat\partial_i] = \delta^{li}\frac{1}{2ia}\left(1 - e^{2ia\hat\partial_n}\right) - \frac{ia}{2}\delta^{li}\widehat\Delta_\kappa + ia\,\hat\partial_l\hat\partial_i, \tag{6.23}$$

$$[N^l, \hat\partial_n] = \hat\partial_l, \tag{6.24}$$

where $\widehat\Delta_\kappa = \sum_i \hat\partial_i\hat\partial_i$. The commutator of derivatives compatible with (6.6) is given by

$$[\hat\partial_\mu, \hat\partial_\nu] = 0. \tag{6.25}$$

The Leibniz rule for non-commutative functions reads

$$\hat\partial_i \hat f(\hat x) = \hat\partial_i \triangleright \hat f(\hat x) + \hat f(\hat x^i, \hat x^n + ia)\,\hat\partial_i. \tag{6.26}$$

$\hat\partial_n$ satisfies the ordinary Leibniz rule. Since we have calculated the action of the symmetry generators on the derivatives, we can also include the derivatives as generators in the symmetry algebra. As a result we obtain the $\kappa-$deformed Poincaré algebra $\mathcal{P}_\kappa$ on $\kappa-$deformed Euclidean space. It is generated by rotations M^{rs}, boosts N^l and translations $\hat\partial_\mu$. The algebra relations are given by (6.12) - (6.14) and (6.21) - (6.24). Note the difference in some minus signs compared to the algebra relations of the Poincaré algebra on $\kappa-$Minkowski space, given in [17]. The co-product of the translation generators reads

$$\Delta\hat\partial_n = \hat\partial_n \otimes \mathbf{1} + \mathbf{1} \otimes \hat\partial_n, \tag{6.27}$$

$$\Delta\hat\partial_i = \hat\partial_i \otimes \mathbf{1} + \exp(ia\hat\partial_n) \otimes \hat\partial_i. \tag{6.28}$$

6.3 Invariants and Wave Equations

First of all, let me construct an invariant I under $\kappa-$Poincaré transformations of the coordinate algebra. Invariant means $h \triangleright I = \epsilon(h)I$, where $h \in \mathcal{P}_\kappa$. This is a kind of radius square, $\sum_\mu x^\mu x^\mu$. However, in the case of $\mathcal{E}_\kappa$, there is a linear correction term and we get

$$I = \sum_{\mu=1}^{n} \hat{x}^\mu \hat{x}^\mu - ia(n-1)\hat{x}^n, \tag{6.29}$$

cf. [17]. However, $[N^l, \widehat{I}] \neq 0$, but

$$[N^l, I] = ia\left(2\hat{x}^n + ia - ia(n-1)\right) N^l - 2ia \sum_{b=1}^{n-1} \hat{x}^b M^{lb}, \tag{6.30}$$

in accordance with (6.18). These terms vanish, if we let them act on $\mathbf{1}$. The substitution

$$\begin{aligned} \hat{x}^i &\mapsto \hat{\hat{x}}^i = \hat{x}^i, \\ \hat{x}^n &\mapsto \hat{\hat{x}}^n = \hat{x}^n - \frac{ia(n-1)}{2}, \end{aligned} \tag{6.31}$$

simplifies the expression for I,

$$I = \sum_\mu \hat{\hat{x}}^\mu \hat{\hat{x}}^\mu + \frac{a^2(n-1)^2}{4}. \tag{6.32}$$

A rescaling leaves us with a quadratic expression and with the usual Euclidean metric $g_{\mu\nu}$,

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu}, \\ \tilde{I} &= g_{\mu\nu} \hat{\hat{x}}^\mu \hat{\hat{x}}^\nu. \end{aligned} \tag{6.33}$$

6.3.1 Klein-Gordon Operator

The Klein-Gordon operator $\widehat{\Box}$ is the Casimir operator of the momentum algebra, i.e., it is an invariant of the algebra generated by derivatives under the adjoint action of the Poincaré algebra, $[N^l, \widehat{\Box}] = [M^{rs}, \widehat{\Box}] = 0$. This is the deformed mass square operator. A general ansatz leads to

$$\widehat{\Box} = e^{-ia\hat{\partial}_n} \widehat{\Delta}_\kappa + \frac{2}{a^2}\Big(1 - \cos(a\hat{\partial}_n)\Big). \tag{6.34}$$

6.3.2 Dirac Operator

Dirac operators for κ-Minkowski space have already been constructed in [77,80]. In [77], the Dirac operator is defined as the square root of the Klein-Gordon operator (6.34), whereas the construction in [80] is similiar to the construction shown here. The Dirac operator D is required to be invariant under κ-Poincaré transformations,

$$[T, D] = 0, \tag{6.35}$$

where $T \in \{N^l, M^{rs}, \hat{\partial}_n\}$ act on functions of coordinates as well as on spinor indices. This approach differs in the way the orbital and spinorial operators add up. In our case, it is just the sum of the operators, cf. (6.47), relying on the fact that the adjoint action of the deformed Lorentz generators on derivatives $\hat{\partial}_\mu$, and therefore on the Dirac operator which is a function of $\hat{\partial}_\mu$, does not involve any Lorentz generators acting on the right. Remarkably, the square of the Dirac operator in [80] agrees with (6.55).

In the classical Minkowski space, we have the following transformations of coordinates x^μ and derivatives ∂_μ under a Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{6.36}$$
$$\partial_\mu = (\Lambda^{-1})^\nu{}_\mu \partial_\nu. \tag{6.37}$$

A spinor ψ transforms accordingly

$$\psi'(x') = S\psi(x), \tag{6.38}$$

where the matrices S build a finite dimensional representation of the Lorentz group. The Dirac operator D is uniquely determined by demanding that D is linear in the derivatives and a scalar under Lorentz transformations, i.e.,

$$D'\psi' = S\, D\psi. \tag{6.39}$$

Consequently, D is given by

$$D = \gamma^\mu \partial_\mu, \tag{6.40}$$

where γ^μ are the Dirac matrices with

$$\Lambda^\mu{}_\nu \gamma^\nu = S^{-1}\gamma^\mu\, S. \tag{6.41}$$

Let us express eqn. (6.39) infinitesimally. Using

$$\Lambda^\mu{}_\nu \approx \delta^\mu{}_\nu + \lambda^\mu{}_\nu, \tag{6.42}$$
$$S \approx 1 + s \tag{6.43}$$

we get

$$[s, D] - [\lambda, D] = 0, \tag{6.44}$$

where $[\lambda, D] = \lambda \triangleright D = \gamma^\mu (\lambda^{-1})^\nu{}_\mu \partial_\nu$. The matrices s are given by the usual γ matrices,

$$s^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{6.45}$$

It can be shown easily that $s^{\mu\nu}$ satisfy the algebra of the Lorentz generators,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i \left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} \right). \tag{6.46}$$

Generalising eqn. (6.44) to κ-Euclidean space we get the defining equation

$$[T, \widehat{D}] + [t, \widehat{D}] = 0, \tag{6.47}$$

where $\widehat{D}$ is the deformed Dirac operator and T some generator of the κ-Poincaré algebra. The spin or internal degree of freedom is undeformed. The operators t generate this representation. As in the classical case the generators are given in terms of the γ matrices in n-dimensional Euclidean space, $\gamma^1, \ldots, \gamma^{n-1}, \gamma^0$,

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}. \tag{6.48}$$

If we denote these classical generators with small letters ($t \in \{m^{rs}, n^r\}$), we have

$$m^{rs} = \frac{1}{4}[\gamma^s, \gamma^r], \tag{6.49}$$

$$\begin{aligned} n^r &= \frac{1}{4}[\gamma^r, \gamma^0] = \frac{1}{2}\gamma^r \gamma^0 \\ &= m^{nr}. \end{aligned} \tag{6.50}$$

The action of the generators on γ matrices is given by the adjoint action,

$$n^r \triangleright \gamma^l = [n^r, \gamma^l] = -\gamma^0 \delta^{rl}, \tag{6.51}$$

$$n^r \triangleright \gamma^0 = \gamma^r. \tag{6.52}$$

The most general ansatz for the Dirac operator can be written as

$$\widehat{D} = \gamma^0 \sum_k b_k \hat{\partial}_n^k + \gamma^0 \widehat{\Delta}_\kappa \frac{ia}{2} \sum_k c_k \hat{\partial}_n^k + \gamma^r \hat{\partial}_r \sum_k d_k \hat{\partial}_n^k, \tag{6.53}$$

where b_k, c_k, d_k are coefficients proportional to a^k. Eqn. (6.47) allows for a unique solution,

$$\widehat{D} = \frac{\gamma^0}{a}\sin(a\hat{\partial}_n) + \frac{ia\gamma^0}{2}\widehat{\Delta}_\kappa \exp(-ia\hat{\partial}_n) + \gamma^r \hat{\partial}_r \exp(-ia\hat{\partial}_n). \tag{6.54}$$

The square of the Dirac operator $\widehat{D}$ does not give the Klein-Gordon operator (6.34). It involves an additional factor which introduces a regularisation mass in the spirit of Pauli-Villars regularisation, c.f. [74],

$$\widehat{D}^2 = \widehat{\Box}\left(\frac{(ia)^2}{4}\widehat{\Box} + 1\right). \tag{6.55}$$

In the limit $a \to 0$, the regulator mass goes to infinity.

6.4 Derivatives and Vector Fields

We have introduced derivatives $\hat{\partial}_\nu$ as translation generators. Their Lorentz transformation property is given by eqns. (6.21) - (6.24). The action of the derivatives on non-commutative coordinates were obtained in eqn. (6.20). However, we find a second differential structure compatible with the algebra relations of the coordinates, using the Dirac operator $\widehat{D}$. Similarly, A. Connes introduces derivatives as the commutator with the Dirac operator. He has a different approach to non-commutative geometry. It is based on the spectral triple, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, see e.g., [81]. The spectral triple consists of the non-commutative algebra of coordinates, $\mathcal{A}$ represented on the Hilbert space $\mathcal{H}$ and the Dirac operator $\mathcal{D}$. In the commutative case, $\mathcal{A}$ is an abelian C^*-algebra of operators on the Hilbert space $\mathcal{H}$ and corresponds to a compact manifold, as shown in Fig. 2.2. $\mathcal{D}$ is a selfadjoint operator on $\mathcal{H}$.

Using

$$\begin{aligned}
\widehat{D} &= \gamma^0 \widehat{D}_n + \gamma^r \widehat{D}_r, \\
\widehat{D}_i &= \hat{\partial}_i\, e^{-ia\hat{\partial}_n}, && (6.56)\\
\widehat{D}_n &= \frac{1}{a}\sin(a\hat{\partial}_n) + \frac{ia}{2}\widehat{\Delta}_\kappa\, e^{-ia\hat{\partial}_n}, && (6.57)
\end{aligned}$$

we can calculate the action of the new derivatives on non-commutative coordinates,

$$[\widehat{D}_n, \hat{x}^i] = ia\hat{\partial}_i \exp(-ia\hat{\partial}_n), \tag{6.58}$$

$$[\widehat{D}_j, \hat{x}^i] = \delta^i_j \exp(-ia\hat{\partial}_n), \tag{6.59}$$

$$[\widehat{D}_n,\, \hat{x}^n] = \cos(a\hat{\partial}_n) + \frac{(ia)^2\widehat{\Delta}_\kappa}{2}\exp(-ia\hat{\partial}_n), \tag{6.60}$$

$$[\widehat{D}_j,\, \hat{x}^n] = 0. \tag{6.61}$$

In order to obtain a proper Leibniz rule, we have to express the rhs. in terms of $\widehat{D}_\mu$, i.e., we have to invert (6.56) and (6.57). The first step of the inversion is to calculate $e^{ia\hat{\partial}_n}$ and its inverse $e^{-ia\hat{\partial}_n}$. Note that replacing a by $-a$ will not lead from $e^{ia\hat{\partial}_n}$ to its inverse, because $\widehat{D}_\mu$ depend on a as well. Inserting

$$\widehat{\Delta}_\kappa = e^{ia\hat{\partial}_n}\widehat{\Box} - \frac{2}{a^2}e^{ia\hat{\partial}_n}\Big(1 - \cos(a\hat{\partial}_n)\Big)$$

into eqn. (6.57) yields

$$e^{-ia\hat{\partial}_n} = 1 - ia\widehat{D}_n + \frac{(ia)^2}{2}\widehat{\Box}. \tag{6.62}$$

(6.55) can be considered as a quadratic equation in $\widehat{\Box}$. The solution is easily obtained, taking into account the proper limit $a \to 0$,

$$\frac{(ia)^2}{2}\widehat{\Box} = -1 + \sqrt{1 + (ia)^2\widehat{D}_\mu\widehat{D}_\mu} \tag{6.63}$$

and hence

$$e^{-ia\hat{\partial}_n} = -ia\widehat{D}_n + \sqrt{1 + (ia)^2\widehat{D}_\mu\widehat{D}_\mu}. \tag{6.64}$$

We imply summation over repeated indices unless stated differently. Multiplying (6.57) with $e^{ia\hat{\partial}_n}$ yields the quadratic equation for $e^{ia\hat{\partial}_n}$

$$e^{2ia\hat{\partial}_n}\Big(1 + (ia)^2\widehat{D}_\mu\widehat{D}_\mu\Big) - 2ia\widehat{D}_n e^{ia\hat{\partial}_n} - 1 = 0 \tag{6.65}$$

with the solution

$$e^{ia\hat{\partial}_n} = \frac{1}{1 + (ia)^2\widehat{D}_j\widehat{D}_j}\left(ia\widehat{D}_n + \sqrt{1 + (ia)^2\widehat{D}_\mu\widehat{D}_\mu}\right). \tag{6.66}$$

The root is selected by demanding that $e^{ia\hat{\partial}_n} \to 1$, for $a \to 0$. By direct computation, we see that it is the inverse of (6.64). Equipped with eqns. (6.64) and (6.66), we can compute the Leibniz rules for $\widehat{D}_\mu$. They read

$$[\widehat{D}_n,\, \hat{x}^i] = ia\widehat{D}_i, \tag{6.67}$$

$$[\widehat{D}_j,\, \hat{x}^i] = -ia\delta^i_j\widehat{D}_n + \delta^i_j\sqrt{1 + (ia)^2\widehat{D}_\mu\widehat{D}_\mu}, \tag{6.68}$$

$$[\widehat{D}_n,\, \hat{x}^n] = \sqrt{1 + (ia)^2\widehat{D}_\mu\widehat{D}_\mu}, \tag{6.69}$$

$$[\widehat{D}_j,\, \hat{x}^n] = 0. \tag{6.70}$$

Furthermore, we can express the derivatives $\hat{\partial}_\mu$ in terms of $\widehat{D}_\mu$,

$$\hat{\partial}_i = \frac{\widehat{D}_i}{1+(ia)^2\widehat{D}_j\widehat{D}_j}\left(ia\widehat{D}_n + \sqrt{1+(ia)^2\widehat{D}_\mu\widehat{D}_\mu}\right) \tag{6.71}$$

$$= \frac{\widehat{D}_i}{1+(ia)^2\widehat{D}_j\widehat{D}_j}\left(1 + ia\widehat{D}_n + \frac{(ia)^2}{2}\widehat{\Box}\right),$$

$$\hat{\partial}_n = -\frac{1}{ia}\ln\left(-ia\widehat{D}_n + \sqrt{1+(ia)^2\widehat{D}_\mu\widehat{D}_\mu}\right). \tag{6.72}$$

In order to assure you that we do not devide by operators, we give you the expansion of the above equations around $a = 0$,

$$\hat{\partial}_i = \widehat{D}_i + ia\widehat{D}_i\widehat{D}_n + \frac{(ia)^2}{2}\left(\widehat{D}_n\widehat{D}_n - \widehat{D}_j\widehat{D}_j\right)\widehat{D}_i + \mathcal{O}(a^3), \tag{6.73}$$

$$\hat{\partial}_n = \widehat{D}_n - \frac{ia}{2}\widehat{D}_i\widehat{D}_i - \frac{(ia)^2}{6}\widehat{D}_n\left(3\widehat{D}_i\widehat{D}_i + \widehat{D}_n\widehat{D}_n\right) + \mathcal{O}(a^3). \tag{6.74}$$

Remarkably, the transformation properties of $\widehat{D}_\mu$ under rotations and boosts are especially simple. They are determined by eqn. (6.47),

$$[N^l, \widehat{D}_n] = \widehat{D}_l, \tag{6.75}$$

$$[N^l, \widehat{D}_i] = -\delta^l_i\widehat{D}_n. \tag{6.76}$$

This is the usual, undeformed transformation property of a vector. The structure under rotation is always the same, since the rotation sector of the Hopf algebra is undeformed. Since $[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0$, we also have

$$[\widehat{D}_\mu, \widehat{D}_\nu] = 0.$$

From (6.56) and (6.57), we can compute the co-product of $\widehat{D}_\mu$ using the fact that the co-product Δ is an algebra homomorphism. We find

$$\Delta\widehat{D}_i = \widehat{D}_i \otimes e^{-ia\hat{\partial}_n} + \mathbf{1} \otimes \widehat{D}_i, \tag{6.77}$$

$$\Delta\widehat{D}_n = \widehat{D}_n \otimes e^{-ia\hat{\partial}_n} + e^{ia\hat{\partial}_n} \otimes \widehat{D}_n + ia\widehat{D}_i e^{ia\hat{\partial}_n} \otimes \widehat{D}_i. \tag{6.78}$$

We have not inserted expressions (6.62) and (6.66) into above equations. We can express $\widehat{D}_\mu$ in terms of $\hat{\partial}_\mu$ and vice versa, in a closed form, i.e., $\widehat{D}_\mu$ are elements of the enveloping algebra generated by $\hat{\partial}_\mu$ and vice versa. The derivatives D_μ are merely a different set of generators of the enveloping algebra. Therefore, we have two derivatives we can gauge. We decided to gauge the Dirac operator $\widehat{D}_\mu$, see Section 6.6, because it transforms linearly under Lorentz transformations. The derivatives $\hat{\partial}_\mu$ have a very involved transformation property.

Let us analyse this model further, and let us consider the structure of vector fields. As there are two derivative structures, there are two vector field structures. Let us first derive the transformation properties of a vector field $\widehat{A}_\mu$ transforming in a similiar way as the translation generators $\hat{\partial}_\mu$. We demand that the transformation is linear in $\widehat{A}_\mu$, that we are back at eqns. (6.21) - (6.24) when we replace $\widehat{A}_\mu$ by $\hat{\partial}_\mu$ and that the relations are respected by the symmetry algebra. These assertions fix the transformation property uniquely. We obtain up to all order in a

$$[M^{rs}, \widehat{A}_i] = \delta^{ri}\widehat{A}_s - \delta^{si}\widehat{A}_r, \tag{6.79}$$

$$[M^{rs}, \widehat{A}_n] = 0, \tag{6.80}$$

$$\begin{aligned}[N^l, \widehat{A}_i] = {} & \delta^l_i \frac{1-e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n}\widehat{A}_n - \frac{ia}{2}\delta^l_i\hat{\partial}_k\widehat{A}_k + \frac{ia}{2}(\hat{\partial}_l\widehat{A}_i + \hat{\partial}_i\widehat{A}_l) \\ & - \frac{a}{2\hat{\partial}_n}\,\delta^l_i\tan(\frac{a\hat{\partial}_n}{2})\,\hat{\partial}_k\left(\hat{\partial}_n\widehat{A}_k - \hat{\partial}_k\widehat{A}_n\right) \\ & + \left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot(\frac{a\hat{\partial}_n}{2})\right)\left(\hat{\partial}_n\hat{\partial}_i\widehat{A}_l + \hat{\partial}_n\hat{\partial}_l\widehat{A}_i - 2\hat{\partial}_l\hat{\partial}_i\widehat{A}_n\right),\end{aligned} \tag{6.81}$$

$$[N^l, \widehat{A}_n] = \widehat{A}_l. \tag{6.82}$$

We will shortly introduce another vector field that transforms similiar to the derivative $\widehat{D}_\mu$. In order to do so, we express the Dirac operator (6.54) in terms of a "Lorentz vielbein" ${e_\mu}^\nu$,

$$\widehat{D} = \gamma^\mu {e_\mu}^\nu\hat{\partial}_\nu. \tag{6.83}$$

We can define another vector field $\widehat{V}_\mu$ by

$$\widehat{V}_\mu = {e_\mu}^\nu\widehat{A}_\nu. \tag{6.84}$$

$\widehat{V}_\mu$ will transform like a classical vector, cf. (6.75) and (6.76). Let us calculate the derivative (symmetry algebra) valued vielbein ${e_\mu}^\nu$. Eqn. (6.83) does not define the vielbein uniquely. The vielbein contains itself derivatives and since there is a derivative at the end of (6.83), in some cases you cannot tell which derivative belongs to the vielbein. Therefore we need to take into account the transformation properties. We know that the Dirac operator $\widehat{D} = \gamma^\mu {e_\mu}^\nu\hat{\partial}_\nu$ satisfies relation (6.47). We also know that $\gamma^\mu {e_\mu}^\nu \widehat{A}_\nu$ satisfies the same relation. From this expression, we can extract the transformation of the vielbein itself. It reads

$$\begin{aligned}\left[N^l, {e_n}^n\right] &= {e_l}^n + {e_n}^l \frac{e^{2ia\hat{\partial}_n} - 1}{2ia\hat{\partial}_n} - {e_n}^l \frac{a}{2\hat{\partial}_n} \tan\left(\frac{a\hat{\partial}_n}{2}\right) \hat{\partial}_j\hat{\partial}_j \\ &\quad +2{e_n}^j \left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n} \cot\left(\frac{a\hat{\partial}_n}{2}\right)\right) \hat{\partial}_l\hat{\partial}_j,\end{aligned} \tag{6.85}$$

$$\begin{aligned}\left[N^l, {e_n}^i\right] &= {e_l}^i - \delta^{li}{e_n}^n + \frac{ia}{2}{e_n}^l\hat{\partial}_i - \frac{ia}{2}\delta^{li}{e_n}^j\hat{\partial}_j - \frac{ia}{2}{e_n}^i\hat{\partial}_l \\ &\quad +\frac{a}{2}{e_n}^l \tan\left(\frac{a\hat{\partial}_n}{2}\right)\hat{\partial}^i - (\delta^{li}{e_n}^j\hat{\partial}_j + {e_n}^i\hat{\partial}_l)\left(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot(\frac{a\hat{\partial}_n}{2})\right),\end{aligned} \tag{6.86}$$

$$\begin{aligned}\left[N^l, {e_i}^n\right] &= -\delta^l_i{e_n}^n + {e_i}^l \frac{e^{2ia\hat{\partial}_n} - 1}{2ia\hat{\partial}_n} - {e_i}^l \frac{a}{2\hat{\partial}_n} \tan\left(\frac{a\hat{\partial}_n}{2}\right) \hat{\partial}_j\hat{\partial}_j \\ &\quad +2{e_i}^j \left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n} \cot(\frac{a\hat{\partial}_n}{2})\right) \hat{\partial}_l\hat{\partial}_j,\end{aligned} \tag{6.87}$$

$$\begin{aligned}\left[N^l, {e_i}^j\right] &= -\delta^l_i{e_n}^j - \delta^{lj}{e_i}^n + \frac{ia}{2}{e_i}^l\hat{\partial}_j - \frac{ia}{2}\delta^{lj}{e_i}^k\hat{\partial}_k - \frac{ia}{2}{e_i}^j\hat{\partial}_l \\ &\quad +\frac{a}{2}{e_i}^l \tan\left(\frac{a\hat{\partial}_n}{2}\right)\hat{\partial}^j - (\delta^{lj}{e_i}^k\hat{\partial}_k + {e_i}^j\hat{\partial}_l)\left(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot(\frac{a\hat{\partial}_n}{2})\right).\end{aligned} \tag{6.88}$$

Using the transformation properties of the vielbein, we can express the vielbein in terms of derivatives,

$${e_n}^n = \frac{1}{a\hat{\partial}_n}\sin(a\hat{\partial}_n) + e^{-ia\hat{\partial}_n}\left(\frac{ia}{2} - \frac{i}{\hat{\partial}_n}\tan(\frac{a\hat{\partial}_n}{2})\right)\frac{\widehat{\Delta}_\kappa}{\hat{\partial}_n}, \tag{6.89}$$

$${e_n}^j = \frac{i}{\hat{\partial}_n}e^{-ia\hat{\partial}_n}\tan(\frac{a\hat{\partial}_n}{2})\,\hat{\partial}_j, \tag{6.90}$$

$${e_i}^n = \left(e^{-ia\hat{\partial}_n} - \frac{1 - e^{-ia\hat{\partial}_n}}{ia\hat{\partial}_n}\right)\frac{\hat{\partial}_i}{\hat{\partial}_n}, \tag{6.91}$$

$${e_i}^j = \frac{1 - e^{-ia\hat{\partial}_n}}{ia\hat{\partial}_n}\delta^j_i. \tag{6.92}$$

We have two different vector structures, one - V_μ - transforming like a classical vector and like $\widehat{D}_\mu$, and one - A_μ - transforming like the derivatives $\hat{\partial}_\mu$. The vielbein ${e_\mu}^\nu$

connects both vector fields, intertwines the different transformation rules. The lower index of the vielbein transforms like a classical vector index, cf. (6.51) and (6.52) - from now on denoted by indices $\alpha, \beta, \gamma, ...$ and $a, b, c, \ldots$, respectively. The upper index and indices transforming like $\hat{\partial}_\mu$ will be denoted from now on by indices $\mu, \nu, \rho, ...$ and $k, l, m, \ldots$, respectively.

In the computation of the transformation property of the vector field $\widehat{A}_\mu$, (6.81) and (6.82), we have put all the derivatives on the left of the field. Since $\widehat{A}_\mu$ is a field and depends on the coordinates, it is very hard to permute the derivatives to the right side of the field. Equally well, we could have started with an ansatz where all the derivatives are on the right side. Therefore we introduce another vector field, $\widehat{\widehat{B}}_\mu$. Similarily to (6.81) and (6.82), we obtain

$$
\begin{aligned}
[N^l, \widehat{\widehat{B}}_i] = {} & \delta_i^l \, \widehat{\widehat{B}}_n \frac{1 - e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n} - \frac{ia}{2} \delta_i^l \, \widehat{\widehat{B}}_k \hat{\partial}_k + \frac{ia}{2} (\widehat{\widehat{B}}_i \hat{\partial}_l + \widehat{\widehat{B}}_l \hat{\partial}_i) \\
& - \frac{a}{2} \, \delta_i^l \left(\widehat{\widehat{B}}_k \hat{\partial}_n - \widehat{\widehat{B}}_n \hat{\partial}_k \right) \frac{\hat{\partial}_k}{\hat{\partial}_n} \tan(\frac{a\hat{\partial}_n}{2}) \\
& + \left(\widehat{\widehat{B}}_l \hat{\partial}_n \hat{\partial}_i + \widehat{\widehat{B}}_i \hat{\partial}_n \hat{\partial}_l - 2\widehat{\widehat{B}}_n \hat{\partial}_l \hat{\partial}_i \right) \left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n} \cot(\frac{a\hat{\partial}_n}{2}) \right),
\end{aligned}
\tag{6.93}
$$

$$
[N^l, \widehat{\widehat{B}}_n] = \widehat{\widehat{B}}_l. \tag{6.94}
$$

We want to construct scalars under Poincaré transformations. Therefore, we have to introduce fields $\widehat{B}^\mu$ and $\widehat{\widehat{A}}^\mu$, such that

$$
\widehat{\widehat{A}}^\mu \widehat{A}_\mu \;\; \text{and} \;\; \widehat{\widehat{B}}_\mu \widehat{B}^\mu
$$

are invariant, i.e., $[N^l, \widehat{\widehat{A}}^\mu \widehat{A}_\mu] = [N^l, \widehat{\widehat{B}}_\mu \widehat{B}^\mu] = 0$. The invariance defines the transformation properties. They are given by

$$
\begin{aligned}
[N^l, \widehat{\widehat{A}}^n] = {} & -\widehat{\widehat{A}}^l \, \frac{1 - e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n} - \widehat{\widehat{A}}^l \, \frac{a}{2\hat{\partial}_n} \tan\left(\frac{a\hat{\partial}_n}{2} \right) \hat{\partial}_j \hat{\partial}_j \\
& + 2\widehat{\widehat{A}}^j \left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n} \cot(\frac{a\hat{\partial}_n}{2}) \right) \hat{\partial}_l \hat{\partial}_j,
\end{aligned}
\tag{6.95}
$$

$$
[N^l, \widehat{\widehat{A}}^i] = -\delta^{li} \widehat{\widehat{A}}^n + \frac{ia}{2} \widehat{\widehat{A}}^l \hat{\partial}_i - \frac{ia}{2} \delta^{li} \widehat{\widehat{A}}^m \hat{\partial}_m - \frac{ia}{2} \widehat{\widehat{A}}^i \hat{\partial}_l \tag{6.96}
$$

$$
\begin{aligned}
&+2\widehat{\widetilde{A}}^l \tan\left(\frac{a\hat{\partial}_n}{2}\right)\hat{\partial}_i - (\delta^{li}\widehat{\widetilde{A}}^j\hat{\partial}_j + \widehat{\widetilde{A}}^i\hat{\partial}_l)\left(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot(\frac{a\hat{\partial}_n}{2})\right),
\end{aligned}
$$

$$
\begin{aligned}
[N^l, \widehat{B}^n] &= -\frac{1-e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n}\widehat{B}^l - \frac{a}{2\hat{\partial}_n}\tan\left(\frac{a\hat{\partial}_n}{2}\right)\hat{\partial}_j\hat{\partial}_j\widehat{B}^l \qquad (6.97)\\
&+2\left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot(\frac{a\hat{\partial}_n}{2})\right)\hat{\partial}_l\hat{\partial}_j\widehat{B}^j,\\
[N^l, \widehat{B}^i] &= -\delta^{li}\widehat{B}^n + \frac{ia}{2}\hat{\partial}_i\widehat{B}^l - \frac{ia}{2}\delta^{li}\hat{\partial}_m\widehat{B}^m - \frac{ia}{2}\hat{\partial}_l\widehat{B}^i \qquad (6.98)\\
&+2\tan\left(\frac{a\hat{\partial}_n}{2}\right)\hat{\partial}_i\widehat{B}^l - \left(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot(\frac{a\hat{\partial}_n}{2})\right)(\delta^{li}\hat{\partial}_j\widehat{B}^j + \hat{\partial}_l\widehat{B}^i).
\end{aligned}
$$

In the same way as the vielbein ${e_\alpha}^\mu$ connects $\widehat{V}_\alpha$ and $\widehat{A}_\mu$, there is another vielbein ${\tilde{e}_\mu}^{\ \alpha}$ connecting $\widehat{\widetilde{A}}^\mu$ and $\widehat{V}^\alpha$,

$$
\widehat{V}^\alpha = \widehat{\widetilde{A}}^\mu\, {\tilde{e}_\mu}^{\ \alpha}, \quad \widehat{\widetilde{A}}^\mu = \widehat{V}^\alpha {\tilde{e}^{-1}}_{\alpha}^{\ \mu}. \qquad (6.99)
$$

We know that $\widehat{V}^\mu\widehat{V}_\mu$ is an invariant, and

$$
\widehat{\widetilde{A}}^\mu\widehat{A}_\mu = \widehat{V}^\alpha\, {\tilde{e}^{-1}}_{\alpha}^{\ \mu}\, {e^{-1}}_{\mu}^{\ \beta}\, \widehat{V}_\beta. \qquad (6.100)
$$

Therefore, we conclude that

$$
{\tilde{e}^{-1}}_{\alpha}^{\ \mu}{e^{-1}}_{\mu}^{\ \beta} = \delta_\alpha^\beta, \qquad (6.101)
$$

and we get

$$
{\tilde{e}^{-1}}_{\alpha}^{\ \mu} = {e_\alpha}^\mu. \qquad (6.102)
$$

Let us calculate the inverse of the vielbein perturbatively. In order to do so, we have to expand ${e_\alpha}^\mu$ around δ_α^μ,

$$
\begin{aligned}
{e_n}^n &= 1 + \frac{(ia)^2}{6}\partial_n^2 + \mathcal{O}(a^3),\\
{e_n}^i &= \frac{ia}{2}\partial_i - \frac{(ia)^2}{2}\partial_n\partial_i + \mathcal{O}(a^3),\\
{e_i}^n &= -\frac{ia}{2}\partial_i + \frac{(ia)^2}{3}\partial_n\partial_i + \mathcal{O}(a^3),\\
{e_i}^j &= \delta_i^j(1 - \frac{ia}{2}\partial_n + \frac{(ia)^2}{6}\partial_n^2) + \mathcal{O}(a^3).
\end{aligned}
$$

Then, the inverse is defined by

$$e_\alpha{}^\nu e^{-1}{}_\nu{}^\beta = \delta_\alpha^\beta. \tag{6.103}$$

We obtain the following expressions,

$$e^{-1}{}_n{}^n = 1 - \frac{(ia)^2}{6}\hat{\partial}_n^2 - \frac{(ia)^2}{4}\widehat{\Delta}_\kappa + \mathcal{O}(a^3), \tag{6.104}$$

$$e^{-1}{}_n{}^i = -\frac{ia}{2}\hat{\partial}_i + \frac{(ia)^2}{4}\hat{\partial}_n\hat{\partial}_i + \mathcal{O}(a^3), \tag{6.105}$$

$$e^{-1}{}_i{}^n = \frac{ia}{2}\hat{\partial}_i - \frac{(ia)^2}{12}\hat{\partial}_n\hat{\partial}_i + \mathcal{O}(a^3), \tag{6.106}$$

$$e^{-1}{}_i{}^j = \delta_i^j\left(1 + \frac{ia}{2}\hat{\partial}_n + \frac{(ia)^2}{12}\hat{\partial}_n^2\right) - \frac{(ia)^2}{4}\hat{\partial}_i\hat{\partial}_j + \mathcal{O}(a^3). \tag{6.107}$$

We may also introduce a conjugation † on the algebra. The coordinates are chosen to be real and we have the following definitions compatible with all the relations

$$\hat{x^\mu}^\dagger = \hat{x}^\mu, \qquad T^\dagger = -T,\, T \in \{M^{rs}, N^l, \hat{\partial}_\mu\}, \tag{6.108}$$

$$\widehat{A}_\mu{}^\dagger = \widehat{\widehat{B}}_\mu, \qquad \widehat{\widehat{A}}^{\mu\dagger} = \widehat{B}^\mu, \qquad \widehat{V}_\mu^\dagger = \widehat{V}_\mu. \tag{6.109}$$

6.5 Representation on Commutative Functions

In this Section, we want to translate our results to the ∗-product formalism introduced in Chapter 3. We will work out the ∗-product and the action of the $\kappa-$Poincaré generators on commutative functions for two different orderings. We will examine symmetrical ordering using CBH quantisation (Section 3.1) and normal ordering. The formulae can be written in a closed form, considering normal ordering.

6.5.1 Symmetric Ordering

The ∗-product of functions in symmetrical ordering is obtained as

$$\begin{aligned} f * g\,(x) =\; & f(x)g(x) + \frac{ia}{2}\left(x^\nu\partial_n f(x)\partial_\nu g(x) - x^\mu\partial_\mu f(x)\partial_n g(x)\right) \\ & + \frac{(ia)^2}{12}\Big(x^\nu(\partial_n^2 f(x)\partial_\nu g(x) + \partial_\nu f(x)\partial_n^2 g(x)) \\ & \quad - x^\mu(\partial_\mu\partial_n f(x)\partial_n g(x) + \partial_n f(x)\partial_\mu\partial_n g(x))\Big) \end{aligned} \tag{6.110}$$

$$+\frac{(ia)^2}{8}x^\mu x^\nu\Big(\partial_n^2 f(x)\partial_\mu\partial_\nu g(x)-\partial_\nu\partial_n f(x)\partial_n\partial_\mu g(x)$$
$$-\partial_\mu\partial_n f(x)\partial_n\partial_\nu g(x)+\partial_\mu\partial_\nu f(x)\partial_n^2 g(x)\Big)+\mathcal{O}(a^3).$$

This $*$-product is hermitian, i.e., $\overline{f * g}=\bar{g} * \bar{f}$. The bar denotes complex conjugation. The representation of the algebra generators on commutative functions will be denoted by $\tilde{M}^{rs}$, $\tilde{N}^l$, $\tilde{\partial}_\mu$. The action of these generators of the $\kappa-$Poincaré algebra on commutative functions is calculated in a similar way as the $*$-product in Section 3.1. First, one has to calculate the action of the generators on symmetrically ordered polynomials of the non-commutative coordinates. Expressing the result again in terms of symmetrically ordered polynomials, it can be translated to the action on the corresponding commutative monomials, see (3.21) and (3.22). In a second step, one has to generalise these expressions to commutative functions, polynomials in the coordinates. The action of the derivatives on commutative functions is given by

$$\tilde{\partial}_n \triangleright f(x)=\partial_n f(x), \tag{6.111}$$

$$\tilde{\partial}_i \triangleright f(x)=\partial_i\frac{\exp(ia\partial_n)-1}{ia\partial_n}f(x). \tag{6.112}$$

The derivatives on the rhs. are ordinary derivatives. We can also evaluate the action of the derivatives on a product of two function, i.e., we calculate the modified Leibniz rule. It is given by

$$\tilde{\partial}_n \triangleright f * g(x)=(\tilde{\partial}_n \triangleright f(x)) * g(x)+f(x) * (\tilde{\partial}_n \triangleright g(x)), \tag{6.113}$$

$$\tilde{\partial}_i \triangleright f * g(x)=\Big(\tilde{\partial}_i \triangleright f(x)\Big) * g(x)+(e^{ia\tilde{\partial}_n}\triangleright f(x)) * \Big(\tilde{\partial}_i \triangleright g(x)\Big). \tag{6.114}$$

The boosts and the rotation generators act on functions in the following way,

$$\tilde{N}^l \triangleright f(x)=\Big(x^l\partial_n-x^n\partial_l \tag{6.115}$$
$$+x^l\partial_\mu\partial_\mu\frac{e^{ia\partial_n}-1}{2\partial_n}+x^\nu\partial_\nu\partial_l\,\frac{a\partial_n+i(e^{ia\partial_n}-1)}{a\partial_n^2}\Big)f,$$

$$\tilde{M}^{rs}\triangleright f(x)=(x^s\partial_r-x^r\partial_s)\,f(x), \tag{6.116}$$

where $\Delta_{cl}=\sum_{i=1}^{n-1}\partial_i\partial_i$. The action of the rotations $\tilde{M}^{rs}$ is easily obtained, since their algebra and co-algebra structure is undeformed. Acting with the generator N^l on symmetrically ordered polynomials in the non-commutative coordinates, we see that $\tilde{N}^l$ must have the form

$$\tilde{N}^l=x^l\partial_n-x^n\partial_l+x^l\partial_\mu\partial_\mu\sum_{i=0}^{\infty}A_i\partial_n^i+x^\mu\partial_\nu\partial_l\sum_{i=0}^{\infty}B_i\partial_n^i. \tag{6.117}$$

The coefficients A_i and B_i are determined by the algebra relations. Since the quantities denoted with a tilde are a representation of the κ-Poincaré algebra, they have to fulfill the same relations. The relation $[\tilde{N}^l, \tilde{\partial}_n] = \tilde{\partial}_l$ determines coefficients B_i. We obtain

$$\begin{aligned} B(\partial_n) &= \sum_m B_m \partial_n^m = \frac{a\partial_n + i(e^{ia\partial_n} - 1)}{a\partial_n^2} \\ &= -\sum_m \frac{(ia)^{m+1}}{(m+2)!}\, \partial_n^m. \end{aligned} \tag{6.118}$$

The relation $[\tilde{N}^l, \tilde{N}^k] = \tilde{M}^{lk}$ provides us with another equation for A_i and B_i, namely

$$A(z)' = 2A(z)^2 + A(z)B(z) + zA'(z)B(z), \tag{6.119}$$

where $A(z) = \sum_i A_i z^i$. Eqn. (6.119) has the solution

$$\begin{aligned} A(z) &= \frac{e^{iaz} - 1}{2z} \\ &= \sum_{m=0}^{\infty} \frac{(ia)^{m+1}}{2(m+1)!}\, \partial_n^m, \end{aligned} \tag{6.120}$$

taking $A_0 = \frac{ia}{2}$ into account, which we know from acting on symmetrically ordered polynomials. The action on a product of functions is defined by the co-product structure,

$$\begin{aligned} \tilde{N}^l \triangleright f * g(x) &= \left(\tilde{N}^l \triangleright f(x)\right) * g(x) + (e^{ia\tilde{\partial}_n} \triangleright f(x)) * (\tilde{N}^l \triangleright g(x)) \\ &\quad -ia\left(\tilde{\partial}_b \triangleright f(x)\right) * (\tilde{M}^{lb} \triangleright g(x)), \end{aligned} \tag{6.121}$$

$$\tilde{M}^{rs} \triangleright f * g(x) = (\tilde{M}^{rs} \triangleright f(x)) * g(x) + f(x) * (\tilde{M}^{rs} \triangleright g(x)). \tag{6.122}$$

The Leibniz rules for the generators have been checked explicitly up to second order. That means, that the $*$-product is covariant with respect to the action of the quantum Poincaré algebra $\mathcal{P}_\kappa$.

Since we have calculated the action of the non-commutative derivatives on commutative functions, we can also rewrite the Klein-Gordon operator (6.34) and the Dirac operator (6.54). They now act on commutative functions and spinors, respectively. The Klein-Gordon operator reads

$$\Box = \frac{2}{(ia\partial_n)^2}(\cos(a\partial_n) - 1)\left(\Delta_{cl} + \partial_n^2\right). \tag{6.123}$$

We drop the hats in order not to forget that we are back in the commutative regime, i.e., the operators act on classical wave functions. For the Dirac operator we obtain

$$D = \frac{\gamma^0}{ia}\left(i\sin(a\partial_n) + \frac{\Delta_{cl}}{\partial_n^2}(\cos(a\partial_n) - 1)\right) + \frac{\gamma^r}{ia}\frac{2i\partial_r}{\partial_n}\, e^{-\frac{ia}{2}\partial_n}\sin(\frac{a}{2}\partial_n), \tag{6.124}$$

$D = \gamma^0 D_n + \gamma^r D_r$. As the Dirac operator is also considered as derivative, we will mention its Leibniz rule. Using (6.77) and (6.78), we find for the action on a product of functions

$$\begin{aligned}\tilde{D}_n \triangleright f * g(x) &= (\tilde{D}_n \triangleright f(x)) * (e^{-ia\tilde{\partial}_n} \triangleright g(x)) + (e^{ia\tilde{\partial}_n} \triangleright f(x)) * (\tilde{D}_n \triangleright g(x)) \\ &\quad + ia\sum_{i=1}^{n-1}\left(\tilde{D}_i e^{ia\tilde{\partial}_n} \triangleright f(x)\right) * (\tilde{D}_i \triangleright g(x)), \qquad (6.125)\\ \tilde{D}_i \triangleright f * g(x) &= (\tilde{D}_i \triangleright f(x)) * (e^{-ia\tilde{\partial}_n} \triangleright g(x)) + f(x) * (\tilde{D}_i \triangleright g(x)), \qquad (6.126)\end{aligned}$$

where $\tilde{D}_\mu \triangleright f(x) = D_\mu f(x)$.

6.5.2 Normal Ordering

Let us compute the $*$-product in normal ordering. By normal ordered monomials we mean monomials like $(x^n)^{i_n}(x^l)^{i_l}$, where all the x^ns are on the left. We have to modify the quantisation rule (3.8) slightly. We define the map W by

$$\hat{f} = W(f) = \frac{1}{(2\pi)^{n/2}}\int d^n k\ : e^{ik_\mu \hat{x}^\mu} : \tilde{f}(k), \tag{6.127}$$

where : : defines the ordering,

$$: e^{ik_\mu \hat{x}^\mu} : = e^{ik_n \hat{x}^n} e^{ik_i \hat{x}^i}, \tag{6.128}$$

compare e.g., [74, 78]. The $*$-product can be computed using eqn. (3.5),

$$W(f *_N g) = \frac{1}{(2\pi)^n}\int d^n k\, d^n p : e^{ik_\mu \hat{x}^\mu} :: e^{ip_\nu \hat{x}^\nu} : \tilde{f}(k)\tilde{g}(p) \tag{6.129}$$

The next step in the computation of the $*$-product is to order the exponentials according to (6.128). We find

$$\begin{aligned}e^{ik_n\hat{x}^n} e^{ik_i\hat{x}^i} e^{ip_n\hat{x}^n} e^{ip_i\hat{x}^i} &= e^{ik_n\hat{x}^n} e^{ip_n\hat{x}^n} \underbrace{e^{-ip_n\hat{x}^n} e^{ik_i\hat{x}^i} e^{ip_n\hat{x}^n}}_{\exp(ik_i\hat{x}^i e^{ap_n})} e^{ip_i\hat{x}^i} \\ &= e^{i(k_n+p_n)\hat{x}^n} e^{i(k_i+p_i)\hat{x}^i} \exp\left(ik_i\hat{x}^i(e^{ap_n} - 1)\right),\end{aligned}$$

where we have used

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \tag{6.130}$$

and

$$e^{-ip_n\hat{x}^n}\hat{x}^i e^{ip_n\hat{x}^n} = \hat{x}^i e^{ap_n}. \tag{6.131}$$

Therefore, we obtain

$$W(f *_N g) = \frac{1}{(2\pi)^n}\int d^nk\, d^np e^{i(k_n+p_n)\hat{x}^n} e^{i(k_i+p_i)\hat{x}^i} \exp\left(ik_i\hat{x}^i(e^{ap_n}-1)\right), \tag{6.132}$$

and the the $*$-product of functions for normal ordering is given by

$$f *_N g\,(x) = \exp\left(z^i\frac{\partial}{\partial x^i}\left(e^{-ia\frac{\partial}{\partial y^n}}-1\right)\right) f(x)g(y)\Big|_{\substack{z\to x\\ y\to x}}. \tag{6.133}$$

We can check the equation for the $*$-product on normal ordered monomials. For monomials we can calculate the $*$-product in the same way as in Section 3.2 and 3.3. We obtain the result

$$\begin{aligned}(\hat{x}^n)^{i_n}(\hat{x}^1)^{i_1}\cdots(\hat{x}^{n-1})^{i_{n-1}}\,(\hat{x}^n)^{j_n}(\hat{x}^1)^{j_1}\cdots(\hat{x}^{n-1})^{j_{n-1}} = \\ = (\hat{x}^n)^{i_n}\left(\hat{x}^n-(i_1+i_2+\ldots+i_{n-1})ia\right)^{j_n}(\hat{x}^1)^{i_1+j_1}\cdots(\hat{x}^{n-1})^{i_{n-1}+j_{n-1}}.\end{aligned} \tag{6.134}$$

And therefore we have

$$\begin{aligned}((x^n)^{i_n}(x^1)^{i_1}\cdots(x^{n-1})^{i_{n-1}})\ \tilde{*}_N\ ((x^n)^{j_n}(x^1)^{j_1}\cdots(x^{n-1})^{j_{n-1}}) = \\ = (x^n)^{i_n}\left(x^n-(i_1+i_2+\ldots+i_{n-1})ia\right)^{j_n}(x^1)^{i_1+j_1}\cdots(x^{n-1})^{i_{n-1}+j_{n-1}}.\end{aligned}$$

It is a partial displacement in x^n. We see that (6.133) leads to the same result,

$$\begin{aligned}(x^n)^{i_n}(x^1)^{i_1}\cdots(x^{n-1})^{i_{n-1}} *_N (x^n)^{j_n}(x^1)^{j_1}\cdots(x^{n-1})^{j_{n-1}} = x^I *_N x^J = \\ = \exp\left(z^i\tfrac{\partial}{\partial x^i}\left(e^{-ia\frac{\partial}{\partial y^n}}-1\right)\right)x^I y^J\Big|_{\substack{z\to x\\ y\to x}} \\ = (x^n)^{i_n}\left(x^1+x^1(e^{-ia\frac{\partial}{\partial y^n}}-1)\right)^{i_1}\cdots\left(x^{n-1}+x^{n-1}(e^{-ia\frac{\partial}{\partial y^n}}-1)\right)^{i_{n-1}} y^J\Big|_{y\to x} \\ = (x^n)^{i_n}(x^1)^{i_1}\cdots(x^{n-1})^{i_{n-1}}\left(e^{-ia\frac{\partial}{\partial y_n}}\right)^{i_1+\ldots+i_{n-1}} y^J\Big|_{y\to x} \\ = (x^n)^{i_n}(x^n-(i_1+\ldots+i_{n-1})ia)^{j_n}(x^1)^{i_1+j_1}\cdots(x^{n-1})^{i_{n-1}+j_{n-1}}.\end{aligned} \tag{6.135}$$

In contrast to the symmetrically ordered $*$-product, this product is not hermitian, i.e., conjugation changes the ordering. Similarily as in the previous Subsection, we can compute the action of the $\kappa-$symmetry generators on commutative functions. For the derivatives we get

$$\tilde{\partial}_i \triangleright f(x) = \partial_i \, e^{ia\partial_n} \, f(x), \tag{6.136}$$

$$\hat{\partial}_n \triangleright f(x) = \partial_n \, f(x). \tag{6.137}$$

The action of the Lorentz generators reads

$$\tilde{N}^l \triangleright f(x) = \left(\frac{x^l}{a} \sin(a\partial_n) - x^n \partial_l \, e^{ia\partial_n} + \frac{ia}{2} x^l \Delta_{cl} \, e^{ia\partial_n} \right) f(x), \tag{6.138}$$

$$\tilde{M}^{rs} \triangleright f(x) = (x^s \partial_r - x^r \partial_s) \, f(x). \tag{6.139}$$

Furthermore, we again rewrite the Klein-Gordon and Dirac operator. We get

$$\Box = e^{ia\partial_n} \Delta_{cl} + \frac{2}{a^2} \left(1 - \cos(a\partial_n)\right), \tag{6.140}$$

$$D = \gamma^0 \left(\frac{\sin(a\partial_n)}{a} + \frac{ia}{2} \Delta_{cl} \, e^{ia\partial_n} \right) + \gamma^r \partial_r. \tag{6.141}$$

The action of the Dirac operator on a product of functions is given by eqn. (6.125), where the action is defined by eqn. (6.141). Comparing (6.141) with (6.138), we see that

$$\tilde{N}^l \triangleright f(x) = \left(x^l D_n - x^n D_l \, e^{ia\partial_n} \right) f(x). \tag{6.142}$$

6.6 Seiberg-Witten Maps

We proceed to gauging the Dirac operator $\widehat{D}$. We define

$$\begin{aligned} \widehat{\mathcal{D}}_\alpha &= e_\alpha{}^\mu \hat{\partial}_\mu - i\widehat{V}_\alpha \\ &= \widehat{D}_\alpha - i\widehat{V}_\alpha, \end{aligned} \tag{6.143}$$

where $\widehat{V}_\alpha$ is the gauge potential. The operator $\gamma^\alpha \widehat{\mathcal{D}}_\alpha$ satisfies by construction the defining eqn. (6.47). The Lorentz transformation of the algebra valued gauge potential $\widehat{V}_\alpha$ is given by eqns. (6.51) and (6.52). Let us introduce a non-commutative field $\widehat{\psi}$ which transforms as follows under a non-commutative gauge transformation,

$$\hat{\delta}_\alpha \widehat{\psi} = i\widehat{\Lambda}_\alpha \widehat{\psi}. \tag{6.144}$$

The gauge transformation of $\widehat{V}_\alpha$ is determined by demanding that $\widehat{\mathcal{D}_\gamma}\psi$ transforms covariantly, i.e.,

$$\hat{\delta}_\alpha \widehat{\mathcal{D}_\gamma}\, \widehat{\psi} = i\widehat{\Lambda}_\alpha \widehat{\mathcal{D}_\gamma}\, \widehat{\psi}. \tag{6.145}$$

Therefore, we have for the gauge transformation of the gauge potential $\widehat{V}_\alpha$

$$(\hat{\delta}_\alpha \widehat{V}_\gamma)\, \widehat{\psi} = [\widehat{D}_\gamma,\, \widehat{\Lambda}_\alpha]\, \widehat{\psi} - i[\widehat{V}_\gamma,\, \widehat{\Lambda}_\alpha]\, \widehat{\psi}. \tag{6.146}$$

Because of the co-product structure of the derivatives $\widehat{D}_\mu$ given in (6.77) and (6.78), the gauge field $\widehat{V}_\alpha$ has to be derivative dependent, i.e., Poincaré symmetry algebra valued. In the quantum group case [82], the co-product of derivatives reads

$$\Delta\check{\partial}_\mu = \check{\partial}_\mu \otimes \mathbf{1} + L_\mu{}^\nu \otimes \check{\partial}_\nu, \tag{6.147}$$

where $L_\mu{}^\nu$ is the L-matrix, cf. (2.39). Here the L-matrix, a linear transformation appears in the first tensor factor of the second term. In this case covariant derivatives are defined introducing a vielbein $E_\mu{}^\nu$ with non-trivial transformation properties,

$$\mathcal{D}_\mu \Psi = E_\mu{}^\nu \left(\check{\partial}_\nu - iA_\nu\right) \Psi. \tag{6.148}$$

In our case, this very factor is non-linear in the derivatives, and it cannot be compensated by a vielbein, but we have to introduce a derivative dependent gauge field. The gauge field $\widehat{V}_\alpha$ can be expanded in terms of derivatives and in terms of the non-commutativity a,

$$\widehat{V}_\alpha = \sum_l \left\{ a^l \widehat{V}_\alpha^{(0,l)} + a^{l+1} \widehat{V}_\alpha^{(1,l)\mu}\, \hat{\partial}_\mu + \ldots + a^{l+n} \widehat{V}_\alpha^{(n,l)\mu_1 \ldots \mu_n}\, \hat{\partial}_{\mu_1} \ldots \hat{\partial}_{\mu_n} + \ldots \right\}, \tag{6.149}$$

where the component fields, e.g., $\widehat{V}_\alpha^{(0,l)}$, $\widehat{V}_\alpha^{(n,l)\mu_1 \ldots \mu_n}$ are still elements of the enveloping algebra of the gauge symmetry, see Section 4.2. We plug (6.149) into eqn. (6.146), in order to compute the gauge transformation property of the component functions. In order to compute the Seiberg-Witten maps, we have to proceed as discussed in Subsection 4.2.3. We have to solve the gauge equivalence conditions. We use the symmetrically ordered $*$-product (6.110). In the last Chapter, quantities with a "hat" denoted quantities that can be expanded in terms of commutative fields and parameters. In this Chapter, however, we have used this notation to denote quantities on the algebra level. In order to avoid unnecessary confusion, quantities with a "tilde" will indicate commutative elements that can be expanded via Seiberg-Witten maps in terms of ordinary fields and gauge parameters. In very much the same way, we have denoted the representation of the κ-Poincaré generators on commutative functions with $\tilde{N}^l$, $\tilde{M}^{rs}$ and $\tilde{\partial}_\mu$.

Let us first consider the Seiberg-Witten map of the gauge parameter $\tilde{\Lambda}$ to first order in a. The first order expression of the gauge equivalence relations (4.32) in case of $\kappa-$Euclidean space reads

$$\begin{aligned} i\delta_\alpha \Lambda^1_\beta[A] - i\delta_\beta \Lambda^1_\alpha[A] + [\alpha,\, \Lambda^1_\beta[A]] + [\Lambda^1_\alpha[A],\, \beta] - \Lambda^1_{[\alpha,\beta]}[A] = \\ = -\tfrac{ia}{2}\left(x^\mu\{\partial_n\alpha,\, \partial_\mu\beta\} - x^\mu\{\partial_n\beta,\, \partial_\mu\alpha\}\right), \end{aligned} \tag{6.150}$$

where $\tilde{\Lambda}_\alpha[A] = \alpha + \Lambda^1_\alpha[A] + \mathcal{O}(a^2)$, and A is the classical gauge field. The right hand side of eqn. (6.150) can be writen more concisely as

$$-\frac{ia}{2}\left(x^\mu\{\partial_n\alpha,\, \partial_\mu\beta\} - x^\mu\{\partial_n\beta,\, \partial_\mu\alpha\}\right) = -\frac{i}{2}x^\lambda C^{\mu,\nu}_\lambda\{\partial_\mu\alpha,\, \partial_\nu\beta\},$$

where $C^{\mu\nu}_\lambda$ are the structure constants of the space algebra,

$$[x^\mu \overset{*}{,} x^\nu] = iC^{\mu\nu}_\lambda x^\lambda$$

with $C^{\mu\nu}_\lambda = a\,(\delta^\mu_n\delta^\nu_\lambda - \delta^\mu_\lambda\delta^\nu_n)$. The solution of (6.150) is given by

$$\Lambda^1_\alpha[A] = \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda\{A_\mu, \partial_\nu\alpha\} = -\frac{a}{4}x^\lambda\left(\{A_n, \partial_\lambda\alpha\} - \{A_\lambda, \partial_n\alpha\}\right). \tag{6.151}$$

Also in higher orders in the expansion, there will occur terms that look similar to terms occuring in the canonical case, replacing $\theta^{\mu\nu}$ by $x^\lambda C^{\mu\nu}_\lambda$ [18]. Both expressions constitute the respective Poisson structure. Let us introduce non-commutative fields $\tilde{\psi}$ which transform as follows under a non-commutative gauge transformation,

$$\hat{\delta}_\alpha\tilde{\psi} = i\tilde{\Lambda}_\alpha * \tilde{\psi}. \tag{6.152}$$

We expand $\tilde{\psi}$ in a,

$$\tilde{\psi}[A] = \psi + \psi^1[A] + \mathcal{O}(a^2).$$

For the consistency relation (4.38) one obtains

$$\delta_\alpha\psi^1[A] = i\Lambda^1_\alpha[A]\,\psi + i\,\alpha\,\psi^1[A] - \frac{1}{2}x^\lambda C^{\mu\nu}_\lambda\partial_\mu\alpha\,\partial_\nu\psi. \tag{6.153}$$

We obtain as solution

$$\psi^1[A] = -\frac{1}{2}x^\lambda C^{\mu\nu}_\lambda A_\mu\,\partial_\nu\psi + \frac{i}{4}x^\lambda C^{\mu\nu}_\lambda A_\mu A_\nu. \tag{6.154}$$

We can always add a solution of the homogeneous part of (6.150) and (6.153) to the solutions (6.151) and (6.154), respectively to obtain another solution. This freedom may be used to make the fields and parameters hermitian, but it is not clear how this works for the gauge field.

For the gauge field $\tilde{V}_\alpha[A]$, the Seiberg-Witten map is much more involved, because of the complicated co-product structure of the derivatives D_μ. I will only briefly sketch the calculations and quote the results, since I have not contributed to these calculations. The solution has been calculated up to third order in a and will also be presented in [18]. The starting point is eqn. (6.146). A close examination of this equation provides us with some help to simplify the expansion (6.149). In the $*$-formalism, eqn. (6.146) reads,

$$(\hat{\delta}_\alpha \tilde{V}_\gamma) * \tilde{\psi} = \tilde{D}_\gamma \triangleright (\tilde{\Lambda}_\alpha * \tilde{\psi}) - \tilde{\Lambda}_\alpha * \tilde{D}_\gamma \triangleright \tilde{\psi} \tag{6.155}$$
$$-i\tilde{V}_\gamma * \tilde{\Lambda}_\alpha * \tilde{\psi} + i\tilde{\Lambda}_\alpha * \tilde{V}_\gamma * \tilde{\psi}. \tag{6.156}$$

Using the co-product of the derivatives $\tilde{D}_\mu$, we can eliminate the fields $\tilde{\psi}$,

$$\hat{\delta}_\alpha \tilde{V}_c = (\tilde{D}_c \triangleright \tilde{\Lambda}_\alpha) * e^{-ia\tilde{\partial}_n} - i\tilde{V}_c * \tilde{\Lambda}_\alpha + i\tilde{\Lambda}_\alpha * \tilde{V}_c, \tag{6.157}$$
$$\hat{\delta}_\alpha \tilde{V}_n = (\tilde{D}_n \triangleright \tilde{\Lambda}_\alpha) * e^{-ia\tilde{\partial}_n} + ia(\tilde{D}_i e^{ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha) * \tilde{D}_i \tag{6.158}$$
$$+ \Big((e^{ia\tilde{\partial}_n} - 1) \triangleright \tilde{\Lambda}_\alpha\Big) * \tilde{D}_n - i\tilde{V}_n * \tilde{\Lambda}_\alpha + i\tilde{\Lambda}_\alpha * \tilde{V}_n.$$

Therefore, we can find for the derivative dependence of the gauge field

$$\tilde{V}_b = \tilde{v}_b e^{-ia\tilde{\partial}_n}, \tag{6.159}$$
$$\tilde{V}_n = \tilde{v}_{\mathrm{I}n}\tilde{\partial}_j\tilde{\partial}_j e^{-ia\tilde{\partial}_n} + \tilde{v}^j_{\mathrm{II}n}\tilde{\partial}_j e^{-ia\tilde{\partial}_n} + \tilde{v}_{\mathrm{III}n} e^{-ia\tilde{\partial}_n} + \tilde{v}_{\mathrm{IV}n}\cos(a\tilde{\partial}_n) \tag{6.160}$$
$$+\tilde{v}_{\mathrm{V}n}\sin(a\tilde{\partial}_n)$$

and the gauge equivalence conditions become

$$\delta_\alpha \tilde{v}_c[A] = \tilde{\partial}_c e^{-ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha[A] - i\tilde{v}_c[A] * (e^{-ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha[A]) + i\tilde{\Lambda}_\alpha[A] * \tilde{v}_c[A], \tag{6.161}$$
$$\delta_\alpha \tilde{v}_{\mathrm{I}n}[A] = \frac{ia}{2}(e^{ia\tilde{\partial}_n} - 1) \triangleright \tilde{\Lambda}_\alpha[A] - i\tilde{v}_{\mathrm{I}n}[A] * (e^{ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha[A]) \tag{6.162}$$
$$+i\tilde{\Lambda}_\alpha[a] * \tilde{v}_{\mathrm{I}n}[A],$$
$$\delta_\alpha \tilde{v}^j_{\mathrm{II}n}[A] = ia\tilde{\partial}_j \triangleright \tilde{\Lambda}_\alpha[A] - i[\tilde{v}^j_{\mathrm{II}n}[A] \overset{*}{,} \tilde{\Lambda}_\alpha[A]] - 2i\tilde{v}_{\mathrm{I}n}[A] * (\tilde{\partial}_j \triangleright \tilde{\Lambda}_\alpha[A]), \tag{6.163}$$
$$\delta_\alpha \tilde{v}_{\mathrm{III}n}[A] = \frac{ia}{2}\tilde{\partial}_j\tilde{\partial}_j e^{-ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha[A] - i\tilde{v}_{\mathrm{III}n}[A] * (e^{ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha[A]) \tag{6.164}$$
$$+i\tilde{\Lambda}_\alpha[A] * \tilde{v}_{\mathrm{III}n}[A] - i\tilde{v}_{\mathrm{I}n}[A] * (\tilde{\partial}_j\tilde{\partial}_j e^{-ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha[A])$$
$$-i\tilde{v}^j_{\mathrm{II}n}[A] * (\tilde{\partial}_j e^{-ia\tilde{\partial}_n} \triangleright \tilde{\Lambda}_\alpha[A]),$$
$$\delta_\alpha \tilde{v}_{\mathrm{IV}n}[A] = \frac{1}{a}\sin(a\tilde{\partial}_n) \triangleright \tilde{\Lambda}_\alpha[A] - i\tilde{v}_{\mathrm{IV}n}[A] * (\cos(a\tilde{\partial}_n) \triangleright \tilde{\Lambda}_\alpha[A]) \tag{6.165}$$

$$
\begin{aligned}
&-i\tilde{v}_{\mathrm{V}n}[A] * (\sin(a\tilde{\partial}_n) \triangleright \tilde{\Lambda}_\alpha[A]) + i\tilde{\Lambda}_\alpha[A] * \tilde{v}_{\mathrm{IV}n}[A], \\
\delta_\alpha \tilde{v}_{\mathrm{V}n}[A] = &\frac{1}{a}(\cos(a\tilde{\partial}_n) - 1) \triangleright \tilde{\Lambda}_\alpha[A] + i\tilde{v}_{\mathrm{IV}n}[A] * (\sin(a\tilde{\partial}_n) \triangleright \tilde{\Lambda}_\alpha[A]) \qquad (6.166) \\
&-i\tilde{v}_{\mathrm{V}n}[A] * (\cos(a\tilde{\partial}_n) \triangleright \tilde{\Lambda}_\alpha[A]) + i\tilde{\Lambda}_\alpha[A] * \tilde{v}_{\mathrm{V}n}[A].
\end{aligned}
$$

Because of the co-product the structure for $\tilde{V}_n$ is far more complicated. We expand the fields $\tilde{v}_\mu[A]$ in terms of the commutative gauge field A_μ,

$$
\tilde{v}_\mu[A] = A_\mu + a\, v^1_\mu[A] + a^2 v^2_\mu[A] + \mathcal{O}(a^3) \tag{6.167}
$$

and can solve (6.159) and (6.160) order by order in a. For the 0^{th} order, we have to demand

$$
\begin{gathered}
v^0_{\mathrm{I}n} = v^{j0}_{\mathrm{II}n} = v^0_{\mathrm{III}n} = v^0_{\mathrm{V}n} = 0, \quad v^0_{\mathrm{IV}n} = A_n, \\
v^0_j = A_j.
\end{gathered} \tag{6.168}
$$

The solution up to first order is given by [18]

$$
\begin{aligned}
v^1_c[A] &= -\frac{i}{2}\partial_n A_c - \frac{1}{2}A_n A_c - \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \left(\{A_\mu, \partial_\nu A_c\} + \{A_\mu, F_{\nu c}\}\right), && (6.169) \\
v^1_{\mathrm{I}n}[A] &= 0, \\
v^{1j}_{\mathrm{II}n}[A] &= iA_j, \\
v^1_{\mathrm{III}n}[A] &= \frac{i}{2}(\partial_j - iA_j)A_j, && (6.170) \\
v^1_{\mathrm{IV}n}[A] &= \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \{A_\nu, \partial_\mu A_n\} + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \{F_{\mu n}, A_\nu\}, && (6.171) \\
v^1_{\mathrm{V}n}[A] &= 0, && (6.172)
\end{aligned}
$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu + ig[A_\mu, A_\nu]$ is the field strength. This solution is not the most general one. The discussion in Section 5.2 applies. It has to be noted that due to $v^j_{\mathrm{II}n}$ the gauge potential is not hermitian.

6.7 Formulation of Models

Let us return to the non-commutative algebra for a moment. Physical fields $\hat{\chi}$ are defined as power series in the non-commutative coordinates,

$$
\hat{\chi}(\hat{x}) = \sum_I c_{i_1 \ldots i_n} : (\hat{x}^1)^{i_1} \ldots (\hat{x}^n)^{i_n} : \quad = \sum_I c_I : \hat{x}^I : \tag{6.173}
$$

with certain properties under Poincaré transformations. A scalar field $\hat{\phi}$ is defined by

$$\hat{\phi}'(\hat{x}') = \hat{\phi}(\hat{x}), \tag{6.174}$$

where e.g., $\hat{x}'^{\mu} = \hat{x}^{\mu} + \zeta(N^l \triangleright \hat{x}^{\mu})$ is the transformation of $\hat{x}$ under a boost. ζ is an infinitesimal parameter. Explicitly, we have

$$\begin{aligned}
\hat{\phi}'(\hat{x}') &= \sum_I c'_I \, :(\hat{x}'^1)^{i_1} \dots (\hat{x}'^n)^{i_n}: \\
&= \sum_I c'_I \, :((1+\zeta N_l) \triangleright \hat{x}^1)^{i_1} \dots ((1+\zeta N_l) \triangleright \hat{x}^n)^{i_n}: \\
&= \sum_J c_J \, :(\hat{x}^1)^{j_1} \dots (\hat{x}^n)^{j_n}: .
\end{aligned} \tag{6.175}$$

Spinor (vector) fields are defined as

$$\hat{\psi}'_{\mu}(\hat{x}') = (1 + \zeta N^l_{\text{rep}})_{\mu}{}^{\nu} \triangleright \hat{\psi}_{\nu}(\hat{x}), \tag{6.176}$$

where N^l_{rep} is a representation of the boost generators acting on spinors (vectors). In the classical case, the transformation only affects the spinor (vector) indices, the coordinate dependence is untouched. In the κ-deformed case, the transformation of a vector field, $N^l_{\text{rep}\,\mu}{}^{\nu} \triangleright \hat{A}_{\nu}$ is given by $[N^l, A_{\mu}]$, eqn. (6.81) and (6.82). The coordinate dependence of the fields are affected by the occuring derivatives.

This leads to Poincaré covariant field equations. The Klein-Gordon equation (6.34) and the Dirac Equation (6.54) are given by

$$(\widehat{\Box} + m^2) \triangleright \hat{\phi}(\hat{x}) = 0, \tag{6.177}$$

$$(\gamma^k \widehat{D}_k - m) \triangleright \hat{\psi}(\hat{x}) = 0. \tag{6.178}$$

They are covariant, i.e.,

$$(\widehat{\Box}' + m^2) \triangleright \hat{\phi}'(\hat{x}') = (\widehat{\Box} + m^2) \triangleright \hat{\phi}(\hat{x}), \tag{6.179}$$

$$(\gamma^k \widehat{D}'_k - m) \triangleright \hat{\psi}'(\hat{x}') = (\gamma^k \widehat{D}_k - m) \triangleright \hat{\psi}(\hat{x}). \tag{6.180}$$

In the $*$-formalism, the Klein-Gordon and the Dirac operator are obtained taking into account eqns. (6.123) and (6.124).

For the construction of Lagrangian models, we need a measure invariant under κ-Poincaré transformations, i.e.,

$$\langle h \triangleright \hat{f}(\hat{x}) \rangle = \epsilon(h) \, \langle \hat{f}(\hat{x}) \rangle, \tag{6.181}$$

$h \in \mathcal{P}_\kappa$, and ϵ denotes the co-unit. The trace property is also a desirable feature for the functional $\langle\,\rangle$,

$$\langle \hat{f}\hat{g} \rangle = \langle \hat{g}\hat{f} \rangle. \tag{6.182}$$

In [83], a measure function is computed for q-deformed 3 and 4 dimensional Euclidean space and 4 dimensional Minkowski space, such that the integral satisfies the trace property, up to first order in $h = \ln q$,

$$\int d^n x\, \mu(x)\, f * g(x) = \int d^n x\, \mu(x)\, g * f(x) + \mathcal{O}(a^2). \tag{6.183}$$

In the same way the weight function can be computed for κ-Euclidean space. We get the defining equations for the measure function $\mu_\kappa(x)$

$$\partial_n\, \mu_\kappa(x) = 0, \qquad x^i \partial_i\, \mu_\kappa(x) = -(n-1)\, \mu(x). \tag{6.184}$$

The solution is given by

$$\mu_\kappa(x) = \frac{1}{x^1 x^2 \ldots x^{n-1}}. \tag{6.185}$$

However, this integral is not invariant under κ-Poincare transformations. The commutative limit seems to be a problem, since to 0^{th} order we have $\int d^n x\, \mu_\kappa(x) f(x)$. We may modify the measure slightly,

$$\mu'_\kappa(x) = \frac{X^1 \ldots X^{n-1}}{x^1 \ldots x^{n-1}}, \tag{6.186}$$

where X^j are the covariant coordinates. In the limit $a \to 0$, we have $\mu'_\kappa(x) \to 1$.

In order to compute the transformation properties of a volume element, we have to construct a covariant differential calculus. This has already been done in [84, 85]. They assumed that the relations between coordinates and differentials are linear in the differentials, that a mixed Jacobi identity holds, and that the exterior derivative d satisfy the classical Leibniz rule. However, they found that, in n-dimensions, there is no n-dimensional covariant differential calculus satisfying all of these relations. The calculus has to be higher dimensional. It nevertheless might be possible to construct a n-dimensional covariant calculus, by allowing a deformed Leibniz rule. We define the exterior derivative

$$\hat{d} = \hat{\xi}^\mu \hat{D}_\mu, \tag{6.187}$$

where $\hat{\xi}^\mu$ are the non-commutative coordinate differentials. Note that we use the Dirac operator, and not the derivatives $\hat{\partial}_\mu$ to define the exterior derivative. The algebra of coordinates can be extended to also include the differentials $\hat{\xi}^\mu$. Relations consistent with (6.6) and covariant with respect to the κ-Poincaré algebra have still to be found. Then, we can map the coordinates and differentials to the commutative calculus via the quantisation isomorphism W, using non-commutative products.

Equipped with a covariant integral, we can write down Lagrangians using the $*$-formalism invariant under both, κ-Poincaré and gauge symmetry. Since the $*$-product is covariant, i.e.,

$$\tilde{h} \triangleright f * g = (\tilde{h}_{(1)} \triangleright f) * (\tilde{h}_{(2)} \triangleright g),$$

$h \in \mathcal{P}_\kappa$, we know how the symmetry generators act on a ($*$-)product of functions or fields. However, if we expand the fields in terms of Seiberg-Witten maps, cf. Section 6.6, and consider only terms up to some power in a, we will break κ-Poincaré invariance. It is present when we consider expressions to all orders of a.

6.8 Minkowski Signature

Let us define the n-dimensional Minkowski signature by its metric $\eta_{\mu\nu}$,

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}. \tag{6.188}$$

κ-Minkowski space $\mathcal{M}_\kappa$ is generated by the coordinates $\hat{x}^0, \hat{x}^1 \dots, \hat{x}^{n-1}$. We want to discuss two different situations, corresponding to different directions of the non-commutativity. First, the non-commutativity points into the time direction, that is the case in κ-Minkowski space, as discussed in [16, 17]. Secondly, we want to rotate the direction of non-commutativity into the $1-$direction. In this case, we can, for example, consider the interesting case of 5 dimensional Minkowski space with the non-commutativity in the fifth direction. The fifth dimension may be compactified. This resembles a very interesting model, which, however, will not be studied here.

6.8.1 Time Non-Commutativity

Let the non-commutativity point into the time-direction. We have the commutation relations

$$[\hat{x}^0, \hat{x}^i] = ia\hat{x}^i, \tag{6.189}$$

where $i = 1, 2, \dots, n-1$. There are two different changes in the algebra relations and in the action of the generators on coordinates. First, we have to replace n by 0. Secondly, we have to change the Euclidean signature to the Minkoski one. The rotational sector is classical, therefore no changes have to be done there. Relation (6.23) has to be altered according to [17],

$$[N^l, \hat{\partial}_i] = \eta_{li}\,\frac{1}{2ia}\left(1 - e^{2ia\hat{\partial}_n}\right) - \frac{ia}{2}\eta_{li}\,\widehat{\Delta}_\kappa + ia\,\hat{\partial}_l\hat{\partial}_i, \tag{6.190}$$

where $\widehat{\Delta}_\kappa = \eta^{km}\hat{\partial}_k\hat{\partial}_m = -\sum_{i=1}^{n-1}\hat{\partial}_i\hat{\partial}_i$. Equation (6.11) has to be changed, also. It now reads

$$N^l\hat{x}^0 = \eta_{l\mu}\hat{x}^\mu + (\hat{x}^0 + ia)\,N^l = -\hat{x}^l + (\hat{x}^0 + ia)\,N^l. \tag{6.191}$$

All other relations are unchanged. Therefore, we get for the invariant of $\mathcal{M}_\kappa$ (6.29)

$$\begin{aligned} I &= \hat{x}^0\hat{x}^0 - \sum_i \hat{x}^i\hat{x}^i - 3ia\hat{x}^0 \\ &= \eta_{\mu\nu}\hat{x}^\mu\hat{x}^\nu - 3ia\hat{x}^0. \end{aligned} \tag{6.192}$$

The Klein-Gordon operator (6.34) becomes

$$\widehat{\Box} = \frac{2}{a^2}\Big(1 - \cos(a\hat{\partial}_n)\Big) + e^{-ia\hat{\partial}_n}\widehat{\Delta}_\kappa, \tag{6.193}$$

where we again define the Laplace operator as $\widehat{\Delta}_\kappa = \eta^{km}\hat{\partial}_k\hat{\partial}_m$. The γ matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{6.194}$$

The action of the boost generators on the γ matrices, (6.51) and (6.52) become

$$\begin{aligned} n^r \triangleright \gamma^l &= \eta^{rl}\gamma^0, \\ n^r \triangleright \gamma^0 &= -\gamma^r. \end{aligned} \tag{6.195}$$

Finally, we obtain for the Dirac Operator the expression

$$\widehat{D} = \frac{\gamma^0}{a}\sin(a\hat{\partial}_0) + \frac{ia\gamma^0}{2}\widehat{\Delta}_\kappa \exp(-ia\hat{\partial}_0) + \gamma^r\hat{\partial}_r \exp(-ia\hat{\partial}_0). \tag{6.196}$$

Bear in mind that $\widehat{\Delta}_\kappa = -\hat{\partial}_i\hat{\partial}_i$.

Symmetric Ordering

We obtain for the action of the boosts on commutative functions

$$\tilde{N}^l \triangleright f(x) = \Big(- x^l \partial_0 - x^0 \partial_l + x^l \eta^{\mu\nu} \, \partial_\nu \partial_\mu \frac{e^{ia\partial_0} - 1}{2\partial_0} \\ + x^\nu \partial_\nu \partial_l \, \frac{a\partial_0 + i(e^{ia\partial_0} - 1)}{2\partial_0^2} \Big) f(x). \tag{6.197}$$

The action of the derivatives on functions is unaltered. Therefore, the Klein-Gordon and Dirac operators read

$$\Box = \frac{2}{(ia\partial_0)^2} \left(\cos(a\partial_0) - 1\right) \left(\partial_0^2 + \Delta_{cl}\right), \tag{6.198}$$

$$D = \frac{\gamma^0}{ia} \left(i \sin(a\partial_0) + \frac{\Delta_{cl}}{\partial_0^2} (\cos(a\partial_0) - 1) \right) + \frac{\gamma^r}{ia} \frac{2i\partial_r}{\partial_0} e^{-\frac{ia}{2}\partial_0} \sin(\frac{a}{2}\partial_0), \tag{6.199}$$

where $\Delta_{cl} = \eta^{km} \partial_k \partial_m$.

Normal Ordering

The action of boosts on commutative functions is given by

$$\tilde{N}^l \triangleright f(x) = \left(-\frac{x^l}{a} \, \sin(a\partial_0) - x^0 \partial_l \, e^{ia\partial_0} - \frac{ia}{2} x^l \Delta_{cl} \, e^{ia\partial_0} \right) f(x). \tag{6.200}$$

For the Klein-Gordon operator we obtain the following expression,

$$\Box = \frac{2}{a^2} \left(1 - \cos(a\partial_0)\right) + e^{ia\partial_0} \Delta_{cl}. \tag{6.201}$$

The Dirac operator becomes

$$D = \gamma^0 \left(\frac{\sin(a\partial_0)}{a} + \frac{ia}{2} \Delta_{cl} \right) + \gamma^r \partial_r. \tag{6.202}$$

6.8.2 Space Non-Commutativity

To remind the reader, the position of the space-time indices are important in this section. The metric η is used to lower and raise indices,

$$A^\mu = \eta^{\mu\nu} A_\nu,$$

and we have $\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^\mu_\sigma = \eta^\mu_\sigma$. For a^μ pointing into the 1−direction, we have

$$[\hat{x}^1, \hat{x}^f] = ia\hat{x}^f, \tag{6.203}$$

where $f = 0, 2, 3, \ldots, n-1$. We proceed as before and obtain for the action of the derivatives on coordinates

$$\begin{aligned} \hat{\partial}_1\hat{x}^f &= \hat{x}^f\hat{\partial}_1, \quad \hat{\partial}_1\hat{x}^1 = 1 + \hat{x}^1\hat{\partial}_1, \\ \hat{\partial}_f\hat{x}^g &= \delta^g_f + \hat{x}^g\hat{\partial}_f, \quad \hat{\partial}_f\hat{x}^1 = (\hat{x}^1 + ia)\,\hat{\partial}_f, \end{aligned} \tag{6.204}$$

where $f, g = 0, 2, 3, \ldots, n-1$. For the other algebra generators, we use the definition

$$N^l \equiv M^{0l}.$$

The algebra relations of $\mathcal{P}_\kappa$ are obtained by first exchanging n with 1 and then changing the signature to the Minkowski one. We get as a result

$$\begin{aligned} [N^i, \hat{\partial}_1] &= \eta^i_1\,\hat{\partial}_0, \\ [N^1, \hat{\partial}_0] &= -\frac{1}{2ia}\left(1 - e^{2ia\hat{\partial}_1}\right) - \frac{ia}{2}(\hat{\partial}_0\hat{\partial}_0 - \sum_{b=2}^{n-1}\hat{\partial}_b\hat{\partial}_b) + ia\hat{\partial}_0\hat{\partial}_0, \\ [N^a, \hat{\partial}_0] &= \hat{\partial}_a, \\ [N^1, \hat{\partial}_a] &= ia\hat{\partial}_0\hat{\partial}_a = -ia\hat{\partial}_0\hat{\partial}^a, \\ [N^b, \hat{\partial}_a] &= \eta^b_a\,\hat{\partial}_0, \end{aligned} \tag{6.205}$$

$$\begin{aligned} [M^{1a}, \hat{\partial}_1] &= \hat{\partial}_a, \\ [M^{ab}, \hat{\partial}_1] &= 0, \\ [M^{1a}, \hat{\partial}_0] &= ia\hat{\partial}_0\hat{\partial}_0, \\ [M^{1a}, \hat{\partial}_c] &= \eta^a_c\,\frac{1}{2ia}\left(1 - e^{2ia\hat{\partial}_i}\right) - \frac{ia}{2}\eta^a_c\,(-\hat{\partial}_0\hat{\partial}_0 + \sum\hat{\partial}_b\hat{\partial}_b) - ia\hat{\partial}_a\hat{\partial}_c, \\ [M^{ab}, \hat{\partial}_c] &= \eta^a_c\,\hat{\partial}_b - \eta^b_c\,\hat{\partial}_a = \eta^b_c\,\hat{\partial}^a - \eta^a_b\,\hat{\partial}^b, \end{aligned} \tag{6.206}$$

where $a, b, c = 2, 3, \ldots, n-1$. All the other relations or unaltered. We can rewrite above relations in a compact form[1] as

$$[M^{\mu\nu}, \hat{\partial}_\sigma] = \eta^\nu_\sigma\,\hat{\partial}^\mu - \eta^\mu_\sigma\,\hat{\partial}^\nu, \tag{6.207}$$

$$[M^{g1}, \hat{\partial}_1] = \hat{\partial}^g = -\hat{\partial}_g, \tag{6.208}$$

$$[M^{g1}, \hat{\partial}_f] = \eta^g_f\,\frac{1}{2ia}\left(e^{2ia\hat{\partial}_1} - 1\right) - \frac{ia}{2}\eta^g_f\,(\hat{\partial}_0\hat{\partial}_0 - \sum\hat{\partial}_b\hat{\partial}_b) + ia\hat{\partial}^g\hat{\partial}_f.$$

[1]this notation has been introduced by M. Dimitrijević

The action of the generators on coordinates is given by

$$[N^1, \hat{x}^0] = -\hat{x}^1, \quad [N^a, \hat{x}^0] = -\hat{x}^a, \tag{6.209}$$
$$[N^1, \hat{x}^1] = -\hat{x}^0 - iaN^1, \quad [N^a, \hat{x}^1] = 0, \tag{6.210}$$
$$[N^1, \hat{x}^a] = iaN^a, \quad [N^b, \hat{x}^a] = -\delta^{ab}\hat{x}^0, \tag{6.211}$$

$$[M^{1a}, \hat{x}^\mu] = \eta^{\mu a}\hat{x}^1 - \eta^{\mu 1}\hat{x}^a - iaM^{a\mu}, \tag{6.212}$$
$$[M^{ab}, \hat{x}^\mu] = \eta^{\mu b}\hat{x}^a - \eta^{\mu a}\hat{x}^b, \tag{6.213}$$

where $i = 1, \ldots, n-1$, $a, b = 2, \ldots, n-1$, $\mu = 0, 1, \ldots, n-1$.
For $\mu = 0$, eqn. (6.212) reads

$$[M^{1a}, \hat{x}^0] = -iaM^{a0} = iaN^a. \tag{6.214}$$

In a more concise form we can rewrite the relations in the following way

$$[M^{fg}, x^\mu] = \eta^{\mu g}\hat{x}^f - \eta^{\mu f}\hat{x}^g, \tag{6.215}$$
$$[M^{f1}, x^\mu] = \eta^{i\mu}\hat{x}^f - \eta^{\mu f}\hat{x}^1 + iaM^{a\mu}, \tag{6.216}$$

where $f, g = 0, 2, 3, \ldots, n-1$. As done in Subsection 6.8.1, one could calculate the $*$-product of functions and the action of the generators on commutative functions. This will not be covered here.

6.9 Conclusions and Remarks

We have constructed a deformed version of the n-dimensional algebra generated by rotations and translations, which we have called $\mathcal{P}_\kappa$. This construction is such that the n-dimensional κ-deformed Euclidean space,

$$[\hat{x}^n, \hat{x}^i] = ia\hat{x}^i$$

constitutes a $\mathcal{P}_\kappa-$module algebra. In the limit briefly discussed in Section 6.8, we recover κ-deformed Minkowski space which serves as a model for space-time at small distances or high energies. The advantage over the canonical deformation of space-time which has been extensively discussed prior in this work is that the symmetry structure of classical space-time is conserved. Space-time is still symmetric under a deformed Poincaré symmetry. $\mathcal{P}_\kappa$ is a quantum group [86].

The structure of κ-Euclidean space-time and its symmetry algebra has been exploited in detail, and a lot of interesting features have shown up. In analogy to [16, 17], we have constructed the invariant of the coordinate algebra under Poincaré transformations (6.29) and the Klein-Gordon operator as Casimir of the symmetry algebra (6.34). The Dirac operator is introduced demanding that $\gamma^\alpha \widehat{D}_\alpha(\hat{\partial})$ is a scalar under κ-Poincaré transformations, where γ^α are the ordinary Dirac matrices. This determines $\widehat{D}_\alpha(\hat{\partial})$ uniquely. The square of the Dirac operator, however, is not equal to the Klein-Gordon operator, but we obtain (6.55). The difference might be interpreted as a regulator, in the spirit of Pauli-Villars. It is interesting and vital for our work, that the Dirac operator introduces additional derivatives into the algebra. The step from $\hat{\partial}_\mu$ to $\widehat{D}_\alpha$ is merely a change of basis in the enveloping algebra of the translations. Therefore, we can also express the derivatives $\hat{\partial}_\mu$ as a function of $\widehat{D}_\alpha$. The formulae are given in eqns. (6.71) and (6.72). Due to the transformation properties of $\widehat{D}$, $\widehat{D}_\alpha$ transforms as an ordinary vector in commutative theory. So we have equipped the algebra with two differential structures, $\hat{\partial}_\mu$ have a complicated structure under Lorentz transformations and a simple action on coordinates, $\widehat{D}_\alpha$ have simple transformation properties, but act in a complicated way on coordinates (6.67) - (6.70). Two different kinds of vector fields are introduced by its transformation properties. $\widehat{V}_\alpha$ transforms ordinarily, i.e., in the same way as $\widehat{D}_\alpha$. $\widehat{A}_\mu$ and $\widehat{\widehat{\widehat{B}}}_\mu$ transform in rather the same way as the derivatives $\hat{\partial}_\mu$. Because of the co-product of the κ-Poincaré generators, $\widehat{A}_\mu \widehat{A}_\mu$ (or $\widehat{A}_\mu^\dagger \widehat{A}_\mu$) does not form a scalar. We have to introduce another vector field $\widehat{\widehat{\widehat{A}}}^\mu$, such that $\widehat{\widehat{\widehat{A}}}^\mu \widehat{A}_\mu$ builds a scalar. The same is true for $\widehat{\widehat{\widehat{B}}}_\mu$, it is a "dual" situation. There is a conjugation † which satisfies,

$$\widehat{A}_\mu{}^\dagger = \widehat{\widehat{\widehat{B}}}_\mu, \qquad \widehat{\widehat{\widehat{A}}}^{\mu\dagger} = \widehat{B}^\mu, \qquad \widehat{V}_\mu^\dagger = \widehat{V}_\mu.$$

Note that $\widehat{\widehat{\widehat{A}}}^\mu \neq \widehat{A}_\mu{}^\dagger$. The vector fields $\widehat{A}_\mu$ and $\widehat{\widehat{\widehat{A}}}^\mu$ are related to the ordinarily transforming vector field $\widehat{V}_\alpha$, in the same as the derivatives $\hat{\partial}_\mu$ are related to $\widehat{D}_\alpha$: via the "Lorentz" vielbein $e_\alpha{}^\mu$. The vielbein is an intertwiner between these two structures.

Now, algebraic field equations covariant under κ-Poincaré transformations can be formulated. This has been done in Section 6.7.

All these structures and formulae can be translated to the commutative regime and formulated within the $*$-formalism. Therefore, we can write down covariant field equations for commutative functions and fields. Gauge fields can be introduced to the algebra as vector fields. By construction they can be added to derivatives, covariantly. Seiberg-Witten maps can be used to express the fields and gauge parameters of the non-commutative theory in terms of commutative gauge fields and commutative gauge parameters. The structure of the gauge fields is somehow peculiar. In canonically non-commutative gauge theory, the gauge parameters and the gauge fields are elements of the enveloping algebra of the gauge group. The co-product of the derivatives is the classical one. In our case the co-product is non-trivial. Non-linear operators apear. As a consequence the gauge equivalence conditions only close, if the gauge field is not only an element of the enveloping algebra of the gauge group, but also an element of the enveloping algebra of the translations. The gauge field has to be Poincaré symmetry valued. Since the gauge field is derivative valued, the field strength $\widehat{F}_{\mu\nu} = [\widehat{D}_\mu, \widehat{D}_\nu]$ is also derivative valued. A possible action functional would therefore contain derivatives. In order to get rid of them, we have to let them act on $\mathbf{1}$.

We need an integration invariant under κ-Poincaré transformations, in order to be able to consider Lagrangian models. These Lagrangian models will then be both, co-variant under κ-Poincaré and under arbitrary gauge symmetry. κ-Poincaré covariance will be broken, if we do not consider the Seiberg-Witten maps to all orders.

To ensure that the invariant integral also possesses the trace property might become a problem.

Once we are able to examine actions, we can study physical models and their predictions, the structure of the newly found interactions. One of the main tasks will be to study renormalisability of such models. The hope is that a non-commutative space-time compatible with a Poincaré symmetry is a good candidate to describe nature in the high energy regime. κ-deformation is a very mild deformation and has already at that early stage revealed a lot of interesting new structures.

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