k-Inflation and k-Essence

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Zusammenfassung

Wir betrachten eine breite Klasse von Skalarfeldern mit nicht-kanonischen Ableitungstermen. Es wird gezeigt, daß solche Skalarfelder im Stande sind, eine Inflationsphase zu treiben, die wir mit "k-Inflation" bezeichnen. Die Bedingungen an den Lagrangian des Skalarfeldes, die das Vorhandensein von Potenzgesetz-, Pol- und de Sitter- Inflationslösungen garantieren werden hergeleitet. Wir zeigen, daß diese Bedingungen allgemein erfüllt werden können, sogar in Abwesenheit eines Potentialtermes. Die entsprechenden Lösungen sind üblicherweise Attraktoren, und deshalb führen ziemlich allgemeine Anfangsbedingungen zu einer Inflationsphase. Bei diesen Lösungen dauert die Inflation lange genug, um die heutige Homogenität und Isotropie unseres Universums zu begründen. Die Existenz eines "Slow-Roll Regime" führt zu einem nahezu Skalen-invarianten Spektrum von Dichtestörungen und die Abweichung vom "Slow-Roll" ermöglicht einen natürlichen Übergang von der Inflationsphase in ein strahlungsdominiertes Friedmann Universum.

Ein Skalarfeld mit nicht-kanonischen Ableitungstermen kann auch erklären, warum das Universum eine Phase beschleunigter Expansion zu einer so späten Zeit seiner Entwicklung eingegangen ist. Früher existierende Modelle, die späte kosmische Beschleunigung unterbringen, müssen sehr genau angepaßt werden. Im Gegensatz dazu führt k-Essenz, ein geeignetes nicht-kanonisches Skalarfeld, zu später kosmischer Beschleunigung ohne diese genaue Anpassung. Da k-Essenz kosmische Beschleunigung vor Equipartition nicht auslösen kann, und da unser Dasein durch Strukturbildung an letzteres gebunden ist, erklärt ein solches Skalarfeld, warum wir kosmische Beschleunigung gerade erst heute beobachten. Wir liefern eine genaue Beschreibung des dynamischen Mechanismusses, der ein solches Verhalten erlaubt, und konstruieren konkrete Beispiele. Wir zeigen, daß es zwei Lösungsklassen gibt. Die eine Klasse sagt ein Ende der Beschleunigunsphase voraus, während bei der anderen die beschleunigte Expansion des Kosmos ewig fortschreitet.

Abstract

We consider a broad class of scalar fields with non-canonical derivative terms. It is shown that such scalar fields may be able to support a stage of inflation we call "k-inflation". The conditions on the scalar field Lagrangian which guarantee the existence of power-law, pole-like and de Sitter inflationary solutions are derived. We demonstrate that these conditions can be generically satisfied even in the absence of a potential term. The corresponding inflationary solutions are generically attractors, and hence inflation starts from rather arbitrary initial conditions. These solutions last long enough to explain the present homogeneity and flatness of the universe. The existence of a slow-roll regime leads to a nearly scale invariant spectrum of density perturbations and the departure from slow-roll induces a natural transition from inflation to a radiation-dominated Friedmann universe.

A scalar field with non-canonical derivative terms can also explain why the universe has entered a period of accelerated expansion at such a late stage of its evolution. Previously existing models that accommodate this sort of late type cosmic acceleration have to be fine-tuned. Unlike these models, an appropriate non-canonical scalar field, k-essence, leads to late time cosmic acceleration without fine-tuning. Because k-essence can not trigger cosmic acceleration before equipartition, and because our own existence through the growth of structure is related to the latter moment of time, such a model naturally explains why we happen to observe cosmic acceleration precisely today. We provide a detailed discussion of the dynamical mechanisms which allow such behavior and construct concrete examples. We also show that there are two classes of solutions which may yield late time cosmic acceleration, one in which the acceleration continues forever and one for which the acceleration has finite duration.

Contents

Introduction

In most theoretical models studied in modern cosmology scalar fields play a prominent role. They can drive inflation [38], they may be responsible for phase transitions in the early universe [38], they have been proposed as dark matter candidates [9], and they could even support the apparent present stage of accelerated expansion of the universe [14]. The consideration of scalar fields in modern cosmology is deeply rooted in theories of elementary particle physics, where scalar fields also play an essential role. As we will discuss, the effective field theories that describe such scalar fields generically contain "non-canonical" derivative terms different from the common squared field gradient $\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$. Usually, often without explicit justification, these terms are assumed to be small and are discarded in cosmological applications. In this thesis we show that they can be determinant, leading to a stage of inflation and an explanation of late time cosmic acceleration.

Certainly, one can naively imagine two situations where non-canonical derivative terms may be relevant. The first one is at high energy densities. It is commonly believed that in the early stages of the universe, its expansion was accelerated. Such a stage of accelerated expansion is known as inflation [38]. In the common inflationary scenarios, typical energy scales are of the order of 10^{14} GeV, much above the energy scales reached in present (and probably near future) accelerators. It is hence possible that non-canonical derivative terms played a significant role then. This is the idea behind "k-inflation". In k-inflation the scalar field which drives inflation is assumed to have a general Lagrangian containing non-canonical derivative terms. The conditions that guarantee the existence of inflation for such a Lagrangian can be satisfied even by a potential-less scalar. Hence, in the latter case inflation is driven by the kinetic energy density of the field, and not by its potential energy density, as in the usual models. Thus, in principle, k-inflation offers the possibility of implementing a successful stage of inflation in models where ordinary potential-driven inflation is not feasible.

The second possible case where non-standard derivative terms may be relevant is when non-linear dynamics forces them to remain large. This is also what happens in k-essence models. Surprisingly, recent measurements of large scale structure, cosmic microwave background anisotropies and type Ia supernovas seem to suggest that the universe is presently undergoing a stage of (late) cosmic accelerated expansion [4]. The straightforward interpretation of this expansion, that the universe is dominated by a cosmological constant, faces a tremendous fine tuning problem. k-Essence is a scalar field—of the same type as in k-inflation—which may effectively act as a cosmological constant. It was introduced to explain late time cosmic acceleration naturally, namely, without explicitly encoding the time of dominance in the model parameters. In fact, due to its special attractor properties, k-essence can only behave as a cosmological constant and dominate the energy density of the universe after equipartition. This is the moment of time radiation ceases to be the most important energy component in the universe and the structures in the universe—galaxies and clusters— start to develop. Because life can not evolve until these structures form, we happen to observe late time cosmic acceleration today both because our own existence and the time of k-essence dominance are related to equipartition.

This thesis is mainly based on work done in collaboration with T. Damour, V. Mukhanov and P. J. Steinhardt and published in [1], [2] and [3]. In order to avoid repeating references we will omit citations of these papers in the future. Regarding our choice of citations, in the context of a doctoral thesis we have opted for mentioning general references rather than original work.

The dissertation is organized as follows. In chapter 1 we shortly summarize the basic notions of cosmology and inflation we shall use in the rest of our work. We also explain why, generically, effective field theories include non-canonical derivative terms, and illustrate our discussion with examples from string theory and non-linear electrodynamics (Born-Infeld theory). In chapter 2 we introduce and define the "k-field", the scalar field which shall be responsible for inflation and an explanation of late time cosmic acceleration. Inflation driven by the k-field, k-inflation for short, is thoroughly discussed in chapter 3. Here we derive the properties the k-field Lagrangian has to satisfy in order to successfully account for the big bang initial conditions and discuss how the three main types of inflation discussed in the literature [40] can be driven by a k-field. In chapter 4 we address late time cosmic acceleration and suggest how it can be related to the onset of matter dominance through a suitable k-field. Finally, in the last "chapter" we draw our conclusions and shortly discuss possible further avenues. Among the appendices, the reader may find the table of symbols and notation in appendix A particularly useful.

Chapter 1

Basics

This introductory chapter consists of two parts. The first one is a short summary of the basic notions of cosmology [57, 33] we shall need in later chapters. We particularly emphasize two aspects: The recently observed late time cosmic acceleration of the universe [45, 51, 4] and inflation as a solution of many of the puzzles of the standard big-bang cosmology [38]. In the second part we present less-known developments about fields with non-canonical derivative terms as they appear in different particle-physics contexts. The material presented in this second part is the link for our later considerations of non canonical scalar fields in a cosmological setting.

1.1 Standard cosmology

1.1.1 The observed universe

Looking at the sky one observes myriads of stars. The most important one for our lives however, the sun, can be observed only during daylight. This daylight needs around 8 minutes to travel from the sun to our eyes. Proxima Centauri, the next closest star, is around 1 pc away from us, and this means that light needs around 3.3 years to reach the earth from that star. Proxima Centauri is not our only companion star: Our galaxy, the Milky Way, contains around $2 \cdot 10^{11}$ stars, and light has to cross 30 kpc in order to travel from one of its ends to the other. This distance is small if we compare it with the distance to the largest of our close galaxy neighbors in the local group, the Andromeda M31 galaxy, which is around 1 Mpc far away from us.

The local group is a rather small cluster of galaxies, containing around 30 galaxies of different types, spanning a distance of around 3 Mpc. It is part of the so called local Virgo supercluster, which contains around 10 clusters of galaxies and is roughly 30 Mpc wide. At this level the hierarchical structure of the universe seems to stop. If one explores even larger distances, one observes that superclusters arrange themselves into filament-like structures, which surround large voids of 50 Mpc typical size. On distances larger than about 200 Mpc, the universe starts to resemble a homogeneous web of filaments (figure 1.1). Our telescopes can not reach however arbitrarily far objects, since light travels at a finite speed and the universe has a finite age, around $1.5 \cdot 10^{10}$ years. During that time light travels around 3000 Mpc, the size of our present so-called "horizon".

As we look to large distances we observe the state of the universe at earlier times in the past, since light travels at a finite speed. The oldest signal from the universe we can observe at present is the cosmic microwave background radiation. It consists of a gas of photons with Planckian spectrum with a temperature of around 2.7 K. This temperature is nearly independent on the direction we look at on the sky; it merely shows small anisotropies of the order of $\Delta T/T \approx 10^{-5}$ (figure 1.2).

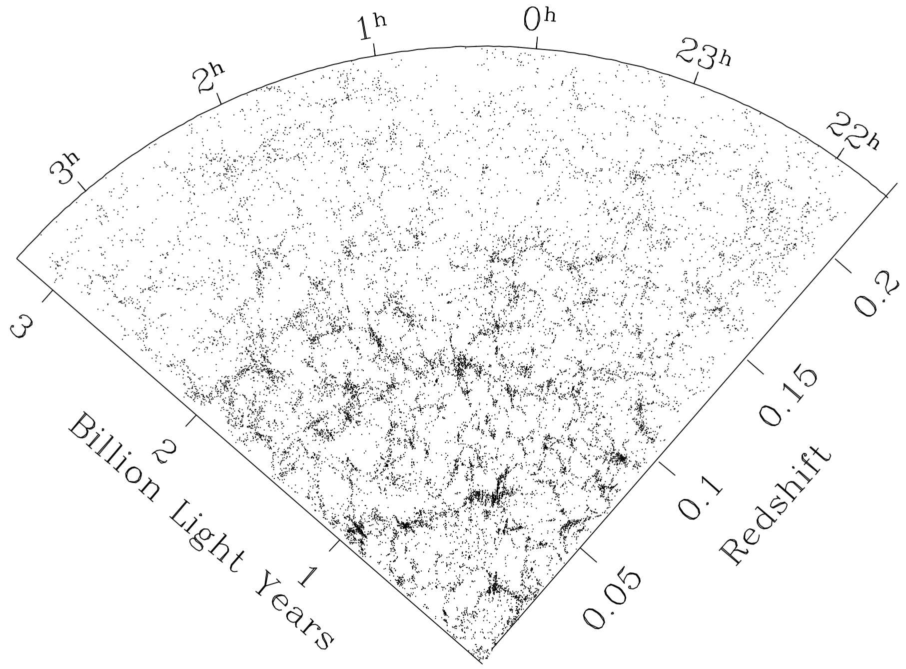

Figure 1.1: A diagram of the galaxy distribution in a sky slice with the Earth at its vertex. The plot shows around 50000 galaxies (dots) reaching out to ~1000 Mpc distance [43].

1.1.2 The Friedmann-Robertson-Walker metric

As we have mentioned, the matter distribution of the universe seems to be homogeneous when averaged over large enough scales and, in addition, the microwave radiation is nearly isotropic around us. In turns out [56] that both properties considerably constrain the possible shapes of our universe[1].

The metric of a homogeneous and isotropic universe can take only three different forms, corresponding to the three different values κ may have in the metric

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right). \quad (1.1)$$

For an open—hyperbolic— universe $\kappa = -1$, for a flat—Euclidean— universe $\kappa = 0$ and for a closed—spherical—universe $\kappa = +1$. Spatial lengths are proportional to the scale factor a, which depends in principle on cosmic time t. Whereas the form of the metric (1.1) is determined solely by the symmetry of the universe, the time evolution of $a(t)$ depends on the underlying theory of gravity. Today, the most compelling, simple and well tested one [61, 62] is general relativity [56], which we shall adopt throughout our whole work. General relativity relates the curvature of spacetime to its content through Einstein's equations. If ε_{tot} and p_{tot} denote the total energy density and the total pressure of the universe's constituents, then Einstein's equations for the metric (1.1) reduce to the set

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\varepsilon_{\text{tot}} \quad (1.2)$$

$$\ddot{a} = -\frac{4\pi G}{3}(\varepsilon_{\text{tot}} + 3\,p_{\text{tot}})\,a, \quad (1.3)$$

where, as in the rest of the thesis, a dot means derivative with respect to time, d/dt.

It can be easily verified that, in general, equations (1.2) and (1.3) do not have static solutions; the scale factor either grows, $\dot{a} > 0$ or decreases, $\dot{a} < 0$. As a matter of fact E. Hubble observed that distant galaxies seemed to recede from us with a speed proportional to their distance, a phenomenon that was interpreted as a consequence of the universe's expansion. The proportionality constant H was called Hubble parameter, and it can be easily shown using (1.1) that

$$H = \frac{\dot{a}}{a}. \quad (1.4)$$

Significant efforts in observational cosmology have been devoted to measure as exactly as possible the present value of the Hubble parameter $H_0 \approx 68 \pm 6$ km sec^{-1} Mpc^{-1} [34]. One of the reasons is that this value determines the "critical energy density"

$$\varepsilon_{\text{crit}} \equiv \frac{3H^2}{8\pi G}, \quad (1.5)$$

[1]All our statements about the universe concern its local structure, i.e. its structure in a region of the size of the horizon. Therefore, they do not necessarily apply to its global properties.

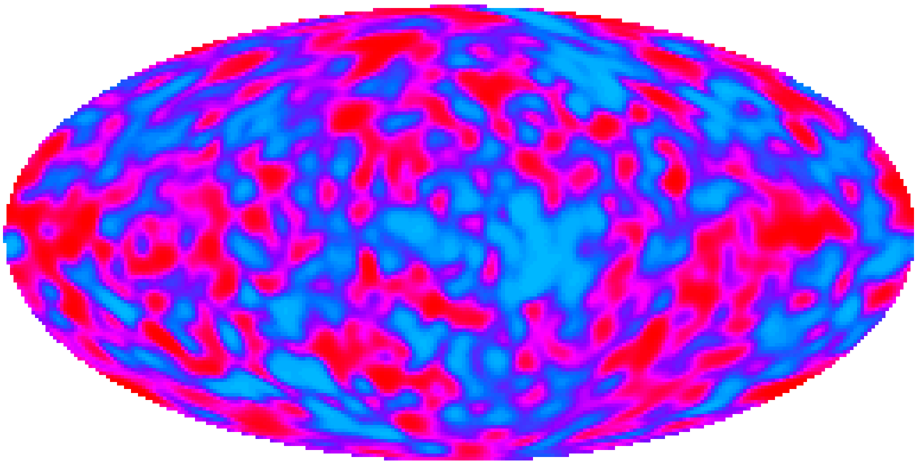

Figure 1.2: Temperature anisotropies in the cosmic microwave background as measured by the COBE [6] satellite experiment. The different colors denote variations in the CMBR temperature of the order $\Delta T \approx 10^{-5}$ K.

which is crucial in order to determine whether our universe is open, flat or closed. In order to see why, let us rewrite equation (1.2) in the form,

$$1 = \Omega_{\rm tot} + \Omega_{\rm curv}, \tag{1.6}$$

where Ω_i, the *density parameter* of the i component, denotes the energy density of any component i measured in units of the critical energy density,

$$\Omega_i = \frac{\varepsilon_i}{\varepsilon_{\rm crit}}, \tag{1.7}$$

and where we have defined the equivalent of an energy density of curvature by the relation $\varepsilon_{\rm curv} = -3\kappa/(8\pi G a^2)$. Therefore, from (1.6), if the total energy density of the universe is critical the energy density in curvature is zero and the universe is flat. If the energy density in the universe is subcritical, the energy density in curvature has to be positive and the universe is hyperbolic. And if the energy density of the universe is bigger than critical, the universe is spheric. To conclude let us notice that since H is time dependent, the critical energy density varies with time; today $\varepsilon_{\rm crit} \approx 10^{-29}$ g cm^{-3}.

1.1.3 The constituents of the universe

Do we live in a flat, open or closed universe? As discussed above, the answer to this question depends on the total energy density of the universe, which is the sum of the energy densities of the different constituents the universe contains. One of them is the matter structures like galaxies and clusters are made of. Surprisingly this matter is mostly not the (baryonic) matter we are familiar with, but a sort of "dark matter". For our purposes it will suffice to know that this matter behaves as a nonrelativistic gas of dust particles, and hence we shall call it just "dust". The most important property of dust is that it does not exert any pressure, or in other words, the ratio of its pressure to its energy density (what we shall call the equation of state w) is zero. It follows from this fact that the energy density of dust ε_d decreases in agreement with what one expects from particle number conservation,

$$\varepsilon_d \propto \frac{1}{a^3}. \tag{1.8}$$

Besides of dust, an important component in the universe is the radiation found in the cosmic microwave background and a (yet unobserved) neutrino background. As opposed to dust, the ratio of its pressure to its energy density is $w_r = 1/3$, and from this it follows that

$$\varepsilon_r \propto \frac{1}{a^4}. \tag{1.9}$$

Finally, a concordance of several recent measurements seems to imply that there is an important additional constituent in the universe, which is sometimes denoted as "dark-energy". Very little is known about this component. The only quite certain fact is that it has a negative equation of state $w_{\text{dark}} < -0.6$ [48]. Theorists have considered for a long time that the universe may contain a sort of dark energy called "cosmological constant" Λ, which has an equation of state $w_\Lambda = -1$, and hence would perfectly fit the behavior of the observed dark energy component. Hence, in most analysis, the dark energy component is assumed to be a cosmological constant. The energy density of a cosmological constant does not change with time,

$$\varepsilon_\Lambda = const. \tag{1.10}$$

The triangle diagram of figure 1.3 shows the experimental constraints on the energy densities of dust, curvature and a cosmological constant. Today, the energy density of radiation is negligible, $\Omega_r \approx 10^{-4}$, and hence it does not appear in the figure. Notice that because of equation (1.6), the contributions of the different components add to one, $\Omega_d + \Omega_\Lambda + \Omega_{\text{curv}} = 1$. The figure clearly shows that observational data favor a universe with $\Omega_d \approx 0.3$, $\Omega_\Lambda \approx 0.7$ and $\Omega_{\text{curv}} \approx 0$. Thus, dark energy is the most important component of the universe today, and the total energy density is critical, the universe is flat. In the next subsections we shall explore the consequences of these results.

1.1.4 Brief thermal history of the universe

Einstein's equations can be used to track the evolution of the universe back in time and describe its history. Our universe is expanding today and hence

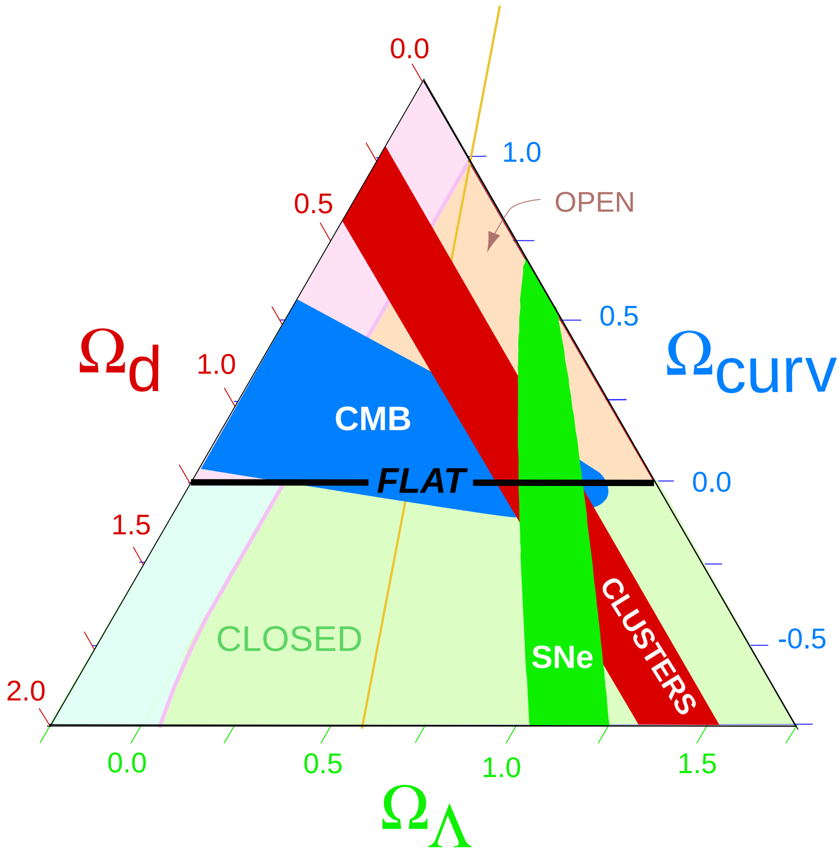

Figure 1.3: The "cosmic triangle" [4] is a convenient way to show the experimental constraints on the energy density fractions of the constituents of the universe: dust (Ω_d), curvature ($\Omega_{\rm curv}$) and cosmological constant (Ω_Λ). Measurements of large scale structure, fundamentally cluster abundances, mainly constrain the amount of dust in the universe (red strip). Measurements of cosmic microwave anisotropy mainly constrain the amount of curvature (blue wedge), and Type I supernova observations mainly constraint the acceleration of the universe (green strip). The three completely independent constraints intersect at a single "point" $\Omega_d \approx 0.3$, $\Omega_\Lambda \approx 0.7$, $\Omega_{\rm curv} \approx 0$.

it was smaller in the past. Because of the different scale factor dependencies of the various energy densities, (1.8), (1.9) and (1.10), their relative contributions to the total one change as one goes back in time. This change is shown in figure 1.4, which plots the of the energy densities of a cosmological constant, dust, curvature and radiation. As a time variable it is convenient to use the scale factor itself, and hence the figure uses "redshift"

$$z \equiv \frac{a_0}{a} - 1 \qquad (1.11)$$

instead of cosmic time t. Observe that the redshift grows as we proceed back in time. In our description of cosmic history we shall proceed from small redshift to large redshift, i.e. from the present to the past.

As shown in figure 1.4, if the universe is presently dominated by a cosmological constant, its energy density soon (at $z \approx 1$) becomes negligible, whereas curvature remains negligible all the way back from the present (measurements show that the universe is nearly flat today). Hence, during most of our recent past, the universe has been dust dominated. At $z \approx 1.1 \cdot 10^3$ the universe's photons decoupled from the electrons and were able to propagate freely. We observe this radiation today as the cosmic microwave background of subsection 1.1.1. Hence, by looking at the microwave background we see a picture of the universe at the time it was 1100 times smaller than today. Because the temperature of radiation in an expanding universe is inversely proportional to the scale factor, this implies that at that time its temperature was around 3000 K. The domination of dust concluded at matter-radiation equipartition, $z \approx 4 \cdot 10^3$, where the energy densities of radiation and dust became equal. For larger redshifts, radiation was the dominant component of the universe.

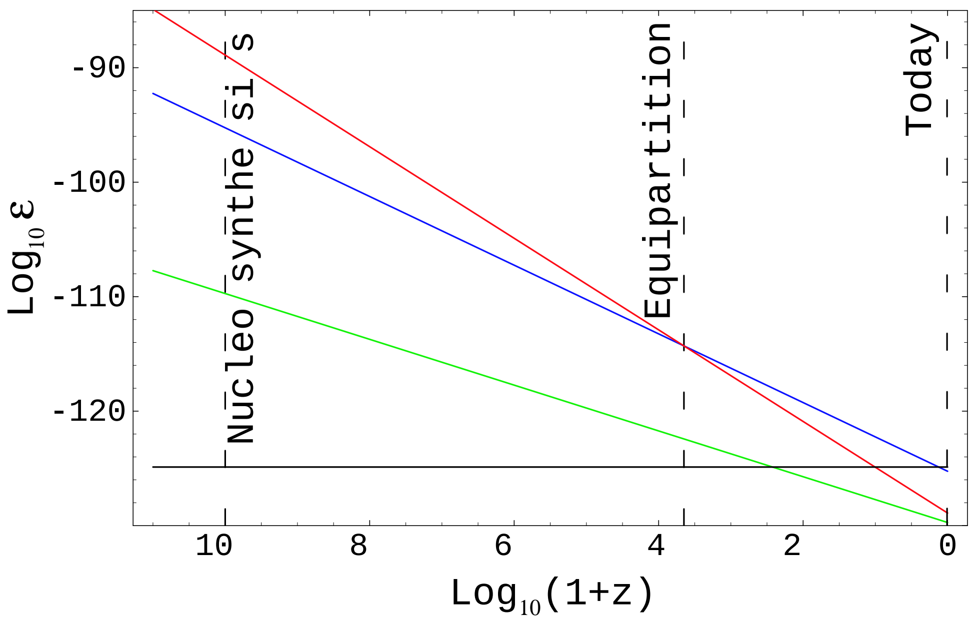

Figure 1.4: Past cosmic evolution of the energy densities in radiation (red), dust (blue), curvature (green) and cosmological constant (black).

As the redshift increases the universe becomes smaller and hotter. When the temperature was about 10^{10} K, i.e. at the time the universe was 10^{10} times smaller than today, the light elements (D, He, Li) were formed from a soup of baryons. The predicted abundances of the different light elements are in perfect agreement with the observed abundances [33, 52], and confirm again the "hot" universe theory.

As we continue back in time we enter the realm of speculation. If we would blindly follow Einstein's equations we would meet a point where the scale factor vanishes and the energy density of the universe becomes infinite. This moment of time is the so-called "Big-Bang". For several reasons however, cosmologists strongly believe that prior to reaching this singularity (i.e. at smaller redshift), the universe underwent a stage of "inflation". What happened prior to inflation is unknown.

1.2 Inflation

1.2.1 Problems of the standard cosmology

The standard hot big bang scenario is not a theory about the "big-bang" itself; it only describes the evolution of the universe from an initial moment of time close to that hypothetical event. Therefore, the initial conditions implicitly present in the scenario can not be explained within the scenario itself. This fact would not be important if the big bang initial conditions were fairly generic. On the contrary, it turns out that these initial conditions are extremely unnatural. Historically, their different unnatural features have been grouped into the following main three problems:

- The flatness problem

  As shown in figure 1.4, the energy density of curvature decreases much more slowly than the energy densities of matter and radiation but nonetheless, it is still negligible today. Therefore, in the early universe the energy density of curvature had to be, seemingly without any explanation, highly suppressed in comparison to the energy density of radiation.

- The horizon problem

  We mentioned previously that the universe seems to be homogeneous on large enough scales. Actually, we know that it was nearly homogeneous already at the time of recombination, since the temperature of the microwave background radiation is nearly the same on all sky directions. It can be shown however that if cosmic evolution proceeded according to the standard cosmology, at the time of recombination the visible universe consisted of 10^6 causally disconnected regions. Within the standard cosmology those regions never had a chance to interact and thermalize to a common temperature, and hence, there is no explanation for this initial homogeneity of the universe.

- The problem of the origin of structure

  Finally, although the early universe was nearly homogeneous, it actually contained small inhomogeneities from which the structures we observe today emerged by gravitational instability. In the context of the standard cosmology, the spectrum of these “primordial” inhomogeneities belongs to the initial conditions. It can be neither explained nor predicted.

1.2.2 Inflation and big bang initial conditions

The initial conditions of the standard hot big-bang scenario can be easily explained by an early stage of accelerated expansion of the universe known as “inflation”. As an example of how inflation solves the problems above, let us first address the flatness problem. Consider the evolution of the curvature density parameter $\Omega_{\rm curv}$. It follows from its definition that

$$\frac{d|\Omega_{\rm curv}|}{dt} = -\frac{2|\Omega_{\rm curv}|}{aH} \cdot \ddot{a}.$$

Therefore during inflation ($\ddot{a} > 0$) $|\Omega_{\rm curv}|$ decreases, whereas if the expansion of the universe is decelerated it increases. This example already shows that not any stage of inflation may be able to solve the problems of the standard scenario. In fact, inflation should last sufficiently long in order to drive $\Omega_{\rm curv}$ to small enough values. The criteria an inflationary scenario should meet in order to successfully explain the big-bang initial conditions are the following:

- Causality

  Inflation should last long enough to inflate an initially causally connected patch into a region that encompasses today’s observable universe. Only then does inflation solve the homogeneity problem, since only a causally connected region may be expected to be initially homogeneous. Generically, a causal inflationary stage automatically solves the flatness problem (see however [12]).

- No hair

  The problems of the standard hot big bang model concern its initial conditions. Inflation would not fix them, if it required itself a fine tuned initial state. Hence, on one hand inflation should emerge from rather generic initial conditions and on the other hand the properties of the inflationary stage should be independent of the conditions of the initial causally connected region our observable universe originated from. Only in that case does inflation have predictive power. The term "no hair" refers to the analogous situation in black hole formation, where the final black hole state is independent of the initial collapse conditions.

- Correct spectrum of density perturbations

  During a stage of accelerated expansion the physical wavelength of energy density perturbations grows faster than the size of the Hubble horizon, which is the size of the region where causal microphysical processes may operate. Hence, inflation provides the necessary conditions to explain the origin of those perturbations. This does not mean however that any inflationary scenario explains the *observed* spectrum of density perturbations. The amplitude of this spectrum should have the right magnitude ($\approx 10^{-5}$) and the right shape (nearly scale invariance).

- Exit and Reheating

  The inflationary scenario should account for the end of inflation and the transition to a radiation-dominated Friedmann universe. In particular it should explain how the universe is "reheated", i.e. it should explain the origin of the hot radiation of the standard cosmology.

1.2.3 Inflationary scenarios

Many inflationary scenarios have been proposed during the past 20 years [38]. Nearly all of them rely on a scalar field to drive the inflationary stage. The reason can be traced back to equation (1.3). From that equation one can namely derive a condition on the equation of state of a component which drives inflation. In fact, accelerated expansion implies that

$$w_{\text{tot}} = \frac{p_{\text{tot}}}{\varepsilon_{\text{tot}}} < -\frac{1}{3}. \tag{1.12}$$

The simplest form of matter which can satisfy the last equation is precisely a scalar field. The equation of state of a homogeneous scalar field is

$$w = \frac{p}{\varepsilon} = \frac{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)}{\frac{1}{2}\dot{\varphi}^2 + V(\varphi)},$$

where $V(\varphi)$ is the scalar field potential. Depending on the value of $\dot{\varphi}^2$, w can take any value in the range $-1 \leq w \leq 1$, which overlaps with the one required by inflation (1.12).

The value that the equation of state of the field does actually take during cosmic evolution is determined by the field equation of motion. The second common feature of most inflationary scenarios is the existence of a "slow-roll" regime (see also [16]): If the potential V is flat enough, i.e if it satisfies the slow-roll conditions

$$\left(\frac{V_{,\varphi}}{V}\right)^2 \ll 1 \quad \text{and} \quad \frac{V_{,\varphi\varphi}}{V_{,\varphi}} \ll 1, \tag{1.13}$$

the field is driven from a fairly big set of initial conditions into a slow-rolling stage ($\dot{\varphi}^2 \ll V(\varphi)$) where its equation of state is "de Sitter-like", $w \approx -1$. During that stage, the scale factor grows nearly exponentially in time, and the expansion rate is determined by the (nearly constant) value of the scalar field potential, $a \propto \exp(\sqrt{V}t)$.

The "slow-roll" conditions (1.13) are satisfied by many different potentials, and in many cases the different inflationary scenarios just differ in the choice of the potential. Particularly natural choices are the potentials of the so-called "chaotic scenarios", such as $V(\varphi) = m^2\varphi^2/2$ or $V(\varphi) = \lambda\varphi^4/4$. The word "chaotic" was intended to mean the "no hair" feature we mentioned above: If the field is big enough, $\varphi \gg 1$, it is soon driven to the slow roll regime where $w \approx -1$ and the universe expands exponentially. During that regime the field slowly rolls down the potential, until it reaches field values $\varphi \approx 1$ and the slow-roll conditions are violated. Afterwards the field starts to oscillate around the potential minimum and due to the couplings of the inflaton field to other matter fields the universe is "reheated", that is, the radiation needed for a smooth transition to the familiar Friedmann radiation dominated universe [36] is generated. A phase diagram of the evolution of the scalar field [5] is shown in figure 1.5 .

1.3 Late time cosmic acceleration

The universe mainly consists today of two thirds of a dark energy component with a negative equation of state, $w_{\text{dark}} < -0.6$ and one third of pressureless dust. Hence, the ratio of total pressure to total energy density in the universe is

$$w_{\text{tot}} = \frac{p_{\text{tot}}}{\varepsilon_{\text{tot}}} \approx \Omega_{\text{dark}} \cdot w_{\text{dark}} < -0.4 < -\frac{1}{3}.$$

Thus, according to equation (1.3), the universe's expansion should be accelerating today, as if the universe had started a late stage of inflation. In fact, the magnitude to redshift relation of Type Ia supernovas [45, 51] seems to directly imply this cosmic acceleration, as we mentioned in subsection 1.1.3.

The simplest candidate for a dark energy component which drives cosmic acceleration is a cosmological constant. This interpretation raises, however, an issue similar to the flatness problem of the standard big bang cosmology (see figure 1.4 and subsection 1.2.1). Indeed, the value of a cosmological constant that has started to dominate the energy density of the universe just

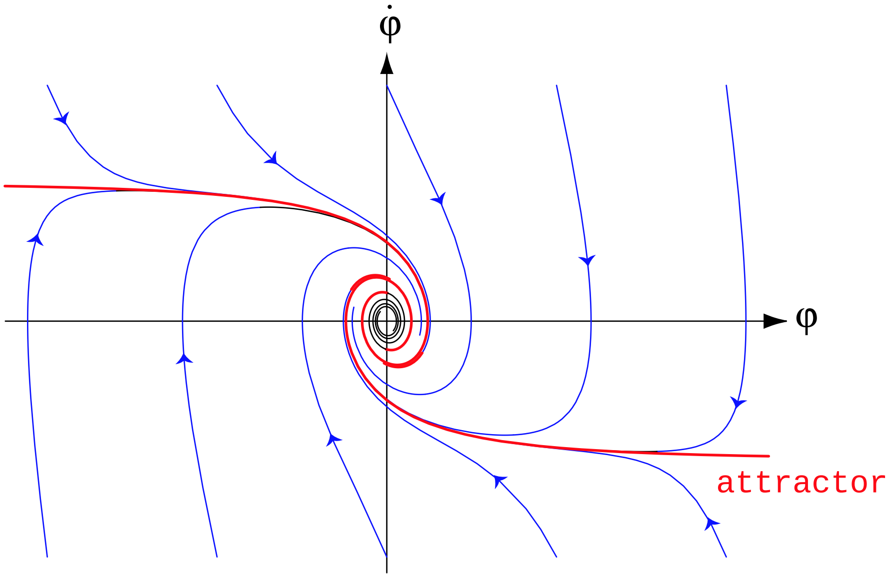

Figure 1.5: The motion of a homogeneous scalar field with potential $V(\varphi) = \frac{1}{2}m^2\varphi^2$ in flat space. At large energies (large values of φ), regardless of initial conditions, the field rapidly approaches the inflating attractor (red) where $\varepsilon + p \approx 0$. During that stage the universe inflates. Once the field reaches lower energies the field and its velocity $\dot{\varphi}$ start oscillating around the origin. During these oscillations particles are produced, the universe is reheated.

recently has to be extraordinarily small, $\varepsilon_\Lambda \approx 10^{-124}$, compared to the naively expected value $\varepsilon_\Lambda \approx 1$ (all quantities in Planckian units). A key challenge for theoretical physics is to address this cosmic coincidence problem: why does the dark energy component have a tiny energy density compared to the naive expectation based on quantum field theory, and why does cosmic acceleration begin at such a late stage in the evolution of the universe [28].

Different proposals have been made to solve this challenge. The extraordinary fine-tuning of a cosmological constant has led some authors to use the anthropic principle as an explanation [58, 28]. Essentially, the anthropic principle states that our own existence explains the fact that the cosmological constant is so small. For if the cosmological constant were significantly larger, cosmic acceleration would have started before structure had ever had a chance to develop, and intelligent beings would never have evolved.

A different alternative consists in introducing "quintessence" [50, 23, 14], a dynamical component which may behave effectively as a cosmological constant. As in inflation, this dynamical component is usually a scalar field with an appropriate potential. It turns out that, still, the moment of the dominance of this field has to be encoded in the scalar field potential, and hence, although quintessence models are a phenomenologically viable alternative to a cosmological constant, they do not explain the cosmic coincidence problem. In chapter 4, we illustrate this fine tuning, and at the same time we propose a quite different alternative which avoids it. First, we will need however some concepts about non-canonical scalar fields.

1.4 Non-canonical scalar fields in particle physics

Quantum field theories [59, 49] provide a very successful framework to describe elementary non-gravitational particle interactions. This success is best exemplified by the standard model, which, up to the recent indications of non-vanishing neutrino masses [24], has been tested repeatedly during the past in a vast number of accelerator experiments up to energies of about 100 GeV or distance scales of 10^{-16} cm.

Calculations in quantum field theories generically yield divergences. These divergences are absent of the physical predictions of the theory as long as the Lagrangian is “renormalizable”. Hence, it was thought for a long time that any satisfactory field theory should be renormalizable. The criterion of renormalizability played a significant role in the development of the standard model since, besides of allowing very precise finite predictions, it strongly restricts the possible terms that may appear in the Lagrangian. Consider a scalar field not coupled to any other field for instance. Among the infinite number of Lorentz-invariant terms which the Lagrangian may contain, renormalizability just allows six of them: $\Box\phi$, $\partial_\mu\phi\partial^\mu\phi$, ϕ, ϕ^2, ϕ^3 and ϕ^4. The term $\Box\phi$ yields a boundary term in the action, and does not contribute to the field equations of motion, and the linear term ϕ can be eliminated by a field redefinition. Thus, the most general renormalizable Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4.$$

In particular, the Lagrangian contains only one derivative term, $\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$, which is (up to field redefinition) uniquely determined by renormalizability. We shall call that derivative term, a “canonical” kinetic term, and a scalar field with a canonical kinetic term a “canonical field”.

The non-renormalizability of Einstein's gravity and the Wilsonian approach to the renormalization group [49] have changed the viewpoint about renormalizability. Currently it is mostly believed, that the field theories we use to describe particle interactions (as for instance, the standard model) are just low energy approximations of a more fundamental theory, which may not even be a field-theory at all [59]. These low-energy approximations are characterized in general by non-renormalizable effective field theories, which may contain any term compatible with the symmetries of the theory. Thus, the Lagrangian of an effective theory describing a scalar field may generically contain, besides the renormalizable ones above, any other term such as, for instance, φ^5, $\varphi^2 \partial_\mu \phi \, \partial^\mu \phi$, $(\partial_\mu \phi \, \partial^\mu \phi)^2$, $\partial_\mu \partial_\nu \phi \, \partial^\mu \partial^\nu \phi$, etc. Symmetries may strongly restrict in that case the terms which do appear in the Lagrangian. For example, if the theory has the symmetry $\phi \to \phi + \lambda$ and we insist upon keeping the field equations second order, the Lagrangian can depend only on the quantity $\partial_\mu \phi \, \partial^\mu \phi$ alone, $\mathcal{L} = \mathcal{L}(\partial_\mu \phi \, \partial^\mu \phi)$. Although effective theories contain every possible term consistent with the symmetries of the theory, renormalizable field theories still retain in this picture a special role. In fact, it can be shown by simple dimensional arguments [59] that at sufficiently low energies, non-renormalizable terms in any effective field-theory yield negligible contributions to scattering amplitudes, and thus at low enough energies, every quantum field theory should look like a renormalizable one.

The infinities appearing in ordinary quantum field theories (say, quantum electrodynamics) are already a hint that, as it is commonly believed, these theories are not fundamental. Some of these divergences reflect the infinite self-energies of the particles they describe. This problem, though, is not exclusive of quantum field theories, and has been known for instance in classical electrodynamics for a long time. Infinities seem to arise because particles are point-like; arbitrarily close to a point-particle the electric field becomes arbitrarily large. Heuristically, there are mainly two conceivable ways of avoiding infinite self-energies: The strength of the electromagnetic field can be forced to have a bound or point-particles can be replaced by higher-dimensional objects (like strings). The first approach was pursued back in the 30's by Born and Infeld, when quantum field theorists were still struggling with infinities, whereas the second leads to string theory. Remarkably, Born-Infeld theory has recently emerged in string theory as describing the low energy dynamics of higher-dimensional objects known as "branes".

1.4.1 String theory

In string theory [31, 47], point particles are replaced by a one-dimensional object, the string. Today, it seems to be the only theory which incorporates a quantum description of gravity and particle gauge interactions. String theory is not a field-theory in the usual sense, but rather a field theory defined on the two-dimensional surface swept by the string. This two-dimensional field theory allows the perturbative computation of scattering amplitudes in an expansion in powers of its coupling constant g^2_{string}. This coupling constant is determined by the expectation value of the dilaton ϕ, a scalar field which invariantly appears in the massless excitations of any string theory, according to the relation $g^2_{\text{string}} = e^\phi$. Once the scattering amplitudes for the different particles are computed, one can write down an effective field theory which reproduces those scattering amplitudes. Because the string is not a point-like object, but rather a one dimensional "string" with tension $1/\alpha'$, such an effective action contains an infinite number of terms which can be organized in a derivative expansion in powers of α' (α' has dimensions of a squared length). This leads to a structure of the type

$$\begin{aligned} S_{\text{eff}} &= \frac{1}{(\alpha')^4}\int d^{10}x\sqrt{-g}\Big\{ - B_g(\phi)R - B_\phi^{(0)}(\phi)(\partial\phi)^2 \\ &\quad +\alpha'\left[c_1^{(1)}B_\phi^{(1)}(\phi)(\partial\phi)^4+\cdots\right]+\mathcal{O}(\alpha'^{\,2})\Big\}, \end{aligned} \tag{1.14}$$

where the ellipsis stands for other four-derivative terms (like $(\Box\phi)^2$, $\hat{R}^2_{\mu\nu\rho\sigma},\ldots$). The coupling functions B can be expressed as an expansion in power of the string coupling $g^2_{\text{string}}=e^{\phi}$,

$$\begin{aligned} B_g(\phi) &= e^{-\phi}+c_{g,0}+c_{g,1}e^{\phi}+\cdots, \\ B_\phi^{(0)}(\phi) &= e^{-\phi}+c_{\phi,0}+c_{\phi,1}e^{\phi}+\cdots, \\ B_\phi^{(1)}(\phi) &= e^{-\phi}+\cdots, \end{aligned}$$

where the ellipses contain higher contributions in g^2_{string}, including non perturbative ones. Observe that at tree level, the coupling of the dilaton is "universal"; all coupling functions are equal $e^{-\phi}$.

The action (1.14) shows several interesting features of string theory. The most striking one is the dimensionality of spacetime; string theory is only consistent if formulated in ten spacetime dimensions. Since our spacetime seems to be four-dimensional, one usually assumes that out of the ten dimensions six are "compactified". This means that the ten-dimensional spacetime $M^{(10)}$ is assumed to have a tensor product structure, $M^{(10)}=M^{(4)}\times K^{(6)}$, where $M^{(4)}$ is our four dimensional world and $K^{(6)}$ are six "small" (say, Planckian size) additional dimensions. Upon compactification, these extra dimensions enter the four dimensional effective action as a set of additional scalar fields describing the shape of the compact manifold $K^{(6)}$. Those fields, called "moduli" fields, are similar to the dilaton. The second remarkable fact is that string theory does not yield Einstein gravity but a scalar-tensor theory. Although the curvature terms has not the conventional form, by a conformal rescaling of the metric, $g^{\text{new}}_{\mu\nu}=B_g(\phi)g^{\text{old}}_{\mu\nu}$ it is always possible to write the four-dimensional gravitational action in Einstein-Hilbert form, although the resulting couplings to matter may not respect Einstein's equivalence principle.

Equation (1.14) nicely illuminates our previous discussion about effective actions. At low energies, the predictions of string theory can be extracted from a low-energy effective action which generically contains any term compatible with general coordinate invariance. These terms can be organized in an expansion in powers of the string tension $1/\alpha'$. On dimensional grounds, a term containing $2n$ derivatives and m powers of a dimensionless field ϕ is (in Fourier space) proportional to $(\alpha' k^2)^n \phi^m$. Thus, whereas for fixed m terms with an increasing number of derivatives become increasingly negligible at low momenta ($k \ll 1/\sqrt{\alpha'}$), for large field or m values they may still be important.

1.4.2 Born-Infeld Action and Branes

Already in 1934 Born and Infeld proposed an alternative theory of electromagnetism which avoids the infinite self-energy of the electron [8]. They argued that every sensible theory should avoid letting physical quantities become infinite. As an example they pointed out how the Lagrangian of a free particle in classical mechanics, $\frac{1}{2}m\dot{x}^2$, which allows infinite speeds, is replaced by $m(1-\sqrt{1-\dot{x}^2})$ in relativistic mechanics, an expression which sets the speed of light as the upper limit of any velocity. Hence, Born and Infeld proposed replacing the usual Lagrangian of the electromagnetic field $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ by $\sqrt{1+\frac{1}{2}F_{\mu\nu}F^{\mu\nu}}-1$. (A similar way of limiting the curvature of spacetime was considered in [10].)

The Born-Infeld action can be easily derived from symmetry arguments. The simplest action invariant under general coordinate transformations has the form $\int d^4x\sqrt{-\det a_{\mu\nu}}$, where $a_{\mu\nu}$ is an arbitrary (covariant) second rank tensor. Such a tensor can be split into a symmetric and an antisymmetric part, $a_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu}$. The symmetric part $g_{\mu\nu}$ can be interpreted as the metric of spacetime, and the antisymmetric part $F_{\mu\nu}$ as the electromagnetic field strength. One arrives hence at the Born-Infeld action

$$S_{\mathrm{Born-Infeld}} = -\int d^{p+1}x\sqrt{-\det(g_{\mu\nu}+F_{\mu\nu})}, \qquad (1.15)$$

where for generality we allow an arbitrary dimensionality of spacetime.

The Born-Infeld action (1.15) also describes the motion of a $p-$dimensional extended object, a $p-$brane, with an electromagnetic field propagating on the "world-volume" swept by it [30]. For a vanishing electromagnetic field (1.15) is equal to the "area" swept by the brane, and the action principle states that this surface is minimal. The motion of the brane is characterized by the embedding of the brane in a higher dimensional "target" spacetime, $z^M = z^M(x^\mu)$. If this target spacetime is flat, the metric on the brane is just given by the induced metric

$$g_{\mu\nu} = \eta_{MN}\frac{\partial z^M}{\partial x^\mu}\frac{\partial z^N}{\partial x^\nu}.$$

The electromagnetic field is characterized by the values of the electromagnetic potential along the brane, $A_\nu(x^\mu)$, and the electromagnetic field strength is as usual related to the potential by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

As a particular example, let us consider a 3-dimensional brane moving in a 5-dimensional flat spacetime with metric $\eta_{MN} = \mathrm{diag}(+1, -1, \cdots, -1)$. The embedding of the brane can be chosen in the following way: $z^\mu = x^\mu$ for $\mu = 0 \ldots 4$ and $\phi \equiv z^5(x^\mu)$ for the fifth dimension. Then, for a vanishing electromagnetic field the Born-Infeld action reads,

$$-\int d^4x \sqrt{1 - \partial_\mu\phi\,\partial^\mu\phi}, \tag{1.16}$$

where the four dimensional metric is $\eta_{\mu\nu} = (+1, -1, -1, -1)$. Thus, from a four-dimensional point of view, the presence of an extra dimension appears in the action as a scalar field with a highly non-standard kinetic term. We shall meet a similar kinetic term much later in our work. At this point let us notice that a kinetic term of the form (1.16) has been shown to have deep connections with fluid-mechanical systems, namely, with the so-called Chaplygin gas [32, 35].

Chapter 2

General Model: The k-field

We have seen that an effective action may contain any term compatible with the symmetries of the underlying theory. In the case of a scalar field minimally coupled to gravity, those terms include any combination of the field itself and its properly contracted derivatives. Among all these infinite number of terms only two are commonly considered in cosmological applications: The canonical kinetic term $\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$ and the field potential $V(\phi)$. In this thesis our aim is to consider more general Lagrangians and verify whether additional non-canonical terms may play any significant role in cosmology. At this point, the success of general relativity to explain a vast wealth of data ranging from Mercury's peryhelium advance to light element abundances shall serve us as a guideline to restrict the type of terms we shall consider among the overwhelming range of possibilities. As hinted in equations (1.2) and (1.3), the vacuum equations of motion in general relativity are second order. As a matter of fact, all equations of motion in nature (at least classically) seem to require just a set of two initial conditions (position and momentum). We are thus naturally lead to the "k-field", the subject this short chapter is devoted to.

The k-field is a scalar field whose Lagrangian contains any term involving maximally one field derivative. The best way to characterize it is to write down its action, which describes not only the k-field self-interactions, but also its interactions with additional matter fields of the theory. This is done in section 2.1. It turns out that the k-field can be described as a perfect fluid with definite pressure and energy density. The perfect fluid analogy is used in section 2.2 to formulate the field equations of motion in a flat, homogeneous universe. Of course, due to the very nature of the k-field, these equations are second order and do not involve higher derivatives. Afterwards, in section 2.3, we consider small perturbations around the homogeneous background in order to find the behavior of small inhomogeneities. Its qualitative evolution is determined by the "speed of sound", which we relate to the stability of the background solutions. We conclude by expressing the k-field action in a different parameterization, which will allow us to formulate certain restrictions on the k-field Lagrangian in a simpler way.

2.1 The Action Principle

We want to consider a scalar field with non-canonical kinetic terms involving at most one field derivative. In a cosmological setting, we should describe how this field is coupled to gravity, and how it interacts with additional matter components in the universe, such as dust particles and radiation. By definition, the k-field φ is described by the action

$$S_{\text{tot}}[g_{\mu\nu}, \varphi, \psi_m] = S_{EH}[g_{\mu\nu}] + S_k[g_{\mu\nu}, \varphi] + S_m[g_{\mu\nu}, \psi_m], \tag{2.1}$$

where $g_{\mu\nu}$ and ψ_m denote the spacetime metric and the remaining matter fields respectively. The total action is divided into different sectors, each one describing a different component. S_{EH} is the Einstein-Hilbert action of general relativity,

$$S_{EH} = -\int d^4x\sqrt{-g}\,\frac{R}{6},$$

so we deal exclusively with Einstein gravity in four spacetime dimensions. Notice that a scalar-tensor type of gravitational action (for instance (1.14)) can be recast into Einstein-Hilbert form by a conformal transformation anyway, although such a conformal transformation may alter the couplings of the scalar field in a way not compatible with (2.1). The reader interested in the relationship between different conformal frames may consult [18]. The factor $1/6$ reflects our unit choice $c = \hbar = 8\pi G/3 = 1$.

The k-field itself is described by the action

$$S_k = \int d^4x\sqrt{-g}\,p(\varphi, X). \tag{2.2}$$

The requirement that the k-field Lagrangian do not involve higher order field derivatives, along with general coordinate invariance, forces p to depend on the field derivatives through the combination

$$X \equiv \frac{1}{2}(\partial\varphi)^2 \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \tag{2.3}$$

which we shall call the "kinetic" variable. The function p contains the kinetic and potential terms of the k-field. By a "canonical scalar field" one understands a field for which the Lagrangian can be cast in the form $p(\varphi, X) = X - V(\varphi)$, where V is the scalar field potential. Here we aim to consider more general, non-canonical forms, and we will assume that p is a general function of its arguments. In particular, later on we will focus on factorizable Lagrangians,

$$p(\varphi, X) = K(\varphi)\tilde{p}(X), \tag{2.4}$$

since as we shall show, the latter form of p turns to be general enough to accommodate a wide class of cosmologically interesting situations. It also appears naturally from the effective actions of string theory and it is able to describe a canonical self interacting scalar field (the Lagrangian $p = X - V$ can be cast in the form (2.4) by the field redefinition $d\varphi_{\text{new}} = d\varphi_{\text{old}}/\sqrt{V(\varphi_{\text{old}})}$).

Finally, the action S_m describes the remaining "matter" fields ψ_m of the theory. For our purposes it will suffice to consider dust—nonrelativistic particles— and radiation. Notice that the matter action does not depend on φ; matter and the k-field are coupled only gravitationally.

The properties of any continuous field or matter distribution are described by the energy momentum tensor. The energy momentum tensor of the k-field is obtained by functional differentiation of the k-field action (2.2) with respect to the metric,

$$T^{(k)}_{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta S_k}{\delta g^{\mu\nu}} = \frac{\partial p(\varphi, X)}{\partial X} \partial_\mu \varphi \partial_\nu \varphi - p(\varphi, X) g_{\mu\nu}. \tag{2.5}$$

An important fact is that the above expression can be put in perfect fluid form. For a perfect fluid

$$T_{\mu\nu} = (\varepsilon + p) u_\mu u_\nu - p\, g_{\mu\nu}, \tag{2.6}$$

where u^μ is the four-velocity of the fluid and where ε and p are the energy density and the pressure measured by an observer at rest with respect to that fluid. Accordingly, by comparing (2.5) with (2.6) one finds that the energy density of the k-field is

$$\varepsilon = 2X p_{,X} - p, \tag{2.7}$$

its pressure is precisely the function $p(\varphi, X)$ and its four velocity is given by

$$u_\mu = \sigma \frac{\partial_\mu \varphi}{\sqrt{2X}}, \tag{2.8}$$

where $\sigma = \text{sgn}(\partial_0 \varphi)$. From equation (2.7) follows that for the Lagrangians (2.4) the energy density can be factorized similarly to the pressure,

$$\varepsilon = K(\varphi) \cdot \tilde{\varepsilon}(X), \quad \tilde{\varepsilon}(X) \equiv 2X \tilde{p}_{,X} - \tilde{p}. \tag{2.9}$$

Consequently, a tilde denotes the X-dependent part of a quantity that can be factorized into φ and X dependent parts. To conclude, let us note that because the four velocity enters (2.6) only quadratically, there is a sign ambiguity in the choice of u_μ. Our sign choice corresponds to a future directed u_μ. In any case, this ambiguity is irrelevant since only the combination $u_\mu u_\nu$ enters the equations of motion.

2.2 Equations of motion

Our universe is isotropic and homogeneous on large enough scales and there is also firm evidence by now that its spatial sections are flat (see chapter 1). Hence, we will consider the motion of a homogeneous k-field $\varphi(t)$ in a homogeneous and isotropic flat universe described by the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\, d\vec{x}^2$. In fact, a homogeneous and isotropic universe is a consequence of a (sufficiently long) stage of inflation, and we will show below how a k-field may drive such a stage and thus justify our assumption.

The equations of motion of our model are obtained by functional differentiation of the total action (2.1) with respect to the different fields of the theory. By differentiating (2.1) with respect to the metric one obtains Einstein's equations $G_{\mu\nu} = 3\left({}^{(k)}T_{\mu\nu} + {}^{(m)}T_{\mu\nu}\right)$, where ${}^{(k)}T_{\mu\nu}$ and ${}^{(m)}T_{\mu\nu}$ are the energy-momentum tensors of the k-field and matter respectively. The 00 component of Einstein's equations yields equation (1.2), which for our unit choice and spatial metric reads

$$H^2 = \varepsilon_{\text{tot}} \equiv \varepsilon_k + \varepsilon_m. \tag{2.10}$$

The matter and k-field equations of motion have the form of the "conservation laws"

$${}^{(i)}T^\mu{}_{\nu;\mu} = 0, \tag{2.11}$$

which also follow from Einstein's equations. Here i is an index denoting matter (m) or k-essence (k). The behavior of a homogeneous perfect fluid—be it dust, radiation or the k-field— in a homogeneous background is completely characterized by the $\nu = 0$ component of equation (2.11), which is an expression of energy conservation, $d\varepsilon_i/dt = -3H(\varepsilon_i + p_i)$. It will be convenient to remove the Hubble parameter from the last equation by a suitable time choice. Introducing the number of "e-foldings" $N = \log a/a_0$ as a new time variable the equation can be rewritten as

$$\frac{d\varepsilon_i}{dN} = -3\varepsilon_i(1 + w_i), \tag{2.12}$$

where we have introduced, as costumary, the ratio of pressure to energy density

$$w_i \equiv \frac{p_i}{\varepsilon_i}, \tag{2.13}$$

which we shall somehow loosely call the "equation of state". Equation (2.12) suffices to settle the behavior of dust and radiation, since their equations of state are constant. Indeed, for a constant equation of state (2.12) can be immediately integrated, yielding the scale factor dependence of the energy density

$$\varepsilon_i \propto \frac{1}{a^{3(1+w_i)}}. \tag{2.14}$$

The equation of state of dust is $w_d = 0$ and hence it follows that $\varepsilon_d \propto 1/a^3$. In a similar way for radiation $w_r = 1/3$ and hence $\varepsilon_r \propto 1/a^4$. However, the equation of state of the k-field is not a constant, since for factorized Lagrangians it depends explicitly on X,

$$w_k = \frac{p}{2Xp_{,X} - p} = \frac{\tilde{p}}{2X\tilde{p}_{,X} - \tilde{p}}. \tag{2.15}$$

In order to describe the evolution of the k-field we should hence know how X changes in time. Substituting the expression of the energy density (2.9) into equation (2.12), and using equation (2.10) to relate the Hubble constant to the total energy density one gets

$$\frac{dX}{dN} = -\frac{\sqrt{8X\tilde{\varepsilon}}}{\tilde{\varepsilon}_{,X}}\left[r(X) + \sigma\frac{K_{,\varphi}}{2K^{3/2}}\sqrt{\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}}\right], \tag{2.16}$$

where $\sigma \equiv \text{sign}(d\varphi/dN)$ and where we have defined the new function

$$r(X) = \sqrt{\frac{9\cdot\tilde{\varepsilon}}{8X}}(1 + w_k(X)). \tag{2.17}$$

Equation (2.16) is one of the main results of this chapter. It is the equation of motion we are going to use to compute the evolution of the k-field. Notice that it becomes singular at $K = 0$, and as a result the sign of K does not change during cosmic evolution. Therefore we shall assume, without loss of generality, that K is positive. An analogous statement holds for $\tilde{\varepsilon}_{,X}$. Since its sign is also conserved during cosmic evolution we shall mostly assume it to be positive, as for a canonical scalar field.

2.3 Stability

Our universe is obviously not completely homogeneous. In order to take into account inhomogeneities of the k-field (which could be in principle responsible for the actual inhomogeneity of the universe) we shall develop first order perturbation theory around the homogeneous and isotropic background we have considered in section 2.2. We start by introducing perturbations in the k-field, $\varphi \to \varphi(t) + \delta\varphi(t, \vec{x})$. These perturbations will induce perturbations in its energy density ($\varepsilon \to \varepsilon + \delta\varepsilon$), pressure ($p \to p + \delta p$) and four-velocity ($u \to u + \delta u$). Because we are exclusively dealing with perturbations of the field, we drop the "k" subindex meanwhile. The perturbations of the energy-momentum tensor are related, through Einstein's equations, to metric perturbations. In longitudinal gauge, the perturbed metric reads

$$ds^2 = (1 + 2\Phi)\, dt^2 - a^2(t)\cdot(1 - 2\Phi)\, d\vec{x}^2, \tag{2.18}$$

where the metric perturbation $\Phi = \Phi(t, \vec{x})$ is a generalization of the Newtonian potential in an expanding universe [42].

The equations of motion of the k-field perturbations can be derived by linearizing (2.11). It was shown in [26] that when the k-field dominates the energy density of the universe, the linearized equations can be reduced to a single equation for a single perturbation variable,

$$\frac{\partial^2 v}{\partial \eta^2} - c_s^2 \Delta v - \frac{1}{z}\frac{\partial^2 z}{\partial \eta^2} v = 0. \tag{2.19}$$

Here η denotes conformal time ($d\eta = dt/a$), v is an expression linear in the perturbations Φ and $\delta\varphi$, and z is a function of time determined by the background evolution. For our purposes, the most important quantity is the squared speed of propagation of perturbations c_s^2, which is given by

$$c_s^2 = \frac{p_{,X}}{\varepsilon_{,X}} = \frac{\tilde{p}_{,X}}{\tilde{\varepsilon}_{,X}}. \tag{2.20}$$

If the squared speed of sound c_s^2 is positive, equation (2.19) describes an oscillator driven by an external force. Hence, solutions are oscillatory and describe "sound" waves propagating with speed c_s. On the other hand, for $c_s^2 < 0$, equation (2.19) has a growing solution that signals the gravitational instability of the background solution. In the latter case the exponential growth of the perturbations hinders the existence of a well-defined, stable, background around which perturbation theory may be developed, and hence, in the following we shall always require $c_s^2 > 0$.

Notice that although the k-field action is manifestly invariant under (local) Lorentz transformations, the speed of sound (2.20) can also take values bigger than 1 [26, 27]. This fact does not seem to contradict any physical law. Indeed, the assertion that "nothing can travel faster than light" follows from causality requirements in flat Minkowski space [54]. We are dealing however with a curved spacetime with a preferred reference frame, the one where the k-field is homogeneous, and the usual considerations do not apply. Similarly, the k-field may also violate any of the standard energy conditions [56]. These are rather "ad-hoc" postulates thought to be valid for all reasonable classical matter, and are essential in the derivation of certain singularity theorems. The weak energy condition states that the energy density measured by any observer should be positive, and implies for a perfect fluid that $\varepsilon \geq 0$ and $\varepsilon + p \geq 0$. When considering flat cosmologies the requirement $\varepsilon_{\text{tot}} \geq 0$ follows from Einstein's equation (2.10), and hence we will impose $\varepsilon \geq 0$ for consistency. The strong energy condition states that for a perfect fluid $\varepsilon + 3p > 0$, and as we have mentioned it has to be violated during inflation. Finally, the dominant energy condition follows by requiring that the speed of energy flow measured by any observer be less than the speed of light, implying $\varepsilon \geq |p|$, a condition which can be also easily violated by the k-field.

2.4 Alternative Lagrangian parametrizations

Up to now, the X-dependent part of the Lagrangian $\tilde{p}$, has been a completely arbitrary function. In this section we want to study how different criteria translate into restrictions on the form of $\tilde{p}$. These restrictions can be stated in a simpler form if the pressure is not parametrized as in (2.4), but as

$$p(\varphi, y) = K(\varphi)\frac{g(y)}{y}, \quad \text{where} \quad y \equiv \frac{1}{\sqrt{X}}. \tag{2.21}$$

In this new parametrization y plays the role of X and the function g plays the role of $\tilde{p}$, with the advantage that conditions on the form of $\tilde{p}$ can be re-expressed as simpler conditions on g. This will be particularly useful in our discussion of the global properties of the k-field Lagrangian in the chapters about k-inflation and k-essence.

Consider for instance the energy density of the k-field, equation (2.7). We want this energy density to be positive during the cosmic evolution of the field, $2X\tilde{p}_{,X} - \tilde{p} > 0$. Because in the new parameterization

$$\tilde{\varepsilon} = -g_{,y} \tag{2.22}$$

positivity of the energy density translates into the simple condition that g be decreasing, $dg/dy < 0$. On the other hand, as stated above, we know that we may restrict our considerations to regions where $\tilde{\varepsilon}_{,X} = 2X\tilde{p}_{,X} - \tilde{p}$ is positive, and because

$$\tilde{\varepsilon}_{,X} = \frac{1}{2}y^3 g_{,yy},$$

that means that we may assume that g is concave, $d^2g/dy^2 > 0$. The last quantity also determines the sign of the speed of sound through equation (2.20), which in the new parameterization can be expressed as

$$c_s^2 = \frac{g - y g_{,y}}{y^2 g_{,yy}}.$$

Stability requires a positive squared speed of sound, implying $g - y g_{,y} > 0$, since we take g to be concave. The last stability condition also has a simple geometrical interpretation: The intersection of the tangent to g at a given point with the g-axis should happen above the y-axis. In particular, the speed of sound is zero at points whose tangent passes through the origin.

Finally, for later purposes, it will be convenient to express the k-field equation of motion also in terms of the new variable y. It can be easily verified from (2.16) and (2.21) that y obeys the equation

$$\frac{dy}{dN} = \frac{3}{2}\frac{w_k(y) - 1}{r_{,y}(y)}\left[r(y) + \sigma\frac{K_{,\varphi}}{2\,K^{3/2}}\sqrt{\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}}\,\right], \tag{2.23}$$

where r and w_k, expressed in terms of g and y, read

$$r(y) \equiv \sqrt{-\frac{9}{8}\frac{dg}{dy}}\, y\,(1+w_k) = \frac{3}{2\sqrt{2}}\frac{g - y g_{,y}}{\sqrt{-g_{,y}}}, \quad w_k(y) = -\frac{g}{y g_{,y}}. \qquad (2.24)$$

Chapter 3

k-Inflation

Inflation explains many of the puzzles of the standard big-bang cosmology. Hence, considerable efforts have been devoted to developing different inflationary scenarios. In most of them inflation is driven by a self-interacting scalar field with Lagrangian $p = X - V(\varphi)$. There are many different models of this kind and all of them rely on some sort of "slow-roll" regime, during which the scalar field slowly rolls down its potential V. Because the speed of the field is proportional to the slope of the potential, slow-roll inflationary scenarios work only if the corresponding potential is sufficiently "flat".

Of course, the choice of the Lagrangian cannot be arbitrary. Ultimately, the Lagrangian responsible for inflation should stem from a fundamental theory of particle interactions. Some of the first inflationary models appeared for instance in the context of grand unified gauge theories, but these scenarios lost their appeal with further developments in the subject. Today many particle physicists consider string theory as a candidate for a truly unified theory of gravitation and gauge interactions. Therefore, it would be natural to look for inflation in this framework. String theory predicts indeed the existence of a whole set of scalar fields known as moduli, which are natural candidates for the inflaton. However, it is difficult to implement inflation with such scalar fields because they remain massless to all orders in perturbation theory and, even if one includes non-perturbative effects, the non-perturbative potentials are not flat enough [11]. There is nevertheless an alternative way to implement an inflationary stage. In fact, as we have seen in chapter 1, the low-energy effective action of string theory contains non-canonical scalar field kinetic terms. Thus, we may think of each of these scalar fields as a k-field. In this chapter we show that, even if the theory does not contain any scalar field potential, under appropriate conditions a k-field may lead to a stage of inflation we call k-inflation [1]. Indeed, k-inflation may be also attractive in a non-stringy context; it provides an in principle totally different way to implement inflation, keeping at the same time its main virtues.

This chapter is organized as follows: In section 3.1 we describe the general properties of a k-field Lagrangian which allows inflation and how its form is constrained once further restrictions (such as, for instance, the absence of a potential term) are imposed. In section 3.2 we find two of the main types of inflation discussed in the literature—power-law and pole-like inflation—by looking at solutions of the equations of motion with constant equation of state. We formulate precisely for what kind of Lagrangians these solutions exist and argue that in the pole-like case, an exit from inflation is impossible. Next, in section 3.3, we discuss de Sitter-like inflation and how it can be driven by a k-field. The existence of de Sitter-like k-inflationary solutions requires certain generic conditions to hold. We show that although they reduce to the known "flatness" conditions on the potentials if the field is a canonical one, they can be satisfied even by potential-less k-field Lagrangians. After briefly presenting in section 3.4 the perturbation spectra predicted by k-inflation and discussing some of its phenomenological consequences we conclude by arguing in section 3.5 that besides of accounting for a sufficient number of e-foldings, the k-field can also describe how inflation ends and how the transition to a radiation dominated universe occurs.

3.1 General properties

During inflation, the universe's expansion is accelerated. The k-field may fuel this acceleration only if its equation takes values smaller than $-1/3$, and in particular, provided its pressure becomes negative while its energy density is positive. Let us first consider what kind of functions $\tilde{p}$ allow such behavior. On general grounds, for small X we expect $\tilde{p}$ to have an expansion in powers of X,

$$\tilde{p}(X) = c_0 + c_1 X + c_2 X^2 + \cdots .$$

The first obvious observation is that if $\tilde{p}$ has to become negative, at least one coefficient c_i should be smaller than zero (recall that we assume $K > 0$). In the standard inflationary scenarios one has in fact $c_0 < 0$, $c_1 > 0$ and the remaining coefficients vanish. The corresponding Lagrangians describe (after a field redefinition) a canonical scalar field with a potential which depends on K. Since during such inflationary stages the energy density is dominated by the potential, we will call this kind of inflation "potential-driven". Different Lagrangian choices may be possible as well. In fact, as we will see, inflationary solutions exist as long as the function $\tilde{p}(X)$ satisfies certain simple conditions at a single point X_*. However, although these local conditions can be easily satisfied, the inclusion of further requirements on the Lagrangian may strongly restrict the global form of the Lagrangian in a non-trivial way.

As a paradigmatic illustration of our last statement let us consider the case of inflation driven by a potential-less k-field. By definition, a Lagrangian has no potential if $\tilde{p}(X=0)=0$. Besides of having no potential, a further natural condition we shall impose is that the Lagrangian should reduce to the canonical one for small values of X, $p(X) \approx X, \quad X \ll 1$. These conditions and its implications can be easily visualized in terms of the function $g(y)$ of (2.21). In order for inflation to proceed, g should become negative, and in order for g to describe a canonical potential-less Lagrangian at small values of X, it should behave as $g(y) \approx 1/y$ for large values of y (see section 2.4). A possible form of such a function is shown in figure 3.1. As clearly seen in that figure, the sign of the slope of g has to change inevitably during a transition from negative pressure to the $1/y$ behavior at large y. But since the energy density of the k-field is $-g_{,y}$, this means that an eventual inflating region (where $g<0$) is separated from the "vacuum" ($y=\infty$) by a region of negative energy densities. Moreover, during the same transition the sign of $g_{,yy}$ also changes, implying that $\tilde{\varepsilon}_{,X}$ also changes its sign. Recall however that during cosmic evolution the sign of $\tilde{\varepsilon}_{,X}$ is conserved, and hence, points where $\tilde{\varepsilon}_{,X}=0$ are non-traversable barriers. Thus, for the class of Lagrangians (2.4), a transition from inflation driven by a k-field to a stage where the k-field behaves as an ordinary potential-less field is not possible. This does not mean of course that a transition is impossible. Below we are going to show that by considering Lagrangians which reduce to (2.4) in certain limits, such a transition may be accomplished.

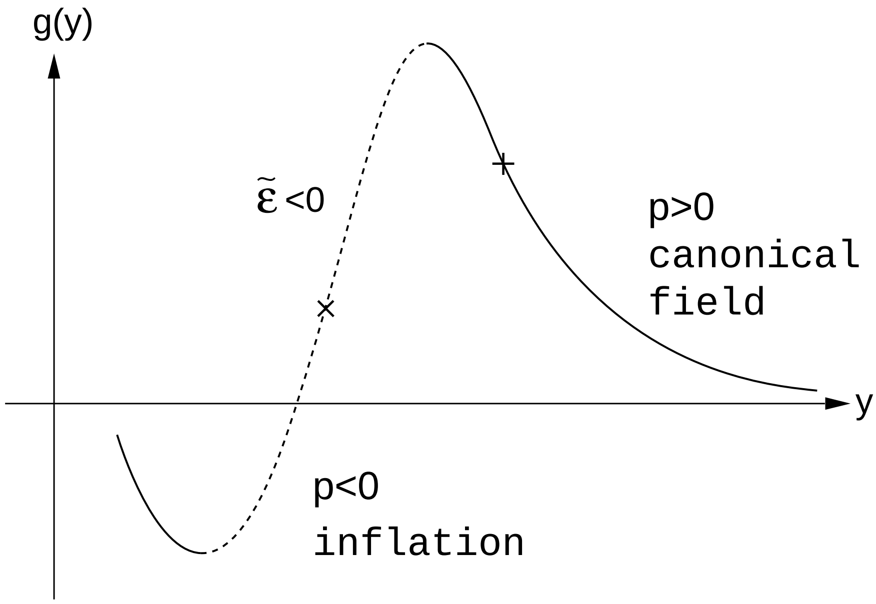

Figure 3.1: A sample function $g(y)$ interpolating between negative pressures and the $1/y$ behavior of a canonical potential-less scalar field at large y. The dashed line corresponds to regions of negative energy density, and the crosses denote unsurmountable barriers.

3.2 Power-law and pole-like k-inflation

In order to look for k-inflationary solutions let us first of all write down the equation of motion of the k-field. At this point, in contrast to our previous discussion, we assume that the energy density of the universe is dominated solely by the k-field. This assumption is self-consistent, since the energy density of the k-field during inflation ($w_k < -1/3$) decreases much slower than the ones of dust ($w_{\rm dust} = 0$) and radiation ($w_{\rm rad} = 1/3$). Thus, setting $\varepsilon_{\rm tot} = \varepsilon_k$ in (2.16) we obtain the "master equation" we are going to use in this chapter,

$$\frac{dX}{dN} = -\frac{\sqrt{8X\tilde{\varepsilon}}}{\tilde{\varepsilon}_{,X}}\left[r(X) + \sigma\frac{K_{,\varphi}}{2K^{3/2}}\right], \tag{3.1}$$

where $r(X)$ is the function defined in (2.17).

We shall find a very important class of inflationary solutions driven by the k-field by studying solutions of (3.1) for which the equation of state w_k is a constant. Because the equation of state of the k-field is only X-dependent, this implies that itself must be constant for those solutions. On the other hand, from the right-hand side of the master equation (3.1) it is evident that non-trivial solutions with $dX/dN = 0$ may exist only if

$$\frac{K_{,\varphi}}{K^{3/2}} = const.$$

Thus, the requirement of a constant equation of state during inflation considerably constraints the possible functions K. As a matter of fact, the most general solution of the previous equation is $K \propto 1/(\varphi - \varphi_*)^2$, and without loss of generality we choose

$$K(\varphi) = \frac{1}{\varphi^2}, \tag{3.2}$$

which yields $K_{,\varphi}/K^{3/2} = -2$. Notice that, we assume the k-field to be positive, $\varphi > 0$. For a negative field, the statements about the sign of $\dot{\varphi}$ below have to be reversed.

Once K is assumed to be given by (3.2), the equation of motion (3.1) implies that there are solutions with constant $X = X_*$ if the condition

$$r(X_*) = \sigma \tag{3.3}$$

is satisfied, where $\sigma = 1$ for a growing field ($d\varphi/dt > 0$) and $\sigma = -1$ for a decreasing field ($d\varphi/dt < 0$). Thus, (3.3) implicitly defines the possible constant values of $X = X_*$ allowed by the equation of motion. The corresponding expansion is inflationary simply if

$$w_k(X_*) < -1/3. \tag{3.4}$$

In principle, conditions (3.3) and (3.4) suffice to guarantee the existence of inflationary solutions. Given $\tilde{p}$ one computes $r(X)$ according to (2.17) and finds the solutions to (3.3). If at any of those solutions (3.4) is satisfied, then the given $\tilde{p}$ admits an inflationary solution. For example, let us consider the Lagrangian of a canonical scalar field, $\tilde{p} = -C + \alpha X$. If and only if $\alpha > 2/9$ the equation $r(X) = 1$ has a (single) solution X_* at which $w(X_*) = 4/9\alpha - 1$. The former restriction on α arises because for such $\tilde{p}$ the equation of state of the k-field can not be bigger than one. Likewise, the requirement of having an inflationary solution translates into $\alpha > 2/3$. After a field redefinition, $\varphi_{\text{old}} = \exp(\varphi_{\text{new}}/\sqrt{\alpha})$, the same Lagrangian reduces to the canonical one $p(\varphi_{\text{new}}, X_{\text{new}}) = -C\exp(-2\varphi_{\text{new}}/\sqrt{\alpha}) + X_{\text{new}}$, and it can be easily verified that our last condition on α is equivalent to the one the exponential potential has to satisfy in order to have an inflationary solution [39]. On the other hand, the equation $r(X) = -1$ has a (single) solution X_* if and only if $\alpha < 0$. The equation of state corresponding to this solution is smaller than -1, $w(X_*) = 4/9\alpha - 1$, and as we shall see, this means that the corresponding solution describes "pole-like" inflation. Observe that since the kinetic term has the "wrong" sign, for negative α the Lagrangian can not be cast in canonical form. This is a consequence of the fact that the equation of state of a canonical field can only take values in the range $-1 \leq w \leq 1$. A scalar field with the "wrong" sign of the kinetic term has been also considered in [13] in the context of late time cosmic acceleration.

In the following it will be more useful to adopt an alternative "constructive" approach. Suppose we want to build a Lagrangian which allows solutions with a constant equation of state given by w_k^* and a constant speed of sound given by c_s^*. It is easy to verify that at any arbitrary point X_* the function $\tilde{p}$ should satisfy

$$\tilde{p}(X_*) = \frac{8X_*}{9}\frac{w_k^*}{(1+w_k^*)^2} \tag{3.5}$$

$$\tilde{p}_{,X}(X_*) = \frac{4}{9}\frac{1}{1+w_k^*} \tag{3.6}$$

$$\tilde{p}_{,XX}(X_*) = \frac{2}{9X_*}\frac{1}{1+w_k^*}\left(\frac{1}{c_s^{*2}} - 1\right). \tag{3.7}$$

For such $\tilde{p}$, $r(X_*) = \text{sgn}(1 + w_k^*)$ and thus, according to (3.3), along an inflationary solution with $w_k < -1$, the field has to decrease, $\sigma = -1$. Because the energy density is (for constant X) proportional to $1/\varphi^2$ it follows that i) that the energy density increases with time—as expected from $w_k < -1$— and ii) that ε_k becomes singular after a finite cosmic time interval. This singularity is also manifest in the functional dependence of the scale factor on t,

$$a(t) \propto (-t)^{\frac{2}{3(1+w_k)}},$$

which has a pole at $t = 0$ (t takes negative values and approaches the pole $t = 0$ as cosmic time elapses). Therefore, inflation with $w_k < -1$ is called pole-like inflation or “super-inflation” [40] . Pole like inflation—in the string conformal frame— also appears in the so-called “pre-big bang” model in string cosmology [29, 37]. In this scenario a pole-like epoch of inflation occurs prior to what is supposed to be the big-bang. One of the main obstacles faced by this scenario is the smooth transition from the inflationary stage to the common radiation dominated Friedmann expansion of the universe. Similarly pole like k-inflation is confronted with an analogous “graceful-exit” problem. After a graceful exit we expect the k-field to smoothly join a non-inflationary branch of the Lagrangian, where $\tilde{\varepsilon}_{,X}$ is positive. On the other hand, it follows from equations (3.5) that for a stable background ($c_s^{*2} > 0$) $\tilde{\varepsilon}_{,X} = 2X\tilde{p}_{,XX} + \tilde{p}_{,X}$ is negative during pole-like inflation. Thus in this case during a graceful exit $\tilde{\varepsilon}_{,X}$ should change sign, what we know can not happen. At this level, pole-like inflation seems unable to account for a successful inflationary scenario.

During an inflationary stage with $w_k > -1$ the field decreases with time, $\sigma = 1$. The scalar factor increases as a power of cosmic time according to

$$a(t) \propto t^{\frac{2}{3(1+w_k)}},$$

and therefore one speaks about power-law inflation. Whereas an exit from pole-like inflation seemed to be impossible, there is no hurdle preventing an exit from power law inflation. In particular, by choosing an appropriate second derivative of $\tilde{p}$ at X_*, the squared speed of sound can be adjusted to *any* arbitrary (positive) value without conflict with $\tilde{\varepsilon}_{,X}(X_*) > 0$. In subsection 3.2.2 we illustrate with an explicit example how a transition from a power-law stage to a radiation-dominated universe may be accomplished.

3.2.1 Attractors and repulsors

Under appropriate conditions there are solutions of the k-field equations of motion for which the equation of state and the kinetic variable X are constant. However, there is no guarantee that these solutions will be ever realized if the initial values of the k-field are not chosen specifically to match them. In this subsection we study how the k-field evolves from generic, non-fine tuned, initial conditions. We shall find that solutions of constant $X = X_*$ can be divided into two groups: attractors and repulsors. The attractor solutions are always reached if the initial value of X is chosen sufficiently close to X_*, whereas repulsor solutions are never reached as long as the initial value of X is not exactly chosen to agree with X_*.

Recall that the inflationary solutions we are discussing only exist for $K \propto 1/\varphi^2$. The motion of the k-field, equation (3.1), is then dictated entirely by the function r, and the possible values of constant $X = X_*$ are determined by the condition $r(X_*) = \pm 1$. Because stability requires $\mathrm{sgn}(\tilde{\varepsilon}_{,X}) = \sigma$, it follows readily by linearizing the equation of motion around X_* that such a solution is an attractor if $\sigma \cdot dr/dX > 0$ and a repulsor if $\sigma \cdot dr/dX < 0$. Since

$$\frac{dr}{dX} = \frac{3}{2}\frac{\tilde{\varepsilon}_{,X}}{\sqrt{8\tilde{\varepsilon}X}}(1-w), \tag{3.8}$$

this means that all solutions with $w(X_*) < 1$, and in particular all inflationary solutions, are attractors. Notice that relation (3.8) also restricts the possible form of $\sigma \cdot r(X)$. It grows for $w < 1$ and decreases for $w > 1$. Figure 3.2 shows two possible forms of $r(X)$ corresponding to two different Lagrangians. The intersection points of the $r(X)$ curve with the lines $r = \pm 1$ denote the solutions of constant equation of state. A Lagrangian which allows power-law inflation ($\tilde{\varepsilon}_{,X} > 0$) may yield several intersections, whereas a one which allows pole-like solutions ($\tilde{\varepsilon}_{,X} < 0$) can yield only one. The global evolution of the k-field proceeds as shown by the arrows in figure 3.2. The arrows point towards the attractors and away from the repulsors. Hence, all intersections where $w_k < 1$ are attractors.

The "no-hair" property of k-inflationary models crucially rests on the existence of these inflationary attractor solutions. For a large set of initial conditions the field is driven to the inflationary solutions discussed above. Once those solutions are reached, the field "forgets" its past, and the properties of the inflationary stage are determined uniquely by the function $\tilde{p}$ at X_*, regardless of initial conditions. Observe however that we have not considered departures from homogeneity in the initial conditions. Although this issue certainly deserves careful consideration, and non-canonical terms may play also a significant role here, such a study would surpass the scope of this work.

3.2.2 Examples

In the previous subsection we have identified the criteria which guarantee the existence of inflationary solutions with constant equation of state. In this subsection we shall illustrate how these criteria can be easily satisfied even in the absence of a potential, and at the same time we shall demonstrate that an exit from such a (power-law) inflationary solution is possible. These examples are based on simple toy models, designed to illustrate the essential features of power-law k-inflation. Certainly, other forms are also possible.

Consider the toy k-field potential-less Lagrangian

$$p(\varphi, X) = \frac{1}{\varphi^2}\left(X + \frac{29}{12} \cdot L(\varphi) \cdot X^2 + \frac{13}{64}X^3\right), \tag{3.9}$$

and let $L(\varphi)$ be any function which becomes $L_- = -1$ for large negative $\varphi - \varphi_{\text{end}}$ and $L_+ = +1$ for large positive $\varphi - \varphi_{\text{end}}$, as shown in figure 3.2.2. When $-\varphi \gg \varphi_{\text{end}} > 0$ the Lagrangian is of the form (2.4). Its coefficients have been chosen in order that in this limit it allows an inflationary solution

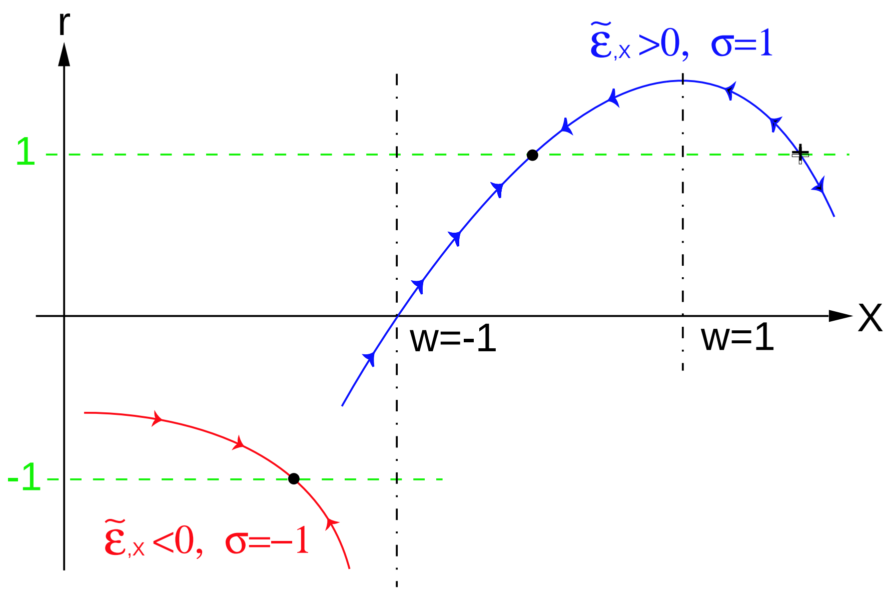

Figure 3.2: Two generic functions $r(X)$. The intersection of the curves with the $r = \pm 1$ lines correspond to solutions of the equations of motion with constant equation of state. Cosmic evolution proceeds along the arrows, and therefore any intersection where $w < 1$ is an attractor.

$X = X_* = 8$ with constant equation of state $w = -2/3$. In the limit $\varphi \gg \varphi_{\rm end} > 0$, the Lagrangian (3.9) is also of the form of (2.4), but in that limit, there are no points where $r(X) = \pm 1$, and $X = 0$ is the only late time attractor of the system.

Consequently, if the initial value of φ is much smaller than $\varphi_{\rm end}$, the field rapidly approaches the power-law inflating attractor solution at $X_* = 8$. At this attractor, the field steadily grows until it reaches the vicinity of $\varphi_{\rm end}$. Then, due to the change in the form of p around $\varphi_{\rm end}$, the power-law attractor solution ceases to exit and the field is forced to approach the remaining late-time attractor of the system, the vacuum $X = 0$. We have solved numerically the equations of motion for a k-field described by (3.9) for different initial conditions. The evolution of the field is depicted figure 3.4 in form of a phase diagram and confirms our previous description of the field motion.

To summarize, our toy model exemplifies that it is possible to construct k-inflationary models which satisfy at once the following requirements: i) have a power-law inflating solution, ii) do not contain any potential term, iii) reduce to the Lagrangian of a canonical scalar field for small values of X and finally iv) allow for a graceful exit from inflation.

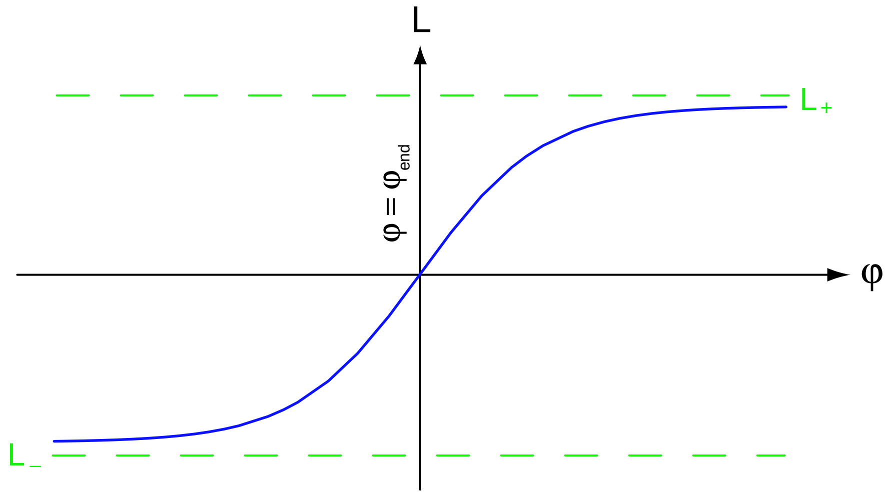

Figure 3.3: Generic form of the function $L(\varphi)$. It approaches L_- for large enough negative values of $\varphi - \varphi_{\text{end}}$ and it approaches L_+ for large enough values of $\varphi - \varphi_{\text{end}}$. The parameter φ_{end} is an arbitrary (positive) field value.

3.3 Slow-roll k-inflation

During power-law and pole-like inflation the equation of state of the k-field is exactly constant. In this section we relax the condition of having a constant equation of state during the inflationary stage. Instead, we look for solutions with a slowly varying w_k close to -1. Our motivations are twofold. On one hand, a stage of accelerated expansion close to de Sitter is important because, generically, its spectrum of density perturbations is nearly scale invariant. On the other hand, a stage of expansion where w is exactly -1 never stops, and hence can not belong to a successful inflationary model. By considering an inflationary stage with a slightly changing de Sitter-like equation of state we cover both issues at the same time.

The equation of state of the k-field depends only on the variable X. Hence, looking at solutions of the equations of motion where w_k is nearly constant is equivalent to looking at solutions with nearly constant X. The equation of motion of the k-field (3.1) can be rewritten as

$$\frac{dr}{dN} = -\frac{3}{2}\left[1 - w(X)\right] \cdot \left[r(X) + \sigma \frac{K_{,\varphi}}{2\, K^{3/2}}\right]. \tag{3.10}$$

Because in order to derive the last equation we multiplied both sides of (3.1) with $1 - w_k(X)$, its apparent solution $w_k(X) = 1$ is an artifact. If X is nearly constant, the X dependent function r should be approximately constant too, and consequently, our desire of a nearly constant equation of state suggests that $dr/dN = (dr/dX)(dX/dN)$ should be negligible.

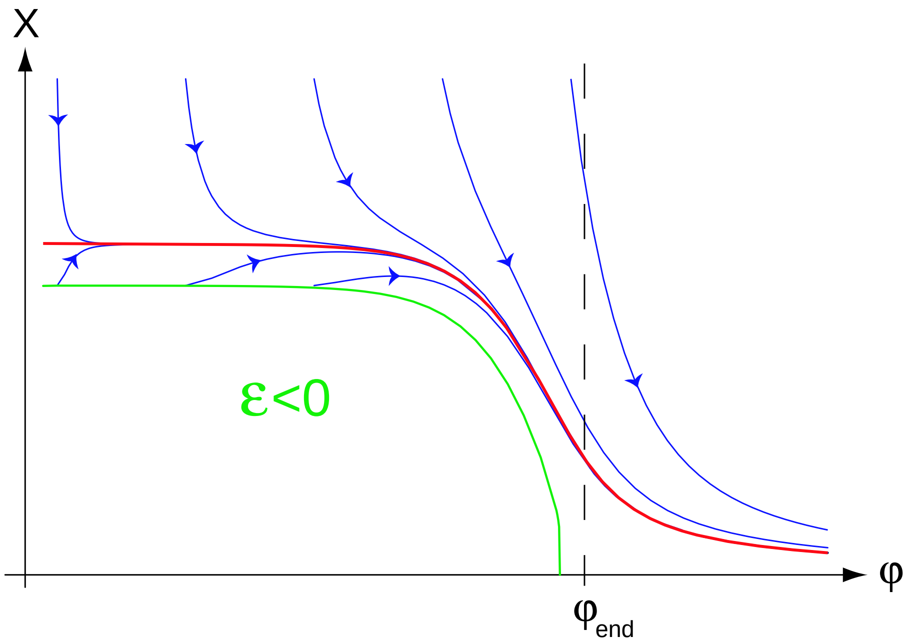

Figure 3.4: Schematic phase diagram of the solutions to the equations of motion derived from (3.9). For $-\varphi \gg \varphi_{\rm end} > 0$ the field rapidly approaches a constant value of X where the equation of state is $-2/3$. Once the field reaches the vicinity of $\varphi_{\rm end}$ it abandons this constant value of X and smoothly approaches the vacuum $X = 0$. Regions with negative energy density are not reachable by flat cosmologies, and are hence excluded.

We denote a regime where dr/dN is negligible as "slow-roll". During slow roll we may neglect the left-hand side of equation (3.10) and hence, the slow-roll approximate solution of the equations of motion $X_{\rm sr}(\varphi)$ is given implicitly by

$$r(X_{\rm sr}) = -\sigma \frac{K_{,\varphi}}{2\,K^{3/2}}. \tag{3.11}$$

Notice that the above slow-roll solution expresses X as a function of φ, and not X as a function of N. At this point we should verify whether our "slow-roll" assumption is consistent. By requiring $dr(X_{\rm sr})/dN$ to be much smaller than any term in the right hand side of (3.10) we obtain the condition

$$\left| \frac{K_{,\varphi\varphi}}{K^{1/2} K_{,\varphi}} - \frac{3}{2} \frac{K_{,\varphi}}{K^{3/2}} \right| \ll \frac{3}{2} \sqrt{\frac{\tilde{\varepsilon}(X_{\rm sr})}{X_{\rm sr}}} \left(1 - w(X_{sr})\right). \tag{3.12}$$

The expression on the right hand side of the inequality is expected to be a number of $\mathcal{O}(1)$, and thus the slow-roll condition (3.12) imposes a restriction on the derivatives of K. Observe that for $K \propto 1/\varphi^2$ equation (3.12) is trivially satisfied, since the left hand side of the inequality exactly vanishes.

This is a consequence of the fact that for such K there are solutions with exactly constant X.

In the following, we could use (3.11) and the inequality (3.12) to study how deviations from the $1/\varphi^2$ dependence may induce power-law inflationary solutions with nearly constant equation of state, and we could also study how these deviations affect the value of the equation of state during inflation. Instead of following that path, for the reason exposed above, we shall *require* that our approximate slow-roll solution describe quasi de Sitter inflation,

$$1 + w(X_{\text{sr}}) \ll 1.$$

Since, $1 + w(X)$ itself is proportional to $r(X)$, it follows automatically from (3.11) that our slow-roll solution corresponds to a quasi de Sitter inflationary stage if the additional "slow-roll" condition

$$\epsilon \equiv \frac{K_{,\varphi}}{K^{3/2}} \ll 1 \tag{3.13}$$

is satisfied. In that case, we may expand the implicit slow-roll solution (3.11) explicitly in powers of the small parameter ϵ. The zeroth order approximation corresponds to the limit $\epsilon \to 0$, where $r(X_{\text{sr}})$ exactly vanishes and $w(X_{\text{sr}})$ is exactly -1. Let us denote by X_* the point where $r(X_*) = 1 + w(X_*) = 0$. Defining δX by the relation $X_{\text{sr}} = X_* + \delta X$ one finds from (3.11) and (3.8) to first order

$$\delta X = -\sigma \frac{\sqrt{2\tilde{\varepsilon} X}}{3\tilde{\varepsilon}_{,X}}\bigg|_{X_*} \cdot \frac{K_{,\varphi}}{K^{3/2}}.$$

The last expressions can be used to compute the value of any X-dependent quantity $Q(X_{\text{sr}})$, as long as X_* is not equal zero. To lowest order $Q^{(0)} = Q(X_*)$ and to first order $Q^{(1)} = Q_{,X}(X_*) \cdot \delta X$. Accordingly, Q is constant to lowest order during quasi de Sitter inflation, whereas the slow-roll condition (3.12) guarantees that the relative change in the first order correction $Q^{(1)}$ during an e-folding is small (this statement will become clearer below). As an example, let us compute the squared speed of sound during slow roll. To lowest order the speed of sound vanishes, and to first order one finds

$$c_s^2 = -\frac{\sigma}{3}\sqrt{\frac{\tilde{\varepsilon}}{2X}}\frac{1}{\tilde{\varepsilon}_{,X}}\bigg|_{X_*} \cdot \frac{K_{,\varphi}}{K^{3/2}}. \tag{3.14}$$

Thus, the stability of the solutions imposes the condition that the function K should decrease as the field evolves, and the slow-roll relation (3.12) implies that the speed of sound is essentially constant on Hubble time scales.

3.3.1 Slow-roll conditions and potential driven inflation

In the previous section we have shown that there are two main criteria which guarantee the existence of slow-roll, quasi de Sitter inflationary solutions.

The first one, (3.12), is the essential part of the slow roll approximation, namely, the neglect of the change of X during an e-folding. The second one,

$$\epsilon \equiv \frac{K_{,\varphi}}{K^{3/2}} \ll 1$$

just fixes the equation of state during slow-roll to a value close to -1. It is interesting to note that the first slow-roll condition (3.12) is practically equivalent to requiring that the relative change in the slow-roll parameter ϵ be small, $d \log \epsilon / dN \ll 1$. Taking into account that $\epsilon \ll 1$ during slow roll inflation the first criterion reduces then to

$$\eta \equiv \frac{K_{,\varphi\varphi}}{K^{1/2} K_{,\varphi}} \ll 1, \tag{3.15}$$

since one expects the right hand side of (3.12) to be a coefficient of $\mathcal{O}(1)$. These flatness criteria— (3.13) and (3.15)—are satisfied by a very wide class of functions. They include

1. Any positive power: $K \propto \varphi^n$, $n > 0$ for $\varphi \gg 1$
2. Any negative power different from -2: $K \propto \varphi^n$, $-2 \neq n < 0$, for $\varphi \ll 1$
3. Any growing exponential: $K \propto \exp(n\varphi)$, $n > 0$ for $\varphi \gg 1$
4. Any function with an asymptotic behavior $K \to const$ and $K_{,\varphi} \to 0$ as $\varphi \to \infty$

In order to better understand the meaning of these slow roll conditions it will be useful to study them in the particular case of a canonical scalar field. The Lagrangian $p = K(-1 + X)$ can be brought into standard form, $p = -V(\varphi_{\text{new}}) + X_{\text{new}}$ by the field redefinition $d\varphi_{\text{new}} = \sqrt{K(\varphi_{\text{old}})}\, d\varphi_{\text{old}}$. The potential V is just the function K expressed in terms of the new variable and hence, in that case, the slow-roll conditions translate into restrictions on the potential. This way, the slow-roll criterion (3.13), which was an expression of the fact that the equation of state should be de-Sitter like, takes the form (dropping the "new"-label)

$$\epsilon = \frac{V_{,\varphi}}{V} \ll 1.$$

The slow-roll criterion (3.15), which was a necessary condition for the slow roll approximation (3.11), needs a modification due to the fact that $X_* = 0$ at the de Sitter point. In the latter case the slow-roll approximation is valid if (dropping again the "new"-label)

$$\frac{V_{,\varphi}}{V} \cdot \eta = \frac{\epsilon}{2} + \frac{V_{,\varphi\varphi}}{V} \ll 1.$$

The reader may have recognized the last two expression as the old flatness conditions of the potential driven scenarios [38]. Our approach shows that there are in a certain sense more universal than originally expected, since they originate from the existence of quasi de Sitter inflationary solutions in a much wider class of Lagrangians.

3.3.2 Attractors and repulsors

Our analysis of slow-roll, quasi de-Sitter inflation has been "local". The existence of an X_* for which $w(X_*) = -1$, along with local conditions (involving derivatives at a single point) on the function K guarantee the existence of de-Sitter like inflationary solutions (at least for an $\mathcal{O}(1)$ number of e-folds). In this subsection we describe the global evolution of the field. By that we mean that we study the evolution of the field away from those inflationary solutions. Since they only exist provided (3.13) and (3.15) are satisfied, for simplicity we restrict ourselves to the lowest order $\epsilon \to 0$, i.e. to a φ-independent K.

For constant K the k-field can be derived directly from the energy conservation equation (2.12), which takes the simple form

$$\tilde{\varepsilon}_{,X} \frac{dX}{dN} = -3\,\tilde{\varepsilon} \cdot (1 + w). \tag{3.16}$$

In the limit $\epsilon \to 0$, the slow-roll solutions we have discussed are just given by $X = X_* = const.$, where X_* is any point where de equation of state is exactly the de Sitter one, $w(X_*) = -1$. In order to study the evolution of the scalar field away from such solutions, it is going to be convenient to consider a parametric plot $(\tilde{\varepsilon}(X), \tilde{p}(X))$ in the $\tilde{p} - \tilde{\varepsilon}$ plane. The concrete form of such a plot depends on the form of the Lagrangian $\tilde{p}(X)$. In figure 3.5 we show the one of a hypothetical generic Lagrangian. The inflationary solutions are located at the intersection of the plot with the $\tilde{\varepsilon} + \tilde{p} = 0$ line and denoted by thick dots. We assume that the Lagrangian has the standard potential-less form for small values of X, and hence, in the vicinity of the origin $\tilde{p} \approx \tilde{\varepsilon}$. Observe that during the transition from the origin to the inflationary solutions, the curve goes through negative energy densities, as pointed out in section (3.1). Nevertheless in flat cosmologies the total energy density cannot become negative (2.10), and hence those regions are excluded (shaded). The arrows on the $(\tilde{p}, \tilde{\varepsilon})$ curve, point along the direction of cosmic evolution of the k-field. Thus, the points where $\tilde{\varepsilon}_{,X} = 0$, denoted by a cross, are repulsors. Because the speed of sound has a different sign at both sides of the crosses, there are stable regions with positive squared speed of sound (denoted by "s") and unstable regions where the squared speed of sound are negative (denoted by "u" and dashed). Since regions where $c_s^2 < 0$ are absolutely unstable, the dashed parts of the plot have no physical meaning. Figure 3.5 also shows that all de Sitter inflationary solutions are attractors of the system (thick dots). Among the latter, the k-field vacuum $X = 0$ plays a special role, since the energy density and the pressure of the k-field vanishes at that point.

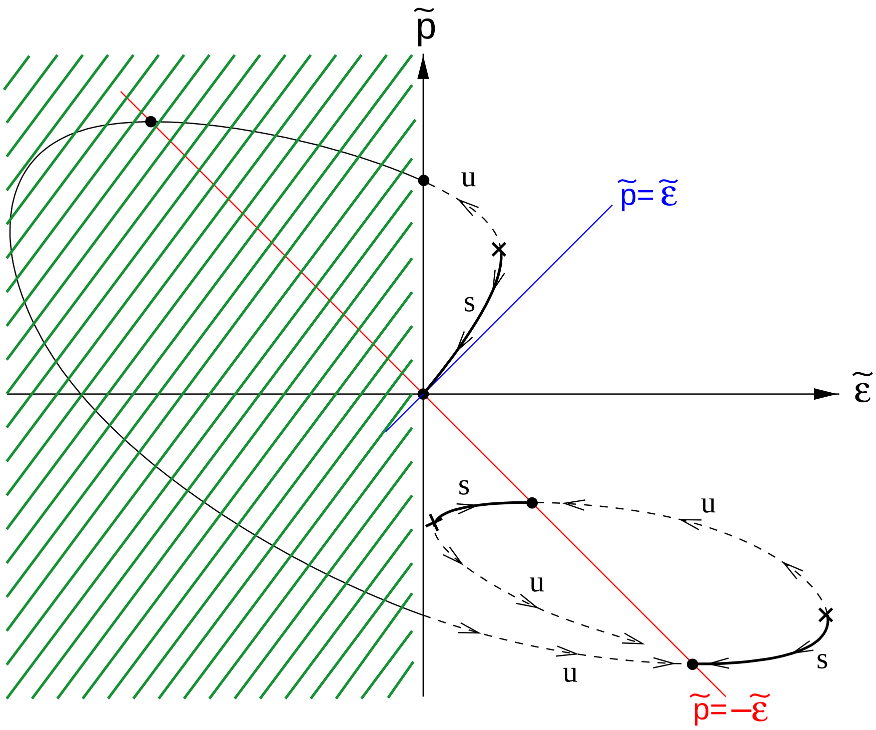

Figure 3.5: Parametric plot $(\tilde{\varepsilon}, \tilde{p})$ for an hypothetical generic Lagrangian $\tilde{p}(X)$. The cosmological evolution proceeds along the arrows. The shaded region excludes the region of negative energy densities; it is unreachable for flat cosmologies. The intersections of the plot with the lines $\tilde{p} = -\tilde{\varepsilon}$ correspond to de Sitter inflating attractor solutions. Those solutions are approached either from regions with positive squared speed of sound (continuous) or from absolute unstable regions (dashed). The crosses denote repulsors of the field evolution.

Because inflating solutions are attractors, if the k-field starts in an appropriate stable branch of figure 3.5, it will be soon driven to an inflationary solution. Once the field reaches that solution, de Sitter inflation will proceed indefinitely into the future. This is the reason why a stage of purely de Sitter stage is not suitable for an inflationary scenario. Ideally, after certain number of e-folds, inflation should stop and approach the vacuum. In the next section we discuss how the violation of the slow-roll conditions may induce such a transition.

3.3.3 Examples

In order to illustrate our discussion about slow-roll k-inflation let us consider simpler versions of our previous examples of subsection 3.2.2. Our purpose is to show that even the most simple potential-less non-canonical Lagrangian, a quadratic one, can support a stage of inflation under appropriate conditions.

The most general quadratic potential-less k-field Lagrangian is $p(\varphi, X) = L(\varphi)X + M(\varphi)X^2$. By a field-redefinition one can eliminate the φ-dependent coefficient M, and hence, without loss of generality the most general quadratic Lagrangian can be written as

$$p(\varphi, X) = L(\varphi)X + X^2. \tag{3.17}$$

By an additional field redefinition, $d\varphi_{\text{new}} = |L(\varphi)|^{-1/2}\, d\varphi$, p can be also cast in factorized form (2.4), though the form of $\tilde{p}$ depends on the sign of L,

$$p(\varphi_{\text{new}}, X_{\text{new}}) = L^2(\varphi_{\text{new}}) \cdot \left[\text{sgn}(L) \cdot X_{\text{new}} + X_{\text{new}}^2\right].$$

If we were only interested in functions L with a definite sign, both parametrizations would be equivalent. However, an essential feature of L in our examples is that it changes sign at certain φ_{end}— Lagrangians of this kind appear for instance in string theory [21]—, and hence, we shall keep the parameterization (3.17) when solving the equations of motion. Nevertheless, as long as L has a definite sign, the factorized form is still valid and useful. In particular, for $L < 0$ there is a de Sitter point at $X_* = 1/2$. Using $K = L^2$ the slow-roll conditions (3.13) and (3.15) can be reformulated in terms of $L(\varphi)$; surprisingly they have the same form as in terms of K,

$$\frac{L_{,\varphi}}{(-L)^{3/2}} \ll 1, \quad \frac{L_{,\varphi\varphi}}{L_{,\varphi}(-L)^{1/2}} \ll 1. \tag{3.18}$$

Let now L be any function which changes sign at arbitrary φ_{end}; it approaches any negative constant L_- for large enough negative values of $\varphi - \varphi_{\text{end}}$ and it approaches a positive constant L_+ for $\varphi \gg \varphi_{\text{end}}$ (figure 3.2.2). Since for such functions the conditions (3.18) are trivially satisfied for big enough negative values of the field φ, it follows that there exists a quasi de Sitter inflationary solution in that range. The squared speed of sound during inflation can be computed using (3.14),

$$c_s^2 = -\frac{\sigma}{6}\frac{L_{,\varphi}}{(-L)^{3/2}},$$

and therefore stability requires the field to grow with time. Thus, during inflation the field unavoidably approaches φ_{end}, where the slow-roll conditions become violated and inflation ends. Once the field crosses that point, the k-field reaches a region where $L > 0$. In this region there is no de Sitter point, and the linear term of the Lagrangian has the usual canonical form.

Here $X = 0$ is the only late time attractor of the system, and therefore as time continues increasing the k-field approaches the vacuum $X = 0$ where its energy density vanishes.

We have solved the field equations of motion for the Lagrangian (3.17) and a function L of the generic form of figure 3.2.2. The numerical solutions, shown in the schematic phase diagram of figure 3.3.3, nicely illustrate all the features we have discussed previously. Regardless of initial conditions cosmic evolution drives the k-field to the de Sitter-like inflationary solution where $X \approx 1/2$. When the field reaches the vicinity of the origin, inflation ceases and the field smoothly approaches the vacuum $X = 0$. The reader should compare that phase diagram with the one of ordinary potential driven inflation of figure 1.5.

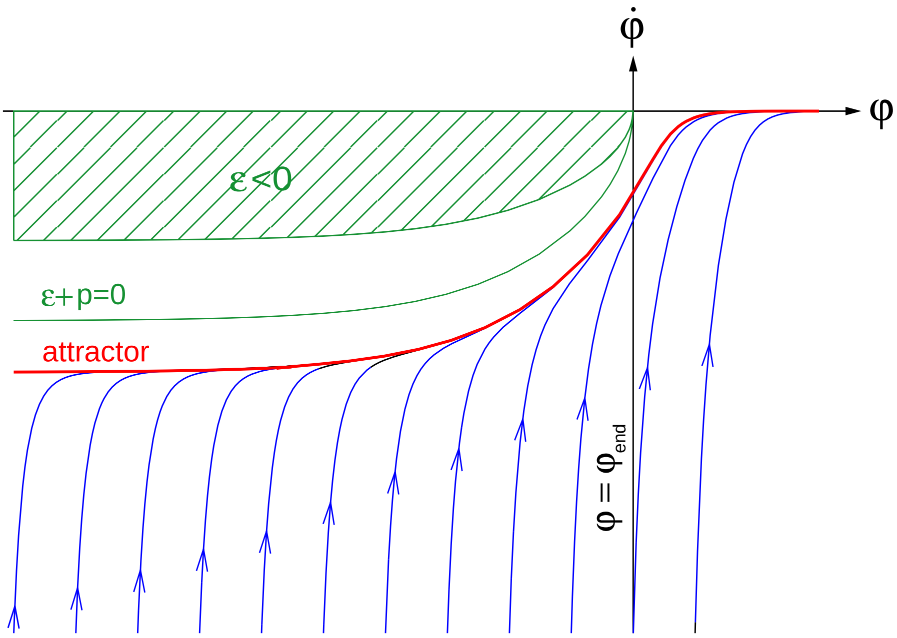

Figure 3.6: Schematic phase diagram of the solutions to the equations of motion derived from (3.17). For $-\varphi \gg \varphi_{\rm end} > 0$ the field rapidly approaches a nearly constant value of X where the equation of state is de-Sitter like. Once the field reaches the vicinity of $\varphi_{\rm end}$ it abandons this value and smoothly approaches the vacuum $X = 0$.

3.4 Power spectra

One of the most important aspects of any inflationary scenario is its predicted primordial spectrum of metric perturbations. These metric perturbations can be directly related to the CMB temperature fluctuations, and thus by measuring these fluctuations, one can discard and distinguish between different inflationary models.

The spectrum of metric perturbations produced during k-inflation was computed in [26] both for slow-roll and power-law models. Here we shall concentrate on slow-roll models, although many of the discussed properties also apply to power-law k-inflation. The spectrum of scalar perturbations is conveniently expressed in terms of the Bardeen variable ζ, which is proportional to the Newtonian potential Φ in (2.18). The latter is at the same time related to the CMB temperature fluctuations on large angular scales through $\delta T/T \approx \Phi/3$ [42]. The power spectrum $\mathcal{P}_k$ of the variable ζ is by definition

$$\mathcal{P}_k^\zeta \equiv \frac{1}{2\pi^2}|\zeta(k)|^2\, k^3,$$

where $\zeta(\vec{k})$ is the Fourier transform of the function $\zeta(\vec{x})$. According to [26] this spectrum is given, for slow roll models, by

$$\mathcal{P}_k^\zeta = \frac{1}{4\pi^2}\frac{\varepsilon}{c_s \cdot (1+w)}\bigg|_s,$$

where $|_s$ means that for given comoving mode k, the corresponding function should be evaluated at "sound-horizon crossing", $c_s k = aH$. The last expression reduces to the standard result for the usual potential-driven slow-roll models, where $c_s = 1$. Generically however, in slow-roll k-inflation the speed of sound is small, and hence, this difference leads to slightly different results. Using equations (3.14) and (3.11) this spectrum can be expressed completely in terms of K. Assuming that the X_* dependent-factors are of $\mathcal{O}(1)$ one gets

$$\mathcal{P}_k^\zeta \approx \frac{K^4}{K_{,\varphi}^2}\bigg|_s.$$

The previous formula gives already a hint on the value of K during inflation. We know that temperature anisotropies of the CMB have typical amplitude

$$\frac{\delta T}{T} \approx \sqrt{P_k^\zeta} \approx 10^{-5}, \tag{3.19}$$

and hence, during inflation $\varphi \cdot K \approx 10^{-5}$. Besides of the amplitude of the spectrum, there are also constraints on its slope. The (scalar) spectral index n_S is defined as

$$n_S - 1 = \frac{d\ln \mathcal{P}^\zeta}{d\ln k}.$$

Its value on large scales has been determined by the COBE experiment to be $n_S \approx 1.2 \pm 0.3$ [6]. A spectrum with $n_S = 1$ does not depend on k, and is called a scale invariant "Harrison-Zeldovich" spectrum. As in the common potential driven models, the spectral index can be expressed in terms of the slow-roll parameters (3.13) and (3.15),

$$n_S - 1 \propto 4\epsilon - 2\eta.$$

Because the slow roll parameters are small, slow-roll models (both potential and kinetic driven) predict a nearly scale invariant spectrum, as anticipated previously.

During k-inflation, a background of gravitational waves is also generated from vacuum tensor metric fluctuations. Its spectrum $\mathcal{P}_T = d\ln\mathcal{P}_T/d\ln k$ agrees with the one of the usual potential driven inflationary scenarios, since the gravitational sectors are the same in both cases. For our purposes the most important fact is that the ratio of the spectrum of scalar perturbations to tensor perturbations is constrained through the so-called "consistency relation"

$$\frac{\mathcal{P}^h}{\mathcal{P}^\zeta} = -8c_s \cdot n_T.$$

In the usual potential driven models $c_s = 1$. Thus, by testing whether $\mathcal{P}^h/\mathcal{P}^\zeta = -8n_T$ one can phenomenologically distinguish between k-inflation and the conventional one.

3.5 Causality, exit and reheating

Among all the different criteria a successful inflationary scenario should satisfy, we have only addressed the no-hair property, the spectrum of density perturbations and, partially, the exit from a k-inflationary stage. Besides, k-inflation should also explain why the observable universe seems to be flat and homogeneous, and how the radiation in contains was produced. In this section we discuss both remaining issues.

k-Inflation proceeds as long as the function K satisfies certain conditions. If $K \propto 1/\varphi^2$, the universe inflates with a constant equation of state, whereas if K satisfies the slow-roll conditions (3.13) and (3.15), the universe undergoes a stage of quasi de Sitter inflation. Therefore, if any of those conditions is violated around certain field φ_{end}, inflation stops when the field reaches the vicinity of that value. During both types of inflation the value of X can be regarded as constant, and hence, the number of e-foldings of inflation can be easily computed from (2.3) by recalling the definition of N,

$$\Delta N \approx \sigma\sqrt{\frac{\tilde{\varepsilon}(X_*)}{2X_*}}\int_{\varphi_{\text{ini}}}^{\varphi_{\text{end}}}\sqrt{K(\varphi)}, \tag{3.20}$$

where φ_{ini} is the value of the k-field at the beginning of inflation.

It is of utmost importance that inflation lasts long enough. In fact, in order to solve the homogeneity problem, an initial horizon-sized patch should be blown up into a region that at least contains today's observable universe. Since the universe has expanded by a factor $T_{\rm rh}/T_0$ since the end of inflation, where $T_{\rm rh}$ is the reheating temperature and $T_0 \approx 3K$, causality requires

$$\frac{1}{H_i}e^{\Delta N}\frac{T_{\rm rh}}{T_0} > \frac{1}{H_0}.$$

Below we argue that in the simplest k-inflationary reheating scenario $T_{\rm rh} \approx 10^{-2}H_{\rm rh}$. Assuming that the Hubble parameter is nearly constant during inflation and because today's horizon is $1/H_0 \approx 10^{28}$ cm., we find that the necessary amount of inflation is

$$\Delta N > 67. \tag{3.21}$$

The same calculation implies that a length scale which has today the size of the observable universe crossed the horizon around 67 e-foldings before the end of inflation. Although inflation may have lasted longer than just 67 e-folds, the modes which crossed the horizon before have lengthscales which are bigger than today's observable universe, and hence are not relevant from an observational point of view. Therefore we shall call the moment of time around 67 e-folds before the end of inflation the "beginning" of inflation.

As an example of how causality constrains the possible k-Lagrangians, let us consider any power-law function $K = \varphi^{2n}$ for $n > 0$. These functions satisfy the slow-roll criteria (3.13) and (3.15) roughly for $\varphi > \varphi_{\rm end} \equiv 1$. In order for the squared speed of sound to be positive during inflation, the field has to decrease (see equation (3.14)) and hence while inflation proceeds the k-field steadily approaches the field value $\varphi_{\rm end}$ where inflation ends. By integrating (3.20) and recalling our definition of the beginning of inflation one finds $K(\varphi_{\rm ini})^{1/2}\varphi_{\rm ini} \approx 67$, where $\varphi_{\rm ini}$ is the value of the field at the beginning of inflation. Combining that equation with the restriction which follows from the amplitude of scalar perturbations (3.19) one can finally solve for the initial value of the field and the initial value of K,

$$K(\varphi_{\rm ini}) \approx 10^{-12} \div 10^{-14}, \quad \varphi_{\rm ini} \approx 10^7 \div 10^9.$$

This example shows that, as in the usual potential driven models, k-inflation generically needs the presence of small parameters, which can be traced back to the smallness of the CMB temperature fluctuations.

Observe that in the previous example the end of inflation is "natural". For the given choice of K, the field approaches the point where the slow-roll conditions are violated and inflation ends. On the other hand, for power-law k-inflation, $K \propto 1/\varphi^2$, inflation never ends and thus the form of the Lagrangian has to change in order for an exit to be possible (see section 3.2.2). The situation is analogous to the one in in conventional potential driven scenarios. Whereas for chaotic power-law potentials inflation ends when the field reaches the vicinity of the potential bottom, for exponential potentials inflation never ends and a change in the form of the potential is needed to force an exit.

The evolution of the k-field after the end of inflation determines the way the universe is reheated. In the following we shall assume that the Lagrangian is such that after inflation the k-field is located around the vicinity of the vacuum $X = 0$, where $\tilde{p}(X) \approx X$ (as in the previous examples in subsections 3.2.2 and 3.3.3). In that case, the equation of state of the k-field is given by $w_k \approx 1$, and hence, the energy-density of the k-field can be easily computed from (2.7) to behave as $\varepsilon_k \propto 1/a^6$. In the conventional potential driven inflationary scenarios, the coupling of the inflaton to other fields [36] produces the radiation needed to reheat the universe. In our models (2.1) the k-field is coupled to the remaining matter fields only gravitationally, and hence, the heating of the universe can only proceed by gravitational particle production [7, 22]. The energy density of particles produced during the transition from de Sitter to a stage of power law expansion was computed in reference [17]. Any scalar particle species whose mass m is much smaller that the Hubble parameter at the end of inflation H_{end} is produced with an energy density given by

$$\varepsilon^{\text{rad}}_{\text{end}} \approx 10^{-2} H^4_{\text{end}} \frac{a^4_{\text{end}}}{a^4}.$$

The last formula can be heuristically understood as stating that one massless particle of energy $\approx H_{\text{end}}$ is produced per Hubble volume at the end of inflation. Moreover, since these particles are massless, their energy density subsequently drops as $1/a^4$. Therefore, because the energy density of the k-field drops as $\varepsilon_k \propto 1/a^6$, the produced particles soon dominate the energy density of the universe, and a smooth transition to a radiation dominated universe occurs.

Gravitational particle production is, in a certain sense, “universal”; any massless scalar particle will be produced at the end at inflation. Hence, it has been objected [19] that in such a way cosmologically “dangerous” moduli particles (like the dilaton of string theory), may be overproduced. In fact, the abundances of these particles are so severely constrained by nucleosynthesis and the observed matter fraction that gravitational particle production would certainly violate those limits. However, up to now the constraints remain speculative. The parameters which enter the different constrains (masses and couplings of these particles) are unknown and vary significantly depending on the model. Hence, those objections are not based on firmly enough ground yet to have any implications for our reheating scenario.

Chapter 4

k-Essence

As we mentioned in the introduction, a new key challenge for theoretical cosmology is to address the cosmic coincidence problem: why does the dark energy component that seems to drive cosmic acceleration have a tiny energy density and why does cosmic acceleration begin at such a late stage in the evolution of the universe. If dark energy is a cosmological constant, its value has to be extraordinarily fine tuned in order to match the present observations. Quintessence models [50, 23, 14] were hoped to alleviate the cosmic coincidence problem, but as we will show using their k-field analogue, kinetically driven quintessence, they are not essentially different from a cosmological constant. Thus, the reason for cosmic coincidence both dark energy forms suggest is either pure-coincidence or the anthropic principle. The purpose of introducing k-essence is to provide a dynamical explanation which does not require fine-tuning of initial conditions or coupling parameters and which is decidedly non-anthropic. In this scenario, cosmic acceleration and human evolution are related because both phenomena are linked to the onset of matter-domination. The k-essence component has the property that it only behaves as a negative pressure component after matter-radiation equality, so that it can only overtake the matter density and induce cosmic acceleration after the matter has dominated the universe for some period, at about the present epoch. And, of course, human evolution is linked to matter-domination because the formation of planets, stars, galaxies and large-scale structure only occurs during this period.

The chapter is organized as follows: In section 4.1 we illustrate the way in which the onset of dark energy domination is contained in the parameters of certain quintessence models. In section 4.2 we explain the basic idea of k-essence and how it differs from ordinary quintessence. The behavior of k-essence is essentially determined by the properties of attractor solutions to the equations of motion, and hence we also summarize the equations of motion presented already in chapter 2. In section 4.3, we classify the possible attractor solutions for k-essence. In some cases, the attractor solution causes

k-essence to mimic the equation of state of the matter energy density; we refer to this as a *tracker* solution. In other cases, k-essence mimics a cosmological constant, quintessence or dust without depending on the presence of any additional cosmic energy density. In section 4.4, we show how these principles can be used to control how k-essence travels through a series of attractor solutions as the universe evolves beginning from general initial conditions. In particular, we show how k-essence can transform automatically into an effective cosmological constant at the onset of matter-domination, as is desired to explain naturally the present-day cosmic acceleration. In section 4.5, we show how to utilize these concepts to design model Lagrangians. We explore two illustrative examples. In one case, the future evolution of k-essence causes the universe to accelerate forever. In the other case, k-essence ultimately approaches an equation of state corresponding to pressureless dust, and the universe returns to a decelerating phase.

4.1 Fine tuning in quintessence models

The reason why a cosmological constant has to be extremely fine tuned to a tiny value in order to accommodate late time cosmic acceleration is that its energy density is constant. Hence, a possible way to circumvent this fine tuning could consist in introducing a dynamical component which may drive cosmic acceleration. This is the idea behind quintessence models [50, 23, 14]. An attractive feature of some of these models is the existence of attractor solutions, along which the equation of state of quintessence follows the equation of state of the dominant component of the universe [53]. During these "tracking" stages the equation of state of quintessence is a constant dictated by the dominant matter equation of state. In these scenarios, quintessence is subdominant in the past, but since its energy density decreases slower than the matter one it eventually dominates sooner or later. As quintessence starts dominating, its equation of state approaches -1, and a late stage of cosmic acceleration begins.

We have already studied the connection between accelerated expansion (inflation) and scalar fields. In particular, we have seen that potential driven inflation is a particular example in the broader context of k-inflation. Hence, it is natural to study up to what extent quintessence models can be recovered from a more general class of k-field models (related work has been done independently by the authors of [15]). Therefore, using the properties of the quintessence models discussed above as guidelines, let us look for solutions of the k-field equations of motion along which w_k is a constant. We shall assume that the total energy density is dominated by a matter component with constant equation of state w_m. In that case, H can be directly computed, from (1.4) and (2.14), to be $H = 2/[3(1+w_m)t]$. For constant X (constant w_k) the equation of motion of the k-field takes the simple form $d \log K =$

$-3(1+w_k)dN$, and hence, using $dN = (H/\dot{\varphi})\, d\varphi$ we find substituting the known behavior of the Hubble parameter that

$$\frac{d \log K}{d\varphi} = -\frac{2(1+w_k)}{(1+w_m)\cdot t \cdot \dot{\varphi}} = -2\frac{1+w_k}{1+w_m}\frac{1}{\varphi - \varphi_*},$$

since $\dot{\varphi} = \sigma\sqrt{2X}$ is constant. Setting for simplicity $\varphi_* = 0$ and integrating the above equation we obtain $K(\varphi) = \lambda^2/\varphi^{2n}$, where λ^2 is the integration constant, and where

$$n = \frac{1+w_k}{1+w_m}. \tag{4.1}$$

For given background and k-essence equations of state, the last formula determines the function K which guarantees the existence of solutions with constant w_k. Alternatively, for given $K \propto 1/\varphi^{2n}$, if there is a point X_m such that $w_k = w_k(X_m)$ satisfies (4.1), $X = X_m = const$ is a solution of the k-field equation of motion in the given matter-dominated background. The relation (4.1) shows that the equation of state of the dominant background component dictates the equation of state of the subdominant k-field component, a behavior known in the context of quintessence models as "tracking". Along these solutions, the ratio of k-essence to total energy ratio is given by,

$$\frac{\varepsilon_k}{\varepsilon_{\text{tot}}} = \lambda^2 \left(\frac{9(1+w_m)^2}{8X_m}\right)^n \tilde{\varepsilon}(X_m)\cdot \varepsilon_{\text{tot}}^{n-1}. \tag{4.2}$$

Obviously our results hold as long as the k-field is subdominant and the Hubble parameter is mainly determined by the matter component. Thus, the tracking solution is valid as long as (4.2) is much smaller than one. Quintessence was introduced to explain late time cosmic acceleration by replacing the cosmological constant by a dynamical component. Therefore at some point it should start to dominate the energy density of the universe, $\varepsilon_k/\varepsilon_{\text{tot}} \approx 1$, and drive cosmic acceleration. This observation restricts the possible values of n to $n < 1$, since the contribution of the k-field should become more important as the universe evolves. Notice that for such values of n, the function K satisfies the slow-roll conditions of subsection 3.3.1. Therefore, as soon as the k-field starts dominating, our considerations about k-inflation apply, and it follows from them that the k-field is attracted to a quasi de Sitter inflationary stage, triggering cosmic acceleration as observed.

The moment of k-field dominance can be estimated for given λ by substituting $\varepsilon_k/\varepsilon_{\text{tot}} \approx 1$ into (4.2). And vice versa, for a known moment of quintessence dominance, (4.2) fixes the value of λ. Since quintessence seems to have started to dominate only recently and $\varepsilon_{\text{tot}}(\text{today}) \approx 10^{-124}$, we find, assuming the remaining parameters to be of $\mathcal{O}(1)$,

$$\lambda^2 \approx 10^{-124\cdot(1-n)}. \tag{4.3}$$

The last formula is the main result of this section. If we set $n = 0$, we are back in the case of a cosmological constant, and the formula exhibits the tremendous fine tuning needed to explain its dominance today. Different choices of n, say $n = 1/2$, do not improve the situation, and only if n is fine tuned to a value close to 1 does λ take "natural" values. On the other hand let us stress that even in that case the formula above shows that quintessence models do not explain why late time cosmic acceleration happens precisely today, since the moment of k-field domination has to be encoded into the values of n and λ.

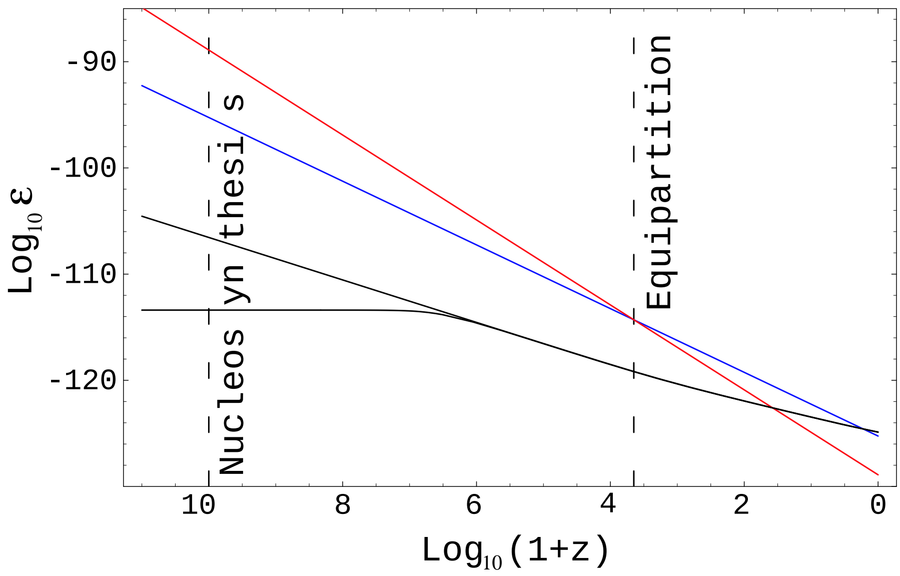

Figure 4.1: Plot of the energy densities of radiation (red), matter (blue) and kinetically driven quintessence (black) versus "time". The energy density of quintessence is plotted for two different sets of initial conditions. Both converge to the tracking attractor, along which the equation of state of quintessence is dictated by the equation of state of matter according to (4.1). Thus, around equipartition, the slope of the quintessence curve slightly changes due to the change of the background total equation of state. Once quintessence starts dominating, the k-field approaches a slow-roll k-inflationary stage, yielding late cosmic acceleration. The moment of dominance is determined by a scaling parameter in the Lagrangian.

Nevertheless, one attractive feature of the solutions we have discussed is that they are attractor solutions, so that the cosmic evolution is insensitive to the initial energy density of the k-field. This feature is apparent in figure 4.1, which shows the result of the numerical integration of the equations of motion of k-essence for different initial conditions. In that particular example the k-field is described by the potential-less Lagrangian

$$p(\varphi, X) = \frac{\lambda^2}{\varphi}(-X + X^2).$$

The results proof that a k-field may accommodate late time cosmic acceleration in exactly the same way as quintessence models, even in the absence of a potential. It is also evident from the structure of the equations that the function that determines the behavior of the solutions is K and not $\tilde{p}$. The role of $\tilde{p}$ is just to allow an equation of state that satisfies (4.1). In that sense we can find what kind of potentials our kinetically driven quintessence models correspond to. By a field redefinition the Lagrangian $p(\varphi, X) = \lambda^2 \varphi^{-2n}(-1 + X)$ can be cast in canonical form, $p = X - V(\varphi)$. The potential has power-law form, $V(\varphi) \propto \varphi^{-2n/(1-n)}$, which not surprisingly is the class of potentials known to yield tracker solutions [53].

4.2 Fundamentals of k-Essence

Our short analysis of kinetically driven quintessence has shown that the value $n = 1$ in (4.1) is somehow special. For this value the equation of state of the k-field exactly matches the one of the matter component, and hence the ratio of k-field to total energy density is a constant during tracking. This behavior is known from quintessence models with an exponential potential. A feature of these models is that the field tracks both radiation (during radiation domination) and matter (during matter domination). However, as long as the field tracks any equation of state, it cannot overtake the matter-density and induce cosmic acceleration. Indeed, for a purely exponential potential, the field never overtakes the matter density and dominates the universe. Hence, this is an unacceptable candidate for the dark energy component.

The distinctive feature of the k-essence models we shall consider is that k-essence only tracks the equation of state of the background during the radiation dominated epoch. A tracking solution during the matter dominated epoch is physically forbidden. Instead, at the onset of matter domination, the k-essence field energy density ε_k drops several orders of magnitude as the field approaches a new attractor solution in which it acts as a cosmological constant with pressure p_k approximately equal to $-\varepsilon_k$. That is, the equation of state is nearly -1. The k-essence energy density catches up and overtakes the matter density, typically several billions of years after matter domination, driving the universe into a period of cosmic acceleration. As it overtakes the energy density of the universe, it begins to approach yet another attractor solution which, depending on details, may correspond to an accelerating universe with $w < -1/3$ or a decelerating or even dust-like solution with $-1/3 < w \leq 0$. In this scenario, we observe cosmic acceleration today because the time for human evolution and the time for k-essence to overtake the matter density are both severals of billions of years due to independent but predictive dynamical reasons. This behavior is summarized in figure (4.2), which shows the cosmic evolution of k-essence compared to the ones of dust and radiation. The reader should compare it with the analogous figures 1.4 and 4.1, which include the energy density of a cosmological constant and quintessence instead.

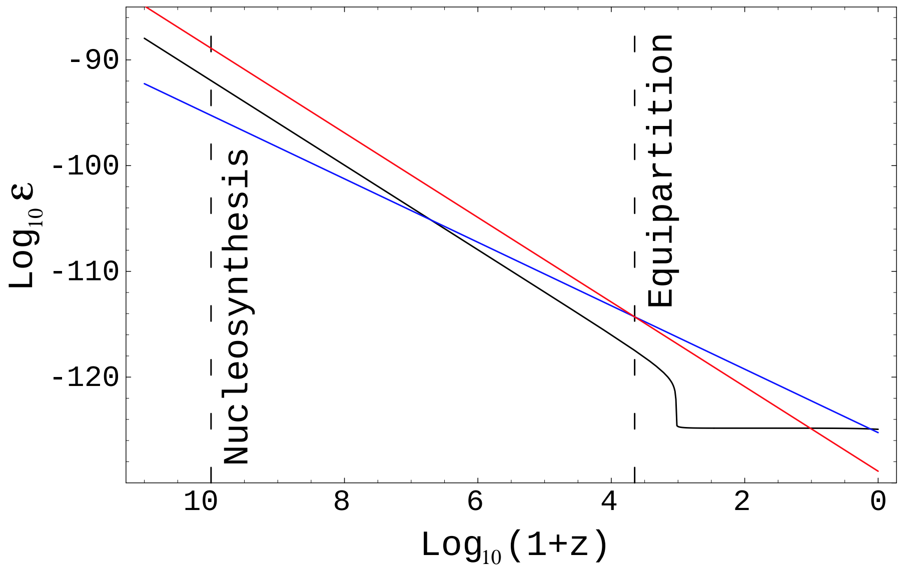

Figure 4.2: Plot of the energy densities of radiation (red), matter (blue) and k-essence (black) versus "time". During radiation domination k-essence behaves as a radiation component, and thus, nucleosynthesis constraints restrict the amount of k-essence present at that time (dashed line at $1+z \approx 10^{10}$). After equipartition (dashed line at $1+z \approx 4 \cdot 10^3$), instead of following the matter track, the energy density of k-essence drops by several orders of magnitude and freezes until it overtakes the energy density of matter.

k-Essence relies on a specific, but broad, class of factorized k-field Lagrangians. In order to study their properties it will be convenient to use the alternative parameterization of the k-field Lagrangian discussed in section 2.4. Recall that in that parameterization $y = 1/\sqrt{X}$ and $\tilde{p} = g(y)/y$. Restrictions on the form of the Lagrangian take then very simple forms. For instance, we shall require positive energy density and positive speed of sound, which translate into the condition that g be a growing convex function respectively,

$$\tilde{\varepsilon} = -g' > 0, \quad \tilde{\varepsilon}_{,X} = \frac{1}{2} y^3 y'' > 0, \tag{4.4}$$

where, as in the rest of this chapter, we use a prime to denote a derivative with respect to y. The k-field equation of motion in terms of y has a similar form to the one in terms of X and reads (we write it down again here for completeness)

$$\frac{dy}{dN} = \frac{3}{2}\frac{w_k(y)-1}{r'(y)}\left[r(y) + \sigma\frac{K_{,\varphi}}{2\,K^{3/2}}\sqrt{\frac{\varepsilon_k}{\varepsilon_{\mathrm{tot}}}}\,\right], \tag{4.5}$$

where the function $r(y)$ is given by

$$r(y) \equiv \sqrt{-\frac{9}{8}\frac{dg}{dy}}\,y\,(1+w_k) = \frac{3}{2\sqrt{2}}\frac{g-yg'}{\sqrt{-g'}}. \tag{4.6}$$

and where the total energy density consists of the energy densities of the k-field and the matter component (dust and radiation). Note that we assume that, because of a yet unknown mechanism, the cosmological constant exactly vanishes.

4.3 Classification of Tracker and Attractors

The attractor solutions for k-essence can be divided into two classes. In one class, k-essence mimics the equation of state of the matter-radiation component in the universe. We refer to these as *trackers* because the cosmic evolution of k-essence follows the track of another energy component. The second class of attractors consists of cases where k-essence is drawn towards an equation of state which is different from matter or radiation. These attractors are important in the limits where k-essence is either a negligibly small or an overwhelming large fraction of the total energy density. The types of attractors available at any given moment in cosmic history depend on whether the universe is radiation or matter dominated. For all types of attractors, there is an associated basin of attraction, a set of initial conditions which evolve towards the attractor.

In the presence of a matter component (dust or radiation) with constant equation of state w_m, equation (4.5) can have tracking solutions for which the k-essence equation of state equals w_m. To reveal when it can happen and to find these solutions explicitly we just need to note that if such solutions exist, they have to be generically of the form $y\,(N) = y_m = \mathrm{const}$, where y_m satisfies the equation

$$w_k\,(y_m) \equiv -\left.\frac{g}{yg'}\right|_{y=y_m} = w_m. \tag{4.7}$$

Substituting this ansatz into equation (4.5) and noting that the ratio $\varepsilon_k/\varepsilon_{\mathrm{tot}}$ should stay constant during the tracking stage, we see that $y\,(N) = y_m$ can be a solution of equation (4.5), only if $K\,(\varphi) \propto 1/\varphi^2$ (we have already observed this fact in our discussions of k-inflation and kinetically driven quintessence). Therefore, we consider from now on only scalar fields with Lagrangian

$$p = \frac{g(y)}{\varphi^2 y}. \tag{4.8}$$

It is worth reminding that this kind of dependence on a scalar field occurs in the string tree-level effective action when expressed in the Einstein frame. In this case, equation (4.5) simplifies to

$$\frac{dy}{dN} = \frac{3}{2}\frac{(w_k(y) - 1)}{r'(y)}\left[r(y) - \sqrt{\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}}\right], \tag{4.9}$$

where we restrict ourselves to the most interesting case of positive σ on the branch of positive φ. To close the system of equations for the two unknown variables y and $\varepsilon_k/\varepsilon_{\text{tot}}$, we use the equation

$$\frac{d(\varepsilon_k/\varepsilon_{\text{tot}})}{dN} = 3\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}\left(1 - \frac{\varepsilon_k}{\varepsilon_{\text{tot}}}\right)(w_m - w_k(y)), \tag{4.10}$$

which immediately follows from equation (2.12). If y_m is a solution of equation (4.7), then $y(N) = y_m = const$, satisfies equations (4.9) and (4.10), provided

$$r^2(y_m) = \left(\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}\right)_m < 1, \tag{4.11}$$

where the inequality is simply the physical constraint that $\varepsilon_k < \varepsilon_{\text{tot}}$ (assuming positive energy densities ε_k and ε_m). If $r(y_m) > 1$, a tracker solution $y(N) = y_m$ is physically forbidden.

4.3.1 When are trackers attractors?

To find out when trackers are stable solutions with a non-trivial basin of attraction, we study the behavior of small deviations from the tracker solution. Substituting $y(N) = y_m + \delta y$ and $\varepsilon_k/\varepsilon_{\text{tot}}(N) = (\varepsilon_k/\varepsilon_{\text{tot}})_m + \delta(\varepsilon_k/\varepsilon_{\text{tot}})$ into equations (4.9) and (4.10) and linearizing, we obtain

$$\frac{d\delta y}{dN} = \frac{3}{2}\frac{(w_k(y_m) - 1)}{r'_m}\left[r'_m \delta y - \frac{\delta(\varepsilon_k/\varepsilon_{\text{tot}})}{2r_m}\right], \tag{4.12}$$

$$\frac{d\delta(\varepsilon_k/\varepsilon_{\text{tot}})}{dN} = -3r_m^2\left(1 - r_m^2\right) w'_k(y_m)\,\delta y, \tag{4.13}$$

where the index "m" denotes evaluation of the appropriate quantities at the tracker point y_m and $(\varepsilon_k/\varepsilon_{\text{tot}})_m$ has been replaced by $r^2(y_m)$ according to equation (4.11). Differentiating equation (4.12) with respect to N and using equation (4.13), one obtains the following closed equation for δy:

$$\frac{d^2\delta y}{dN^2} + \frac{3}{2}(1 - w_m)\frac{d\delta y}{dN} + \frac{9}{2}(1 - r_m^2)(1 + w_m)(c_s^2 - w_m)\delta y = 0. \tag{4.14}$$

Here c_s^2 is the squared "speed of sound" of k- essence at the tracker point and we took into account that $w_k(y_m) = w_m$. equation (4.14) is a second order differential equation with constant coefficients and has two exponential solutions. It is easy to see that for $|w_m| < 1$ both solutions decay if

$$c_s^2 > w_m. \tag{4.15}$$

Therefore, since $c_s^2 = (g - g'y)/g''y^2$, any tracker can be easily made an attractor by arranging a small second derivative of g at the tracker point.

As important examples, let us consider the two most interesting cases, namely, trackers in the presence of radiation (labeled "r" in the equations below) and cold matter (labeled "D" for "dust").

4.3.2 Radiation trackers

For radiation trackers, $w_m \equiv w_r = 1/3$ and equation (4.7), which defines the location of the radiation trackers ($y_m \equiv y_r$), reduces to

$$y_r g'(y_r) = -3g(y_r).$$

The ratio of the energy densities is given by

$$\left(\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}\right)_r = r^2(y_r) \equiv -2g'(y_r)y_r^2 \tag{4.16}$$

and radiation trackers exist only if at the points y_r satisfying equation (4.3.2), $r^2(y_r) < 1$. These trackers are stable attractors only if $g''(y_r) < -4g'(y_r)/y_r$. Radiation trackers are always located in the region where $g > 0$ (positive pressure), corresponding to $y < y_D$ in figure 4.3. For a given $g(y)$, there can be more than one radiation tracker. For each of them, the geometrical way of finding the value of y corresponding to the tracker is given in figure 4.3. These trackers can have different values of $r^2(y_r) = (\varepsilon_k/\varepsilon_{\text{tot}})_r$. Numerically, a likely range for $r^2(y)$ is $10^{-1} \div 10^{-2}$. This is also the range we wish to have in order that cosmic acceleration begin at roughly the present epoch. We label the radiation tracker with the desired value of $r^2(y_R)$ as **R**, and a second possible radiation tracker with a different value of $r^2(y_r)$ (the one closest to y_D) as **r**(?) in figure 4.3. If $r^2(y_r)$ is much smaller than 10^{-2}, the energy density falls so much at the onset of matter-domination (before it freezes at a constant value) that it would not yet have overtaken the matter density today. If $r^2(y_r)$ is much greater than 10^{-1}, then the contribution of k-essence to the total energy density would change the expansion rate in the early universe and adversely affect the predictions of primordial nucleosynthesis. The current constraints on $r^2(y_r)$ from nucleosynthesis vary from 4 per cent [55] to 20 per cent [44], depending on how the observations are weighted.

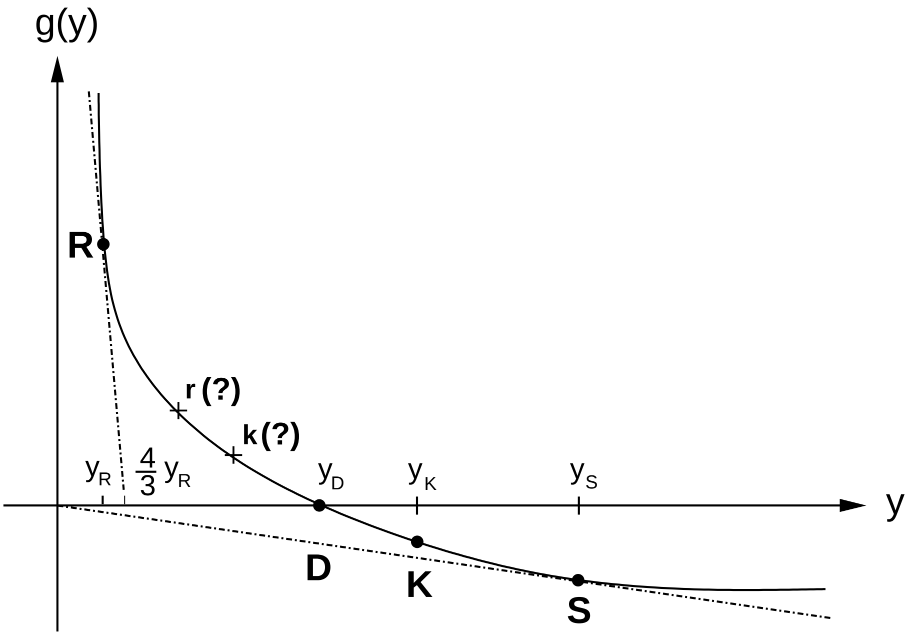

Figure 4.3: A sample function $g(y)$. Boldface letters denote the corresponding attractors; their positions are given on the y-axis. The tangent to the curve at a radiation tracker, such as **R**, goes through $4y_R/3$, whereas the tangent to the curve at the de Sitter point **S** goes through the origin.

4.3.3 Dust trackers

k-Essence can also track dust ($w_D = 0$) in the (cold) matter dominated universe. Since the pressure is proportional to $g(y)$ and is zero for dust, it must be that

$$g\left(y_D\right) = 0 \tag{4.17}$$

at the dust attractor point, $y = y_D$. An additional condition for the existence of the dust tracker is that $r\left(y_D\right) < 1$ (see discussion following equation (4.11)). In this case the ratio of energy densities at the dust tracker is given by

$$\left(\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}\right)_D = r^2\left(y_D\right) = -\frac{9}{8}\, g'(y_D)\, y_D^2. \tag{4.18}$$

If a dust tracker exists then it is always an attractor, since the stability condition equation (4.15) just means here that the "speed of sound" of $k-$essence should be positive. Note, that for the monotonically decreasing convex functions g under consideration only a maximum of one dust attractor can exist (see figure 4.3) since g has only one zero. It is very important to point out that one can easily avoid a dust tracker by considering functions g such that $r^2\left(y_D\right) = -\frac{9}{8}\, g'(y_D)\, y_D^2 > 1$ at y_D.

4.3.4 De Sitter Attractors

We have noted that k-essence can have attractor solutions which are not trackers in that they do not mimic dust or radiation. These attractor solutions play an important role in two extreme cases, namely, when the energy density of matter or radiation is either much bigger or much smaller than the energy density of k-essence. In this subsection, we study the case when the background is dominated by matter-radiation and k-essence is an insignificant component, $\varepsilon_k \ll \varepsilon_m$. In this case, if $g(y)$ satisfies some simple properties, k-essence has an attractor solution in which it behaves like a cosmological constant ($w_k \to -1$). We refer to this solution as the de Sitter attractor (labeled "**S**").

Our purpose is to construct models in which k-essence has a positive pressure, radiation tracker solution (**R**) during the radiation-dominated phase and approaches a state with negative pressure shortly after the onset of the matter dominated phase. At the very least, it is necessary that $g(y)$ be positive for some range of y and negative for another range since the pressure is proportional to $g(y)$. This simple condition is generically sufficient to produce a de Sitter attractor solution: Since g' must be negative (the positive energy condition, equation (4.4)), it follows that g must have a unique zero, y_D, the only dust attractor possible. Furthermore, $g(y)$ is positive for $y < y_D$, a range which must include the radiation tracker, $y = y_R$. For $y > y_D$, the pressure ($\propto g$) and, correspondingly, $w_k = -g/yg'$ are negative. From this observation, combined with the stability condition ($g'' > 0$), it follows that the derivative of $r(y)$ (see definition (4.6))

$$r' = \frac{3}{4\sqrt{2}} \frac{g''y}{\sqrt{-g'}} (w_k - 1) \tag{4.19}$$

must be negative for $y > y_D$. Since $r(y)$ is positive at $y = y_D$ and has a negative derivative for $y > y_D$, generically (provided r' does not approach zero too rapidly) $r(y)$ should vanish at some point $y = y_S > y_D$ and then become negative. As immediately follows from the definition of r, the equation of state of k-essence at $y = y_S$ (point S in figure 4.3) corresponds to a cosmological term: $w_k(y_S) = -1$. Hence, we see that de Sitter attractors exist for a very wide class of $g(y)$ and are a generic feature of k-essence models.

In the absence of matter, $y(N) = y_S = const$ is not a solution of the equations of motion. However, when matter strongly dominates over k-essence ($\varepsilon_k/\varepsilon_{\text{tot}} \ll 1$), there exists a solution in the vicinity of this point. (Formally, in the limit $\varepsilon_k/\varepsilon_{\text{tot}} \to 0$, $y(N) \to y_S$ is an exact solution of equations (4.9) and (4.10).) Setting $w_m = w_k = -1$ in equation (4.12) it can be also verified that this is a stable attractor. For finite, but very small ratio $\varepsilon_k/\varepsilon_{\text{tot}} \ll 1$, the approximate solution, corresponding to $w \approx -1$, is located in the vicinity of y_S and has the form:

$$\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}(N) \propto \exp\left(3\left(1+w_m\right)N\right) \tag{4.20}$$

and

$$y(N) \approx y_S + \frac{2}{\left(3+w_m\right)r'\left(y_S\right)}\left(\frac{\varepsilon_k}{\varepsilon_{\text{tot}}}(N)\right)^{1/2}. \tag{4.21}$$

As shown below, if at any moment of time $\varepsilon_k/\varepsilon_{\text{tot}}$ lies below the basin of attraction of the tracker solutions, k-essence will be driven first to the de Sitter attractor and stay in its vicinity as long as $\varepsilon_k/\varepsilon_{\text{tot}}$ is sufficiently small. We will utilize this property at the transition from the radiation- to the matter-dominated phase.

4.3.5 k-Attractors

Whereas the de Sitter attractors are important when k-essence is an insignificant contribution to the total energy density, the k-attractors arise when k-essence is the dominant energy component. In the absence of matter ($\varepsilon_k/\varepsilon_{\text{tot}} = 1$), the function $y(N) = y_k = const$, where y_k satisfies the equation

$$r(y_k) = 1, \tag{4.22}$$

is a solution of equation (4.9), while equation (4.10) is satisfied identically. This solution describes a power-law expanding universe as in our discussion of power-law k-inflation, section 3.2. The equation of state can be easily obtained from equations (4.6) and (4.22):

$$1 + w_k(y_k) = \frac{2\sqrt{2}}{3}\frac{1}{\sqrt{-g_k' y_k^2}} = const, \tag{4.23}$$

and the scale factor is

$$a \propto t^{\frac{2}{3(1+w_k)}} = t^{\sqrt{-g_k' y_k^2/2}}. \tag{4.24}$$

If $-g_k' y_k^2/2 > 1$ the solution describes power law inflation, which is an attractor of the system provided that $r'(y_k) < 0$. (See section 3.2, in particular figure 3.2, for the analogous condition in terms of the variable X.)

The existence of a k-attractor depends mainly on the form of the function $r(y)$. A k-attractor corresponds to $r(y_k) \to 1$ (*i.e.*, the limit where the energy density is totally dominated by k-essence). In general, if $r(y_*) > 1$ for some y_* and there exists an **S**-attractor ($r(y_S) = 0$), then there must exist a k-attractor somewhere between them, $y_* < y_k < y_S$, simply because $r(y)$ is a continuous function.

In particular, we are interested in the case where there is no dust attractor because $r(y_D) > 1$, and yet there is a de Sitter attractor with $r(y_S) = 0$. In this case, not only must there exist a k-attractor at some $y_D < y_K < y_S$, but we know that it has *negative pressure* (since $g(y_K) < 0$), is *stable* (since $w_k < 1$, see equation (4.19)) and is the *unique* k-attractor with negative pressure (since r' is monotonically decreasing in this $y-$interval).

Note also that this negative-pressure k-attractor only exists if there is no dust tracker solution, that is, $r(y_D) > 1$. If there is a dust tracker, $(r\,(y_D) < 1)$, then, since $r'(y) < 0$ for $y > y_D$, there is no point $y = y_K > y_D$ where $r(y) = 1$ and, hence, there is no k-attractor at $y_D < y < y_S$.

It is possible to have other k-attractors with positive pressure at $y < y_D$ (the closest one to y_D is denoted by $\mathbf{k}(?)$ in figure 4.3), but they will prove to be irrelevant in our scenario.

4.4 Cosmic Evolution and Attractor Solutions

Once all possible attractors for k-essence have been identified, it is easy to understand the evolution of the k-field as a voyage from one attractor solution to another as different phases of cosmic evolution proceed. For both the radiation- and matter-dominated phases, there are several possible configurations of relevant attractor solutions. In this section, we systematically classify the attractor configurations for each phase and their consequences for cosmic evolution.

4.4.1 Radiation-Domination

We assume that $g(y)$ has been chosen so that there exists an attractor solution ($\mathbf{R}$) at $y = y_R$ such that $r^2(y_R) \equiv (\varepsilon_k/\varepsilon_{\rm tot})_R$ is in the range one to ten percent. This energy ratio leads most naturally to a matter-dominated epoch that lasts a few billion years and cosmic acceleration beginning at about the present epoch. Depending on the form of $r^2(y)$, which is determined by $g(y)$ in the Lagrangian, there will be additional attractors during the radiation epoch. Whether y is drawn to the correct attractor y_R depends on initial conditions and the other attractors. Ideally, we want $y = y_R$ to have the largest basin of attraction so that most initial conditions join onto the desired cosmic track. The combination of cosmologically relevant attractors during the radiation-dominated phase can be one of three types:

A_r) **R, S** and **no** other attractors at $y_S > y > y_R$. This occurs only if the function $r\,(y)$ decreases for $y_R < y < y_S$. Conversely, if $r\,(y)$ increases somewhere in the range $y > y_R$ then it inevitably leads to the appearance of an extra k and/or r attractor at $y > y_R$. Let us prove it.

If the function $r\,(y)$ increases within some interval, it means that the derivative $r'\,(y)$ is positive there. On the other hand, as it follows from (4.19), $r'(y)$ is positive only if $w_k > 1$. Since $w_k(y_R) = 1/3$, $w_k(y_S) = -1$ and $w_k(y) > 1$ somewhere in the interval $y_R < y < y_D$, there must be another point $\bar{y}$ within this interval, where $w_k(\bar{y}) = 1/3$. If $r(\bar{y}) < 1$, this point is a radiation tracker different from **R** with a different value of $r^2(y)$. If $r^2(\bar{y}) > 1$, then $\bar{y}$ is not a tracker at all; but, since $r(y_S) = -1$, there must exist a point in the interval $y_S > y_k > \bar{y}$ where $r(y_k) = 1$, which corresponds to a k-attractor. That is, either there is a an extra radiation tracker or there is an extra k-attractor.

For models of type A_r where $r^2(y)$ is monotonically decreasing, a dust tracker solution with $r(y_D) < r(y_R)$ is inevitable and $k-$essence will be attracted immediately to it after matter-radiation equality, a situation we are trying to avoid in order to explain the present-day cosmic acceleration. The model $\tilde{p}(X) = -1 + X$ falls in the above category; with a field redefinition, the action can be recast into the model of a field with canonical kinetic energy rolling down an exponential potential [60, 20], an example which is well-known to track in both the radiation and matter dominated epochs.

B_r) **R, S, K** plus possibly other attractors at $y < y_D$. This situation takes place when there is no dust tracker solution ($r(y_D) > 1$)

C_r) **R, S (no K** attractor**)** and at least one additional attractors **r(?)** or **k(?)**. This case occurs whenever there is a dust tracker solution ($r(y_D) < 1$)) with the property that $r(y_D) > r(y_R)$ or, in other words, $(\varepsilon_k/\varepsilon_{\text{tot}})_D > (\varepsilon_k/\varepsilon_{\text{tot}})_R$. Even though there exists a dust tracker solution, we will show it is nevertheless possible to have a finite period of cosmic acceleration at the present epoch before k-essence reaches the dust tracker solution in the future. For this to occur, the function $r(y)$ must increase somewhere in the interval $y_R < y < y_D$. This is precisely the case considered above (see discussion of case A_r), where we argued that there must be an extra r and/or k-attractor in the interval $y_D < y < y_R$. Furthermore, the attractor closest to y_D must have $r(y_{r/k}) > r(y_D) > r(y_R)$; otherwise, we could find another attractor in the interval $y_{r/k} < y < y_k$, as can be shown by repeating the argument presented under A_r for this interval. If $r(y_{r/k}) > r(y_D) > r(y_R)$, this second tracker has a larger fraction of k-essence.

A phase diagram of the system of equations (4.9)-(4.10) describing the global evolution of the k-field during radiation domination is shown in figures 4.4, 4.5 and 4.6 for each of the cases A_r, B_r and C_r respectively. Phase trajectories cannot cross the lines where $\varepsilon_k/\varepsilon_{\text{tot}}$ is equal to zero or one, and, hence, their tangents are horizontal there. The position of the radiation tracker **R** is fixed by the intersection of the $y = y_R$ line (dashed) and the $r^2(y)$ curve (dotted). If $r^2(y)$ is bigger than one at the intersection point, the tracker does not exist. Notice that the phase trajectories go in the direction of increasing (decreasing) $\varepsilon_k/\varepsilon_{\text{tot}}$ for $w_k(y) < 1/3$ ($w_k(y) > 1/3$) and therefore, their tangents are horizontal at the points where $w_k(y) = 1/3$. On the other hand, phase trajectories evolve in the direction of increasing (decreasing) y for $\varepsilon_k/\varepsilon_{\text{tot}} < r^2(y)$ ($\varepsilon_k/\varepsilon_{\text{tot}} > r^2(y)$) and at the points where these phase lines cross the curve $r^2(y)$ their tangents are horizontal (see equation (4.5)).

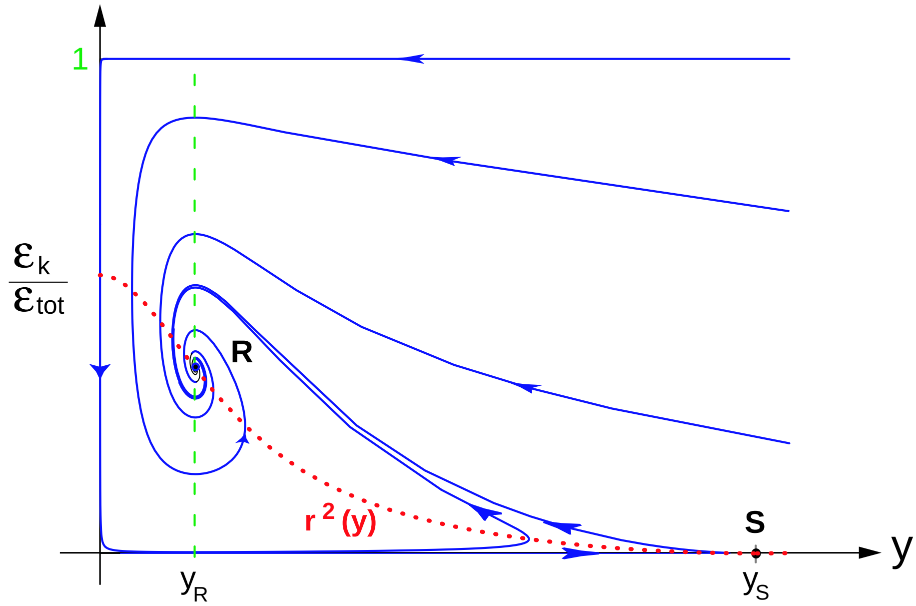

Figure 4.4: Phase diagram for case A_r during the radiation-dominated epoch. Phase lines flow in the direction shown by the arrows, dashed horizontal lines determine the y coordinate of attractor solutions and boldface labels the corresponding attractor points. The dotted line shows the points where $\varepsilon_k/\varepsilon_{\rm tot} = r^2(y)$.

The form of $r(y)$ also gives a clue about the equation of state $w_k(y)$: in the region where $r(y)$ is an increasing function of y we have $w_k(y) > 1$ and where it decreases $w_k(y) < 1$. Hence, as noted previously, $r(y)$ is what mainly determines the structure of the phase diagram.

As clearly seen in the figures in all cases, if the k-field is initially located near the **R**-tracker, it converges to it. Therefore, the basin of attraction is non-zero in all three cases. The attraction region can include equipartition initial conditions, the most natural possibility.

For A_r, figure 4.4, the **R**-attractor has the largest basin of attraction, the complete phase plane. If one starts, e.g., at $(\varepsilon_k/\varepsilon_{\rm tot})_i = \exp(-30)\,(\varepsilon_k/\varepsilon_{\rm tot})_R$, then the k-field rapidly reaches the vicinity of the de Sitter point **S** and joins the attractor connecting this point to the **R**-tracker.

The cases B_r and C_r have limited basins of attraction, and so are not as favorable from the point of view of initial conditions. If the energy density of the k-field is much smaller than the value at the **R**-tracker, the k-field travels first to the vicinity of the **S**-attractor, where it meets the phase trajectory that connects it to the **K**-attractor (case B_r) or the **r**-attractor (case C_r). In either situation, the field never reaches the **R**-tracker. Although the latter two cases have smaller basins of attraction than case A_r, only cases B_r and

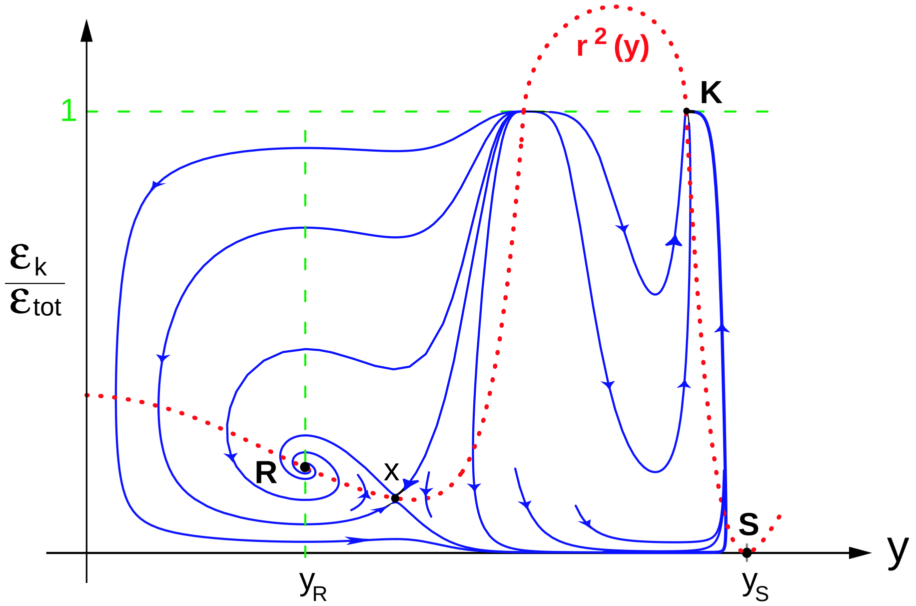

Figure 4.5: Phase diagram of a model of the type B_r during the radiation-dominated phase. In the relevant region of the diagram all trajectories can be traced back to a common origin. Some of the phase trajectories converge to the radiation tracker **R**, while others, after approaching the de Sitter point **S** finally reach the **K**-attractor. The saddle point **x** "separates" both types of trajectories.

C_r can produce cosmic acceleration today. One can simply assume that the initial value of the k-field lies somewhere in the basin of attraction, a reasonable possibility. An alternative is to introduce additional $\varphi-$dependence in the Lagrangian, as for instance, $L = g(y,\varphi)/y\varphi^2$, where $g(y,\varphi) \to g_1(y)$ at high energies (φ is smaller than some φ_*) and $g(y,\varphi) \to g_2(y)$ at relatively low energies (φ is bigger than φ_*), such that $g_1(y)$ has an A_r-set of attractors and $g_2(y)$ has a B_r/C_r-set of attractors. Note, that the exact value of φ_* is not important at all, we only have to be sure that the transition from one regime to the other happens before equipartition. Although modifying the Lagrangian may seem more complicated, it has the advantage that it removes nearly altogether dependence on initial conditions.

4.4.2 Matter Domination

We have shown that it is possible to choose a wide range of models and initial conditions for which the k-field converges to the **R**-tracker during the radiation-dominated epoch. The goal is to produce a scenario in which k-essence overtakes the matter density and induces cosmic acceleration today.

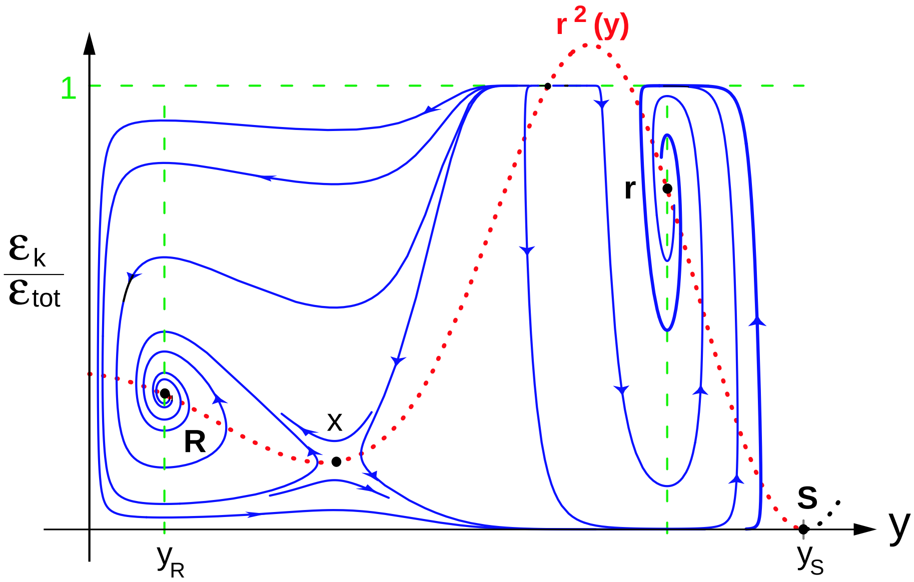

Figure 4.6: Phase diagram of a model of the type C_r during radiation domination, with same notation as in figure 4.5.

Yet, the contribution of k-essence to the total energy density must not spoil big bang nucleosynthesis or dominate over the matter density at the end of the radiation-dominated epoch (see subsection 4.3.2). To satisfy these conditions, it typically suffices if the **R**-tracker satisfies

$$(\varepsilon_k/\varepsilon_{\rm tot})_R = r^2\,(y_R) \simeq 10^{-2} \div 10^{-1}. \tag{4.25}$$

In this subsection, we study the evolution as the universe enters the matter-dominated epoch and the k-field is forced to leave the radiation tracker. In a dust dominated epoch the relevant attractors can appear in the following two possible sets: A_d) **S, K** and B_d) **S, D**.

In both cases successful k-essence models are possible. In the case A_d there is no dust tracker solution, $(r\,(y_D) > 1)$. Therefore, as seen in the phase diagram of figure 4.7, when the radiation-dominated epoch is over, k-essence approaches first the **S-**attractor; afterwards, when its energy density has increased significantly, it moves to the **K-** attractor (a state with negative pressure but $w_k > -1$). If $w_k\,(y_K) < -1/3$, the expansion rate accelerates for the indefinite future; if $-1/3 < w_k\,(y_K) < 0$, the expansion rate decelerates. Either way, the matter-radiation density is increasingly negligible compared to k-essence in the far future.

In the second case (B_d), there is a dust tracker solution. If $(\varepsilon_k/\varepsilon_{\rm tot})_D \ll 1$, k-essence contributes only a small fraction of the total energy density at this attractor, and it approaches this attractor almost immediately after matter-radiation equality. This is not desirable since then k-essence cannot dominate today or cause cosmic acceleration. However, if $(\varepsilon_k/\varepsilon_{\rm tot})_D = r^2\,(y_D) \to 1$ or $(\varepsilon_k/\varepsilon_d)_D \gg 1$, there can be a period of cosmic acceleration before the k-field reaches the dust attractor since it can first approach the **S**-attractor and remain there for a finite time, see figure 4.8. Ultimately, though, the acceleration is temporary; the k-field proceeds to the dust tracker, the expansion of the universe begins to decelerate, and the ordinary and (cold) dark matter density approaches a fixed, finite fraction of the total energy. We refer to the scenario as a "late dust tracker" because the dust attractor is reached long after matter-domination has begun.

Taking into account that $r(y_D)$ is near unity or greater for both case A_d and B_d, we obtain from equations (4.6) and (4.25):

$$\frac{g'_R\, y_R^2}{g'_D\, y_D^2} \le \frac{9}{16}\left(\frac{\varepsilon_k}{\varepsilon_{\rm tot}}\right)_R \simeq 5\cdot(10^{-3} \div 10^{-2}). \tag{4.26}$$

We can also infer from figure 4.3 that $g'_D\cdot(y_R - y_D) \le g\,(y_R) = -y_R\, g'_R/3$ and, therefore, for $(\varepsilon_k/\varepsilon_{\rm tot}) \ll 1$,

$$\frac{y_R}{y_D} \le \frac{3}{16}\left(\frac{\varepsilon_k}{\varepsilon_{\rm tot}}\right)_R \simeq 2\cdot(10^{-3} \div 10^{-2})$$

and

$$\frac{g'_D}{g'_R} \le \frac{1}{16}\left(\frac{\varepsilon_k}{\varepsilon_{\rm tot}}\right)_R \simeq 6\cdot(10^{-4} \div 10^{-3}). \tag{4.27}$$

Since $\varepsilon_k = -g'/\varphi^2$ and $|g'\,(y_S)| \le |g'\,(y_D)|$, we conclude that after radiation domination, when the k-field reaches the vicinity of the **S**-attractor, the ratio of energy densities in k-essence and dust can not exceed $\varepsilon_k/\varepsilon_d < (\varepsilon_k/\varepsilon_{\rm tot})_R^2/16 \simeq 6\cdot(10^{-6}\div 10^{-4})$. This is the nadir of k-essence; once k-essence approaches the **S**-attractor, its contribution to the cosmic density increases again until it becomes comparable to the matter density. In case A_d, the k-field will evolve further to the **K**-attractor and the k-essence energy will increasingly dominate over the matter density. In case B_d, the k-field approaches the **D**-tracker where the ratio of k-essence to the matter density approaches some fixed positive value.

The statements above are generic and do not depend significantly on the concrete model as long as it satisfies the simple criteria formulated above. Let us stress that the only **"small"** parameter used is the ratio $(\varepsilon_k/\varepsilon_{\rm tot})_R$, which has to be of the order of $10^{-2} \div 10^{-1}$, a very natural range for these models and one that satisfies constraints of big bang nucleosynthesis (see subsection 4.3.2). For this range, the present moment is approximately the earliest possible time when cosmic acceleration could occur.

Finally note that, during the transition from the radiation tracker **R** to the de Sitter attractor **S**, the equation of state of k-essence has to take values bigger than one, and hence the dominant energy condition $\varepsilon_k > |p_k|$ is

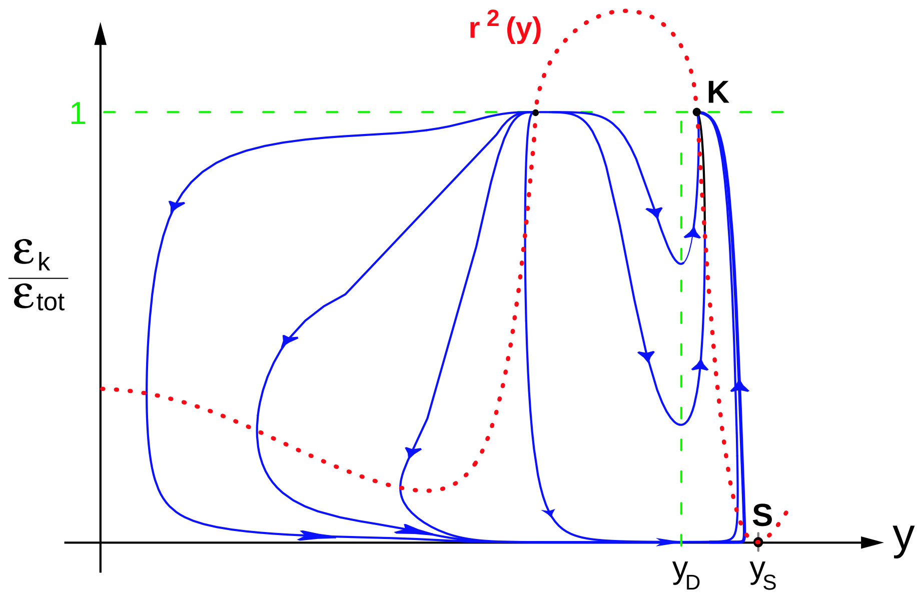

Figure 4.7: Phase diagram of a model of type A_d during the matter-dominated epoch. All trajectories have a common origin and all of them finally reach the **K**-tracker. Trajectories which "skim" the line $\varepsilon_k/\varepsilon_{\text{tot}} \approx 0$ reach this attractor after going through a nearly de Sitter stage (the **S**-attractor).

violated during a certain finite time interval. This violation implies that k-essence can be observed to travel at superluminal speeds, as already remarked in section 2.3.

4.5 Constructing Models

In previous sections, we have presented a general theoretical treatment of the attractor behavior of k-essence fields in a cosmological background. We have emphasized the properties needed to formulate models which will lead naturally to cosmic acceleration at the present epoch. In this section, we discuss how to apply the general principles to construct illustrative toy models.

Let us summarize the conditions we have derived for building viable Lagrangians. First, we must satisfy the general positive energy and stability conditions in equation (4.4). If g takes positive and negative values, they already suffice to guarantee generically the existence of a radiation point y_R where $w(y_R) = 1/3$, a unique dust point y_D where $w(y_D) = 0$, and a unique de Sitter point y_S where $w(y_S) = -1$. The radiation point is an attractor if

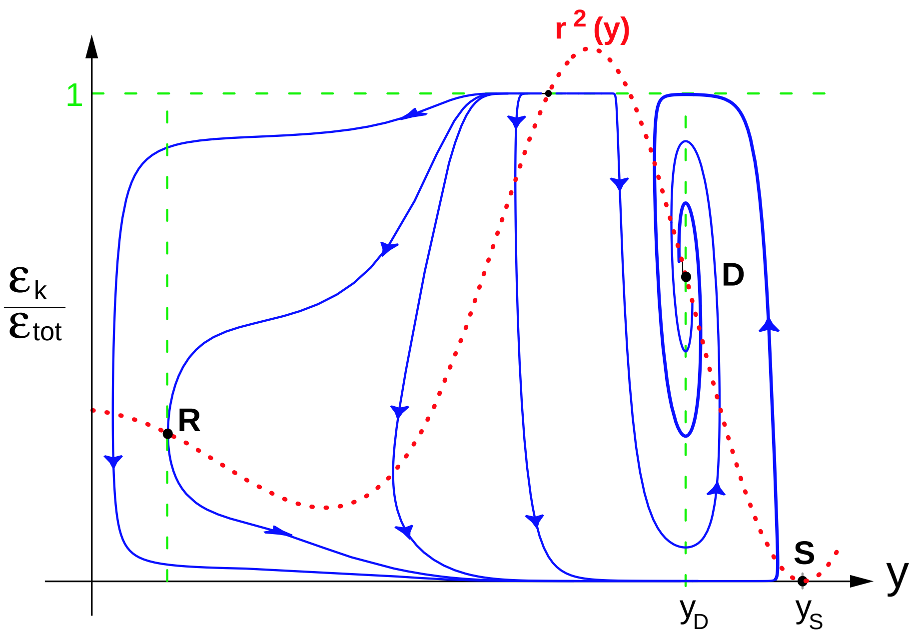

Figure 4.8: Phase diagram of a model of type B_d during the matter-dominated epoch. All trajectories have a common origin and all of them finally reach the **D**-tracker. Trajectories which "skim" the line $\varepsilon_k/\varepsilon_{\text{tot}} \approx 0$ reach this attractor after going through a nearly de Sitter stage.

$g''(y_R)$ is sufficiently small,

$$g''(y_R) < -4\frac{g'(y_R)}{y_R}, \tag{4.28}$$

and the remaining prerequisites needed to ensure a successful scenario are then reduced to simple restrictions on the derivative of g at two separate values of y:

i) At y_R, $r_R^2 = -2g'(y_R)y_R^2 \simeq 10^{-2} \div 10^{-1}$.

ii) At y_D either $r_D^2 = -9\,y_D^2\,g'(y_D)/8 > 1$ or $1-r_D^2 = 1+9\,y_D^2\,g'(y_D)/8 \ll 1$.

The first condition in ii) corresponds to cases where there is no dust attractor, and the second condition to cases where there is a dust attractor with a small matter to k-essence energy density ratio.

A straightforward way of constructing a function with given derivatives at two points is to glue two linear functions with the required slopes, as shown in figure 4.9. Observe that if $g(y)$ is linear around the radiation point the attractor requirement (4.28) is automatically fulfilled. In order to have a finite c_s^2, it suffices to introduce small quadratic corrections to the glued linear functions. We implement this procedure to build a toy model expressed in terms of artificial parameters (from the point of view of fundamental physics) that can be simply related to figure 4.9 and our earlier discussion of attractor solutions. One should appreciate that, for this pedagogical purpose, we have "overparameterized" the problem—the outcome is rather insensitive to most parameters as long as they obey certain simple general conditions. Simpler forms with fewer parameters are certainly possible.

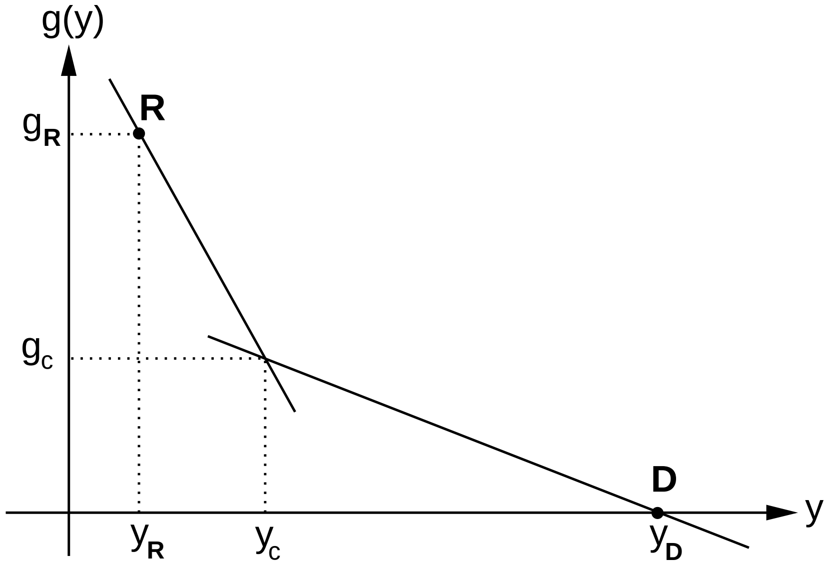

Figure 4.9: A simple toy model for $g(y)$ consisting of two linear pieces meeting at the "crossing point" y_c. Here y_R and y_D are the radiation and the dust attractor values, and the derivatives of g at these points are g'_R and g'_D, respectively.

Let $g_{\rm glue}(y)$ be any smooth function constructed by gluing the two linear pieces of figure 4.9. The function $g_{\rm glue}$ depends on y and has y_R, g'_R, y_D and g'_D as parameters, where y_R and y_D are the radiation and the dust attractor values, and the derivatives of g at these points are g'_R and g'_D respectively. Our toy model corresponds to

$$g(y) \equiv g_{\rm glue}(y)\left(1 - \frac{y}{s^2 \cdot y_D}\right). \tag{4.29}$$

The factor $g_{\rm glue}$ describes the function in figure 4.9 and the factor in parenthesis provides the quadratic corrections needed to have a positive speed of sound. It so happens that the latter factor also shifts the de Sitter point from $y = \infty$, as it would be for purely linear functions, to finite y, although this is not crucial for our purpose. For $s \gg 1$ the de Sitter point is located at $y_S \approx s \cdot y_D$ and $g \approx g_{\rm glue}$.

Once a general form for g is known, such as the example above, one can study how the model parameters affect the resulting cosmology. Our conclusion is that the predictions of the toy model are relatively insensitive to the gluing function or to the particular values of y_R, y_D, g'_R, g'_D and y_S as long as they satisfy certain simple relations. For instance, what sets the values of Ω_k and w_k today? Do these depend on the precise form of the interpolating function? We have solved numerically the equations of motion for a wide range of gluing functions $g_{\rm glue}$ in equation (4.29). For a typical parameter choice, the final value of Ω_k does not depend on the particular gluing function as long as $g_{\rm glue}$ conforms closely enough to figure 4.9.

The value of Ω_k today does depend on the evolution of $\varepsilon_k/\varepsilon_m$. At early times the field is locked at the radiation tracker, and its fractional energy density ratio is given by $-2g'_R y_R^2$. After radiation-matter equality the field can not follow the radiation tracker anymore and its energy density drops by several orders of magnitude until $\varepsilon_k/\varepsilon_m$ reaches a minimum value at the time w_k falls below zero. We shall label this minimum value with the subscript "min". The energy density at this minimum is roughly given by

$$\left(\frac{\varepsilon_k}{\varepsilon_m}\right)_{\rm min} \approx r_R^2 \frac{g'_D}{g'_R}. \tag{4.30}$$

The position of the minimum in time only depends on the distance between the radiation and crossing point $y_c - y_R$. As $y_c - y_R$ increases from zero, the minimum is shifted from matter-radiation equality to later times. After reaching the minimum, the field moves on to the de Sitter attractor and $\varepsilon_k/\varepsilon_m$ grows as $(z+1)^{-3}$, where z is the redshift. In order to have k-essence dominate today, it must be that $\varepsilon_k/\varepsilon_m$ during the radiation epoch lies roughly between 10^{-1} and 10^{-2}. Then, $(\varepsilon_k/\varepsilon_m)_{\rm min}$ lies in the range $10^{-4} \div 10^{-6}$ and, provided y_c is chosen appropriately, this has k-essence dominating at about the present epoch. One can see these conditions impose constraints on certain combinations of our parameters, although in a fairly natural range not very far from unity.

As discussed in subsection 4.4.2, there are two possible future fates for the universe depending upon whether there is a "late dust tracker" solution or not. By requiring $r_D^2 > 1$ we avoid a dust tracker and, therefore, insure that the k-field approaches the k-attractor when k-essence starts to dominate. The equation of state of k-essence at the k-attractor depends on the parameter s. By increasing s the equation of state w_k at the k-attractor approaches -1, and in the limit $s \to \infty$, $w_k(y_K) \to -1$. If $w_k < -1/3$, the expansion rate of the universe accelerates forever. Using the maximal value of w_k at the present epoch as allowed by supernovae observations, say, s can be simply adjusted to insure that w_k at the k-attractor is less than or comparable to this value. In this case, the equation of state of k-essence today will be less than or equal to $w_k(y_K)$, which is set by s, as described above.

If $r_D^2 < 1$, it is possible to have successful models if r_D^2 is sufficiently close to 1. In such a model the equation of state of k-essence will finally reach $w_k = 0$ in the far future; so, ultimately, cosmic acceleration ceases and the expansion begins to decelerate again. Nevertheless, it is still possible to have a finite period in which the equation of state is negative and which includes the present epoch. It is worth noting that models without a dust attractor are more generic and natural, since they do not require a special tuning of $r(y_D)$ to a value close but smaller than unity at the dust point. Below we illustrate examples of both types.

4.5.1 Model without dust attractor

Models that belong to the general class A_d illustrated in figure 4.7 do not have dust attractor solutions because $r(y_D) > 1$. Choosing the following values of the parameters, $y_R = 0.1$, $g'_R = -5$, $y_D = 17$, $g'_D = -5 \cdot 10^{-3}$ and $s^2 \cdot y_D = 135$, we have $r(y_D) \approx 1.2$. Therefore, there has to be a **K**-inflationary attractor, which is located for our parameter choice at $y_K \approx 28$. At the **K**-attractor, k-essence has the equation of state $w_k(y_K) \simeq -0.43$. The ratio of the energy densities at the **R**-tracker in this model is $(\varepsilon_k/\varepsilon_{\mathrm{tot}})_R = 0.1$. The results of the numerical calculations are presented in figures 4.10 and 4.11. We see that during the radiation stage k-essence quickly reaches the radiation tracker, in particular, the oscillations of the equation of state w_k in figure 4.11 around $w_k = 1/3$ decay exponentially rapidly. The k-field has the same equation of state as radiation until the moment when dust starts to dominate. Around this time the energy density of k-essence suddenly drops by three orders of magnitude and the equation of state, after a very short period of increase, drops down to $w_k \simeq -1$, the value of the equation of state along the **S**-attractor. After that, when the energy density of k-essence becomes significant, w_k starts to increase towards the **K**-attractor value, -0.43. Since Ω_k is not yet unity, the current value is somewhere between the **K**-attractor value and -1; in this example, the value today ($z = 0$) is $w_k \simeq -0.69$. The energy density of k-essence today is $\Omega_k \approx 0.65$, and because we assumed a flat universe, $\Omega_m = 0.35$. For completeness let us mention that we have defined "today" ($z = 0$) to be the moment when the matter-radiation energy density ratio is given by $(\varepsilon_r/\varepsilon_m)_{\mathrm{today}} \equiv 4.307 \cdot 10^{-5}/(\Omega_m h^2)$.

4.5.2 Model with a late dust attractor

Taking $y_R = 11 \cdot 10^{-3}$, $g'_R = -34$, $y_D = 11$, $g'_D = -8 \cdot 10^{-3}$ and $s^2 \cdot y_D = 56$, we can construct a model with a "late dust tracker", corresponding to the phase diagram in figure 4.8. The parameters have been deliberately chosen to differ significantly from the ones in the model without dust attractor in order to illustrate that fine tuning is not necessary.

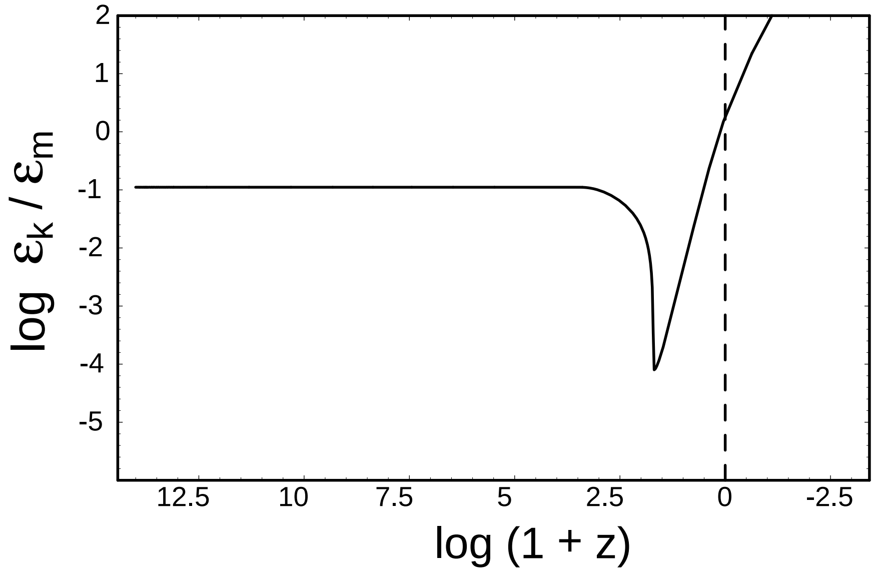

Figure 4.10: The ratio of k-essence to matter energy density, $\varepsilon_k/\varepsilon_m$, vs. $1+z$ for a model with a k-attractor.

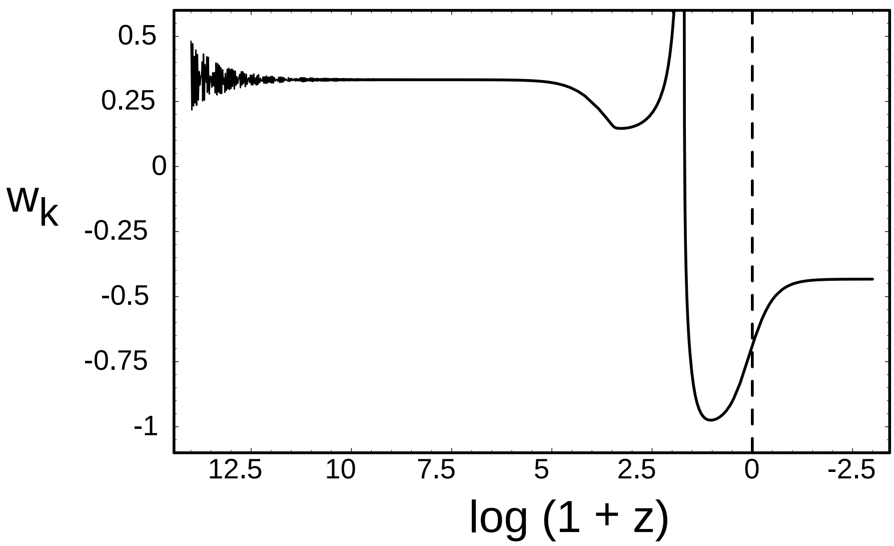

Figure 4.11: The equation of state w_k vs. $1+z$ for a model with k-attractor.

The late dust attractor is reached after k-essence passes near the de Sitter attractor following matter-radiation equality. At the late dust tracker $(\varepsilon_k/\varepsilon_{\rm tot})_D = r^2\,(y_D) \simeq 0.88$ and, correspondingly, $(\varepsilon_k/\varepsilon_d)_D \simeq 7$. Hence, the fractional contribution of the matter density is small but remains finite in the indefinite future. The ratio of energies at the **R**-tracker is $(\varepsilon_k/\varepsilon_{\rm tot})_R \simeq$

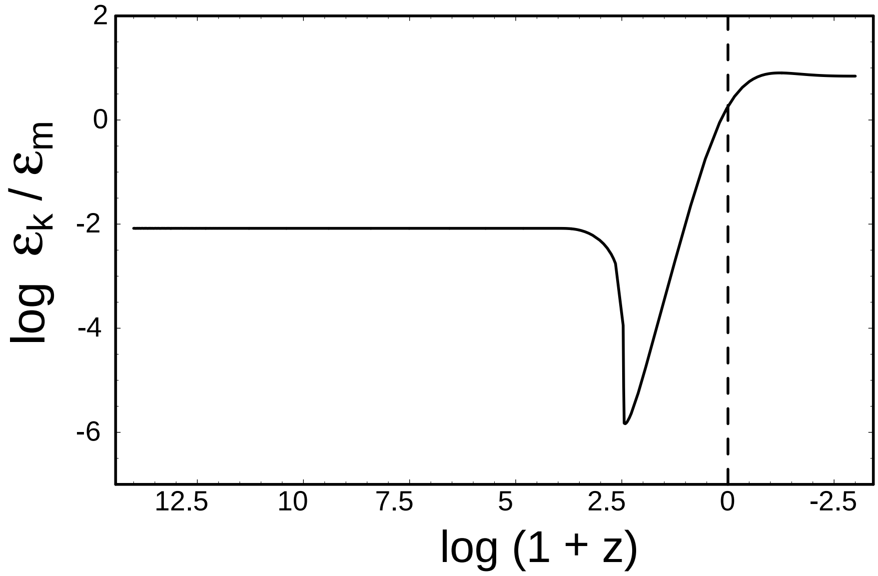

Figure 4.12: The ratio of k-essence to matter energy density, $\varepsilon_k/\varepsilon_m$, vs. $1+z$ for a model with a late dust tracker solution. In this type of model, $w_k \to 0$ in the far future and the ratio of k-essence to matter energy density approaches a constant.

$8.3 \cdot 10^{-3}$. The results of the numerical calculations are presented in figures. 4.12 and 4.13. The evolution of the k-field here is very similar to the one we described in the previous case; the differences between both models occur at small red-shifts. The fraction of the critical energy density of k-essence today is in this model also $\Omega_k = 0.65$ and the equation of state w_k takes the value -0.4. The future evolution of the model with a late dust attractor is completely different from what we found in the previous one. Here the ratio of the energy densities of k-essence and dust will continue growing in the future only until it becomes approximately 7. After that it will start to oscillate around this value with exponentially decaying amplitude while the pressure approaches the dust point, where $w_k = 0$.

4.5.3 Examples

The toy models presented thus far are all built on the ansatz shown in figure 4.9, which entails numerous parameters. We have pointed out that the large number of parameters is not a necessary feature. We have introduced this form for pedagogical purposes, since it enables one to study directly the relation between the attractor solutions and cosmic evolution. Indeed, our analysis showed that the cosmological solution is relatively insensitive to most of the parameters provided they obey a few broad conditions.

To emphasize the point, consider a model of the form

$$\tilde{p}(X) = -b + 2\sqrt{1 + X \cdot h(aX)}, \tag{4.31}$$

where $h(aX)$ is some smooth function that can be expanded in a power series in X. This particular form is reminiscent of the Born-Infeld action (1.16) in which $h(aX)$ could represent higher order corrections in X. (This choice of a square-root form is not essential—simply an example.) As a specific case, for $b = -2.05$ and $X \cdot h(aX) = X - (aX)^2 + (aX)^3 - (aX)^4 + (aX)^5 - (aX/2)^6$ the Lagrangian defined by (4.31) satisfies all constraints and produces $\Omega_m = 0.3$ and $w_k = -0.8$ today if one chooses $a = 10^{-4}$. This particular example has a cosmic evolution similar to the one described in subsection 4.5.1 (no dust attractor). We see that in this case, as with a wide range of other functional forms, the condition $b > 2$ and the choice of the single parameter a suffices to satisfy all of the conditions of the multi-parameter toy models.

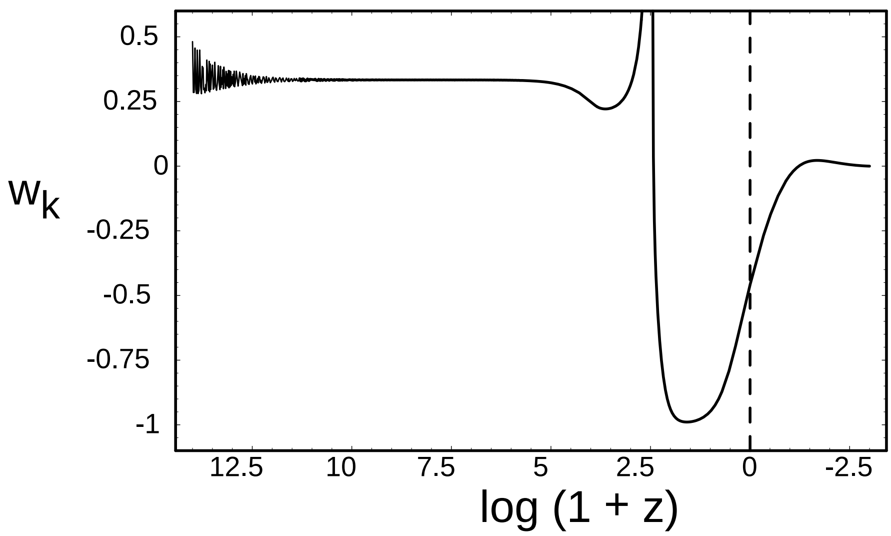

Figure 4.13: The equation of state w_k vs. $1 + z$ for a model with a late dust tracker solution.

Conclusions and Outlook

The inclusion of non-canonical kinetic terms in the Lagrangian of a scalar field can have quite non-trivial and unexpected cosmological effects. Technically, the k-field approach, at least in the examples we have constructed, relies on attractor properties that naturally arise if the action contains terms that depend non-linearly on the gradients of the k-field. Non-linear terms of this type are expected to appear in any effective field theory an do indeed arise in most models unifying gravity with other particle forces, including supergravity and superstring models. In the past, these contributions have been ignored for reasons of "simplicity". The examples of k-inflation and k-essence demonstrate that the effects of non-linear dynamics can be dramatic. The non-linear dynamics is totally missed if the kinetic energy terms are truncated at the lowest order contributions. Hence, the kinds of attractor effects discussed in this work have gone unnoticed in most treatments of field theory.

With k-essence, we have provided a novel explanation of the late time cosmic acceleration problem. Introducing a dark energy component with negative pressure has resolved many observational problems with the cold dark matter model including the recent evidence from supernovae searches that the universe is undergoing cosmic acceleration. At the same time, the dark energy component presents a profound challenge to cosmology and fundamental physics. What is its composition and why has it become an important contribution to the energy density of the universe only recently? The example of k-essence shows that it is possible to find a predictive, dynamical explanation that does not rely on coincidence or the anthropic principle. Unlike a cosmological constant or quintessence models of the past, the energy density today is not fixed by fine-tuning the vacuum density or other model parameters. Rather, the energy density today is forced to be comparable to the matter density today because of the dynamical interaction between the k-essence field and the cosmological background. An important question to consider is whether there are observational tests to distinguish k-essence from alternative explanations. One notable feature of k-essence models compared to the more general tracker quintessence models [63, 53] is that the equation of state w_k is increasing at the present epoch. For quintessence scalar fields rolling down tracker potentials, the quintessence tracks the matter density

($w = 0$) during most of the matter-dominated epoch, and only recently has begun to decrease towards $w = -1$. Hence, measurements of dw/dz for the dark energy [25] would distinguish these two possibilities from one another and from a cosmological constant. However, this test would not distinguish k-essence from more contrived quintessence models that can also be tuned so that w_k is increasing today as well. A second feature of k-essence is the non-linear kinetic energy contribution. A consequence is that the effective sound speed c_s^2 is generically different from unity, whereas $c_s^2 = 1$ for a scalar field rolling down a potential. Depending on the model, the distinctive sound speed can have subtle or significant effects on the cosmic microwave background anisotropy. As regards the future of the universe, our work here offers a new, perhaps pleasant possibility. In previous models with cosmological constant or quintessence, the acceleration of the universe continues forever and ordinary matter—that composes stars, planets and life as we know it— becomes a rapidly shrinking fraction of the energy density of the universe. In the "late dust tracker"scenario which we have introduced here, the acceleration is temporary and the matter density approaches a fixed, finite fraction of the total.

From a different perspective, our analysis of k-inflation has shown that inflation is an even more generic phenomenon than originally suspected. While most inflationary scenarios rely on an appropriate scalar field potential, k-inflation demonstrates that inflation can be implemented in a much larger class of Lagrangians, a class that contains even potential-less fields. In our treatment, the dependence of the Lagrangian on the scalar field on one hand, and its derivatives on the other, has been decoupled. Inflation proceeds if the field derivative dependent part $\tilde{p}(X)$ allows an inflationary equation of state and if the field-dependent part $K(\varphi)$ satisfies certain flatness conditions, which for a canonical Lagrangian reduce to the familiar slow-roll conditions. The conditions on $\tilde{p}$ are easily met by a wide class of Lagrangians, including those that do not contain the equivalent of a potential term. Alternatively, if a potential term is responsible for the inflationary stage, our considerations imply that the addition of higher order non-canonical kinetic terms (such as X^2) to the Lagrangian will not spoil the inflationary solutions. It is also important to note, that although k-inflation and potential driven inflation share most of the main desirable features of an inflationary scenario and are similar in many aspects, they can be phenomenologically distinguished through the influence of the non-canonical kinetic terms on the speed of sound of the k-field perturbations. This speed of sound may be in fact determined in future experiments from the consistency relation of scalar and tensor perturbation spectra.

Since both k-essence and k-inflation are based on the same idea, the k-field, it is natural to wonder whether both can be accommodated in a single model. Such a k-field would provide a unified account of phenomena at such different energy scales as inflation and late-time cosmic acceleration, and relate times as different as 10^{-34} sec and 10^{10} years. Moreover, it could eventually link the amplitude of the primordial density fluctuations—and hence the time of structure formation— to the moment of late cosmic acceleration[1] through the dependence on these numbers on a single parameter.

A k-field may also play a significantly different role in cosmology, as suggested by the considerations of Born and Infeld on non-canonical electromagnetic fields. As we mentioned, the Born-Infeld action was introduced to prevent field strengths from becoming singular. Singularities on the other hand do indeed generically appear in cosmological solutions, and they are believed to signal the breakdown of the underlying theories used to describe our universe. Yet, an appropriate k-field Lagrangian may restrict the possible values of the scalar so as to avoid these singularities and yield a viable cosmology by explaining both inflation and late time cosmic acceleration. Certainly, though it may be regarded as wishful thinking, such a k-field would incorporate much of our understanding of the universe.

[1]We thank A. Vilenkin for raising this issue.

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I am deeply indebted to Slava Mukhanov for his guidance and support during all these years. Under his supervision, I've also had the great opportunity of working with Thibault Damour and Paul Steinhardt, from whom this work has also enormously benefitted. Raul Abramo, Johannes Martin, Ivo Sachs, Christian Sicka and Oliver Zahn have also somehow shaped the original manuscript by suggesting improvements (easy work!) and they also deserve my special recognition. I want to thank all the colleagues in the theory group I've interacted with in the past. I'm sorry not to mention all of them by name— Studenten, Diplomanden, Doktoranden, Postdocs und Professoren, the list would be too large— though everyone I spoken with about physics during this years should be acknowledged.

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Appendix A

Symbols and Notation

Symbol	Meaning	Defined in
a	Scale factor	Page 3
CMBR	Cosmic microwave background radiation	
$const$	A constant value	
c_s^2	Squared speed of sound of the k-field	Equation (2.20)
$\mathbf{D}$	Dust attractor	Section 4.3
g	Enters an alternative field parametrization	Equation (2.21)
g	Determinant of the metric $g = \det g_{\mu\nu}$	
$g_{\mu\nu}$	Spacetime metric.	
G	Newton's gravitational constant.	
	We work in units where $8\pi G/3 = 1$	Page 20
h	Hubble parameter in units	
	of 100 km $\sec^{-1}$ Mpc^{-1}	
k	Comoving wave number	
K	φ-dependent part of the k-Lagrangian	Equation (2.4)
$\mathbf{K}$	k-Attractor	Section 4.3
H	Hubble parameter	Equation (1.4)
N	Number of e-foldings	Page 22
p	Pressure	
pc	1 Parsec, 3.3 light years	
$\tilde{p}$	X-dependent part of the k-field pressure	Equation (2.4)
r	Enters the k-field equation of motion	Eqs. (2.17), (2.24)
R	Scalar curvature	
$\mathbf{R}$	Radiation attractor	
S	Action functional	
$\mathbf{S}$	de Sitter attractor	Section 4.3
T	Temperature	
$T_{\mu\nu}$	Energy momentum tensor	Page 22

Symbol	Meaning	Defined in
t	Cosmic time	Page 3
v	Perturbation variable	Equation (2.19)
V	Scalar field potential	
w	Equation of state	Equation (2.13)
$\vec{x}$	Comoving coordinates	
X	Squared gradient of the k-field	Equation (2.3)
y	Alternative kinetic variable	Equation (2.21)
z	Redshift	Equation (1.11)
δA	Perturbation of the variable A	Page 23
∂_μ	$\partial/\partial x^\mu$	
ΔA	Flat Laplacian $(\partial_x^2 + \partial_y^2 + \partial_z^2)A$ or	
ΔA	Change of the variable A	
ϵ	1st Slow roll parameter	Equation (3.13)
ε	Energy density	
$\tilde{\varepsilon}$	X-dependent part of the k-field energy density	Equation (2.9)
$\varepsilon_{\text{crit}}$	Critical energy density	Equation (1.5)
φ	The k-field	Page 20
ϕ	A generic scalar field	
Φ	The Newtonian potential	Equation (2.18)
κ	Labels the three possible geometries of space	Page 3
Λ	Cosmological constant	
η	2nd Slow roll parameter	Equation (3.15)
η_{MN}	Flat spacetime metric, diag$(+1, -1, \ldots, -1)$.	
Ω	Density parameter	Equation (1.7)
σ	Sign of the k-field "velocity"	Page 21
Subindices	**Meaning**	**Defined in**
0	Denotes the value of a quantity today	
i	Labels any component	Page 22
m	Labels matter (radiation and dust)	Page 22
r	Labels radiation	
d	Labels dust	
k	Labels the k-field	Page 22
tot	Total	
$,x$	Derivative with respect to the variable x	
Others	**Meaning**	
$\dot{q}$	Time derivative of q, dq/dt	
q'	Derivative of q with respect to y, dq/dy	
$A \div B$	Between A and B	

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