Estimation of MIMO Transmit-Antenna Number Using Higher-Order Moments Based Hypothesis Testing

Tao Li, Yongzhao Li, *Senior Member, IEEE,* Yunfei Chen, *Senior Member, IEEE,* Leonard J. Cimini, Jr., *Fellow, IEEE,* and Hailin Zhang, *Member, IEEE*

***Abstract*—This letter proposes a higher-order-moment based hypothesis testing algorithm to estimate the transmit-antenna number for multiple-input multiple-output (MIMO) systems. Exploiting the asymptotic normal distribution of the moments composed by noise eigenvalues, the proposed algorithm improves the estimation performance for low signal-to-noise ratios (SNRs). Moreover, since the empirical distribution of the moments converges quickly to the normal distribution when the number of samples increases, our algorithm can make a reliable estimation in a sample starved condition. Computer simulations are provided to demonstrate that the proposed algorithm outperforms the conventional algorithms.**

***Index Terms*—Higher-order moments, hypothesis testing, MIMO, number of transmit antennas.**

I. Introduction

BLIND identification of signal parameters is a vital technology in intelligent multiple-input multiple-output (MIMO) scenarios (e.g., cognitive radio, software defined radio, surveillance, and security-monitoring) [1]. In particular, the estimation of the number of transmit antennas is often a prerequisite for identifying other signal parameters.

The representative works on the estimation of the number of transmit antennas can be mainly categorized as information theoretic criteria based algorithms [2]–[5] and hypothesis testing based algorithms [6]–[10]. The former transforms the estimation problem into a model selection problem according to the Akaike information criterion (AIC) or the minimum description length (MDL) [2]. However, the AIC algorithm is inconsistent, and the MDL algorithm underestimates at a low signal-to-noise ratio (SNR) for a small sample size. For example, in [5], the statistical properties of the higher-order moments were exploited to derive an improved MDL algorithm, which requires less samples than MDL to achieve the same probability of correct estimation. The latter transforms the estimation problem into a series of hypothesis tests by comparing test statistics with thresholds. They can achieve a tradeoff between the AIC and MDL algorithms. Hence, the hypothesis testing based approaches have attracted much attention. Nevertheless, the conventional hypothesis testing algorithms do not perform well at a low SNR for a small sample size either, since the distributions of the test statistics in the conventional hypothesis testing algorithms do not converge fast when the sample size increases. Owing to the fast-converged distribution property of higher-order moments composed by noise eigenvalues [4], [5], in this letter, we propose a hypothesis testing algorithm by exploiting the distribution properties of higher-order moments.

To our best knowledge, no previous hypothesis testing based algorithm utilizes the distribution properties of higher-order moments of noise eigenvalues to construct the test statistics. As the empirical distributions of the test statistics converge fast to the normal distribution, the proposed algorithm can perform well with a reduced resolution SNR threshold in a sample starved condition. The main contributions of this letter are twofold. i) According to the asymptotic normal distribution of the higher-order moments, the thresholds of the test statistics are derived for the estimator. ii) A hypothesis testing method based on the test statistic constructed by the higher-order moments is used to determine the number of transmit antennas. Simulation results show that, at a low SNR and a sample starved condition, the proposed algorithm compares favorably with the conventional algorithms.

II. System Model

We consider a MIMO communication system, where an unauthorized receiver with N_r antennas is intercepting the signals from a transmitter with N_t antennas ($N_r > N_t$). The received signals at the i-th time instant are described by

$$\mathbf{y}(i) = \mathbf{H}\,\mathbf{x}(i) + \mathbf{n}(i) \quad \text{for} \;\; i = 1, ..., N, \tag{1}$$

where $\mathbf{y}(i) \in \mathbb{C}^{N_r \times 1}$ denotes the received signals, $\mathbf{x}(i) \in \mathbb{C}^{N_t \times 1}$ denotes the transmitted signals, $\mathbf{n}(i) \in \mathbb{C}^{N_r \times 1}$ represents the complex additive white Gaussian noise with mean zero and variance σ_n^2, and $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ stands for the unknown channel matrix which is assumed to be of full rank.

The population covariance matrix of $\mathbf{y}(i)$ is

$$\mathbf{\Sigma} = \mathbb{E}[\mathbf{y}(i)\mathbf{y}^{\dagger}(i)], \tag{2}$$

where $(\cdot)^{\dagger}$ denotes the Hermitian transpose. After eigenvalue decomposition (EVD), the eigenvalues of $\mathbf{\Sigma}$ are arranged in descending order as

$$\lambda_1 > \lambda_2 > \cdots > \lambda_{N_t} > \lambda_{N_t+1} = \cdots = \lambda_{N_r} = \sigma_n^2. \tag{3}$$

From (3), the number of transmit antennas can be determined from the cardinality of the smallest eigenvalues of $\mathbf{\Sigma}$.

This work was supported in part by the National Key R&D Program of China (2016YF-B1200202), the National Natural Science Foundation of China (61771365), the Natural Science Foundation of Shaanxi Province (2017JZ022), and the 111 Project (B08038). *(Corresponding author: Yongzhao Li)*

T. Li, Y. Li, and H. Zhang are with the State Key Laboratory of Integrated Services Networks, Xidian University, Xi'an 710071, China (e-mail: tli@stu.xidian.edu.cn; yzhli@xidian.edu.cn; hlzhang@xidian.edu.cn).

Y. Chen is with the School of Engineering, University of Warwick, Coventry CV4 7AL, U.K. (e-mail: Yunfei.Chen@warwick.ac.uk).

L. J. Cimini, Jr., is with the Department of Electrical and Computer Engineering, University of Delaware, Newark, DE 19716 USA (e-mail: cimini@udel.edu).

However, the population covariance matrix $\mathbf{\Sigma}$ is unavailable at the unauthorized receiver. It can only be approximated by a sample covariance matrix (SCM) as

$$\mathbf{R} = \frac{1}{N}\sum\nolimits_{i=1}^{N}\mathbf{y}(i)\mathbf{y}(i)^{\dagger}, \tag{4}$$

where N denotes the number of samples. The sample eigenvalues of $\mathbf{R}$ are

$$l_1 \geq l_2 \geq \cdots \geq l_{N_t} \geq l_{N_t+1} \geq \cdots \geq l_{N_r}. \tag{5}$$

Therefore, it is difficult to determine the number of transmit antennas by merely "observing" the sample eigenvalues $\{l_j\}_{j=1}^{N_r}$.

III. Higher-Order-Moment Based Hypothesis Testing Algorithm

A. Pertinent Statistical Properties of Higher-Order Moments

In this section, we introduce some pertinent statistical properties of higher-order moments of eigenvalues according to the random matrix theory (RMT) [11].

Let $\mathbf{R}_{\rm sf}$ denote a signal-free SCM formed from an $N_r \times N$ matrix of observations with independent and identically distributed (i.i.d.) Gaussian samples of mean zero and variance σ^2. The eigenvalues of the signal-free SCM $\mathbf{R}_{\rm sf}$ are

$$d_1 \geq d_2 \geq \cdots \geq d_{N_r-1} \geq d_{N_r}. \tag{6}$$

Then, the celebrated result from Marčenko and Pastur [12] states that, in the asympotic regime, i.e.,

$$N_r, N \to \infty, c = N_r/N, \tag{7}$$

$F^{\mathbf{R}_{\rm sf}}(x)$ associated with the empirical distribution of the eigenvalues converges to a Marčenko-Pastur distribution $F^{W}(x)$ with density

$$\begin{aligned} dF^W(x) = &\max\left[0, \left(1-\tfrac{1}{c}\right)\right]\delta(x) \\ &+\tfrac{\sqrt{(x-a_-)(a_+-x)}}{2\pi\sigma^2 xc}\Pi_{[a_-,a_+]}(x)\,dx, \end{aligned} \tag{8}$$

where $a_{\pm} = \sigma^2\left(1 \pm \sqrt{c}\right)^2$, $\Pi_{[a_-,a_+]}(x) = 1$ when $a \leq x \leq b$ and zero otherwise, and $\delta\left(x\right)$ is the Dirac delta function.

Hence, the rth-order moment of the eigenvalues of the signal-free SCM $\mathbf{R}_{\rm sf}$ converges almost surely as

$$M_r^{\mathbf{R}_{\rm sf}} = \frac{1}{N_r}\sum\nolimits_{j=1}^{N_r} {d_j}^r \xrightarrow{a.s.} \int x^r dF^W(x) =: M_r^W. \tag{9}$$

In the presence of signals, the eigenvalues of the SCM $\mathbf{R}$ are different from the eigenvalues of $\mathbf{R}_{\rm sf}$. Nevertheless, Johnstone [13] has shown that the distribution of the $N_r - N_t$ noise eigenvalues of $\mathbf{R}$ is closely approximated by a Marčenko-Pastur distribution, as in the signal-free case [4], [13]. The rth-order moment of the noise eigenvalues of the SCM $\mathbf{R}$ is

$$M_r^{\mathbf{R}}(N_t) = \frac{1}{N_r - N_t}\sum\nolimits_{j=N_t+1}^{N_r} {l_j}^r. \tag{10}$$

Following the method in [4], [5], the rth-order moment based test statistic T_r is constructed as

$$T_r(N_t) = \frac{M_r^{\mathbf{R}}(N_t)}{\left[\widehat{\sigma}^2(N_t)\right]^r} = \frac{\frac{1}{N_r-N_t}\sum_{j=N_t+1}^{N_r} {l_j}^r}{\left[\widehat{\sigma}^2(N_t)\right]^r}, \tag{11}$$

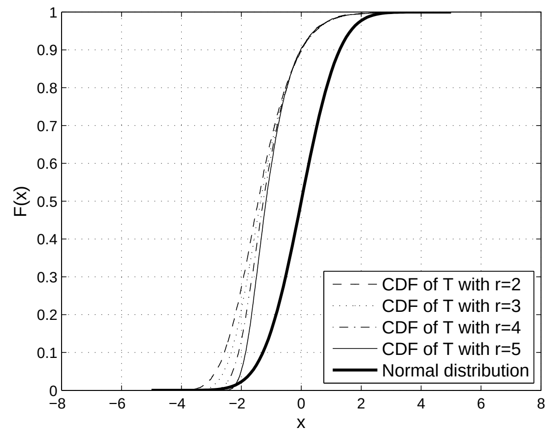

Fig. 1. CDF of the normalized $T_r(N_t)$ with $N_t = 2$, $N_r = 8$, $N = 32$ and SNR$= 0$ dB.

where $\widehat{\sigma}^2(N_t)$ is the maximum likelihood (ML) estimate of the noise variance given by [7]

$$\widehat{\sigma}^2(N_t) = \frac{1}{N_r - N_t}\sum\nolimits_{j=N_t+1}^{N_r} l_j. \tag{12}$$

According to [4], [5], the distribution of $T_r(N_t)$ converges to a normal distribution as

$$T_r(N_t) \xrightarrow{a.s.} \mathcal{CN}\left(\alpha_r, \frac{\beta_r}{{N_r}^2}\right). \tag{13}$$

For $r = 2$, 3, 4, and 5, the parameters α_r and β_r are $\alpha_2 = c+1$, $\alpha_3 = c^2 + 3c + 1$, $\alpha_4 = c^3 + 6c^2 + 6c + 1$, $\alpha_5 = c^4 + 10c^3 + 20c^2 + 10c + 1$, and $\beta_2 = 2c^2$, $\beta_3 = 3c^2(6c^2 + 13c + 6)$, $\beta_4 = 36c^2(2c^2 + 4c + 1)(c^2 + 4c + 2)$, $\beta_5 = 20c^2(2c^3 + 15c^2 + 20c + 5)(5c^3 + 20c^2 + 15c + 2)$, where $c = N_r/N$. Therefore, the normalized test statistic $T_r(N_t)$ asymptotically has a standard normal distribution, which is expressed as

$$\frac{T_r(N_t) - \alpha_r}{\sqrt{\beta_r/{N_r}^2}} \xrightarrow{a.s.} \mathcal{CN}\left(0, 1\right). \tag{14}$$

To demonstrate that (14) is reasonable, curves for the empirical cumulative distribution function (CDF) of the normalized test statistic $T_r(N_t)$ are shown in Fig. 1 when $N_t = 2$, $N_r = 8$, $N = 32$, and SNR$= 0$ dB. For comparison, we also plot the CDF curve of the standard normal distribution.

As shown in Fig. 1, for $r = 2$, 3, 4, and 5, the distribution of the normalized test statistic $T_r(N_t)$ is close to the standard normal distribution, demonstrating that (14) is reasonable, and it has strong reliability under sample-starved conditions.

B. Hypothesis Testing and Threshold

Using the theoretical distribution of the test statistic in (13), we can estimate the number of transmit antennas using a serial binary hypothesis test as

$$\begin{cases} T_r(k) > \gamma_r, & \text{under } \mathcal{H}_1, \\ T_r(k) \leq \gamma_r, & \text{under } \mathcal{H}_0, \end{cases} \tag{15}$$

where $T_r(k) = \frac{1}{N_r-k}\sum_{j=k+1}^{N_r} {l_j}^r \Big/ \widehat{\sigma}^{2r}(k)$ denotes the rth-order moment based test statistic, and γ_r denotes the threshold with $k = 1, 2, \ldots, N_r - 1$. The alternative hypothesis $\mathcal{H}_1$ represents the case when the eigenvalue l_{k+1} is a signal eigenvalue, while the null hypothesis $\mathcal{H}_0$ represents the case when the eigenvalue l_{k+1} is a noise eigenvalue.

In blind estimation, since the unauthorized receiver cannot obtain enough information about the transmitted signals, it is difficult to obtain the distribution function of $T_r(k)$ under $\mathcal{H}_1$. On the other hand, (13) provides the distribution function of $T_r(k)$ under $\mathcal{H}_0$. Therefore, the value of the threshold γ_r can be calculated under $\mathcal{H}_0$.

For the standard normal distribution $\mathcal{CN}(0,1)$, we can set a double-sided threshold t. According to (14),

$$-t \le \frac{T_r(N_t) - \alpha_r}{\sqrt{\beta_r / {N_r}^2}} \le t. \tag{16}$$

Then, the threshold γ_r can be derived as

$$\gamma_r = \alpha_r + t\sqrt{\beta_r / {N_r}^2}, \tag{17}$$

with $r = 2$, 3, 4, and 5.

The main procedures of the proposed higher-order moments based hypothesis testing (HOM-HT) algorithm are summarized in **Algorithm 1**.

Algorithm 1 HOM-HT

Obtain the eigenvalues of the SCM $\mathbf{R}$ by EVD:
$$l_1 > l_2 > \cdots > l_{N_t} > l_{N_t+1} > \cdots > l_{N_r};$$
for $k = 1$ to $N_r - 1$ **do**
  Calculate the value of the test statistic
$$T_r(k) = \frac{1}{N_r - k}\sum\nolimits_{j=k+1}^{N_r} {l_j}^r \Big/ \widehat{\sigma}^{2r}(k)\ ;$$
  Calculate the value of the threshold $\gamma_r(k)$ using (17);
  if $T_r(k) \le \gamma_r(k)$ **then**
    break;
  end if
end for
Output: $\widehat{K}_{\mathrm{HOM-HT}} = k$.

In addition, as this algorithm exploits the distribution properties of higher-order moments composed by the eigenvalues from noise subspace, it is only valid for $N_r > N_t$.

Theorem 1 (Asymptotic consistency): The HOM-HT algorithm is an asymptotically consistent estimator when the sample size N goes to infinity with N_t and N_r fixed, i.e.,

$$\lim_{N\to\infty} P\left\{\widehat{K}_{\mathrm{HOM-HT}} = N_t\right\} = 1, \tag{18}$$

Proof: See Appendix A. ∎

IV. Numerical Results and Discussion

In this section, we compare the empirical performance of the proposed HOM-HT algorithm with the conventional algorithms, and then illustrate the impact of the sample length on the performance of the HOM-HT algorithm. The simulations are implemented in an i.i.d. flat Rayleigh fading MIMO channel with complex valued signals and complex Gaussian noise. The double-sided threshold of the HOM-HT algorithm is set to $t = 1.6$. The transmitted data are modulated into QPSK symbols and the MIMO space-time mode is BLAST using spatial multiplexing. The performance metric is probability of detection, which is calculated over 10000 trials of Monte Carlo simulations.

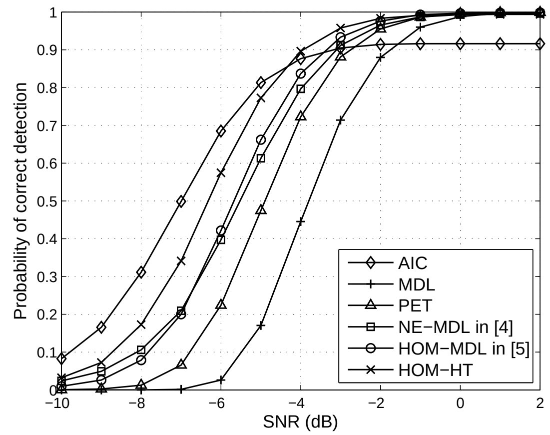

Fig. 2. Comparison of the empirical performance of the HOM-HT algorithm with the AIC, MDL, PET, NE-MDL, and HOM-MDL algorithms ($N_t = 2$, $N_r = 8$, $N = 100$, and $r = 5$).

Fig. 2 shows the detection probability of the HOM-HT algorithm with $r = 5$, the conventional AIC, MDL algorithms [2], and the predicted eigenvalue threshold (PET) based algorithms [6], the secondary moment based improved MDL algorithm [4], and the higher-order moments based improved MDL algorithm [5]. The results demonstrate that the HOM-HT algorithm reaches 100% correct estimation the fastest. Specifically, when $N_t = 2$, $N_r = 8$ and $N = 100$, the HOM-HT algorithm can achieve 98% correct estimation. The AIC algorithm cannot make a consistent estimation despite the fact that its probability of detection is higher than the other algorithms at low SNRs.

Impact of the sample length on the performance of the HOM-HT algorithm is shown in Fig. 3. Obviously, the probability of correct estimation can be improved by increasing the number of samples from $N = 32$ to $N = 128$. Specifically, when $N_t = 8$ and $N_r = 16$, the improvement is 4 dB. In Fig. 4, the impact of the receive-antenna number on the performance of the HOM-HT algorithm is illustrated. The probability of correct estimation can be improved by increasing the receive-antenna number from $N_r = 8$ to $N_r = 12$. Specifically, when $N_t = 4$ and $N = 128$, the improvement is 2 dB. On the other hand, the order of moments also impacts the performance slightly. As can be seen from Figs. 3 and 4, the 5th-order moment has the fastest convergence speed. As a consequence, simulation results show that the HOM-HT algorithm compares favorably with the conventional algorithms. With the larger noise subspace, a higher detection probability can be achieved, since the implementation of the HOM-HT algorithm is based on the properties of the noise subspace.

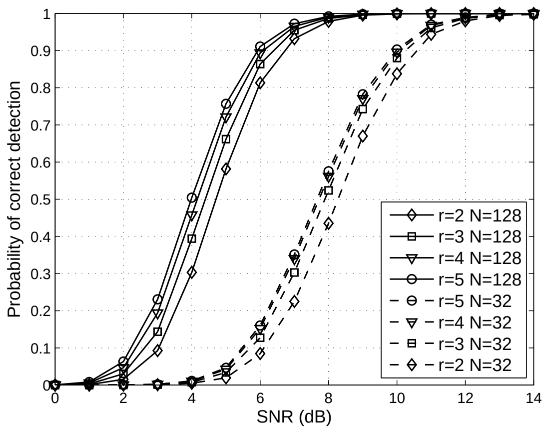

Fig. 3. Comparison of the empirical performance of the HOM-HT algorithm with different values of N ($r = 2$, 3, 4, and 5, $N_t = 8$, and $N_r = 16$).

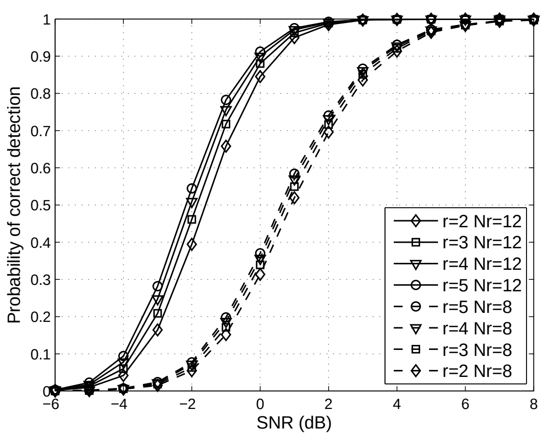

Fig. 4. Comparison of the empirical performance of the HOM-HT algorithm with different values of N_r ($r = 2$, 3, 4, and 5, $N_t = 4$, and $N = 128$).

V. Conclusion

In this letter, we have proposed a higher-order moments based hypothesis testing algorithm to estimate the transmit-antenna number for non-cooperative MIMO systems. Theoretical analysis and simulation results have shown that the proposed algorithms can obtain a reliable estimate at a relatively low SNR, and a consistent estimation. In engineering practice, the algorithm can provide significant technical support for various civil and military wireless communication scenarios, such as cognitive radio, eavesdropping, countermeasure, and civilian security-monitoring.

Appendix A
Proof of Theorem 1

When $N \to \infty$, the SCM tends to the population covariance matrix, i.e., $\mathbf{R} \to \mathbf{\Sigma}$. Furthermore, the eigenvalues of $\mathbf{R}$ in (5) tend to the eigenvalues of $\mathbf{\Sigma}$ in (3) with probability one.

When $N \to \infty$, we obtain $\alpha_r \to 1$ and $\beta_r \to 0$. Then, from (17), we have

$$\lim_{N\to\infty} \gamma_r = 1^+. \tag{19}$$

For hypothesis $\mathcal{H}_1$, when $1 \le k \le N_t - 1$, from (11) we have

$$\lim_{N\to\infty} T_r(k) = \frac{M_r^{\mathbf{R}}(k)}{\widehat{\sigma}^{2r}(k)} > 1. \tag{20}$$

Hence, when $N \to \infty$, $T_r(k) > \gamma_r$ is always true. According to the decision criterion in (15), we obtain

$$\lim_{N\to\infty} P\left\{\widehat{K}_{\mathrm{HOM-HT}} \ge N_t\right\} = 1. \tag{21}$$

For hypothesis $\mathcal{H}_0$, when $N_t \le k \le N_r - 1$, from (11), we have

$$\lim_{N\to\infty} T_r(k) = \frac{M_r^{\mathbf{R}}(k)}{\widehat{\sigma}^{2r}(k)} = 1. \tag{22}$$

Hence, when $N \to \infty$, $T_r(k) < \gamma_r$ is always true. According to the decision criterion in (15), we obtain

$$\lim_{N\to\infty} P\left\{\widehat{K}_{\mathrm{HOM-HT}} > N_t\right\} = 0. \tag{23}$$

Therefore, with (21) and (23), we obtain

$$\lim_{N\to\infty} P\left\{\widehat{K}_{\mathrm{HOM-HT}} = N_t\right\} = 1, \tag{24}$$

and the asymptotic consistency of the HOM-HT algorithm is proved.

References

[1] Y. A. Eldemerdash, O. A. Dobre, and M. Öner, "Signal identification for multiple-antenna wireless systems: achievements and challenges," *IEEE Commun. Surveys Tuts.*, vol. 18, no. 3, pp. 1524–1551, thirdquarter 2016.

[2] O. Somekh, O. Simeone, Y. Bar-Ness, and W. Su, "Detecting the number of transmit antennas with unauthorized or cognitive receivers in MIMO systems," in *Proc. IEEE MILCOM*, 2007, pp. 1–5.

[3] B. Nadler, "Nonparametric detection of signals by information theoretic criteria: performance analysis and an improved estimator," *IEEE Trans. Signal Process.*, vol. 58, no. 5, pp. 2746–2756, May 2010.

[4] R. R. Nadakuditi and A. Edelman, "Sample eigenvalue based detection of high-dimensional signals in white noise using relatively few samples," *IEEE Trans. Signal Process.*, vol. 56, no. 7, pp. 2625–2638, July 2008.

[5] E. Yazdian, M. H. Bastani, and S. Gazor, "Source enumeration in large arrays based on moments of eigenvalues in sample starved conditions," in *Proc. IEEE SiPS*, 2012, pp. 79–84.

[6] K. Hassan, C. N. Nzéza, R. Gautier, E. Radoi, M. Berbineau, and I. Dayoub, "Blind detection of the number of transmitting antennas for spatially-correlated MIMO systems," in *Proc. IEEE ITST*, 2011, pp. 458–462.

[7] S. Kritchman and B. Nadler, "Non-parametric detection of the number of signals: hypothesis testing and random matrix theory," *IEEE Trans. Signal Process.*, vol. 57, no. 10, pp. 3930–3941, Oct. 2009.

[8] R. R. Nadakuditi and J. W. Silverstein, "Fundamental limit of sample generalized eigenvalue based detection of signals in noise using relatively few signal-bearing and noise-only samples," *IEEE J. Sel. Topics Signal Process.*, vol. 4, no. 3, pp. 468–480, June 2010.

[9] M. Mohammadkarimi, E. Karami, O. A. Dobre, and M. Win, "Number of transmit antennas detection using time-diversity of the fading channel," *IEEE Trans. Signal Process.*, vol. PP, no. 99, pp. 1–1, Aug. 2017.

[10] Y. Li, T. Li, M. Hu, J. Hu, and H. Zhang, "Wishart-matrix based blind estimation of transmit-antenna number for non-collaborated MIMO systems," in *Proc. IEEE MILCOM*, 2015, pp. 884–888.

[11] A. Edelman and N. R. Rao, "Random matrix theory," *Acta Numerica*, vol. 14, no. 14, pp. 233–297, 2005.

[12] V. A. Marčenko and L. A. Pastur, "Distribution of eigenvalues for some sets of random matrices," *Math. USSR-Sbornik*, vol. 1, no. 1, pp. 507–536, 1967.

[13] I. M. Johnstone, "On the distribution of the largest eigenvalue in principal components analysis," *Ann. Stat.*, pp. 295–327, 2001.