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DECISION MAKERS BEST CHOICE: A COMPARATIVE INVESTIGATION INTO THE EFFICIENCY OF SEARCH STRATEGIES BASED ON RANKS

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1 Introduction

Since Stigler's well known article [Stigler, 1961, McQueen, 1960] a variety of papers have been published on the economics of information. Important and well known applications of this theory are consumer search for information in price and quality of goods, search of unemployed workers for a job in the labor market (see for example [Lippmann, 1980]) and the search of unwed individuals searching for a marriage partner (see for example [Becker and Landes, 1977]). In economics less well known is the application of search models in the design of (randomized) algorithms [Janko, 1976] and online algorithms. In this application, for example, an algorithm searches within a pre-specified set of algorithms solving the same problem such that the total loss of the algorithm finally chosen to solve the problem plus the search cost is minimized. Rather independently of the majority of the literature on optimal search several articles have been published analyzing the optimal policy of search if we assume that only ordinal utility only can be assigned to the objects found. The early work usually did not consider search costs [Chow et al., 1964, Chow et al., 1971] and concentrated on asymptotic considerations. The results were thus hardly to compare with the results in the economics of information literature. Just recently in the late nineties authors tried to use these models and compare it with the decision makers behavior [Seale and Rapoport, 1997, Seale and Rapoport, 2000].
Parallel to these research investigations in computer sciences online and offline search problems were investigated especially in their relation to trading, portfolio selection and online-decision problems [Borodin and El-Yaniv, 1998]. There is still an ongoing discussion on Money-Making-algorithms, portfolio selection

algorithms and related online decision problems which is reinforced by the increasing real-time nature of today's decision making in business. Also recent developments in some businesses like the increase of algorithmic trading and the development of dark pools in finance increase the relevance of further such research.

We shall in this paper try to compare the efficiency of search strategies. To compare ordinal utility based cost measures with cardinal utility oriented methods we have to violate some of the basic assumptions of ordinal utility theory introducing search costs which are deductible from the ordinal utility indices. Despite this effect which makes the interpretation of the results more difficult these results yield some insight into the efficiency of drawing without and with recall having linear utility functions and a limited number of observations of discrete and uniformly distributed utility (loss) without being able to observe the actual value of an observation.

For exploratory purposes we shall describe the search model in terms of a consumer searching for the lowest cost of the alternative offered plus search costs. In the description of the problem we follow closely the description given recently in [Bearden et al., 2006].

Decision makers (DMs) must usually choose among alternatives. Alternatives very often are presented sequentially, one at a time. Consequently, in many situations, they must choose options without knowing the full choice set. Consider the problem of deciding when to sell an asset on an open market. On any given day, a trader observes a selling price and must decide whether to sell without knowing what the price will be on the following day. Similarly, in tight markets like real estate, stock, currency, job markets and internet offers - when offers may disappear from the market soon after they appear, potential buyers must quite often make irrevocable decisions without knowing what options will appear on the horizon.

Sequential decision problems have been specified formally in a number of ways. Problems in which the DM is assumed to have complete information about the distribution from which the observations are sampled are referred to as *full-information problems*. There are also *partial-information problems* in which the DM is assumed to know certain features of the distribution from which the observations are drawn (e.g., that it is Gaussian or multinomial) but not others (e.g., the mean and variance of the distribution). The class of problems requiring the weakest assumptions about the DM's state of knowledge regarding the distribution from which the observations are drawn are known as *no-information problems*. As models of real-world sequential decision problems, this latter class seems to have a number of virtues. We will illustrate these by first noting the drawbacks of the full- and partial-information problems.

Consider the full-information sequential search problems. It seems doubtful that a DM trying to decide when to accept an asset has always perfect information about the distribution of price changes for that asset. To model the asset-selling decision problem as one involving full information is therefore often unrealistic. One might argue that a partial-information formulation might be more defensible. One can assume that the DM is well informed and knows that the returns

are lognormally distributed, for example, but that he or she does not know the parameters of the distribution. This formulation may be realistic for the asset selling problem. However, both the full- and partial-information formulations become difficult when decision alternatives cannot be characterized by single attributes like in multi-attribute decision making situations or decisions are made very rarely or only once. Even it is apparent in multi-attribute situations that we attribute some utility to the alternatives it is much easier to construct an ordinal utility index than a cardinal utility. An ordinal utility index requires only a weak preference ordering between every pair in the set of alternatives and a ordinal utility index can be found which is invariant to monotone transformations. (In the case of indifference we have to build quotient sets, which allow a strong preference relation to be found on them [Ferschl, 1975]). Such it is considerably easier to match them according to their "quality" In the no-information problems it is only necessary to be able to rank the alternatives according to their quality and it is not necessary that the DM knows the (multivariate) distributions and eventually some properties of these. In investigating these problems it is relative difficult to search "optimally". This especially because search cost has to be put into relation to the ranks of the alternatives , or in other words some ordinal utility index has to be compared with actual search cost, which are typical cardinal measures. Recent experiments [Zwick et al., 2003] did just this. Therefore we have for reasons of comparison also introduced search cost in rank models.
Rank models of search have been investigated since Gardner described the classical secretary problem (CSP) in his contribution to mathematical games in the Scientific American 1960 [Gardner, 1960]. Since than many contributions investigated this problem with various scientific achievements and results. Especially the case where not only the best solution was at least after some time considered successful or the alternatives were appearing following some stochastic distribution in their time of appearance. Many of the problems considered the limiting case when the number of choices goes to infinity. Most recently authors around Rapoport [Bearden et al., 2006, Zwick et al., 2003, Seale and Rapoport, 1997, Seale and Rapoport, 2000] investigated the behavior of DM in such varying contexts assuming a "secretary problem"-like environment. In these papers the behavior of DMs is experimentally investigated. In the first paper the DM is compared with the theoretical results of a generalized secretary problem where single alternatives are presented in random order, one at a time, and only the rank order of the current alternatives relative to the ones that already have been observed can be ascertained. The DM may in each period accept an alternative or observe a new alternative at a fixed cost or recall an alternative already observed. This alternative is assumed to be available with a known probability which decreases with age. The DMs goal is to select the overall best alternative from a fixed set. Experimentally the set size and the search cost were varied. On account of the observed behavior of either searching not enough (for example [Newmann, 1977, Dickson and Sawyer, 1990]) with no search costs or searching too much under the cost conditions, they propose as immediate contribution to investigate the result that consumers under certain circumstances search to

much or too little. They propose further in their paper to investigate situations where fixed cost per inspection, recall with a geometric decay function and an objective of selecting nothing but the best should be altered to more realistic everyday assumptions where only “satisfactory” results are searched for.
In the research of Bearden [Bearden et al., 2006] exactly this is done using the analytical results of Yeo and Yeo [Yeo and Yeo, 1994].Starting from a general discussion of different papers relaxing the usual CSP assumptions it is assumed that the DM’s payoff increases monotonically in the quality of the selected applicant so that the lower the rank the higher the payoff leaving out the case where many ranks can eventually have equal value (like a problem given in Janko 1978 [Janko, 1978]). Bearden [Bearden et al., 2006] chose a multi-threshold rule with monotone payoffs $w_1 \geq w_2 \geq w_3 \geq ... \geq w_k$ assigned to the k best applicants. Mucci showed that this rule is optimal when $r_1 \leq r_2 \leq ... \leq r_n = n$. If we denote by $r = (r_1, r_2, ..., r_k)$ the multi threshold rule threshold values for any monotone order k stopping rule. Yeo and Yeo (1994) showed that the choice of an optimal stopping rule $\mathbf{r*}$ consists in finding the vector $\mathbf{r}$ which maximizes $Q(\mathbf{r}) = \sum_{a=1}^{k} w_a P(a|\mathbf{r})$ where $P(a|\mathbf{r})$ is the probability of selecting the $a-$th best applicant given policy $\mathbf{r}$. Its value is given by [Yeo and Yeo, 1994]

$$P(a|\mathbf{r}) = \frac{1}{n\binom{n-1}{a-1}} \sum_{d=1}^{k} \sum_{j=r_d}^{r_{d+1}-1} \prod_{i=1}^{d} \frac{r_i - 1}{j - 1} \sum_{s=1}^{min(d,a)} \binom{j-1}{s-1}\binom{n-j}{a-s} \qquad (1)$$

for $a = 1, 2, ..., n - r_1$, where s denotes the relative rank of the $j - th$ applicant. Using n-dimensional lattices the policies can be calculated.
An example of an optimal policy is given for $n = 50$, $k = 5$ and payoffs $w_1 = 16, w_2 = 8, w_3 = 4, w_4 = 2$ and $w_5 = 1$ by the optimal threshold values $r_1^* = 17, r_2^* = 36, r_3^* = 44, r_4^* = 47$ and $r_5^* = 49$.
The probabilities of selecting the first, second, etc. best applicant under the optimal rule are $0.351, 0.227, 0.128, 0.067$ and 0.033. The probability of no selection is 0.166 and the expected payoff with the optimal policy is 8.11. This setting of the problems seems to include some pitfalls.
This latter policy implies that we assign “cardinal utility” to ranks as we calculate average payoffs. It also implies that a DM, who assigns utilities to certain ranks does have an idea of an herewith implied probability distribution. In the case of the above example - assuming for simplicity sampling with replacement - we get $p(X = i) = 1/50$ for $i = 16, 8, 4, 2, 1$ and $p(X = 0) = 45/50$. (We easily can reformulate this for a policy without replacement using the multidimensional hypergeometric distribution). The expected value of the outcome Y is than observed in a sample of size m is $E(Y) = m \sum ip_i$. Therefore for example the expected value of a sample of size 14 is $\sum E(Y) = \frac{1}{50}(16+8+4+2+1)*14 \simeq 8.6$. Thus a DM would be better off with such a sample than in the multi-threshold policy assuming he is able to understand the correct distribution. Otherwise he seems also not to be able to assign utilities.
In the following pages we want to compare alternative decision rules under the *no-information* case and to discuss their relevance.

2 Comparison of No-Information Search Strategies

Using ordinal utility means that we assume the existence of a weak preference ordering on the set of consequences of the alternatives only. After putting all consequences to which we are indifferent in quotient classes we are able to introduce an ordinal utility function on the quotient set. We assume here that the quotient set consists of a finite number of elements only and the ordinal utility index is constructed by sorting the equivalence classes. (This is possible in 0(n log n) time with n equivalence classes.) We shall furthermore for reasons of clarity and simplicity assume here that every equivalence class contains only one element[1]. The resulting problem is then to draw with or without replacement from a finite set of the first n natural numbers {1,2,...,n}, which represent the ordinal indices. Drawing an offer means that we are able to determine the rank of the utility of the offer within the offers we already got. We are not able to determine the ordinal utility index of the utility of the offer in the quotient set until we have drawn all offers. The efficiency of rank oriented search strategies can be compared with distribution oriented search strategies only if we assume search costs of zero or if we introduce search costs which are deductible from the ordinal utility index. We shall choose this latter more realistic possibility and assume fixed search costs c for every observation throughout. We shall compare the results we get investigating these rank-oriented strategies without recall with the results in a search with recall. For reasons of simplicity we shall assume a linear cardinal utility function of the observer. For the consumer search example chosen we assume whenever it is plausible – that the goods are described by the characteristic, 'price'. Sampling from this distribution is assumed to be costly. The cost of observation is constant and equal to c. Once an observation is drawn at cost c, the price can be observed without cost.

2.1 The Optimal Strategy for a Finite Sets of Offers without Recall

Rank oriented stopping strategies have been studied extensively in literature. Only recently search costs c were included into the considerations [Seale and Rapoport, 1997, Seale and Rapoport, 2000, Zwick et al., 2003, Bearden et al., 2006].

Suppose now that a consumer is offered n price quotations. The consumer can observe the offers only one at a time as he has no prior information on their true rank in sorting order (according to their negative utility $\ell(i)$). (With sorting order we mean that $\ell(1) \leq \ell(2) \leq ... \leq \ell(i) \leq \ell(i+1) \leq ... \leq \ell(n)$ is valid.) The only information the searcher can rely on, is the information about the relative rank of the latest offer observed by the searcher within the offers already observed before. We assume in the following considerations that the searcher

[1] We such avoid the necessity to consider the problem of drawing from multi-sets of ordinal indexes (with and without replacement).

will not be able to observe – whether intentionally or unintentionally – an offer already observed previously. We will furthermore in the derivation assume that the searcher, once he has decided not to accept a particular object, can never go back and select it at a later stage[2]. Now let us assume that Y_m denotes the random variable of the relative rank of the $m-th$ observed offer within the observations already drawn. The random variables

$$Y_1, Y_2, ..., Y_n \tag{2}$$

are independently distributed and we get for the probability

$$W(Y_m = j) = 1/m \text{ for j=1,2,...,m.} \tag{3}$$

Now let R denote the true rank of the offer observed and let R_m be the random variable "true" rank when we assume that the $m-th$ element drawn has the relative rank j within the offers observed so far. Let $E(R_m|Y_m = j)$ be the expected true rank assuming $Y_m = j$. Now obviously the following relation is valid:

$$E(R_m|Y_m = j) = \sum_{b=j}^{n-m+j} \ell(b) W(R_m = b|Y_m = j) \tag{4}$$

where $\ell(b)$ is the loss function.
For the unconditional expectation $E(R_m)$ we get therefore:

$$E(R_m) = \sum_{i=1}^{m} E(R_m|Y_m = i) W(Y_m = i) \tag{5}$$

Assuming that the true rank is equal to the loss index, that means $l(i) = i$, we get

$$E(R_m|Y_m = j) = \sum_{r=j}^{n-m+j} W(R_m = r|Y_m = j) r. \tag{6}$$

An offer with true rank r has the relative rank j iff the $j-1$ offers are drawn out of $r-1$ offers with the true ranks $1, 2, ...r-1$ and $m-j$ offers were drawn out of the $n-r$ offers with the true rank $r+1, r+2, ..., n$.
Therefore we get

$$W(R_m = r|Y_m = j) = \frac{\binom{r-1}{j-1}\binom{n-r}{m-j}}{\binom{n}{m}} \tag{7}$$

and from the definition above

$$E(R_m|Y_m = j) = \sum_{r=j}^{n-m+j} \frac{r\binom{r-1}{j-1}\binom{n-r}{m-j}}{\binom{n}{m}} = \frac{(n+1)}{m+1} j \tag{8}$$

[2] This is usually called "searching without recall."

Using this result we get for the expected true rank of the $m-th$ offer observed

$$E(R_m) = E(\frac{n+1}{m+1}Y_m) = \frac{n+1}{m+1}E(Y_m) \tag{9}$$

Using the principle of backward induction (see for example [Chow et al., 1971]) we get for the expected rank when only one offer is left:

$$E(R_n) = E(\frac{n+1}{n+1}j) = \frac{1}{n}\sum_{i=1}^{n} j = \frac{n+1}{2} \tag{10}$$

Assuming now that we have drawn $n-1$ observations we should only observe the $n-th$ observations if the expected rank plus the search cost, which must in our derivation be expressable in units of ranks is lower than the expected rank at the $(n-1)-th$ step; we get therefore

$$v_{n-1} = E(min(\frac{n+1}{n}Y_{n-1}, v_n + c)) = \frac{1}{n-1}\sum_{j=1}^{n-1} min(\frac{n+1}{n}j, v_n + c) \tag{11}$$

where v_k denotes the expected loss (expressed in rank units) at the $k-th$ observation and $v_n = E(R_n) = \frac{n+1}{2}$.
Similarly we get for the expected loss at the $i-th$ observation:

$$v_i = E(min(\frac{n+1}{i+1}Y_i, v_{i+1} + c)) = \frac{1}{i}\sum_{j=1}^{i} min(\frac{n+1}{i+1}j, v_{i+1} + c) \tag{12}$$

Computing successively the values of $v_{n-1}, v_{n-2}, ..., v_1$, we get the expected value of the strategy v_1.
This can be simplified if we use the for practical purposes indeed necessary reservation index s_i for stopping with a relative rank $\leq s_i$ at the $i-th$ observation. We get the reservation index vector, which is an integer valued function of the number of observations drawn by the following considerations: Stopping with the $i-th$ observation implies

$$\frac{n+1}{i+1}Y_i \leq v_i + 1 + c \tag{13}$$

or in terms of the implicit relative rank which implies stopping at the $i-th$ observation

$$s_i = [\frac{i+1}{n+1}(v_{i+1} + c)] \text{ for i=n-1,...,2,1.} \tag{14}$$

Using s_i and the fact that the relation

$$v_i = \frac{n+1}{i+1}E(Y_i|Y_i \leq s_i) + W(Y_i > s_i)(v_{i+1} + c) \tag{15}$$

is valid we get:

$$v_i = \frac{1}{i}(\frac{n+1}{i+1}(1 + 2 + ... + s_i) + (i - s_i)(v_{i+1} + c)) = \tag{16}$$

$$= \frac{1}{i}(\frac{n+1}{i+1}\frac{s_i(s_i+1)}{2} + (i - s_i)(v_{i+1} + c)) \quad (17)$$

Using $v_n = \frac{n+1}{2}$ we can easily calculate the reservation rank vector and the expected value of this strategy v_1.
For the following calculation we used search costs of $c = 1$ rank units and price offers for goods of identical utility assuming that the price represents the loss.

The matrix $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 10 \\ 2 & 5 & 7 & 8 & 9 & 10 \end{smallmatrix}\right)$ expresses the decision rule that until a sample size of 2 you should stop with relative rank 1, with the sample sizes 3, 4, 5 you should stop with a relative rank equal to or better than 2, and so on. If you do this your expected cost for searching and purchasing would be 5.5 rank units assuming search costs of $c = 1$ (rank units).

2.2 The "Classical Secretary Problem" as Decision Rule

As we can see by the numerical examples given the unlimited acceptance of offers of any relative rank using their true rank as a representation of their (cardinal) utility involves for a large number of offers some calculation and permanent sorting and memorizing of offers already observed, although these offers are not valid anymore. The other extreme would be to accept only offers with the possible true rank 1. These offers must naturally also have the relative rank of 1. As DeGroot'[DeGroot, 1970] shows, this is the other extreme to the strategy considered above. The problem of decision rules given for example in [Bearden et al., 2006, Zwick et al., 2003] is that many variations are possible. Therefore we consider here only the extreme cases of rank based decision rules without recall. The strategy considered below is equivalent to a strategy which maximizes the probability to find the offer with true rank 1. If search costs are 0 it is well known that asymptotically the observer should initially observe n/e offers and then stop with an offer which is relatively better than the best of these n/e offers already observed in a learning phase. If we introduce search costs of c the behavior of this strategy has to be reconsidered. The problem is formulated as a rank maximizing problem for decreasing sorting order $\ell(n), \ell(n-1), ..., \ell(1)$.

Lemma: The probability, that the true rank of the offer chosen is v and the number of the initially drawn subset of the total sample (learning set) is k, is equal to

$$W(N = v, L = \ell) = \frac{k}{\ell - 1}\frac{(v-1)!(n-\ell)!}{(v-\ell)!n!} \quad (18)$$

There is a probability that the offer with true rank one was already observed within the first k offers.

n	Search Cost Plus Price	Relative Rank to Stop (Sample Size until which relative rank above has to be applied)
3	2.0	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
4	2.5	$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \end{pmatrix}$
5	3.0	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix}$
6	3.5	$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 4 & 5 & 6 \end{pmatrix}$
7	4.0	$\begin{pmatrix} 1 & 2 & 4 & 7 \\ 2 & 4 & 6 & 7 \end{pmatrix}$
8	4.5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 8 \\ 2 & 4 & 6 & 7 & 8 \end{pmatrix}$
9	5.0	$\begin{pmatrix} 1 & 2 & 3 & 5 & 9 \\ 2 & 4 & 7 & 8 & 9 \end{pmatrix}$
10	5.5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 10 \\ 2 & 5 & 7 & 8 & 9 & 10 \end{pmatrix}$
15	7.03	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 15 \\ 2 & 5 & 9 & 11 & 12 & 13 & 14 & 15 \end{pmatrix}$
20	8.44	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 10 & 20 \\ 3 & 6 & 10 & 13 & 15 & 17 & 18 & 19 & 20 \end{pmatrix}$
25	9.81	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 13 & 25 \\ 3 & 7 & 11 & 15 & 18 & 20 & 21 & 22 & 23 & 24 & 25 \end{pmatrix}$
30	10.99	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 & 15 & 30 \\ 3 & 7 & 12 & 16 & 20 & 23 & 25 & 26 & 27 & 28 & 29 & 30 \end{pmatrix}$
35	12.0	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 & 13 & 18 & 35 \\ 3 & 8 & 12 & 17 & 21 & 25 & 28 & 30 & 31 & 32 & 33 & 34 & 35 \end{pmatrix}$
40	12.98	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 & 12 & 15 & 20 & 40 \\ 3 & 8 & 13 & 18 & 23 & 27 & 31 & 33 & 34 & 36 & 37 & 38 & 39 & 40 \end{pmatrix}$
45	13.95	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 14 & 17 & 23 & 45 \\ 4 & 9 & 14 & 19 & 24 & 29 & 33 & 36 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 \end{pmatrix}$
50	14.89	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 19 & 25 & 50 \\ 4 & 9 & 15 & 20 & 25 & 31 & 35 & 39 & 41 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 \end{pmatrix}$
55	15.75	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 17 & 21 & 28 & 55 \\ 4 & 9 & 15 & 21 & 27 & 32 & 37 & 42 & 45 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 \end{pmatrix}$
60	16.51	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 16 & 19 & 23 & 30 & 60 \\ 4 & 10 & 16 & 22 & 28 & 34 & 39 & 44 & 48 & 50 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 \end{pmatrix}$
65	17.27	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 17 & 20 & 25 & 33 & 65 \\ 4 & 10 & 16 & 22 & 29 & 35 & 41 & 46 & 50 & 54 & 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 & 65 \end{pmatrix}$
70	18.02	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 16 & 18 & 22 & 26 & 35 & 70 \\ 4 & 11 & 17 & 23 & 30 & 36 & 42 & 48 & 53 & 57 & 59 & 61 & 62 & 64 & 65 & 66 & 67 & 68 & 69 & 70 \end{pmatrix}$
75	18.75	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 16 & 17 & 20 & 23 & 28 & 38 & 75 \\ 5 & 11 & 17 & 24 & 31 & 37 & 44 & 50 & 55 & 60 & 63 & 65 & 66 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 \end{pmatrix}$
80	19.48	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 17 & 19 & 21 & 25 & 30 & 40 & 80 \\ 5 & 11 & 18 & 25 & 32 & 38 & 45 & 52 & 58 & 62 & 66 & 68 & 70 & 72 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 \end{pmatrix}$
85	20.20	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 18 & 20 & 22 & 26 & 32 & 43 & 85 \\ 5 & 12 & 18 & 25 & 32 & 39 & 46 & 53 & 60 & 65 & 69 & 72 & 74 & 76 & 77 & 78 & 79 & 80 & 81 & 82 & 83 & 84 & 85 \end{pmatrix}$
90	20.83	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 19 & 21 & 24 & 28 & 34 & 45 & 90 \\ 5 & 12 & 19 & 26 & 33 & 40 & 48 & 55 & 62 & 67 & 72 & 75 & 78 & 80 & 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 \end{pmatrix}$
95	21.44	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 20 & 22 & 25 & 29 & 36 & 48 & 95 \\ 5 & 12 & 19 & 27 & 34 & 41 & 49 & 56 & 63 & 70 & 75 & 79 & 82 & 84 & 85 & 86 & 87 & 88 & 89 & 90 & 91 & 92 & 93 & 94 & 95 \end{pmatrix}$
100	22.06	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 18 & 19 & 21 & 23 & 26 & 31 & 38 & 50 & 100 \\ 5 & 12 & 20 & 27 & 35 & 42 & 50 & 58 & 65 & 72 & 78 & 82 & 85 & 87 & 89 & 90 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 \end{pmatrix}$

$$\text{The sum} \sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} \frac{(v-1)!k(n-\ell)!}{(v-\ell)!(\ell-1)n!}$$

therefore is equal to $1 - \frac{k}{n}$. It is important to recognize that this strategy will be formulated as a rank maximizing strategy. Using a different sorting order this does not cause a problem. If we denote the event, that the best offer is not observed within the sample of the first k offers with T, we get the following theorem.

Theorem 1[3]*: The conditional expectation $E(R|T)$ of the true rank of the offer accepted by the classical policy assuming that the best offer is not within the first k offers observed is given by:*

$$E(R|T) = \frac{1}{2}(\frac{(2k+1)(n+1)}{(k+1)} - \frac{(2n+1)k}{n})\frac{n}{n-k} \tag{19}$$

The variance of this mean value $\sigma^2_{(R|T)}$ is equal to

$$\sigma^2_{(R|T)} = (\frac{2n+1}{3})E(R|T) - E^2(R|T) + \frac{n(n+1)k}{3(k+2)} \tag{20}$$

The expected number of offers observed such is

$$E(L|T) = \frac{nk}{(n-k)} \sum_{\ell=k}^{n-1} \frac{1}{\ell} \tag{21}$$

and the variance of this mean value is

$$\sigma^2_{(L|T)} = nk + E(L|T)(1 - E(L|T)) \tag{22}$$

If we can assume that the relation $k << n$ is valid we can use the approximation

$$E(L|T) = \frac{nk}{n-k} \sum_{\ell=k}^{n-1} \frac{1}{\ell} \simeq \frac{nk}{(n-k)} log\frac{n}{k} + 0.5 \tag{23}$$

As the reader can easily verify the unconditional expectation of $E_k(R)$ can be interpreted as a continuous, differentiable and concave function of k and we get for the maximum

$k_{max} \simeq [(\frac{n^2+n}{2n+1})^{0.5}]$ for $k << n$ and $0 \leq k \leq n$ using the equation $\frac{dE_k(R)}{dk} = 0$.

The search process considered has two phases:

[3]The proof of this theorem is lengthy and can be found in Appendix A.

- the learning phase, which consists of drawing k observations initially and memorizing the best observation $o*$;
- the actual search phase, which leads to stopping as soon as an observation is made which is better than $o*$.

The choice the searcher has to make is the choice of the number of observations k he wants to draw in the learning phase. To choose an optimal value k^+ of k with respect to fixed search costs c he has to act such that an increase of k^+ by one observation increases the expected rank of the observation stopped with by less than c per unit of search cost. We get the increased search costs implied by an increase of the learning phase (with k observations) by one observation (to a total of $k+1$ observations) by considering the difference $E_{k+1}(L|T) - E_k(L|T)$ $= \Delta E_k(L/T)$. The gain in ranks is equal to

$$E_{k+1}(R|T) - E_k(R|T) = \Delta_k E(R|T) \tag{24}$$

conditional on the event T that the best offer is not observed within the learning phase. So we have to consider the case of the complimentary event $\bar{T}$. $\bar{T}$ denotes the event that the best offer was already observed in the learning phase. We assume that we continue in this case until we have observed all offers realizing then that the best offer in the learning phase must have been the best offer of all offers. We get therefore the expected rank of our rank maximizing strategy using a learning phase of k observations by

$$E_k(R|Pol) = E_k(R|T) + n\frac{k}{n} \tag{25}$$

Similar considerations hold for the value of the total expected search costs of our policy:

$$E_k(L|Pol) = cE_k(L|T) + cn\frac{k}{n} \tag{26}$$

We denote the differences of expected search cost

$$E_{k+1}(L|Pol) - E_k(L|Pol) \tag{27}$$

by $\Delta E_{k+1}(L|Pol)$ and the difference of expected rank

$$E_{k+1}(R|pol) - E_k(R|Pol) \tag{28}$$

by $\Delta E_{k+1}(R|Pol)$.
Therefore we should choose an optimal value k^+ such that

$$\frac{\Delta E_k + (R|Pol)}{\Delta E_k + (L|Pol)} \leq c \leq \frac{\Delta E_{k+1} + (R|Pol)}{\Delta E_{k+1} + (L|Pol)} \tag{29}$$

is valid. (It should be remembered that for $k \simeq 1, 2, ..., [(\frac{n^2+n}{2n+1})^{0.5}]$ the value of $E_k(R)$ is increasing monotonically; for larger k these values decrease. Furthermore $E_k(L|T)$ and $E_k(R|T)$ are increasing monotonically for all values of

$k(k = 1, 2, ..., n-1))$. Our strategy was formulated as a rank maximizing strategy; it suffices now to think of a decreasing sorting order

$$\ell(n), \ell(n-1), ..., \ell(1) \tag{30}$$

of the prices for the different offers. We thus get the ranks for ascending sorting order $E(PR|Pol)$ from $E(R|Pol)$ simply through

$$E(PR|Pol) = n + 1 - E(R|Pol). \tag{31}$$

Obviously it suffices to use $\Delta E(R|Pol)$ as the relation

$$|\Delta E(R|Pol)| = |\Delta(PR|Pol)| \tag{32}$$

is valid.
Now let us assume $c = 1$. Using this value and using above considerations to get optimal values of k for various values of n we get:

k^+	for n between 3 and 1000
1	3-25
2	25-56
3	57-103
4	104-167
5	168-248
6	249-348
7	349-468
8	469-607
9	608-767
10	768-948
11	948-

Using these values of k^+ and calculating the expected search costs and the expected rank of the price we get for selected values of n and $c = 1$:

n	$E(PR\|Pol) + k$	$cE(L\|Pol) + ck$
20	6.5	4.734
30	7.5	8.346
40	9.167	8.85
50	10.83	9.248
60	11	12.989
70	12.25	13.401
80	13.5	13.763
90	14.75	14.084
100	16	14.373
200	22.167	24.454
500	36.278	42.13
1000	53.167	61.668

These results are obviously valid for the case that we assume that we are searching from the finite set {0,1,2,... ,n-1} of price offers without visiting one offer twice[4] and without recall if we reduce the value of $E(PR|Pol) + k$ by one. The effect of not allowing recall in this case is not the only difference. In the. strategies described here we made the following assumptions:

1. The number of offers which will ultimately become available is known precisely.
2. There is no recall of offers once they have been passed over.
3. There is no possibility to evaluate any offers on a priori grounds apart from ranking it relative to other offers already observed.

The difference between the first strategy given and the latter one is that in the latter one we simply maximize the probability of selecting the best alternative or, to put it into other words, the second best is not more acceptable than the worst. The difference between these strategies and the strategies so far described primarily in the economics of search literature lies in the abilities we require from our searcher. In our case we only require that he is able to sort the alternatives observed. Besides we require however for the calculation of the optimal policy that search costs c can be expressed in true rank units or in other words and that the difference in utility between two alternatives is proportional (at least in the average) to the difference in true rank. But we are not restricted in the application of these models to those situations where we observe a random variables value, with a distribution with known or unknown parameters. Even the true rank is in this policies not supposed to be a directly observable quantity. We only assume sortability of the observed alternatives.

2.3 The Strategy with Several Improvements of Offers

The formula for the probability of finding the best solution L_n of n potential solutions is applicable to the case of drawing not more than $m < n$ solutions with regard to these m solutions $(L_1 < L_2 < ... < L_m)$. $L_i < L_k$ means here that L_k is "better than" L_i with regard to some sorting criterion. Using the arguments given we can analyze the case where we find exactly i solutions during the continuation of random sampling *which improve each other successively*; the first one improves the best solution found in the sample of k_o.

Theorem 2: The probability $W(i|k_o, m)$ of having found the best solution L_m in the set of possible solutions $\alpha = \{L_i | i = 1, 2, ..., m; L_1 < L_2 < ... < L_m\}$ and an initial sample of k_o is given by:

[4] Sampling without replacement.

$$W(i|k_o,m) = \frac{1}{m} \sum_{k_i=k_{i-1}+1}^{m} \frac{1}{k_i-1} \sum_{k_{i-1}=k_{i-2}+1}^{k_i-1} \frac{1}{k_{i-1}-1} \cdots \sum_{k_1=k_o+1}^{k_2-1} \frac{k_o}{k_1-1}; \quad (33)$$

$$m \geq k_i > k_{i-1} > k_{i-2} > ... > k_1 > k_o; k_i \geq k_o + 1. \quad (34)$$

Proof:
Drawing randomly from a finite set without replacement can be considered as taking the elements from a specified permutation of the set in the order given by the permutation. Drawing randomly means that each permutation is initially equally likely to appear. Let us assume that the random sample initially drawn is equal to

$$\{L_{j_1}, L_{j_2}, ..., L_{k_o}\} \quad (35)$$

Let the best solution found in the set be L^o. Let us denote by $L_{k_q}(q = 1, 2, ..., i)$ the i solution of α found in the continuation of random sampling, which are each better than all solutions found so far:

$$L^o < L_{k_1} < L_{k_2} < ... < L_{k_i} \; ; \quad (36)$$

$k_q(q = 1, 2, ..., i)$ denotes the $k_q - th$ solution randomly drawn. The probability that the solution out of the first $k_2 - 1$ solutions drawn is the solution $L_{k_1}(k_1 > k_o)$ is $1/(k_2 - 1)$. The probability, that the best solution of the first $k_1 - 1$ solutions is the best solution of the k_o solutions initially drawn, is equal to $k_o/(k_1 - 1)$; summing up the product of these probabilities over the possible values $k_o + 1, k_o + 2, ..., k_2 - 1$ of k_1 we get the probability $W(1|k_o, k_2 - 1)$ that the first improvement is the best solution within $k_2 - 1$ solutions:

$$W(1|k_o, k_2 - 1) = \frac{1}{k_2 - 1} \sum_{k_1=k_o+1}^{k_2-1} \frac{k_o}{k_1 - 1} \; . \quad (37)$$

The probability $W(2|k_o, k_3 - 1)$, that the second improvement of the best solution found so far is the best solution within the first $k_3 - 1$ solutions is equal to the product

$$\frac{1}{k_3 - 1} W(1|k_o, k_2 - 1) \quad (38)$$

summed up over the possible values of k_2 which are given by the sequence $k_1 + 1, k_1 + 2, ..., k_3 - 1$:

$$W(2|k_o, k_3 - 1) = \frac{1}{k_3 - 1} \sum_{k_2=k_1+1}^{k_3-1} W(1|k_o, k_2 - 1) \; . \quad (39)$$

We apply this argument repeatedly and get finally:

$$W(i|k_o,m) = \frac{1}{m} \sum_{k_i=k_{i-1}+1}^{m} W(i-1|k_o,k_i-1) \tag{40}$$

For $i+1 \geq j \geq 1$ and $k_{i+1}-1 = m$ the following relation is valid:

$$W(j-1|k_o,k_j-1) = \frac{1}{k_j-1} \sum_{k_{j-1}=k_{j-2}+1}^{k_j-1} W(j-2|k_o,k_{j-1}-1) \tag{41}$$

with

$$W(0|k_o,k_1-1) = \frac{k_o}{k_1-1}(k_o < k_1 <,...,k_i \leq m) \ . \tag{42}$$

Q.E.D.
Let us apply theorem 2 using a simple example.
For $m=5, k_o=1, i=2$ we get

$$W(2|1,5) = \frac{1}{5} \sum_{k_2=3}^{5} \frac{1}{k_2-1} \sum_{k_1=2}^{k_2-1} \frac{1}{k_1-1} = \frac{35}{120} = 0,292 \ . \tag{43}$$

The 35 permutations where after two improvements and starting with the best solution out of a random sample of size $k_o=1$ the optimal solution is the best solution found are given:

12534	14532	23154	24315	32415
12543	14352	23541	21354	34512
13524	14325	24513	31245	34152
13254	14235	24153	31452	34125
13542	21453	24135	31425	34521
14523	21435	24531	32145	34251
14253	23514	24351	32451	34215

Another example where we can simply check the result using the 24 permutations of the numbers 1234 is given by $b=4$ and $i=2$. For $k_o=1,2$ we get $W(2|1,4) = 1/4$ and $W(2|2,4) = 1/12$.
From the theorem given above we can deduce the following result.

Theorem 3:The probability $W(i|k_o,m)$ is approximately equal to the probability given by the Poisson distribution function with mode $\lfloor\lambda\rfloor$ [5], mean λ and variance λ with the parameter $\lambda = \log(\frac{m}{k_o})$ for $i = 0,1,...(k_o >> i, m >> k_o)$:

$$W(i|k_o,m) \simeq W'(i|k_o,m) = \frac{e^{-\lambda}\lambda^i}{i!} \ . \tag{44}$$

[5] $\lfloor\lambda\rfloor$ stands here for the rounded lower integer in case λ is not an integer. $\lfloor\lambda\rfloor$ is $\lambda-1$ if λ is an integer.

Proof:
The proof can be given by applying Euler's summation formula and simplifying the results. Obviously

$$W'(i|k_o, m) = \frac{k_o \log^i(m/k_o)}{i!\ m} \ . \tag{45}$$

We have to show that $W'(i|k_o, m)$ can be derived from $W(i|k_o, m)$:

$$W(i|k_o, m) = \frac{1}{m} \sum_{k_i=k_o+1}^{m} ... \frac{1}{k_3 - 1} \sum_{k_2=k_o+2}^{k_3-1} \frac{k_o}{k_2 - 1} \sum_{k_1=k_o+1}^{k_2-1} \frac{1}{k_1 - 1} \simeq \tag{46}$$

$$\simeq \frac{1}{m} \sum_{k_i=k_o+i}^{m} ... \frac{1}{k_3 - 1} \sum_{k_2=k_o+2}^{k_3-1} \frac{k_o}{k_2} \log(\frac{k_2}{k_o}) \ . \tag{47}$$

We get

$$\frac{1}{k_{j+1}} \sum_{k_j=k_{j-1}+1}^{k_{j+1}-1} (\frac{k_o}{k_j}) \frac{\log^{j-1}(\frac{k_j}{k_o})}{(j-1)!} \simeq \tag{48}$$

$$\simeq \frac{1}{k_{j+1}} \int_{k_o+j}^{k_{j+1}-1} \frac{1}{(j-1)!} (\frac{k_o}{k_j}) \log^{j-1}(\frac{k_j}{k_o}) dk_j \simeq \tag{49}$$

$$\simeq \frac{k_o}{k_{j+1}(j-1)!} \int_1^{k_{j+1}/k_o} \frac{\log^{j-1} u}{u} du \simeq \frac{k_o}{j!\ k_{j+1}} log^j(\frac{k_{j+1}}{k_o}); \tag{50}$$

$$W(2|1, 6) = \frac{1}{6} \sum_{k_2=3}^{6} \frac{1}{k_2 - 1} \sum_{k_1=2}^{k_2-1} \frac{k_o}{k_1 - 1} = 0,3125. \tag{51}$$

(The probability $W'(2|1, 6)$ is far from being a good approximation in this case as m is only 6).

2.4 The Secretary Problem with an Unknown Number of Applicants

A drawback in applications of the secretary problem is the necessity to know the number of applicants in advance. To avoid this we can assume that the number of applicants is a random variable N with a known distribution $W(N = k)$, $k = 1, 2, ...$ [Presman and Sonin, 1973].
The results in this case are more complicated and this policy gives relatively low optimal win probabilities compared with the CSP. For applications very interesting is the later result of Bruss [Bruss, 1984] called the unified approach. In this case it is assumed that an applicant must be selected on some time interval $[0, T]$ from an unknown number N of rankable applicants. If all arrival permutations are equally likely, suppose that all applicants have the same arrival density f on $[0, T]$ and $F(t)$ is the corresponding arrival time d.f. Let τ be such that $F(\tau) = \frac{1}{e}$. Than consider a strategy where you wait and observe all

applicants up to the time τ and then select the first candidate which is better than the preceding candidates - if possible. This strategy is called the $\frac{1}{e}$-strategy and has the following properties:

a) yields for all N a success probability of at least $\frac{1}{e}$

b) guarantees a lower success probability bound $\frac{1}{e}$ and the bound is optimal

c) selects, if there is at least an applicant, none with probability $\frac{1}{e}$ [Bruss, 1984].

The basic idea of the theorem can be easily described. Instead of using permutations a change of time is used:

$$x = F(t), t\epsilon[0, T] \; . \tag{52}$$

Such each distribution function describing the frequency of arrivals $F(t)$ (unimodal, multimodal, etc.) is transformed to x which is uniformly distributed in $[0, 1]$. If N is the number of arrivals the probability of being optimal is for $N = 1$ equal to 1 and the optimal x is given by $x = 0$. For $N \geq 2$ the strategy yields a success if the best candidate arrives in $]x, 1]$ before all other candidates arrive in $]x, 1]$ which are better than the best of those which arrive in $[0, x]$. Now lets consider the best $k + 1$ candidates. According to the assumptions the $(k+1)$st arrives in $[0, x]$ and the k best ones in $]x, 1]$ with probability $x(1-x)^k$. As the best arrives before the second, third, ..., kth best with probability $\frac{1}{k}$ we get

$$p_n(x) = P(\frac{\text{success of the } x\text{-strategy}}{N = n}) \tag{53}$$

$$= x \sum_{k=1}^{n-1} \frac{1}{k}(1-x)^k + \frac{(1-x)^n}{n} \quad \text{for } n\text{=2,3,...} \tag{54}$$

As we know the taylor series expansion of $\log x$ is $-(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{3}(x-1)^3+\frac{1}{4}(x-1)^4-\frac{1}{5}(x-1)^5+0((x-1)^6)$ we get $\lim_{n\to\infty} p_n(x) = -x \log x$. This function has its unique maximum for $x\epsilon[0, 1]$ at $x = e^{-1}$. Instead of combinatorial arguments probability proportions of a rather arbitrary distribution of arrival were used.(For further discussions of the properties of this policy the reader is refered to Bruss [Bruss, 1984]).

2.5 Rank Oriented Search with Recall

The models presented so far only constitute two extreme examples of a pattern of strategies. In several papers given by Rapaport with coauthors and others (see [Bearden et al., 2006] and the literature cited there) various decision making situations were investigated and empirically compared. To see the efficiency of no-information decision rules we confined ourself to extreme cases without recall. In the following we investigate a simple policy with recall. The models given above concentrate on observations without recall. At least limited recall is in many practical cases available. So we should compare the results derived in a search without recall with the strategies selecting alternatives according to their rank with unlimited[6] recall. A non-sequential rule is investigated. In a sequential policy we would indeed be able to determine a reservation rank but we would not be able to determine in general whether or whether not an offer observed has a true rank lower or equal to this reservation rank. Therefore we shall consider here only a non-sequential strategy. This strategy shall: simply consist in drawing a sample of a priori fixed size and determining the best alternative drawn. This is the alternative to be chosen. To reduce the side effects we have to determine the optimal number of observations, in observations without replacement We assume again that the n alternatives will be sorted in decreasing order of the price of the offers:

$$l(n) \geq l(n-1) \geq ... \geq l(1) \tag{55}$$

Our aim is to maximize the rank in this sorting order; that means $\ell(n)$ has rank 1, $\ell(n-i)$ rank 2 and so on.

Lemma: $X_k = max\{y_1, y_2, ..., y_k\}$ *and the set* $\{y_1, y_2, ..., y_k\}$ *is a random sample of offers drawn without replacement from* $\{1, 2, ..., n\}$*. The probability for drawing an offer with rank* x *is*

$$P(X_k = x) = \frac{\binom{x-1}{k-1}}{\binom{n}{k}}, \; k \leq x \leq n \tag{56}$$

The expected rank observed in a sample of k *alternatives is*

$$E(X_k) = \frac{(n+1)k}{k+1} \tag{57}$$

and its variance $\sigma^2_{X_k}$ *is equal to*

$$\sigma^2_{X_k} = E(X_k)(u - E(X_k)) \tag{58}$$

with $u = \frac{(n+2)(k+1)}{k+2} - 1$
Using now the difference $E(X_{k+1}) - E(X_k)$ we get for the optimal value of k the equation:

$$0 = c(k^2 + 3k) - (n-1) \tag{59}$$

or

$$k_{1,2} = \frac{-3c \mp \sqrt{c2 - 4c(n-1)}}{2c} \tag{60}$$

[6]Limited recall is to the author's knowledge rarely considered in the literature so far. Exceptions are Young, M.C.K. and Ewick, R., et al.

For the special case $c = 1$ we get for the optimal value k^+

$$k^+ = [-1.5 + \sqrt{5/4 + n}] \tag{61}$$

which for large n is approximately equal to $\sqrt{n}$.
The expected true rank will then be

$$E(n + 1 - X_k) = \frac{n + 1}{k^+ + 1} \tag{62}$$

For larger values of $\sqrt{n}$ this will not deviate much from $\sqrt{n}$ giving a total cost of approximately $2\sqrt{n}$ for larger n and $c = 1$. Using our numerical calculations it can be seen that this is less than the policies assuming no recall. This is not only true for the expected cost of the offer plus the search cost but also for the variance of the distribution of cost of the alternative stopped with.

2.6 Conclusions

Several classes of search strategies with and without cost were discussed, which basically do not assume knowledge of the distribution of the observations value or cardinal utility. Rather we assumed in this class of strategies that we are able to sort the offers observed. We are able to distinguish a pattern of different search strategies between these two extreme versions and even with sampling with recall (see for example [Yeo and Yeo, 1994, Yang, 1974]). From these possible strategies, we considered from the CSP only two extreme strategies, one with nothing but the best and the other one with multiple thresholds. All strategies deduced here from the CSP do not allow recall. One observation was allowed per offer and the number n of offers available was assumed to be known in advance. This last assumption at least could be dropped if we follow an alternate approach. It was shown that if we drop the assumption of no recall using the simple fixed sample size strategy, we get a strategy which seems to has less costs. This proposition is however based only on numerical calculations.

Basically the rank oriented strategies without recall, seem to be less efficient than strategies where we assume that the distribution is known and the offers utility (or loss) can be recognized. What we can easily see from our investigation that using search cost (expression in ranks) the success of search is worst (expressed in ranks minus cost) in the CSP.
If we use an optimal decision rule with a finite number of offers and using backward induction we get better results.
In both cases we do not allow recall. When we allow unlimited recall we do not consider decay. In this case the simplest decision rule is to sample from a uniform distribution of ranks and to take the best offer. With a search cost of one we get an expected value of $2\sqrt{n}$. This policy seems to be superior to the optimal policy with backward induction using search cost.
If we attribute cardinal utility values to ranks like it is proposed in Bearden et al. [Bearden et al., 2006] we need not to use rank oriented models. In this case

we can use multinomial or multidimensional hypergeometric models (sampling with or without recall). If again for example we use a cardinal utility value which is equal to the rank (based on a monotone function of the ranks, which we can recognize) we are much better off using a sequential policy based on optimal stopping. With a search cost of $c = 1$ we get for example $\sim \sqrt{2n}$.If we cannot observe the value of the offer in advance we can eventually use correlated attributes to get more information by testing these, if this is feasible [Janko and Hartmann, 1985] . This shall be discussed in a further discussion paper. Two policies based on ranks were further described here allowing in one case for several improvements and in the other case for an unknown number of applicants. No cost comparisons were made in these cases.

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Appendix A

First we show that the probability that the true rank N finally chosen is v when the size of the learning set is k and the number L of observed alternatives ℓ is equal to:

$$W(N=v, L=\ell) = \frac{k}{\ell-1}\frac{(v-1)!(n-\ell)!}{(v-\ell)!n!} \tag{63}$$

If the $\ell-th$ alternative observed has a rank of v and we stop with this alternative implies that $\ell-1$ alternatives observed before were of a rank lower than v. As there are $v-1$ such alternatives we have $(v-1)!/(v-\ell)!$ ordered subsets of $(\ell-1)$ alternatives drawn from $(v-1)$ alternatives. Now the second best alternative was within the first k alternatives observed. As this alternative is observed with equal probability as first, second, ..., $k-th$ alternative, the probability for the second best to be within the first k observations is such $k/(\ell-1)$. Recognizing that there are exactly $n!/(n-\ell)!$ possibilities of drawing ℓ alternatives (sorted) out of n alternatives we get the lemma. We show now that the relation

$$\sum_{\ell=k+1}^{n}\sum_{v=\ell}^{n}\frac{(v-1)!k(n-\ell)!}{(v-\ell)!(\ell-1)n!} = 1-\frac{k}{n} \tag{64}$$

is correct.
We show the correctness by calculation:

$$\sum_{\ell=k+1}^{n}\sum_{v=\ell}^{n}\frac{(v-1)!k(n-\ell)!}{v(v-\ell)!(\ell-1)} = \sum_{\ell=k+1}^{n}\frac{k(n-\ell)!(\ell-1)!}{(\ell-1)n!}\sum_{v=\ell}^{n}\binom{v-1}{\ell-1} =$$

$$\sum_{\ell=k+1}^{n}\frac{k}{(\ell-1)n}\binom{n-1}{\ell-1}^{-1}\binom{n}{\ell} = \sum_{\ell=k+1}^{n}\frac{k}{(\ell-1)\ell} =$$

$$k\sum_{\ell=k+1}^{n}\left(\frac{1}{\ell-1}-\frac{1}{\ell}\right) = k\left(\frac{1}{k}-\frac{1}{n}\right) = 1-\frac{k}{n}$$

We now show the correctness of the formulas in our theorem 1.

$$E(R|T) = b\sum_{\ell=k+1}^{n}\sum_{v=\ell}^{n}v\frac{(v-1)!k(n-\ell)!}{(v-\ell)!(\ell-1)n!} = \frac{bk}{n!}\sum_{\ell=k+1}^{n}\frac{(n-\ell)!}{(\ell-1)}\sum_{v=\ell}^{n}\frac{v!}{(v-\ell)!} =$$

$$\frac{bk}{n!}\sum_{\ell=k+1}^{n}\frac{(n-\ell)!\ell!}{(\ell-1)}\sum_{v=\ell}^{n}\binom{v}{\ell} = b\frac{k}{n!}\sum_{\ell=k+1}^{n}\frac{(n-\ell)!\ell!(n+1)!}{(\ell-1)(\ell+1)!(n-\ell)!} =$$

$$bk(n+1)\sum_{\ell=k+1}^{n}\frac{1}{(\ell-1)(\ell+1)} = \frac{k(n+1)}{2}b\sum_{\ell=k+1}^{n}\left(\frac{1}{\ell-1}-\frac{1}{(\ell+1)}\right) =$$

$$b\frac{k(n+1)}{2}\left(\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{n-1}-\frac{1}{(k+2)}-\frac{1}{(k+3)}-\ldots-\frac{1}{(n+1)}\right)$$

$$b\frac{1}{2}((n+1)+\frac{k(n+1)}{k-1}-\frac{k(n-1)}{n}-k)=$$

$$(\frac{(2k+1)(n+1)}{k+1}-\frac{(2n+1)k}{n})(\frac{n}{2(n-k)})$$

$$E(L|T)=b\sum_{\ell=k+1}^{n}\sum_{v=\ell}^{n}\ell\frac{(v-1)!k(n-\ell)!}{(v-\ell)!(\ell-1)n!}=$$

$$\frac{k}{n!}b\sum_{\ell=k+1}^{n}\frac{\ell(n-\ell)!}{(\ell-1)}\sum_{v=\ell}^{n}\frac{(v-1)!}{(v-\ell)!}=$$

$$\frac{k}{n!}b\sum_{\ell=k+1}^{n}\frac{\ell(n-\ell)!(\ell-1)!}{(\ell-1)}\sum_{v=\ell}^{n}\frac{(v-1)!}{(v-\ell)!(\ell-1)!}=$$

$$\frac{k}{n!}b\sum_{\ell=k+1}^{n}\frac{\ell(n-\ell)!(\ell-1)!}{(\ell-1)}\sum_{v=\ell}^{n}\binom{v-1}{\ell-1}=\frac{k}{n!}b\sum_{\ell=k+1}^{n}\frac{\ell(n-\ell)!(\ell-1)!}{(\ell-1)}\binom{n}{\ell}=$$

$$kb\sum_{\ell=k+1}^{n}\frac{1}{\ell-1}=\frac{nk}{(n-k)}\sum_{\ell=k}^{n-1}\ell^{-1}$$

We have to show

$$E(R^2|T)=\frac{k}{n!}b\sum_{\ell=k+1}^{n}\frac{(n-\ell)!\ell!}{(\ell-1)}\sum_{v=\ell}^{n}\binom{v}{\ell}v.$$

The following relation is valid:

$$\sum_{v=\ell}^{n}v\binom{v}{\ell}=\sum_{v=k}^{n}((v+1)\binom{v}{\ell}-\binom{v}{\ell})=\sum_{v=k}^{n}((\ell+1)\binom{v+1}{\ell+1}-\binom{v}{\ell})=$$

$$(\ell+1)\sum_{v=\ell}^{n}\binom{v+1}{\ell+1}-\sum_{v=\ell}^{n}\binom{v}{\ell}=(\ell+1)\binom{n+2}{\ell+2}-\binom{n+1}{\ell+1}=$$

$$(\frac{\ell+(\ell+1)n}{(\ell+2)}\binom{n+1}{\ell+1}.$$

Using this relation we get:

$$E(R^2|T)=\frac{k}{n!}b\sum_{\ell=k+1}^{n}\frac{(n-\ell)!\ell!(n+1)!}{(\ell-1)(\ell+1)!(n-\ell)!}-\frac{(\ell+1)n+\ell}{(\ell+2)}$$

or

$$E(R^2|T)=\frac{k}{n!}b\sum_{\ell=k+1}^{n}\frac{(n-\ell)!\ell!(n+1)!}{(\ell-1)(\ell+1)!(n-\ell)!}(\frac{(n+2)(\ell+1)}{(\ell+2)}-1)=$$

$$= (n+1)(n+2)kb \sum_{\ell=k+1}^{n} \frac{1}{(\ell-1)(\ell+2)} - bk(n+1) \sum_{\ell=k+1}^{n} \frac{1}{(\ell-1)(\ell+1)} =$$

$$\frac{k(n+1)(n+2)}{3} b \sum_{\ell=k+1}^{n} (\frac{1}{(\ell-1)} - \frac{1}{(\ell+2)}) - \frac{k(n+1)b}{2} \sum_{\ell=k+1}^{n} (\frac{1}{(\ell-1)} - \frac{1}{(\ell+1)}) =$$

$$\frac{2(n+2)}{3} b (\frac{(n+1)k}{2}((\frac{1}{k} + \frac{1}{(k+1)} - \frac{1}{(n+1)} - \frac{1}{n}) + (\frac{1}{(k+2)} - \frac{1}{(n+2)}))) -$$

$$-b\frac{1}{2}(\frac{(2k+1)(n+1)}{(k+1)} - \frac{(2n+1)k}{n}).$$

From this we get

$$E(R|T) = b\frac{(n+1)k}{2}(\frac{1}{k} + \frac{1}{(k+1)} - \frac{1}{(n+1)} - \frac{1}{n}) =$$

$$= \frac{b}{2}(\frac{(2k+1)(n+1)}{k+1} - \frac{(2n+1)k}{n}).$$

We get

$$E(R^2|T) = \frac{2(n+2)}{3}(E(R|T) + \frac{b(n+1)k}{2(k+2)} - \frac{b(n-1)k}{2(n+2)}) - E(R|T) =$$

$$= \frac{2(n+2)}{3}(E(R|T) + \frac{b(n+1)k(n-k)}{2(n+2)(k+2)}) - E(R|T)$$

and using the lemma

$$E((R - E(R|T))^2|T) = E(R^2|T) - E^2(R|T)$$

we get finally:

$$\sigma^2(R|T) = (\frac{2n+1}{3})E(R|T) - E^2(R|T) + \frac{n(n+1)k}{3(k+2)}.$$

Similarly we get the variance $E(L|T)$.

$$E(L^2|T) = \sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} \ell^2 \frac{(v-1)!k(n-\ell)!}{(v-\ell)!(\ell-1)n!} b =$$

$$\frac{k}{n!} b \sum_{\ell=k+1}^{n} \frac{\ell\ell(n-\ell)!(\ell-1)!}{(\ell-1)} \sum_{v=\ell}^{n} \binom{v-1}{\ell-1} = kb \sum_{\ell=k+1}^{n} \frac{\ell}{(\ell-1)} =$$

$$= bk \sum_{l=k}^{n-1} (1 + \frac{1}{\ell}) = nk + E(L|T).$$

Using the lemma above we get

$$\sigma_L^2 = nk + E(L|T) - E^2(L|T).$$