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The evolution of a discontinuity**

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Leland's approach to option pricing: The evolution of a discontinuity

Peter Grandits[*] and Werner Schachinger[†]

Abstract

A claim of Leland (1985) states that in the presence of transaction costs a call option on a stock S, described by geometric Brownian motion, can be perfectly hedged using Black-Scholes delta hedging with a modified volatility. Recently Kabanov and Safarian (1997) disproved this claim, giving an explicit (up to an integral) expression of the limiting hedging error, which appears to be strictly negative and depends on the path of the stock price only via the stock price at expiry S_T. We prove in this paper that the limiting hedging error, considered as a function of S_T, exhibits a removable discontinuity at the exercise price. Furthermore, we provide a quantitative result describing the evolution of the discontinuity, which shows that its precursors can very well be observed also in cases of reasonable length of revision intervals.

Key words: Transaction costs, Hedging

1 Introduction

The proof of the celebrated Black-Scholes formula for option pricing relies basically on two assumptions on the stock market. On the one hand the model for the discounted price process is geometric Brownian motion, and on the other hand transaction costs are neglected. Therefore perfect hedging is possible, and we obtain a unique price for a derivative security, say a European Call option.

In practice one has of course to take into account these "market frictions". A very interesting approach to the problem in the literature is Leland's (1985) . He claims that the price of a call option should be given by the Black-Scholes price with a modified volatility, which depends on the transaction costs, the original volatility and the time interval between successive adjustments of the portfolio. He claimed also that the hedging error can be made arbitrarily small, if the length of the revision intervals tends to zero, and if one uses Black-Scholes delta-hedging with the modified volatility.

In a remarkable recent paper Kabanov and Safarian (1997) showed that this claim is not true, and they were able to compute the *limiting hedging error* (number of revision intervals tending to infinity). The resulting function is a rather involved integral, depending on the path of the stock price only via its value S_T at the expiration time T. Kabanov and Safarian also provided a plot of the result, which insinuates that the limiting hedging error is a continuous function with respect to S_T. To see what happens for a finite number of revision intervals, we have run a Monte Carlo simulation of Leland's strategy, and the very surprising result is provided in Fig. 1. The plot shows

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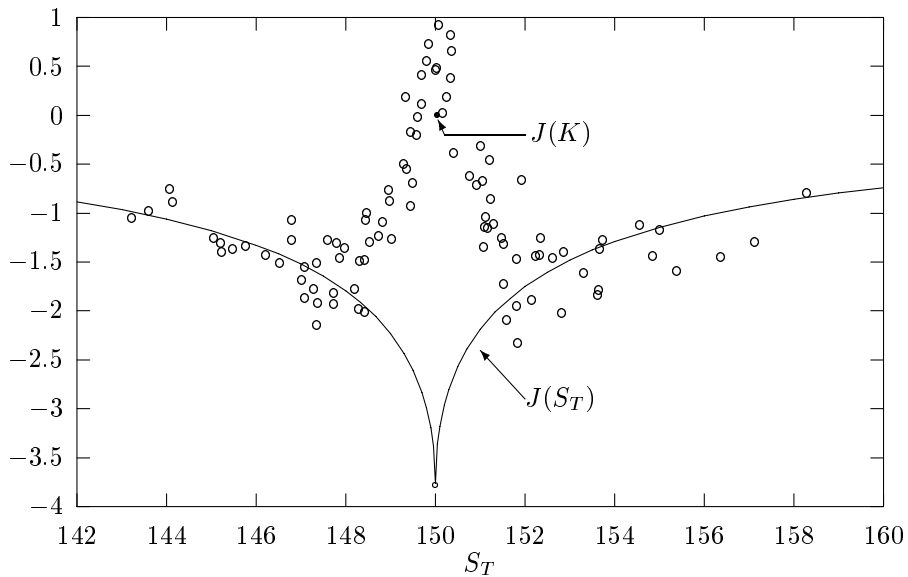
Figure 1: The limiting hedging error J, and the outcomes of 100 Monte Carlo simulations of the hedging error J^n for $n = 1000$, plotted over S_T with parameters: $T = 1$, $K = S_0 = 150$, $\mu = 0$, $\sigma = 0.02$ and $k = 0.05$.

the value of the hedging error for a certain fixed number of revision intervals and the indicated set of parameters. For the definition of these parameters we refer to Section 2.

The aim of the present paper is to find an explanation of the striking peak near $S_T = K$, where K is the exercise price. It will turn out that the limiting hedging error has a removable discontinuity at $S_T = K$. A plot of the limiting hedging error is also provided in Fig. 1. One could think that this is a rather academic problem, because after all the set $\{S_T = K\}$ has probability zero. But a glance on Fig. 1 reveals that this has certain impacts on the hedging result for a fixed (finite) number of revision intervals. Our second result will give an asymptotic estimate of the extent of this remarkable peak, when the length of the revision intervals approaches zero. This will give us an idea in which region of terminal values of the stock price the hedger will feel the influence of the discontinuity.

2   Main results

We start with a description of the model and our notation, which we have chosen close to the one of Kabanov and Safarian. So the stock price movement is given by geometric Brownian motion on the time interval $[0, T]$, and for convenience we set $T = 1$. Therefore S_t is given by

$$(2.1) \qquad\qquad S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t},$$

where W_t is the standard Wiener process. As Kabanov and Safarian did, we also assume that the bond price is constant, and we shall work with the risk neutral measure, i.e. $\mu = 0$. The derivative security we want to hedge is a European Call option with terminal payoff $H = (S_1 - K)^+$. We assume also that the transaction costs are a fixed fraction k of the trading volume. The trading strategy suggested by Leland is the following. Denote by ξ_t^n the number of shares of the stock in the

portfolio at time t, where n denotes the number of revision intervals. Then ξ_t^n is given by

$$\xi_t^n = \sum_{i=1}^n \widehat{C}_x(t_{i-1}, S_{t_{i-1}}, \widehat{\sigma}) I_{]t_{i-1}, t_i]}(t)$$

with $t_i = i/n$,

$$\widehat{C}(t, x, \widehat{\sigma}) = x\Phi(\widehat{d}) - K\Phi(\widehat{d} - \widehat{\sigma}\sqrt{1-t})$$

and

$$\widehat{d}(x, \widehat{\sigma}) = \frac{\ln(\frac{x}{K})}{\widehat{\sigma}\sqrt{1-t}} + \frac{1}{2}\widehat{\sigma}\sqrt{1-t}, \qquad \widehat{\sigma}^2 = \sigma^2(1 + \frac{\gamma}{\sigma}), \qquad \gamma = 2\sqrt{\frac{2}{\pi}}k\sqrt{n}.$$

As usual Φ denotes the standard normal distribution function with density ϕ. Since the initial endowment is given by $\widehat{C}(0, S_0, \widehat{\sigma})$, we end up with the following value process

$$V_t(\xi^n) = \widehat{C}(0, S_0, \widehat{\sigma}) + \int_0^t \xi_u^n dS_u - k\sum_{t_i \le t} S_{t_i}|\xi_{t_i}^n - \xi_{t_{i-1}}^n|.$$

Denoting $J^n = V_1(\xi^n) - H$, by Theorem 2 of Kabanov and Safarian (1997) the limiting hedging error $J = \lim_{n\to\infty} J^n$ is given by $J = J_1 - J_2$, where

$$
\begin{aligned}
J_1(S_1) &= \min\{K, S_1\}, \\
J_2(S_1) &= \frac{1}{4}\int_0^\infty \frac{S_1}{\sqrt{v}} G(S_1, v, k) \exp\left(-\frac{v}{2}\left(\frac{\ln(\frac{S_1}{K})}{v} + \frac{1}{2}\right)^2\right) dv, \\
G(S_1, v, k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left| x - \frac{2k\ln(\frac{S_1}{K})}{\sqrt{2\pi}v} + \frac{k}{\sqrt{2\pi}}\right| e^{-\frac{x^2}{2}} dx.
\end{aligned}
$$

Our first theorem claims that the function $J_2(\cdot)$ has a removable discontinuity at K.

Theorem 2.1

$$\lim_{s\searrow K} J_2(s) = \lim_{s\nearrow K} J_2(s) = J_2(K) + \alpha,$$

where α is given by

$$\alpha = \frac{Kk}{2}.$$

Proof: We restrict ourselves to the proof of $\lim_{s\searrow K} J_2(s) = J_2(K) + \alpha$, since the other limit can be calculated analogously. Defining $\ln(\frac{s}{K}) = \epsilon$, we first compute

$$\lim_{\epsilon\to 0} \frac{1}{4}\int_{\sqrt{\epsilon}}^\infty \frac{s}{\sqrt{v}} G(s, v, k) \exp\left(-\frac{v}{2}\left(\frac{\ln(\frac{s}{K})}{v} + \frac{1}{2}\right)^2\right) dv.$$

A simple application of the dominated convergence theorem yields $J_2(K)$ as limit. On the other interval we get

$$
\begin{aligned}
&\frac{1}{4}\int_0^{\sqrt{\epsilon}} \frac{s}{\sqrt{v}} G(s, v, k) \exp\left(-\frac{v}{2}\left(\frac{\ln(\frac{s}{K})}{v} + \frac{1}{2}\right)^2\right) dv \\
&= \frac{1}{4}\int_0^{\sqrt{\epsilon}} \frac{e^\epsilon K}{\sqrt{v}} G(e^\epsilon K, v, k) \exp\left(-\frac{v}{2}\left(\frac{\epsilon}{v} + \frac{1}{2}\right)^2\right) dv \\
&= \frac{K}{4}\int_0^{\epsilon^{-\frac{3}{2}}} \frac{e^\epsilon}{\sqrt{r}} \epsilon G(e^\epsilon K, r\epsilon^2, k) \exp\left(-\frac{1}{2r} - \frac{\epsilon}{2} - \frac{r\epsilon^2}{8}\right) dr
\end{aligned}
$$

by the substitution $v = r\epsilon^2$. As $\epsilon G(e^\epsilon K, r\epsilon^2, k)$ tends to $\sqrt{\frac{2}{\pi}}\frac{k}{r}$ for $\epsilon \to 0$, we get by dominated convergence

$$\frac{Kk}{4}\sqrt{\frac{2}{\pi}}\int_0^\infty r^{-\frac{3}{2}}e^{-\frac{1}{2r}}dr = \frac{Kk}{2}$$

as limiting value for the integral, which concludes our proof. $\qquad\square$

In the sequel we will use the standard asymptotic notation, which we want to recall briefly: for two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ we write $a_n = \mathcal{O}(b_n)$ (resp. $a_n = \Omega(b_n)$), if there exist absolute constants $c > 0$, $C > 0$ and $N \in \mathbb{N}$ such that $|a_n| \leq c|b_n|$ (resp. $|a_n| \geq C|b_n|$) for $n \geq N$. Moreover, we write $a_n \sim_d b_n$ if $\frac{a_n}{b_n} \to 1$ in distribution. For instance, a statement like "$X_n = Y_n + \mathcal{O}(a_n)$ *holds with probability* $1 - e^{-\Omega(b_n)}$" means that there exist $c, C > 0$ and $N \in \mathbb{N}$ such that $\mathbb{P}\left[|X_n - Y_n| \leq ca_n\right] \geq 1 - e^{-Cb_n}$ for $n \geq N$.

Our second result clarifies the coming into being of the discontinuity of $J_2(s)$ at $s = K$. From Kabanov and Safarian (1997), Appendix B, we know that J_2 is the limit in probability of

$$(2.2) \qquad \Sigma^n := k\sum_{t_i \leq 1} S_{t_i}\left|\xi_{t_i}^n - \xi_{t_{i-1}}^n\right|.$$

Clearly, this does not answer the question, if for any fixed s conditionally on $\{S_1 = s\}$, we have $\Sigma^n \to J_2$ in probability. In particular, let $\bar{J}_2(s)$ be the continuous function that coincides with $J_2(s)$ everywhere but in $s = K$. Then $\bar{J}_2(S_1)$ is also a limit in probability of Σ^n. The following theorem shows, that the whole story about the discontinuity is not just a story about choosing some peculiar representative from some equivalence class of real functions.

Theorem 2.2 *Denoting* $\Sigma^n(s) := (\Sigma^n|S_1 = s)$, *we have*

$$\Sigma^n(s_n) \sim_d J_2(K) + kK\left|\Phi\left(\frac{n^{\frac{1}{4}}}{C}\ln\left(\frac{s_n}{K}\right)\right) - \frac{1}{2}\right|$$

as $n \to \infty$, *for any sequence* $(s_n)_{n\geq 0}$ *satisfying* $s_n = K + \mathcal{O}\left(n^{-\frac{1}{4}}\right)$.

Here $C = \sqrt{2\sqrt{\frac{2}{\pi}}\sigma k}$ *and* Φ *is the standard normal distribution function.*

Theorem 2.2 shows in particular that we have $\Sigma^n(K) \to J_2(K)$ in probability, (since all the $\Sigma^n(K)$ live on the same probability space,) and $\Sigma^n(Ke^{\lambda n^{-\frac{1}{4}}}) \to J_2(K) + kK\left|\Phi\left(\frac{\lambda}{C}\right) - \frac{1}{2}\right|$ in distribution. The latter limit is a peak-shaped function of λ turned upside down, with values close to $J_2(K) + \frac{kK}{2}$, when λ is large in absolute value. Moreover, $J(K) - kK\left|\Phi\left(\frac{\lambda}{C}\right) - \frac{1}{2}\right|$ is a good "approximation" of the peak observable in the data points of simulations of the hedging error J^n, the better, the larger n is, however valid only in domains that shrink like $n^{-\frac{1}{4}}$. For the hedger this means that he will feel the influence of the discontinuity on his hedging result for values of S_1 fulfilling $S_1 - K = \mathcal{O}(n^{-\frac{1}{4}})$.

We will actually prove the following more precise result of which Theorem 2.2 is a corollary. To facilitate reading, we have chosen to defer the more technical parts of the proof (i.e., the proofs of Lemmas 2.1, 2.2 and 2.4,) to Appendix A.

Theorem 2.3 *Let* $\Lambda > 0$ *and* $\varepsilon \in]0, \frac{1}{10}]$ *be given, and denote* $\psi(\lambda) = kK\left|\Phi\left(\frac{\lambda}{C}\right) - \Phi(0)\right|$. *Then there exists* $d = d(\Lambda, \varepsilon) > 0$ *such that*

$$\mathbb{P}\left[\left|\Sigma^n - J_2(K) - \psi(\lambda)\right| > dn^{-\frac{\varepsilon}{2}}\,\Big|\,S_1 = Ke^{\lambda n^{-\frac{1}{4}}}\right] \leq dn^{2\varepsilon - \frac{1}{2}}\ln n,$$

holds for $\lambda \in [-\Lambda, \Lambda]$.

Proof: We will see that in (2.2) only a small proportion of terms, corresponding to t_i in the neighborhood of 1, contributes essentially to the sum, and that in this neighborhood we can safely replace S_{t_i} by S_1. We choose $\beta = \frac{1}{2} + \varepsilon$ and split Σ^n as follows

$$
\Sigma^n = k \sum_{m=0}^{n-1} S_{1-\frac{m}{n}} \left| \xi^n_{1-\frac{m}{n}} - \xi^n_{1-\frac{m+1}{n}} \right|
$$

$$
= kS_1 \sum_{m=0}^{\lfloor n^\beta \rfloor} |\cdot| + k \sum_{m=0}^{\lfloor n^\beta \rfloor} \left(S_{1-\frac{m}{n}} - S_1 \right) |\cdot| + k \sum_{m=\lfloor n^\beta \rfloor + 1}^{n-1} S_{1-\frac{m}{n}} |\cdot| =: \Sigma^n_1 + \Sigma^n_2 + \Sigma^n_3,
$$

where $\xi^n_{1-\frac{m}{n}} = \Phi(\widehat{d}_m)$, and $\widehat{d}_m$ is given by

$$
(2.3) \qquad \widehat{d}_{m-1} = \frac{\ln(S_{1-\frac{m}{n}}/K)}{\widehat{\sigma}\sqrt{\frac{m}{n}}} + \frac{1}{2}\widehat{\sigma}\sqrt{\frac{m}{n}}, \quad 1 \le m \le n.
$$

Moreover $\widehat{\sigma}^2 = \sigma^2 + C^2\sqrt{n}$, thus
$$
(2.4) \qquad \widehat{\sigma} = Cn^{\frac{1}{4}} + \mathcal{O}(n^{-\frac{1}{4}}).
$$

The next two lemmas are concerned with the estimation of the terms Σ^n_2 and Σ^n_3. By $\mathbb{P}_{\lambda,n}$ we will denote probability, conditional on the event $S_1 = Ke^{\lambda n^{-\frac{1}{4}}}$.

Lemma 2.1 *Let $0 < \alpha < \frac{1-\beta}{2}$. Then*

$$
(2.5) \qquad \mathbb{P}_{\lambda,n}\left[|\Sigma^n_2| > n^{-\alpha}\Sigma^n_1\right] = e^{-\Omega(n^{1-\beta-2\alpha})},
$$

uniformly in $\lambda \in [-\Lambda, \Lambda]$.

Lemma 2.2
$$
(2.6) \qquad \mathbb{P}_{\lambda,n}\left[|\Sigma^n_3| > e^{-\frac{C^2}{10}n^{\beta-\frac{1}{2}}}\right] = e^{-\Omega(n^{\beta-\frac{1}{2}})},
$$

uniformly in $\lambda \in [-\Lambda, \Lambda]$, where C is the constant defined in Theorem 2.2.

Having shown that Σ^n_2 and Σ^n_3 contribute very little to Σ^n, we split Σ^n_1 even further, using the following sets of indices, given as intersections of real intervals and the set $\mathbb{N}$ of natural numbers:
$$
I^n_{11} := [0, |\lambda|\, n^{\frac{1}{4}-\varepsilon}[\cap \mathbb{N}, \quad I^n_{12} := [|\lambda|\, n^{\frac{1}{4}-\varepsilon}, n^{\frac{1}{2}-2\varepsilon}[\cap \mathbb{N}, \quad I^n_{13} := [n^{\frac{1}{2}-2\varepsilon}, n^\beta]\cap \mathbb{N}.
$$

$$
\Sigma^n_1 = kS_1 \sum_{m=0}^{\lfloor n^\beta \rfloor} \left| \xi^n_{1-\frac{m}{n}} - \xi^n_{1-\frac{m+1}{n}} \right|
$$

$$
= kS_1 \sum_{m \in I^n_{11}} |\cdot| + kS_1 \sum_{m \in I^n_{12}} |\cdot| + kS_1 \sum_{m \in I^n_{13}} |\cdot| =: \Sigma^n_{11} + \Sigma^n_{12} + \Sigma^n_{13}.
$$

These sums are dealt with in the following lemma:

Lemma 2.3 *There exists $\delta = \delta(\Lambda) > 0$ such that the following hold uniformly in $\lambda \in [-\Lambda, \Lambda]$:*

$$
\mathbb{P}_{\lambda,n}\left[|\Sigma^n_{11} - \psi(\lambda)| > \delta(\Lambda^{\frac{1}{2}} n^{\frac{\varepsilon}{2}-\frac{1}{8}} + \Lambda n^{-\frac{1}{4}})\right] = e^{-\Omega(n^\varepsilon)},
$$

$$
\mathbb{P}_{\lambda,n}\left[|\Sigma^n_{12}| > \delta(\Lambda^{\frac{1}{2}} n^{\frac{\varepsilon}{2}-\frac{1}{8}} + n^{-\frac{\varepsilon}{2}})\right] = e^{-\Omega(n^\varepsilon)},
$$

$$
(2.7) \qquad \mathbb{P}_{\lambda,n}\left[|\Sigma^n_{13} - J_2(K)| > \delta(\Lambda n^{2\varepsilon-\frac{1}{4}} + n^{-\varepsilon})\right] = \mathcal{O}\left(n^{2\varepsilon-\frac{1}{2}}\ln n\right).
$$

Note that for fixed Λ the quantities $\Lambda^{\frac{1}{2}}n^{\frac{\varepsilon}{2}-\frac{1}{8}}+\Lambda n^{-\frac{1}{4}}$, $\Lambda^{\frac{1}{2}}n^{\frac{\varepsilon}{2}-\frac{1}{8}}+n^{-\frac{\varepsilon}{2}}$, and $\Lambda n^{2\varepsilon-\frac{1}{4}}+n^{-\varepsilon}$, appearing in the preceding lemma, are all of order $\mathcal{O}(n^{-\frac{\varepsilon}{2}})$, if $\varepsilon\in]0,\frac{1}{10}]$. For the proof of Lemma 2.3 we make use of the following lemma, where we list some properties of the sequence $(\widehat{d}_m)_{m=0}^{n-1}$.

Lemma 2.4 *Let $\varepsilon\in]0,\frac{1}{10}]$ be given. Then the following hold simultaneously with $I\!P_{\lambda,n}$-probability $1-e^{-\Omega(n^\varepsilon)}$ and uniformly in $\lambda\in[-\Lambda,\Lambda]$.*

   i) $(\widehat{d}_m)$ is monotonically increasing resp. decreasing in m in the range $0\le m<|\lambda|\,n^{\frac{1}{4}-\varepsilon}$ according to the sign of λ.

   ii) $\left|\Delta\widehat{d}_{m-1}\right|\le\frac{\Lambda n^{\frac{1}{4}}}{\widehat{\sigma}m^{\frac{3}{2}}}+\frac{\widehat{\sigma}}{m^{\frac{1}{2}}n^{\frac{1}{2}-\frac{\varepsilon}{2}}}$ in the range $1\le m<n^{\frac{1}{2}-2\varepsilon}$.

   iii) $\widehat{d}_{m-1}=\frac{\widehat{\sigma}}{2}m^{\frac{1}{2}}n^{-\frac{1}{2}}+\mathcal{O}\left(\Lambda n^{2\varepsilon-\frac{1}{4}}\right)$ in the range $n^{\frac{1}{2}-2\varepsilon}\le m\le n^{\frac{1}{2}+\varepsilon}$.

   iv) $\Delta\widehat{d}_{m-1}=\frac{\widehat{\sigma}}{4m^{\frac{1}{2}}n^{\frac{1}{2}}}+\frac{\sigma n^{\frac{1}{4}}}{Cm^{\frac{1}{2}}}\left(W_{1-\frac{m}{n}}-W_{1-\frac{m+1}{n}}\right)+\mathcal{O}\left(\Lambda n^{4\varepsilon-\frac{3}{4}}\right)$ in the range $n^{\frac{1}{2}-2\varepsilon}\le m\le n^{\frac{1}{2}+\varepsilon}$.

We are now ready to supply the proof of Lemma 2.3.

Proof of Lemma 2.3: By Lemma 2.4 *i)*, with probability $1-e^{-\Omega(n^\varepsilon)}$, the sequence $\left(\Phi(\widehat{d}_m)\right)_{m\in I_{11}^n}$ is monotone. Telescoping yields

$$\Sigma_{11}^n=kS_1\sum_{m\in I_{11}^n}\left|\Phi(\widehat{d}_m)-\Phi(\widehat{d}_{m+1})\right|=kS_1\left|\Phi(\widehat{d}_0)-\Phi(\widehat{d}_{\lceil|\lambda|n^{\frac{1}{4}-\varepsilon}\rceil})\right|$$

$$=kK\left|\Phi(\frac{\lambda}{C})-\Phi(0)\right|+\mathcal{O}\left(\Lambda^{\frac{1}{2}}n^{\frac{\varepsilon}{2}-\frac{1}{8}}+\Lambda n^{-\frac{1}{4}}\right),$$

since $\widehat{d}_0=\frac{\lambda}{C}+\mathcal{O}\left(n^{-\frac{1}{4}}+\Lambda n^{-\frac{3}{4}}\right)$, $\widehat{d}_{\lceil|\lambda|n^{\frac{1}{4}-\varepsilon}\rceil}=\mathcal{O}\left(\Lambda^{\frac{1}{2}}n^{\frac{\varepsilon}{2}-\frac{1}{8}}\right)$ and $S_1=K+\mathcal{O}\left(\Lambda n^{-\frac{1}{4}}\right)$.

We turn to Σ_{12}^n, where we employ Lemma 2.4 *ii)*, and the simple inequality $|\Phi(x)-\Phi(y)|\le\frac{1}{\sqrt{2\pi}}|x-y|$. With probability $1-e^{-\Omega(n^\varepsilon)}$ we have

$$\Sigma_{12}^n=kS_1\sum_{m\in I_{12}^n}\left|\Phi(\widehat{d}_m)-\Phi(\widehat{d}_{m+1})\right|\le\frac{kS_1}{\sqrt{2\pi}}\sum_{m\in I_{12}^n}\left|\widehat{d}_m-\widehat{d}_{m+1}\right|$$

$$\le\frac{kS_1}{\sqrt{2\pi}}\sum_{m\in I_{12}^n}\left(\frac{\Lambda n^{\frac{1}{4}}}{\widehat{\sigma}m^{\frac{3}{2}}}+\frac{\widehat{\sigma}}{m^{\frac{1}{2}}n^{\frac{1}{2}-\frac{\varepsilon}{2}}}\right)=\mathcal{O}\left(\Lambda^{\frac{1}{2}}n^{\frac{\varepsilon}{2}-\frac{1}{8}}+n^{-\frac{\varepsilon}{2}}\right).$$

Using Lemma 2.4 *iii)* and *iv)*, we can evaluate Σ_{13}^n on a set of probability $1-e^{-\Omega(n^\varepsilon)}$. We abbreviate $Y_m^n=W_{1-\frac{m}{n}}-W_{1-\frac{m+1}{n}}$.

$$\Sigma_{13}^n=kS_1\sum_{m\in I_{13}^n}\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)\left|\frac{C}{4m^{\frac{1}{2}}n^{\frac{1}{4}}}+\frac{\sigma n^{\frac{1}{4}}}{Cm^{\frac{1}{2}}}Y_m^n\right|\left(1+\mathcal{O}\left(\Lambda n^{2\varepsilon-\frac{1}{4}}\right)\right)$$

$$(2.8)\qquad=kK\frac{\sigma}{C}n^{-\frac{1}{4}}\sum_{m\in I_{13}^n}\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{\sqrt{m}}\left|\frac{C^2}{4\sigma}+\sqrt{n}Y_m^n\right|+\mathcal{O}\left(\Lambda n^{2\varepsilon-\frac{1}{4}}\right)$$

Denoting now $U_m^n=\frac{C^2}{4\sigma}+\sqrt{n}Y_m^n$, we are going to show

$$(2.9)\quad\left\|n^{-\frac{1}{4}}\sum_{m\in I_{13}^n}\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{\sqrt{m}}|U_m^n|-n^{-\frac{1}{4}}\sum_{m\in I_{13}^n}\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{\sqrt{m}}I\!E\,|U_m^n|\right\|_{L^2(I\!P_{\lambda,n})}^2=\mathcal{O}\left(\frac{\ln n}{\sqrt{n}}\right).$$

In the sequel expectations, (co-)variances and correlations are always computed w.r.t. $\mathbb{P}_{\lambda,n}$, we omit subscripts λ, n.

$$\mathbb{E}\left[n^{-\frac{1}{4}}\sum_{m\in I_{13}^n}\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{\sqrt{m}}(|U_m^n|-\mathbb{E}\,|U_m^n|)\right]^2$$

$$(2.10)\qquad =n^{-\frac{1}{2}}\sum_m\frac{\phi^2\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{m}\mathrm{Var}\,|U_m^n|+n^{-\frac{1}{2}}\sum_{m\neq\ell}\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{\ell}{n}}\right)}{\sqrt{m\ell}}\mathrm{Cov}\,(|U_m^n|,|U_\ell^n|).$$

Using the fact that the random variables Y_m^n are increments over intervals of length $\frac{1}{n}$ of a Brownian Bridge from 0 to $w_{\lambda,n}:=\frac{1}{\sigma}\left(\ln\frac{K}{S_0}+\frac{\sigma^2}{2}+\lambda n^{-\frac{1}{4}}\right)$ of length 1, we see that $\sqrt{n}Y_m^n$ is Gaussian with mean $\frac{w_{\lambda,n}}{\sqrt{n}}$ and variance $\mathrm{Var}\,U_m^n=n\mathrm{Var}\,Y_m^n=1-\frac{1}{n}$, which implies

$$(2.11)\qquad\qquad\qquad\qquad\qquad \mathrm{Var}\,|U_m^n|=\mathcal{O}(1),$$

uniformly in m. We can for $m\neq\ell$ also compute $\mathrm{Cov}\,(U_m^n,U_\ell^n)=n\mathrm{Cov}\,(Y_m^n,Y_\ell^n)=-\frac{1}{n}$. This implies that the correlation $\rho=\mathrm{corr}\,(U_m^n,U_\ell^n)$ satisfies $|\rho|=\mathcal{O}\left(\frac{1}{n}\right)$. Now $\mathrm{corr}\,(|U_m^n|,|U_\ell^n|)$ depends on ρ analytically (we fix the marginal distributions of the Gaussian vector (U_m^n,U_ℓ^n), but let ρ vary) and equals 0 for $\rho=0$ ($|U_m^n|$ and $|U_\ell^n|$ are independent for $\rho=0$). Therefore $\mathrm{corr}\,(|U_m^n|,|U_\ell^n|)=\mathcal{O}\left(\frac{1}{n}\right)$ and by (2.11) also

$$\mathrm{Cov}\,(|U_m^n|,|U_\ell^n|)=\mathcal{O}\left(\frac{1}{n}\right),$$

uniformly in m, ℓ. The Var–sum in (2.10) is bounded by $\frac{\mathcal{O}(1)}{\sqrt{n}}\sum_{m=1}^n\frac{1}{m}=\mathcal{O}\left(\frac{\ln n}{\sqrt{n}}\right)$, and the Cov–sum in (2.10) is bounded by $\frac{1}{\sqrt{n}}\mathcal{O}\left(\frac{1}{n}\right)\sum_{m,\ell=1}^n\frac{1}{\sqrt{m\ell}}=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. This proves (2.9). Knowing that $\sqrt{n}Y_m^n$ is distributed $\mathcal{N}\left(\frac{w_{\lambda,n}}{\sqrt{n}},1-\frac{1}{n}\right)$ enables us to derive

$$\mathbb{E}\left|\frac{C^2}{4\sigma}+\sqrt{n}Y_m^n\right|=\mathbb{E}\left|\frac{C^2}{4\sigma}+Z\right|+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

where Z is distributed $\mathcal{N}(0,1)$. Next we return to (2.8) computing $n^{-\frac{1}{4}}\sum_{m\in I_{13}^n}\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{\sqrt{m}}$. (Finiteness of this term justifies the error term in (2.8).) We replace this term by

$$n^{-\frac{1}{4}}\sum_{m=1}^\infty\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{\sqrt{m}}\quad\text{and finally by}\quad n^{-\frac{1}{4}}\int_1^\infty\frac{\phi\left(\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}\right)}{\sqrt{m}}dm,$$

thus trading in error terms of order $\mathcal{O}\left(n^{-\varepsilon}\right)$ and $\mathcal{O}\left(n^{-\frac{1}{4}}\right)$. Change of variables $v=\widehat{\sigma}^2\frac{m}{n}$ leads to

$$\frac{n^{\frac{1}{4}}}{\widehat{\sigma}}\int_{\widehat{\sigma}^2/n}^\infty v^{-\frac{1}{2}}\phi\left(\frac{\sqrt{v}}{2}\right)dv=\frac{1}{C\sqrt{2\pi}}\int_0^\infty v^{-\frac{1}{2}}\exp\left(-\frac{v}{8}\right)dv+\mathcal{O}\left(n^{-\frac{1}{4}}\right).$$

After final simplifications, using the definition of C given in Theorem 2.2, and application of Chebyshev's inequality to (2.9), we obtain

$$\mathbb{P}_{\lambda,n}\left[\left|\Sigma_{13}^n-\frac{K}{4}\mathbb{E}\left[\left|\frac{k}{\sqrt{2\pi}}+Z\right|\right]\int_0^\infty v^{-\frac{1}{2}}\exp\left(-\frac{v}{8}\right)dv\right|>n^{-\varepsilon}\right]=\mathcal{O}\left(n^{2\varepsilon-\frac{1}{2}}\ln n\right),$$

which implies (2.7). The proof of Lemma 2.3 is finished. $\qquad\square$
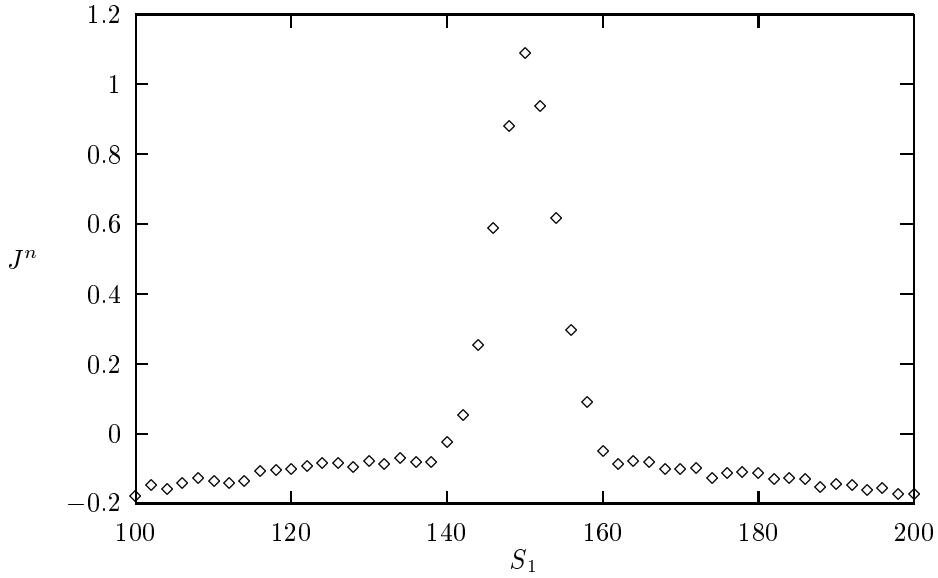
Figure 2: Averages of 5000 Monte Carlo simulations of the hedging error J^n, $n = 128$, conditioned on each of 51 values of S_1, with parameters: $K = S_0 = 150$, $\mu = 0$, $\sigma = 0.2$ and $k = 0.01$.

Putting together the results of Lemmas 2.1, 2.2 and 2.3, choosing $\alpha \geq \frac{\varepsilon}{2}$ in Lemma 2.1, completes the proof of Theorem 2.3. $\quad \square$

Remarks: 1. A close inspection of the proof of Theorem 2.3 shows that the weaker statement

$$\forall \varepsilon > 0 : \ \mathbb{P}\left[\left|\Sigma^n - J_2(K) - \psi(\lambda)\right| > \varepsilon \Big| S_1 = K e^{\lambda n^{-\frac{1}{4}}}\right] \to 0, \ \text{for } n \to \infty, \ \text{uniformly in } \lambda \in [-\Lambda, \Lambda]$$

holds even in the case where we let Λ depend on n, as long as $\Lambda = \Lambda_n = \mathcal{O}\left(n^{\frac{1}{4}-\gamma}\right)$ for some $\gamma > 0$.

2. Since σ enters into $\psi(\lambda)$ as $\sigma^{-\frac{1}{2}}$, halving σ has roughly the same effect on the hedging error J^n in the case that $S_1 = K + \mathcal{O}(n^{-\frac{1}{4}})$, as using 4 times as many revision intervals.

3. In order to see the effect of the removable discontinuity more clearly, we have chosen slightly unrealistic parameters for the simulations depicted in Fig. 1. However, also in more realistic cases the peak is present (not to say overwhelming), as can be seen in Fig. 2: For the indicated parameters and each of the values $S_1 = 100 + 2k$, $k = 0, \ldots, 50$, we ran 5000 simulations and plotted the averages.

4. As it is mentioned in Kabanov and Safarian (1997), the limiting hedging error has always negative sign. Observe however that Fig. 2 shows that the average (conditioned on S_1) of the hedging error for finite n is positive in a neighborhood of $S_1 = K$ (at least for this special choice of parameters. However, also in Fig. 1 outcomes near K tend to lead to positive hedging errors.) This observation makes Leland's strategy more reasonable than one would have guessed not knowing about the discontinuity in the limiting hedging error and the influence it has on cases of reasonable length of revision intervals.

Appendix A

Proof of Lemma 2.1: We are going to prove

$$\mathbb{P}_{\lambda,n}\left[\sup_{0\le m\le\lfloor n^\beta\rfloor}\frac{|S_{1-\frac{m}{n}}-S_1|}{S_1}>n^{-\alpha}\right]=e^{-\Omega(n^{1-\beta-2\alpha})},$$

which at once implies (2.5). From (2.1) we obtain

$$\frac{|S_{1-t}-S_1|}{S_1}=\left|e^{\frac{\sigma^2}{2}t-\sigma(W_1-W_{1-t})}-1\right|,$$

and since $|e^x-1|>y$ implies $|x|>\frac{y}{2}$, for $y\in[0,1]$, we derive

(A.1) $\mathbb{P}_{\lambda,n}\left[\sup\limits_{0\le m\le\lfloor n^\beta\rfloor}\dfrac{|S_{1-\frac{m}{n}}-S_1|}{S_1}>n^{-\alpha}\right]\le\mathbb{P}\left[\sup\limits_{0\le t\le n^{\beta-1}}|W_1-W_{1-t}|>x_n\Big|W_1=w_{\lambda,n}\right],$

where

$$w_{\lambda,n}=\frac{1}{\sigma}\left(\ln\frac{K}{S_0}+\frac{\sigma^2}{2}+\lambda n^{-\frac{1}{4}}\right)\qquad\text{and}\qquad x_n=\frac{1}{2\sigma}n^{-\alpha}-\frac{\sigma}{2}n^{\beta-1}.$$

There is need for some calculations involving Brownian Bridge: We deduce from Karatzas and Shreve (1991), p256, eq3.40, that

(A.2) $$\mathbb{P}_T^{0\to a}\left[\sup_{0\le t\le T}|W_t|\ge x\right]\le2e^{-2\frac{(x-|a|)^2}{T}}$$

holds for $T,x>0$ and $|a|\le x$, where by $\mathbb{P}_T^{0\to a}$ we denote the measure, under which $(W_t)_{t\in[0,T]}$ is a Brownian Bridge from 0 to a of length T. For $0<s<1$, we also want to recall the following fact, which is a special case of Karatzas and Shreve (1991), p359, eq6.28:

(A.3) $$\mathbb{P}_1^{0\to w}[W_s\in da]=\frac{1}{\sqrt{2\pi s(1-s)}}e^{-\frac{(a-sw)^2}{2s(1-s)}}.$$

Equations (A.2) and (A.3) enable us to derive

$$\mathbb{P}_1^{0\to w}\left[\sup_{0\le t\le s}|W_t|\ge x\right]$$

(A.4) $$\le\int_{|a|<x}2e^{-2\frac{(x-|a|)^2}{s}}\mathbb{P}_1^{0\to w}[W_s\in da]+\int_{|a|\ge x}\mathbb{P}_1^{0\to w}[W_s\in da].$$

If $|sw|\le x$, the second term in (A.4) is dominated by $e^{-\frac{(x-|sw|)^2}{2s(1-s)}}$, by the elementary estimate $1-\Phi(x)\le\frac{1}{2}e^{-\frac{x^2}{2}},x\ge0$. Under the further assumption $s\le\frac{3}{4}$, the first term in (A.4) is dominated by

$$\frac{2}{\sqrt{2\pi s(1-s)}}\int_{|a|<x}e^{-\frac{(x-|a|)^2}{2s(1-s)}-\frac{(a-sw)^2}{2s(1-s)}}da$$

$$\le\frac{4}{\sqrt{2\pi s(1-s)}}\int_{-\infty}^\infty e^{-\frac{(x-a)^2}{2s(1-s)}-\frac{(a-|sw|)^2}{2s(1-s)}}da=2\sqrt{2}e^{-\frac{(x-|sw|)^2}{4s(1-s)}}.$$

Therefore

(A.5) $$\mathbb{P}_1^{0\to w}\left[\sup_{0\le t\le s}|W_t|\ge x\right]\le(1+2\sqrt{2})e^{-\frac{(x-|sw|)^2}{4s(1-s)}}\mathbb{I}_{\{x\ge|sw|\}}+\mathbb{I}_{\{x<|sw|\}}$$

holds for $0 < s \le \frac{3}{4}$.

We continue estimating the right hand side of (A.1), observing that for large enough n we have $n^{\beta-1} \le \frac{3}{4}$ and $(x_n - n^{\beta-1}|w_{\lambda,n}|)^2 \ge \frac{x_n^2}{2} \ge \frac{1}{3}\left(\frac{n^{-\alpha}}{2\sigma}\right)^2$, uniformly in $\lambda \in [-\Lambda, \Lambda]$, and therefore, employing (A.5) with $s = n^{\beta-1}$, $x = x_n$ and $w = w_{\lambda,n}$, we obtain

$$\mathbb{P}_1^{0 \to w_{\lambda,n}}\left[\sup_{0 \le t \le n^{\beta-1}}|W_1 - W_{1-t}| > x_n\right] = \mathbb{P}_1^{0 \to w_{\lambda,n}}\left[\sup_{0 \le t \le n^{\beta-1}}|W_t| > x_n\right] \le (1 + 2\sqrt{2})e^{-\frac{1}{48\sigma^2}n^{1-2\alpha-\beta}}.$$

(Here we used that, if $(B_t)_{t \in [0,1]}$ is a Brownian Bridge from 0 to a, then $(B_1 - B_{1-t})_{t \in [0,1]}$ is also a Brownian Bridge from 0 to a.) This completes the proof. $\qquad\square$

Proof of Lemma 2.2: We just use the following two estimates, which are simple applications of (A.2),

$$\mathbb{P}_{\lambda,n}\left[\sup_{t \in [0,1]}S_t > e^L\right] \le \mathbb{P}_1^{0 \to w_{\lambda,n}}\left[\sup_{t \in [0,1]}W_t > \frac{1}{\sigma}(L - \ln S_0)\right] = e^{-\Omega(L^2)}, \text{ as } L \to \infty,$$

$$\mathbb{P}_{\lambda,n}\left[\sup_{m \in [n^\beta, n]}\left|\widehat{d}_{m-1} - \frac{1}{2}\widehat{\sigma}\sqrt{\frac{m}{n}}\right| > 1\right] = \mathbb{P}_{\lambda,n}\left[\sup_{m \in [n^\beta, n]}\left|\frac{\ln(S_{1-\frac{m}{n}}/K)}{\widehat{\sigma}\sqrt{\frac{m}{n}}}\right| > 1\right]$$

$$(A.6) \qquad\qquad \le \mathbb{P}_{\lambda,n}\left[\sup_{m \in [n^\beta, n]}\left|\ln S_{1-\frac{m}{n}}\right| > \widehat{\sigma}n^{\frac{\beta-1}{2}} - |\ln K|\right] = e^{-\Omega\left(n^{\beta-\frac{1}{2}}\right)}.$$

Since $\left|\widehat{d}_m - \widehat{d}_{m-1}\right| \le \left|\widehat{d}_{m-1} - \frac{1}{2}\widehat{\sigma}\sqrt{\frac{m}{n}}\right| + \left|\frac{1}{2}\widehat{\sigma}\sqrt{\frac{m}{n}} - \frac{1}{2}\widehat{\sigma}\sqrt{\frac{m+1}{n}}\right| + \left|\widehat{d}_m - \frac{1}{2}\widehat{\sigma}\sqrt{\frac{m+1}{n}}\right|$, equation (A.6) implies

$$\mathbb{P}_{\lambda,n}\left[\sup_{m \in [n^\beta, n-1]}\left|\widehat{d}_m - \widehat{d}_{m-1}\right| > 3\right] = e^{-\Omega\left(n^{\beta-\frac{1}{2}}\right)}.$$

We now turn to Σ_3^n, majorizing $S_{1-\frac{m}{n}}$ by e^L and applying the mean value theorem to the terms $\left|\Phi(\widehat{d}_m) - \Phi(\widehat{d}_{m+1})\right|$, noting that $\widehat{\sigma} > Cn^{\frac{1}{4}}$,

$$\Sigma_3^n = k\sum_{m=\lfloor n^\beta \rfloor+1}^{n-1} S_{1-\frac{m}{n}}\left|\Phi(\widehat{d}_m) - \Phi(\widehat{d}_{m+1})\right| \le ke^L\sum_{m=\lfloor n^\beta \rfloor+1}^{n-1}\phi\left(\frac{1}{2}\widehat{\sigma}\sqrt{\frac{m}{n}} - 1\right)\left|\widehat{d}_m - \widehat{d}_{m+1}\right|$$

$$\le 3ke^L\sum_{m=\lfloor n^\beta \rfloor+1}^{n-1}\phi\left(\frac{C}{2}m^{\frac{1}{2}}n^{-\frac{1}{4}} - 1\right) = \mathcal{O}\left(e^{L-\frac{C^2}{9}n^{\beta-\frac{1}{2}}}\right),$$

with probability at least $1 - e^{-\Omega(L^2)} - e^{-\Omega(n^{\beta-\frac{1}{2}})}$, and taking $L = n^{\frac{\beta}{2}-\frac{1}{4}}$ yields (2.6) and thus completes the proof. $\qquad\square$

The following lemma will be applied to increments of Brownian Bridge in the proof of Lemma 2.4.

Lemma 2.5 *Let $s, x \in \mathbb{R}$ and $n \in \mathbb{N}$ satisfy $x > \frac{|s|}{n}$ and let $(X_i)_{i \ge 1}$ be an i.i.d. sequence of zero mean Gaussian random variables with variance $\frac{1}{n}$. Then*

$$\mathbb{P}\left[\sup_{1 \le i \le n}|X_i| > x \,\Big|\, \sum_{i=1}^n X_i = s\right] \le 2n\exp\left(-\frac{n}{2}\left(x - \frac{|s|}{n}\right)^2\right).$$

Proof: First we note that, by the independence of the X_i, we have for $x \ge 0$

$$\mathbb{P}\left[\sup_{1 \le i \le n}X_i \le x\right] = \left(\Phi\left(x\sqrt{n}\right)\right)^n \ge \left(1 - \frac{1}{2}e^{-\frac{nx^2}{2}}\right)^n \ge 1 - \frac{n}{2}e^{-\frac{nx^2}{2}}.$$

Moreover

$$\mathbb{P}\left[\sup_{1\leq i\leq n}|X_i|>x\,\Big|\,\sum_{i=1}^n X_i=s\right]\leq\mathbb{P}\left[\sup_{1\leq i\leq n}X_i>x\,\Big|\,\sum_{i=1}^n X_i=s\right]+\mathbb{P}\left[\sup_{1\leq i\leq n}-X_i>x\,\Big|\,\sum_{i=1}^n X_i=s\right]$$

$$\leq2\mathbb{P}\left[\sup_{1\leq i\leq n}X_i>x\,\Big|\,\sum_{i=1}^n X_i=|s|\right]=2\mathbb{P}\left[\sup_{1\leq i\leq n}X_i>x-\frac{|s|}{n}\,\Big|\,\sum_{i=1}^n X_i=0\right]$$

$$\leq2\mathbb{P}\left[\sup_{1\leq i\leq n}X_i>x-\frac{|s|}{n}\,\Big|\,\sum_{i=1}^n X_i\geq0\right]+2\mathbb{P}\left[\sup_{1\leq i\leq n}X_i>x-\frac{|s|}{n}\,\Big|\,\sum_{i=1}^n X_i<0\right]$$

$$=4\mathbb{P}\left[\sup_{1\leq i\leq n}X_i>x-\frac{|s|}{n}\right]\leq2n\exp\left(-\frac{n}{2}\left(x-\frac{|s|}{n}\right)^2\right).$$

At several places we made use of the fact that $\left\{(X_i)_{i=1}^n\,\big|\,\sum_{i=1}^n X_i=a\right\}$, $\left\{(X_i+\frac{a}{n})_{i=1}^n\,\big|\,\sum_{i=1}^n X_i=0\right\}$ and $\left\{(-X_i)_{i=1}^n\,\big|\,\sum_{i=1}^n X_i=-a\right\}$ are equal in law, for i.i.d. Gaussian random variables X_i, $i\geq1$ and $a\in\mathbb{R}$. $\qquad\square$

Proof of Lemma 2.4: Assuming $S_1=Ke^{\lambda n^{-\frac{1}{4}}}$, we derive from (2.3)

$$(A.7)\qquad\qquad\widehat{d}_{m-1}=\frac{\lambda n^{-\frac{1}{4}}}{\widehat{\sigma}\sqrt{\frac{m}{n}}}+\frac{\ln\left(\frac{S_{1-\frac{m}{n}}}{S_1}\right)}{\widehat{\sigma}\sqrt{\frac{m}{n}}}+\frac{\widehat{\sigma}}{2}\sqrt{\frac{m}{n}}.$$

We prove *i)* by first observing that $\frac{|\lambda|n^{-\frac{1}{4}}}{\widehat{\sigma}\sqrt{\frac{m}{n}}}$ is decreasing in m. Employing the mean value theorem, there are numbers $\tilde{m}\in]m-1,m[$, such that the sequence of differences can be expressed as

$$(A.8)\qquad\Delta\widehat{d}_{m-1}=\widehat{d}_m-\widehat{d}_{m-1}=-\frac{\lambda n^{\frac{1}{4}}}{2\widehat{\sigma}\tilde{m}^{\frac{3}{2}}}-\frac{\ln\left(\frac{S_{1-\frac{m+1}{n}}}{S_1}\right)}{2\widehat{\sigma}\tilde{m}^{\frac{3}{2}}n^{-\frac{1}{2}}}+\frac{\ln\left(\frac{S_{1-\frac{m+1}{n}}}{S_{1-\frac{m}{n}}}\right)}{\widehat{\sigma}m^{\frac{1}{2}}n^{-\frac{1}{2}}}+\frac{\widehat{\sigma}}{4\tilde{m}^{\frac{1}{2}}n^{\frac{1}{2}}}.$$

We are now going to show that in the range $1\leq m<|\lambda|\,n^{\frac{1}{4}-\varepsilon}$ the first term on the r.h.s. of (A.8), which is $-\frac{\lambda n^{\frac{1}{4}}}{2\widehat{\sigma}\tilde{m}^{\frac{3}{2}}}$, dominates in absolute value the other terms with probability $1-e^{-\Omega(n^{2\varepsilon})}$. Indeed,

$$\mathbb{P}_{\lambda,n}\left[\sup_{m\in I_{11}^n}\left(\left|\frac{\ln\left(S_{1-\frac{m+1}{n}}/S_1\right)}{2\widehat{\sigma}\tilde{m}^{\frac{3}{2}}n^{-\frac{1}{2}}}\right|-\frac{1}{3}\frac{|\lambda|\,n^{\frac{1}{4}}}{2\widehat{\sigma}\tilde{m}^{\frac{3}{2}}}\right)\geq0\right]$$

$$\leq\mathbb{P}_{\lambda,n}\left[\sup_{m\in I_{11}^n}\left|W_1-W_{1-\frac{m+1}{n}}\right|\geq\frac{|\lambda|}{\sigma}\left(\frac{n^{-\frac{1}{4}}}{3}-\frac{\sigma^2}{2}n^{-\frac{3}{4}-\varepsilon}\right)\right]=e^{-\Omega(|\lambda|n^{\frac{1}{4}+\varepsilon})}=e^{-\Omega(n^{2\varepsilon})}$$

follows from (A.5) with $s=|\lambda|\,n^{-\frac{3}{4}-\varepsilon}$, $x=\frac{|\lambda|}{\sigma}n^{-\frac{1}{4}}$ and $w=w_{\lambda,n}$, and

$$\mathbb{P}_{\lambda,n}\left[\sup_{m\in I_{11}^n}\left(\left|\frac{\ln\left(S_{1-\frac{m+1}{n}}/S_{1-\frac{m}{n}}\right)}{\widehat{\sigma}m^{\frac{1}{2}}n^{-\frac{1}{2}}}\right|-\frac{1}{3}\frac{|\lambda|\,n^{\frac{1}{4}}}{2\widehat{\sigma}\tilde{m}^{\frac{3}{2}}}\right)\geq0\right]$$

$$\leq\mathbb{P}_{\lambda,n}\left[\sup_{m\in I_{11}^n}m\left|W_{1-\frac{m}{n}}-W_{1-\frac{m+1}{n}}\right|\geq\frac{|\lambda|}{\sigma}\left(\frac{n^{-\frac{1}{4}}}{6}-\frac{\sigma^2}{2}n^{-\frac{3}{4}-\varepsilon}\right)\right]$$

$$\leq\mathbb{P}_{\lambda,n}\left[\sup_{m\in I_{11}^n}\left|W_{1-\frac{m}{n}}-W_{1-\frac{m+1}{n}}\right|\geq\frac{n^{-\frac{1}{2}+\varepsilon}}{6\sigma}-\frac{\sigma}{2n}\right]=e^{-\Omega(n^{2\varepsilon})},$$

follows from Lemma 2.5. Finally, $\frac{1}{3}\frac{|\lambda|n^{\frac{1}{4}}}{2\widehat{\sigma}\tilde{m}^{\frac{3}{2}}}\geq\frac{\widehat{\sigma}}{4\tilde{m}^{\frac{1}{2}}n^{\frac{1}{2}}}$ is the same as $\tilde{m}\leq\frac{2|\lambda|n^{\frac{3}{4}}}{3\widehat{\sigma}^2}$, and this is, by (2.4), true for all large enough n and all m in the range $1\leq m<|\lambda|\,n^{\frac{1}{4}-\varepsilon}$.

The proof of *ii)* is similar: Here the first and the second term (resp. the third and fourth term) on the r.h.s. of (A.8) are dominated by $\frac{\Lambda n^{\frac{1}{4}}}{2\widehat{\sigma}m^{\frac{3}{2}}}$ (resp. $\frac{\widehat{\sigma}}{2m^{\frac{1}{2}}n^{\frac{1}{2}-\frac{\varepsilon}{2}}}$) with probability $1 - e^{-\Omega(n^{\varepsilon})}$. This is obvious for the first and the fourth term. The second term is settled by another application of (A.5), now with $s = n^{-\frac{1}{2}-2\varepsilon}$ and $x = \frac{\Lambda}{2\sigma}n^{-\frac{1}{4}}$:

$$\mathbb{P}_{\lambda,n}\left[\sup_{1\le m<n^{\frac{1}{2}-2\varepsilon}}\left|W_1 - W_{1-\frac{m+1}{n}}\right| \ge \frac{\Lambda}{\sigma}n^{-\frac{1}{4}} - \frac{\sigma}{2}n^{-\frac{1}{2}-2\varepsilon}\right] = e^{-\Omega(\Lambda^2 n^{2\varepsilon})}.$$

The third term uses again Lemma 2.5:

$$\mathbb{P}_{\lambda,n}\left[\sup_{1\le m<n^{\frac{1}{2}-2\varepsilon}}\left|W_{1-\frac{m}{n}} - W_{1-\frac{m+1}{n}}\right| \ge \frac{\widehat{\sigma}^2 n^{-1+\frac{\varepsilon}{2}}}{2\sigma} - \frac{\sigma}{2n}\right] = e^{-\Omega(n^{\varepsilon})}.$$

To prove *iii)*, we have to show, that the first and second term on the r.h.s. of (A.7) are of order $\mathcal{O}\left(\Lambda n^{2\varepsilon-\frac{1}{4}}\right)$. This is obvious for the first term. The second term once more uses (A.5), with $s = n^{-\frac{1}{2}+\varepsilon}$ and $x = \frac{\Lambda}{\sigma}n^{\varepsilon-\frac{1}{4}}$:

$$\mathbb{P}_{\lambda,n}\left[\sup_{m\in I^n_{13}}\frac{\left|W_1 - W_{1-\frac{m}{n}}\right|}{m^{\frac{1}{2}}} \ge \frac{\Lambda}{\sigma}n^{2\varepsilon-\frac{1}{2}}\right] \le \mathbb{P}_{\lambda,n}\left[\sup_{m\in I^n_{13}}\left|W_1 - W_{1-\frac{m}{n}}\right| \ge \frac{\Lambda}{\sigma}n^{\varepsilon-\frac{1}{4}}\right] = e^{-\Omega(\Lambda^2 n^{\varepsilon})}.$$

To prove *iv)*, we have to show, that the first and second term on the r.h.s. of (A.8) are of order $\mathcal{O}\left(\Lambda n^{4\varepsilon-\frac{3}{4}}\right)$. This is obvious for the first term. The second term once more uses (A.5):

$$\mathbb{P}_{\lambda,n}\left[\sup_{m\in I^n_{13}}\frac{\left|W_1 - W_{1-\frac{m}{n}}\right|}{m^{\frac{3}{2}}} \ge \frac{\Lambda}{\sigma}n^{4\varepsilon-1}\right] \le \mathbb{P}_{\lambda,n}\left[\sup_{m\in I^n_{13}}\left|W_1 - W_{1-\frac{m}{n}}\right| \ge \frac{\Lambda}{\sigma}n^{\varepsilon-\frac{1}{4}}\right] = e^{-\Omega(\Lambda^2 n^{\varepsilon})}.$$

The proof of Lemma 2.4 is now complete. $\square$

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