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The modified homotopy perturbation method for solving strongly nonlinear oscillators

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ABSTRACT

In this paper we propose a reliable algorithm for the solution of nonlinear oscillators. Our algorithm is based upon the homotopy perturbation method (HPM), Laplace transforms, and Padé approximants. This *modified homotopy perturbation method* (MHPM) utilizes an alternative framework to capture the periodic behavior of the solution, which is characteristic of oscillator equations, and to give a good approximation to the true solution in a very large region. The current results are compared with those derived from the established Runge–Kutta method in order to verify the accuracy of the MHPM. It is shown that there is excellent agreement between the two sets of results. Results also show that the numerical scheme is very effective and convenient for solving strongly nonlinear oscillators.

1. Introduction

Considerable attention has been directed toward the study of strongly nonlinear oscillators, and several methods have been used to find approximate solutions to these nonlinear problems. Such methods include the harmonic balance method (HB) [1], the elliptic Lindstedt–Poincare method (LP) [1–3], the Krylov–Bogolioubov–Mitropolsky method (KBM) [4,5], averaging [1,6], and the multiple scales method (MSM) [2], He's parameter-expanding methods [7], Adomian decomposition method [8] and an equivalent nonlinearization method [9]. These methods are all widely used to obtain approximate solutions of nonlinear oscillators. A feature common to all of these methods is that they solve weakly nonlinear systems by using perturbation techniques to reduce these systems to simpler equations. These procedures change the original problem to make it tractable by conventional methods. In short, the physical problem is transformed into a purely mathematical one for which a solution is readily available. Recently, He [10,11] has proposed a new perturbation technique which eliminates the "small parameter" assumption. This method, *the homotopy perturbation method* (HPM), is applied to various linear and nonlinear problems [12–16]. Recently, Beléndez and his coworkers [17–20] implemented the homotopy perturbation method to solve several oscillator equations. For more details, see [12–20] and the references therein.

Our concern in this work is the derivation of approximate analytical oscillatory solutions for the nonlinear oscillator equation

$$y''(t) + c_1 y(t) + c_2 y^2(t) + c_3 y^3(t) = \varepsilon f(y(t), y'(t)), \quad y(0) = a, \quad y'(0) = b, \tag{1}$$

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where $c_i, i = 1, 2, 3$, are positive real numbers and ε is a parameter (not necessarily small). We assume that the function $f(y(t), y'(t))$ is an arbitrary nonlinear function of its arguments. Our modified homotopy perturbation method will be employed in a straightforward manner that does not necessitate any linearization or smallness assumptions.

The HPM for solving differential and integral equations, linear and nonlinear, was first proposed by He [10,11], and has been the subject of extensive analytical and numerical studies. This method, which combines the traditional perturbation method with topological homotopy theory, deforms our problem continuously into a simple problem which is easily solved. This method does not require a small parameter in the equation and has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. However, HPM also has some drawbacks. By using HPM, we get a series solution, which in practice is a truncated series solution. This series solution does not exhibit the periodic behavior characteristic of oscillator equations, and it gives a good approximation to the true solution only in a very small region.

In order to improve the accuracy of HPM, we introduce an alternative technique which modifies the series solution for nonlinear oscillatory systems as follows: we first apply a Laplace transformation to the truncated series obtained by HPM; next, we convert the transformed series into a meromorphic function by forming its Padé approximants; and then we utilize an inverse Laplace transformation to get an analytic solution, which may be periodic or it may be a better approximate solution than HPM truncated series solution. Finally, we make a numerical comparison between our method and the fourth-order Runge–Kutta method. We note that the first connections between series solution methods and Padé approximants were established in [21–24].

This paper is organized as follows: In Section 2, we describe the homotopy perturbation method and briefly discuss Padé approximants. In Section 3, our method is applied to a variety of examples to demonstrate the efficiency and simplicity of the method. We conclude with a summary.

2. The Homotopy perturbation method

The HPM was first proposed by the Chinese mathematician He [10,11]. The essential idea of this method is to introduce a homotopy parameter, say p, which varies from 0 to 1. At $p = 0$, the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As p gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of the deformation gives the desired solution.

For the convenience of the reader, we will first review the HPM [10,11], and then apply it to solve a nonlinear problem (1). Consider the nonlinear differential equation

$$L(u) + N(u) = f(r), \quad r \in \Omega, \tag{2}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{3}$$

where L is a linear operator, $N$is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω, and $f(r)$ is a known analytic function.

He's homotopy perturbation technique [10,11] defines a homotopy $v(r, p) : \Omega \times [0, 1] \to R$ so that

$$\mathrm{H}(v, p) = (1 - p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)], \tag{4}$$

or

$$\mathrm{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)], \tag{5}$$

where $r \in \Omega$, $p \in [0, 1]$ is an impeding parameter, and u_0 is an initial approximation of Eq. (2) which satisfies the boundary conditions. Obviously, from Eq. (4) and Eq. (5), we have

$$\mathrm{H}(v, 0) = L(v) - L(u_0), \tag{6}$$

$$\mathrm{H}(v, 1) = L(v) + N(v) - f(r). \tag{7}$$

As p moves from 0 to 1, $v(r, p)$ moves from $u_0(r)$ to $u(r)$. In topology, this called a *deformation* and $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are said to be *homotopic*. Our basic assumption is that the solution of Eq. (4) and Eq. (5) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{8}$$

The approximate solution of Eq. (2), therefore, can be readily obtained as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{9}$$

The convergence of the series (8) has been proved in [10,11].

Using the HPM, we construct the following homotopy for Eq. (1):

$$y''(x) + c_1 y(t) = -p(c_2 y^2(t) + c_3 y^3(t) - \varepsilon f(y, y')), \quad p \in [0, 1], \tag{10}$$

or

$$y''(x) = -p(c_1 y(t) + c_2 y^2(t) + c_3 y^3(t) - \varepsilon f(y, y')), \quad p \in [0, 1], \tag{11}$$

The homotopy parameter p always ranges from 0 to 1. At $p = 0$, Eq. (10) is linearized as

$$y''(x) + c_1 y(t) = 0, \tag{12}$$

and Eq. (11) is linearized as

$$y''(x) = 0. \tag{13}$$

At $p = 1$ Eq. (10) or (11) turns out to be the original differential equation (1). The basic assumption is that the solution of Eq. (10) or (11) can be written as a power series in p

$$y = y_0 + p y_1 + p^2 y_2 + p^3 y_3 + \cdots. \tag{14}$$

Substituting Eq. (14) into Eq. (10) or (11) and equating the terms with identical powers of p, we can obtain a series of linear equations of the form

$$\begin{aligned}
&p^0 : y_0'' + c_1 y_0 = 0, \qquad y_0(0) = a, \qquad y_0'(0) = b,\\
&p^1 : y_1'' + c_1 y_1 = -c_2 y_0^2 - c_3 y_0^3 + \varepsilon f(y_0, y_0'), \qquad y_1(0) = 0, \qquad y_1'(0) = 0,\\
&p^2 : y_2'' + c_1 y_2 = -c_2 y_1^2 - c_3 y_1^3 + \varepsilon f(y_1, y_1'), \qquad y_2(0) = 0, \qquad y_2'(0) = 0,\\
&p^3 : y_3'' + c_1 y_3 = -c_2 y_2^2 - c_3 y_2^3 + \varepsilon f(y_2, y_2'), \qquad y_3(0) = 0, \qquad y_3'(0) = 0,\\
&\vdots
\end{aligned} \tag{15}$$

or of the form

$$\begin{aligned}
&p^0 : y_0'' = 0, \qquad y_0(0) = a, \qquad y_0'(0) = b,\\
&p^1 : y_1'' = -c_1 y_0 - c_2 y_0^2 - c_3 y_0^3 + \varepsilon f(y_0, y_0'), \qquad y_1(0) = 0, \qquad y_1'(0) = 0,\\
&p^2 : y_2'' = -c_1 y_1 - c_2 y_1^2 - c_3 y_1^3 + \varepsilon f(y_1, y_1'), \qquad y_2(0) = 0, \qquad y_2'(0) = 0,\\
&p^3 : y_3'' = -c_1 y_2 - c_2 y_2^2 - c_3 y_2^3 + \varepsilon f(y_2, y_2'), \qquad y_3(0) = 0, \qquad y_3'(0) = 0,\\
&\vdots
\end{aligned} \tag{16}$$

It obvious that the linear equations in (15) or (16) are easy to solve, the components y_n, $n \geq 0$ of the homotopy perturbation method can be completely determined, and the series solutions are thus entirely determined. Finally, we approximate the solution $y(x) = \sum_{n=0}^{\infty} y_n(x)$ by the truncated series

$$\Phi_N(t) = \sum_{n=0}^{N-1} y_n(t). \tag{17}$$

2.1. Padé approximants

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $y(x)$. The [L/M] Padé approximants to a function $y(x)$ is given by [21,25]:

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}, \tag{18}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^j, \tag{19}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}), \tag{20}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$.

Since we can obviously multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we impose the normalization condition

$$Q_M(0) = 1.0. \tag{21}$$

Finally we require that the $P_L(x)$ and $Q_M(x)$ have no common factors.

If we write the coefficients of $P_L(x)$ and $Q_M(x)$ as

$$\left.\begin{aligned} P_L(x) &= p_0 + p_1x + p_2x^2 + \cdots + p_Lx^L, \\ Q_M(x) &= q_0 + q_1x + q_2x^2 + \cdots + q_Mx^M, \end{aligned}\right\} \tag{22}$$

then by (21) and (22) we may multiply (20) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (20) in more detail as

$$\left.\begin{aligned} &a_{L+1} + a_Lq_1 + \cdots + a_{L-M+1}q_M = 0, \\ &a_{L+2} + a_{L+1}q_1 + \cdots + a_{L-M+2}q_M = 0, \\ &\vdots \\ &a_{L+M} + a_{L+M-1}q_1 + \cdots + a_Lq_M = 0, \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} &a_0 = p_0, \\ &a_0 + a_0q_1 = p_1, \\ &a_2 + a_1q_1 + a_0q_2 = p_2, \\ &\vdots \\ &a_L + a_{L-1}q_1 + \cdots + a_0q_L = p_L. \end{aligned}\right\} \tag{24}$$

To solve these equations, we begin by solving Eqs. (23), a set of linear equations, for all of the unknown q's. Once the q's are known, Eqs. (24) give an explicit formula for the unknown p's, which completes the solution.

If Eqs. (23) and (24) are nonsingular, then we can solve them directly and obtain Eq. (25) [21], where Eq. (25) holds and, if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum\limits_{j=M}^{L} a_{j-M}x^j & \sum\limits_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum\limits_{j=0}^{L} a_jx^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}. \tag{25}$$

We use Mathematica to obtain diagonal Padé approximants of various orders, such as [2/2], [4/4] or [6/6].

3. Examples

In order to assess the advantages and the accuracy of our method for solving nonlinear oscillatory systems, we have applied it to a variety of initial-value problems arising in nonlinear dynamics. All results are calculated by using the Mathematica symbolic calculus software.

Example 1. Consider the following Helmholtz equation:

$$y''(x) + 2y(x) + y^2(x) = 0, \tag{26}$$

subject to the initial conditions

$$y(0) = 0.1, \qquad y'(0) = 0. \tag{27}$$

Using the homotopy perturbation method, the homotopy for Eq. (26) is constructed as

$$y''(t) = p[-2y(t) - y^2(t)]. \tag{28}$$

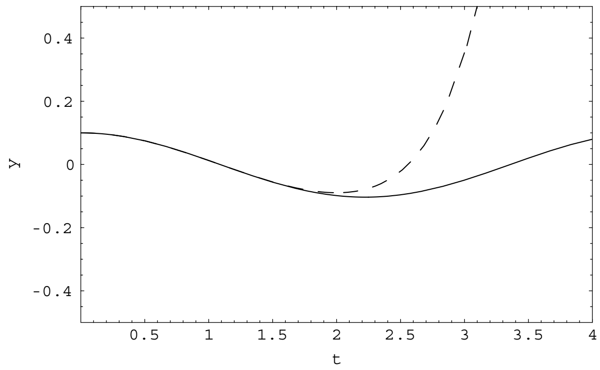

Fig. 1. Plots of displacement y versus time t. Solid line: Runge–Kutta method; dashed line: HPM.

Substituting (14) and the initial conditions (27) into (28), we obtain the following set of linear differential equations:

$$\begin{aligned}
&y_0'' = 0, \qquad y_0(0) = 0.1, \qquad y_0'(0) = 0,\\
&y_1'' = -2y_0 - y_0^2, \qquad y_1(0) = 0, \qquad y_1'(0) = 0,\\
&y_2'' = -2y_1 - 2y_0y_1, \qquad y_2(0) = 0, \qquad y_2'(0) = 0,\\
&y_3'' = -2y_2 - (y_1^2 + 2y_0y_2), \qquad y_3(0) = 0, \qquad y_3'(0) = 0,\\
&\vdots
\end{aligned} \tag{29}$$

Consequently, the first few components of the homotopy perturbation solution for Eq. (26) are as follows:

$$\begin{aligned}
y_0 &= 0.1,\\
y_1 &= -0.105t^2,\\
y_2 &= 0.01925t^4,\\
y_3 &= -0.00177917t^6,\\
&\vdots
\end{aligned} \tag{30}$$

and so on, and in this manner the rest of the components of the homotopy perturbation solution can be obtained. The fifth-term approximate solution for Eq. (26) is given by

$$y(t) = 0.1 - 0.105t^2 + 0.01925t^4 - 0.00177917t^6 + 0.000142083t^8. \tag{31}$$

This series solution does not exhibit the periodic behavior which is characteristic of the oscillatory system (26) and (27). Comparison of the approximate solution (31) and the solution obtained by the fourth-order Runge–Kutta method in Fig. 1 shows that it converges in a small region but yields an incorrect solution in a wider region.

In order to improve the accuracy of the homotopy perturbation solution (31), we implement the modified homotopy perturbation method as follows:

Applying the Laplace transformation to the series solution (31) yields

$$L[y(t)] = \frac{0.1}{s} - \frac{0.21}{s^3} + \frac{0.462}{s^5} - \frac{1.281}{s^7} + \frac{5.7288}{s^9}. \tag{32}$$

For simplicity, let $s = 1/t$; then

$$L[y(t)] = 0.1t - 0.21t^3 + 0.462t^5 - 1.281t^7 + 5.7288t^9. \tag{33}$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.1t + 1.27t^3}{1 + 14.8t^2 + 26.46t^4}. \tag{34}$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{1.27s + 0.1s^3}{26.46 + 14.8s^2 + s^4}. \tag{35}$$

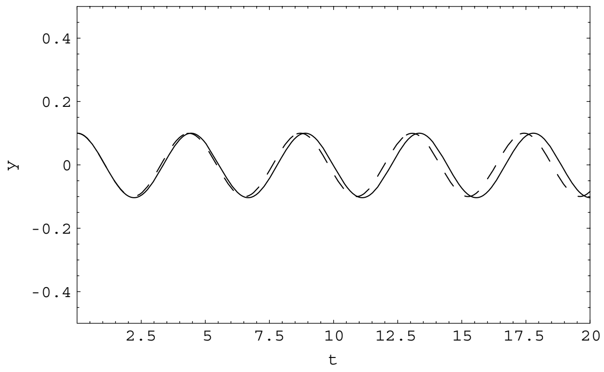

Fig. 2. Plots of displacement y versus time t. Solid line: Runge–Kutta method; dashed line: MHPM.

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain the modified homotopy perturbation solution

$$y(t) = 0.0998141\cos(1.4423t) + 0.000185858\cos(3.56648t). \tag{36}$$

Comparison of the modified approximate solution (34) and the solution obtained by fourth-order Runge–Kutta method in Fig. 2 shows that the modified homotopy perturbation method greatly improves upon the differential transform truncated series (31) in both convergence rate and accuracy.

Example 2. Consider the nonlinear equation

$$y''(t) + y(t) = -\varepsilon y^2(t)y'(t), \tag{37}$$

subject to the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{38}$$

This equation can be appropriately called the "unplugged" van der Pol equation, and all of its solutions are expected to oscillate with decreasing amplitude to zero. Momani [22] and Momani and Ertürk [23] derived numerical solutions for the above equation using the modified decomposition method and the modified differential transform method, respectively, when $\varepsilon = 0.1$. For comparison with the solution obtained in [22,23], we set the parameter $\varepsilon = 0.1$ in this example.

Using the modified homotopy perturbation method, the homotopy for Eq. (37) is constructed as

$$y''(t) = p[-y(t) - 0.1y^2(t)y'(t)]. \tag{39}$$

Substituting (14) and the initial conditions (38) into (39), we obtain the following set of linear differential equations:

$$\begin{aligned}
&y_0'' = 0, \qquad y_0(0) = 1, \qquad y_0'(0) = 0,\\
&y_1'' = -y_0 - 0.1y_0^2y_0', \qquad y_1(0) = 0, \qquad y_1'(0) = 0,\\
&y_2'' = -y_1 - 0.1(y_0^2y_1' + 2y_0y_1y_0'), \qquad y_2(0) = 0, \qquad y_2'(0) = 0,\\
&y_3'' = -y_2 - 0.1(y_1^2y_1' + 2y_0y_2y_0' + y_0^2y_2' + 2y_0y_1y_1'), \qquad y_3(0) = 0, \qquad y_3'(0) = 0,\\
&\vdots
\end{aligned} \tag{40}$$

Consequently, the first few components of the homotopy perturbation solution for Eq. (37) are as follows:

$$\begin{aligned}
&y_0 = 1,\\
&y_1 = \frac{-t^2}{2},\\
&y_2 = 0.0166667t^3 + 0.0416667t^4,\\
&y_3 = -0.000416667t^4 - 0.0066667t^5 - 0.001388889t^6,\\
&\vdots
\end{aligned} \tag{41}$$

and so on, and in this manner the rest of the components of the homotopy perturbation solution can be obtained. The fifth-term approximate solution for Eq. (37) is given by

$$y(t) = 1 - 0.5t^2 + 0.0166667t^3 + 0.04125t^4 - 0.00665833t^5 - 0.0009861t^6 + 0.00136905t^7 + O(t^8). \tag{42}$$

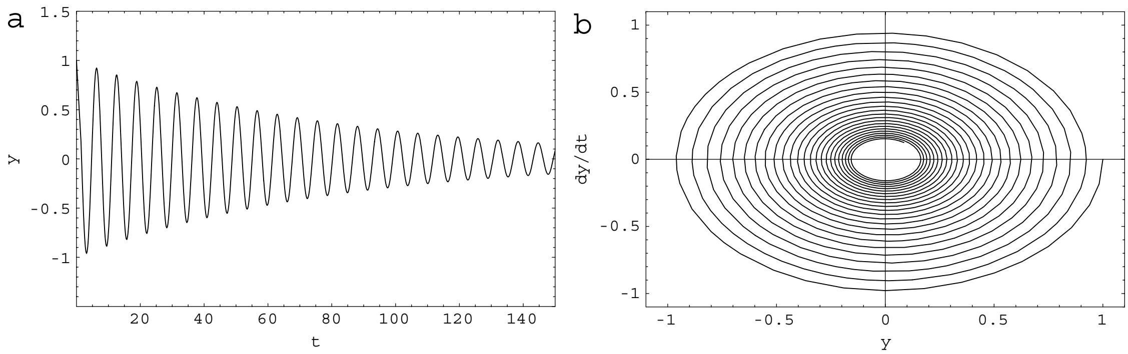

Fig. 3. Plots of Eq. (47): (a) displacement y versus time t and (b) phase plane diagram.

Because Eqs. (37) and (38) form an oscillatory system, we apply the Laplace transformation to the series solution (42), which yields

$$L[y(t)] = \frac{1}{s} - \frac{1}{s^3} + \frac{0.1}{s^4} + \frac{0.99}{s^5} - \frac{0.799}{s^6} + \frac{0.71}{s^7} + \frac{6.9}{s^8} - \cdots. \tag{43}$$

For simplicity, let $s = 1/t$; then

$$L[y(t)] = t - t^3 + 0.1t^4 + 0.99t^5 - 0.799t^6 - 0.71t^7 + 6.9t^8 - \cdots. \tag{44}$$

The [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right] = \frac{t + 0.32183t^2 + 9.1524t^3 + 0.313t^4}{1 + 0.321833t + 10.152t^2 + 0.534t^3 + 9.13t^4} \tag{45}$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{s^3 + 0.32183s^2 + 9.1524s + 0.313}{s^4 + 0.321833s^3 + 10.152s^2 + 0.534s + 9.13}. \tag{46}$$

By applying the inverse Laplace transformation to the [4/4] Padé approximant, we obtain the modified homotopy perturbation solution

$$y(t) = a_1 e^{(-0.154136-2.99971i)t} + a_2 e^{(-0.154136+2.99971i)t} + a_3 e^{(-0.0125308-0.998948i)t} + a_4 e^{(-0.0125308+0.998948i)t}, \tag{47}$$

where

$$a_1 = 0.000284635 - 0.00152866i, \qquad a_2 = 0.000284635 + 0.00152866i,$$
$$a_3 = 0.499715 + 0.0109027i, \qquad a_4 = 0.499715 - 0.0109027i.$$

Fig. 3 shows the displacement and phase diagram of the modified approximate solution (47). The results of our computations are in excellent agreement with the results obtained by the numerical solution of Momani [22,23] using the modified decomposition method.

Example 3. Consider the following initial-value problem

$$y''(t) + y(t) + 0.45y^2(t) = y(t)y'(t), \tag{48}$$

subject to the initial conditions

$$y(0) = 0.1, \qquad y'(0) = 0. \tag{49}$$

Using the modified homotopy perturbation method, the homotopy for Eq. (48) is constructed as

$$y''(t) = p[-y(t) - 0.45y^2(t) + y(t)y'(t)]. \tag{50}$$

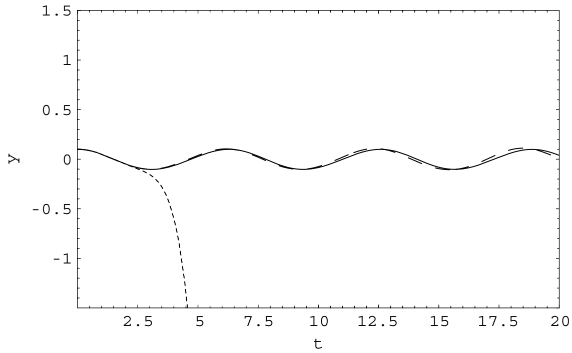

Fig. 4. Plots of displacement y versus time t. Solid line: Runge–Kutta method; dashed line: MHPM; dotted line: HPM.

Substituting (14) and the initial condition (49) into (48) we obtain the following set of linear differential equations:

$$\begin{aligned}
&y_0'' = 0, \qquad y_0(0) = 0.1, \qquad y_0'(0) = 0,\\
&y_1'' = -y_0 - 0.45y_0^2 + y_0y_0', \qquad y_1(0) = 0, \qquad y_1'(0) = 0,\\
&y_2'' = -y_1 - 0.45(2y_0y_1) + y_1y_0' + y_0y_1', \qquad y_2(0) = 0, \qquad y_2'(0) = 0,\\
&y_3'' = -y_2 - 0.45(2y_0y_2 + y_1^2) + y_2y_0' + y_1y_1' + y_0y_2', \quad y_3(0) = 0, \quad y_3'(0) = 0,\\
&\vdots
\end{aligned} \tag{51}$$

Consequently, the first few components of the homotopy perturbation solution for Eq. (48) are as follows:

$$\begin{aligned}
y_0 &= 0.1,\\
y_1 &= -0.05225t^2,\\
y_2 &= -0.00174167t^3 + 0.00474604t^4,\\
y_3 &= \; -0.0000435417t^4 + 0.000462846t^5 - 0.000254341t^6,\\
&\vdots
\end{aligned} \tag{52}$$

and so on, and in this manner the rest of components of the homotopy perturbation solution can be obtained. The fifth-term approximate solution for Eq. (48) is given by

$$\begin{aligned}
y(t) \;=\; & 0.1 - 0.05225t^2 - 0.00174167t^3 + 0.0047025t^4 + 0.000461977t^5 - 0.000229878t^6\\
& - 0.0000530213t^7 + 8.93598 \times 10^{-6}t^8.
\end{aligned} \tag{53}$$

Applying the Laplace transformation to the series solution (53), yields

$$L[y(t)] = \frac{0.1}{s} - \frac{0.1045}{s^3} - \frac{0.01045}{s^4} + \frac{0.11286}{s^5} + \frac{0.0554373}{s^6} - \frac{0.165512}{s^7} - \frac{0.267227}{s^8} + \frac{0.360299}{s^9}. \tag{54}$$

Setting $s = 1/t$ in Eq. (54) and calculating the [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right] = \frac{0.1t + 0.0643186t^2 + 0.570538t^3 - 0.0226526t^4}{1 + 0.643186t + 6.75038t^2 + 0.550103t^3 + 5.99275t^4}. \tag{55}$$

Recalling $t = 1/s$, and by using the inverse Laplace transformation to the [4/4] Padé approximant, we obtain the modified solution

$$y(t) = a_1\mathrm{e}^{(-0.33482802-2.368212\mathrm{i})t} + a_2\mathrm{e}^{(-0.33482802+2.368212\mathrm{i})t} + a_3\mathrm{e}^{(0.013235-1.02343\mathrm{i})t} + a_4\mathrm{e}^{(0.013235+1.02343\mathrm{i})t} \tag{56}$$

where

$$\begin{aligned}
&a_1 = 0.1(-0.00143534 + 0.00307974\mathrm{i}), \qquad a_2 = 0.1(-0.00143534 - 0.00307974\mathrm{i}),\\
&a_3 = 0.1(0.501435 - 0.0140806\mathrm{i}), \qquad a_4 = 0.1(0.501435 + 0.0140806\mathrm{i}).
\end{aligned}$$

The graph of the displacement is sketched in Fig. 4 and is compared with the numerical solution of Runge–Kutta method.

Example 4. Consider the following initial-value problem

$$y''(x) + y(x) + y^3(x) = y(x)y'(x) \tag{57}$$

subject to the initial conditions

$$y(0) = 0.1, \qquad y'(0) = 0. \tag{58}$$

Using the modified homotopy perturbation method, the homotopy for Eq. (57) is constructed as

$$y''(t) = p[-y(t) - y^3(t) + y(t)y'(t)]. \tag{59}$$

Substituting (14) and the initial conditions (58) into (59) we obtain the following set of linear differential equations:

$$\begin{aligned}
&y_0'' = 0, \qquad y_0(0) = 0.1, \qquad y_0'(0) = 0,\\
&y_1'' = -y_0 - y_0^3 + y_0y_0', \qquad y_1(0) = 0, \qquad y_1'(0) = 0,\\
&y_2'' = -y_1 - 3y_0^2y_1 + y_0'y_1 + y_0y_1', \qquad y_2(0) = 0, \qquad y_2'(0) = 0,\\
&y_3'' = -y_2 - 3y_0^2y_2 - 3y_0y_1^2 + y_0'y_2 + y_1y_1' + y_2'y_0, \qquad y_3(0) = 0, \quad y_3'(0) = 0,\\
&\vdots
\end{aligned} \tag{60}$$

Consequently, the first few components of the homotopy perturbation solution for Eq. (57) are as follows:

$$\begin{aligned}
&y_0 = 0.1,\\
&y_1 = -0.0505t^2,\\
&y_2 = -0.00168333t^3 + 0.00433458t^4,\\
&y_3 = -0.0000420833t^4 + 0.000428408t^5 - 0.000174323t^6,\\
&\vdots
\end{aligned} \tag{61}$$

and so on, and in this manner the rest of components of the homotopy perturbation solution can be obtained. The fourth-term approximate solution for Eq. (57) is given by

$$\begin{aligned}
y(t) &= 0.1 - 0.0505t^2 - 0.00168333t^3 + 0.0042925t^4 + 0.000427567t^5\\
&\quad - 0.000151584t^6 - 0.0000454819t^7 + 7.8514 \times 10^{-6}t^8.
\end{aligned} \tag{62}$$

Applying the Laplace transformation to the series solution (62), yields

$$L[y(t)] = \frac{0.1}{s} - \frac{0.101}{s^3} - \frac{0.0101}{s^4} + \frac{0.10302}{s^5} + \frac{0.051308}{s^6} - \frac{0.109131}{s^7} - \frac{0.229229}{s^8} + \frac{0.316568}{s^9}. \tag{63}$$

For simplicity, let $s = 1/t$; then

$$L[y(t)] = 0.1t - 0.101t^3 - 0.0101t^4 + 0.10302t^5 + 0.051308t^6 - 0.109131t^7 - 0.229229t^8 + 0.316568t^9. \tag{64}$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.1t - 0.0226813t^2 + 0.439089t^3 - 0.0133316t^4}{1 - 0.0226813t + 5.40089t^2 - 0.0552239t^3 + 4.42241t^4}. \tag{65}$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{0.1(-0.0303603 + s)(4.39112 + 0.007679s + s^2)}{4.2241 - 0.0552239s + 5.40089s^2 - 0.0226813s^3 + s^4}. \tag{66}$$

By using the inverse Laplace transformation to the [4/4] Padé approximant, we obtain the modified approximate solution

$$Y(t) = a_1 e^{(0.00478127-1.00317i)t} + a_2 e^{(0.00478127+1.00317i)t} + a_3 e^{(0.00655938-2.09627i)t} + a_4 e^{(0.00655938+2.09627i)t}. \tag{67}$$

where,

$$\begin{aligned}
a_1 &= 0.1(0.499451 - 0.0158157i), \qquad & a_3 &= 0.1(0.000548934 + 0.00642771i)\\
a_2 &= 0.1(0.499451 + 0.0158157i), \qquad & a_4 &= 0.1(0.000548934 - 0.00642771i).
\end{aligned}$$

The graph of the displacement is sketched in Fig. 5 and is compared with the numerical solution of the fourth-order Runge–Kutta method.

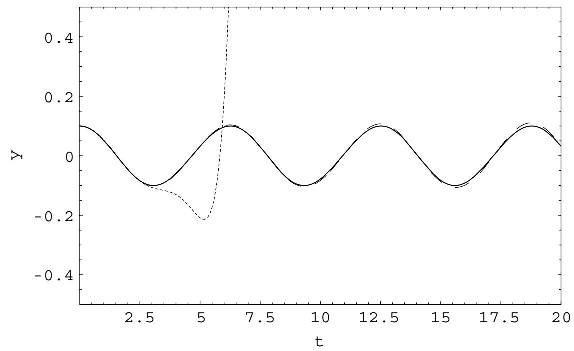

Fig. 5. Plots of displacement y versus time t. Solid line: Runge–Kutta method; dashed line: MHPM; dotted line: HPM.

Example 5. Consider the following initial-value problem

$$y''(x) + y(x) + y^2(x) + y^3(x) = -\varepsilon y y' \tag{68}$$

subject to the initial conditions

$$y(0) = 0.1, \qquad y'(0) = 0. \tag{69}$$

Using the modified homotopy perturbation method, the homotopy for Eq. (68) is constructed as

$$y''(t) = p[-y(t) - y^2(t) - y^3(t) - \varepsilon y(t)y'(t)]. \tag{70}$$

Substituting (14) and the initial condition (69) into (70) we obtain the following set of linear differential equations:

$$\begin{aligned}
&y_0'' = 0, \qquad y_0(0) = 0.1, \qquad y_0'(0) = 0,\\
&y_1'' = -y_0 - y_0^2 - y_0^3 - \varepsilon y_0 y_0', \qquad y_1(0) = 0, \qquad y_1'(0) = 0,\\
&y_2'' = -y_1 - 2y_0y_1 - 3y_0^2y_1 - \varepsilon(y_0'y_1 + y_0y_1'), \qquad y_2(0) = 0, \qquad y_2'(0) = 0,\\
&y_3'' = -y_2 - 2y_0y_2 - 2y_1^2 - 3y_0^2y_2 - 3y_0y_1^2 - \varepsilon(y_0'y_2 + y_1y_1' + y_2'y_0), \quad y_3(0) = 0, \quad y_3'(0) = 0,\\
&\vdots
\end{aligned} \tag{71}$$

Consequently, the first few components of the homotopy perturbation solution for Eq. (68) are as follows:

$$\begin{aligned}
y_0 &= 0.1,\\
y_1 &= -0.0505t^2,\\
y_2 &= -0.00185\varepsilon t^3 + 0.00568875t^4,\\
y_3 &= -0.00004625\varepsilon^2 t^4 - 0.000535575\varepsilon t^5 - 0.000469391t^6,\\
&\vdots
\end{aligned} \tag{72}$$

and so on, and in this manner the rest of components of the homotopy perturbation solution can be obtained. The fourth-term approximate solution for Eq. (68) is given by

$$\begin{aligned}
y(t) = {} & 0.1 - 0.0555t^2 + 0.00568875t^4 - 0.000366716t^6 + 0.0000257661t^8\\
& + \varepsilon(-0.00185t^3 - 0.000535575t^5 + 0.0000723832t^7) + \cdots.
\end{aligned} \tag{73}$$

Because Eqs. (68) and (69) form an oscillatory system, we apply the Laplace transform to the series solution (60), which yields

$$\begin{aligned}
L[y(t)] = {} & \frac{0.1}{s} - \frac{0.111}{s^3} + \frac{0.13653}{s^5} - \frac{0.26403552}{s^7} + \frac{1.038889152}{s^9}\\
& + \varepsilon\left(\frac{0.0111}{s^4} - \frac{0.064269}{s^6} + \frac{0.364811328}{s^8}\right) + \cdots.
\end{aligned} \tag{74}$$

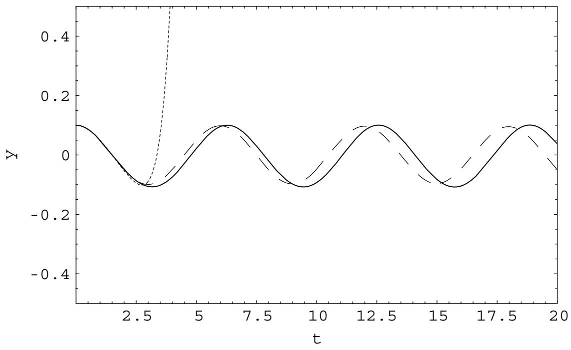

Fig. 6. Plots of displacement y versus time t when $\varepsilon = 0.1$. Solid line: Runge–Kutta method; dashed line: MHPM; dotted line: HPM.

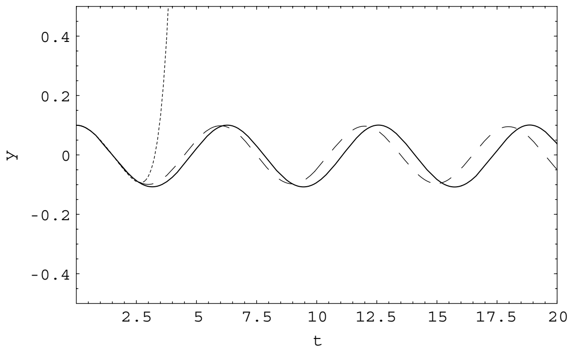

Fig. 7. Plots of displacement y versus time t when $\varepsilon = 0.3$. Solid line: Runge–Kutta method; dashed line: MHPM; dotted line: HPM.

For simplicity, let $s = 1/t$; then

$$\begin{aligned} L[y(t)] &= 0.1t - 0.111t^3 + 0.13653t^5 - 0.26403552t^7 + 1.038889152t^9 \\ &\quad + \varepsilon(0.0111t^4 - 0.064269t^6 + 0.364811328t^8) + \cdots . \end{aligned} \tag{75}$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.1t + 0.01t^2\varepsilon}{1 + 1.11t^2 + 0.1t\varepsilon} \tag{76}$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{0.1s + 0.01\varepsilon}{1.11 + s^2 + 0.1s\varepsilon}. \tag{77}$$

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain the modified approximate solution

$$\begin{aligned} y(t) &= (e^{-0.05t\varepsilon - 0.05t\sqrt{-21.0713+\varepsilon}}(-0.05\varepsilon + 0.05\sqrt{-21.0713+\varepsilon}\sqrt{21.0713+\varepsilon} \\ &\quad + e^{0.1t\sqrt{-21.0713+\varepsilon}}(0.05\varepsilon + 0.05\sqrt{-21.0713+\varepsilon}\sqrt{21.0713+\varepsilon})))/(\sqrt{-21.0713+\varepsilon}\sqrt{21.0713+\varepsilon}). \end{aligned} \tag{78}$$

The modified approximate solution (78) and the solution obtained by fourth-order Runge–Kutta method are compared in Figs. 6 and 7 when $\varepsilon = 0.3$ and 0.1, respectively. The results of our computations are in excellent agreement with the results obtained in [24] using the modified differential transform method.

4. Conclusion

The modified homotopy perturbation method suggested in this paper is an efficient tool for calculating periodic solutions to nonlinear oscillatory systems. All of the examples show that the results of the present method are in excellent agreement with those obtained by the fourth-order Runge–Kutta method, even for moderately large values of the parameter ε. These examples indicate that the modified homotopy perturbation method greatly improves HPM's truncated series solution in rate of convergence and that it often yields the true analytic solution. The basic idea described in this paper is expected to be further employed to solve other similar strongly nonlinear oscillators.

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