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# POINT-WISE SUMMABILITY OF FOURIER SERIES AND ITS RELATED SERIES IN THE NORLUND SENSE

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## ABSTRACT

In this note we consider the point-wise summability of Fourier series in the Norlund sense by extending a theorem of Lebesgue [6]. We also consider analogous theorems of our extension to derived Fourier series, and conjugate Fourier series. We finally prove some general theorem on the above topic.

## INTRODUCTION

1. Let $\sum_{k=o}^{\infty} u_k$ be a given series, and let $\{ S_n \}$ denote the sequence of its partial sums. Let $\{ q_n \}$ be a sequence of real numbers such that $Q_n = q_0 + q_1 + \ldots\ldots + q_n \neq 0$ $(n \geqslant 0)$, $Q_n = q_n = 0$ $(n < 0)$. Define $t_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k$.

If $\lim_{n \to \infty} t_n = S$ we say that $\sum_{k=0}^{\infty} u_k$ is summable in the Norlund sense or $S(N, q_n)$. we note that when $q_n = 1$ for $n = 0, 1, 2, \ldots..$, then Norlund's method of summability reduces to Cesaro's method of summability or (C, 1) summability.

The necessary and sufficient conditions for the regularity of the $S(N,q_n)$ method are [2]:

$$(1) \quad \frac{q_n}{Q_n} = o\ (1) \text{ as } n \to \infty \quad , \text{ and}$$

$$(2) \quad \sum_{k=0}^{n} q_k = 0\ (\ |\ Q_n\ |\ ) \text{ as } n \to \infty \quad .$$

If $q_0 > 0$, and $\{ q_n \}$ is non-negative for $n = 1, 2, \ldots.$, then clearly condition (1) above only is necessary and sufficient for the regularity of the $S(N,q_n)$ method; furthermore if in addition $\{q_n\}$ in non-increasing then the $S(N,q_n)$ method is automatically regular since $\frac{q_n}{Q_n} = 0\ (\frac{1}{n+1})$ .

2. Let f be a periodic function with period $2\pi$, integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let the Fourier series S [ f ] of f at t=x be given by:

$$S\ [f] = \frac{1}{2} a_0 + \sum_{j=1}^{\infty} a_j \cos jx + bj \sin jx.$$

Then the derived Fourier series of f $S\ [f']$ is given by:

$$S\ [f'] = \sum_{J=1}^{\infty} j\ b_j \cos jz - a_j \sin jx = \sum_{j=1}^{\infty} j\ B_j\ (x),$$

and the cojugate Fourier series of f $\tilde{S}\ [f]$ is given by:

$$\tilde{S}\ [f] = \sum_{j=1}^{\infty} b_j \cos jx - a_j \sin jx.$$

Let $\phi\ (t) = f(x+t) + f(x-t) - 2f(x),$

$\psi\ (t) = f(x+t) - f(x-t),$

$r\ (t) = f(x+t) - f(x-t) - 2\ t\ f'(x),$

$\Phi\ (t) = \int_0^t |\ \phi(u)\ |\ du,$

$\Psi\ (t) = \int_0^t |\ \psi\ (u)\ |\ du\ ,$ and

$R(t) = \int_0^t |\ r'(u)\ |\ du\ .$

In [6] we have:

Theorem 2.1 (Lebesgue): S [ f ] is summable (C, 1) to f(x) at each point x where $\Phi\ (t) = o(t)$.

3 We consider the following lemmas:

Lemma 3.1 (Tamarkin and Hille [3] ): Let $\{ q_n \}$ be non-increasing sequence of non-negative real numbers. Then for any a such that $0 \leqslant a \leqslant b \leqslant \infty$ , $0 \leqslant t \leqslant \pi$ and any n,

$$\left| \sum_{k=a}^{b} q_k \ e^{i(n-k)t} \right| \leqslant K\, Q_\tau \quad \text{, where}$$

K is an absolute constant, $\tau = [\frac{1}{t}]$ the integral part of $\frac{1}{t}$ , and $Q_n = q_0 + q_1 + \ldots.\ q_n$.

Lemma 3.2 (Pati [ 5 ] ) : Let $\{ q_n \}$ be a sequence of non-negative real numbers, and let $Q_n = q_0 + q_1 + \ldots. +q_n$ . Then for $0 \leqslant t \leqslant \frac{1}{n}$ we have :

$$\frac{1}{2\pi Q_n} \sum_{k=o}^{n} q_k \frac{\sin(2n-2k+2)\frac{t}{2}}{\sin\frac{t}{2}} = 0(n) \text{ as } n \to \infty .$$

Lemma 3.3 (Dikshit [ 1 ] ) : Let $\{ q_n \}$ be a sequence of non negative real numbers, and let $Q_n = q_0 + q_1 + \ldots\ldots + q_n$. Then for $0 \leqslant t \leqslant \pi$ we have :

$$\frac{1}{2\pi Q_n} \sum_{k=o}^{n} q_k \frac{\cos\frac{t}{2} - \cos(n-k+\frac{1}{2})t}{\sin\frac{t}{2}} = 0(n) \text{ as } n \to \infty.$$

4. Let g be a positive function defined for $x > x_0$. Then g is said to be slowly varying [ 6 ] if for $\propto > 0$, $g(x).\,x$ is an increasing, and $\frac{g(x)}{x^{\propto}}$ is a decreasing function of x for x sufficiently large. Accordingly the sequence $q_n = \frac{g(n)}{(n+1)^{\propto}}$, where g is slowly varying, and $\infty > 0$ is non-negative, and non-increasing (for n large enough); furthermore if $\propto < 1$ then by [ 6 ] $Q_n = \sum_{k=o}^{n} q_k \sim \frac{n^{1-\propto}}{1-\propto} \cdot g(n)$ as $n \to \infty$.

In the first part of our note we consider the following :

Let $q_n = \frac{g(n)}{(n+1)^{\propto}}$, where $0 < \propto < 1$, and g is slowly varying.

Then we have the following theorems:

Theorem 1: S [ f ] is summable $S(N, \frac{g(n)}{(n+1)^{\propto}})$ to f (x) at each point x where $\Phi(t) = o(t)$ .

Analogously we have:

Theorem 2: S [f'] is summable $S(N, \frac{g(n)}{(n+1)^{\propto}})$ to f'(x) at each x where $R(t) = o(t)$.

Theorem 3: $\tilde{S}$ [ f ] is summable $S(N, \frac{g(n)}{(n+1)^{\propto}})$ to

$$\frac{1}{\pi} \int_{o}^{\pi} \frac{\phi(t)}{2\tan\frac{t}{2}} dt$$ at each point x where $\Psi(t) = o(t)$.

The proof of the above theorem is contained with some slight modifications (see remarks on p. 5) in the proof of theorems 4, 5 and 6 below, and hence is omitted. We note further that the choice $\propto = 0$ with $g(x) = 1$ for all x reduces to Lebesgue's theorem [6].

In the second part of our note we consider the following :

Let $q_n$ be a non-negative, non-increasing sequence of real numbers ($q_0>0$) and let $Q_n = q_0 + q_1 + .... + q_n$ be such that $Q_n \to \infty$ as $n \to \infty$. Similarly let $a_n$ be a non-negative, nn-increasing sequence of real numbers with $a_1 < 0$, and let $A_n = a_1+a_2 + ...... +a_n$ be such that $A_n \to \infty$ as $n \to \infty_n$ .

Assume now that $\frac{1}{Q_n} \sum_{k=1}^{n} \frac{a_k, Q_k}{a_k} = 0(1)$ as $n \to \infty$ .

Then we have the following theorems:

Theorem 4 : S [f] is summable $S(N,q_n)$ to $f(x)$ at each point x

where $\phi(t) = o\left(\frac{a_\tau}{A_\tau}\right)$

Analogously we have :

Theorem 5: S [f'] is summable $S(N, q_n)$ to $f'(x)$ at each point x

where $R(t) = o\left(\frac{a_\tau}{A_\tau}\right)$ .

Therome 6: $\tilde{S}$ [ f ] is summable $S(N,q_n)$ to $\frac{1}{\pi} \int_0^\pi \frac{\psi(t)}{2 \tan \frac{t}{2}} dt$

at each point x where $\psi(t) = o\left(\frac{a_\tau}{A_\tau}\right)$ .

Proof of theorem 4: Let $S_n(x)$ denote the sequence of partial sums of S [ f ] at t=x. Then

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\operatorname{Sin}\,(n+\frac{t}{2})t}{\sin \frac{t}{2}} dt.$$

Hence

$$t_n(x) - f(x) = \int_o^{\pi} \phi(t) \frac{1}{2\pi Q_n} \sum_{k=o}^{n} q_k \frac{\sin(n-k+\frac{1}{2})t}{\sin\frac{t}{2}} dt,$$

$$= \int_o^{\pi} \phi(t) K_n(t) dt \quad , \qquad \text{say.}$$

In order to prove the theorem we show

$$\int_o^{\pi} \phi(t) K_n(t) dt = o(1) \qquad \text{as } n \to \infty .$$

Now for a suitable choice of $\delta$ such that $0 < \frac{1}{n} < \delta < \pi$ we have:

$$(t) K_n(t) dt = \left( \int_o^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi} \right) \phi(t) K_n(t) dt ,$$

$$= I_1 + I_2 + I_3 \text{ , say .}$$

First by lemma 3.2 above we have:

$$I_1 = \int_o^{\frac{1}{n}} \phi(t) K_n(t) dt$$

$$= O(n \int_o^{\frac{1}{n}} |\phi(t)| dt) = o\left(\frac{n.\ a_n}{A_n}\right) = o(1) \text{ as } n \to \infty,$$

since $n.a_n \leqslant A_n$.

Second, clearly the method $S(N,q_n)$ is regular. Hence by the Riemann-Lebesgue theorem and the regularity of the method $S(N,q_n)$

we have : $I_3 = \int_{\delta}^{\pi} \phi(t) K_n(t) dt = o(1) \quad \text{as } n \to \infty$

Third by lemma 3.1 we have:

$$I_2 = \int_{\frac{1}{n}}^{\delta} \phi(t) K_n(t) dt = O\left(\frac{1}{Q_n} \int_{\frac{1}{n}}^{\delta} |\phi(t)| \frac{Q_{\tau}}{t} dt\right).$$

Now we can easily show (see [1], [4], [5]) that:

$$\frac{1}{Q_n}\int_{\frac{1}{n}}^{\delta}|\phi(t)|\frac{Q_\tau}{t}\,dt=\left[\frac{1}{Q_n}\Phi(t)\frac{Q_\tau}{t}\right]_{\frac{1}{n}}^{\delta}\frac{1}{Q_n}\int_{\frac{1}{n}}^{\delta}\Phi(t)\frac{dQ_\tau}{t}+\frac{1}{Q_n}\int_{\frac{1}{n}}^{\delta}\Phi(t)\frac{Q_\tau}{t^2}\,dt,$$

$= I_{2,1}+I_{2,2}+I_{2,3}$, say.

$$I_{2,1}=0\left(\frac{1}{Q_n}\right)+o\left(\frac{Q_n\,.\,n\,.\,a_n}{Q_n\,,\,A_n}\right)=o(1)+o(1)=o(1)\text{ as } n\to\infty\,.$$

$$I_{2,2}=0\left(\frac{1}{Q_n}\int_{\frac{1}{\delta}}^{n}s.\ \Phi\left(\frac{1}{s}\right)dQ_{[s]}\right)$$

$$=o\left(\frac{1}{Q_n}\sum_{k=1}^{n}\frac{k\,.\,a_k\,(Q_k-Q_{k-1})}{A_k}\right)+o(1)\,,$$

$$=o\left(\frac{1}{Q_n}\sum_{k=1}^{n}\frac{a_k\,.\,k\,.\,q_k}{A_k}\right)+o(1)=o\left(\frac{1}{Q_n}\sum_{k=1}^{n}\frac{a_k\,.\,Q_k}{A_k}\right)+o(1)$$

$$=o(1)\text{ as } n\to\infty\,.$$

$$I_{2},3=o\left(\int_{\frac{1}{\delta}}^{n}\frac{a_{[s]}\,Q_{[s]}}{Q_n-A_{[s]}}\,ds\right)$$

$$=o\left(\frac{1}{Q_n}\sum_{k=1}^{n}\frac{a_k-Q_k}{A_k}\right)+o(1)\text{ as } n\to\infty.$$

Hence $I_2 = o(1)$ as $n \to \infty$. This completes the proof of theorem 4.

Proof of theorem 5: Let $S_n(x)$ denote the sequence of partial sums of the derived series of a Fourier series. Then

$$S_n(x)=-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}\left(\frac{\sin(n+\frac{1}{2})\,t}{\sin\frac{t}{2}}\right)dt\,,\text{ and}$$

hence by integration by parts and simplifying we have :

$$S_n(x)-f'(x)=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,r'(t)\,dt.\text{ Therefore}$$

$$t_n(x) - f'(x) = \frac{1}{2\pi Q_n} \int_o^{\pi} r'(t) \sum_{k=o}^{n} q_k \frac{\sin (n-k+\frac{1}{2})t}{\sin \frac{t}{2}} \, dt$$

$$= \int_o^{\pi} r'(t) \; K_n(t) \; dt = 0(1) \text{ as in theorem 4 above.}$$

Proof of theorem 6: Let $S_n$ (x) denote the sequence of partial sums of the conjugate series of Fourier series. Then it is easily seen that:

$$t_n(x) \frac{1}{\pi} \int_o^{\pi} \frac{(t)}{2 \tan \frac{t}{2}} \, dt = - \int_o^{\pi} \psi(t) \; K_n \; (t) \; dt \text{ , where}$$

$$K_n \; (t) = \frac{1}{2\pi Q_n} \sum_{k=0}^{h} q_k \frac{\cos(n-k+\frac{1}{2}) \; t}{\sin \frac{t}{2}}$$

Now the proof follows as in theorem 4 above.

## REMARKS

1. If $a_k = . q_K = 1$, then $A_n = n$, and $Q_n = n+1 \sim A_n$. We have:

$$\Phi \; (t) = 0( \frac{a_\tau}{A_\tau} ) = o \; (t), \text{ and } \frac{1}{Q_n} \sum_{k=o}^{n} \frac{a_k \; Q_k}{A_k} = 0(1) \text{ as } n \to \infty \; .$$

Clearly this case represents Lebesgue's theorem [ 6 ].

2. If $a_k = 1$, and $q_k = \frac{g(k)}{(K+1)^{\propto}}$ $o < \cdot \propto < 1$ . Then $A_k = k$, and $Q_k \sim \frac{k^{1-\propto}}{1-\propto} g \; (k)$

as $k \to \infty$ ; furthermore $\Phi$ (t) = o(t), and

$$\frac{1}{Q_n} \sum_{k=o}^{n} \frac{a_k \cdot Q_k}{A_k} \sim \frac{1}{n^{1-\propto} g(n)} \sum_{k=o}^{n} k^{-\propto} g(k) = 0 \; (1) \text{ as } n \to \propto.$$

Clearly this case represents theorems 1, 2, and 3.

3. Assume that $\frac{g(n)}{(n+1)^{\propto}}$ is non-increasing for $n > n_1$. Then it is easily seen that application

of lemma 3.1 is possible in the proofs of theorems 1, 2, and 3.

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