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A REMARK ON PROPER LEFT H* — ALGEBRAS

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ABSTRACT

W. Ambrose gave the theory of proper H* -algebras and M. Smiley in (2) gave an example of a left H* -algebra which is not a two-sided H* -algebra. Then he modified some of the arguments of Ambrose which yield the structure of proper right H*-algebras. In fact he proved that a proper right H*-algebra is merely a proper H*-algebra in which the norm has been changed to a certain equivalent norm in each of the simple components.

In this short paper, we define proper left H*-algebras and give two lemmas for these classes. Then we prove the main result that every proper left H*-algebra is a proper H*-algebra.

Thus, in this paper, we prove that the following are equivalent:

(i) Proper left H*-algebras.
(ii) Proper right H*-algebras.
(iii) Proper H*-algebras.

INTRODUCTION

An H*-algebra is a Banach*-algebra A which satisfies the following further conditions:

(1) The underlying Banach space of A is a Hilbert space (of arbitrary dimension) with inner product $< , >$;

(2) for each x∈A there is an element x* (called the adjoint of x) in A, such that for all y,z in A we have both

$<xy, z> = <y, x^*z>$ .... (*)
and $<yx, z> = <y, zx^*>$ .... (**)

This definition of H*-algebras was given by W. Ambrose (1) who gave a full construction of the theory of H*-algebras, and in 1953 M.F. Smiley introduced the concept of right H*-algebras by imposing only the condition (**) without the other condition (*). In this paper we define the left H*-algebras by imposing the condition (*) only and a two-sided H*-algebra is a left and right H*-algebra. Smiley in (2) proved the results on proper two-sided H*-algebras for proper right H*-algebras: by a proper H*-algebra A we mean Ax = 0 implies that x = 0.

Thus, to say that an H*-algebra is proper is the same by saying that it is semisimple.

Throughout the paper, let A be a proper left H*-algebra and for every ideal I in A let I^p be the orthogonal complement of I.

MAIN THEORY

We start with the following two lemmas which are similar to lemmas 2.4 and 2.5 in (1).

Lemma 1

If R is a right ideal of A and $xA \subseteq R$, then $x \in R$.

Proof

Let $x = x_1 + x_2$ with $x_1 \in R$ and $x_2 \in R^p$. Then for all z in A we have $x_2 z = xz - x_1 z$. Since $xz \in R$ and $x_1 z \in R$, $x_2 z \in R$ for all $z \in A$. But $x_2 \in R^p$, so $<x_2 z, x_2> = 0$. Also $0 = <x_2 z, x_2> = <z, x^*{}_2 x>$ $\forall$ $z \in A$. This implies that $x^*{}_2 x_2 = 0$ and so $x_2 = 0$ (since A is semisimple) and this implies $x = x_1 \in R$.

Lemma 2

If I is a two-sided ideal in A then $I = I^*$.

Proof

If $x \in I$ and $y \in I^p$ (the orthogonal complement of I), then $xy=0$. Hence for all $z \in A$ we have $<xy, z> = 0$. Thus $<y, x^*z> = 0$ for all $z \in A$; i.e. x^*z is orthogonal to I^p; hence $x^*z \in I$ for all $z \in A$; i.e. $x^*A \subseteq I$ and by Lemma 1, $x^* \in I$; i.e. $x \in I \rightarrow x^* \in I \rightarrow x^{**} \in I \rightarrow I = I^*$.

Remark

In Lemmas 1 and 2 if we assume that A is a proper right H*-algebra then it was proved in (2) the following:

(i) If L is a left ideal of A and $Ax \subseteq L$, then $x \in L$.

(ii) If I is a two-sided ideal of A, then $I = I^*$.

Now we are able to prove the main theorem.

Theorem

If A is a proper left H*-algebra, then it is a two-sided H*-algebra.

Proof

Since A is a left H*-algebra, we have: for all $x,y,z \in A$, $<xy, z> = <y, x^*z>$. In order to prove the theorem we need to show that $<yx, z> = <y, zx^*>$. To do this:

Let $x \in A$ and $N = \text{lin}\{x\}$ - the linear space spanned by x, $M_1 = N^p$ (the orthogonal complement of N), and $M_2 = \{y \in A: y^* \in M_1\}$.

For any $y,z \in A$, $y^*z = \lambda x + v$ where $\lambda \in C$ and $v \in M_1 = N^p$ (Note that $\lambda x \in N$) and so $z^*y = \bar{\lambda} x^* + v^*$ with $v^* \in M_2$.

$$<yx, z> = <x, y^*z> = <x, \bar{\lambda} x + v>$$
$$= <x, \lambda x> + <x, v>$$

$$
\begin{aligned}
&= \bar{\lambda} \langle x, x \rangle \text{ since } \langle x, v \rangle = 0 \\
&= \bar{\lambda} \langle x^*, x^* \rangle = \langle \bar{\lambda} x^*, x^* \rangle \\
&= \langle \bar{\lambda} x^*, x^* \rangle + \langle v^*, x^* \rangle \text{ since } \langle v^*, x^* \rangle = 0 \\
&= \langle \bar{\lambda} x^* + v^*, x^* \rangle = \langle z^* y, x^* \rangle \\
&= \langle y, z x^* \rangle \text{ for all } y, z \in A,
\end{aligned}
$$

and the proof is complete.

Remark

The above theorem is true for proper right H*-algebras also and the proof is exactly by the same method.

Thus proper left H*-algebras, proper right H*-algebras and proper two-sided H*-algebras coincide.

REFERENCES

(1) **Ambrose, W.** 1945, "Structure theorems for a special class of Banach algebras" Trans. Amer. Math. Soc. Vol. 57, pp. 364-386.

(2) **Smiley, M.F.** 1953, "Right H*-algebras" Proc. Amer. Math. Soc. Vol. 4, pp. 1-4.