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Numerical techniques for the variable order time fractional diffusion equation [*]

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Abstract

In this paper we consider the variable order time fractional diffusion equation. We adopt the Coimbra variable order (VO) time fractional operator, which defines a consistent method for VO differentiation of physical variables. The Coimbra variable order fractional operator also can be viewed as a Caputo-type definition. Although this definition is the most appropriate definition having fundamental characteristics that are desirable for physical modeling, numerical methods for fractional partial differential equations using this definition have not yet appeared in the literature. Here an approximate scheme is first proposed. The stability, convergence and solvability of this numerical scheme are discussed via the technique of Fourier analysis. Numerical examples are provided to show that the numerical method is computationally efficient.

Key words: Numerical method, variable order time fractional diffusion equation, stability, convergence, solvability, Fourier analysis.
MSC(2000) 26A33, 34K28, 65M12, 60J70

1 Introduction

The number of scientific and engineering problems involving fractional calculus is already very large and still growing. One of the main advantages of the fractional calculus is that the fractional derivatives provide an excellent approach for the description of memory and hereditary properties of various materials and processes [6]. Many of the numerical methods using different kinds of fractional derivative operators for solving fractional partial differential equations have been proposed [13–18,28]. Anh and Leonenko presented a spectral representation of the mean-square solution of the fractional diffusion equation with random initial condition, from which the Caputo-Djrbashian regularied fractional derivative was adopted [1]. Odibat proposed two algorithms for numerical fractional integration and Caputo fractional differentiation. Using the new modification derive an algorithm to approximate fractional derivatives of arbitrary order for a given function by a weighted sum of function and its ordinary derivative values at specified points [21]. Blaszczyk focused on a numerical scheme applied for a fractional oscillator equation which includes a complex form of left- and right-sided fractional derivatives in a finite time interval [2].

Recently, more and more researchers are finding that a variety of important dynamical problems exhibit fractional order behavior that may vary with time or space. This fact indicates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex dynamical problems. The concept of a variable order operator is a much more recent development, which is a new paradigm in science. Samko and Ross [24,25] directly generalized the Riemann-Liouvile and Marchaud fractional integration and differentiation of the case of variable order, and then showed some properties and an inversion formula. Lorenzo and Hartley [19,20] suggested the concept of a variable order operator is allowed to vary either as a function of the independent variable of integration or differentiation (t), or as a function of some other (perhaps spatial) variable (x), they also explored more deeply the concept of variable order integration and differentiation and sought the relationship between the mathematical concepts and physical processes. Different authors have proposed different definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. Coimbra [4] took the Laplace-transform of Caputo's definition of the fractional derivative as the starting point to suggest a novel definition for the variable order differential operator. Because of its meaningful physical interpretation, Coimbra's definition is better suited for modeling physical problems. Ingman et al. [8,9] employed the time dependent variable order operator to model the viscoelastic deformation process. Pedro et al. [22] studied the motion of particles suspended in a viscous fluid with drag force using variable order calculus. Sun et al. [27] introduced a classification of variable-order fractional diffusion

models based on the possible physical origins that prompt the variable-order.

The variable order operator definitions recently proposed in the literature include the Riemann-Liouvile-definition, Caputo-definition, Marchaud-definition, Coimbra-definition and Grünwald-definition [4,19,20,24,25]. However, to the best of the authors knowledge, detailed studies of the Grünwald-type variable order operator have not yet been performed. Samko et al. [24] compared the Riemann-Liouvile-definition and Marchaud-definition variable order operators, and noted the loss of certain properties of the Riemann-Liouvile definitions, with the Marchaud-definition being more suitable than the Riemann-Liouvile-type. Ramirez et al. [23] also compared the Riemann-Liouvile-definition, Caputo-definition, Marchaud-definition and Coimbra-definition variable order operators based on a very simple criteria: the variable order operator must return the correct fractional derivative that corresponds to the argument of the functional order. Ramirez et al. found that only the Marchaud-definition and Coimbra-definition satisfy the above elementary requirement, and the Coimbra-definition variable order operator is more efficient from the numerical standpoint. Soon et al. [26] also showed that the Coimbra-definition variable order operator satisfies a mapping requirement, and it is the only definition that correctly describes position-dependent transitions between elastic and viscous regimes because it correctly returns the appropriate derivatives as a function of $x(t)$. Ramirez [23] showed that the Coimbra definition is the most appropriate definition having fundamental characteristics that are desirable for physical modeling.

Since the kernel of the variable order operators has a variable-exponent, analytical solutions to variable order fractional differential equations are more difficult to obtain, and have not been the the focus of much attention. However, the development of numerical techniques to solve variable order fractional differential equations are at the early stage of development. Coimbra [4] proposed a consistent (first-order accurate) approximation for the solution of variable order differential equations. Soon et al. [26] employed a second-order Runge-Kutta method consisting of an explicit Euler predictor step followed by an implicit Euler corrector step to numerically integrate the variable order differential equation. Sun et al. [27] introduced a classification of the variable-order fractional diffusion models based on the possible physical origins that motivated the variable-order, and employed the Crank-Nicholson scheme to get the diffusion curve of the variable order differential operator model. However, many of these authors [4,5,26,27] haven't discussed the stability and convergence of the numerical solutions. Lin et al. [11] investigated stability and convergence of an explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. Zhuang et al. [29] proposed explicit and implicit Euler approximations for the variable-order fractional advection-diffusion equation with a nonlinear source term. Chen et al. [3] proposed two numerical schemes for a variable-order anomalous subdiffusion equation, one

with first order temporal accuracy and fourth order spatial accuracy, the other with second order temporal accuracy and fourth order spatial accuracy. However these authors [3,11,29] considered Riemann-Liouville variable-order fracctional derivatives, or the Riesz variable-order fractional derivative.

In this paper, we consider the following variable order time fractional diffusion equation(VOTFDE):

$$
{}_0D_t^{q(x,t)}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),
$$
$$
(x,t) \in \Omega = [0,L] \times [0,T],
$$
(1)

with initial and boundary conditions

$$
u(x,0) = g(x),\ 0 \le x \le L,
$$
(2)

$$
u(0,t) = u(L,t) = 0,\ 0 < t < T.
$$
(3)

where $0 < \underline{q} \le q(x,t) \le \overline{q} < 1$ and ${}_0D_t^{q(x,t)}$ denotes the variable order time fractional derivative defined by Coimbra [4]:

$$
{}_0D_t^{q(x,t)}u(x,t) = \frac{1}{\Gamma(1-q(x,t))} \int_{0^+}^{t} (t-\sigma)^{-q(x,t)} \frac{\partial u(x,\sigma)}{\partial \sigma} d\sigma
$$
$$
+ \frac{(u(x,0^+) - u(x,0^-))t^{-q(x,t)}}{\Gamma(1-q(x,t))}.
$$
(4)

The definition (4) is particularly useful for the solution of well-posed physical problems. In addition, the differential operator (4) requires only one initial condition. We adopt the Coimbra-definition variable order operator in this work. For the sake of simplicity, assuming $u(x,0^+) = u(x,0^-)$, then the Coimbra definition can be viewed as the following Caputo-type definition

$$
{}_0D_t^{q(x,t)}u(x,t) = \frac{1}{\Gamma(1-q(x,t))} \int_{0^+}^{t} (t-\sigma)^{-q(x,t)} \frac{\partial u(x,\sigma)}{\partial \sigma} d\sigma.
$$
(5)

To the best of the authors knowledge, numerical schemes via the above Caputo-type definition that is investigated here have not appeared in the literature.

2 Approximate scheme for the variable order time fractional diffusion equation

Let $\Omega = [0,L] \times [0,T]$, we define the function space

$$
G(\Omega) = \left\{ u(x,t) \Big| \frac{\partial^4 u(x,t)}{\partial x^4}, \frac{\partial^2 u(x,t)}{\partial t^2} \in C(\Omega) \right\}.
$$

In this paper, we suppose the continuous problem (1)-(3) has a smooth solution $u(x,t) \in G(\Omega)$.

We take an equally spaced mesh of M points for the spatial domain $0 \leq x \leq L$, N constant time steps for the temporal domain $0 \leq t \leq T$, and denote the spatial grid points by

$$x_i = ih, i = 0, 1, \ldots, M,$$

and the temporal grid points by

$$t_n = n\tau, n = 0, 1, \ldots, N,$$

where the grid spacing is simply $h = L/M$ in the spatial domain and $\tau = T/N$ in the temporal domain.

At the grid point (x_i, t_n), Eq. (1) becomes

$$_0D_t^{q(x_i,t_n)}u(x_i, t_n) = \frac{\partial^2 u(x_i, t_n)}{\partial x^2} + f_i^n, \tag{6}$$

where $f_i^n \equiv f(x_i, t_n)$.

The second-order spatial derivative can be approximated by the following expression:

$$\frac{\partial^2 u(x_i, t_n)}{\partial x^2} = \frac{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n)}{h^2} + O(h^2). \tag{7}$$

Adopting the discrete scheme given in [10,12], we discretize the variable order time fractional derivative as

$$
\begin{aligned}
&_0D_t^{q(x_i,t_n)}u(x_i, t_n) \\
&= \frac{1}{\Gamma(1 - q(x_i, t_n))} \int_{0^+}^{t_n} (t_n - \sigma)^{-q(x_i,t_n)} \frac{\partial u(x_i, \sigma)}{\partial \sigma} d\sigma \\
&= \frac{1}{\Gamma(1 - q(x_i, t_n))} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - \sigma)^{-q(x_i,t_n)} \frac{\partial u(x_i, \sigma)}{\partial \sigma} d\sigma \\
&= \frac{1}{\Gamma(1 - q(x_i, t_n))} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - \sigma)^{-q(x_i,t_n)} \\
&\quad \times \left(\frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} + C \cdot \tau \right) d\sigma \\
&= \frac{1}{\Gamma(1 - q(x_i, t_n))} \sum_{j=0}^{n-1} \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{t_j}^{t_{j+1}} (t_n - \sigma)^{-q(x_i,t_n)} d\sigma + r_i^n \\
&= \frac{\tau^{-q(x_i,t_n)}}{\Gamma(2 - q(x_i, t_n))} \sum_{j=0}^{n-1} d_{i,j,n}[u(x_i, t_{j+1}) - u(x_i, t_j)] + r_i^n, \tag{8}
\end{aligned}
$$

where

$$d_{i,j,n} = (n-j)^{1-q(x_i,t_n)} - (n-j-1)^{1-q(x_i,t_n)}, j = 0, 1, \ldots, n-1;$$

$$\begin{aligned}
r_i^n &= \frac{1}{\Gamma(1-q(x_i,t_n))} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} C \cdot \tau (t_n - \sigma)^{-q(x_i,t_n)} d\sigma \\
&= \frac{C \cdot \tau}{\Gamma(2-q(x_i,t_n))} \sum_{j=0}^{n-1} \left[(t_n - t_j)^{1-q(x_i,t_n)} - (t_n - t_{j+1})^{1-q(x_i,t_n)} \right] \\
&= \frac{C \cdot \tau^{2-q(x_i,t_n)}}{\Gamma(2-q(x_i,t_n))} \sum_{j=0}^{n-1} \left[(n-j)^{1-q(x_i,t_n)} - (n-j-1)^{1-q(x_i,t_n)} \right] \\
&= \frac{C \cdot \tau^{2-q(x_i,t_n)}}{\Gamma(2-q(x_i,t_n))} n^{1-q(x_i,t_n)} \\
&\leq \frac{C \cdot T^{1-q(x_i,t_n)}}{\Gamma(2-q(x_i,t_n))} \cdot \tau \\
&\leq \tilde{C} \cdot \tau.
\end{aligned}$$

We denote u_i^n for the numerical approximation to $u(x_i, t_n)$. From Eq. (7) and Eq. (8), we obtain the following approximate scheme for Eq. (1):

$$\frac{\tau^{-q(x_i,t_n)}}{\Gamma(2-q(x_i,t_n))} \sum_{j=0}^{n-1} d_{i,j,n}(u_i^{j+1} - u_i^j) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + f_i^n. \qquad (9)$$

Let $s(x_i, t_n) = \frac{h^2}{\tau^{q(x_i,t_n)}\Gamma(2-q(x_i,t_n))}$, taking into account $d_{i,n-1,n} = 1$, by rearrangement Eq. (9) can be rewritten as

$$-u_{i+1}^n + (2 + s(x_i,t_n))u_i^n - u_{i-1}^n = s(x_i,t_n) \sum_{k=1}^{n-1} (d_{i,n-k,n} - d_{i,n-k-1,n})u_i^{n-k}$$

$$+ s(x_i,t_n)d_{i,0,n}u_i^0 + h^2 f_i^n,$$

$$i = 1, 2, \ldots, M-1; n = 1, 2, \ldots, N.$$

$$(10)$$

The initial and boundary conditions are

$$u_i^0 = g(x_i), \ i = 0, 1, \ldots, M; \qquad (11)$$

$$u_0^n = u_M^n = 0, \ n = 1, 2, \ldots, N. \qquad (12)$$

Note that the coefficients possess the following properties:

Lemma 1: The coefficients $d_{i,j,n}$ $(j = 0, 1, \ldots, n-1; i = 1, 2, \ldots, M-1; n = 1, 2, \ldots, N)$ satisfy

(1) $d_{i,j,n} > 0; d_{i,0,1} = d_{i,n-1,n} = 1; \sum_{k=1}^{n-1} (d_{i,n-k,n} - d_{i,n-k-1,n}) + d_{i,0,n} = 1;$

(2) $d_{i,j,n}$ are increase monotonically with j increases.

Proof:

(1) This conclusion can be obtained by a straight forward calculation.
(2) Let
$$f(j) = d_{i,j,n}, j = 0, 1, \ldots, n - 1.$$
It follows that
$$f'(j) = (1 - q(x_i, t_n)) \left[\frac{1}{(n-j-1)^{q(x_i,t_n)}} - \frac{1}{(n-j)^{q(x_i,t_n)}} \right] > 0, \; j \neq n - 1.$$

Thus, the conclusion can be obtained.

Lemma 2 (Discrete Gronwall Lemma): Let $\{y_n\}$ and $\{g_n\}$ be nonnegative sequences and c a nonnegative constant. If

$$y_n \leq c + \sum_{0 \leq k < n} g_k y_k, \; n \geq 0,$$

then

$$y_n \leq c \prod_{0 \leq j < n} (1 + g_j) \leq c \exp\left(\sum_{0 \leq j < n} g_j \right), \; n \geq 0.$$

Proof: See [7].

3 Stability of the approximate scheme

In this section, we use the technique of Fourier analysis to discuss the stability of the approximate scheme (10)-(12). Consider the following equation

$$-u_{j+1}^n + (2 + s(x_j, t_n))u_j^n - u_{j-1}^n = s(x_j, t_n) \sum_{k=1}^{n-1} (d_{j,n-k,n} - d_{j,n-k-1,n})u_j^{n-k}$$
$$+ s(x_j, t_n)d_{j,0,n}u_j^0$$
$$j = 1, 2, \ldots, M - 1; n = 1, 2, \ldots, N.$$
$$(13)$$

For $n = 0, 1, \ldots, N$, we define the following grid function:

$$u^n(x) = \begin{cases} u_j^n, & x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], j = 1, 2, \ldots, M - 1; \\ 0, & x \in [0, \frac{h}{2}] \cup (L - \frac{h}{2}, L). \end{cases}$$

Then $u^n(x)$ has the Fourier series expansion

$$u^n(x) = \sum_{l=-\infty}^{\infty} \xi_n(l) e^{i2\pi lx/L}, \; n = 0, 1, \ldots, N,$$

where

$$\xi_n(l) = \frac{1}{L} \int_0^L u^n(x) e^{-i2\pi lx/L} dx.$$

Let
$$u^n = [u_1^n, u_2^n, \ldots, u_{M-1}^n],$$
then, using the Parseval identities
$$\int_0^L |u^n(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\xi_n(l)|^2, \ n = 0, 1, \ldots, N,$$

and
$$\int_0^L |u^n(x)|^2 dx = \sum_{j=1}^{M-1} h|u_j^n|^2, \ n = 0, 1, \ldots, N,$$

we obtain
$$\|u^n\|_2 \equiv \left(\sum_{j=1}^{M-1} h|u_j^n|^2\right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\xi_n(l)|^2\right)^{\frac{1}{2}}, \ n = 0, 1, \ldots, N. \quad (14)$$

Assume that the solution of the equation (13) has the form
$$u_j^n = \xi_n e^{i\sigma j h},$$

where $\sigma = 2\pi l/L$. Substituting the above expression into (13) gives
$$[-e^{i\sigma h} + (2 + s(x_j, t_n)) - e^{-i\sigma h}]\xi_n$$
$$= s(x_j, t_n) \sum_{k=1}^{n-1} (d_{j,n-k,n} - d_{j,n-k-1,n})\xi_{n-k} + s(x_j, t_n)d_{j,0,n}\xi_0. \quad (15)$$

Using the identify
$$\sin^2 z = -\frac{1}{4}(e^{i2z} - 2 + e^{-i2z}),$$

Eq. (15) can be rewritten as
$$\xi_n = \frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)} \sum_{k=1}^{n-1}(d_{j,n-k,n} - d_{j,n-k-1,n})\xi_{n-k} + \frac{s(x_j, t_n)d_{j,0,n}}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}\xi_0,$$
$$n = 1, 2, \ldots, N.$$
$$(16)$$

Lemma 3: Let ξ_n $(n = 1, 2, \ldots, N)$ be the solution of Eq. (16), then

$$|\xi_n| \leq |\xi_0|, \ n = 1, 2, \ldots, N.$$

Proof: For $n = 1$, in view of (16) and Lemma 1, we obtain

$$\xi_1 = \frac{s(x_j, t_1)d_{j,0,1}}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_1)}\xi_0 = \frac{s(x_j, t_1)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_1)}\xi_0.$$

Therefore, we obtain

$$|\xi_1| = \left|\frac{s(x_j, t_1)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_1)}\xi_0\right| \leq |\xi_0|.$$

8

Suppose that

$$|\xi_k| \le |\xi_0|, k = 1, 2, \ldots, n-1.$$

According to (16) and Lemma 1,

$$
\begin{aligned}
|\xi_n| &\le \frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)} \sum_{k=1}^{n-1} |d_{j,n-k,n} - d_{j,n-k-1,n}||\xi_{n-k}| \\
&\quad + \frac{s(x_j, t_n)d_{j,0,n}}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}|\xi_0| \\
&= \frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)} \sum_{k=1}^{n-1} (d_{j,n-k,n} - d_{j,n-k-1,n})|\xi_{n-k}| \\
&\quad + \frac{s(x_j, t_n)d_{j,0,n}}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}|\xi_0| \\
&\le \frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)} \left\{ \sum_{k=1}^{n-1}(d_{j,n-k,n} - d_{j,n-k-1,n}) + d_{j,0,n} \right\} |\xi_0| \\
&= \frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}|\xi_0| \\
&\le |\xi_0|.
\end{aligned}
$$

The proof of Lemma 3 is completed via mathematical induction.

According to (14) and Lemma 3, it can be obtained that the solution of equation (13) satisfies

$$\|u^n\|_2 \le \|u^0\|_2, n = 1, 2, \ldots, N.$$

Thus, we have the following result:

Theorem 1. The approximate scheme (10)-(12) is unconditionally stable.

4 Convergence of the approximate scheme

In this section, we use the technique of Fourier analysis to discuss the convergence of the approximate scheme (10)-(12). In view of (7)-(9), we have

$$
\begin{aligned}
&\frac{\tau^{-q(x_j, t_n)}}{\Gamma(2 - q(x_j, t_n))} \sum_{k=0}^{n-1} d_{j,k,n}[u(x_j, t_{k+1}) - u(x_j, t_k)] \\
&= \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{h^2} + f_j^n + O(\tau) + O(h^2).
\end{aligned}
\tag{17}
$$

9

Multiplying by h^2 on both sides of (17), and by simple calculation, we have that

$$
\begin{aligned}
-u(x_{j+1}, t_n) &+ (2 + s(x_j, t_n))u(x_j, t_n) - u(x_{j-1}, t_n) \\
&= s(x_j, t_n) \sum_{k=1}^{n-1} (d_{j,n-k,n} - d_{j,n-k-1,n})u(x_j, t_{n-k}) + s(x_j, t_n)d_{j,0,n}u(x_j, t_0) \\
&\quad + h^2 f_j^n + R_j^n,
\end{aligned} \tag{18}
$$

$$ j = 1, 2, \ldots, M - 1; n = 1, 2, \ldots, N, $$

where $s(x_j, t_n) = \frac{h^2}{\tau^{q(x_j,t_n)}\Gamma(2-q(x_j,t_n))}$, $R_j^n = h^2 \left[O(\tau) + O(h^2)\right]$.

Subtracting (10) from (18), we obtain the following error equation

$$
\begin{aligned}
-E_{j+1}^n &+ (2 + s(x_j, t_n))E_j^n - E_{j-1}^n \\
&= s(x_j, t_n) \sum_{k=1}^{n-1} (d_{j,n-k,n} - d_{j,n-k-1,n})E_j^{n-k} + s(x_j, t_n)d_{j,0,n}E_j^0 + R_j^n,
\end{aligned} \tag{19}
$$

where $E_j^k = u(x_j, t_k) - u_j^k$.

Since $E_j^0 = 0$, we can rewrite (19) as

$$
\begin{aligned}
-E_{j+1}^n &+ (2 + s(x_j, t_n))E_j^n - E_{j-1}^n \\
&= s(x_j, t_n) \sum_{k=1}^{n-1} (d_{j,n-k,n} - d_{j,n-k-1,n})E_j^{n-k} + R_j^n,
\end{aligned} \tag{20}
$$

$$ j = 1, 2, \ldots, M - 1; n = 1, 2, \ldots, N. $$

For $n = 0, 1, 2, \ldots, N$, we define the following grid functions, respectively

$$
E^n(x) = \begin{cases} E_j^n, & x \in \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right], \ j = 1, 2, \ldots, M - 1, \\ 0, & x \in [0, \frac{h}{2}] \cup (L - \frac{h}{2}, L], \end{cases}
$$

and

$$
R^n(x) = \begin{cases} R_j^n, & x \in \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right], \ j = 1, 2, \ldots, M - 1, \\ 0, & x \in [0, \frac{h}{2}] \cup (L - \frac{h}{2}, L]. \end{cases}
$$

Then, $E^n(x)$ and $R^n(x)$ have the Fourier series expansions

$$
E^n(x) = \sum_{l=-\infty}^{\infty} \alpha_n(l)e^{i2\pi lx/L}, \ n = 0, 1, \ldots, N,
$$

and

$$
R^n(x) = \sum_{l=-\infty}^{\infty} \beta_n(l)e^{i2\pi lx/L}, \ n = 0, 1, \ldots, N,
$$

where

$$
\alpha_n(l) = \frac{1}{L}\int_0^L E^n(x)e^{-i2\pi lx/L}dx, \ \beta_n(l) = \frac{1}{L}\int_0^L R^n(x)e^{-i2\pi lx/L}dx.
$$

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Letting

$$E^n = [E_1^n, E_2^n, \ldots, E_{M-1}^n]^T, \; R^n = [R_1^n, R_2^n, \ldots, R_{M-1}^n]^T,$$

and applying the Parseval identities

$$\int_0^L |E^n(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\alpha_n(l)|^2, \; n = 0, 1, \ldots, N,$$

$$\int_0^L |R^n(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\beta_n(l)|^2, \; n = 0, 1, \ldots, N,$$

and

$$\int_0^L |E^n(x)|^2 dx = \sum_{j=1}^{M-1} h|E_j^n|^2, \; n = 0, 1, \ldots, N,$$

$$\int_0^L |R^n(x)|^2 dx = \sum_{j=1}^{M-1} h|R_j^n|^2, \; n = 0, 1, \ldots, N,$$

we have, respectively

$$\|E^n\|_2 \equiv \left(\sum_{j=1}^{M-1} h|E_j^n|^2 \right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\alpha_n(l)|^2 \right)^{\frac{1}{2}}, \; n = 0, 1, \ldots, N, \qquad (21)$$

and

$$\|R^n\|_2 \equiv \left(\sum_{j=1}^{M-1} h|R_j^n|^2 \right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\beta_n(l)|^2 \right)^{\frac{1}{2}}, \; n = 0, 1, \ldots, N. \qquad (22)$$

Since j, n are finite, there is a positive constant C_1 for all j, n such that

$$|R_j^n| \leq C_1 h^2(\tau + h^2), \; j = 1, 2, \ldots, M, \; n = 1, 2, \ldots, N. \qquad (23)$$

Further, by the first equality of (22) we have

$$\|R^n\|_2 \leq C_1 \sqrt{L} h^2(\tau + h^2), \; n = 1, 2, \ldots, N. \qquad (24)$$

By the convergence of the series on the right-hand side of (22), there is a positive constant C_2 such that

$$|\beta_n| \equiv |\beta_n(l)| \leq C_2 |\beta_1(l)| \equiv C_2|\beta_1|, \; n = 1, 2, \ldots, N. \qquad (25)$$

We now assume that E_j^n and R_j^n have the following form:

$$E_j^n = \alpha_n e^{i\sigma j h}, \; R_j^n = \beta_n e^{i\sigma j h}$$

where $\sigma = 2\pi l/L$. Substituting the above expressions into (20), we have

$$-\alpha_n e^{i\sigma(j+1)h} + (2 + s(x_j, t_n))\alpha_n e^{i\sigma jh} - \alpha_n e^{i\sigma(j-1)h}$$
$$= s(x_j, t_n) \sum_{k=1}^{n-1}(d_{j,n-k,n} - d_{j,n-k-1,n})\alpha_{n-k}e^{i\sigma jh} + \beta_n e^{i\sigma jh}. \tag{26}$$

Multiplying (26) by $e^{-i\sigma jh}$ and again using the identify that

$$\sin^2 z = -\frac{1}{4}(e^{i2z} - 2 + e^{-i2z}),$$

we obtain that

$$\alpha_n = \frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}\sum_{k=1}^{n-1}(d_{j,n-k,n} - d_{j,n-k-1,n})\alpha_{n-k}$$
$$+ \frac{1}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}\beta_n. \tag{27}$$

Lemma 4: Let α_n $(n = 1, 2, \ldots, N)$ be the solution of Eq. (27), then there is a positive constant C_3 such that

$$|\alpha_n| \le C_3 h^{-2}|\beta_1|, \ n = 1, 2, \ldots, N.$$

Proof: Since $0 < q(x, t) < 1$, we can suppose that

$$\frac{1}{s(x_j, t_n)} \le C_0 \tau^{q(x_j, t_n)} h^{-2}, \ j = 1, 2, \ldots, M, \ n = 1, 2, \ldots, N,$$

where C_0 is a positive constant.

According to (27), (25) and Lemma 1, for fixed j and n, we have

$$|\alpha_n| \le \left|\frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}\right|\sum_{k=1}^{n-1}|d_{j,n-k,n} - d_{j,n-k-1,n}||\alpha_{n-k}|$$
$$+ \left|\frac{\beta_n}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}\right|$$
$$= \frac{s(x_j, t_n)}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}\sum_{k=1}^{n-1}(d_{j,n-k,n} - d_{j,n-k-1,n})|\alpha_{n-k}|$$
$$\frac{1}{4\sin^2\frac{\sigma h}{2} + s(x_j, t_n)}|\beta_n|$$
$$\le \sum_{k=1}^{n-1}(d_{j,n-k,n} - d_{j,n-k-1,n})|\alpha_{n-k}| + \frac{1}{s(x_j, t_n)}|\beta_n|$$
$$\le C_0 C_2 h^{-2}|\beta_1| + \sum_{k=1}^{n-1}(d_{j,n-k,n} - d_{j,n-k-1,n})|\alpha_{n-k}|$$
$$= C_0 C_2 h^{-2}|\beta_1| + \sum_{k=1}^{n-1}(d_{j,k,n} - d_{j,k-1,n})|\alpha_k|$$
$$= C_0 C_2 h^{-2}|\beta_1| + \sum_{k=1}^{n-1}g_k|\alpha_k|,$$

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where $g_k = d_{j,k,n} - d_{j,k-1,n}$ and $|\alpha_k|$ are nonnegative sequences. Using lemma 2, we have

$$
\begin{aligned}
|\alpha_n| &\leq C_0 C_2 h^{-2} |\beta_1| \cdot \exp\Big(\sum_{k=1}^{n-1} g_k\Big) \\
&\leq C_0 C_2 h^{-2} |\beta_1| \cdot \exp\Big(\sum_{k=1}^{n-1} (d_{j,n-k,n} - d_{j,n-k-1,n})\Big) \\
&\leq C_0 C_2 h^{-2} |\beta_1| \cdot \exp(d_{j,n-1,n} - d_{j,0,n}) \\
&= C_0 C_2 h^{-2} |\beta_1| \cdot \exp(1 - d_{j,0,n}) \\
&\leq C_0 C_2 \exp(1) h^{-2} |\beta_1| \\
&= C_3 h^{-2} |\beta_1|,
\end{aligned}
$$

where $C_3 = C_0 C_2 \exp(1)$.

Thus, the proof of Lemma 4 is completed.

Using (21), (22), (24) and Lemma 4, we obtain

$$
\|E^n\|_2 \leq C_3 h^{-2} \|R^1\|_2 \leq C_3 h^{-2} \cdot C_1 \sqrt{L} h^2 (\tau + h^2) = C(\tau + h^2),
$$

where $C = C_3 C_1 \sqrt{L}$.

Now we can get the following theorem of convergence:

Theorem 2. Suppose that the continuous problem (1) has a smooth solution $u(x,t) \in G(\Omega)$, then the approximate scheme (10) is convergent, and with order $O(\tau + h^2)$.

5   Solvability of the approximate scheme

It can be seen that the corresponding homogeneous linear algebraic equations for the approximate scheme (10)-(12) are

$$
-u_{i+1}^n + (2 + s(x_i, t_n))u_i^n - u_{i-1}^n = s(x_i, t_n) \sum_{k=1}^{n-1} (d_{i,n-k,n} - d_{i,n-k-1,n})u_i^{n-k}
$$
$$
- s(x_i, t_n) d_{i,0,n} u_i^0,
$$
$$
i = 1, 2, \ldots, M - 1; n = 1, 2, \ldots, N. \tag{28}
$$
$$
u_i^0 = 0, \ i = 0, 1, \ldots, M; \tag{29}
$$
$$
u_0^n = u_M^n = 0, \ n = 1, 2, \ldots, N. \tag{30}
$$

Similar to the proof of Theorem 1, we can also verify the solutions of the equations (28)-(30) satisfy

$$\|u^n\|_2 \leq \|u^0\|_2, n = 1, 2, \ldots, N.$$

Since $u^0 = 0$, we have that

$$u^n = 0, n = 1, 2, \ldots, N,$$

which indicates that the equations (28)-(30) have only zero solutions. Thus, we can obtain the following theorem:

Theorem 3. The approximate scheme (10)-(12) is uniquely solvable.

6 Numerical results

In this section, the following variable order time fractional diffusion equation is considered:

$$\begin{cases} {_0}D_t^{q(x,t)}u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \ (x,t) \in \Omega = [0,1] \times [0,1], \\ u(x,0) = 10x^2(1-x), \qquad\qquad 0 \leq x \leq 1, \\ u(0,t) = u(1,t) = 0, \qquad\qquad 0 < t < 1. \end{cases} \tag{31}$$

where $q(x,t) = \dfrac{2 + \sin(xt)}{4}$ (satisfies $0 < q(x,t) < 1$) and

$$f(x,t) = 20x^2(1-x)\left[\frac{t^{2-q(x,t)}}{\Gamma(3 - q(x,t))} + \frac{t^{1-q(x,t)}}{\Gamma(2 - q(x,t))}\right] - 20(t+1)^2(1-3x). \tag{32}$$

The exact solution is

$$u(x,t) = 10x^2(1-x)(t+1)^2. \tag{33}$$

A comparison of the numerical solution and the exact solution is provided in Table 1.

Table 2 shows that when we take a fixed value $h = 0.01$, then as the number of time steps of our approximate scheme is decreased, a reduction in the maximum error is observed, as expected and the convergence order of time is $O(\tau)$, where the convergence order is calculated by the following formula:

$$\text{Convergence order} = \log_{\frac{\tau_1}{\tau_2}} \frac{e_1}{e_2}.$$

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Table 1

The error, numerical solution and exact solution, when $t = 1$, $h = 1/10$, $\tau = 1/100$.

Space (x_i)	Numerical solution	Exact solution	Error
0.1000	0.36002996	0.36000000	0.00002996
0.2000	1.28005972	1.28000000	0.00005972
0.3000	2.52008803	2.52000000	0.00008803
0.4000	3.84011251	3.84000000	0.00011251
0.5000	5.00012981	5.00000000	0.00012981
0.6000	5.76013595	5.76000000	0.00013595
0.7000	5.88012705	5.88000000	0.00012705
0.8000	5.12010048	5.12000000	0.00010048
0.9000	3.24005643	3.24000000	0.00005643

Table 2

Maximum error behavior versus time grid size reduction at $t = 1$ when $h = 0.01$.

τ	Maximum error	Convergence order
1/4	0.01040868	
1/8	0.00410863	1.34
1/16	0.00160268	1.36
1/32	0.00061591	1.38

Table 3 shows that when we take $h^2 = \tau$, as the numbers of spatial subintervals/time steps is decreased, a reduction in the maximum error is observed, as expected the convergence order of the approximate scheme is $O(h^2 + \tau)$, where the convergence order is calculated by the following formula:

$$\text{Convergence order} = \log_{\frac{h_1}{h_2}} \frac{e_1}{e_2}.$$

Table 3

Maximum error behavior versus grid size reduction at $t = 1$ when $h^2 = \tau$.

h	Maximum error	Convergence order
1/4	0.00162521	
1/8	0.00024926	2.70
1/16	0.00003720	2.74

7 Conclusions

In this paper, a new numerical scheme for the variable order time fractional diffusion equation with the Coimbra variable order time fractional operator has been proposed. The convergence, stability and solvability of the numerical scheme have been discussed via the technique of Fourier analysis. Some numerical examples have been given and the results have demonstrated the effectiveness of the theoretical analysis.

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