Computationally efficient methods for solving time-variable-order time-space fractional reaction-diffusion equation ⋆

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Abstract:
Fractional differential equations are becoming more widely accepted as a powerful tool in modelling anomalous diffusion, which is exhibited by various materials and processes. Recently, researchers have suggested that rather than using constant order fractional operators, some processes are more accurately modelled using fractional orders that vary with time and/or space.
In this paper we develop computationally efficient techniques for solving time-variable-order time-space fractional reaction-diffusion equations (TSFRDE) using the finite difference scheme. We adopt the Coimbra variable order time fractional operator and variable order fractional Laplacian operator in space where both orders are functions of time. Because the fractional operator is nonlocal, it is challenging to efficiently deal with its long range dependence when using classical numerical techniques to solve such equations.
The novelty of our method is that the numerical solution of the time-variable-order TSFRDE is written in terms of a matrix function vector product at each time step. This product is approximated efficiently by the Lanczos method, which is a powerful iterative technique for approximating the action of a matrix function by projecting onto a Krylov subspace. Furthermore an adaptive preconditioner is constructed that dramatically reduces the size of the required Krylov subspaces and hence the overall computational cost. Numerical examples, including the variable-order fractional Fisher equation, are presented to demonstrate the accuracy and efficiency of the approach.

Keywords: fractional derivative of variable order, time-space fractional reaction-diffusion equation, preconditioned Lanczos method, matrix transfer technique, Krylov subspace

1. INTRODUCTION

In the last decades, constant-order fractional differential equations have won considerable popularity because of their promising applications in various areas (Baleanu et al., 2010; Sabatier et al., 2007; Klages et al., 2008; Podlubny, 1999). More recently, to better describe the behaviour of some heterogeneous diffusion processes, the concept of variable-order fractional differential equations has been investigated by numerous authors (Samko and Ross, 1993; Samko, 1995; Lorenzo and Hartley, 1998, 2002; Coimbra, 2003; Ramirez and Coimbra, 2010). This idea allows us to consider the order of the fractional derivative to vary either as a function of time (t), space (x) or some other variables.

Since the kernel of the variable-order operator has a variable exponent, analytical solutions to variable order fractional differential equations are more difficult to obtain. Hence, to illustrate the solution behaviours of such equations and further explore their applications, the numerical approximation of these equations plays an important role in the development of these new type of variable-order fractional models.

The research on variable-order fractional partial differential equations is relatively new, and hence numerical approximation of these equations is still at an early stage of development. Coimbra (2003) proposed a first-order accurate approximation for the solution of variable-order differential equations. Soon et al. (2005) employed a second-order Runge-Kutta method consisting of an explicit Euler predictor step followed by an implicit Euler corrector step to numerically integrate variable-order differential equations. Sun et al. (2009) introduced a classification of variable-order fractional diffusion models based on the possible physical origins that motivate the variable order, and used the Crank-Nicolson scheme to get the diffusion curve of the variable-order fractional models. Zhuang et al. (2009) presented explicit and implicit Euler approximations for the variable-order fractional advection-diffusion

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equation with a nonlinear source term, and also provided a rigorous stability and convergence analysis. Chen et al. (2010) proposed two numerical schemes for a variable-order anomalous subdiffusion equation: one with first order temporal accuracy and fourth order spatial accuracy, and the other with second order temporal accuracy and fourth order spacial accuracy. Shen et al. (2012) presented numerical techniques for the variable-order time fractional diffusion equation and discussed the stability and convergence.

Due to the nonlocal property of the fractional derivative of either constant or variable order, it is challenging to efficiently deal with its long range dependence using classical numerical techniques. Recent advances in computationally efficiently solving constant-order fractional differential equations equations have been made by Yang et al. (2011a,b). Using the matrix transfer technique (Ilić et al., 2006), Yang et al. (2011a,b) write the numerical solution of the time-space fractional diffusion (or reaction-diffusion) equations in two dimensions as a matrix function vector product $f(\mathbf{A})\mathbf{b}$ at each time step, where $\mathbf{A}$ is an approximate matrix representation of the standard Laplacian. Depending on the methods to generate the matrix $\mathbf{A}$, i.e. the finite difference/element/volume methods, this product is then approximated efficiently by the Lanczos method or the M-Lanczos method, which are powerful iterative techniques for approximating the action of a matrix function by projecting onto a Krylov subspace.

In this paper, we extend this Krylov subspace technique including adaptive preconditioning to solve the following time-variable-order time-space fractional reaction-diffusion equation (TSFRDE) with homogeneous Dirichlet or Neumann boundary conditions

$$\begin{aligned} {}_0D_t^{q(t)}u(x,t) &= -K_\alpha(-\Delta)^{\alpha(t)/2}u(x,t) + g(u,x,t), \quad (1)\\ u(x,0) &= u_0(x), \end{aligned}$$

where $0 \le x \le L$, $t > 0$; $u(x,t)$ is (for example) a concentration and K_α is the diffusion coefficient. The time fractional derivative of variable order $q(t)$ $(0 < q(t) < 1)$ is defined by Coimbra (2003) as

$$\begin{aligned} &{}_0D_t^{q(t)}u(x,t)\\ =&\frac{1}{\Gamma(1-q(t))}\int_{0^+}^{t}(t-\tau)^{-q(t)}\frac{\partial u(x,\tau)}{\partial\tau}d\tau\\ &+\frac{(u(x,0^+)-u(x,0^-))\,t^{-q(t)}}{\Gamma(1-q(t))}. \end{aligned} \quad (2)$$

For $q(t) = 1$, ${}_0D_t^{q(t)}u = \partial u/\partial t$. For the sake of simplicity, assuming $u(x,0^+) = u(x,0^-)$, then the Coimbra definition can be viewed as the following Caputo-type definition

$$\begin{aligned} &{}_0D_t^{q(t)}u(x,t)\\ =&\frac{1}{\Gamma(1-q(t))}\int_{0^+}^{t}(t-\tau)^{-q(t)}\frac{\partial u(x,\tau)}{\partial\tau}d\tau. \end{aligned} \quad (3)$$

The space fractional derivative $-(-\Delta)^{\alpha(t)/2}$ is the fractional Laplacian operator of variable order $\alpha(t)$ $(1 < \alpha(t) \le 2)$, which is defined through its eigenfunction expansion on the finite domain $[0, L]$ (Ilić et al., 2006).

Definition 1. Suppose the Laplacian $(-\Delta)$ has a complete set of orthonormal eigenfunctions φ_n corresponding to eigenvalues λ_n^2 on a bounded region $\mathcal{D}$, i.e. $(-\Delta)\varphi_n = \lambda_n^2\varphi$ on $\mathcal{D}$; $\mathcal{B}(\varphi) = 0$ on $\partial\mathcal{D}$, where $\mathcal{B}(\varphi)$ is one of the standard three homogeneous boundary conditions. Let

$$\begin{aligned} \mathcal{F} = \Bigg\{ f = \sum_{n=1}^{\infty} c_n\varphi_n, \quad c_n = \langle f, \varphi_n\rangle,\\ \sum_{n=1}^{\infty}|c_n|^2|\lambda_n|^{\alpha(t)} < \infty, \quad 1 < \alpha(t) \le 2 \Bigg\}, \end{aligned}$$

then for any $f \in \mathcal{F}$, the time-variable-order fractional Laplacian $(-\Delta)^{\alpha(t)/2}$ is defined by

$$(-\Delta)^{\alpha(t)/2}f = \sum_{n=1}^{\infty} c_n(\lambda_n)^{\alpha(t)}\varphi_n.$$

The nonlinear reaction term $g(u,x,t)$ is assumed to be Lipschitz continuous.

2. NUMERICAL SCHEME

In this section, we present the numerical scheme to simulate the solution behaviour of the time-variable-order TSFRDE (1). Let $x_i := ih$, $i = 0, 1, 2, \ldots, M$, where $h := L/M$ is the spatial step; $t_n := n\tau$, $n = 0, 1, 2, \ldots, N$, where $\tau := T/N$ is the time step; u_i^n denote the numerical approximation of $u(x_i, t_n)$; and $\mathbf{u}^n$ denote vectors of such values.

First, according to the matrix transfer technique proposed by Ilić et al. (2006), the fractional Laplacian operator of constant order α, i.e., $(-\Delta)^{\alpha/2}$, can be approximated by the matrix representation of the standard Laplacian operator raised to the same fractional order, i.e., $\mathbf{A}^{\alpha/2}$. This technique can be extended to approximate the fractional Laplacian operator of variable order $\alpha(t)$ as follows

$$-(-\Delta)^{\alpha(t_n)/2}u(x,t_n) \approx -\mathbf{A}^{\alpha(t_n)/2}\mathbf{u}^n, \quad (4)$$

where

$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(M-1)\times(M-1)}$$

under homogeneous Dirichlet conditions, or

$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$$

under homogeneous Neumann conditions.

Next, adapting the finite difference scheme for discretising the constant-order time fractional derivative in Lin and Xu (2007), similarly we discretise the variable-order time fractional derivative as

$${}_0D_t^{q(t_n)}u(t_n) = \frac{1}{s_n}\sum_{k=0}^{n-1} d_k^n\left[u(t_{n-k}) - u(t_{n-k-1})\right] + O(\tau^{2-q(t_n)}), \quad (5)$$

where

$$s_n = \tau^{q(t_n)}\,\Gamma(2-q(t_n))$$

and

$$d_k^n = (k+1)^{1-q(t_n)} - k^{1-q(t_n)}, \; k = 0, 1, \ldots, n-1.$$

Finally, combining (4) and (5) together and treating the nonlinear reaction term explicitly (Yang et al., 2010), we obtain the numerical approximation of the time-variable-order TSFRDE (1) in vector form

$$\frac{1}{s_n}\sum_{k=0}^{n-1} d_k^n \left[\mathbf{u}^{n-k} - \mathbf{u}^{n-k-1}\right] = -K_\alpha \mathbf{A}^{\alpha_n/2}\mathbf{u}^n + \mathbf{g}^{n-1}, \quad (6)$$

where $\alpha_n = \alpha(t_n)$ and $\mathbf{g}^{n-1} = g(\mathbf{u}^{n-1}, \mathbf{x}, t_{n-1})$. After some further manipulations of (6), we write the numerical solution of the time-variable-order TSFRDE (1) in terms of a matrix function vector product at each time step

$$\mathbf{u}^n = f_n(\mathbf{A})\,\mathbf{b}^n, \quad (7)$$

where

$$f_n(\mathbf{A}) = \left[\mathbf{I} + s_n K_\alpha \mathbf{A}^{\alpha_n/2}\right]^{-1}$$

and

$$\mathbf{b}^n = \sum_{k=0}^{n-2}(d_k^n - d_{k+1}^n)\mathbf{u}^{n-k-1} + d_{n-1}^n\mathbf{u}^0 + s_n\mathbf{g}^{n-1}.$$

To compute $\mathbf{u}^n$, the traditional method requires us to solve a $(M-1)\times(M-1)$ linear system (or $(M+1)\times(M+1)$ depending on which boundary condition is used) at each time step t_n. Although the matrix $\mathbf{A}$ generated from the finite difference method is symmetric and tridiagonal, the matrix function $f_n(\mathbf{A})$ will be a dense matrix due to raising $\mathbf{A}$ to a fractional index. When the number of spatial steps M is large, it is very time consuming to solve a large dense system at each time step. Hence, it is challenging to solve for $\mathbf{u}^n$ efficiently.

3. MATRIX FUNCTION APPROXIMATION

In this section, we devise a novel and efficient algorithm to approximate the matrix function vector product $\mathbf{u}^n = f_n(\mathbf{A})\,\mathbf{b}^n$ without needing to form the dense matrix $f_n(\mathbf{A})$. The prevailing method in the literature for approximating the matrix-vector product $f(\mathbf{A})\mathbf{b}$ for a scalar, analytic function $f : D \subset \mathbb{C} \to \mathbb{C}$ is the Lanczos approximation

$$f(\mathbf{A})\mathbf{b} \approx \|\mathbf{b}\|\mathbf{V}_m\, f(\mathbf{T}_m)\,\mathbf{e}_1, \quad \mathbf{b} = \|\mathbf{b}\|\mathbf{V}_m\,\mathbf{e}_1, \quad (8)$$

where

$$\mathbf{A}\mathbf{V}_m = \mathbf{V}_m\mathbf{T}_m + \beta_m\,\mathbf{v}_{m+1}\,\mathbf{e}_m^T$$

is the Lanczos decomposition with the columns of $\mathbf{V}_m$ forming an orthonormal basis for the Krylov subspace $\mathcal{K}_m(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \ldots, \mathbf{A}^{m-1}\mathbf{b}\}$ and $\mathbf{T}_m$ symmetric and tridiagonal [see van der Vorst (1987) and Saad (1992)].

This method is particularly attractive when a good preconditioner is available. A good preconditioner can significantly improve the rate of convergence of the standard Lanczos method. We now propose an adaptively preconditioned Lanczos method. The idea is to construct a matrix $\mathbf{Z}^{-1}$ and work with the matrix function $f(\mathbf{A}\mathbf{Z}^{-1})$ rather than $f(\mathbf{A})$ itself. The matrix $\mathbf{Z}^{-1}$ should be such that $f(\mathbf{A}\mathbf{Z}^{-1})\mathbf{b}$ is in some sense easier to compute than $f(\mathbf{A})\mathbf{b}$, but at the same time there must be a known relationship between $f(\mathbf{A}\mathbf{Z}^{-1})\mathbf{b}$ and $f(\mathbf{A})\mathbf{b}$, since the latter is what is actually required in the numerical scheme.

It is known that small eigenvalues hinder the convergence of the Lanczos method, and several authors have proposed preconditioners to deal directly with this problem. Suppose we compute the k smallest eigenvalues $\{\lambda_i\}_{i=1}^k$ and corresponding eigenvectors $\{\mathbf{q}_i\}_{i=1}^k$ of the matrix $\mathbf{A}$. Lehoucq and Sorensen (1996)'s implicitly restarted Arnoldi method can be used to do this efficiently. Then setting $\mathbf{Q}_k = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k]$ and $\mathbf{\Lambda}_k = \text{diag}\{\lambda_1, \ldots, \lambda_k\}$ we therefore have $\mathbf{A}\mathbf{Q}_k = \mathbf{Q}_k\mathbf{\Lambda}_k$. Baglama et al. (1998) and Erhel et al. (1996) have both proposed the preconditioner $\mathbf{Z}^{-1}$ taking the form

$$\mathbf{Z}^{-1} = \lambda^*\mathbf{Q}_k\mathbf{\Lambda}_k^{-1}\mathbf{Q}_k^T + \mathbf{I} - \mathbf{Q}_k\mathbf{Q}_k^T, \quad (9)$$

where $\lambda^* = \frac{\lambda_{min}+\lambda_{max}}{2}$. Here λ_{min}, λ_{max} are the smallest and largest eigenvalues of $\mathbf{A}$ respectively, obtained from the implicit restarted Arnoldi process (Saad, 1992). It is easy to show that the product $\mathbf{A}\mathbf{Z}^{-1}$ has the same eigenvectors as $\mathbf{A}$ but its k smallest eigenvalues $\{\lambda_i\}_{i=1}^k$ are all mapped to λ^* [see Baglama et al. (1998); Erhel et al. (1996)]. Hence this preconditioner eliminates the influence of the k smallest eigenvalues on the convergence of the Lanczos method.

The important observation at this point is the following relationship between $f(\mathbf{A})$ and $f(\mathbf{A}\mathbf{Z}^{-1})$ (Ilić et al., 2008)

$$f(\mathbf{A})\mathbf{b} = \mathbf{Q}_k f(\mathbf{\Lambda}_k)\mathbf{Q}_k^T\mathbf{b} + f(\mathbf{A}\mathbf{Z}^{-1})\hat{\mathbf{b}}, \quad (10)$$

where $\hat{\mathbf{b}} = (\mathbf{I} - \mathbf{Q}_k\mathbf{Q}_k^T)\mathbf{b}$.

Note that if $\mathbf{A}$ is symmetric then so too is $\mathbf{A}\mathbf{Z}^{-1}$ (Ilić et al., 2008). Hence, we can apply the standard Lanczos decomposition to $\mathbf{A}\mathbf{Z}^{-1}$, i.e.,

$$\mathbf{A}\mathbf{Z}^{-1}\mathbf{V}_m = \mathbf{V}_m\mathbf{T}_m + \beta_m\mathbf{v}_{m+1}\mathbf{e}_m^T,$$

where $\mathbf{v}_1 = \hat{\mathbf{b}}/\|\hat{\mathbf{b}}\|$ and the columns of $\mathbf{V}_m$ form an orthonormal basis for the Krylov subspace $\mathcal{K}_m(\mathbf{A}\mathbf{Z}^{-1}, \hat{\mathbf{b}}) = \text{span}\{\hat{\mathbf{b}}, \mathbf{A}\mathbf{Z}^{-1}\hat{\mathbf{b}}, \ldots, (\mathbf{A}\mathbf{Z}^{-1})^{m-1}\hat{\mathbf{b}}\}$. The Lanczos approximation (8) then gives

$$f(\mathbf{A}\mathbf{Z}^{-1})\hat{\mathbf{b}} \approx \mathbf{V}_m f(\mathbf{T}_m)\mathbf{V}_m^T\hat{\mathbf{b}} \quad (11)$$

where the much smaller matrix function $f(\mathbf{T}_m)$ can be easily calculated by finding the diagonalisation of $\mathbf{T}_m$. To this end, the equations (9)–(11) form the adaptively preconditioned Lanczos method for approximating $f(\mathbf{A})\mathbf{b}$ when $\mathbf{A}$ is non-singular (under homogeneous Dirichlet boundary conditions).

However, when the matrix $\mathbf{A}$ is obtained from (1) with homogeneous Neumann boundary conditions, we notice that there appears to be a problem building the Krylov subspace $\mathcal{K}_m(\mathbf{A}\mathbf{Z}^{-1}, \hat{\mathbf{b}})$. In this case $\mathbf{A}$ is singular (it has a single zero eigenvalue), and hence we cannot perform $\mathbf{\Lambda}^{-1}$ to form $\mathbf{Z}^{-1}$ as required in (9).

But in fact, we do not actually need to form the matrix $\mathbf{Z}^{-1}$. We only ever need the product $\mathbf{A}\mathbf{Z}^{-1}$. So now, we illustrate a clever way to express $\mathbf{A}\mathbf{Z}^{-1}$ so that the division by zero eigenvalue is avoided.

Since $\mathbf{A}\mathbf{Q}_k = \mathbf{Q}_k\mathbf{\Lambda}_k$, multiplying $\mathbf{A}$ on the both sides of (9) gives

$$\begin{aligned}
\mathbf{A}\mathbf{Z}^{-1} &= \lambda^*\mathbf{A}\mathbf{Q}_k\mathbf{\Lambda}_k^{-1}\mathbf{Q}_k^T + \mathbf{A} - \mathbf{A}\mathbf{Q}_k\mathbf{Q}_k^T \\
&= \lambda^*\mathbf{Q}_k\mathbf{\Lambda}_k\mathbf{\Lambda}_k^{-1}\mathbf{Q}_k^T + \mathbf{A} - \mathbf{Q}_k\mathbf{\Lambda}_k\mathbf{Q}_k^T \\
&= \lambda^*\mathbf{Q}_k\mathbf{Q}_k^T + \mathbf{A} - \mathbf{Q}_k\mathbf{\Lambda}_k\mathbf{Q}_k^T \\
&= \mathbf{A} + \mathbf{Q}_k(\lambda^*\mathbf{I}_k - \mathbf{\Lambda}_k)\mathbf{Q}_k^T \\
&= \mathbf{A} + \mathbf{Q}_k\mathbf{\Omega}_k\mathbf{Q}_k^T, \quad (12)
\end{aligned}$$

where $\mathbf{\Omega}_k = \lambda^* \mathbf{I}_k - \mathbf{\Lambda}_k$. Hence, rather than building the Krylov subspace $\mathcal{K}_m(\mathbf{A}\mathbf{Z}^{-1}, \hat{\mathbf{b}})$, we build

$$\mathcal{K}_m(\mathbf{A}_*, \hat{\mathbf{b}}) = \text{span}\{\hat{\mathbf{b}}, \mathbf{A}_*\hat{\mathbf{b}}, \ldots, {\mathbf{A}_*}^{m-1}\hat{\mathbf{b}}\},$$

where $\mathbf{A}_* = \mathbf{A} + \mathbf{Q}_k\mathbf{\Omega}_k\mathbf{Q}_k^T$.

This way we extend the application of the preconditioner $\mathbf{Z}^{-1}$ defined in (9) to cover all cases, including when $\mathbf{A}$ is singular (under homogeneous Neumann conditions).

We remark that the matrix $\mathbf{A}_*$ is not formed explicitly anywhere either. Only its action on a vector is required when building the Krylov subspace.

According to the theory presented to this point, we propose the following Algorithm 1 to approximate the solution of the time-variable-order TSFRDE (1). The error bound used in this algorithm is derived in Ilić et al. (2008).

Input: Discrete Laplacian matrix $\mathbf{A}$, right hand side vector $\mathbf{b}$, tolerance τ, number of time steps n, number of stored eigenpairs k, and maximum size of Krylov subspace *maxiter*.
Output: $\mathbf{u}^n$

Compute Λ_k and Q_k;
for *time step j=1:n* **do**
  Set $\hat{b} = (I - Q_kQ_k^T)b$;
  Set $v_1 = \hat{b}/\|\hat{b}\|_2$;
  for $m = 1 : maxiter$ **do**
    Set $w = (A + Q_k\Omega_kQ_k^T)v_m$;
    if $m \neq 1$ **then**
      $w = w - \beta_{m-1}v_{m-1}$;
    end
    $\alpha_m = v_m^T w$;
    $w = w - \alpha_m v_m$;
    $\beta_m = \|w\|_2$;
    $v_{m+1} = w/\beta_m$;
    Compute linear system residual
    $\|r_m\|_2 = \|\hat{b}\|_2 \, |\beta_m e_m^T T_m^{-1} e_1|$;
    Compute μ_{min} – the smallest eigenvalue of T_m;
    Compute error bound as $f(\mu_{min})\|r_m\|_2$;
    if *error bound* $< \tau$ **then**
      break;
    end
  end
  Compute $u^j = Q_k f(\Lambda_k)Q_k^T b + \|\hat{b}\|_2 V_m f(T_m)e_1$;
end

Algorithm 1: Lanczos approximation to $\mathbf{u}^n = f_n(\mathbf{A})\mathbf{b}^n$ with adaptive preconditioning, where $\mathbf{A}$ is symmetric.

4. NUMERICAL RESULTS

In this section, we present two numerical examples to demonstrate the accuracy and efficiency of our proposed approach.

Example 1: Take $\alpha(t) = 2$ (standard diffusion in space) and $K_\alpha = 1$. Consider the following time-variable-order time fractional diffusion equation with homogeneous Dirichlet boundary conditions:

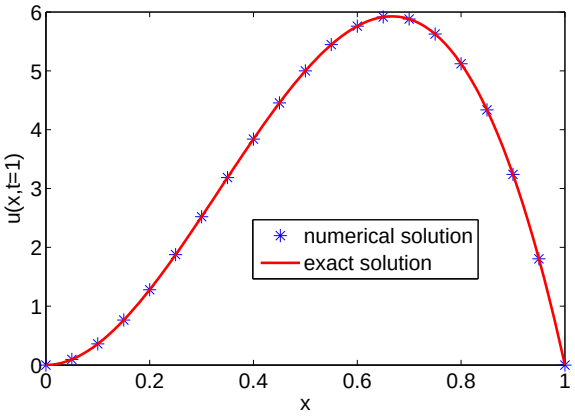

Fig. 1. Comparison of numerical solution and exact solution for $t = 1$ with $\tau = 0.01$, $h = 0.01$ and $q(1) = 0.7104$ in Example 1.

Table 1. Temporal errors at $t = 1$ with $h = 0.005$ and $q(1) = 0.7104$ in Example 1.

τ	Maximum errors
0.01	2.0948e-004
0.005	8.5430e-005
0.0025	3.4906e-005
0.00125	1.4353e-005
0.000625	5.9846e-006
Order	1.28

$$
\begin{aligned}
&{}_0D_t^{q(t)}u(x,t) = \Delta u(x,t) + g(x,t), \qquad (13)\\
&u(x,0) = 10x^2(1-x), \quad 0 \le x \le 1,\\
&u(0,t) = u(1,t) = 0, \quad t > 0,
\end{aligned}
$$

where $q(t) = \frac{2+\sin(t)}{4}$ (satisfies $0 < q(t) < 1$) and

$$
\begin{aligned}
g(x,t) =& 20x^2(1-x)\left[\frac{t^{2-q(t)}}{\Gamma(3-q(t))} + \frac{t^{1-q(t)}}{\Gamma(2-q(t))}\right]\\
&- 20(t+1)^2(1-3x).
\end{aligned}
$$

The exact solution is

$$u(x,t) = 10x^2(1-x)(t+1)^2.$$

A comparison of the exact and numerical solutions for Example 1 at time $t = 1$ is presented in Figure 1. It is apparent that the numerical solutions are in excellent agreement with the exact solution.

To identify the order of convergence in time for the numerical scheme (7), we compute the maximum error in the numerical solution at $t = 1$ with fixed $h = 0.005$ and $q(1) = 0.7104$ for a sequence of reduced temporal steps in Table 1. The order of convergence in time is estimated to be $O(\tau^{1.28})$, which is consistent with the theoretical claim $O(\tau^{2-q(1)})$ in (5).

Another highlight of our proposed numerical method is the excellent performance of the preconditioner on accelerating the rate of convergence of the standard Lanczos method. In Figure 2 and Table 2, we illustrate the impact of the preconditioner on the size of the Krylov subspace m when k smallest approximate eigenpairs are used. This includes the case $k = 0$, where no preconditioning was applied. We see that the average subspace size m is greatly reduced as we increase the number of eigenpairs from $k = 5$

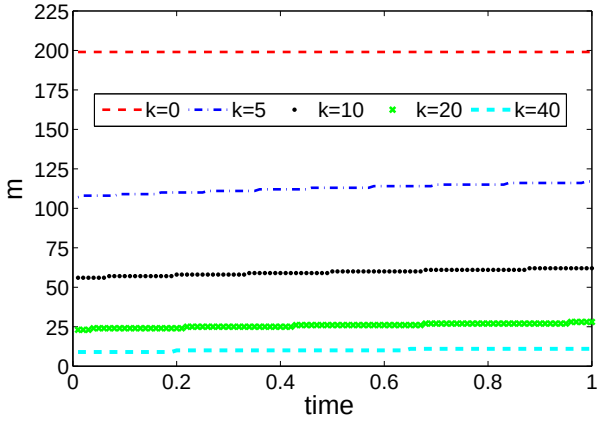

Fig. 2. Subspace size m at each time step for $t_{final} = 1$ with $\tau = 0.01$, $h = 0.005$ in Example 1.

Table 2. Number of smallest eigenpairs used k and the corresponding average subspace size m in Example 1.

$k = 0$	$k = 5$	$k = 10$	$k = 20$	$k = 40$
$m = 199$	$m = 112.63$	$m = 59.39$	$m = 25.72$	$m = 10.17$

to $k = 40$. Hence, not only solving fractional models accurately, our proposed adaptively preconditioned Lanczos method also enables us to solve test problems much more efficiently.

To further illustrate the effect of the variable order in both time and space, we present another example with the Fisher reaction term leading to logistic growth, with $u = 0$ being an unstable equilibrium point.

Example 2: Take $K_\alpha = 0.01$. Consider the following time-variable-order time-space fractional reaction-diffusion equation with homogeneous Neumann boundary conditions:

$$
\begin{aligned}
&{}_0D_t^{q(t)}u(x,t) = -K_\alpha(-\Delta)^{\alpha(t)/2}u(x,t) + u(x,t)(1-u(x,t)),\\
&u(x,0) = e^{-5x}, \quad 0 \le x \le 10, \qquad (14)\\
&\frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = 10, \quad 0 \le t \le 20,
\end{aligned}
$$

where $\alpha(t) = 1.7 + 0.5e^{-\frac{t}{50}-1}$ (satisfies $1 < \alpha(t) \le 2$) and $q(t) = \frac{2+\sin(t)}{4}$ (satisfies $0 < q(t) < 1$). The exact solution is not available for this example.

In Figure 3, we show several evenly spaced time slices for the evolution of the fractional Fisher model (14) in four cases:

(a) the standard diffusion with $\alpha(t) = 2$ and $q(t)$=1;
(b) the time-variable-order space fractional diffusion with $1.82 < \alpha(t) < 1.89$ and $q(t)$=1;
(c) the time-variable-order time fractional diffusion with $\alpha(t) = 2$ and $0.25 \le q(t) \le 0.75$; and
(d) the time-variable-order time-space fractional diffusion with $1.82 < \alpha(t) < 1.89$ and $0.25 \le q(t) \le 0.75$.

We can see that the system exhibits various different anomalous diffusion behaviours when the fractional derivative is involved in space and/or time. From Figure 3(a) the steady wave speed of the standard Fisher equation with $\alpha = 2$ and $q = 1$ is visible. Figure 3(b) shows the effects

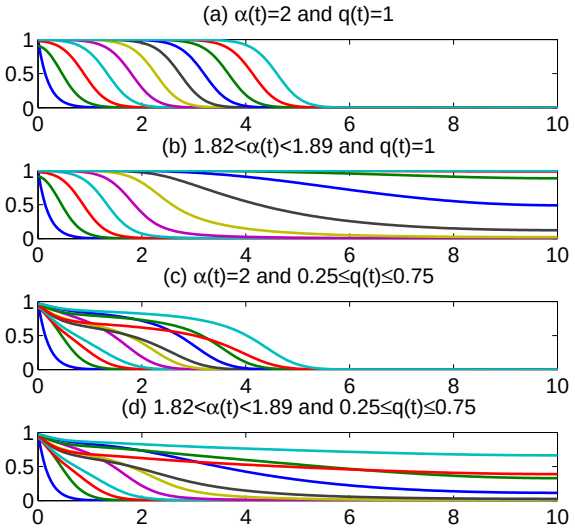

Fig. 3. Fisher reaction for (a) standard diffusion, (b) time-variable-order space fractional diffusion, (c) time-variable-order time fractional diffusion, and (d) time-variable-order time-space fractional diffusion in Example 2.

Table 3. Number of smallest eigenpairs used k and its corresponding average subspace size m in Example 2.

$k = 0$	$k = 5$	$k = 10$	$k = 20$	$k = 40$
$m = 201$	$m = 90.46$	$m = 40.08$	$m = 12.29$	$m = 2.44$

of space fractional diffusion, in this case $q(t) = 1$ and $\alpha(t)$ is varying between 1.82 and 1.89. This is consistent with the exponential spread derived by Engler (2010). Figure 3(c) presents the effects of time fractional diffusion or subdiffusion, in this case $\alpha(t) = 2$ and $q(t)$ is changing between 0.25 and 0.75. The effect of the variable order on the shape of the travelling wave is clearly visible. Finally, Figure 3(d) presents the mixed effects of time fractional diffusion and space fractional diffusion. In this case $\alpha(t)$ is varying between 1.82 and 1.89 while $q(t)$ is changing between 0.25 and 0.75.

Again, to showcase the excellent performance of the adaptively preconditioned Lanczos method, we compute the numerical solution for the case (b) at $t_{final} = 20$ over $x \in [0, 10]$ with $\tau = 0.2$ and $h = 0.05$. In Figure 4 and Table 3, we illustrate the impact of the preconditioner on the size of the Krylov subspace m when k smallest approximate eigenpairs are used. We see that the average subspace size m is greatly reduced as we increase the number of eigenpairs from $k = 5$ to $k = 40$. Especially, when $k = 40$ the average subspace size m is only 2.44, compared with $m = 201$ when $k = 0$ (i.e. no preconditioning). Hence, we would only need to compute a 3×3 matrix function ($k = 40$) instead of a 201×201 matrix function ($k = 0$) at each time step. Note that the cost of using MATLAB's `eigs` function to compute the smallest eigenpairs of the matrix $\mathbf{A}$ is trivial, owing to its tridiagonal structure.

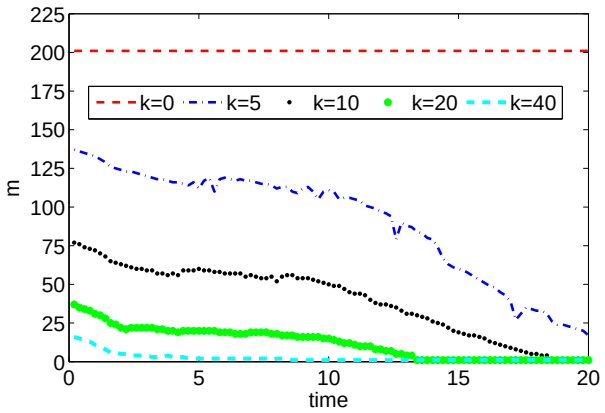

Fig. 4. Subspace size m at each time step for $t_{final} = 20$ with $\tau = 0.2$, $h = 0.05$ in Example 2.

5. CONCLUSIONS

In this paper, a finite difference scheme with an adaptively preconditioned Lanczos algorithm is constructed for simulating the solution behaviours of the time-variable-order time-space fractional reaction diffusion equation. The excellent numerical performance of the proposed technique is described and demonstrated. The highlights of this technique include its efficiency and the generalisation of its application to both cases where $\mathbf{A}$ is non-singular (under homogeneous Dirichlet boundary condition) and singular (under homogeneous Neumann boundary condition). The proposed techniques can be applied to solve higher dimensional time-variable-order fractional differential equations and this will be explored in future work.

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