Ergodic Capacity of the Exponentially Correlated Slotted Amplify and Forward Relay Channel

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***Abstract*—In this paper we analyze the performance degradation of slotted amplify-and-forward protocol in wireless environments with high node density where the number of relays grows asymptotically large. Channel gains between source-destination pairs in such networks can no longer be independent. We analyze the degradation of performance in such wireless environments where channel gains are exponentially correlated by looking at the capacity per channel use. Theoretical results for eigenvalue distribution and the capacity are derived and compared with the simulation results. Both analytical and simulated results show that the capacity given by the asymptotic mutual information decreases with the network density.**

I. Introduction

Wireless communications have facilitated an important development in personal communications thanks to its ability of providing seamless communications and connectivity. However, wireless systems have to deal with two inherent drawbacks: the scarcity of the available radio spectrum and channel impairments. Thus the wireless networks should be designed to exhibit high spectral efficiency and to combat adverse propagation effects. Diversity techniques have long been used to mitigate these problems. In the past two decades, space and polarization diversities obtained by using multiple antennas at the transmitter and the receiver have been attracting intensive research interest in both academia and industry. These systems offer multiplexing gain which increases of the total mutual information and diversity gain which improves the robustness of the communication link to undesirable channel fading effects.

Cooperative communication, which can significantly improve the efficiency and robustness of wireless communication systems, has attracted attention of many researchers around the globe [1]. Many relaying strategies for cooperative networks with various topologies have been proposed with the trend of allowing the source nodes to transmit data using non-orthogonal signals [2]. In Cooperative diversity systems, users create a virtual array through distributed transmission and reception, making it effectively a MIMO (Multiple Input, Multiple Output) system. Cooperative node can either amplify and forward the received signal (dummy-forwarding) or decode re-encode and forward the received signal (smart-forwarding). Because of practical considerations in the design of the radio devices, cooperation is usually proposed with half-duplex relays, enforcing two phases of the communication to be based on the state of the relay: broadcast (the relay-receive) phase and cooperative (relay-transmit) phase. Cooperative strategies are often compared using the diversity and multiplexing trade-off (DMT) [3], which is a fundamental measure that characterizes the throughput and the error performance simultaneously. The reason behind using a trade-off measure to analyze the performance is the equilibrium that exists between throughput and error performance. An increase in one results in a decrease in the other as characterized by the DMT.

The simplest cooperative diversity protocols are the Amplify-and-Forward (AF) protocols, where the cooperative terminals scale the received signal and retransmit it. The AF protocols are attractive for their low complexity, since in practice, it would be unrealistic for the cooperative terminals to lend much resource to perform power-consuming signal processing. Depending on whether the source keeps transmitting during the forwarding of the cooperative terminals, the AF schemes can be categorized into two classes: the orthogonal AF and the non-orthogonal AF (NAF). Orthogonal AF protocols are proposed and analyzed by Laneman et al. [4] while the NAF scheme is first proposed by Nabar et al. in [5]. It is shown in [5] that the NAF scheme outperforms the orthogonal AF protocols in terms of the received SNR and the ergodic capacity. Furthermore a network with single source node, a single destination node and multiple dedicated relay nodes, sequential slotted amplify and forward strategy has been shown to be asymptotically optimal [6].

In this paper we extend the work of [6] to a protocol applicable to a dense network where a large number of relays are available to relay source data in an environment that the channel gains between network nodes are not necessarily uncorrelated. An interesting aspect that we would like to point out here is that analysis of capacity or more specifically mutual information of MIMO systems with correlated channel gains between input and output antenna arrays is not a new topic in wireless communications but capacity analysis accommodating channel correlations in cooperative systems is quite rare. Hence the purpose of the work presented in this paper is to evaluate the ergodic capacity dense network where correlation in channel gains between nodes does not vanish.

The structure of the paper is as follows. Section II presents

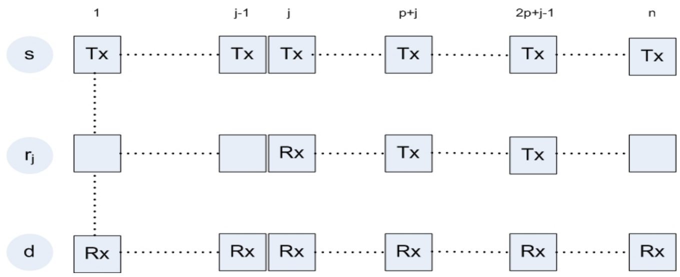


Fig. 1. Cooperative Slot Structure

the case of the dense relay networks. In Section III we present the cooperative slot structures, the assumptions of our extended SAF protocol and the relevant channel matrix. The outline of the analysis procedure is presented and a closed from expression for the ergodic capacity of the exponentially correlated SAF relay channel is derived in Section IV. Section V compares the theoretical and simulation results while Section VI concludes the paper.

II. Channels in Dense Relay Networks

In relation to MIMO systems the channel correlation between transmit and receive antenna arrays can be investigated by introducing the "Kronecker model", where the transmit and receive covariance matrices Ψ_t and Ψ_r model the non-independent behavior of the channel gains such that the channel gain becomes $\mathbf{H} = \Psi_t \hat{\mathbf{H}} \Psi_r$ where $\hat{\mathbf{H}}$ is a matrix with independent random variables.

The "Kronecker model" cannot be applied in the same way to analyze the SAF protocol presented in [6], since the system is multiple-input, single-output (MISO) and in the analysis of the relay channel gains between source to relay links (and relay-destination links) have been assumed to be independent. Furthermore with respect to the SAF protocol analyzed in [6] channel gains are assumed to be "quasi static". This is due to the fact in most cases the coherence time of the wireless channel between two terminals are assumed to be large to the extent that slow-fading occurs. Yet in a network where the terminals are randomly moving and not necessarily slowly, the "quasi static" assumption may not be valid.

In addition to the above, if the network is dense the channel gains between network elements can no longer be assumed to be independent - a certain amount of spatial correlation exists. Hence, we assume that the correlation coefficient between two nodes nodes a and b, say ρ_{ab} is given by $\rho_{ab} = \exp(-\lambda d_{ab})$, where λ is a medium dependent correlation factor and d_{ab} is the distance between nodes a and b allowing a high degree of correlation. According to the transmission model for spatial data gathering from a cluster of nodes to a certain center node used in [7] and [8], ρ_{ab} in general can be modeled as $\rho_{ab} = \exp(-\lambda f(d_{ab}))$ where $f(d_{ab})$ is an increasing function of d_{ab}.

III. System Description and Transmission Model

The following assumptions for the SAF protocol are made in this paper:

- One cooperation frame consists of M subframes.
- In each subframe s will be transmitting and d will be receiving.
- Any arbitrary relay $\mathsf{r_j}$ listens to a total of p sequential source transmissions, starting from subframe j continuously. Then starting from the $(p+j)^{th}$ subframe $\mathsf{r_j}$ begins its transmission of p slots of data. The transmission and reception pattern of all relays are being sequentially continuous, the relay $\mathsf{r_j}+1$ - starts reception from $(j+1)$th subframe and begins its transmission from $(p+j+1)^{th}$ slot. The total number of subframes is $M = pq$ - where q is the number of relays that participate in the transmission. The structure of the cooperative slots thus formed is shown in Fig. III.
- Destination starts decoding its desired messages after receiving the whole cooperation frame.
- Following the same procedure presented in [6], we can write the expressions for the received signals at the destination d and the relay $\mathsf{r_j}$ during the i^{th} subframe as:

$$y_{d,i} = \sqrt{\mathsf{SNR}} g_0^{(i)} x_i + \frac{1}{\sqrt{p}} \sqrt{\mathsf{SNR}} \sum_{k=i-p-1}^{i-1} b_k^{(i)} x_{r_k,i} + z_{d,i}$$
$$y_{r_j,i} = \sqrt{\mathsf{SNR}} h_j^{(i)} x_i + \sqrt{\mathsf{SNR}} \frac{1}{p} \sum_{u=1}^{i-1} \sum_{v=u-p-1}^{u-2} \gamma_{v-1,v}^{(u)} b_{v-1}^{(u)} y_{r_u,v} + z_{r_j,i} \tag{1}$$

Here x_i denotes the source transmitted i^{th} symbol, $x_{r_k,i}$, the "unamplified" i^{th} symbol of relay $\mathsf{r_{k,i}}$, and y_d, y_{r_j} denote the symbol received at the destination and the r_j^{th} relay. The channel gains between destination d and relay $\mathsf{r_j}$ and the relay $\mathsf{r_j}$ and the source are represented by $g_j^{(i)}$ and $h_j^{(i)}$ respectively, while $g_0^{(i)}$ stands for the source to destination gain. The amplification factor at the relay r_j is given by $b_j^{(i)}$ while $\gamma_{j,j'}^{(i)}$ stands for the inter-relay channel gain between $\mathsf{r_j}$ and $\mathsf{r_{j'}}$ relays.The additive white Gaussian noise components at the destination and the relay is given by $z_{d,i}$, and $z_{r_j,i}$ and SNR denotes the signal to noise ration of the source to destination link. Now we make the following assumptions.

- All channel gains described above, those between source, destination and the relays are modeled as complex Gaussian random variables with zero mean and unit variance. Furthermore we assume that the relays are spatially isolated such that the inter-relay gain, between relays $\mathsf{r_j}$ and $\mathsf{r_{j'}}$ to be $\gamma_{j,j'}^{(i)} = 0$.
- We assume the additive noise Gaussian elements at the destination and the relays, $z_{d,i}$ and $z_{r_j,i}$ to be circular symmetric complex Gaussian variables with zero mean and unit variance.
- We shall consider the amplification gain given by $b_j^{(i)}$ at the j^{th} relay to be a constant for a particular p.
- The relays are assumed to be arranged in an equal spaced rectangular grid. Now in this setting, we assume that the source to relay gains and relay to destination

gains are *exponentially correlated* according to the relationship, $\mathbf{E}\big[\mathfrak{Re}(g_j^{(i)})\mathfrak{Re}(g_{j'}^{(i)})\big] = \mathbf{E}\big[\mathfrak{Re}(h_j^{(i)})\mathfrak{Re}(h_{j'}^{(i)})\big] = \mathbf{E}\big[\mathfrak{Im}(g_j^{(i)})\mathfrak{Im}(g_{j'}^{(i)})\big] = \mathbf{E}\big[\mathfrak{Im}(h_j^{(i)})\mathfrak{Im}(h_{j'}^{(i)})\big] = \exp\big(-\lambda|d_{j,j'}|\big)$, where $d_{j,j'}$ denote the distance between relays j and j' and $\mathfrak{Re}$ and $\mathfrak{Im}$ stand for real and imaginary parts. Further we assume that, $g_{j'}^{(i)}$'s and $h_{j'}^{(i)}$'s to be uncorrelated.

- We assume that the amplification factors obey the power normalization given by,

$$\mathbb{E}\Big[\frac{1}{\sqrt{p}}\sum_{v=u-p-1}^{u-2}|b_{v-1}y_{r_u,v}|^2\Big] \leq 1 \tag{2}$$

This condition assures that the total power utilized by the relays in transmitting source information is normalized to unity.

After several manipulations we obtain the following transmission model input-output relation in the vector form.

$$\mathbf{y}_\mathsf{d} = \sqrt{\mathsf{SNR}}\mathsf{diag}(\mathbf{g})\cdot\mathbf{x} + \sum_{j=1}^{p}\mathbf{U}_\mathsf{c}\mathbf{y}_{\mathsf{r_j}} + \mathbf{z}_\mathsf{d} \tag{3}$$

$$\mathbf{y}_{\mathsf{r_j}} = \sqrt{\mathsf{SNR}}\mathsf{diag}(\mathbf{d_j})\mathbf{x} + \mathbf{z}_\mathsf{r} \tag{4}$$

where, components of $\mathbf{g} \in \mathbb{C}^{M\times 1}$ are defined by $g(i) = g_0^{(i)}$ and $\mathbf{U}_{\mathsf{c},\mathbf{j}}$, are $(M)\times(M)$ matrices defined as,

$$\mathbf{U}_{\mathsf{c,j}} \triangleq \begin{pmatrix} \mathbf{0^T} & 0 \\ \mathsf{diag}(\mathbf{c_j}) & \mathbf{0} \end{pmatrix} \tag{5}$$

with diagonal components of $\mathbf{c_j}, \in \mathbb{C}^{(M-1)\times(M-1)}$ and $\mathbf{d_j} \in \mathbb{C}^{M\times M}$ being defined by,

$$c_j(i) = \{g_j^{(i)}\}_{i=j+p+1}^{j+2p} \tag{6}$$

$$d_j(i) = \{h_j^{(i)}\}_{i=j}^{j+p} \tag{7}$$

Now the channel matrix for the transmission model is of the form:

$$\mathbf{H} = \mathsf{diag}(\mathbf{g})\mathbf{I} + \sum_{j=1}^{p}\mathbf{U}_{\mathsf{c,j}}\mathsf{diag}(\mathbf{d_j}) \tag{8}$$

Hence we derive $\mathbf{H}$ as,

$$\mathbf{H} = \frac{1}{\sqrt{p}}\begin{pmatrix} g_0^{(1)} & 0 & \cdots & 0 \\ g_1^{(1)}h_1^{(1)} & g_0^{(2)} & \cdots & 0 \\ g_2^{(1)}h_2^{(1)} & g_1^{(2)}h_1^{(2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ g_{p-1}^{(1)}h_{p-1}^{(1)} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & g_1^{(N-1)}h_1^{(N-1)} & g_0^{(N)} \end{pmatrix} \tag{9}$$

It is observed that the above channel matrix is random *Gram type matrix*. This fact is used in the next section to find the eigen value distribution and the a formula to estimate the ergodic capacity of the SAF channel.

IV. Eigen value distribution and the Ergodic Capacity

In this section, we briefly describe the steps and present the relevant expressions pertaining to the derivation of the eigenvalue distribution for the infinite version of the matrix $\mathbf{HH}^\dagger$ and derive a close form expression for the eigenvalue density to estimate the channel capacity given by,

$$C = (1/M)\log\det\big[\mathbf{I} + \mathsf{SNR}\mathbf{HH}^\dagger\big] \tag{10}$$

At this point it is important to present a result related to the eigenvalue distribution of *random Gram* type matrices [9]. The eigenvalue distribution of a random Gram type matrix can be found as a solution to an integral equation. More specifically, if the entries of the covariance matrix $\mathbf{V}$ satisfy the summability condition,

$$\sum_{x,j\in\mathbb{Z}}|V_j(x)| = V_{max} < \infty \tag{11}$$

then, using a similar methodology adopted in [9] it can be proven with regard to matrix $\mathbf{HH}^\dagger$ that,

1) the sequence $\sigma_M(\lambda) = \frac{1}{M}\sharp\{\lambda_i^{(M)} \leq \lambda\}$, called the normalized eigenvalue counting function converges in probability to a non-random function $\sigma(\lambda)$, in the limit $M, p \to \infty$, $M/p \to q > 1$
2) the Stieltjes transform $f(z) = \int(\lambda - z)^{-1}d\sigma(\lambda)$, $z \in \mathbb{C}\backslash\mathbb{R}$ can be found from the following relation,

$$f(z) = \frac{1}{2\pi}\int_0^1 e^{2\pi i\tau}\tilde{g}(\tau, z)d\tau \tag{12}$$

where $\tilde{g}(q, z)$ is a solution to the equation,

$$\tilde{g}(\tau, z) = \Bigg(-z + q\frac{\tilde{V}_r(\tau)}{1 + \int_0^1\tilde{V}_r(s)\tilde{g}(s, z)ds}\Bigg)^{-1} \tag{13}$$

and

$$\tilde{V}_r(\tau) = \sum_{x\in\mathbb{Z}} w^2V^2(x)\exp\{-2\pi ix\tau\} \tag{14}$$

where, $w = 1/2$, is the variance of the real (or imaginary) parts of the channel gains.

Applying (12),(13) and (14) to the matrix $\mathbf{HH}^\dagger$ with $\mathbf{H}$ defined as in (9) and $V(x) = \exp(-\lambda x)$, we obtain the following integral equation.

$$\int_0^1\big\{z\tilde{g}(r, z) + 1\big\}V(\tau)\tilde{g}(\tau, z)d\tau = \tilde{g}(\tau, z)\big\{qV(r) - z\big\} - 1 \tag{15}$$

Note that, $V(\tau) = (1/4)\{1 - \rho\exp(-2\pi i\tau)\}^{-1}$ with $\rho = \exp(-2\lambda)$. From [10] we note that a solution $y(x) = \tilde{g}(r, z)$ to the integral equation of the form of (15) can be given by;

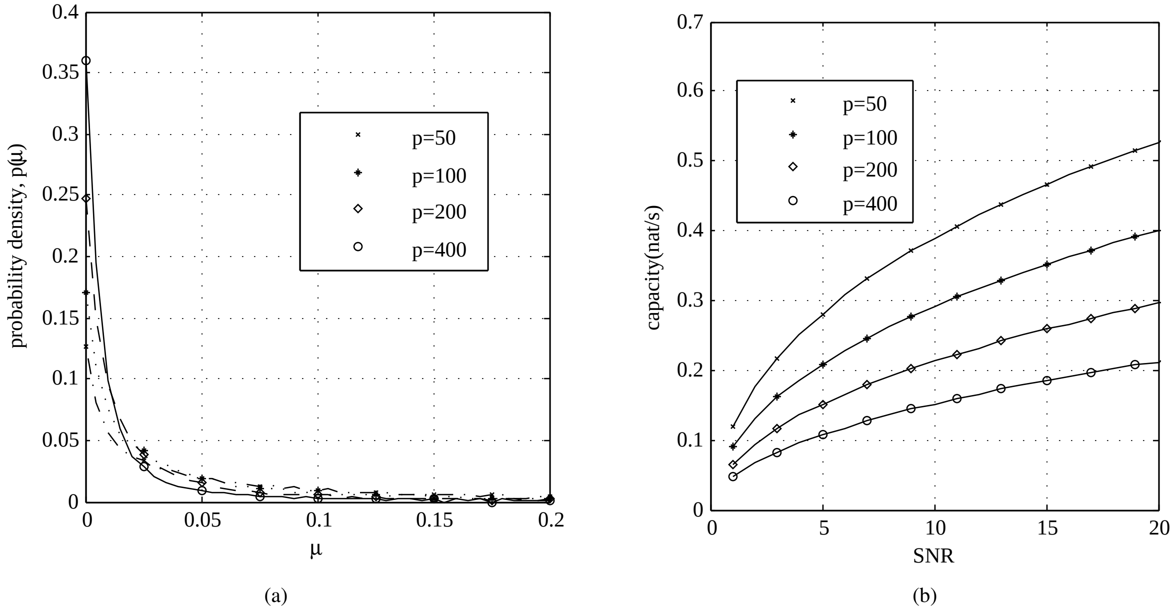


Fig. 2. (a) Eigenvalue distributions for exponentially correlated system $p = 50, 100, 200, 400$ relays and M=2000 number of subframes; $\mu \in \mathcal{R}^+$ (b) Capacity per channel use for exponentially correlated system, $p = 50, 100, 200, 400$ number of relays and M=2000 number of subframes

$$\mu\varphi(y(x)) - h(x) = 0 \tag{16}$$

$$\mu - F(\mu) = 0; \quad F(\mu) = \int_a^b G(t, y(t)) \tag{17}$$

where,

$$\varphi(\tilde{g}(\tau, z)) = z\tilde{g}(\tau, z) + 1 \tag{18}$$

$$h(\tau, z) = \tilde{g}(\tau, z)\big\{qV(\tau) - z\big\} - 1 \tag{19}$$

$$G(t, g(t, z)) = V(t)\tilde{g}(t, z) \tag{20}$$

From (15), (17) and (20) we obtain,

$$F(\mu) = \int_0^1 \frac{(1+\mu)/4}{q/4 - (1 - \rho e^{-2\pi i t})(z + \mu z)} dt \tag{21}$$

Hence $F(\mu)$ can be evaluated to be,

$$F(\mu) = \frac{1+\mu}{q - 4z - 4\mu z} \tag{22}$$

According to (17), μ has two solutions. Let $\mu_k, k = 1, 2$ be the solutions obtained in this way. Now, from (18) and (19) we obtain $\tilde{g}(r, z)$ to be,

$$\tilde{g}(r, z) = \frac{\mu_k + 1}{qV(r) - z - \mu_k z} \tag{23}$$

Now substituting (23) to (12) we obtain,

$$f(z) = \frac{\rho}{8\pi} \int_0^1 \frac{(1+\mu_k)(e^{2\pi i r} - \rho)}{q/4 - (z + \mu_k z)(1 - \rho e^{-2\pi i r})} dr \tag{24}$$

Eq. (24) can be solved to obtain the solution in the form of,

$$f(z) = \frac{\rho q}{32\pi} \frac{\mu_k^2}{1 + \mu_k} \tag{25}$$

To recover the p.d.f one may use the Stietjes-Perron inversion formula [11] (note that $f(z)$ is continuous) and we obtain,

$$p_\Omega(\omega) = \rho \frac{(q - 4\omega)^+ \sqrt{[(4\omega - q + 1)^2 - 16\omega]^+}}{1024\pi^2\omega} \tag{26}$$

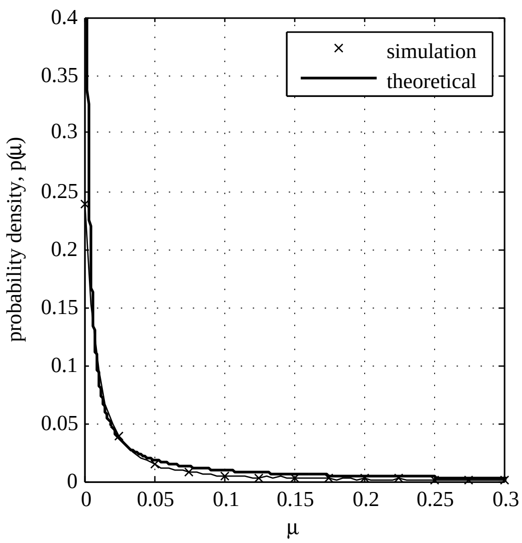
Finally, the ergodic capacity of the system can be estimated by evaluating the following integral [11],

$$C = \int_0^\infty \ln(1 + \mathsf{SNR}.\omega) p_\Omega(\omega) d\omega \tag{27}$$

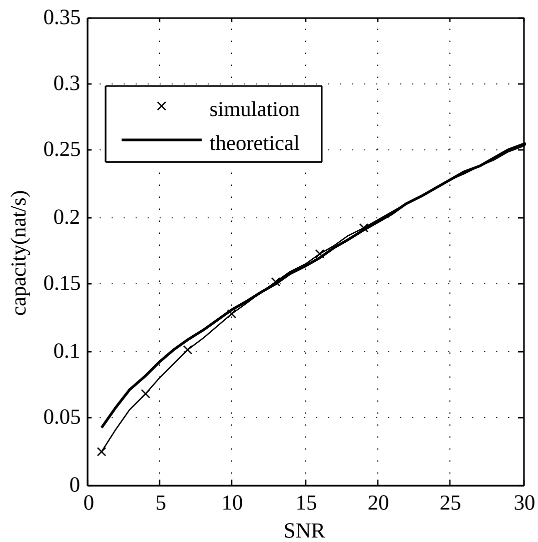
(27) can be evaluated numerically with the results obtained are discussed (along with other simulation results) in the next section.

V. Simulation Results

In this section we present the simulation results for eigenvalue distribution and capacity results for the SAF model with correlated channel gain environment outlined in section II. We also compare the simulation results with the theoretical results derived in section IV. Fig. 2a and Fig. 2b results of the eigenvalue distribution and the capacity per channel use for relay numbers of $p = 50, 100, 200, 400$ and subframe size of $M = 2000$. The latter information completely determines the parameters of (9). In Fig. 3a and Fig. 3b we compare the simulation results with the theoretically obtained results for the eigenvalue distribution and the channel capacity for an SAF system with $p = 200$ and subframe size of $M = 2000$. λ is set to 1 in both cases and we note that in case of both of eigenvalue distribution and the capacity the theoretical expression is in good agreement with the simulation results.



(a)



(b)

Fig. 3. Capacity per channel use for: (a) Theoretical and simulation eigenvalue distribution comparison for exponentially correlated system $p = 200$, $M = 2000$; $\mu \in \mathcal{R}^+$ (b) Theoretical and simulation capacity per channel use comparison for exponentially correlated system

From Fig. 2b we clearly see that, as the network density or the number of nodes increase, the capacity per channel use deceases. This can be clearly established and verified by using the eigenvalue distributions depicted in Fig. 2a. As highlighted in Fig. 2a we see that as the number of nodes within a given subframe, p, increases the singularity of the spectrum at the origin increases thus increasing the possibility of outage. From (26). Furthermore, according to (23) we may observe that the degradation in capacity is proportional to $\rho = e^{-2\lambda}$.

VI. Conclusion

In this paper we derived analytical expressions for the asymptotic eigenvalue distributions and the capacity per channel use of the transmission model of the generalized slotted-amplify-and-forward protocol, assuming an exponentially spatially correlated channel gain environment. We have shown analytically and verified by simulation that the capacity given by asymptotic mutual information does decrease with network density.

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