Strategic Project Evaluation for Open Pit Mining Ventures Using Real Options and Allied Econometric Techniques

Luis Alberto Martínez Tipe
Bachelor of Engineering (Faculty of Mining Engineering), Universidad Nacional de Ingenieria–Lima-Peru.
MDip in Engineering (WH Bryan Research Centre), The University of Queensland.
MPhil in Engineering (WH Bryan Research Centre), The University of Queensland.

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CANDIDATE NAME: Luis A. Martínez

PRINCIPAL SUPERVISOR: Professor Rodney Wolff

ASSOCIATE SUPERVISOR: Professor Chris Alford

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Name ________________________ Signature______________________

Name ________________________ Signature______________________

Name ________________________ Signature______________________

Name ________________________ Signature______________________

Under the requirements of PhD regulations, Section 16, it is hereby certified that the thesis of the above-named candidate has been examined. I recommend on behalf of the Examination Committee that the thesis be accepted in fulfilment of the conditions for the award of the degree of Doctor of Philosophy.

Name ________________________ Signature______________________

Chair of Examiners

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Abstract

Open pit mine operations are complex businesses that demand a constant assessment of risk. This is because the value of a mine project is typically influenced by many underlying economic and physical uncertainties, such as metal prices, metal grades, costs, schedules, quantities, and environmental issues, among others, which are not known with much certainty at the beginning of the project. Hence, mining projects present a considerable challenge to those involved in associated investment decisions, such as the owners of the mine and other stakeholders.

In general terms, when an option exists to acquire a new or operating mining project, , the owners and stock holders of the mine project need to know the value of the mining project, which is the fundamental criterion for making final decisions about going ahead with the venture capital. However, obtaining the mine project's value is not an easy task. The reason for this is that sophisticated valuation and mine optimisation techniques, which combine advanced theories in geostatistics, statistics, engineering, economics and finance, among others, need to be used by the mine analyst or mine planner in order to assess and quantify the existing uncertainty and, consequently, the risk involved in the project investment.

Furthermore, current valuation and mine optimisation techniques do not complement each other. That is valuation techniques based on real options (RO) analysis assume an expected (constant) metal grade and ore tonnage during a specified period, while mine optimisation (MO) techniques assume expected (constant) metal prices and mining costs. These assumptions are not totally correct since both sources of uncertainty—that of the orebody (metal grade and reserves of mineral), and that about the future behaviour of metal prices and mining costs—are the ones that have great impact on the value of any mining project.

Consequently, the key objective of this thesis is twofold. The first objective consists of analysing and understanding the main sources of uncertainty in an open pit mining project, such as the orebody (*in situ* metal grade), mining costs and metal price uncertainties, and their effect on the final project value. The second objective consists of breaking down the wall of isolation between economic valuation and mine optimisation techniques in order to generate a novel open pit mine evaluation framework called the "Integrated Valuation / Optimisation Framework (IVOF)". One important characteristic of this new framework is that it incorporates the RO and MO valuation techniques into a single integrated process that quantifies and describes uncertainty and risk in a mine project evaluation process, giving a more realistic estimate of the project's value. To achieve this, novel and advanced engineering and econometric methods are used to integrate financial and geological uncertainty into dynamic risk forecasting measures.

The proposed mine valuation/optimisation technique is then applied to a real gold disseminated open pit mine deposit to estimate its value in the face of orebody, mining costs and metal price uncertainties.

Keywords: *Decision making criteria, dynamic cash flow, flexibility value, metal price forecasting, multi-parametric orebody modelling, net present value, open pit mine design and planning, orebody modelling, project value, risk analysis, and real options.*

TABLE OF CONTENTS

Chapter 1 General Introduction..21

1.1 Introduction...21
1.1.1 Uncertainty in orebody modelling 22
1.1.2 Uncertainty in metal prices and costs 23
1.1.3 Uncertainty and risk in open pit mine planning and design........................24

1.2 Uncertainty and risk in open pit mine project evaluation: statement of the problem 28

1.3 Thesis aim and objectives 34

1.4 Relevance of the research to the mining industry...36

1.5 Scope and limitations of the research 36

1.6 Structure of the thesis...38

Chapter 2 Literature Review 41

2.1 Introduction to open pit orebody modelling 41

2.2 Introduction to discounted cash flow, real options and mine optimisation techniques 42

2.3 Organisation of the literature review 43

2.4 Discrete representation of spatial random variables and random functions 44

2.5 Ordinary Kriging (OK) and Sequential Gaussian Simulation (SGS) 46
2.5.1 Ordinary Kriging (OK) 46
2.5.2 Sequential Gaussian Simulation (SGS) 48

2.6 Estimation vs. Simulation 50
2.6.1 How many simulations need to be generated to accurately characterise the uncertainty of the orebody model? 63

2.7 Open pit capital investment project: from the traditional DCF to the RO techniques 64
2.7.1 Certainty equivalent, forward contracts and the real options analysis for open pit project evaluation...68
2.7.2 Partial differential equations 71

2.7.3 The Binomial lattice 80
2.7.4 Monte Carlo simulations 86
2.7.5 Managerial strategies 87
2.7.5.1 The option to close the mine at any time 87
2.7.5.2 The option to defer initial project investment 87

2.8 Open pit mine optimisation techniques based on orebody uncertainty 90

2.9 Summary and conclusions 97

Chapter 3 Building the Base-case Open Pit Mine Plan and Design: Preliminaries for IVOF 100

3.1 Introduction 100

3.2 Building the Base-case plan and design: The virtual 2D Gold Mine 103

3.3 Evaluating the 2D gold mine problem 104

3.4 Conclusion 112

Chapter 4 The Integrated Valuation Optimisation Framework (IVOF) 114

4.1 Introduction 114

4.2 The *"flaw of averages"* in mine project evaluation 114

4.3 A road to improve: Developing a novel integrated mine evaluation framework (IVOF) that accounts for uncertainty and risk 119

4.4 The Novel Integrated Valuation/Optimisation Framework 124
4.4.1 Algorithm for estimating the open pit project value using the IVOF technology 132

4.5 Conclusions and comments 141

Chapter 5 Simulation–Based Geological Risk Assessment: The Foundations of IVOF 143

5.1 Assessing the risk in an open pit mine project under geological uncertainty .. 143
5.1.1 Selecting a suitable simulation orebody model(s) for mine design and valuation 143
5.1.2 Risk assessment of a gold deposit using the uncertainty of the orebody .. 149

5.2 Comments and conclusions 156

Chapter 6 Understanding and Modelling Future Metal Prices in Open Pit Mine Project Evaluation 159

6.1 Introduction..159

6.2 Traditional ways of modelling metal prices in the mining industry 161

6.3 Modelling future metal prices as stochastic processes 164

6.4 Modelling future metal prices as risk-neutral stochastic processes.................170

6.5 A simple example: the gold mine project evaluation problem 175

6.6 General comments and conclusions..181

Chapter 7 Case study: Valuing a Gold Mine Project Using the Proposed IVOF 184

7.1 Introduction..184

7.2 Stage 1: The base-case mine design..188

7.3 Stage 2: What if metal grades vary? Assessing the risk of the base-case mine design indicators due to metal grade variation: An Upside/Downside-potential analysis..191

7.4 Stage 3: What if metal price varies? Assessing the risk of the base-case mine design indicators due to both metal grade and metal price variation.....................194

7.5 Stage 4: Considering managerial flexibility: closing the mine in the presence of adverse technical and economic conditions..196

7.6 Stage 5: Building a risk-robust open pit mine design based on uncertainty and risk analysis..197

7.7 General comments and conclusions..200

Chapter 8 Conclusions and future directions...202

8.1 Introduction..202

8.2 Future directions 204

References...206

Appendix A Independence and Conditional Expectation 221

Appendix B Financial options analysis - the basics 222

Appendix C Selected open pit valuation techniques 224

C.1 The CA/BDH/Smith real options model 224

C.2 The Longstaff-Schwartz Least Square Monte Carlo 228

C.3 The Upside/Downside potential mine optimisation 231

C.4 Lane's cut-off grade optimisation 233

Appendix D Papers published from this dissertation 235

LIST OF FIGURES

Figure 1. Schematic representation of: A) orebody and the ultimate pit; B) long-term scheduling composed of the ultimate pit and cutbacks 1, 2 and 3; C) short-term scheduling for cutback 1, where a, b, c, and d represent the extraction sequence; and D) plan view of the short-term scheduling for cutback 1, where a, b, c and d represent the extraction sequence (adapted from Martinez, 2003). 27

Figure 2. The current market value of a mining project investment opportunity. V is the present value of the mine project and I is the initial capital investment. The straight line indicates the variation of the real option value of the mine when flexibility is not taken into account throughout the operating life of the project, while the curve indicates the variation of the real option mine's value due to the introduction of flexibility throughout the operating life of the mine. The dot-dashed line indicates the traditional static NPV. As it will be shown in Chapter 2, real options theory has its foundations in the theory of financial options (see Appendix A for details of financial options). Consequently, the real option value is always greater than or equal to zero while the traditional NPV can have negative values. 30

Figure 3. Total risk and RO project value. As observed in the diagram, an increase in total risk widens the distribution of outcomes (left diagram), creating more outcomes with a positive payoff and consequently an increase flexibility value (right diagram). Adapted from Amram and Kulatilaka (1999, p35). 32

Figure 4. A comparison of idealised profiles of true grade (solid line), simulated grade (dot-dashed lines), and kriged estimated grade (dashed line) on a common *profile. Note the similar variability of the true and simulated grades in contrast to the smoothed pattern of estimated grades. Also, note that all profiles pass through the known data points. (Reproduced from Journel, 1975).* 52

Figure 5. Diagram showing the exhaustive data set containing 78000 samples, and its histogram and respective experimental and fitted model variograms. In the figure, the bar next to the map of exhaustive data set indicates the variation in metal grade: dark colours indicate high grade. 57

Figure 6 Diagram showing the sample data set containing 800 samples, and its histogram and respective experimental and fitted model variograms. In the figure, the bar next to the map of sample data set indicates the variation in metal grade: dark colours indicate high grade. 58

Figure 7. Diagram showing the estimated model of the deposit with its respective histogram, experimental and fitted model variograms. In the figure, the bar next to the map of estimated values indicates the variation in metal grade: dark colours indicate high grade. 59

Figure 8. Diagram showing the reduction of variance that the estimation performs with respect to the sample data. 60

*Figure 9. Diagram showing a sampled zone (in the square) where ordinary kriging gives a correct estimation and an unsampled zone (inside the circle) where ordinary kriging does not give an accurate estimation. Note that, in the middle diagram, the circles indicates drill sample sites, where the darker the circle the higher the grade.*61

Figure 10. Diagram showing two simulations of the deposit with their respective histograms and variograms. In the figure, the bars next to the map of simulated values indicate the variation in metal grade: dark colours indicate high grade. 62

Figure 11. Diagram showing the differences in the continuity of the cdf curve of a specific geological attribute, e.g., ore tonnes and metal quantities above a given cut-off grade, when using one, three,..., and several (>100) simulations of an orebody model. Note that there is not a specific rule that can be used to know the number of simulations necessary to achieve accuracy. In practice, it is common to generate

$N \geq 100$ *simulations since it gives a good representation of the continuity of the cumulative distribution function (cdf).* 64

Figure 12. The two-period open pit mine project evaluation problem. In the figure (left part) V_0 *is the current value of the open pit project,* V_u *and* V_d *are an up or down movement of the open pit project's value at period* $T > t_0$*, occurring with the objective probabilities* q *and* $(1-q)$*, respectively. The problem consists of estimating the current extended value of the mine,* EV_{t_0}*, that includes the option of implementing an operational strategy at period* $T > t_0$*. In this case the current extended value of the mine, or payoff, at period* $T > t_0$ *will depend on the extended future value of the mine project, that is,* EV_u^T*, if it goes up, or* EV_d^T*, if it goes down, and the cost,* K*, of implementing the operational strategy.* 83

Figure 13. Generalised scheme of the two-date binomial call real option pricing problem stated in Figure 2. To estimate the extended value of the call option EV_{t_0}*, a replicating portfolio of the stock, e.g., metal produced by the mine, and bonds, is built. In this case, the portfolio is composed of* A *quantity of the metal produced by the mine, whose current price is* S_0*, and* B *bonds. In the figure,* r *is the risk-free rate of return,* p *is the risk neutral probability and* m*,* $(1-m)$ *are the objective probabilities of an up (*S_u*) or down (*S_d*) movement, respectively, on the metal price in one period ahead. Observe that the extended value of the mine does not depend on the objective probability,* m*, but the risk neutral probability* p*. Observe in the figure that for this example,* $T = 1$ *period ahead, and both* EV_u *and* EV_d *are as in Figure 4.* 84

Figure 14. Schematic representation of the general Monte Carlo simulation and risk assessment framework. In this framework, each image of the orebody gives a realistic representation of the variability of metal grade throughout the deposit. Then, by inputting these images into the optimisation and mine design process (non-linear transfer function), the orebody uncertainty is transferred onto the response parameters (reproduced from Martinez (2003)). 92

Figure 15 Diagram showing the open pit mine operation process. 103

Figure 16. Table showing the 2D Gold Mine project-economic and technical parameters. Slope angle is assumed to be 45°...104

Figure 17. The 2D gold orebody model. Gold grades given in Oz/t. Grade relates to the "temperature" of the colours where hot colours indicate a high grade. 104

Figure 18. Mining cost model ($/block).Colours still relate to grade but now costs are shown, and they increase with depth. 106

Figure 19. Metal recovered model (Oz/block). Colours still relate to grade but now metal recovery is shown...106

Figure 20. Block revenue model ($/block). Colours still relate to grade but now block revenue are shown. 106

Figure 21. Value of block if milled ($/block). Colours still relate to grade but now block value are shown...106

Figure 22. Economic block model $\max\left\{BlockVal_{Milled}, MiningCost\right\}$. 107

Figure 23. The gold mine project ultimate pit limits (in red colour). This was obtained using the simple floating cone algorithm (see Martinez, 2003 for more details about the floating cone algorithm)....107

Figure 24. Nested pit shells generation due to different gold prices. Determined using the floating cone algorithm...108

Figure 25. The gold mine project key indicators for each pit shell..........................108

Figure 26. The Gold Mine open pit mine project set up on the 900$/Oz pit shell limits. 109

Figure 27. The gold mine project: bench-by-bench production schedule.................110

Figure 28. The gold mine project - bench-by-bench production schedule, key project indicators (10% WACC). 110

Figure 29. Gold mine project: practical production schedule...................................111

Figure 30 The gold mine project- practical production schedule, key project indicators (@10% WACC).......111

Figure 31 Ore vs. Waste curve for the practical and bench-by-bench production schedules.......112

Figure 32. Non-cumulative and cumulative Discounted Cash Flow (DCF) analyses for the gold mine project considering the practical and bench-by-bench production schedules.......112

Figure 33. Scheme showing that "average inputs do not always yield average outputs" when dealing with uncertainty and non-linear processes.......116

Figure 34. A sobering example of the Flaw of Averages (from Savage, 2003).......117

Figure 35. A simple explanation of the real options concept: The profitable contract signed on a BBQ area.......121

Figure 36. Graph showing the main difference between traditional mine evaluation techniques and the IVOF process.......123

Figure 37 Schematic showing the j *plan and design,* ζ_j*, of an open pit mine project. As observed,* ζ_j *is composed of the union of the production scheduling plan and designs,* ς_t^j $(t = \{t_0 = 0, t_1 = \Delta t, ..., t_T = T\Delta t\}$*. Also observe that the shape of the production scheduling plan and design is given by the blocks,* $\upsilon_{x_i}^t$ *, to be extracted at each production period* t *and which obey technical constraints such as slope angle. The value of each block is given by Equations 3.4-3.6.*......126

Figure 38. Diagram showing the base-case open pit mine plan and design, ζ_j *, composed of an ultimate pit and a production scheduling (see Figure 1 for more details). Observe that the final project economic and technical indicators are single estimates.*......133

Figure 39. Diagram showing IVOF stage 2 where the base-case open pit mine plan and design is assessed under in situ metal grade uncertainty. Observe that in this *case, contrary to Stage 1, the final project indicators are characterised by a distribution of values.* 134

Figure 40. Diagram showing IVOF Stage 3: Incorporating the effect of future metal prices (upper diagram) in the current mine plan and design and estimated current market value of the open pit mine project (lower diagram). 136

Figure 41 Diagram summarising the Integrated Valuation/Optimisation Framework. 137

Figure 42 Traditional mine evaluation process where the aggregate discount rate is considered as the Atlas holding the entire risk and uncertainty of a mine project. 139

Figure 43 Traditional mine evaluation techniques can over-estimate the discount rate which will minimise the mine project's value. 139

Figure 44 Traditional mine evaluation techniques can under-estimate the discount rate which will not be able to support the heavy load of risk of a mine project, leading to its collapse. 140

Figure 45 IVOF perspective of mine project evaluation under uncertainty and risk. In this case the estimation of the discount rate, r *, (represented here as the Atlas holding the entire mine project risk) is not a difficult task since it does not account for technical and economic risk, as they are accounted for at their source of uncertainty.* 140

Figure 46. When applied correctly, the IVOF is able to output a map of the technical and economic mine project indicator performance, in this case the cash flow distribution generated at each production period, where high risk areas can be identified via a risk analysis, such as Value@Risk (VaR) and the Upside/downside potential. 142

Figure 48. Diagram showing two simulations of an orebody with a common inner zone A. In the figure, Q_{Sim1} *and* Q_{Sim2} *denote the total ore tonnes of each simulated model while* q_{Sim1}*, and* q_{Sim2} *denote the ore tonnes inside the common zone A.* 146

Figure 49. Diagram showing the process of comparing the probability distribution of the resulting NPVs of three pit designs. Observe that this comparison is based on the *downside-risk and upside-potential of obtaining values below or above a minimum amount return (MAR).* 148

Figure 51. Schematic diagram showing the traditional pit design process: estimation of an orebody model; pit optimisation process; DCF analysis by pit shell; finding the ultimate pit (pit shell 41). 151

Figure 52. Schematic diagram showing the traditional cutback design, and the single value for each of the project indicators considered in the analysis. 151

Figure 53. Base-case mine design dynamic distribution of ore tonnes, gold quantity, waste tonnes and discounted cash flows for each production period or cutback (after Martinez, 2003). Observe that the bars with different colours indicate the result obtained for the base-case mine design without considering the uncertainty of the orebody. 155

Figure 54. Scheme showing Kriging's property of overestimated or underestimated ore tonnes (adapted from Journel and Huibregts, 1978). 156

Figure 55 Chart showing the different copper price values that a copper mine will assume when evaluating the mine project using traditional metal price forecasting (in the figure, the historical copper spot price are for the period 1-04-1986 – 1-04-2006). As observed in the figure, mine projects started in1986 (point A) will underestimate its market value, while mine projects started in 1989 will overestimate its value for periods 1989-2004 and underestimate its value for periods 2004-2006. 164

Figure 56. Twenty sample paths generated by a Monte Carlo simulation of a geometric Brownian process. The parameters for the GBM process are: $S_0 = 50$, $\mu = 10\%$, *and* $\sigma = 30\%$. 166

Figure 57. Fifty sample paths generated by Monte Carlo simulation of a Mean Reverting process. The MR parameters are $S_0 = 50$, $\sigma = 30\%$, $M = 35$, *and* $\eta = 0.8605$. *Observe that the Mean Reverting process tends to revert to the long-term value of* $M = 35$. 168

Figure 58 Chart indicating a jump in the spot price of copper in 2006. 169

Figure 59 Fifty sample paths generated by Monte Carlo simulation of a Mean Reverting with jump process. The MR with jumps model's parameters are $S_0 = 50$, $\sigma = 30\%$, $M = 35$, *and* $\eta = 0.8605$, *and 10% jump rate with sizes equal to 1.5.* ...170

Figure 60. Technical and financial parameters used to evaluate the gold mine project (see Chapter 4 for definitions of the different mining parameters). 175

Figure 61. Static discounted cash flow analysis of the underground gold mining project. This is shown as the traditional DCF spreadsheet analysis performed when evaluating a mine project...178

Figure 62. Sensitivity analysis, with respect to gold price variation, of the underground gold mining project. 179

Figure 63. Historical gold price (top) and gold price return (bottom) for the period 28-06-1985 to 30-06-2005. 179

Figure 64. Probability distribution of NPV (top) and IRR (bottom) of the underground gold mining project using the simple MC (GBM) analysis (after 5000 simulations). The results were obtained using the Crystall Ball software. 180

Figure 65. Probability distribution of NPV (top) and IRR (bottom) of the underground gold mining project using the risk-neutral MC (GBM) analysis (after 5000 simulations). The results were obtained using the Crystall Ball software. 181

Figure 66. East view of the gold mine project drill hole composite data set showing the topography of the deposit (developed in SURPAC)....186

Figure 67. East view of the gold mine deposit showing the oxide, transitional and primary rock types (developed in SURPAC). 186

Figure 68. North-east view of the gold mine deposit showing the eight geotechnical zones (developed in SURPAC)...187

Figure 69. Variogram analysis performed on the gold mine deposit (SURPAC-geostatistical tools). 189

Figure 70. South-east view of the estimated gold mine orebody model. In this case the ordinary Kriging technique (SURPAC-geostatistical tools) was used for generating this model. 189

Figure 71. North view of the gold mine Project base-case open pit mine design showing the selected cutbacks. 190

Figure 72. North view of the gold mine project base-case open pit mine design showing the long-term production scheduling. 190

Figure 73. Summary of the gold mine project indicators. 191

Figure 74. Assessing the risk of the gold mine project base-case mine project indicators in the face of in situ metal grade uncertainty. 193

Figure 75. Chart showing the effect of metal grade uncertainty on the gold mine project key project indicators. 193

Figure 76. Quantification of future gold prices via a binomial lattice. In this case the price model used to describe the price behaviour is the Geometric Brownian Model (GBM). 195

Figure 77. Estimation of the Gold Mine Project value in the face of metal grade and metal price uncertainties using the IVOF technique. 195

Figure 78. IVOF analysis of the Gold Mine Project base-case design with the option to close the mine - without salvage value. As observed in the figure, the (put) option to close the mine, with zero value, is exercised in periods 3 and 4, when gold prices fall to equal to or lower than Au$11.18/gr. and Au$8.28/gr., respectively. This is because the option value, zero, is greater than the negative cash flows that could be generated otherwise, that is, max (0, negative CF). 197

Figure 79. IVOF analysis of the Gold Mine Project base-case design with the option to close the mine - with a salvage value of Au$60M. It is interesting to observe that in this case, the option to close the mine in the 4th production period and receive the Au$60M salvage value is exercised for gold prices equal to or lower than Au$15.09/gr. The figure also suggests that the option to close the mine and receive *the salvage value in year three is exercised for gold prices equal to or lower than Au$11.18/gr.* 197

Figure 81. Chart summarising the IVOF's Stage 5: Selecting the best open pit mine design based on in situ metal grade and metal price uncertainties. The results suggest that the initial base-case mine design generates the highest expected mine value in the face of uncertainty....200

Figure 82. Diagram showing six potential simulated stock price paths. Note that at different stages or periods different prices are at the in-the-money region (for a put option). For example, at time t_1, prices S_6, S_3, and S_5 are at the in-the-money region. 229

Figure 83. Original time series $\{X_t\}$ *and its respective time-dependent volatility.* 254

Figure 84. Original time series $\{Y_t\}$ *and its respective time-dependent volatility.* .255

Figure 85. Original time series $\{Z_t\}$ *and its respective time-dependent volatility.* 255

Figure 86. Analysis of surrogate series and adaptive volatility of time series $\{X_t\}$ *using the SKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).* 256

Figure 87. Analysis of surrogate series and adaptive volatility of time series $\{X_t\}$ *using the BKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).* 256

Figure 88. Analysis of surrogate series and adaptive volatility of time series $\{Y_t\}$ *using the SKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).* 257

Figure 89. Analysis of surrogate series and adaptive volatility of time series $\{Y_t\}$ *using the BKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).* 257

Figure 90. Analysis of surrogate series and adaptive volatility of time series $\{Z_t\}$ *using the SKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).* 258

Figure 91. Analysis of surrogate series and adaptive volatility of time series $\{Z_t\}$ *using the BKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).* 258

Figure 92. Adaptive distribution function of the time series $\{X_t\}$ *for points* $t = 1,100,250,600,800,1000$ *(SKS technique).* 259

Figure 93. Adaptive distribution function of the time series $\{Y_t\}$ *for points* $t = 1,100,250,600,800,1000$ *(SKS technique).* 259

Figure 94. Adaptive distribution function of the time series $\{Y_t\}$ *for points* $t = 1,100,250,600,800,$ *(BKS technique).*..260

Figure 95. Diagram showing the current value x_k *(black circle), its neighbourhood* h *(rectangle), and its nearest neighbours* $\{x_j, x_p, \cdots, x_l, x_n\}$ *(hollow circles).*.............291

Figure 96. Figure showing the ellipse or joint neighbourhood of the current value x_k *and* y_k *of a bivariate process.* 292

Figure 97. Time series of the daily log-returns of both FFR/USD and DEM/USD exchange rates, respectively, for the period 01/05/82 – 17/08/84. The data comprises 600 daily values. 298

Figure 98. Plot showing the dynamic volatility and de-volatised series of the FFR/USD exchange rates for period 01/05/82 – 17/08/84. 299

Figure 99. Plot showing the dynamic volatility and de-volatised series of the DEM/USD exchange rates for period 01/05/82 – 17/08/84. 299

Figure 100. Dynamic Kendall's tau coefficient of correlation (above) and dynamic Archimedean copula (below) between FFR/USD and DEM/USD exchange rates. ..300

Figure 101. Dynamic Archimedean copula parameter. 300

Figure 102. Original time series (top) and 100 surrogate series (bottom) of the FFR/USD exchange rate. 301

Figure 103. Original time series (top) and 100 surrogate series (bottom) of the DEM/USD exchange rate. 301

Figure 104. Empirical distribution function of original (black) and 4 surrogate series (red) of FFR/USD exchange rate. 302

Figure 105. Empirical distribution function of original (black) and 4 surrogate series (red) of DEM/USD exchange rate. 302

Figure 106. Linear cross-correlation between original bivariate process and 100 bivariate surrogate series of FFR/USD and DEM/USD exchange rates. 303

LIST OF TABLES

Table 1. Univariate statistics associated with the exhaustive and sample data sets. Observe that, in general terms, the sample data set provides a realistic approximation of the distribution of the exhaustive data set. 55

Table 2. General similarities and differences between financial and real options (adapted from Bradley, 2004). 70

Table 3. Comparison between a call option on a stock and a real option on the acquisition of a mining project 71

Table 4. Technical and economic parameters considered when performing the process of pit design on the small disseminated gold deposit. 150

Table 5. Results of the risk analysis performed by cutback for key project indicators for the base-case design of the gold deposit under study (after Martinez, 2003). 158

Table 6. Table showing some of the financial and technical parameter values used to evaluate the gold mine project. 187

Table 7. Descriptive statistics for the FFR/USD and DEM/USD FX indexes. 298

Chapter 1

General Introduction

1.1 Introduction

When the possibility of acquiring either a new or operating mine exists–in the case of this thesis, an open pit mine operation–owners and stakeholders of the project need to know the value of the mining project and the cash flow that the mine will generate over its operating life. These are the fundamental bases for making final irreversible decisions about going ahead with the project investment. The choice over such a decision is formally referred to as an option.

However, mine projects are complex businesses that demand a constant assessment of risk. This is because the value of a mine project is typically influenced by many underlying economic and physical uncertainties, such as metal prices, metal grades, costs, schedules, quantities, and environmental issues, among others, which are not known with absolute certainty. Hence, mining projects present a considerable challenge, especially in effective assessments of capital expenditure, to those involved in associated investment decisions.

To estimate a mine project's value, the analyst or mine planner needs to use sophisticated valuation techniques to assist them in assessing and quantifying the existing uncertainty and risk involved in the project investment. In fact, the results obtained from the evaluation process decisively guide owners and stakeholders in their decision as to whether to start or abandon the project, and assist mine planners and engineers to plan and design the entire mine operation.

The estimation of the mine project's value, however, is not an easy task. The reason for this is that, in general, an open pit mine project is based on two main considerations: the geological model of the mineralised deposit, and the economic model of future metal prices and costs that the mine will incur throughout its operating life.

Unfortunately, at the beginning of a project, the necessary information to build the geological and economic models is insufficient, since: i) the geological information of the mineralised deposit, which is obtained from a limited number of samples (drill-holes), is not necessarily a good representative of the entire deposit; and ii) the only information available about future metal price behaviour and costs are based on historical data, which in most cases is not a good forecast of the future.

This lack of information generates uncertainty about the value of the different underlying variables that take part in the evaluation process and, consequently, the value of the mine project.

These main sources of uncertainty are explored in the following subsections.

1.1.1 Uncertainty in orebody modelling

As mentioned in the introduction to this chapter, one of the most critical sources of technical risk in an open pit mine project lies in the geology of the orebody (Dimitrakopoulos, 1998; Dimitrakopoulos, Farrelly and Godoy, 2002). The orebody, which is the material from which minerals and metals of economic value can be extracted, is directly related to the ore grade and tonnes. Hence, an ore reserve statement should not be merely an estimate of what is in the ground, but should be a prediction of what will be fed to the mill.

In the pre-feasibility stage of a mining project, the geology and ore distribution in the mineral deposit are estimated from the information derived from the exploration drilling samples. Consequently, uncertainty arises due to the fact that the information obtained from the samples is not representative of the entire (3-dimensional) ore deposit. It is then correct to state that, if an ore deposit is yet to be mined, the knowledge of its geological characteristics, including tonnage and metal grade, is limited. One consequence of this lack of information is the misclassification of resources, where economic ore can be dispatched to the waste dump, and non-economic ore can be sent to mill (Journel and Kyriakidis, 2004).

To minimise the misclassification of resources, estimation techniques based on stochastic models are commonly used to estimate the geological information at non-sampled locations. This is done by interpolating the data from the few exploration samples.

1.1.2 Uncertainty in metal prices and costs

Another important source of uncertainty that has a critical impact on open pit mine project evaluation is that associated with the economic environment where the mine project is developed. Within this economic environment, future metal prices and costs are the chief sources uncertainty.

Metal prices are one of the most significant sources of uncertainty in an open pit project (Brennan and Schwartz, 1985). In fact, any variation from the expected metal price may considerably modify the results of the entire project evaluation. For example, an overestimated metal price may result in a favourable rate of return to a project, which is otherwise doubtful and, conversely, an underestimated metal price may result in an unfavourable return for the project, which is otherwise profitable.

The price of the metal is the real cash settlement that represents the equilibrium or disequilibrium of the metal market. Since this market is based on demand, supply and other factors, such as speculation, news events, and dividend payouts (Fanning and Parekh, 2004; Case and Fair, 1989; Taylor, Moosa and Cowling, 2000), uncertainty over future metal prices arises because of two main reasons. The first reason consists of the lack of exact knowledge of those factors leading to the increase/decrease in metal supply and demand, and the second reason is made up of the actions that producers or consumers perform in the face of powerful speculative and political motives (MacAvoy, 1988).

In the mining industry, metal prices are normally modelled as the average price of the last three years (Rendu, 2006), especially for those commodities whose prices are listed on open markets, such as precious and base metals. Even though the use of single commodity prices makes comparison between companies easy, preventing the use of excessively optimistic prices, it is also recognised that this method could be misleading when evaluating mining projects. For example, if the mine project is evaluated in a period with high/low metal prices, then the estimated average commodity price will be set up to be high/low throughout the operating life of the mine project, which of course will over/under estimate the value of the mine project.

In traditional finance, commodity (metal) prices are modelled as random variables, which follow stochastic processes over time. This is done in order to capture the complexities of futures markets and metal production as well as other non-measurable factors such as speculation. The purpose of using stochastic processes for modelling future metal price behaviour is not the estimation of an exact future price path but a distribution of paths which are expected to capture the true future behaviour of the metal. This is done because the process of forecasting metal prices for long term periods, which is common for mine operations, is not accurate.

Costs are other sources of uncertainty when evaluating an open pit mine project. The reason for this is that the economic evaluation component of the feasibility study is based on information which provides an answer to the question: “what is it going to cost?” (Gentry and O’Neil, 1998).

In a mining project, costs are normally categorised as *capital costs*, which refers to the investment required for the mine and mill plant; *operating costs*, which refers to the costs incurred in the operating activities such as drilling and blasting; and *general and administrative costs*, which refers to the costs incurred in administrative and other activities related to the mine (Camus, 2002). The costs that are independent of the production level are regarded as *fixed costs*, while the costs that depend directly on the production level are regarded as *variable costs*.

Since estimation of capital and operating costs are an important requirement for open pit mine evaluation, uncertainty in costs arises due to the lack of the engineering or economic information at the beginning of the mine project. Simply put, mining firms do not know with absolute certainty today how much they will need to spend tomorrow, let alone next month or next year.

1.1.3 Uncertainty and risk in open pit mine planning and design

McCarter (1992) defines open pit mining as a surface method in which reclamation is deferred until all or nearly all of the deposit is removed within its economic limits. The objective of open pit design is to determine the projected final pit limits of an orebody and its associated projected grade and ore tonnage in oder to maximise the economic value of the mine[1] while satisfying operational constraints such as mill capacity and slope angles (Hustrulid and Kuchta, 1995; Kim, 1978; Dowd and Onur, 1993; Wright and Mbirikira, 1993).

The complete process of designing an open pit mine consists of two principal stages: mine design, where the ultimate pit limits are found and contoured, and production scheduling, where the sequence of extraction over time is planned.

The ultimate pit limit is the widest possible boundaries within which all subsequent mine planning works are performed while maximising NPV[2]. Figure 1A shows an example of an orebody and the ultimate pit contour.

Production scheduling is the development of a sequence of depletion schedules leading from the initial condition of the deposit to the ultimate pit limits. According to the duration of the scheduling periods, production scheduling can be classified as long-term or short-term scheduling.

Long-term scheduling is the schedule determined by cash flow analysis, and it provides a guide to a more detailed mine design and development. Figure 1B shows the representation of the long-term scheduling composed of the ultimate pit and the cutbacks 1, 2 and 3. A *cutback* is a part of the ore-deposit mined on its own. In other words, a cutback is a small “independent” mine within the mine, with its own working zone, which directly influences the ore production of the mine per period before its depletion.

Short-term scheduling is the development of a sequence of depletion schedules on a daily, weekly or monthly basis within the layout of the mine (cutbacks and pit limit). It is important to note that, in short-term scheduling, the main constraints are the ore tonnage and the ore grade, as well as the requirements of the processing plant, that is, the short term takes care of achieving ore/waste production targets established in the long-term plan. Figure 1C shows the cross-section where the short-term extraction sequence for cutback 1 is performed. Thus, in the figure, the zones denoted by a, b, c and d, represent the sequence of extraction, in that order, of the mineral in cutback 1. Figure 1D shows the plan view of the short-term plan for cutback 1. Since both the ultimate pit and the production scheduling depend directly on the orebody model and future metal price and costs, uncertainty and risk in open pit mine planning and design arise due to the uncertain nature of the underlying variables that take part in the designing and planning process. In this context, the allocation of the physical limits of both the ultimate pit and long-term production sequence (cutbacks) on the orebody turns into a complex and uncertain process since it depends on both the uncertainty of future metal prices and the orebody model uncertainty.

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[1] Normally, the economic value of the mine is expressed as the Net Present Value (NPV) of the project. The NPV will be formally defined in Section 2.2.

[2] The Net Present Value (NPV) is an economic indicator widely used in open pit projects to make the decision of capital investment. Section 2.1 gives a detailed introduction to NPV.

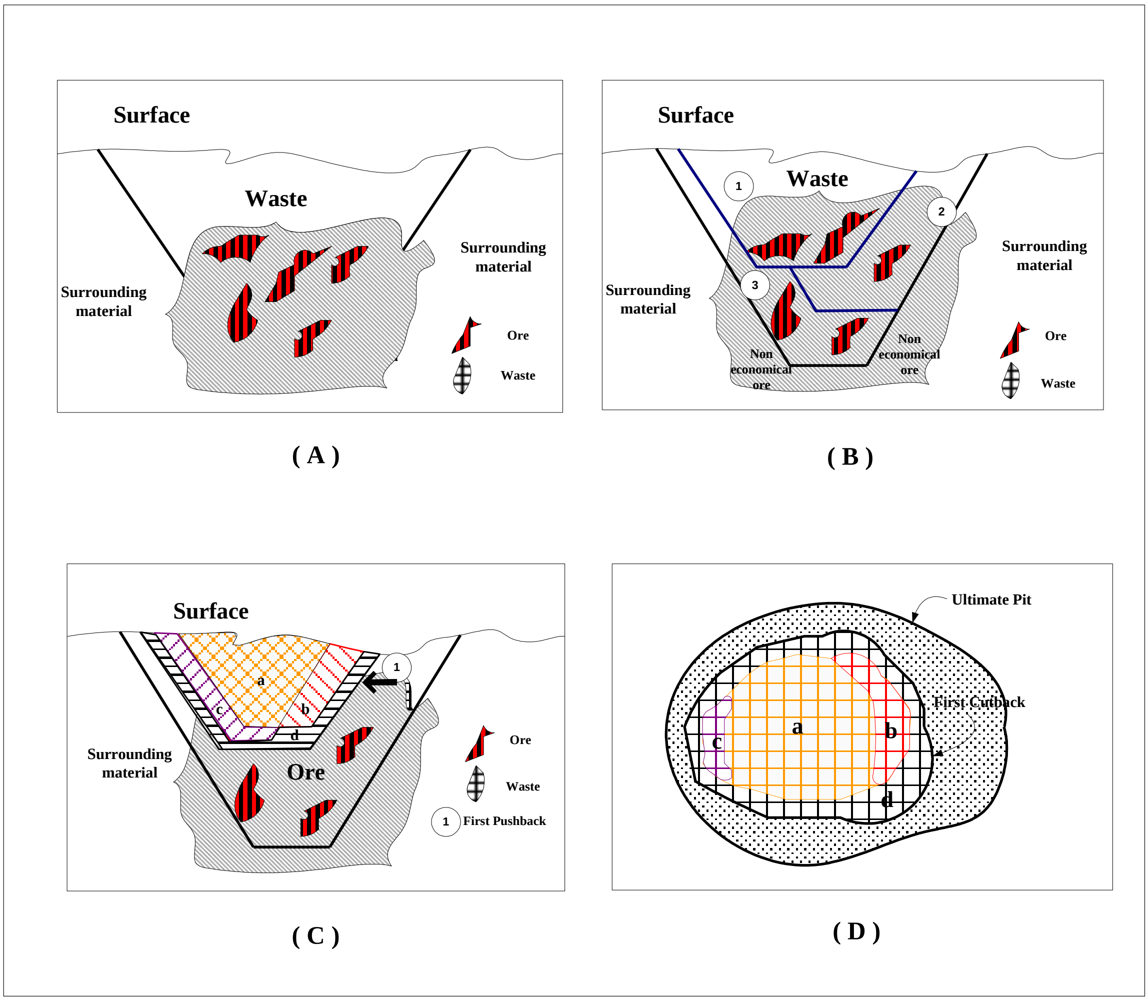
Surface
Waste
Surrounding material
Surrounding material
Ore
Waste
(A)
Surface
Waste
1
2
3
Surrounding material
Surrounding material
Non economical ore
Non economical ore
Ore
Waste
(B)
Surface
a
b
c
d
1
Surrounding material
Ore
Ore
Waste
1 First Pushback
(C)
Ultimate Pit
First Cutback
a
b
c
d
(D)

Figure 1. Schematic representation of: A) orebody and the ultimate pit; B) long-term scheduling composed of the ultimate pit and cutbacks 1, 2 and 3; C) short-term scheduling for cutback 1, where a, b, c, and d represent the extraction sequence; and D) plan view of the short-term scheduling for cutback 1, where a, b, c and d represent the extraction sequence (adapted from Martinez, 2003).

1.2 Uncertainty and risk in open pit mine project evaluation: statement of the problem

As discussed in Section 1.1, the evaluation of an open pit mine project is a complex process. The reason for this is that it involves not only the economic uncertainty of future metal prices and costs, but also the technical uncertainties in which both the orebody model and the designing and planning of the pit are considered. Furthermore, the evaluation of an open pit mine also involves the different operational strategies that are adopted throughout the operating life of the mine to maximise the present value of the project. This is achieved by capturing future potential for profit while minimising the risk of future loses.

Traditional (current) open pit mine valuation techniques[3], which combine quantitative (static) discounted cash flow (DCF) techniques with various sensitivity analyses, do not account for uncertainty and risk in key project indicators such as ore and waste tonnes and cash flows, and consequently, in project evaluation. As reported by many authors (Amram and Kulatilaka, 1999; Armstrong and Galli, 1997; Brennan and Schwartz, 1985; Burmeister, 1989; Carvalho, Remacre and Suslick, 2000; Cortazar, 1999, 2001; Dixit and Pyndick, 1994; Frimpong, 1992; Frimpong and Whiting, 1997; McKnight, 1999; Moyen, Slade and Uppal, 1996; Paddock, Siegel and Smith, 1988; Samis, Laughton and Poulin, 2001; McCarthy and Monkhouse, 2003) the main consequence of not including uncertainty in open pit project evaluation is that it misleads mine project decision-makers and investors to a static description[4] of the economic and technical risk of the open pit mine project.

Even though alternative modelling techniques such as decision analysis (DA) have been developed, which attempt to mimic management decisions in the face of uncertainty, they still only evaluate expected projected cash flows in a static environment based on expected key project indicator values, similar to the DCF technique.

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[3] A review of these techniques is given in Section 2.2.

[4] In this case, the term "*static*" refers to the lack of managerial/operational flexibility (see footnote 6).

One major criticism of DCF techniques is the implicit assumption that a mining project's outcome will be unaffected by future decisions of the firm, thereby ignoring any value that comes from managerial flexibility (Smith and McCardle, 1999). Managerial flexibility is the ability to make decisions during the execution of a mining project so that the upside potential of the mine is maximised while the downside risk is minimised: in other words, expected returns are maximised and expected losses are minimised. Examples of project flexibility include expanding operations in response to positive market conditions, abandoning a project that is underperforming, deferring investment for a period of time, suspending operations temporarily, switching inputs and outputs, reducing the project scale, or resuming operations after a temporary shutdown .

As an alternative to the traditional DCF model, modern project valuation techniques apply modern asset pricing (MAP) theory (Black and Scholes, 1973; Merton, 1973) to solve real problems such as mining ventures. This technique is called Real Options (RO) and refers to choices, and their costs, about whether and how to proceed with mining business investments (Samis, Laughton and Davis, 2005; Samis, Laughton and Poulin, 2003; Schwartz, 1997; Slade, 2001).

In the RO context, a mine project is seen as a compound American option (Bermudan type[5]) in which, at each stage of the project, the holders of the mine have the right but not the obligation to make strategic decisions such as expanding, stopping, and closing the mine operation at a predetermined cost for a predetermined period of time. In particular, the opportunity to invest in an open pit mining project is seen as an American call option in which at any time during the option period the holders of the option have the right but not the obligation to exercise the investment capital.

A typical representation of the RO value of a mine project can be observed in Figure 2. In the figure, V is the present value of the mine project and I is the initial capital investment. The dashed curve represents the current market value of the project with flexibility, the thick line represents the current value of the project without flexibility, defined as $max\{(V-I),0\}$, and the dot-dashed line represents the traditional NPV of the project.

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[5] A Bermudan option is an option where the buyer has the right to exercise at a set (always discretely spaced) number of times.

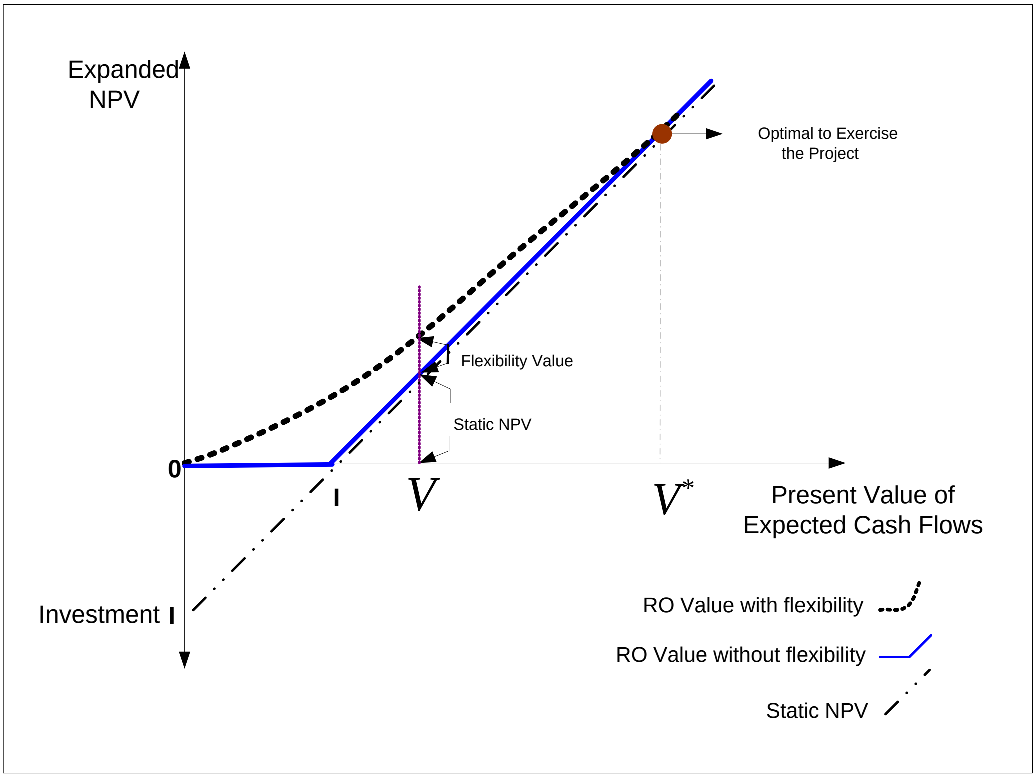

Figure 2. The current market value of a mining project investment opportunity. V *is the present value of the mine project and* I *is the initial capital investment. The straight line indicates the variation of the real option value of the mine when flexibility is not taken into account throughout the operating life of the project, while the curve indicates the variation of the real option mine's value due to the introduction of flexibility throughout the operating life of the mine. The dot-dashed line indicates the traditional static NPV. As it will be shown in Chapter 2, real options theory has its foundations in the theory of financial options (see Appendix A for details of financial options). Consequently, the real option value is always greater than or equal to zero while the traditional NPV can have negative values.*

One important result that can be extracted from Figure 2 is that the RO value of a project with flexibility, commonly called the expanded value of the project or

Expanded NPV (ENPV) (Trigeorgis, 1996; Smith, 2005), can be defined in terms of the traditional NPV as (see Figure 2)

$$ENPV = NPV + FVal, \tag{1.1}$$

where $FVal$ is the *value of (operational/managerial) flexibility*. As observed in Equation 1.1, one important characteristic of the *Expanded NPV* is that it captures the incremental value of operational and managerial flexibility, giving the mine analyst a more realistic overview of the value of the mine project in the face of uncertainty.

Another important result that can be extracted from Figure 2 is that the *ENPV* will always be greater than or equal to the static *NPV, that is*, flexibility does not have a negative value (Slade, 2001). This can be visualised by re-defining Equation 1.1 as

$$ENPV = \begin{cases} NPV + FVal; & FVal > 0; \\ NPV; & FVal = 0. \end{cases} \tag{1.2}$$

Equation 1.2 indicates that if the information obtained from the option to implement a specific operational or economic strategy in the mine project plan is valuable, then the option value, *FVal*, must be added to the direct cost of investing, otherwise the project value remains the same.

But how does flexibility add value to a project investment?

Flexibility has several strategic forms, which, if implemented, generate an aggregate value to the project by either reacting to good news in the future or by minimising future risk (or both). For example, value can be generated if the evaluation process allows the mine analyst to have the flexibility to:

- invest now and make follow-up investments later if the original project is a success (a growth option);
- abandon the project if it is unsuccessful (an abandoning option); or
- wait and learn, resolving uncertainty, before investing (a deferring option).

Observe in Figure 2 that the value obtained from the traditional (static) NPV is represented by a straight line, while the value obtained from the RO technique is represented by a concave curve. The reason for this is that the NPV value is a linear function of the expected cash flows generated over the operating life of the mine, while the RO is a non-linear function of these cash flows. In fact, as shown in Figure 3, the term that makes the RO approach a non-linear function of cash flows is the flexibility value (see Equations 1.1 and 1.2), which interweaves the effects of time and uncertainty on valuation and decision making, and is a function of the volatility of the project's value. Specifically, if two projects have the same payoff function but different risk in their values, the project with more risk will generate more outcomes with a positive payoff than the one with low risk in its value.

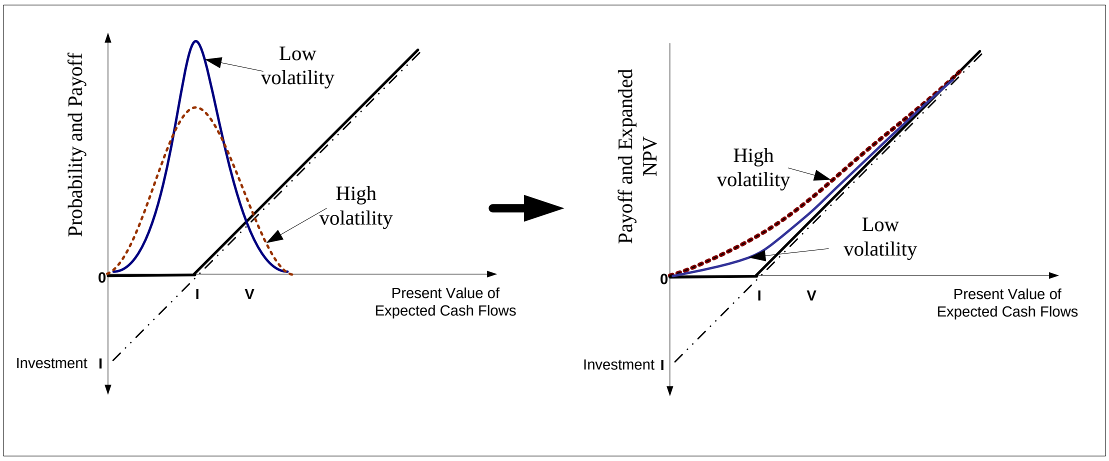
Probability and Payoff
Low volatility
High volatility
0
I
V
Present Value of Expected Cash Flows
Investment I
Payoff and Expanded NPV
High volatility
Low volatility
0
I
V
Present Value of Expected Cash Flows
Investment I

Figure 3. Total risk and RO project value. As observed in the diagram, an increase in total risk widens the distribution of outcomes (left diagram), creating more outcomes with a positive payoff and consequently an increase flexibility value (right diagram). Adapted from Amram and Kulatilaka (1999, p35).

Something interesting to note is that both the RO and DCF valuation techniques have many similarities. Both techniques examine cash flows, focusing on the effects of cash flows' timing and uncertainty on project value (Samis, Laughton and Davis, 2005). However, the method of determining the effect of cash-flow uncertainty on project value is the principal difference between these two valuation techniques (Samis, Laughton and Poulin, 2003).

One key element of RO techniques is that they incorporate the uncertainty of the underlying variables of the mine project into the valuation process. This procedure allows the mine analyst to value operational flexibility and the mine planner to build robust mine designs in the face of uncertainty (Martinez, 2003).

One characteristic of real options is that it works only when costs are sunk[6] and returns uncertain. So, the option's value is determined by the difference of cash flows (returns) to investment cost or exercise price (the less you pay the better).

Another characteristic of RO is that its holder does not lose from increased uncertainty if things turn out wrong, but gains if they turn out right; that is more uncertainty increases the likelihood of larger positive payoffs, and therefore the value of an option, as larger down-sides can be avoided (see Figure 3). This trade-off between risk and profit is achieved by implementing flexible investment strategies such as ending the project if future conditions are unfavourable or expanding production capacity if future conditions are favourable, over the operating life of the project.

However, despite the theoretical attractiveness of the RO techniques, its use by managers and corporate members of mining firms is still limited (Armstrong and Galli, 1997; Amram and Kulatilaka, 1999). One of the reasons for this limitation is that, contrary to the DCF technique, the RO technique is difficult to implement due to the advanced mathematical and statistical concepts that need to be used when dealing with uncertainty and risk (Blais, Poulin and Samis, 2004). For this reason, during the last twenty years, new methodologies based on numerical approaches such as the Binomial model (Cox, Ross and Rubinstein, 1979) and Monte Carlo simulation techniques (Tilley, 1993; Barraquand and Martineau, 1995; Gravet, 2003) have been developed in order to facilitate the implementation of RO in project valuations.

Perhaps the most significant reason that limits the application of the RO techniques in open pit mine project evaluation is that this method considers metal prices as the only source of uncertainty, assuming that technical aspects such as the geology of the orebody and the planning and design of the open pit mine are fixed, well defined and known *ex-ante*.

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[6] Investment expenditures are sunk costs when they are firm or industry specific which, once incurred, cannot be recovered. A sunk cost cannot be altered and is therefore irrelevant for decision-making purposes (Dixit and Pindyck, 1994, p8; Bilodeau, 2000, p16).

In light of the above information, it is clear that there is an urgent need to develop a new practical framework for open pit project capital budgeting and decision making, which is able to account for technical and economic uncertainties, as well as operational strategies. Furthermore, in order to enable owners and stakeholders of mining firms to make rational strategic investment decisions in the face of uncertainty, this new open pit project evaluation framework needs to have the following characteristics:

- It needs to be practical and relatively easy to implement, so it can be used as the standard tool for project evaluation;
- It needs to be flexible and generic in its structure, so that the different sources of uncertainties that affect the value of an open pit mine project can be incorporated in the evaluation process;
- It needs to be easy to update so new technologies for dealing with economic or geological uncertainties can be used when they supersede current ones; and
- It needs to be auditable, reportable, and repeatable. That is it has to be a *white box* where the internal transfer function’s process is available for inspection.

1.3 Thesis aim and objectives

In Section 1.2, it was outlined that current approaches for valuing open pit mine projects are limited by the rigidity of their structure, which does not allow them to account for 1) the geological uncertainty of the orebody and its effect on open pit mine planning and design, and 2) the uncertainty in economic variability of metal prices and production costs at the same time. It was concluded that, in order to allow the owners and stakeholders of an open pit mine project to make rational strategic decisions, new practical evaluation frameworks with the characteristics mentioned at the end of Section 1.2 need to be developed.

Consequently, the main objective of this dissertation is to develop a novel open pit mine evaluation framework (formally introduced in Chapter 4), called the Integrated Valuation/Optimisation Framework (IVOF), in which operational and managerial strategies, the uncertainties of the orebody model, metal prices and costs, and their effect on open pit mine planning and design are integrated into the evaluation process. This will be a significant advance on current methods which are "static" in nature. This contrast will be argued throughout the thesis.

As it will be shown, this new evaluation approach is founded on three key components. The first component includes the key project indicators to be considered in the evaluation process, such as ore tonnes, amount of metal, or cash flow generated at each production period. The second component includes the decision-making criteria, which are based on the performance of key project indicators, such as minimum acceptable risk in meeting given production targets, or minimum acceptable cash flow generated at each production period. The third component includes the operational and managerial strategies that can be adopted throughout the operating life of the open pit mine such as optimising cut-off grades and pit slope angles or deferring the initial capital investment, abandoning, expanding, or selling the project depending of the future technical and economic conditions, respectively.

In addition, one important characteristic that the proposed IVOF has besides the characteristics mentioned at the end of Section 1.2 is that it deals with operating and managerial flexibility and alternative strategic decision-making criteria in the context of *"optimal"* project value, where:

- The open pit mine design, the long-term production schedules, and the operational and managerial strategies are optimised in the face of uncertainty; and
- The cash flow downside risk is minimised in the short-term, while maximising the upside potential for profits in the long-term.

To achieve this objective, this research considers recent techniques developed for open pit mine project evaluation that are based on the optimisation of the open pit mine planning and design in the face of geological (metal grade) uncertainty. The purpose of doing this is to incorporate these techniques – that is, open pit mine optimisation techniques – with RO techniques into a single integrated evaluation framework that allows the mine analyst to assess and manage risk throughout the project's operating life.

Another important characteristic of the proposed evaluation framework is that it is not a black box; that is it is a system which is not only characterised by its inputs and outputs, but by the procedures in which the inputs are assessed, analysed and transformed to render the outputs of the system.

1.4 Relevance of the research to the mining industry

Today's open pit mine operations are characterised by having: i) low metal grade; ii) large capital requirement and reduced operating costs; and iii) large-scale production (Martinez, 2003; Ravenscroft, 1992). Consequently, when the option to invest in an open pit mine project exists, there is little room for inefficiency in mine planning and production scheduling and, consequently, in the estimation of the mine project's value. In fact, a doubtful and badly implemented mine evaluation process could misclassify a non-profitable venture as being profitable with appreciable probability, and *vice versa*. In fact, the consequences of a badly implemented mine evaluation process can be translated into the loss of millions or even billions of dollars for both the owners and stakeholders of the mine project.

The importance of developing a new evaluation approach, such as the one proposed in this research, resides in bringing to the mining industry a practical evaluation framework that allow the mine planner or analyst to obtain a more realistic estimate of an open pit project value in which technical and economic uncertainties are accounted for. Furthermore, this new approach will allow the owners and stakeholders of the mine to make rational investment decisions in the face of uncertainty by assessing and minimising the risk of the mine project while taking advantage of future potential for profit.

1.5 Scope and limitations of the research

As mentioned in Sections 1.2 and 1.3, one important characteristic of the proposed open pit mine evaluation framework, IVOF, is that it is a generic procedure, which is flexible in working with any specific model adopted for the underlying variables that take part in the evaluation process. However, as it is intended that this research is the first in its field, it does not cover all the problems inherent with the evaluation of an open pit mine project, leaving the door open for more improvement, and limiting its scope to the following assumptions.

- The uncertainty in the orebody model arises due to metal grade variation only.
- The algorithm used for pit design is the Lerchs-Grossman algorithm (Lerchs and Grossmann, 1965), which is based on graph theory (see, for example, Martinez, 2003, for a practical explanation of the algorithm).
- The mine operation is seen as a discrete event in which the duration of the production periods given by the long-term extraction sequence are the periods to be used in the evaluation analysis.
- Mill and mine capacity are estimated at the beginning and considered to remain constant over time.
- Although open pit mine projects potentially embody many types of operational flexibility, this research deals mainly with the flexibility of closing or abandoning the mine project at any time if future technical and economic conditions are unfavourable.
- Although mine, processing, refining and fixed costs are assumed to be known, the total production cost is seen as a random variable that changes over time[7].
- The mine cannot sell the final product at any time but at the specified contract period[8], that is, at the end of the production period.

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[7] In Section 3.4.1 it is explained how to determine the total production cost and why it is considered a random variable, with more detail.

[8] These types of contracts are known as forward or future contracts (see Appendix A for more details). As it will be explained in Chapter 5, a direct consequence of this constraint is that metal price uncertainty is only considered at the end of each production period, that is, the metal extracted during

- Cut-off grade optimisation is considered only at the base-case mine design (see final comments in Chapter 6 for more details).

- Stock piling is considered as an operational flexibility during the open pit mine evaluation process.

Although our results work with the above assumptions, the relaxation of these assumptions to more general and complete cases is indicated as future directions for research in the last chapter of this thesis.

Due to the complexity of the open pit evaluation problem, which involves advanced topics such as geostatistics, statistics, economics, finance, and mining engineering techniques, this research is also limited by the demands of time to develop appropriate computer programs for the different tasks involved in evaluating an open pit and in implementing the proposed technique appropriately. For this reason, a simplified example of the application of the IVOF is given in Chapter 7. Again, we discuss extensions at the conclusions of this thesis. Another reason for giving a simplified example was the lack of mining software tools which were (and still are) not available to us when developing this thesis.

1.6 Structure of the thesis

This thesis comprises eight chapters and an appendix section.

Chapter 2 reviews the literature relevant to the IVOF model. To give a comprehensive presentation of this literature review, this chapter is divided into three main parts. The first part explains the paradigm shift from the traditional capital budgeting technique, based on the NPV model, to the real options approach with special emphasis of its application to mine project evaluation. The second part examines in detail open pit evaluation techniques based on the optimisation of the mine in the face of orebody uncertainty and cut-off grade selection. The final part of this chapter summarises the topics discussed in the literature review and concludes the chapter.

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the early stage of a production period will have the same selling price than the metal extracted at the end of the same production period.

Chapter 4 derives and explains the foundations of a novel open pit mine evaluation framework called the IVOF. As it will be shown in this chapter, this new framework is composed of the following stages: i) data analysis and base-case mine design; ii) effect of orebody uncertainties, including production cost uncertainty, on base-case mine design; iii) effect of future metal price uncertainty on base-case mine design; iv) practical optimal mine design based on uncertainty and risk; and v) decision-making based on final results.

As it will be shown in Chapter 4, the key part in the IVOF process is the generation of the long term (strategic) plan and design of the open pit mine project, formally referred to as the base-case mine plan and design. This corresponds to Stage 1 of the IVOF. Although the base-case mine plan and design is based on traditional mine evaluation techniques, Chapter 3 gives a summary of this process highlighting common mistakes that are incurred throughout the process.

Chapter 5 provides a comprehensive explanation of Stage 2 of the IVOF - the effect of orebody uncertainties, including production costs uncertainty, on base-case mine design. Furthermore, this section also shows how to assess the effect of orebody modelling uncertainty on the base-case mine planning and design. As it will be demonstrated in this chapter, both OK and SGCS use the initial drill-hole data set to infer the probability distribution of metal grade values at each specific location within an orebody model.

Chapter 6 provides a comprehensive explanation of Stage 3 of the IVOF - the effect of future metal price uncertainty on base-case mine design. This chapter shows how to model metal price uncertainty in the mining industry. To achieve this, two parametric techniques widely used in the industry for forecasting metal prices are described: Geometric Brownian motion (GBM) and the mean reversion (MR) process. As it will be discussed in this section, these parametric techniques are very useful to model future metal prices when the values of their parameters are well known (the estimation of the parameters of each model is briefly discussed in the Chapter).

Chapter 7 presents a case study where the profitability of a gold mine project is estimated under uncertainty and risk, to decide if it is worth investing or not in the mine venture. In this case, the drill-hole data of a disseminated gold deposit, called the

“Gold-Mine Project”, is used as the case-study. As it will be shown in this chapter, the Gold-Mine Project will be evaluated considering the flexibility of abandoning the project at any time with and without a salvage value.

Chapter 8 gives concluding remarks on the use of the proposed valuation framework and its importance in open pit mine project evaluation, and recommendations for future research directions.

Appendix A explains the fundamentals of stochastic processes with emphasis on expectations calculated from conditional distributions.

Appendix B explains the fundamentals of financial option pricing theory.

Appendix C gives a summary of some selected mine valuation and optimisation procedures which are: C.1) the CA/BDH/Smith real options model; C.2) the Longstaff-Schwartz Least Square Monte Carlo; C.3) the Upside/Downside optimisation model; and C.4) Lane’s cut-off grade optimisation.

Appendix D gives two unpublished manuscripts which are technically but not directly related to this thesis, and which were completed during the candidature.

The following papers have arisen from this thesis.

- Modelling the initial pit design: The first step for mine project valuation (2006). 6th International Mining geology Conference pp. 269-281. Darwin-Australia
- Understanding real options in mine project evaluation: A simple perspective (2010). Minin 2010 pp. 223-234. Santiago de Chile
- Orebody Modelling-Estimation vs. Simulation – A practical viewpoint. 33 Application of Computers and Operation Research in the Mineral Industry APCOM (2007). Pp.721-728
- Designing, planning and evaluating and open pit gold mine under *in situ* metal grade and future gold price uncertainties. Orebody Modelling 2009 Conference pp. 225-234 (The Australasian Institute of Mining and Metallurgy, Melbourne).
- Why Accounting for uncertainty and risk can improve final decision making in strategic open pit mine project evaluation. Project Evaluation 2009 conference pp. 1-12, Melbourne- Victoria.

Chapter 2

Literature Review

2.1 Introduction to open pit orebody modelling

As it was outlined in Section 1.1, the description and characterisation of a mine deposit represents one of the most critical sources of technical risk in an open pit mine project and in a mine project in general.

Normally, the geological characterisation of a mineral deposit is represented in a model of the orebody. The orebody model and surrounding waste material is modelled by dividing it into blocks of a certain size, or Selective Mining Units (SMUs), so that the flow and form of material in the mining process can be representative of reality. In the orebody model, each block contains important information from limited drilling that will be used later in developing the future mine operation. A 3D multi-parametric orebody model (Bye, 2006) is populated with geotechnical, metallurgical, blast index data[9] and metal grade information.

To estimate an orebody model, in general it is required to have information of what the orebody characteristics (such as metal grade) may be at unsampled locations. However, as the information from the drill holes is limited, it is impossible to know with certainty the mineral content in each block. This is the reason why, in any mine operation, there will always exist a divergence between estimated and real values. One consequence of these divergences is that the mineral deposit will be represented with the uncertainty that a given block can be mined for a profit. This unavoidable discrepancy is what is meant by that the saying “expectations sometimes may become a sad reality” (Sinclair and Blackwell, 2002; Krige, 1951).

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[9] Current multi-parametric orebody models are constructed assuming independency between metal grade, geotechnical and metallurgy data.

To reduce the differences between estimated and real values, the geologist normally resorts to techniques based on spatial statistics (geostatistics) and computational methods, which model the spatial uncertainty of the orebody as a stochastic process. These techniques can be categorised into two groups. The first group corresponds to the estimation method based on deterministic linear techniques, such as Ordinary Kriging (OK) (David, 1973; David, Dowd and Korobov, 1974), which is a conditional expectation and accounts for local variability. The second group corresponds to the estimation method based on conditional simulation techniques, such as Sequential Gaussian Simulation (SGS), which is based on the Monte Carlo method and accounts for global variability. Some early investigations of this came from the French school, such as Deraisme and Dumay (1981) and Thwaites (1998), although these did not consider prices and costs to be uncertain.

But, which technique is the best for orebody modelling? Is one technique better than the other? These are precisely the questions that the topic of this chapter analyses and discusses from a simple and practical view point. The chapter will also show that the use of conditional simulation not only assists the geologist to model the orebody, but it also assists the mine analyst or planner to both integrate the orebody model uncertainty in the mine evaluation process and to perform a risk analysis of the mining project, enabling strategic investment to be sheltered while exposing the potential of the mine project.

2.2 Introduction to discounted cash flow, real options and mine optimisation techniques

In Chapter 1, a new open pit mine project evaluation framework, called the integrated valuation/optimisation framework (IVOF), was foreshadowed as an alternative to current evaluation techniques, which are limited in giving realistic estimates of the value of an open pit mine project.

The current approach to open pit mine project evaluation is as follows. In high level terms, it commences with an estimated block model. Typically, the blocks will have parameters for grade and tonnes, with very few other parameters, which are normally point estimates. These are then used together with estimated costs and metal prices, and other engineering factors such as slope angles and bench design – again usually all point estimates – to plan and design the base case open pit mine. What we are emphasising is that variability modelled at each stage of the value chain is very often sacrificed, just leaving mean values; or at best over-simplified information about error is carried forward for use in limited sensitivity analyses. Hence, integration of information (including risk factors) is compromised by "silo-like" operations especially when it comes to preserving and valuing decision options.

The final stage is the long term production schedule where the project NPV is obtained. As a consequence of the above scenario, NPV is static in as far as it does not (and cannot) reflect risk and flexibility of decision making. This is the basis for reviewing antecedents of the IVOF in the present chapter.

2.3 Organisation of the literature review

The chapter is organised as follows. The first part of our review concerns geostatistical methods and their relationship with mine planning. Section 2.4 gives a basic definition of random variables and random functions. The objective of this section is to equip the reader with basic concepts that will be used later on when defining the concepts of Ordinary Kriging and Sequential Conditional Simulation, given in Section 2.5. Section 2.6 presents a simple example in which it is shown why conditional simulation techniques give a better characterisation of the variability of the orebody properties than the estimation technique. (We show in Chapter 5 how both estimation and simulations of an orebody can be used to mine plan and design the open pit mine and to assess the risk of the mine project, which will allow the mine analyst to make strategic decisions that maximise the potential of the mine while minimising the risk of future losses).

We continue our review with the relevant background and literature for the IVOF, which will be defined and discussed formally in Chapter 4. We first explain in Section 2.7 the paradigm shift from the traditional capital budgeting technique, based on the DCF model, to the RO approach, with special emphasis of its application to open pit mine project evaluation. Then Section 2.8 examines open pit evaluation techniques based on the optimisation of the mine in the face of orebody uncertainty and metal grade selection.

Section 2.9 summarises the topics discussed in the literature review, highlighting the advantages of using all these techniques as an integrated process, and concludes the chapter.

2.4 Discrete representation of spatial random variables and random functions

A random variable (RV) Z_i is a function that assigns a real number, z_i, to each outcome *i* in the sample space of a random experiment. In other words, Z_i is a real valued function defined over the elements of a sample space based on some probability distribution. A random variable Z_i is discrete if it assumes only a finite or countable infinite number of values, and is fully characterised by its cumulative density function (*cdf*) $F(z) = P\{Z_i \leq z\}$, which gives the probability that the random variable Z_i at location *i* is not greater than any given limit z.

A spatial random function (RF) $Z\left(x_i\right)$ is defined as the collection of random variables Z_i defined at each location $x_i \in \mathbb{R}^k$ over some domain $A \in \mathbb{R}^k$ of interest, that is, $\left\{Z\left(x_i\right)\right\}_{i=1}^{n}, \forall x_i \in A$. In simple words, a random function is a multivariate process composed of n random variables Z_i ($i = 1, 2, .., n$). Random functions are totally characterised by their multivariate cumulative density function (cdf), given by

$$F\left(z_1, z_2, ..., z_n; x_1, x_2, ..., x_n\right) = \Pr\left\{Z\left(x_1\right) \leq z_1, ..., Z\left(x_n\right) \leq z_n\right\} \tag{1.1}$$

The RF $Z\left(x_i\right)$ is said to be “strictly stationary” if all the RV’s, $Z_i, \forall i = 1, 2, ...n$, in the space *A*, have the same distribution, $F_Z\{z; x \mid (n)\}$. In earth science, a stationary RF normally refers to a second order stationary (weak stationary) RF, that is, the mean is constant and the covariance function only depends on the separation, h, between two points and not on the location x_i.

$Z\left(x_i\right)$ is an intrinsic, zero-drift, random function if

$$\begin{aligned}&E\left\{Z\left(x_i+h\right)-Z\left(x_i\right)\right\}=0\\&\frac{1}{2}Var\left\{Z\left(x_i+h\right)-Z\left(x_i\right)\right\}=\gamma\left(h\right),\end{aligned}\tag{1.2}$$

where the function $\gamma\left(h\right)$ is called the "variogram". A spatial stationary RF $Z\left(x_i\right)$ is said to be "ergodic" in the parameter m, usually taken as the stationary mean of the RF, if the average of its corresponding n realisation statistics $m^{(l)},\forall l$, converges to m as the size of the domain space A increases. For more details about stationary and ergodic random functions, the reader is referred to general books on geostatistics, such as Goovaerts (Goovaerts, 1997), and Journel and Huijbregts (Journel and Huijbregts, 1978).

For a RF $Z\left(x_i\right)$ with multivariate probability density function as in Equation 5.1 the following general expression is valid:

$$\begin{aligned}F(z_1,z_2,..,z_n;x_1,x_2,...,x_n)=F(z_1;x_1)&\\&\times F(z_2;x_2\mid Z(x_1)=z_1)\\&\times F(z_3;x_3\mid Z(x_1)=z_1,Z(x_2)=z_2)\\&\times..........\\&\times F(z_n;x_n\mid Z(x_1)=z_1,Z(x_2)=z_2,.....,Z(x_{n-1})=z_{n-1}),\end{aligned}\tag{1.3}$$

where the conditional n-variate distribution function is the product of its univariate marginal conditional distribution functions. Then, the n-variate cdf of a RF $Z\left(x_i\right)$ conditioned to a specific set of data N is :

$$\begin{aligned}F(z_1,z_2,..,z_n;x_1,x_2,...,x_n\mid N)=F(z_1;x_1\mid N)&\\&\times F(z_2;x_2\mid Z(x_1)=z_1;N+1)\\&\times F(z_3;x_3\mid Z(x_1)=z_1,Z(x_2)=z_2;N+2)\\&\times..........\\&\times F(z_n;x_n\mid Z(x_1)=z_1,.....,Z(x_{n-1})=z_{n-1};N+n-1).\end{aligned}\tag{1.4}$$

The first moment of a spatial stationary and ergodic RF is given by

$$E\left\{Z\left(x_i\right)\right\}=m;\forall x,\tag{1.5}$$

and the second moment is

$$C(h) = E\{Z(x), Z(x+h)\} - m^2; \forall x. \tag{1.6}$$

Normally, in earth sciences, the random variable, Z_{x_0}, expressing the value of an specific attribute at an unsampled location x_0 is modelled as the sum of a smooth deterministic function, $Z^*_{x_0}$, describing the systematic aspect of the phenomenon, and a zero-mean random function, called the error or residual, $R(x_0)$, that is:

$$Z_{x_0} = Z^*_{x_0} + R(x_0). \tag{1.7}$$

Then, from Equation 5.7, it is observed that to have information about the behaviour of the random variable Z_{x_0} it is necessary first to generate a model for both $Z^*_{x_0}$ and $R(x_0)$; observe that the error $R(x_0)$ is a state-dependent error. This is the topic of the next section.

2.5 Ordinary Kriging (OK) and Sequential Gaussian Simulation (SGS)

2.5.1 Ordinary Kriging (OK)

The OK is an unbiased linear estimation technique used in orebody modelling to estimate the value, $Z^*_{x_0}$, of a specific geological attribute, such as metal grade, at an un-sampled location x_0 (see Equation 5.8). One important characteristic of this technique is that it is an exact interpolator which honours the data values at data locations, that is, if one of the known samples Z_{x_j} is considered to be unknown, OK will successfully identify the true value of the observation with variance equal to zero (Journel, 1989; Wackernagel, 2003). Another characteristic of OK is that it accounts for localised variations in the mean by limiting the domain of stationarity to the mean of the local neighbourhood D. The local neighbourhood D is the set of points, x_j, where the samples, Z_{x_j}, of RF $Z(x_i)$ are used to estimate $Z^*_{x_0}$, that is,

$D = \{x_j : j = 1,..,N\} \subset A \in \mathbb{R}^n$. OK is an estimator for which the mean is unknown but it is assumed to be constant within the local domain D.

In this technique, the estimated value, $Z^*_{x_0}$, is expressed as a weighted linear combination of the data, expressed as random variables Z_{x_j} ($\forall x_j \in D$) available at the sample locations x_j, that is,

$$Z^*_{x_0} = \sum_{j=1}^{n} \lambda_j Z_{x_j} . \tag{1.8}$$

In Equation 5.8, λ_j are the weight coefficients assigned to each of the $j = 1,2,..,n$ available observations inside the domain D. One characteristic of the coefficients λ_j is that because

$$E\{R(x_0)\} = E\{Z_{x_0}\} - E\{Z^*_{x_0}\} = 0 , \tag{1.9}$$

the sum $\sum_j \lambda_j$ is equal to the unity and $\lambda_j \geq 0; \forall j$. This can be verified by taking expectation of both sides of Equation 4.8 and using Equation 4.9, that is,

$$\begin{aligned}
&E\{Z^*_{x_0}\} = E\left\{\sum_{j=1}^{n} \lambda_j Z_{x_j}\right\} \\
&E\{Z^*_{x_0}\} = \sum_{j=1}^{n} \lambda_j \cdot E\{Z_{x_j}\} \\
&E\{Z^*_{x_0}\} = E\{Z_{x_j}\} \sum_{j=1}^{n} \lambda_j \\
&\sum_{j=1}^{n} \lambda_j = 1.
\end{aligned} \tag{1.10}$$

As observed in Equation 5.8, to find the estimated grade, $Z^*_{x_0}$, as well as the respective error variance, $Var\{R_{x_0}\}$, which characterise the distribution of the random variable Z_{x_0}, it is necessary to know the values of the ordinary kriging weight coefficients $\lambda_j, \forall j = 1,2,3,...,n$.

To estimate the weight coefficients, λ_j, the OK technique minimises the error variance, $Var\{R_{x_0}\}$, constrained as in Equation 5.10. The variance of the error or residual $R(x_0)$, defined as (see Equation 5.7):

$$R(x_0) = Z_{x_0} - Z^*_{x_0}, \tag{1.11}$$

is expressed as:

$$\begin{aligned} Var\{R(x_0)\} &= Var\{Z_{x_0} - \sum_{j=1}^{n} \lambda_j Z_{x_j}\} \\ Var\{R(x_0)\} &= \left(\bar{\sigma} Z_{x_0}\right)^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j Cov\left\{Z_{x_i} Z_{x_j}\right\} - 2\sum_{j=1}^{n} \lambda_j Cov\left\{Z_{x_j} Z_{x_0}\right\}, \end{aligned} \tag{1.12}$$

where $\left(\bar{\sigma} Z_{x_0}\right)^2 = Var\left\{Z_{x_0}\right\}$.

To minimise the error variance, OK uses the technique of Lagrange multipliers, which introduces a new variable, μ, into the error variance equation and then minimises it. Note that the new expression of the error variance after introducing the Lagrange variable is expressed as

$$Var\{R(x_0)\} = \left(\bar{\sigma} Z_0\right)^2 - \sum_{j=1}^{n}\sum_{i=1}^{n} \lambda_j \lambda_i Cov\left\{Z_{x_j} Z_{x_i}\right\} - 2\sum_{j=1}^{n} \lambda_j Cov\left\{Z_{x_j} Z_{x_0}\right\} + 2\mu\left(\sum_{j=1}^{n} \lambda_j - 1\right). \tag{1.13}$$

The result of using kriging for orebody modelling is an estimated model in which each block has an estimated value of the pertinent attribute, such as metal grade, with a minimum variance.

2.5.2 Sequential Gaussian Simulation (SGS)

SGS simulation is a technique which is based on the decomposition of the multivariate conditional distribution function of the RF $Z(x_j) = \left\{Z(x_j)\right\}_{j=1}^{n}$ (Martinez, 2003), which is defined as :

$$F(z_1, z_2, .., z_N; x_1, x_2, ..., x_N \mid n) = F(z_1; x_1 \mid n) \tag{1.14}$$

$$\begin{aligned}
&\times F(z_2; x_2 \mid Z(x_1) = z_1 \mid (n+1)) \\
&\times F(z_3; x_3 \mid Z(x_1) = z_1, Z(x_2) = z_2 \mid (n+2)) \\
&\times .... \\
&\times F(z_N; x_N \mid Z(x_1) = z_1, ..., Z(x_{N-1}) = z_{N-1} \mid (n+N-1)).
\end{aligned}$$

Observe that the process indicated in Equation 5.14 is a dynamic process in which the simulated value is aggregated to the original data set before performing the next simulation (see Algorithm 1).

In this context, to generate N simulations of the ore deposit, the technique samples the multivariate distribution given in Equation 5.14 in a sequential fashion. The practical difficulty is that, in general, we do not know how to calculate the conditional probabilities involved in Equation 5.14, except in the ideal case of a random Gaussian world. Then, for a Gaussian RF with known mean, the conditional distribution of Z_{x_i} is Gaussian and given by

$$Z_{x_i} = Z^*_{x_i} + \sigma_{x_i} \varepsilon_{x_i}, \tag{1.15}$$

where $Z^*_{x_i}$ is the ordinary kriging estimator of Z_{x_i} (see Equation 5.8), $\sigma^2_{x_i}$ is the associated kriging variance (see Equation 5.12), and ε_{x_i} are independent and identical standard normal distributed innovations, that is, $E\left\{\varepsilon_{x_i}\right\} = 0$ and $Var\left\{\varepsilon_{x_i}\right\} = 1$.

Then, if the domain $A \in \mathbb{R}^3$ is discretised in a fine grid of M nodes, the general algorithm to simulate the M nodes in the domain A is as follows.

Algorithm 1

1) Define a random path, to be followed, visiting every location (node) $x_{(i)}$ being simulated. At this stage, there are M grid nodes to be simulated and n observations available.

2) At each location x_i, build the conditional cumulative distribution function (ccdf) of Z_{x_i} given the initial n observations and all $i-1$ previously simulated values, and draw a simulated value, z_i, from this distribution. This is done by randomly sampling the distribution of the innovation ε in Equation 5.15.

3) Include this new value (z_i) into the initial data set and loop through steps 2 and 3 until the M nodes are visited and simulated.

4) Repeat steps 1, 2, and 3, N times to render N simulations or images of the mineral deposit.

The result of using simulation techniques for orebody modelling is a set of equally probable images of the deposit, which will contain more information about the characteristics of the orebody model, at each location x_i, than the estimated model resulting from using linear estimators, such as kriging. That is now each block inside the block model will be characterised by a probability distribution function rather than a single estimated value. As it will be shown later in Sections 5.4 and 5.5, the additional information about the uncertainty of the geology of an orebody, generated by simulation techniques, will be of great assistance to the mine planner when assessing the risk on the economic and technical indicators of the resulting mine project.

2.6 Estimation vs. Simulation

As outlined in the previous section, kriging will always provide a single estimated model of an orebody. One characteristic of this estimated model is that each block will be characterised by an expected value of the pertinent geological attribute, such as metal grade and rock porosity, among others. Another characteristic of this estimated model is that it will not reproduce the model of spatial variability (covariance model) inferred from the data set. The reason for this is that kriging, or linear interpolator techniques in general, always estimate the conditional mean of the pertinent geological attribute by minimising the local error variance given in Equation 1.13. That is these linear interpolator techniques try to give an exact value of the geological attribute at each location x_i, which in reality has zero variance. This result, however, is at the expense of providing a measure of the spatial variability.

On the contrary, simulation techniques always provide, at each location, a set of alternative equally probable realisations of the pertinent geological attribute spatial distribution. Similarly to OK, simulation techniques also reproduce the data values at their respective locations; however, contrary to OK, simulation techniques also reproduce the data histogram and variogram models inferred from the drill-hole data set. This result, however, is at expense of providing a measure of the estimate or mean. In theory, the average of many simulations will converge to the result given by kriging.

A diagram showing the typical differences between estimation and simulation is given in Figure 4. In the figure the dots represent the given data, such as metal grades throughout a mineral deposit; the continuous line represents the true metal grade; the dashed line represents the best estimation given by kriging; and the point-dashed lines represent two simulations based on the original data. As seen in the figure, the estimation curve is, on average, closer to the real curve (smoothed version of the curve), but it tends to underestimate high values and overestimate low values. This is because estimation is concerned with the minimisation of the error variance and since it is based on regressions it gives the best line that minimises the error variance.

In the figure it is seen that the simulation curves give a better reproduction of the variability of the data than the estimation techniques. In fact, it is seen that simulations give a more realistic panorama about the dispersion variance of the real data, and consequently a better reproduction of the uncertainty of the metal grade throughout the mineral deposit. Observe, however, that since a simulated curve can still over-estimate or under-estimate the true value, it is very important to generate many simulations and analyse the results from a probabilistic viewpoint, that is based on probabilities of occurrence.

Due to the fact that the estimation curve generated by linear interpolation techniques such as Kriging gives a model that is the closest, in terms of error variance, to the original (unknown) true geological attribute, it is normally preferable to locate and estimate mineral reserves; while the simulation curve is preferred for studying the dispersion of the characteristics of these reserves, remembering that, in practice, the real values are known only at the experimental data points x_i.

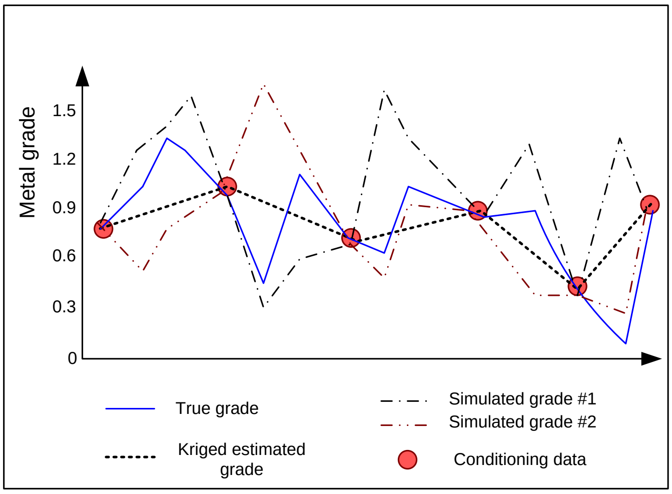


Figure 4. A comparison of idealised profiles of true grade (solid line), simulated grade (dot-dashed lines), and kriged estimated grade (dashed line) on a common profile. Note the similar variability of the true and simulated grades in contrast to the smoothed pattern of estimated grades. Also, note that all profiles pass through the known data points. (Reproduced from Journel, 1975).

But, which one is the best for orebody modelling and mine planning and design?

The answer to the previous question is not straightforward and normally depends on the attitude of both the mine planner and the owners of the mine in the face of uncertainty. If they prefer risk-neutral behaviour, then kriging is the best technique to model the mineral deposit. The reason for this is that, in a risk-neutral world, risk is not an issue; consequently the average model, given by kriging, is the best to characterise the deposit. Contrary to risk-neutral behaviour, if the attitudes of the mine planner and owners are risk-seeking or risk-averse, the simulation technique is the best for modelling the mineral deposit. The reason for this is that a risk-seeking or risk-averse attitude will use a specific model, selected from the simulations of the deposit, which is highly risky/safe in achieving specific targets such as ore tonnes, metal quantities and cash flows. Indeed, a highly risky/safe model also has large/small potential for future rewards, that is, there is a trade-off between rewards and risk.

However, regardless of the attitude of the mine planner and owners of the mine, modern mine valuation techniques such as real options require the use of simulations of the orebody rather than a single estimate. The reason is that most of these techniques are based on stochastic frameworks in which the value of a mine project is estimated under uncertainty, that is, the economic and technical project indicators, such as ore and waste tonnes, costs and cash flows, which are characterised by a distribution of probabilities rather than a single estimated value. It is important to mention that since both estimation and simulation are based on the same data (initial drill-hole data) both give the same information about the uncertainty of the deposit: the difference is that estimation techniques give information about the mean while simulations give information about variability.

The proposed IVOF uses both estimation and simulation techniques when evaluating an open pit mine project. In this context an estimated orebody model is used to build the initial open pit mine project, named the base-case mine plan and design (see Chapter 3), while simulations are used to assess the risk in the base-case design and to, if necessary, re-design the initial base-case mine plan and design.

To demonstrate the advantages and disadvantages of estimation by linear interpolation techniques, such as OK, and non-linear techniques, such as the SGS, a small example in which a 2D data set containing 78000 gold grades data points (see Figure 5), allocated to a 1m x 1m grid, is used as the true gold deposit (normally unknown at the beginning of the project) of a virtual mining project. This data set is called hereafter the exhaustive data set, is the classical Walker Lake standard data set: see, for example, Journel and Kyriakidis (2004), who have their own analysis of this data set as have other authors. From the exhaustive data set, 800 samples are collected, using a random sample grid of 20m x 20m (see Figure 6), and are assumed to represent the drill holes data, which are normally performed in a mine project to obtain information about the true value of the deposit (in this case the exhaustive data set). This drill hole data set is called hereafter the sample data set, normally known when starting a mining project.

The idea is to use this sample data set to assess the resource/reserve variability of the entire unknown gold deposit (exhaustive data set) in terms of metal grade based on two conditional simulation runs (observe that in this case two simulations have been selected for the sake of simplicity), and to compare the results with the ordinary kriging resource estimates. Figure 5 shows the exhaustive data values with their respective histogram and variogram. Figure 6 shows the location map of the (drill-hole) data set as well as its histogram and variogram. From Figures 5 and 6, it can be seen that the experimental variograms, which are plotted in two directions (NE and NW), show a range of approximately 20m for both the exhaustive and sample data sets respectively. In this case, the spherical variogram (see the end of this Chapter) model is seen to give a good representation of the spatial variability. Table 1 presents summary statistics for both the exhaustive and sample data sets. These summary statistics indicate that, in general terms, the univariate statistics inferred from the sample data set provide a realistic approximation of the distribution of the exhaustive data set.

Based on the sample data set, kriging is used to estimate the values at unsampled locations. The result is displayed in Figure 7. As shown in the figure, point ordinary kriging gives a smooth representation of the initial real deposit with well-delimited high-grade zones (in dark colour) and following a 45° North-East trend, which is also observed in the exhaustive data set (see Figure 5). These high-grades zones are due to the influence of high-grade samples located inside the range of the variogram (in this case around 20m).

From Figure 7, it can also be seen that the mean of the estimated values is 0.85 and the standard deviation is 0.81, having a maximum value of grade metal of 13.69 gr/t. Doing a comparison with the sample data set, ordinary kriging estimated the values without varying the mean much, but reducing the variance between samples by almost 54%. This smoothness, or reduction of variance, can be visualised better in Figure 8, where the variograms of both the data sample and ordinary kriging are plotted together.

A comparison of the estimated model, generated by kriging, and the real deposit (exhaustive data set) shows that, in this case, kriging succeeded in generating high-grade zones in places where high-grade samples were available at the sample data set. On the contrary, the previous comparison also shows that kriging was not able to reproduce high-grade zones at locations were no high-grade samples were available. This comparison can be visualised better in Figure 9, where the maps of the real (unknown) deposit, the sample data, and the OK estimated models are shown. In the figure, it can be seen that OK reproduced most of the values (high and low grade) surrounding the values obtained from the sample data set. For example, the zone inside the square (shown in the three maps) is a high-grade zone that was successfully reproduced by kriging due to the presence of high grade samples; consequently, this ore block will be sent to the mill. On the contrary, the zone inside the circle, which is a medium-high grade zone, was not reproduced by kriging because of the lack of medium-high grade samples inside the circle area in the sample data set; this block will be sent to the dump even though it is high grade. Thus, it is seen that ordinary kriging is locally accurate but does not reproduce the variability of the metal grade in unsampled zones.

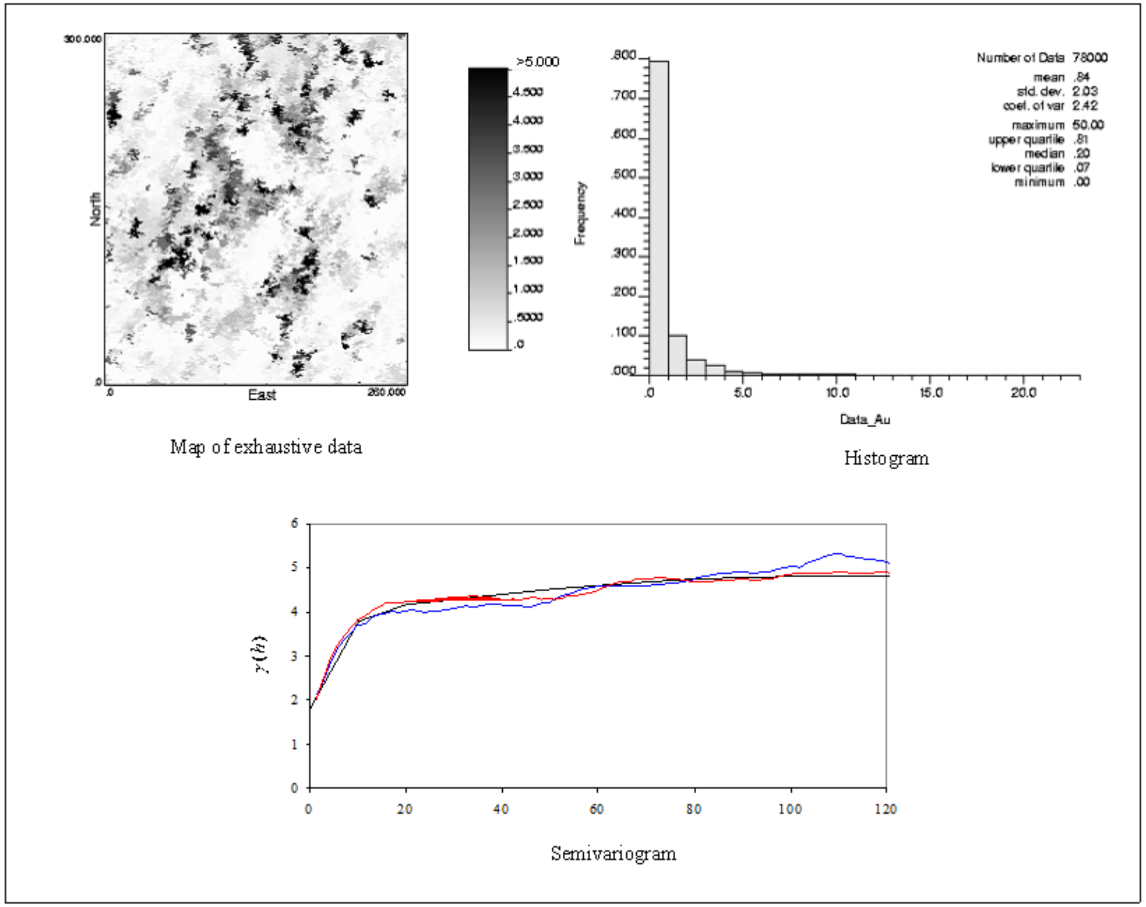
	Exhaustive data set	**Sample data set**
Number Samples	78,000	800
Mean	0.84	0.86
Quartile 25%	0.07	0.07
Median	0.20	0.19
Quartile 75%	0.81	0.79
Std. Deviation	2.03	1.79
Minimum	0.00	0.00
Maximum	50.00	13.69
CV	2.42	2.07

Table 1. Univariate statistics associated with the exhaustive and sample data sets. Observe that, in general terms, the sample data set provides a realistic approximation of the distribution of the exhaustive data set.

Conditional simulation is next used for generating the exhaustive data based on the sample data set. In this case, two simulations or images of the exhaustive data are generated using the sequential Gaussian simulation technique. Figure 10 shows the map, histograms and variograms of both simulations where the mean and the standard deviation of each simulation are seen to be 0.85, 0.87, and 1.78, 1.81, respectively.

From the figure it is clear that, unlike ordinary kriging, simulations reproduce the histograms and variograms of the data sample, giving more accurate values inside the study area.

On the other hand, simulations keep both the spatial correlation and variance of the data sample, giving a better reproduction of the characteristics of the real deposit than kriging. Thus, it is illustrated that conditional simulation techniques represent a potential tool for quantifying and assessing the uncertainty of the orebody via alternative realisations or images of the deposit.

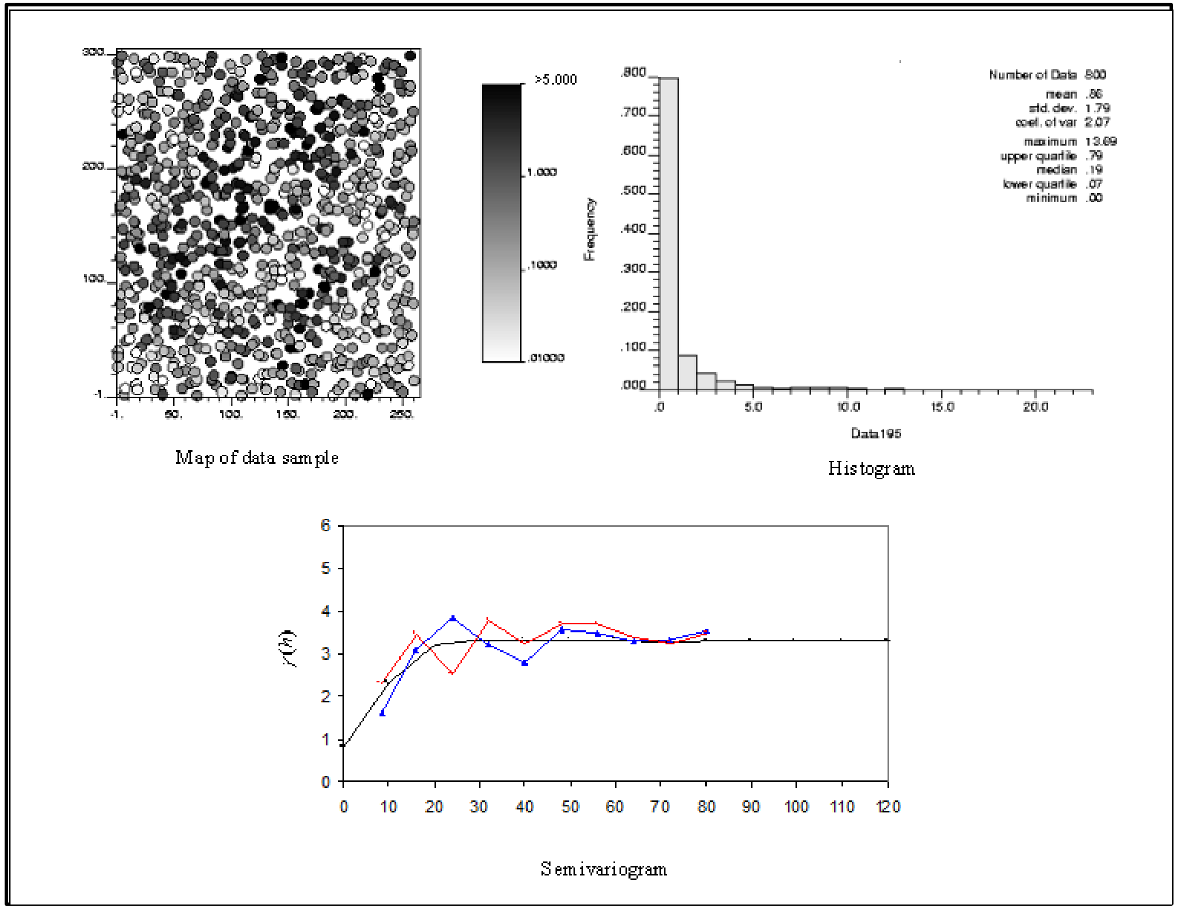


Map of exhaustive data

Histogram

Semivariogram

Figure 5. Diagram showing the exhaustive data set containing 78000 samples, and its histogram and respective experimental and fitted model variograms. In the figure, the bar next to the map of exhaustive data set indicates the variation in metal grade: dark colours indicate high grade.

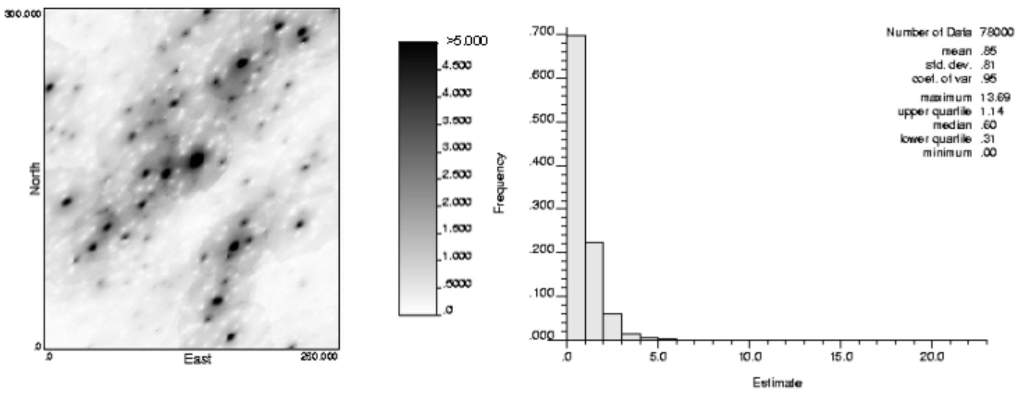


Map of data sample

Histogram

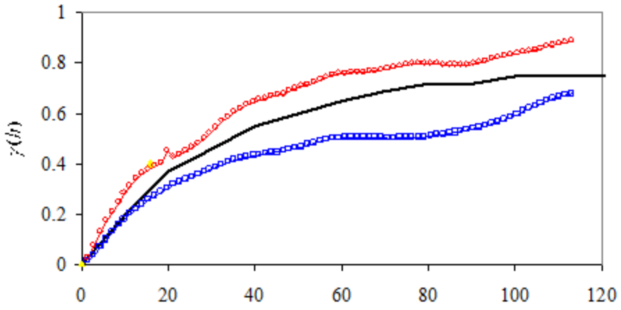
Semivariogram

Figure 6 Diagram showing the sample data set containing 800 samples, and its histogram and respective experimental and fitted model variograms. In the figure, the bar next to the map of sample data set indicates the variation in metal grade: dark colours indicate high grade.



Map of estimated values

Histogram



Semivariogram

Figure 7. Diagram showing the estimated model of the deposit with its respective histogram, experimental and fitted model variograms. In the figure, the bar next to the map of estimated values indicates the variation in metal grade: dark colours indicate high grade.

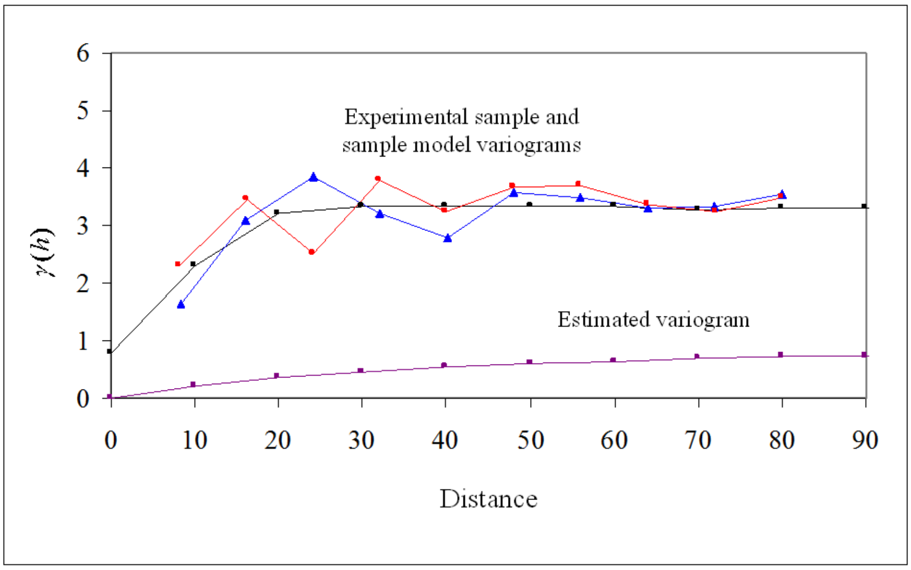
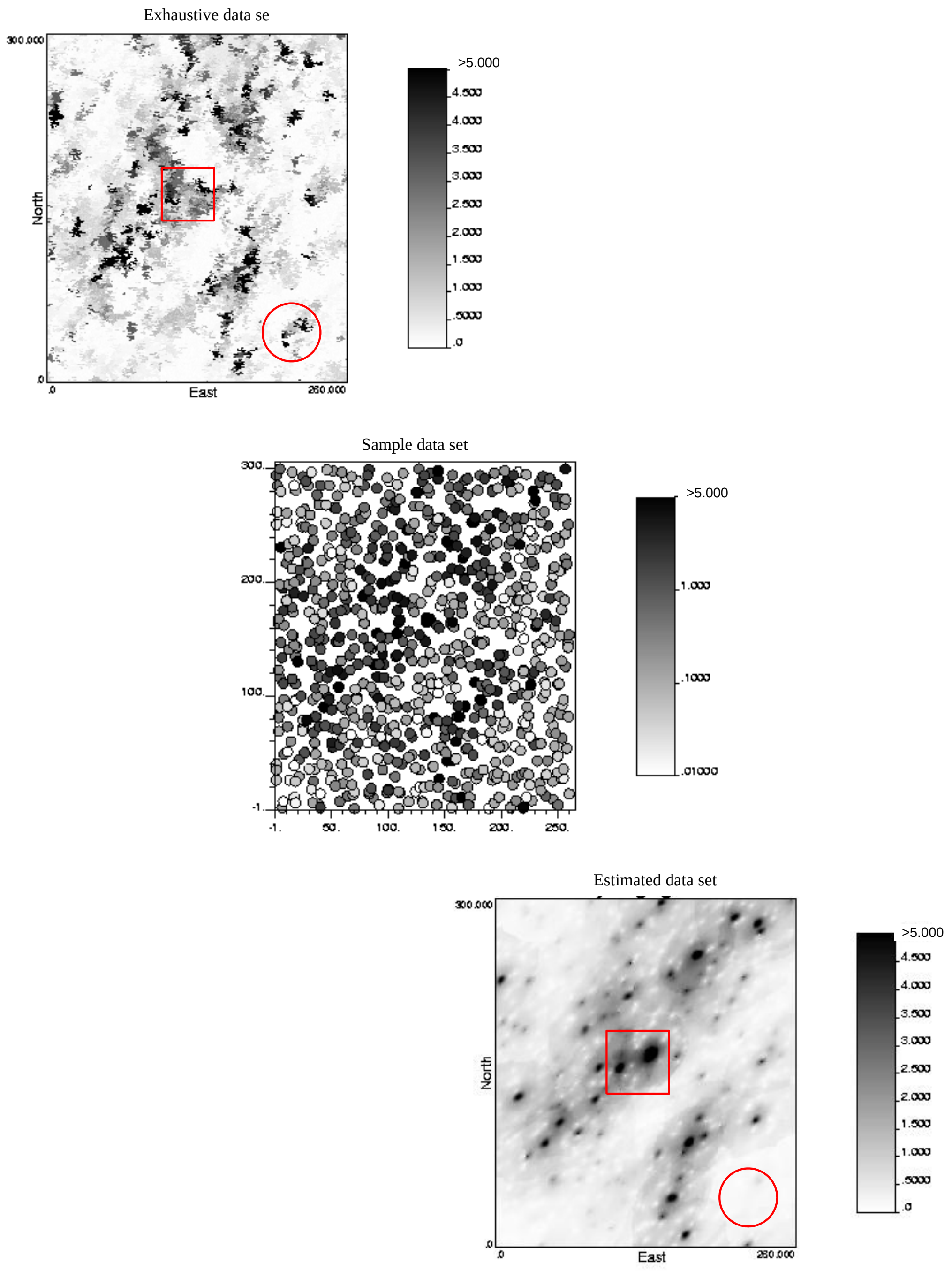


Figure 8. Diagram showing the reduction of variance that the estimation performs with respect to the sample data.


Exhaustive data se
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North
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Sample data set
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Figure 9. Diagram showing a sampled zone (in the square) where ordinary kriging gives a correct estimation and an unsampled zone (inside the circle) where ordinary kriging does not give an accurate estimation. Note that, in the middle diagram, the circles indicates drill sample sites, where the darker the circle the higher the grade.

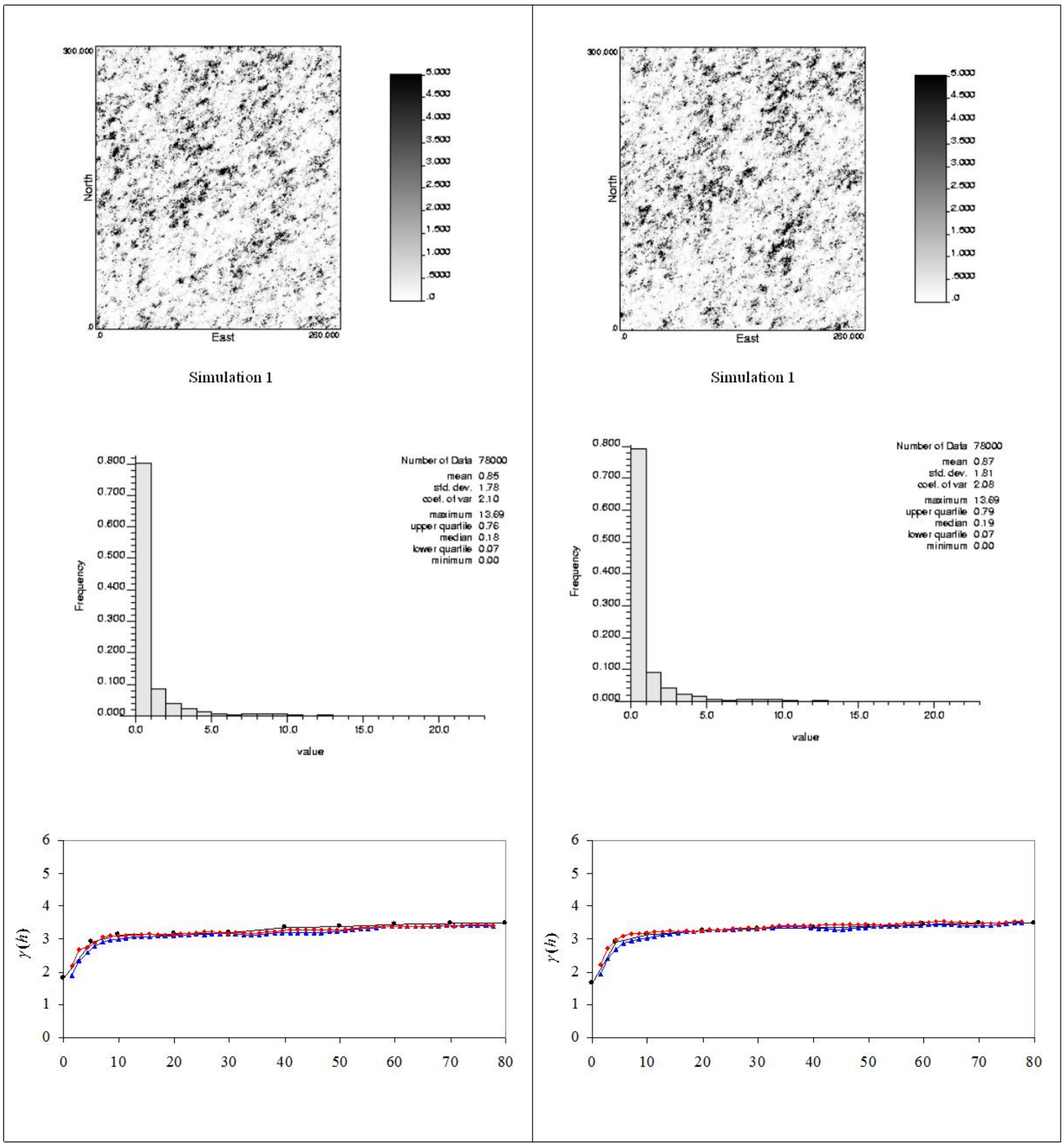

Figure 10. Diagram showing two simulations of the deposit with their respective histograms and variograms. In the figure, the bars next to the map of simulated values indicate the variation in metal grade: dark colours indicate high grade.

2.6.1 How many simulations need to be generated to accurately characterise the uncertainty of the orebody model?

When dealing with simulation techniques to assess the uncertainty of an orebody model, the following general question is frequently formulated: How many simulations should be generated to quantify accurately the orebody uncertainty?

The answer to this question is not straightforward. Even though some books specialising in Monte Carlo and Bootstrap techniques (see for example Efron, 1979) recommend to generate at least 200 simulations, a more realistic answer is to generate enough simulations of the orebody model which can approximate the continuity of the cumulative distribution function (cdf) of the geological attribute under study, for example, ore tonnes and metal content above a given cut-off grade. Figure 11 displays a scheme of the differences in generating cdf curves using one, three and many (>100) simulations of a specified geological attribute. As seen in Figure 11-A, if one simulation of the orebody model is generated (as it is normally done when using a linear estimator), then all the information about the behaviour of the attribute under study throughout the orebody is just a single value, which needs to be accepted as representative of the orebody model. If three simulations are now generated (Figure 11-B) then the cumulative density function will be characterised by a step function, which does not ensure the continuity of the resulting cdf, and consequently it will not give an accurate characterisation of the stochastic behaviour of the process under study. However, if many simulations of the deposit are generated (say >100), as shown in Figures 11-C and 11-D, then the resulting cdf will have better continuity than the cdf curves generated with few models, giving a more accurate characterisation of the stochastic behaviour of the process under study.

In practice, the decision of how many simulations need to be generated will depend on the geologist's or geostatistician's attitude to uncertainty—that is they will decide if a certain number of simulations characterise the orebody well based on experience and visual analysis (core logging and *in situ* analysis). Other factors such as human and computer demand have also some influence on the decision of the number of simulations to be used in the analysis.

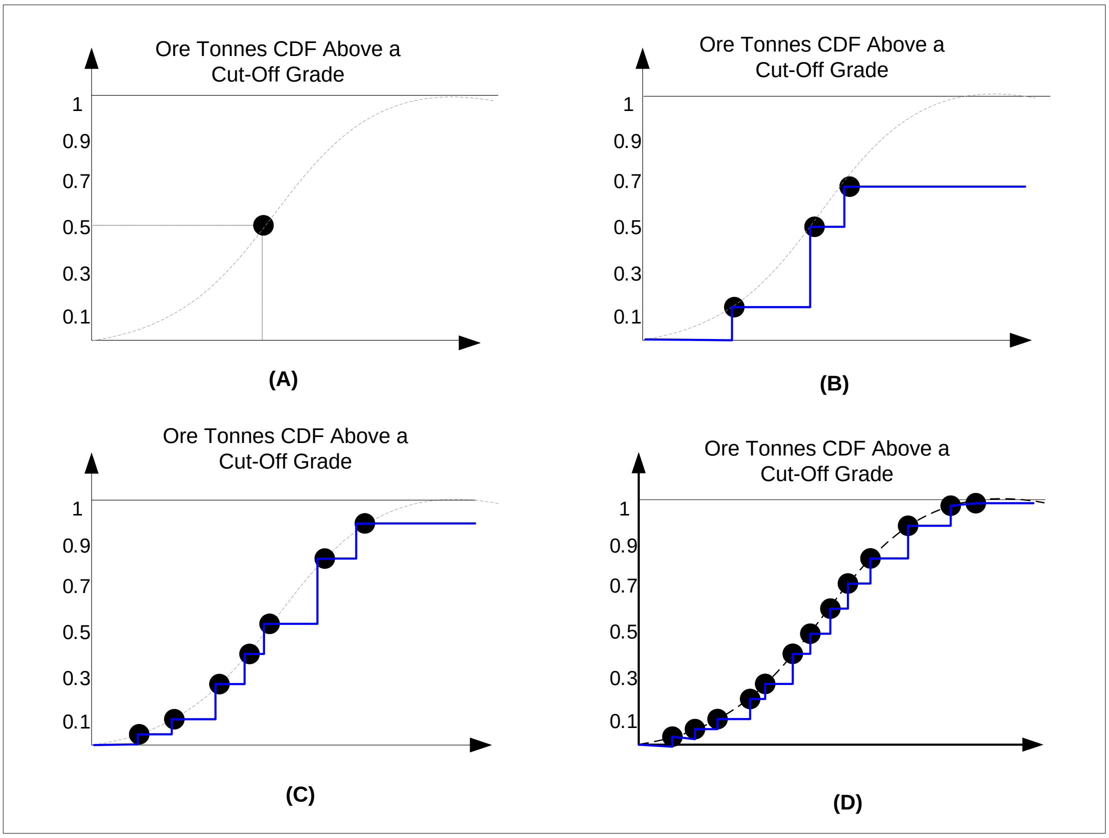

Figure 11. Diagram showing the differences in the continuity of the cdf curve of a specific geological attribute, e.g., ore tonnes and metal quantities above a given cut-off grade, when using one, three,…, and several (>100) simulations of an orebody model. Note that there is not a specific rule that can be used to know the number of simulations necessary to achieve accuracy. In practice, it is common to generate $N \geq 100$ *simulations since it gives a good representation of the continuity of the cumulative distribution function (cdf).*

2.7 Open pit capital investment project: from the traditional DCF to the RO techniques

Traditionally, mine organisations use various types of quantitative methods to estimate costs and values associated with a proposed mine project. Among these measures of profitability, the NPV (Gentry and O'Neil, 1984) is the most used in the mining industry because it recognises the time value of money and accounts for risk *via* an adjusted interest rate, R (see Equation 2.1), giving the analyst a tool for making financial investment and dividend decisions.

More formally, the NPV technique consists of subtracting the capital investment, $CapInv$, incurred at the beginning of the mining project (assumed to be period t_0) from the sum of the present values of the expected net cash flows at time *t* (CF_t) generated throughout the operating life ($t = 1, 2, .., T$) of the open pit mine, given by

$$NPV = \sum_{t=1}^{T} \frac{E\left\{CF_t\right\}}{(1+R)^t} - CapInv\,. \tag{2.1}$$

In practice, the expected cash flows generated at each production period, defined in general terms as

$$E\left\{CF_t\right\} = E\left\{q_t\right\} \times E\left\{S_t\right\} - E\left\{ProdCost_t\right\}, \tag{2.2}$$

are estimated using expected values for the underlying variables such as the metal price, S_t, total production cost, $ProdCost_t$, and metal quantity produced, q_t, at each production period $t = 1, 2, .., T$. Throughout this thesis, we shall consider the calculation of the above expectations in detail.

As mentioned at the beginning of this section, one important characteristic of the NPV technique is that a single adjusted interest rate, R, is usually applied to all future cash flows (see Equation 2.1) to account for both the risk generated by the uncertainty of the future values of the economic and technical underlying variables, such as metal prices, the shareholder’s expectation of returns, ore tonnes, and metal quantities, respectively. Normally, this adjusted interest rate, R, is estimated as the mining company’s Weighted Average Cost of Capital (WACC) (see Equation 2.3). In this way corporate management preferences for cash flows over time are also achieved. If it is considered that the mining company has equity and debt only, the WACC interest rate, R, is determined as the weighted average of the cost of debt and the cost of equity, that is,

$$R_{WACC} = (D/V)(1-T_c)r_d + (M/V)r_e\,, \tag{2.3}$$

where the market value of the mining company, $V = D + M$, is the sum of the firm's interest-bearing debt, D, and the market value of the equity, M; T_c is the corporate tax rate; r_d is the pre-tax yield on the company's debt; and r_e is the company's expected return on equity as given by the capital asset pricing model (CAPM) (Sharpe, 1964; Lintner, 1965). In this case, r_e is defined as

$$\begin{aligned} r_e &= r_f + r_{premium} \\ r_{premium} &= (r_m - r_f)\beta, \end{aligned} \tag{2.4}$$

where r_f is the risk-free rate (usually given as the yield on government bonds), and $r_{premium}$ is the risk premium composed of the expected return on the market portfolio, r_m, and the "beta" of the company, β, which is a measure of the correlation between the return of the company's stock and the return on the market portfolio (Peirson et al., 2001). Observe that if the company is composed of only equities, as it is normally assumed in mining project evaluations after Modigliani and Miller (Modigliani and Miller, 1958)[10], then by replacing $D = 0$ in Equation 2.3, it is observed that the risk adjusted interest rate, R, is equal to the company's expected return on equity, that is,

$$R = r_f + r_{premium}. \tag{2.5}$$

One problem arising from using the WACC risk adjusted rate of return, R, for a mining company is that, because it is a global indicator of risk, that is, the mining company uses this WACC for all projects and for all scenarios, it could lead to an incorrect perception of risk when applied to projects that are significantly different from the firm as a whole. This is the case for different mining projects in which the uncertainty of the orebody is different, that is, metal grade distribution and other geological, geotechnical and geometallurgical properties throughout the orebody vary, among others. In fact, different risk-adjusted discount rates should be used for the different mining projects of the company, rendering each their own cost of capital. To do this, it is necessary somehow to estimate the correlation between the specific project returns and the market as a whole, either by identifying betas from firms that are “similar in risk” to the project or by making a difficult subjective estimate of the beta. In this way, a risk premium $\left(r_m - r_f\right)$ in Equation 2.4 could be estimated and used in the valuation process.

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[10] Modigliani and Miller (1958) have shown that the value of a firm or project is unaffected by its financing decisions. That is, in frictionless markets, the risk-adjusted return, R, expected by both shareholders and lenders would gradually adjust upwards as the level of debt increases, and that, for this reason, in the final analysis the actual level of debt used to fund a project should not matter.

The determination of a single risk premium, which is able to aggregate the risk of all variables input into the valuation process, is, however, a difficult task to achieve. The reason for this is that the estimation of the beta and risk premium only considers variables traded in the market, e.g., metal prices, but not the variables that depend only on the nature of the mining project, such as production costs and metal quantities, among others.

Another problem arising from using the WACC risk adjusted rate of return of the mining company, R, is that given a flexible project in which decisions can be made throughout the operating life of the mining project, it might be necessary to use different discount rates for different production periods, that is, a dynamic discount rate. The risk of the project may change over time depending on how uncertainties unfold and management reacts (Smith and McCardle, 1999). However, the task of estimating an appropriate single dynamic WACC risk-adjusted rate of return is also very difficult to achieve.

Nevertheless, due to its simplicity, the traditional NPV is still the widely accepted way of making practical investment decisions in the mining industry. Observe that the simplicity of the traditional NPV could be an inconvenience for new evaluation techniques to be accepted as standard open pit mine evaluation tools. However, since mine planners, mine managers, and investors have started to ask questions about the performance of their mine projects in the face of uncertainty, it is only a matter of time until a new evaluation technique that is able to answer these questions will emerge as the standard tool for evaluation of open pit mine projects. This is precisely the objective of this dissertation, that is, to give the mining industry an alternative technique for open pit mine project evaluation in which uncertainty, operational flexibility, and planning, design and optimisation procedures are considered.

2.7.1 Certainty equivalent, forward contracts and the real options analysis for open pit project evaluation

One interesting result that can be observed from Equation 2.1 is that, due to the fact that each future cash flow, CF_t, generated at each production period $t = 1, 2, .., T$ is uncertain, its present value is estimated by discounting at the adjusted discount rate, R (see Equation 2.3). However, if each cash flow CF_t were certain then its present value could be estimated by discounting at the risk-free rate r_f, which is only a time adjustment. It is logical, then, to think that there must be a *certainty equivalent* cash flow CF_t^{CE}, for each generated expected cash flow, $E\{CF_t\}$, such that its present value can be expressed as

$$PV = \frac{E\{CF_t\}}{(1+R)^t} = \frac{CF_t^{CE}}{\left(1+r_f\right)^t}, \tag{2.6}$$

where the present value is estimated using just the risk-free discount rate, which is constant over time. This, of course, means that because the same discount rate is used for every future cash flow, the certainty equivalent must decline steadily as a fraction of the expected cash flow, that is,

$$CF_t^{CE} = E\{CF_t\}\left[\frac{\left(1+r_f\right)}{(1+R)}\right]^t, with \quad (1+R) > \left(1+r_f\right). \tag{2.7}$$

One problem with Equation 2.7 is that the estimation of certainty equivalent cash flows using traditional methods, such as the utility theory (see for example Guj, 2006, pp131-132), is not an easy task when risk fluctuates over the life of the project (Brennan and Trigeorgis, 2000). The reason for this is that it requires the estimation of a dynamic risk premium, which is also a difficult task (see, for example, Zhang, (2004) and Berk et al. (1999)).

In the mining industry, however, mineral commodities' forward sales are a form of certainty equivalent because of the binding nature of forward contracts (Guj, 2006; Cortazar and Schwartz, 1998). In this context, a forward contract is an agreement written at period t_0 to buy or sell an underlying asset, in this case the metal produced by a mining project, at a predetermined price, that is, the forward metal price $F_{t_0}^T$, at a specified future delivered period $T > t_0$, where terms are initially set such that the contract is costless (Rubinstein, 1999); that is at period t_0 (normally the beginning of a year) the mine company (the seller) agrees to deliver, at period $T > t_0$ (normally the end of the year), a specific quantity, q, of the asset (the metal(s)) to the buyer, at the specific unit price, $F_{t_0}^T$, which indicates that it is the unit price of the asset agreed at period t_0 to be paid at period $T > t_0$. In other words, the seller will receive the total agreed price, $qF_{t_0}^T$, in cash and with certainty at the time of delivery.

The *real options (RO) technique,* which is based on *modern asset pricing* (MAP) theory, makes use of the concept of forward contracts to evaluate mining projects, as an alternative to the DCF, by understanding and controlling the effect of uncertainty and risk in the project. This is possible because of the major advances in asset pricing theory made in the last three decades.

"The long history of the theory of option pricing began in 1900 when the French mathematician Louis Bachelier deduced an option pricing formula based on the assumption that the price of the underlying asset follows a continuous random walk. Sixty-five years later, Samuelson replaced Bachelier's assumption and stated that the price of the underlying asset follows a geometric continuous random walk" (extracted from Merton, R., 1973- Theory of rational option pricing) .

The key propositions on which these techniques are based are as follows:

1. The evaluation may be carried out, to a good approximation, in a perfect market (free of transaction barriers). In such a market, different assets which produce the same cash flow result have the same price. Additionally, in such a market, it is possible to replicate the cash flow results, and thus the value of a complex asset (in a mining venture), by executing a trading strategy in a portfolio of simple assets, such as riskless bonds and metal future contracts.

2. All asset prices are determined by the investor's risk preferences. Thus, those assets that have direct interaction with future macroeconomic variables are the ones that provide information about risk discounting. In the mining context, the

basic assets are the metal future contracts that are related to the corresponding future metal prices.

Similar to financial option analysis (see Appendix B), real options analysis is a valuation and strategic decision paradigm that applies financial option theory to real assets (Rogers, 2002). So they differ from financial options in that they deal with tangibles rather than financial underlyings. But the concepts underlying their usefulness as a tool for dealing with uncertainty are the same. The similitude and differences between financial (American) options and real options are given in Table 2, while in the mining context, the comparison between a real options on a mining project and a call option on a stock is given in Table 3.

Financial Options	**Real Options**
Short maturity often quoted in months	**Longer maturity usually in years**
Underlying variable driving the value is the equity price or price of a financial product	**Underlying variables are cash flows, which in turn are driven by competition, demand, management, and other real (technical)**
Because of legal restraints cannot control value of option except through the market	**Can increase strategic option value by management decision and flexibility**
Values are usually small	**Values are very large, usually in the millions**
Competition and market effects are irrelevant to its exercise price	**Competition and market drive the value of a strategic option**
Usually solved using closed form partial differential equations and binomial lattices with simulation of the underlying variables	**Usually solved using closed form partial differential equations and or binomial lattices with simulation of the underlying variables**
Marketable with comparables and pricing information	**Not traded and proprietary in nature with no market comparables**
Management assumptions and actions have no bearing on valuation	**Management actions drive the value of a real option**

Table 2. General similarities and differences between financial and real options (adapted from Bradley, 2004).

American Call Option	**Real Option On Mining Project**
Current value of stock	**(Gross) PV of expected cash flows**
Exercise price	**Investment cost**
Time to expiration	**Time until opportunity disappears**
Stock value uncertainty	**Project value uncertainty**
Riskless interest rate	**Riskless interest rate**

Table 3. Comparison between a call option on a stock and a real option on the acquisition of a mining project .

2.7.2 Partial differential equations

Quantitative RO valuation originates from the seminal work of Black and Scholes (BS) (1973) and Merton (1973) in the pricing of financial options. Basically, the authors derived an equilibrium partial differential equation, given by

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0\,, \tag{2.8}$$

which satisfies the price of the call option, C, as a function of the price, S, the volatility, σ, of the underlying stock[11], and the risk free rate of return, r. This was done using the Ito's differential equation (Oksendal, 2003).

The closed solution estimated by the authors is well known as the Black-Scholes option pricing formula (Walls and Eggert, 1996; Rubinstein, 1999; Hull, 1989).

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[11] In this case the authors assumed that the underlying stock followed a stochastic Geometric Brownian Model (GBM) given by $dS_t = \alpha S_t dt + \sigma dZ_t$, with dZ_t being an increment of the standard Wiener process.

The idea of risk neutral valuation in which all assets should have an expected return equal to the risk-free rate of return was introduced by Cox and Ross (1976), which asserted that the BS formula (see Equation 2.8) could be used for pricing options regardless of the risk attitude of the investor after observing that the BS formula did not include the expected return of the stock.

In essence, valuation of financial options is based on the construction of an equivalent portfolio that replicates the return of a financial option. This synthetic equivalent is constructed as a portfolio of the underlying asset and risk-free borrowing and lending. The option and the replicated portfolio must then have the same value at the expiration period to avoid arbitrage[12] (Billingsley, 2006; Hull, 1989). In this context, Harrison and Kreps (1979) demonstrated that the absence of arbitrage implies the existence of risk neutral probabilities associated with each possible payoff of the option at maturity.

One of the first mine valuation models based on financial options was the one developed by Tourihno (1979). In his thesis, Tourihno demonstrated that it is never optimal to extract the reserve (in the ground) of a mining project if there is no time limit for extraction, calling this result the *extraction paradox*. Then, using option theory and assuming that there are costs associated with holding the reserves and storage of the extracted resource, he solves the extraction paradox using partial differential equations and the Ito's lemma (Hull, 1989; Dixit and Pyndick, 1994; Chan and Wong, 2006).

Although this model gave the fundamentals of the application of option theory in solving real problems, such as a natural resource problem, it was given in a general context in which technical aspects of the deposit, such as *in situ* reserves uncertainty and the design of the project, were not considered in the valuation problem. Further, Tourinho's model did not recognise the value that could be added to the estimated project's value by reacting to future good or bad news, that is, the flexibility value.

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[12] Arbitrage is the process of buying assets in one market and selling them in another to profit from unjustifiable price difference, that is,, it is about identifying mispricing and developing strategies to profit from it free of risk.

Brennan and Schwartz (1985) were the first to show how to use the continuous time arbitrage and stochastic control theory to value mining and other natural projects. They established the companion investment rule where the project could be in one of three stages: open, temporarily shut (closed), or permanently abandoned.

In this case, following the work done by Merton (1973) and Cox and Ross (1976), the authors derived an equilibrium partial differential equation for the value of a mine, H, based on future prices given by (after Ross (1978))

$$F\left(S,\tau\right)=Se^{(r-\delta)\tau}, \tag{2.9}$$

where δ is the convenience yield that accrues to the holder of the physical asset and not to the holder of a contract for future delivery of the commodity. A detailed derivation of Equation 2.9 can be found in Hull (1989) p83. Then, the instantaneous change in future prices, dF_t, is given by applying Ito's lemma to Equation 2.9, that is,

$$dF_t = S_t F_s \left[\alpha - r + \delta\right] dt + F_s S_t \sigma dZ_t ; \quad with \quad F_s = \frac{\partial F}{\partial S}. \tag{2.10}$$

Changes in the value of the mine, $H_t = H\left(S,Q,t,j,\phi\right)$, which depends on the metal price, S_t, the physical inventory in the mine, Q_t, both at period *t*, the condition of the mine, that is, $j=1$ if open or $j=0$ otherwise, and the operating policy of the mine ϕ, can now be expressed (applying Ito's lemma) as

$$dH = H_S dS + H_Q dQ + H_t dt + \frac{1}{2} H_{SS} \left(dS\right)^2, \tag{2.11}$$

with

$$dQ_t = -q_0 dt, \quad H_s = \frac{\partial H}{\partial S}, \quad H_Q = \frac{\partial H}{\partial Q}, \quad H_{ss} = \frac{\partial^2 H}{\partial S^2}, \tag{2.12}$$

where q_0 is the production rate of the mine, assumed to be a costless variable between two fixed limits. The solution of Equation 2.11 is then estimated by using appropriate boundary conditions (see Brennan and Schwartz, 1985, for more details).

Other authors also used the PDE to apply real options to mining projects. For example, Palm, Pearson and Read (1986) valued the option to shut down and reopen low-cost and high-cost copper mines. Cavender (1992) valued the option to shut down a small open pit gold mine. Mardones (1993) analysed flexibility to adjust cut-off grade, Frimpong and Whiting (1997) extended the Brennan and Schwartz (1985) mineral resource model to develop the derivative mineral valuation (DMV), which was based on the dynamic arbitrage theory. In another paper, Frimpong and Whiting (1998; 1997) developed the dynamic risk model (DRM) for valuing long term, multi-phase mining ventures, which overcame the limitations of conventional methods. One important feature of this model was that it used the variance sensitivity ratio (VSR) in the capital market to derive the expected venture return and values from the venture and the market risk structures.

Although the key insight of the Brennan and Schwartz model was to stress the functional equivalence between the mine project and a portfolio of traded claims that allow the mine analyst to replicate the untraded project with traded assets, their work was not complete since many assumptions were adopted. Armstrong and Galli (1997) analysed the approach developed by Brennan and Schwartz (1985), highlighting that the weaknesses of their approach (from a mining point of view) were the assumptions of known costs, perfect homogenous ore and known ore quantities, which are untenable since ore bodies are always heterogeneous. They highlighted the importance of reserve evaluation and cut-off grade in project evaluation, suggesting that more work was required to make this part of the model more realistic. Further, the authors also indicated that the assumption of stopping and re-starting a mine just because of unfavourable commodity prices is unrealistic, suggesting that care would be needed to work out the real costs of stopping a mine operation in order to incorporate it into the valuation model. The importance of the inclusion of the uncertainty of the geology of the orebody into the evaluation process was also recognised by Hurn and Wright (1994) after testing the main implications of models of irreversible investment using real data coming from oil operations in the North Sea.

Up to this point it was clear that the assumption of heterogenous and known mineral reserves in the valuation of mining projects was not correct. For this reason, subsequent works done in mine project evaluation tried to account for orebody uncertainty in different ways.

For example, Samis (2000) proposed a model of project structure in which the mineral deposit is divided into multiple ore reserve zones, each of them indicated by an index zone. In this model, the mineral deposit is seen as a portfolio of real assets in which each zone constitutes a fraction of the entire portfolio. Each zone has its own technical characteristics, such as quantity of reserves, quality of mineral, exploitation plan and capacity for ore and waste, and its own grade uncertainty characteristics. Further, the grade uncertainty of each zone is assumed to be independent of the grade uncertainty of other zones. One interesting feature of this model is that it introduces the uncertainty of the mineral deposit into the valuation process by defining, *ex-ante*, a set of possible grade multiplier outcomes for each specified mineralised zone. Once the zones and their respective technical characteristics are established, management operates the project for discrete intervals of time by choosing, at the start of each interval of time, an operating mode from a set of competing operating modes. Each mode specifies the combination of zones that will be active and the amount of mineral processing capacity that is built, abandoned or temporarily closed during the next period.

Although the model proposed by Samis tried to account for all the economic and operating possibilities that could occur in an open pit mine operation, using a combination of decision trees, dynamic programming and partial differential equations for making operational strategies, it also makes big assumptions. For example, it assumes a free mine design selection, that is, the model does not consider a specific mine design or scheduling constraint over time in which it is not always feasible to mine different zones of the pit. As a matter of fact, it assumes that a mineralised deposit can be mined completely, which is not true. In reality in an open pit mine operation the material to be mined is the one inside specific limits, that is, the ultimate pit and the production sequence limits. Consequently, the free selection assumption is not a realistic representation of a surface mine operation, where different zones cannot be exploited as free selection but follow a specific technical constraint, that is, the production schedule plan.

Another assumption of Samis's model is the independence of metal grade uncertainty between the specified mineralised zones, which is not correct since correlation exists throughout the ore deposit. It could happen that the metal grade uncertainty of two or more zones may behave as independent processes; however, the identification of these zones based on the initial data (drill-hole data) is not an easy task without a further analysis (e.g., using geostatistics, as shown in Chapter 5).

Even though an attempt for assessing risk in metal grades is done by incorporating grade multipliers in the model, the model proposed by Samis considers only expected metal grades values in the valuation process due to the fact that geological uncertainty is diversifiable, or unsystematic, risk since it is not correlated with the market uncertainty. Although Samis is correct in that geological uncertainty is an unsystematic risk, the geological-metal grade uncertainty should not be represented merely by a risk factor but a distribution of probability: in this case the practical (real) distribution of probability should be used instead of the adjusted one (in the sense of risk neutral measures). Furthermore, in his conclusions Samis states that geological uncertainty has small impact on development decisions, when these decisions are associated with large expenditures, but big effects when development expenditure is smaller. This conclusion is also not totally accurate since many authors such as Rossi and Van Brunt (1997), Farrelly (2002), Martinez (2003), and Godoy and Dimitrakopoulos (2001), among others, have shown that geological uncertainty plays an important role in mining project evaluation, especially in small projects in which it is important to have a good estimate of the mineral resources since it is recovered in a short period of time and the owners of the mine expect to recover their investment with a profit.

Other authors such as Moel and Tufano (1998), Cortazar et al. (2001) and Gloria (2004) have included the uncertainty of the reserves in the valuation process by considering three discrete orebody models: a rich, an average and a poor model (apparently based on metal grade distribution). The objective of using this procedure was to characterise the effect of the orebody uncertainty on the open pit project mine project evaluation. In this case, each pit design has an associated probability of occurrence, which is the same as the probability of occurrence of their corresponding orebody model. The value of each pit design scenario is then estimated by applying the Brennan and Schwartz model (1985) to each of them. Finally, the expected value of the mine project is obtained by multiplying the estimated value of each pit mine scenario with its respective probability, and summing.

Although this approach for including ore reserve uncertainty in the valuation process, that is, selecting a rich, average or poor orebody model, is an improvement on previous techniques in which either a deterministic model was assumed for the depletion of the reserve or different zones of the deposit were characterised with a range of grade multipliers, its fundamentals are not correct. The reason for this is that this process makes the following assumptions: i) the grade distribution throughout an orebody is linear; ii) there is a direct relationship between the metal content of an orebody and the metal content inside the pit design; and iii) the value of an open pit mine project does not depend on the design of the mine. As it will be shown in Chapter 4, these assumptions are not totally correct and could lead to a wrong perception of the value of the mine.

A more realistic treatment of the problem of including the uncertainty of the orebody in the evaluation process has been given first by Carvalho, Remacre and Suslick (2000) and later on by Henry, Marcotte, and Samis (2005). In their work, the authors explore the possibility of linking geostatistical simulation techniques and real options methods to obtain more reliable estimates of the value of the mineral reserve. To achieve this, the authors make use of simulated orebody models to quantify the uncertainty in metal grades throughout the orebody and to assess their effect on the mineral reserves' value. In this context, the mining of a block with economic mineral, allocated inside an already defined working zone[13], and to be extracted in a specific period T , is seen as a European option. In this case, the uncertainty of metal price is characterised by a Geometrical Brownian Motion (GBM) and the uncertainty in metal grade is characterised by the simulated models, while the cost of extracting the block becomes the strike price of the option. In their conclusions, the authors highlighted the importance of including the uncertainty of metal grades in the evaluation process since it has a great impact in the final reserve value. One drawback of this approach, however, is that it did not consider the effect of orebody uncertainty in open pit mine planning and design, considering that there was already a mine design, as is the case of Henry, Marcotte and Samis' work, or a free selection mining process in which all the blocks inside the deposit are considered for the economic analysis, as is the case of Carvalho, Remacre and Suslick's work. However, this technique sees the mine evaluation problem in a static fashion and can be used as part of a general evaluation framework, such as the proposed IVOF. That is, it could be used to optimise just a given cutback or long term scheduling of a mine project. In this case the cutback could be viewed as the block to be extracted. Although not applicable in open pit project evaluation, the idea of using simulated models of the orebody for quantifying and assessing the effect of metal grade uncertainty on the reserves is certainly correct and more realistic than previous techniques.

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[13] In this case a working zone can be represented by a production bench inside the limits of the open pit mine or a cutback. A production bench is composed of a set of blocks which are drilled, blasted and the material hauled either to the dump (if waste) or to the mill (if ore).

A recent approach that attempts to solve the mine project evaluation problem in the face of mineral reserves and metal prices uncertainties is that of Dogbe (2006). Dogbe improved the work done by Brennan and Schwartz (1985) by including the variation of mineral reserves into the valuation process as a stochastic process. One important characteristic of Dogbe's work is that it is the first to recognise the different effects that metal grade and metal price variations have on the variation of mineral reserves. That is the variation in the mineral reserves of an open pit project is mainly due to two independent factors: i) the variation in metal prices, which is considered as an exogenous geometric Brownian process; and ii) the variation in metal grades, which is considered as an endogenous Wiener process. In this context, the value of the mine is formulated as a complex stochastic partial differential equation that depends upon the amount of reserves in the project, the metal price, and the current state of the mine, e.g., opened or closed.

Even though this model is an improvement on previous works, in which it tries to account for the uncertainty of the risk factors—that is, metal price, reserves of mineral and metal grade—modelling them as stochastic processes embedded in the mine value, it also suffers from the same drawbacks as the previously reviewed models. When valuing an open pit mine project, this work does not account for the orebody uncertainty and its effect in open pit mine plan, design and production scheduling. Furthermore, this work assumes a parametric approach for characterising and quantifying over time both the mineral reserves and the metal grade distribution in the deposit. For example, in his work Dogbe characterised the variation of metal grades as a Wiener process by stating that there is no abnormal information revealed during the mining phase to cause a sudden jump in reserves. He also suggested that as an alternative a combination of jump and diffusion processes could be used to characterise abrupt changes in ore reserves. These assumptions are not realistic since a mineral deposit is a complex three dimensional system that is characterised by several geological and geomorphologic features such as joints and faults, which in most cases control the distribution of mineralisation throughout the orebody. In other words, it is a non-linear spatial system in which the changes in trends and variance of the geological properties do not follow a specific time structure but a spatial non-linear structure, which cannot be characterised by a one-dimensional time diffusion model. An orebody depends on both space and time.

Furthermore, normally, the information collected from drill-holes is not enough to estimate the correct values of the parameters of these assumed models. Consequently, the characterisation of these events *via* an *ex-ante* stochastic one-dimension Wiener or jump-Wiener model is not an accurate process to forecast changes in mineral reserves above a certain cut-off grade, and metal grades. As it was shown in Section 2.7, there are other techniques such as geostatistics that make use of the spatial properties of the orebody are used instead to characterise changes in reserves and metal grades throughout the mineralised orebody.

Another drawback of the model proposed by Dogbe, which is also the main problem of using partial differential equations to solve real options problems, is that his model is not practical and turns the real option problem into a non-tractable complex partial differential equation process in which not only the stochastic process of each risky factor is accounted, but also the managerial flexibility that can be adopted at each production period. The main limitation of real options models based on partial differential equations is the estimation of realistic boundary conditions that allow the estimation of closed-form solutions. The author does recognise this issue[14] and states that more research is needed.

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[14] That is, the Wiener model and variations cannot characterise appropriately the geological properties of a mineral deposit without making strong assumptions, such as linearity, stationarity and elliptical processes, about the variability of metal grade throughout the mineral deposit.

The real options problem using partial differential equations also becomes more complicated if two or more metals are present in the mine project (as is common in reality). In this case the stochastic models assumed for all the risky factors, such as metal prices, reserves of mineral, and metal grade, have to be modelled as a multi-variate process turning the differential equation ruling the mine project's value into a very complex high dimensional process.

Seeing that partial differential equations were not tractable for dealing with complex real options problems, Cortazar (1999, 2001) suggested that other techniques based on the binomial lattice developed by Cox, Ross and Rubinstein (1979), and Monte Carlo simulations such as the Least-Squares Monte Carlo simulation (Longstaff and Schwartz, 2001), and the stratification method proposed by Barraquand and Martineau (1995), could be used instead of differential equations to solve complex real options problems. The reason for this was that most real options problems do not have closed-form analytical solutions.

2.7.3 The Binomial lattice

Similar to the partial differential equation technique, the binomial lattice technique uses a replicating portfolio to solve real options problems such as the open pit mine project evaluation problem. In this context, as suggested by authors such as Copeland and Tufano (2004), Copeland and Antikarov (2001), Brandao et al. (2005), Smith and McCardle (1999), Smith and Nau (1995), McCarthy and Monkhouse (2003), and Guj (2006), among others, the estimated present value of the project itself, which is the NPV without flexibility, is used as a realistic and unbiased estimate of the market value of the project. This is known as the Marketed Asset Disclaimer (MAD) .

In its simple form, the MAD model assumes that the value of the project will evolve over time following a GBM process. To show how this GBM assumption is utilised, let V_t be the value of an open pit project at time t and $R_t = \frac{V_{t+1}}{V_t}$ be its return[15] over the time period $[(t+1),t]$. Under the GBM the logarithm of the returns, $z_t = \ln(R_t)$, is normally distributed with mean μ_z and variance σ_z, which implies that the distribution of the logarithm of the project's return at any time is lognormally distributed. Therefore, V_t will be lognormally distributed and can be modelled as a GBM stochastic process in the form

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[15] Observe that in this case $R_t = \left(1 + r_t^*\right)$, where $r_t^* = \frac{V_{t+1} - V_t}{V_t}$.

$$dV_t = \alpha V_t dt + \sigma V_t dW_t, \tag{2.13}$$

where $\alpha = \mu_z + \frac{1}{2}\sigma^2$ and $dW_t = \varepsilon_t \sqrt{dt}$ is a standard Wiener process where $\varepsilon \sim N(0,1)$. Then, based on this model, a Binomial lattice can be built in order to solve a real options problem as follows.

Assume, for example, that V_{t_0} is the unknown present value (period t_0) of the open pit mine project, without flexibility, which at period $T > t_0$ increases to $V_u = V_{t_0} e^{\sigma\sqrt{\Delta t}}$ with an objective probability q, or decreases to $V_d = V_{t_0} e^{-\sigma\sqrt{\Delta t}}$ with probability $(1-q)$ (see Figure 12).

The problem consists of estimating the value of the mine when the option to implement an operational strategy, such as closing or selling the mine, incurring a cost K at period $T > t_0$, exists. This new value is normally called the extended present value of the mine, EV_{t_0}.

To solve the problem, a replicating portfolio of an amount A of a market traded stock, such as the metal that the mine produces, with current price S_0 and of B dollars invested in a risk-free bond that pays an interest rate r (see Figure 13), is generated. Then, it can be shown that, similar to financial options (see Appendix A), the current extended value, EV_{t_0}, of the project can be expressed as

$$EV_{t_0} = \left(e^{-rT}\right)\left\{pEV_u + (1-p)EV_d\right\}, \tag{2.14}$$

where p and $\left(1-p\right)$ are the probabilities of an upward (EV_u) and downward (EV_d) movement in the expanded project's value, respectively, at period $T > t_0$, under the risk neutral valuation.

Observe that in this case Equation 2.14 does not depend on the objective probability q but on the risk neutral probability p. This is the result obtained by Cox and Ross (1976) in which they state that the pricing of financial options does not depend on the risk attitude of the investor and that the analysis could be performed in a risk neutral world in which a risk-free or risk neutral rate of return can be used.

As a matter of fact, Equation 2.14 can be written as

$$EV_{t_0} = \left(e^{-rT}\right)\left\{pEV_u + \left(1-p\right)EV_d\right\} = \left(e^{-RT}\right)\left\{qEV_u + \left(1-q\right)EV_d\right\}, \qquad (2.15)$$

where the difference in using the adjusted rate of return, R, and the risk-free rate of return, r, resides in the use of the objective probability q or the risk neutral probability p, respectively.

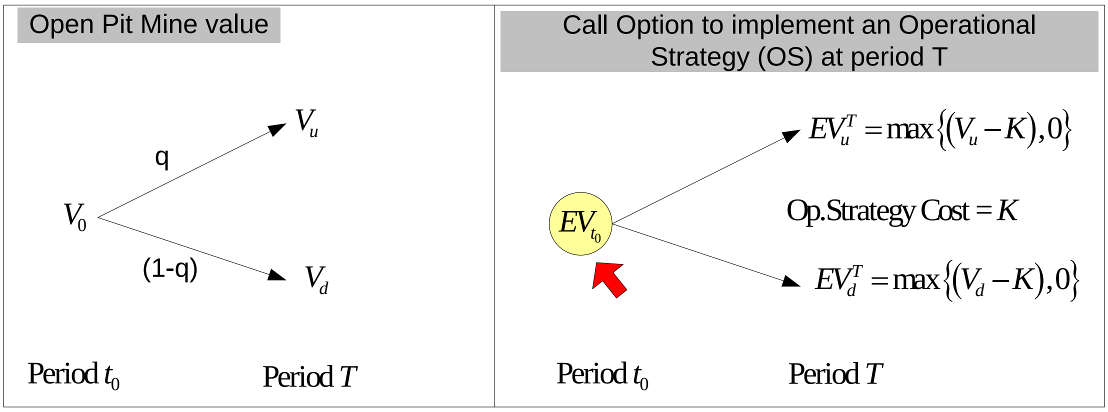
Open Pit Mine value
V_u
q
V_0
(1-q)
V_d
Period t_0
Period T
Call Option to implement an Operational Strategy (OS) at period T
$EV_u^T = \max\{(V_u - K), 0\}$
EV_{t_0}
Op.Strategy Cost $= K$
$EV_d^T = \max\{(V_d - K), 0\}$
Period t_0
Period T

Figure 12. The two-period open pit mine project evaluation problem. In the figure (left part) V_0 *is the current value of the open pit project,* V_u *and* V_d *are an up or down movement of the open pit project's value at period* $T > t_0$*, occurring with the objective probabilities* q *and* $(1-q)$*, respectively. The problem consists of estimating the current extended value of the mine,* EV_{t_0}*, that includes the option of implementing an operational strategy at period* $T > t_0$*. In this case the current extended value of the mine, or payoff, at period* $T > t_0$ *will depend on the extended future value of the mine project, that is,* EV_u^T*, if it goes up, or* EV_d^T*, if it goes down, and the cost,* K*, of implementing the operational strategy.*

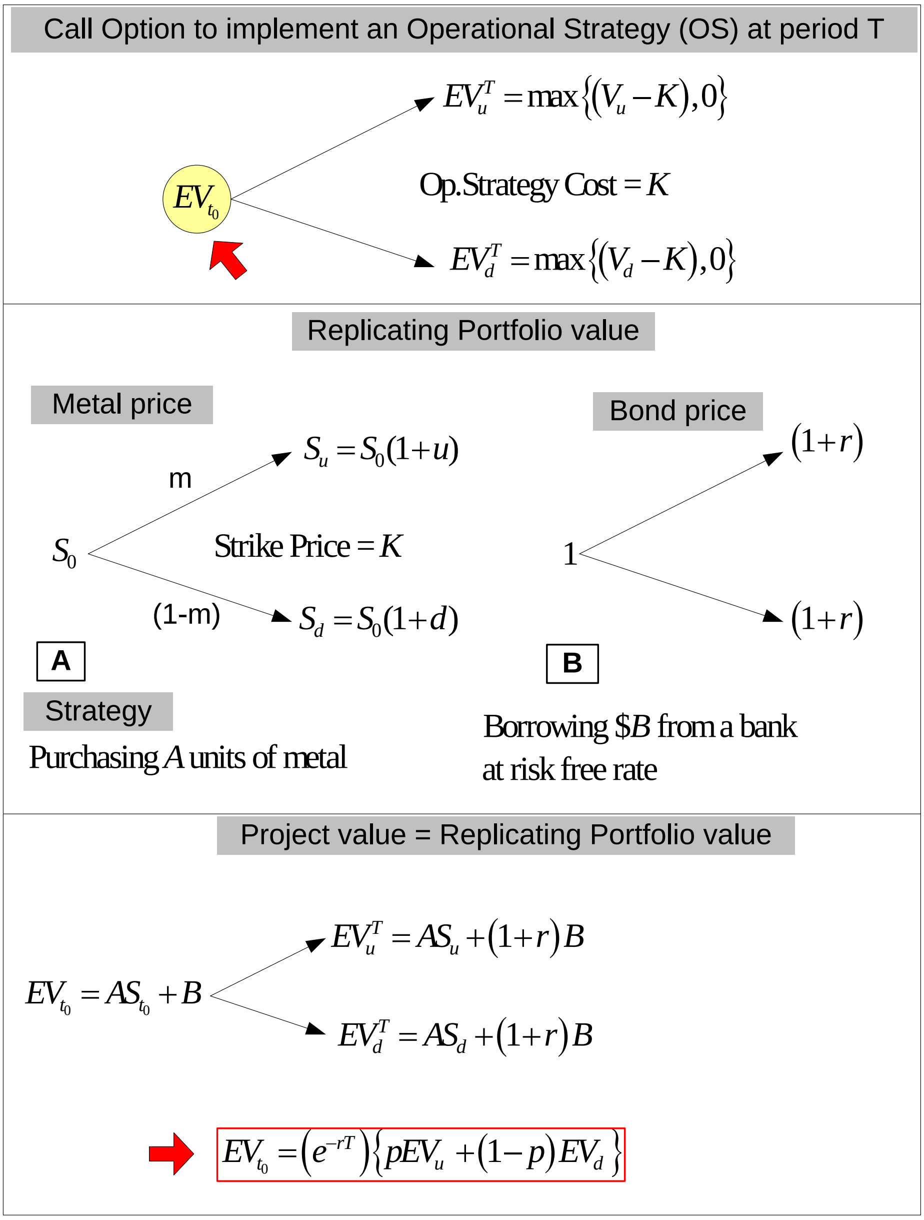
Call Option to implement an Operational Strategy (OS) at period T
$EV_u^T = \max\{(V_u - K), 0\}$
EV_{t_0}
Op.Strategy Cost $= K$
$EV_d^T = \max\{(V_d - K), 0\}$
Replicating Portfolio value
Metal price
$S_u = S_0(1+u)$
m
S_0
Strike Price $= K$
(1-m)
$S_d = S_0(1+d)$
A
Strategy
Purchasing A units of metal
Bond price
$(1+r)$
1
$(1+r)$
B
Borrowing \$$B$ from a bank at risk free rate
Project value = Replicating Portfolio value
$EV_u^T = AS_u + (1+r)B$
$EV_{t_0} = AS_{t_0} + B$
$EV_d^T = AS_d + (1+r)B$
$EV_{t_0} = (e^{-rT})\{pEV_u + (1-p)EV_d\}$

Figure 13. Generalised scheme of the two-date binomial call real option pricing problem stated in Figure 2. To estimate the extended value of the call option EV_{t_0}*, a replicating portfolio of the stock, e.g., metal produced by the mine, and bonds, is built. In this case, the portfolio is composed of* A *quantity of the metal produced by the mine, whose current price is* S_0*, and* B *bonds. In the figure,* r *is the risk-free rate of return,* p *is the risk neutral probability and* m*,* $(1-m)$ *are the objective probabilities of an up (*S_u*) or down (*S_d*) movement, respectively, on the metal price in one period ahead. Observe that the extended value of the mine does not depend on the objective probability,* m*, but the risk neutral probability* p*. Observe in the figure that for this example,* $T=1$ *period ahead, and both* EV_u *and* EV_d *are as in Figure 4.*

Observe that the real options models based on the assumption of GBM requires the estimation of the volatility, σ_V, as well as the dividend yield, δ_V, of the project's value, which, in most cases, are not observable *via* market data. Even though it is mathematically possible through complex stochastic calculus and partial differential equations to build specific project volatility and dividend yield models that take into account and combine the volatility of the various project inputs, as the number of inputs increases the construction of a realistic mathematical model becomes very hard to implement in practice.

Copeland and Antikarov (2001), Brandao et al. (2005), and Guj (2006), have suggested, however, a more friendly way of constructing a real option model of a project by using the volatility of the after-tax cash flows, rather than the individual volatilities of various inputs. After all, the volatility of the project cash flows is a complex weighted function of the volatility of all the project inputs. This is done using Monte Carlo simulation (see, for example, Mun (2006) pp. 109-119). This is precisely the technique on which the MAD model is based.

Smith (2005) suggested, however, being cautious when implementing the MAD model in real options problems. In his work, he suggested that the risk neutral approach given by CA and BDH, which uses a combination of DCF and the risk neutral valuation method[16], could be replaced for a totally risk-neutral approach in which expected values of cash flows are calculated using risk-neutral values for variables that are market risks such as the metal price, and true probability values for variables that are not market risks (private risk) such as operating costs and metal quantity variation, all discounted at the risk-free rate of return. This technique is explained in more detail in Appendix C.1.

Examples of the application of Binomial lattices to solve real options problems in the mining industry are given by McCarthy and Monkhouse (2003) and Pietro Guj (2006) (pp. 144-149), which give a comprehensive explanation of the application of real options in mineral economics. In their work, the authors stated that the Binomial lattice technique and its variations are suitable for mine practices in which friendly tractable real options tools are required for mine project evaluation.

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[16] The risk neutral approach suggested by CA and BDH consisted in using risk adjusted input variables into the DCF analysis, which uses an adjusted discount rate.

2.7.4 Monte Carlo simulations

Despite the beneficial properties that the Binomial lattice provides for solving real options, Smith (2005) stated that even though the MAD model has the advantage of reducing a potentially complex multidimensional problem to a univariate problem, it is apparently limited in solving some types of real options such as the option to wait and learn (deferral option). He suggested the use of Monte Carlo simulation techniques which have been developed for valuing high dimensional financial options as an alternative.

Conversely with the MAD model, when valuing investment projects, Monte Carlo techniques consider the evolution of the underlying variables directly. In this context, a Monte Carlo simulation model is built that takes into account all the uncertainties in the problem. Based on this model, expected NPVs for any given exercise policy can be calculated. If it is desirable to use risk neutral valuation, then the risk neutral probabilities for the uncertainties are used discounted at the risk-free rate of return. The optimal policy is then approximated by using different methods, such as Longstaff and Schwartz’s (2001) least squares method (LSMC), which estimates the required continuation values using linear regression. Finally, given this near-optimal exercise policy, the expected NPV of the project using this exercise policy is calculated providing a lower bound on the project value that would be found using a truly optimal policy.

Although Monte Carlo techniques are being recognised as flexible tools for dealing with many uncertainties when evaluating a mining project (see, for example, the work done by Gravet (2004), Blais, Poulin and Samis (2004), Smith (2005) and Laughton et al. (2000)), these techniques are still new in the mining industry and more research needs to be done to improve them in order to deal with all the uncertainties that affect the value of an open pit mine project. For the sake of completeness, the LSMC is described in Appendix C.2.

2.7.5 Managerial strategies

As outlined throughout this section, RO provides a helpful tool for evaluating open pit mine projects. Firstly, it enhances NPV to capture the value of managerial decision flexibilities in implementing different strategies. Secondly, it enables taking a complex uncertain managerial situation and reducing it to a simpler analytical structure made up of basic types of real options .

As mentioned in Section 1.5, this dissertation will deal mainly with two types of real options: the option to close the mine project at any time if future technical and economic conditions are unfavourable, and the option to defer the initial project investment in order to wait and learn from the arrival of new information.

2.7.5.1 The option to close the mine at any time

The abandonment option is one of the most common strategies considered in project evaluation (see for example pp. 174-175 of Mun, 2006). In mining, it consists of having the flexibility of closing and abandoning the open pit mining project permanently if market conditions or mineral resources decline severely. This strategy normally incurs costs for mine closure and land rehabilitation, while it also realises on secondary markets the resale value of capital equipment and other assets: this is known as the salvage value of the project.

Something that has not been examined in the literature yet is the importance of this option to allocate the ultimate pit of a surface mining operation. In fact, as it will be explained in Chapter 6, the proposed open pit mine evaluation framework (IVOF) sees the allocation of the ultimate pit limits of an open pit mine project as an American abandonment put option in which the mine can be abandoned at any period for a price which will maximise the value of the project.

2.7.5.2 The option to defer initial project investment

The deferral option is a complex strategy which needs to be considered carefully when evaluating an open pit mine project. Normally, when the option to invest in a mining project exists, the holders of the option (the owners and stakeholders of a mining firm) own the investment rights over the mineral deposit, which carries technical uncertainty on the tonnage and grade of the reserve[17]. In addition, the long run metal price follows a stochastic process. The deferral option is the flexibility of waiting and learning from the arrival of new information. Normally, there is a time to expiration of the rights for the option to develop, so the holders of the option need to make a decision during the option period (τ_{Defer}).

To make the investment decision, management needs to observe the new information arriving during the deferral period. The objective is to see if this new information about future parameter outcomes, such as metal prices and metal production, can reduce the uncertainty of the project value to levels that can justify the investment in the mining project[18].

To obtain new information about future metal prices, it suffices to wait and observe the behaviour of metal prices during the deferral option period τ_{Defer} (see, for example, McDonald and Siegel (1986), McCardle (1985), Guj (2006) and Smith and Trigeorgis (2004) pp.127-133, among others).

Contrary to economic information, to obtain more information about the uncertainty of the orebody the mining firm needs to make further investment for allocating extra drill holes to the deposit. The problem arising from this action is to determine the optimal number of additional drill holes, and their respective locations, that maximise the NPV of the project. Logically, the cost of the additional data plus the cost of holding the lease on the project should be less than the expected increment in NPV that can be obtained with this new data, otherwise the mine analyst would construct a mine plan with the data already available.

Different techniques have been used for solving this problem. For example, Guimaraes (2003), following the work done by Martzoukos and Trigeorgis (2001), uses a Bayesian approach for incorporating the new information into the RO valuation process. In this context, the conditional distribution of the value of the reserves, given the new information, is called the revelation distribution, and is used to evaluate the value of the project. One problem with this model is that it relies on the information given by a technical expert about the total uncertainty of the existing reserves, and the expected percentage of reduction of technical uncertainty due to the arrival of new information. In fact, in the mining or petroleum industries, the allocation of extra drill holes or wells, respectively, needs to be planned carefully since it is a very expensive activity. Further, due to the nature of the deposit, it is not an easy task to predict an expected percentage reduction in reserve uncertainty arising from the allocation of extra drill holes or wells, which is in itself a problem that is still a topic of research studies (see, for example, the work done by Froyland et al. (2004) and Boucher et al. (2003)). Gloria (2004) improved the model of Guimaraes (2003) by using conditional simulation techniques to assess the effect of extra drill holes in the valuation project. In her work, extra drill holes were simulated and the mine plan was evaluated considering the orebody models generated using the new data set, which included the original drill-hole data set plus the simulated drill-hole data set. Even though the estimation of the mine based on the original and new data was not totally complete (see Section 2.2.2), the idea of simulating drill-holes in order to have more information about the uncertainty of the orebody is very good and more realistic than assuming expected values. Most importantly, Gloria’s work shows that, contrary to the belief of most people that the variability of orebody uncertainty will decrease as additional information becomes available, additional information increases the uncertainty even when the model is correctly specified[19]. This increment in uncertainty due to the arrival of new information was also demonstrated by Galli et al. (2001).

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[17] As mentioned in Section 1.5, it is assumed that the mine firm has no competition and is the only bidder in the acquisition of the undeveloped mineral deposit.

[18] This reduction of uncertainty can be conveniently expressed as the percentage of variance reduction in the project value.

Froyland et al. (2004) also used conditional simulation techniques to establish an upper bound for the NPV increment (not including drilling costs) achievable through additional drilling, called Value of Infill Drilling Information (VOIDI). Even though it is an interesting model in terms of using stochastic linear programming for optimising production scheduling, it requires the assumptions of pre-established economic parameters, such as metal prices, production cost, as well as technical parameters such as free selection in which all blocks are included in the valuation analysis, which otherwise are not available at the beginning of the project.

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[19] It is believed that the only reason for which uncertainty can increase when new information is available is the misspecification of the initial model.

2.8 Open pit mine optimisation techniques based on orebody uncertainty

Mine Optimisation (MO) procedures are a set of techniques that introduce analytical methods into mine planning and design to evaluate open pit mining projects. These techniques are used by engineers to optimise the value of an open pit mine operation by focusing on the effect of the orebody uncertainty, such as metal grade, on the planning and design, and on the open pit mine project.

One important characteristic of these techniques is that they account for both the orebody uncertainty and the optimisation of the design and planning of the project. In its essence, mine optimisation techniques are based on: i) conditional simulation techniques (Chan and Wong, 2006; Halton, 1970) as methods able to provide a quantitative assessment of the uncertainty over the attributes of the orebody (e.g., grade, ore type, metallurgy recovery, etc.); and ii) Monte Carlo simulation techniques (Glasserman, 2004), which combined with conditional simulation assess the sensitivity of the overall pit economics, long-term mine planning and production scheduling to grade uncertainty.

Mackenzie, Bilodeau and Mascall (1974) wrote one of the first works on mine valuation and optimisation. They described a simplified mine development decision model, using Monte Carlo simulation to optimise mine capacity and cut-off metal grade under conditions of uncertainty. The steps of their process are as follows. Firstly, the sources of uncertainty of the process were identified. In this case, total capital cost, annual operating cost, metal price, and average grade of the deposit, were identified as sources of uncertainty. Secondly, a specific model for each of the identified variables was assumed, and, finally, Monte Carlo simulation was used for sampling the distribution generated for each of the variables that are present in the optimisation process.

Although this model established the fundamentals of open pit mine optimisation in the face of uncertainty, this approach made several assumptions, such as the absent of the metal grade, ore reserves and metal price uncertainties. One of the reasons for these assumptions was the lack of technology necessary for describing or measuring the uncertainty of these variables, specifically the uncertainty of the orebody model and metal price. Furthermore, this model did not consider the design and plan of the mine in the optimisation process.

David (1973) and David et al. (1974) introduced the idea of risk analysis using conditional simulation in mine planning and proposed this technique as a way of modelling possible real values rather than estimates. Key points discussed in their work were: i) the smoothness of estimates, while mining highly variable ore bodies; and ii) undervaluation of deposits because of smoothness of estimates. More specifically, the idea of using simulation in the analysis of open pit optimisation was established for the purpose of predicting the deviations from the forecasted results of a mine plan (obtained by using estimation techniques) and the real values.

Years later, Dowd (1992), Ravenscroft (1992) and Dimitrakopoulos (1998) presented a general framework for dealing with geological uncertainty and risk in open pit mine design and production scheduling, which was based on the work done by David (1973, 1974). The parts of this framework were (see Figure 14):

i) Stochastic conditional simulations, where equally probable orebody models are generated to represent the uncertainty in orebody modelling;

ii) Transfer functions, defined as a 3D non-linear mining process or mining sequence; and

iii) Uncertainty modelling and risk assessment.

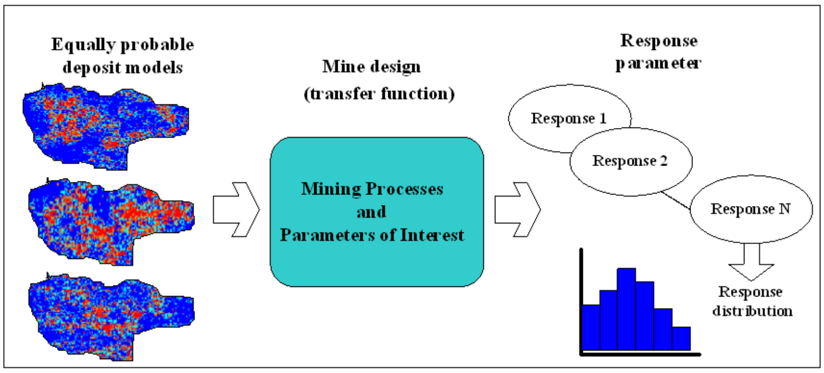

Figure 14. Schematic representation of the general Monte Carlo simulation and risk assessment framework. In this framework, each image of the orebody gives a realistic representation of the variability of metal grade throughout the deposit. Then, by inputting these images into the optimisation and mine design process (non-linear transfer function), the orebody uncertainty is transferred onto the response parameters (reproduced from Martinez (2003)).

Rossi (1997), Van Brunt (1999), Godoy (2003), and Farrelly (2002), among others, used the general framework for dealing with geological uncertainty and risk in open pit mine design (see Figure 14) to model and integrate geological uncertainty in the planning process of a mine. The authors concluded that the inclusion of geological uncertainty into the planning and design of an open pit mine provides a more informed approach to designing and managing the project. Furthermore, these works showed that simulations can be used not only in assessing and quantifying the uncertainty of the orebody, but can also give the starting point for improvement of the initial mine design.

Martinez (2003) extended the framework created by Ravenscroft (1992) and Dimitrakopoulos (1998) and the works done by Farrelly (2002) and Godoy (2002) to develop a new approach to mine design based on risk quantification and alternative strategic decision-making criteria. After revising the concepts of risk integration and quantification in decision-making for both new and operating surface mines, Martinez demonstrated that economic mine evaluation is a critical part of assessing the real value of mining assets, in this case of an open pit mine project. One important characteristic of this model was its ability to deal with quantified geological and grade uncertainty in the context of optimal pit design, where the designs and long-term production schedules were optimised under orebody uncertainty.

Another important characteristic of Martinez's model was the use of both key economic and technical project indicators, as well as the mine planner's decision-making criteria, to optimise the value of an open pit mine project. In this context, the cash flow risk in the short-term was minimised while the potential for profits in the future was maximised. The economic and technical project indicators used in Martinez's work were: the minimum annual ore production, the amount of metal produced in given mining periods, and the discounted cash flows over the life of the mine. Similarly, the mine planner's decision-making criteria used in Martinez's work were the minimum acceptable project DCF (normally given by the corporative group of the mining firm) and the minimum acceptable risk in meeting given production targets.

One drawback of Martinez's model, however, was the time required to perform the optimisation process. The reason for this was the iterative process in which the economic risk of an open pit mine design was assessed. Another drawback of this model was that it did optimise an open pit mine project in the face of orebody uncertainty, keeping costs and prices constant over time.

One key factor in the work done by Martinez (2003) is the open pit design algorithm used to generate the Base Case Pit Design (BCPD), which is composed of a set of nested pit shells (see Section 2.4.3 for more details). The technique used for this purpose was the Nested Lerchs-Grossmann (NLG) algorithm implemented in the Whittle Four-X software (Whittle, 1998; Whittle and Rozman, 1991), which is based on the Lerchs-Grossman algorithm[20] and the parameterisation model developed by Matheron (1975). In this context, the NLG algorithm considers a series of revenue factors (RFs), normally ranging between 0.3 and 2, which multiply the actual metal price, generating a list of different metal prices. The result of this is the generation of optimum nested pit shell surfaces for each price. The approach starts by finding the largest pit using the highest metal price (highest revenue factor)[21]. Next, the pits for the remaining metal prices in the list are generated. This process is repeated until all prices have been dealt with. There is no specific rule for setting up the list of revenue factors. Rather, it depends on the precision of the analysis. An important characteristic is that, in each optimisation, Whittle's approach only considers the blocks placed between the shells for the nearest prices above and below the input metal price. Thus, the program adjusts the values of these blocks to allow for the new price and then carries out a scan through the arcs (see foot note 21), which apply to these blocks until there is no further change.

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[20] The Lerchs-Grossmann algorithm is an optimisation process based on graph theory. In this context, the orebody model is seen as a graph that is a direct tree in which a block inside the deposit is defined as a node and the connection between nodes are arcs. Then, the pit design limits are found as the contour that maximises the economic value obtained from the extracted blocks (see Martinez 2003, Chapter 3 for more details about the Lerchs-Grossmann algorithm).

Another technique based on the parameterisation model, which even though developed almost twenty years ago has not yet received all the attention it deserves, is the Reserve Parameterisation Method (RPM) developed by Francois-Bongarcon (Francois-Bongarcon and Laille, 1984). One important characteristic of this model is that the only quantities that need to be input into the optimisation process are grades, tonnes and metal quantities. Profit formulae are purposely eliminated. As a consequence of this, the pit design process is a purely technical optimisation and the output of the process is not a single optimal pit outline, but rather a series of pit outlines representing the range of truly optimum projects for the full range of possible economic conditions and management policies. In this theory, any feasible project outline can be characterised by a triple $\left(Q,V,T\right)$, where T is the tonnage of recoverable ore to be processed, Q is the corresponding recoverable quantity of metal of the project, and V is the total tonnage to be extracted[22]. For each possible pair $\left(V,T\right)$, a set of technically optimal projects $\left(Q_{\max},V',T'\right)$ are generated such that $\left(V'<V\right)$ and $\left(T'<T\right)$. This is achieved by controlling the variations in V and T. In

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[21] By doing this, all blocks outside this pit are excluded and will not be considered any further, which translates into the saving of computing time.

[22] Defined as the quantity of ore and waste to be removed from the ground.

the general theory, these variations are equivalently replaced by mathematically duality with the variations of two parameters λ and θ, which are the coefficients of the equation of a generic plane in the space $\left(Q,V,T\right)$. Then, the quantity to maximise is $Q-\lambda V-\theta T$. Optimal contours are obtained using classical algorithms of maximisation like the Lerchs-Grossmann algorithm in which a linear additive profit function like the cash flow $CF=aQ-bT-cV$, where a (the price of the metal), b (the processing costs) and c (the extraction costs) are fixed. Observe that the cash flow equation can be re-written as $\frac{CF}{a}=Q-\frac{b}{a}T-\frac{c}{a}V$ which is equivalent to $Q-\lambda V-\theta T$ with $\lambda=\frac{c}{a}$ and $\theta=\frac{c}{a}$ (see, for example, Coléou (1989) and the references therein for more details).

As a matter of fact, the NLG algorithm was initially based on the RPM model (in his initial version 4D) but was later on changed to introduce revenue factors as a strategy for dealing with ore deposit with the presence of two or more economic metals[23].

Many authors such as Ramazan and Dimitrakopoulos (2004), Menabde et al. (2004), and Stone et al. (2004), among others, have tried to improve the model developed by Martinez (2003) in the context of performing open pit mine optimisation under geological uncertainty in a reduced time and using many simulations instead of selected ones (as used in Martinez, 2003). In these cases, stochastic models based on linear programming were used as the pit optimisation technique. Despite the fact that time processing was improved, the results obtained about open pit mine design and optimisation in the face of orebody uncertainty were seen to be incomplete and not optimal, as is the case of the models developed by Ramazan and Dimitrakopoulos (2004a, b)[24] Indeed, the final design obtained from their model is seen to be not optimal, since in some cases pit slopes or mining working zones' width were not achieved while, in other cases, mining constraints such as production scheduling were not respected, having the presence of a block that is supposed to be mined in the first year at the bottom of the pit.

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[23] In this case, Whittle took advantage of the methodology in which an orebody with the presence of two or more economic metals can be analysed using an equivalent metal grade based on equivalent metal prices. Then, by controlling revenues factors affecting metal prices, different pit shells are generated regardless of the number of economic metals in the deposit.

[24] The information about the performance of the models developed by Menabde et al. (2004) and Stone et al. (2004) is not available since the software that implements the models is not in the public domain.

Even though stochastic programming is a strong tool that can help with the problem of optimising an open pit mine in the face of uncertainty, it is still in its early development and more research needs to be done in order to implement it appropriately.

Another important technique used to optimise an open pit mine project is the selection of an optimum cut-off grade policy for the mine operation. Cut-off grade is any grade that, for any specific reason, is used to separate two courses of action, e.g., to mine or to leave, to mill or to dump (Taylor, 1985). The most used technique for this is the *Lane's theory of cut-off grade optimisation* developed by Lane (Lane, 1988; Lane, 1999) (see also King, 2000). This technique determines an optimal cut-off grade strategy which maximises the NPV of the mine as an iterative process based on the production capacities of the mining, milling, and refining stages.

As it will be shown with more detail in Chapter 4, the model developed by Lane (given in more detail in Appendix C.4), has many similarities with the real options theory in which the mine project's value is optimised based on an operational strategy over time which implies an opportunity cost, which is an unknown variable in the model, in this case the selection of the best cut off grade for each production period. Authors such as Mardones (1993) and Sagi (2004) have applied option pricing techniques to improve the selection of cut-off grade in a mine operation. On one hand, Mardones developed a valuation model where the cut-off grade is first calculated using Lane's theory, that is based on the discounted cash flow approach, and the optimised mine is then evaluated using real options. Although cut-off grade is included in the real valuation model developed by Mardones, this model is not fully consistent with contingent claims since the cut-off grade is not optimised along the main objective function, that is, the value of the mine and operational and managerial flexibility. On the other hand, Sagi's work illustrated the trade-off between extraction policy and quality control from a financial view point. The main contributions of Sagi's work were the identification of the opportunity costs of extraction as the crucial factor in determining an optimal cut off grade policy, and the association of the opportunity costs with a European call option (see Appendix B for details in European options) written on the marginal extracted unit of mineral. To our knowledge, the work done by Sagi (2004) is the first in introducing cut-off grade optimisation in a real options mine valuation context. Although its model does not consider the mine design and planning, the association of the opportunity cost of extracting a unit of mineral, with a given grade, with a European option is considered by the proposed IVOF (see Chapter 8) as a future extension.

In other works, Baird and Satchwell (2001) and Osanloo and Ataei (2003), among others, extended Lane's cut-off grade optimisation to deal with a deposit with multiple metals by using equivalent grade factors to optimise the cut-off grades.

Looking for a practical way of implementing Lane's theory in open pit project evaluation, Whittle and Wharton (1995) replaced the unknown opportunity cost variable in Lane's model by two pseudo costs, the delay costs and the change costs, for which the NPV of the project is maximised by moving the milling of ore forward (or backwards) in time by changing the cut-off grade. Normally, these costs, that is, the delay and change costs, are large at the beginning of the project, reducing in amount as the resource is depleted and opportunities reduced. The objective of doing this is to feed the mill with high grade material at the beginning of the project which means greater profit at the start. Other works that look for a practical way of implementing Lane's theory are the ones developed by King (King, 2000b; King, 2000a) who analyses the theory in detail and gives alternatives for implementing it. One of these alternatives is an algorithm to solve the maximisation problem starting from the first production period instead of the last one.

2.9 Summary and conclusions

We have presented a synthesis of the literature concerning open pit mine evaluation procedures was given, having connected it with the geostatistical methods which underlie the planning process. This literature review for valuation focused on two main philosophies for valuing an open pit mining project. The first philosophy was based on the techniques that use economic and financial analysis to estimate the value of the mine. Examples of these techniques are both the DCF and RO techniques. One interesting feature of these techniques is that they consider future metal prices as the main source of uncertainty in an open pit mine project, and assume that the mine plan and design of the project is known and well defined.

The second philosophy was based on the techniques that use a set of procedures to optimise the plan and design of the mine. Examples of these techniques are the cut-off grade optimisation developed by Lane (1988) and the optimisation technique developed by Martínez (2003). Contrary to the first philosophy, these techniques consider the uncertainty of the orebody as the main source of uncertainty of an open pit project, and assume that the economic environment in which the project is developed, e.g., future metal prices, are known and well defined.

In addition, the literature review also shows that the problem of determining more information about the orebody uncertainty and using it to make strategic decisions, such as deferring the project investment, has not been solved appropriately, mainly because of the lack of an appropriate evaluation framework in which the uncertainty of the orebody can be accounted for together with economic uncertainty.

Clearly, the preceding literature review indicates that, in isolation, each of these philosophies, that is, financial and mine optimisation techniques, cannot solve the open pit mine project evaluation problem correctly. However, the literature review clearly exposes that both philosophies are capable of solving the open pit mine project evaluation in conjunction with each other. The reason for this is that both philosophies complement each other: MO techniques solve the problem that RO has, by including the orebody uncertainty as well as its effect on mine planning and design in the evaluation process, and *vice versa*. This is precisely the strategy used by the novel evaluation framework (IVOF) proposed in this dissertation; namely, to integrate both philosophies for evaluating an open pit mine project into an integrated evaluation process, ensuring a complete and more realistic estimate of the value of an open pit project in the face of both metal prices and orebody uncertainties, while considering an optimum open pit mine planning and design. That is the IVOF is a technology that applies real options “on” the mine project evaluation rather than “in” the mine project evaluation. Indeed, as defined by De-Neufville (2002) and Wang and De-Neufville (2006) real options “in” projects are created by changing the actual design of the technical system, while real options “on” projects treat the technology behind the project system as a black box.

In this context, the proposed mine (open pit) project evaluation framework, IVOF, does not *“invent”* a new mathematical evaluation technique, but uses and integrates existing valuation and mine optimisation methodologies to deal with the problem of open pit mine evaluation in the face of orebody and metal price uncertainties, and considering mine plan and design flexibility.

In general, in view of the literature review presented in this chapter, the proposed IVOF is seen to offer a new approach to strategic open pit project evaluation. It is important to observe that the IVOF is not a substitute for any of the techniques discussed in this chapter. In fact, it uses the DCF and mine optimisation techniques as a building block and allows the mine analyst to integrate existing real options techniques into a sophisticated framework that provides more meaningful information about the value of an open pit mine project in the face of uncertainty.

Furthermore, from the literature it can also be concluded that the main problem with current open pit evaluation techniques is the lack of knowledge about the different stages of an open pit mine project, as well as their respective theoretical backgrounds. For instance, the financial analyst performing real options and DCF analysis on an open pit mine project does not know about orebody modelling and mine design and planning, and *vice versa*. In fact, this is one of the reasons why real options is slowly being accepted in the mining industry as a valuation tool. Due to the theoretical complexity involved and the lack of suitable analysts that can appropriately apply this technique, most mining companies see the real options technique as a black-box and resist adopting it in their valuation procedures. Consequently, the objective of the IVOF is to break the wall that isolates financial analyst from mine analyst when evaluating an open pit mine project and integrates the best of both technologies to give a generalised framework for mine project evaluation.

Chapter 3

Building the Base-case Open Pit Mine Plan and Design: Preliminaries for IVOF

3.1 Introduction

This chapter will give, in a general fashion, the steps for building the base-case mine plan and design. This will be explained using a simple example where a virtual two-dimensional mine project is used to plan and design the respective base-case (more complete details of this process can be found in books of mine planning such as Hustralid and Kuchta (1995) or theses such as Martinez, 2003).

The purpose of this illustrated explanation is to provide details of mine planning against which IVOF can be compared and contrasted. In itself this chapter does not contain new material but is essential to highlight the novelty of IVOF over the remaining chapters of the thesis.

Before giving the general steps for building the base-case mine design, it is very important firstly to give the definition of some terms related to the open pit mine optimisation process that will be used throughout this chapter. Although some of these definitions were already given in Chapter 1, they are repeated in this section for the sake of completeness (refer to Figure 15). Also observe that for some definitions lower-case letters indicate deterministic values while capital letters indicate random variables.

Mine definitions

Cut-off grade, Ore and Waste: *Cut-off grade* is the grade level to decide which material is sent from the mine to the treatment plant (defined as *ore),* and which material is dumped (defined as *waste).*

Break-even cut-off grade, (deterministic/random variable) (g_{BE}/G_{BE})*-(units):* defined as that grade for which the net value (profit-costs) of some material mined is zero. This is obtained by equating the value at the mill to the value at the dump. Normally this cut-off grade defines the *economic material* that is contained within the layout of the ultimate pit limits in an open pit mine operation.

Optimum Cut-off grade, g^{op} *:* defined as the minimum grade level that decides which material is sent from the mine to the treatment plant (defined as *ore),* and which material is dumped (defined as *waste),* while maximising the expected cash flow.

Average grade, (deterministic/random variable) $(\bar{g}/\bar{G})$*-(units):* defined as the ratio of total metal quantity and the total ore tonnes.

Production period, t *(year):* the time length of a production period used for economic analysis (although here specified as years, it could also be other lengths such as halves, or quarters of a year).

Mine operating life, T *(years):* the required time for depleting the ore in a mine project.

Profit or cash flow, (deterministic/random variable) (cf/CF) *($/year):* defined as the revenue generated from selling the final product minus the cost incurred in the process.

Mining Capacity, (min)*-(tons/year):* defined as the total material (ore + waste) removed in a production period. It mostly depends on the machinery available for this operation.

Milling (concentration) capacity, $(mill)$ *(tons/year):* defined as the total material sent to the mill in a production period. It mostly depends on mill concentrator facilities.

Refining capacity, ref *(tons, or, oz, or, lb, or units of product/year):* defined as the total units of mill product sent to the refining facilities.

Selling (product) price, (deterministic/random variable) (s/S) *($/unit of product, e.g., $/tonne, or, $/gr, or, $/Oz):* defined as the expected price of a unit of the final product, normally given by the international metal markets.

Refining cost, , $c_{refining}$ *, ($/unit of product, e.g., $/tonnes, or, $/gr, or, $/Oz):* total cost incurred in refining activities.

Mining cost, c_{mining} *, ($/tonnes of material mined):* total cost incurred in the mining activity.

Milling (concentrating) cost, $c_{milling}$ *, ($/tonne of ore):* total cost incurred in the concentrating- milling activities.

Stockpile handling cost, sh *($/tonne of material moved):* total cost incurred in the movement of material from stock piles to the mill or concentrator facilities.

Fixed costs, c_{fixed} *, ($/year):* costs incurred in a production period that are not directly related to mining activities, e.g., general and administrative costs.

Metallurgical recovery, y *(%):* factor that affects the recovery of final product from the ore sent to the mill or concentrator.

Yearly discount rate, R *(%):* normally defined as the Weighted Average Cost of Capital (WACC) of the company owning the mine project.

Risk-free discount rate, r *(%):* normally defined as the discount rate given by government bonuses.

Ore, q *(tonnes/year):* total ore processed in a production period.

Waste, w *(tonnes/year):* total material dumped in a production period.

Material mined, u *(tonnes/year):* total material (ore + waste) removed in a production period, that is,

$$u = q + w$$

Stripping ratio, sr *(no units):* defined as the ratio between the quantity of waste and ore mined in a production period, that is,

$$sr = w / q$$

Product refined, (deterministic/random variable)(m/M) (tonnes, lb or Oz of product/year): total units of material sent to the refining facilities in a production period, where

$$M_t = q_t y \overline{G}_t$$

Salvage value, SVal ($): the value obtained at the end of a mine operation resulting from selling all equipment and mining/processing/refining facilities.

Mine closure costs, Clc ($): the cost incurred when a mine operation is closed indefinitely, which includes mine rehabilitation.

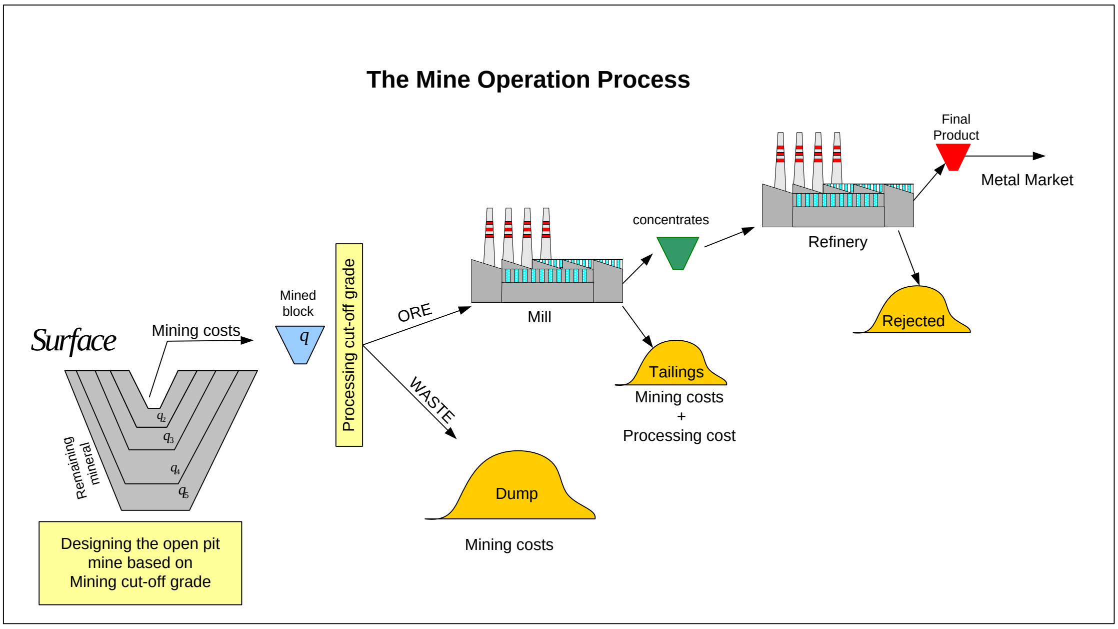
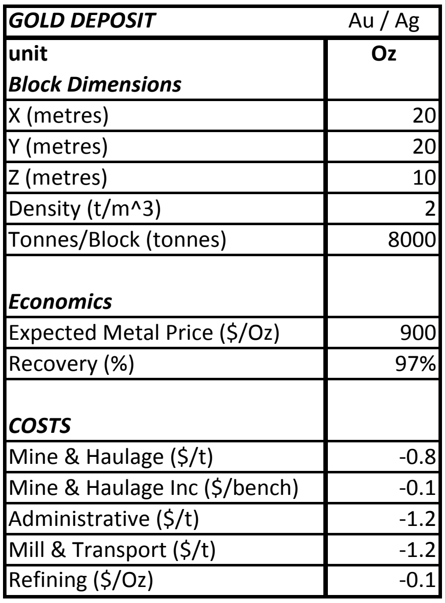

The Mine Operation Process
Final Product
Metal Market
concentrates
Refinery
Rejected
Mill
ORE
Mined block
q
Processing cut-off grade
Surface
Mining costs
WASTE
Tailings
Mining costs
+
Processing cost
q_2
q_3
q_4
q_5
Remaining mineral
Dump
Mining costs
Designing the open pit mine based on Mining cut-off grade

Figure 15 Diagram showing the open pit mine operation process.

3.2 Building the Base-case plan and design: The virtual 2D Gold Mine

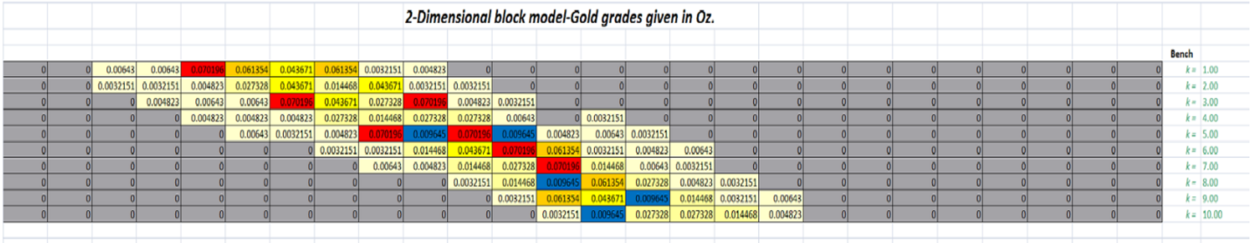
The 2D gold mine problem is a virtual gold project generated for the purpose of understanding the principles of building the base-case open pit mine plan, design and evaluation. The problem consists of planning, designing and estimating the value of a

2D orebody model containing gold metal. The data given for evaluating the 2D gold mine project is composed of two sets. Data Set 1 is related to technical/operational and economic data and is shown in Figure 16. Data Set 2 is the virtual 2-dimensional metal grade block model containing gold in ounces per tonne (see Figure 17). Observe that here it is assumed that the metal grade was estimated using a linear interpolator such as ordinary Kriging (see Chapter 2 for more details).



GOLD DEPOSIT	Au / Ag
unit ***Block Dimensions***	**Oz**
X (metres)	20
Y (metres)	20
Z (metres)	10
Density (t/m^3)	2
Tonnes/Block (tonnes)	8000
Economics	
Expected Metal Price ($/Oz)	900
Recovery (%)	97%
COSTS	
Mine & Haulage ($/t)	-0.8
Mine & Haulage Inc ($/bench)	-0.1
Administrative ($/t)	-1.2
Mill & Transport ($/t)	-1.2
Refining ($/Oz)	-0.1

Figure 16. Table showing the 2D Gold Mine project-economic and technical parameters. Slope angle is assumed to be 45°.



2-Dimensional block model-Gold grades given in Oz.

																										Bench
0	0	0.00643	0.00643	0.070196	0.061354	0.043671	0.061354	0.0032151	0.004823	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 1.00
0	0	0.0032151	0.0032151	0.004823	0.027328	0.043671	0.014468	0.043671	0.0032151	0.0032151	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 2.00
0	0	0	0.004823	0.00643	0.00643	0.070196	0.043671	0.027328	0.070196	0.004823	0.0032151	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 3.00
0	0	0	0	0.004823	0.004823	0.004823	0.027328	0.014468	0.027328	0.027328	0.00643	0	0.0032151	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 4.00
0	0	0	0	0	0.00643	0.0032151	0.004823	0.070196	0.009645	0.070196	0.009645	0.004823	0.00643	0.0032151	0	0	0	0	0	0	0	0	0	0	0	*k* = 5.00
0	0	0	0	0	0	0	0.0032151	0.0032151	0.014468	0.043671	0.070196	0.061354	0.0032151	0.004823	0.00643	0	0	0	0	0	0	0	0	0	0	*k* = 6.00
0	0	0	0	0	0	0	0	0.00643	0.004823	0.014468	0.027328	0.070196	0.014468	0.00643	0.0032151	0	0	0	0	0	0	0	0	0	0	*k* = 7.00
0	0	0	0	0	0	0	0	0	0	0.0032151	0.014468	0.009645	0.061354	0.027328	0.004823	0.0032151	0	0	0	0	0	0	0	0	0	*k* = 8.00
0	0	0	0	0	0	0	0	0	0	0	0.0032151	0.061354	0.043671	0.009645	0.014468	0.0032151	0.00643	0	0	0	0	0	0	0	0	*k* = 9.00
0	0	0	0	0	0	0	0	0	0	0	0	0.0032151	0.009645	0.027328	0.027328	0.014468	0.004823	0	0	0	0	0	0	0	0	*k* = 10.00

Figure 17. The 2D gold orebody model. Gold grades given in Oz/t. Grade relates to the "temperature" of the colours where hot colours indicate a high grade.

3.3 Evaluating the 2D gold mine problem

I. The first part in the evaluation process of the gold mine project is the generation of an economic block model. The steps are:

Estimating the mining cost model ($/block) as (see Figure 18):

$$MiningCost = ((Bench - 1)M\ \&\ HIncrement + M\ \&\ HCost + Admin)\frac{Tonnes}{block},$$

where $Bench$ refers to a mining bench or working zone where drilling and blasting operations are performed; $M\ \&\ HIncrement$ refers to the increment in mining and hauling costs for lower benches; $M\ \&\ HCost$ refers to the cost of mining and hauling material; $Admin$ refers to the administration cost incurred in the mine operation; and $\frac{Tonnes}{block}$ refers to the tonnes of a production block.

Observe that in this case the mining cost is increased with depth in a factor of “M&H Increment” (see Figure 18).

Estimating the metal recovered model (Oz/block) as (see Figure 19):

$$MetalRecov = Metalgrade \times \frac{Tonnes}{block} \times Recovery\ .$$

Estimating the block revenue model ($/block) as (see Figure 20):

$$BlockRevenue = \left(Price - SellingCost\right) MetalRecov\ .$$

Estimating the block value if milled ($/block) as (see Figure 21):

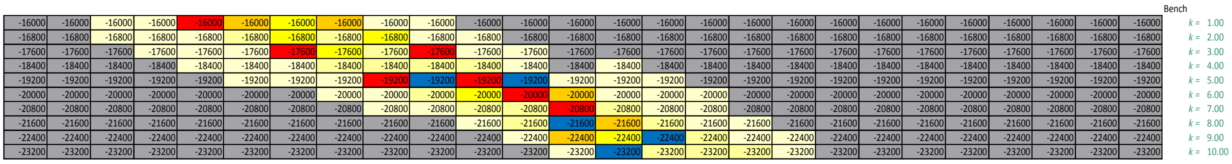
$$BlockVal_{Milled} = BlockRevenue + \left(Mill\ \&\ TransCost \times \frac{Tonnes}{block}\right) + MiningCost\ ,$$

where $Mill\ \&\ TransCost$ refers to the cost of milling and transporting final product.

Generating the block economic model as (see Figure 22):

$$BlockEconomic = \max\left\{BlockVal_{Milled}, MiningCost\right\}.$$

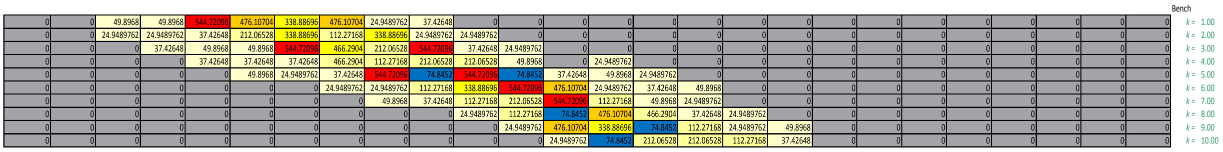
Mining Costs ($/Block)



																										Bench
-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	-16000	*k* = 1.00
-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	-16800	*k* = 2.00
-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	-17600	*k* = 3.00
-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	-18400	*k* = 4.00
-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	-19200	*k* = 5.00
-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	-20000	*k* = 6.00
-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	-20800	*k* = 7.00
-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	-21600	*k* = 8.00
-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	-22400	*k* = 9.00
-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	-23200	*k* = 10.00

Figure 18. Mining cost model ($/block).Colours still relate to grade but now costs are shown, and they increase with depth.

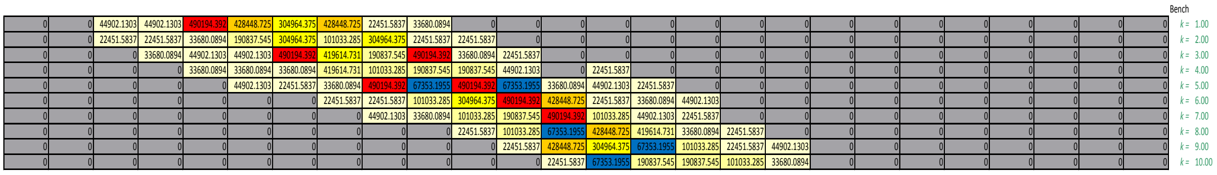
Metal recovered (Oz/Block)



																										Bench
0	0	49.8968	49.8968	544.72096	476.10704	338.88696	476.10704	24.9489762	37.42648	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 1.00
0	0	24.9489762	24.9489762	37.42648	212.06528	338.88696	112.27168	338.88696	24.9489762	24.9489762	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 2.00
0	0	0	37.42648	49.8968	49.8968	544.72096	466.2904	212.06528	544.72096	37.42648	24.9489762	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 3.00
0	0	0	0	37.42648	37.42648	37.42648	466.2904	112.27168	212.06528	212.06528	49.8968	0	24.9489762	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 4.00
0	0	0	0	0	49.8968	24.9489762	37.42648	544.72096	74.8452	544.72096	74.8452	37.42648	49.8968	24.9489762	0	0	0	0	0	0	0	0	0	0	0	*k* = 5.00
0	0	0	0	0	0	0	24.9489762	24.9489762	112.27168	338.88696	544.72096	476.10704	24.9489762	37.42648	49.8968	0	0	0	0	0	0	0	0	0	0	*k* = 6.00
0	0	0	0	0	0	0	0	49.8968	37.42648	112.27168	212.06528	544.72096	112.27168	49.8968	24.9489762	0	0	0	0	0	0	0	0	0	0	*k* = 7.00
0	0	0	0	0	0	0	0	0	0	24.9489762	112.27168	74.8452	476.10704	466.2904	37.42648	24.9489762	0	0	0	0	0	0	0	0	0	*k* = 8.00
0	0	0	0	0	0	0	0	0	0	0	24.9489762	476.10704	338.88696	74.8452	112.27168	24.9489762	49.8968	0	0	0	0	0	0	0	0	*k* = 9.00
0	0	0	0	0	0	0	0	0	0	0	0	24.9489762	74.8452	212.06528	212.06528	112.27168	37.42648	0	0	0	0	0	0	0	0	*k* = 10.00

Figure 19. Metal recovered model (Oz/block). Colours still relate to grade but now metal recovery is shown.

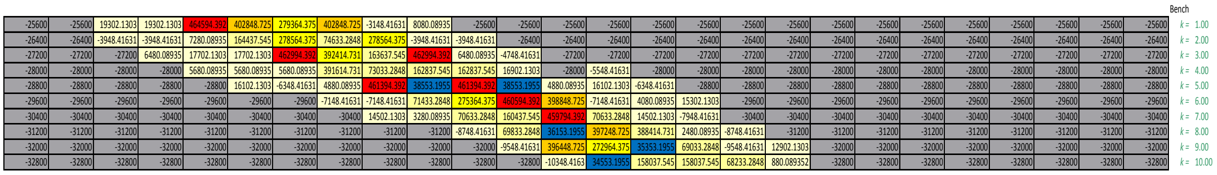
Block Revenue ($/Block)



																										Bench
0	0	44902.1303	44902.1303	490194.392	428448.725	304964.375	428448.725	22451.5837	33680.0894	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 1.00
0	0	22451.5837	22451.5837	33680.0894	190837.545	304964.375	101033.285	304964.375	22451.5837	22451.5837	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 2.00
0	0	0	33680.0894	44902.1303	44902.1303	490194.392	419614.731	190837.545	490194.392	33680.0894	22451.5837	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 3.00
0	0	0	0	33680.0894	33680.0894	33680.0894	419614.731	101033.285	190837.545	190837.545	44902.1303	0	22451.5837	0	0	0	0	0	0	0	0	0	0	0	0	*k* = 4.00
0	0	0	0	0	44902.1303	22451.5837	33680.0894	490194.392	67353.1955	490194.392	67353.1955	33680.0894	44902.1303	22451.5837	0	0	0	0	0	0	0	0	0	0	0	*k* = 5.00
0	0	0	0	0	0	0	22451.5837	22451.5837	101033.285	304964.375	490194.392	428448.725	22451.5837	33680.0894	44902.1303	0	0	0	0	0	0	0	0	0	0	*k* = 6.00
0	0	0	0	0	0	0	0	44902.1303	33680.0894	101033.285	190837.545	490194.392	101033.285	44902.1303	22451.5837	0	0	0	0	0	0	0	0	0	0	*k* = 7.00
0	0	0	0	0	0	0	0	0	0	22451.5837	101033.285	67353.1955	428448.725	419614.731	33680.0894	22451.5837	0	0	0	0	0	0	0	0	0	*k* = 8.00
0	0	0	0	0	0	0	0	0	0	0	22451.5837	428448.725	304964.375	67353.1955	101033.285	22451.5837	44902.1303	0	0	0	0	0	0	0	0	*k* = 9.00
0	0	0	0	0	0	0	0	0	0	0	0	22451.5837	67353.1955	190837.545	190837.545	101033.285	33680.0894	0	0	0	0	0	0	0	0	*k* = 10.00

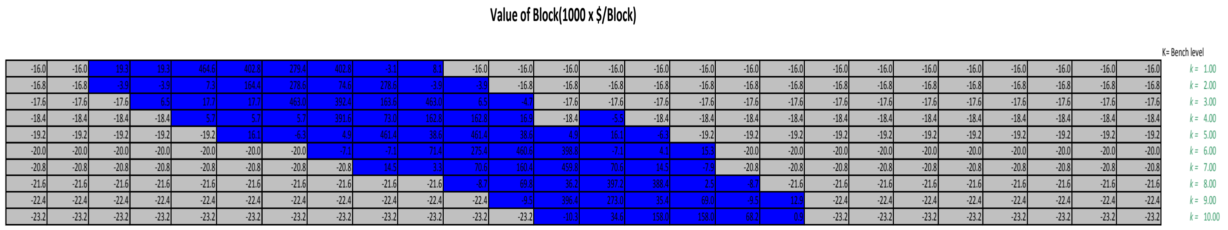
Figure 20. Block revenue model ($/block). Colours still relate to grade but now block revenue are shown.

Value of Block if Milled ($/Block)



																										Bench
-25600	-25600	19302.1303	19302.1303	464594.392	402848.725	279364.375	402848.725	-3148.41631	8080.08935	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	-25600	*k* = 1.00
-26400	-26400	-3948.41631	-3948.41631	7280.08935	164437.545	278564.375	74633.2848	278564.375	-3948.41631	-3948.41631	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	-26400	*k* = 2.00
-27200	-27200	-27200	6480.08935	17702.1303	17702.1303	462994.392	392414.731	163637.545	462994.392	6480.08935	-4748.41631	-27200	-27200	-27200	-27200	-27200	-27200	-27200	-27200	-27200	-27200	-27200	-27200	-27200	-27200	*k* = 3.00
-28000	-28000	-28000	-28000	5680.08935	5680.08935	5680.08935	391614.731	73033.2848	162837.545	162837.545	16902.1303	-28000	-5548.41631	-28000	-28000	-28000	-28000	-28000	-28000	-28000	-28000	-28000	-28000	-28000	-28000	*k* = 4.00
-28800	-28800	-28800	-28800	-28800	16102.1303	-6348.41631	4880.08935	461394.392	38553.1955	461394.392	38553.1955	4880.08935	16102.1303	-6348.41631	-28800	-28800	-28800	-28800	-28800	-28800	-28800	-28800	-28800	-28800	-28800	*k* = 5.00
-29600	-29600	-29600	-29600	-29600	-29600	-29600	-7148.41631	-7148.41631	71433.2848	275364.375	460594.392	398848.725	-7148.41631	4080.08935	15302.1303	-29600	-29600	-29600	-29600	-29600	-29600	-29600	-29600	-29600	-29600	*k* = 6.00
-30400	-30400	-30400	-30400	-30400	-30400	-30400	-30400	14502.1303	3280.08935	70633.2848	160437.545	459794.392	70633.2848	14502.1303	-7948.41631	-30400	-30400	-30400	-30400	-30400	-30400	-30400	-30400	-30400	-30400	*k* = 7.00
-31200	-31200	-31200	-31200	-31200	-31200	-31200	-31200	-31200	-31200	-8748.41631	69833.2848	36153.1955	397248.725	388414.731	2480.08935	-8748.41631	-31200	-31200	-31200	-31200	-31200	-31200	-31200	-31200	-31200	*k* = 8.00
-32000	-32000	-32000	-32000	-32000	-32000	-32000	-32000	-32000	-32000	-32000	-9548.41631	396448.725	272964.375	35353.1955	69033.2848	-9548.41631	12902.1303	-32000	-32000	-32000	-32000	-32000	-32000	-32000	-32000	*k* = 9.00
-32800	-32800	-32800	-32800	-32800	-32800	-32800	-32800	-32800	-32800	-32800	-32800	-10348.4163	34553.1955	158037.545	158037.545	68233.2848	880.089352	-32800	-32800	-32800	-32800	-32800	-32800	-32800	-32800	*k* = 10.00

Figure 21. Value of block if milled ($/block). Colours still relate to grade but now block value are shown.

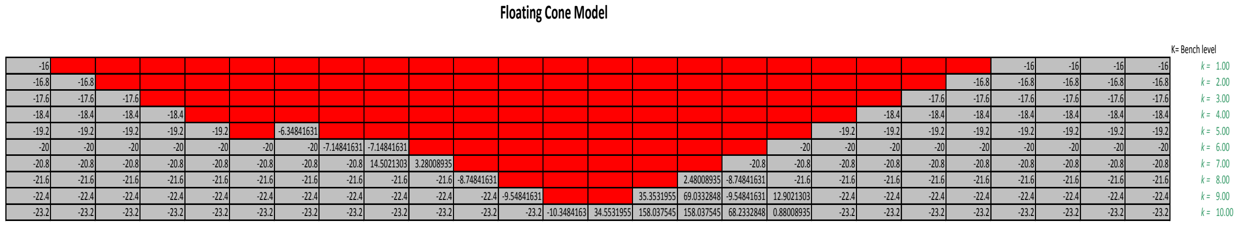


Value of Block(1000 x $/Block)

																										K= Bench level
-16.0	-16.0	19.3	19.3	464.6	402.8	279.4	402.8	-3.1	8.1	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	-16.0	*k* = 1.00
-16.8	-16.8	-3.9	-3.9	7.3	164.4	278.6	74.6	278.6	-3.9	-3.9	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	*k* = 2.00
-17.6	-17.6	-17.6	6.5	17.7	17.7	463.0	392.4	163.6	463.0	6.5	-4.7	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	*k* = 3.00
-18.4	-18.4	-18.4	-18.4	5.7	5.7	5.7	391.6	73.0	162.8	162.8	16.9	-18.4	-5.5	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	*k* = 4.00
-19.2	-19.2	-19.2	-19.2	-19.2	16.1	-6.3	4.9	461.4	38.6	461.4	38.6	4.9	16.1	-6.3	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	*k* = 5.00
-20.0	-20.0	-20.0	-20.0	-20.0	-20.0	-20.0	-7.1	-7.1	71.4	275.4	460.6	398.8	-7.1	4.1	15.3	-20.0	-20.0	-20.0	-20.0	-20.0	-20.0	-20.0	-20.0	-20.0	-20.0	*k* = 6.00
-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	14.5	3.3	70.6	160.4	459.8	70.6	14.5	-7.9	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	*k* = 7.00
-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-8.7	69.8	36.2	397.2	388.4	2.5	-8.7	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	*k* = 8.00
-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-9.5	396.4	273.0	35.4	69.0	-9.5	12.9	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	*k* = 9.00
-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-10.3	34.6	158.0	158.0	68.2	0.9	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	*k* = 10.00

Figure 22. Economic block model $\max\left\{BlockVal_{Milled}, MiningCost\right\}$.

II. The second part in the evaluation process of the gold mine project is finding the ultimate pit limits for the given technical and economic parameters. The result is displayed in Figure 23.



Floating Cone Model

																										K= Bench level
-16	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	-16	-16	-16	-16	*k* = 1.00
-16.8	-16.8	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	-16.8	-16.8	-16.8	-16.8	-16.8	*k* = 2.00
-17.6	-17.6	-17.6	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	*k* = 3.00
-18.4	-18.4	-18.4	-18.4	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	*k* = 4.00
-19.2	-19.2	-19.2	-19.2	-19.2	🟥	-6.34841631	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	*k* = 5.00
-20	-20	-20	-20	-20	-20	-20	-7.14841631	-7.14841631	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥	-20	-20	-20	-20	-20	-20	-20	-20	-20	*k* = 6.00
-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	14.5021303	3.28008935	🟥	🟥	🟥	🟥	🟥	🟥	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	*k* = 7.00
-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-8.74841631	🟥	🟥	🟥	🟥	2.48008935	-8.74841631	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	*k* = 8.00
-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-9.54841631	🟥	🟥	35.3531955	69.0332848	-9.54841631	12.9021303	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	*k* = 9.00
-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-10.3484163	34.5531955	158.037545	158.037545	68.2332848	0.88008935	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	*k* = 10.00

Figure 23. The gold mine project ultimate pit limits (in red colour). This was obtained using the simple floating cone algorithm (see Martinez, 2003 for more details about the floating cone algorithm[25]).

An analysis of the ultimate pit indicators suggests that there are 504k tonnes of ore and 328K tonnes of waste. The total undiscounted mine value is US$8.6M with a total production of 12.32k Ounces of gold with an average grade of 0.244Oz/t.

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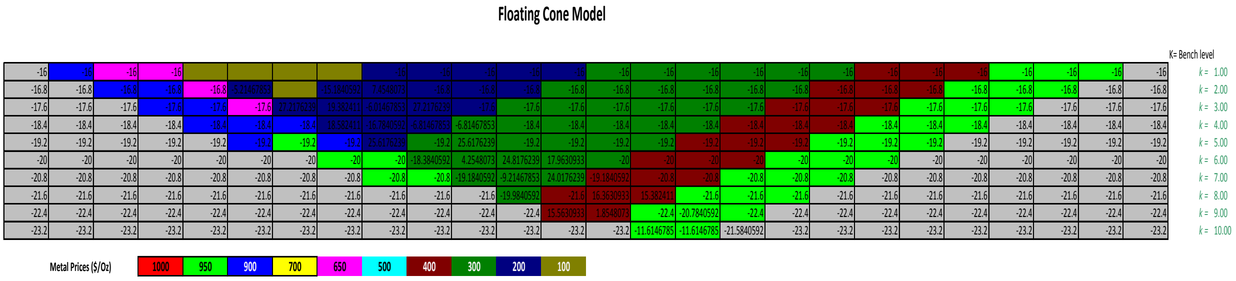
[25] In simple terms, the floating cone method entails approximating the reserve by an inverted cone. The height and radius of the cone are varied as parameters in the search algorithm, also using constraints, e.g., the angle of the side of the cone not exceeding maximum pit slope angles. For each choice of cone shape and location, the NPV for the chosen blocks is calculated, and the blocks are extracted if the NPV is positive. The algorithm continues until the ultimate pit has been explored and all economic blocks have been extracted.

III. The third part in the evaluation process of the gold mine project is to find the appropriate place for starting the mine operation, that is, the starting pit allocation, as well as more information about ore and waste tonnes, grades, and quantity of metal throughout the deposit. To achieve this target, a set of pit shells are generated by changing the gold price as follows: 1000, 950, 900, 700, 650, 500, 400, 300, 200, and 100 dollars per ounce. The results are shown in Figures 24 and 25.



Floating Cone Model

K= Bench level

-16	-16	-16	-16					-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	-16	k = 1.00
-16.8	-16.8	-16.8	-16.8	-16.8	-5.21467853		-15.1840592	7.4548073	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	-16.8	k = 2.00
-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	27.2176239	19.382411	-6.01467853	27.2176239	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	k = 3.00
-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	18.582411	-16.7840592	-6.81467853	-6.81467853	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	k = 4.00
-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	25.6176239	-19.2	25.6176239	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	k = 5.00
-20	-20	-20	-20	-20	-20	-20	-20	-20	-18.3840592	4.2548073	24.8176239	17.9630933	-20	-20	-20	-20	-20	-20	-20	-20	-20	-20	-20	-20	-20	k = 6.00
-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-19.1840592	-9.21467853	24.0176239	-19.1840592	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	k = 7.00
-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-19.9840592	-21.6	16.3630933	15.382411	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	k = 8.00
-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	15.5630933	1.8548073	-22.4	-20.7840592	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	k = 9.00
-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-11.6146785	-11.6146785	-21.5840592	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	k = 10.00

Metal Prices ($/Oz)	1000	950	900	700	650	500	400	300	200	100

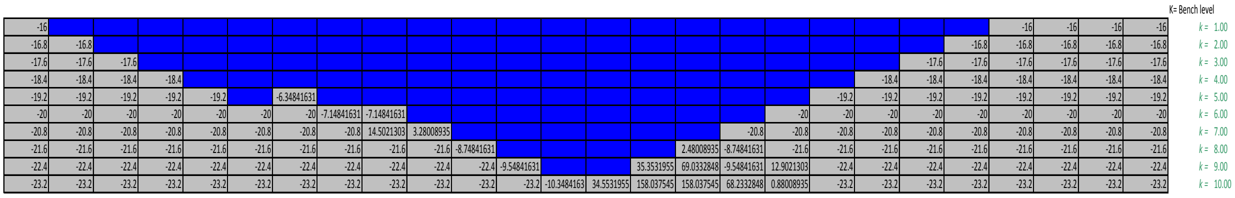
Figure 24. Nested pit shells generation due to different gold prices. Determined using the floating cone algorithm.

Metal Prices ($/Oz)	1000	950	900	700	650	500	400	300	200	100
Mining Value (x1000$)	9915.23	9256.28	8603.55	6168.97	5569.90	3783.95	2604.53	1400.13	610.02	88.45
Mining Cost ($)	-2542400.00	-2542400.00	-1876800.00	-1698400.00	-1698400.00	-1632000.00	-1632000.00	-1136000.00	-424000.00	-80800.00
Ore (Tn)	600000.00	600000.00	504000.00	432000.00	432000.00	400000.00	400000.00	264000.00	128000.00	40000.00
Waste (Tn)	504000.00	504000.00	328000.00	320000.00	320000.00	320000.00	320000.00	248000.00	72000.00	0.00
Metal Qty (Oz)	13178.95	13178.95	12318.21	11981.38	11981.38	11794.26	11794.26	9687.59	6090.78	2174.71
Avg. Grade (Oz/t)	0.02196	0.02196	0.02444	0.02773	0.02773	0.02949	0.02949	0.03670	0.04758	0.05437

Figure 25. The gold mine project key indicators for each pit shell.

Based on the results displayed in Figure 24, it is found that the 100$/Oz pit shell (grey colour) indicates the best zone to start the mine operation since it indicates the presence of high gold grade. It is important to remark that, as observed in Figures 24 and 25, two or more metal prices can generate the same pit shell limits. For example, it is observed that the 1000$/Oz pit shell and the 950$/Oz pit shell have the same limits and consequently the same value for the indicators of the project, except the mine value. This is because the mine value is estimated using different gold prices.

Since the given expected gold price was 900$/Oz, the ultimate pit limits of the 900$/Oz pit shell are considered to be the ultimate pit limits of the Gold Mine project (see Figure 26).



																										K= Bench level
-16	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	-16	-16	-16	-16	*k* = 1.00
-16.8	-16.8	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	-16.8	-16.8	-16.8	-16.8	-16.8	*k* = 2.00
-17.6	-17.6	-17.6	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	-17.6	-17.6	-17.6	-17.6	-17.6	-17.6	*k* = 3.00
-18.4	-18.4	-18.4	-18.4	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	-18.4	*k* = 4.00
-19.2	-19.2	-19.2	-19.2	-19.2	🟦	-6.34841631	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	-19.2	*k* = 5.00
-20	-20	-20	-20	-20	-20	-20	-7.14841631	-7.14841631	🟦	🟦	🟦	🟦	🟦	🟦	🟦	🟦	-20	-20	-20	-20	-20	-20	-20	-20	-20	*k* = 6.00
-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	14.5021303	3.28008935	🟦	🟦	🟦	🟦	🟦	🟦	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	-20.8	*k* = 7.00
-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-8.74841631	🟦	🟦	🟦	🟦	2.48008935	-8.74841631	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	-21.6	*k* = 8.00
-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-9.54841631	🟦	🟦	35.3531955	69.0332848	-9.54841631	12.9021303	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	-22.4	*k* = 9.00
-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-10.3484163	34.5531955	158.037545	158.037545	68.2332848	0.88008935	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	-23.2	*k* = 10.00

Figure 26. The Gold Mine open pit mine project set up on the 900$/Oz pit shell limits.

IV. The fourth part in the evaluation process of the gold mine project consists of establishing the long-term production scheduling of the gold project. To achieve this, the mill target of 80000 tonnes of ore (or equivalently 10 ore blocks) per production period is considered to be the main technical constraint (observe that in this specific case waste tonnes is not considered as a constraint), while the maximisation of cash flow is considered to be the main economic constraint.

At this stage of the evaluation process two production schedules are generated: the bench-by-bench and the practical production schedules. The results are displayed in Figures 27 and 28 for the bench-by-bench schedule, and Figures 29 and 30 for the practical schedule.

As observed in Figure 27, the **bench-by-bench production schedule** is the sequence where the mine is depleted by mining one bench at a time until meeting mill targets (in this case 10 blocks or 80000 tonnes of ore). Observe that this schedule does not consider the waste that needs to be removed in order to achieve mill targets. For example, as displayed in Figure 28, during the first production period the entire first bench was mined before mining the second bench in order to meet mill demand. Then, the expected stripping ratio of this first production period is 1.6 (16/10). A similar procedure was followed in the remaining production periods. The final results (shown in Figure 28) suggest that the gold mine project has an expected Life Of Mine (LOM) of 6 production periods (depending on the periods it could be months, years or any other period of time) and will generate on average a total revenue of $6.36 million (as observed today).

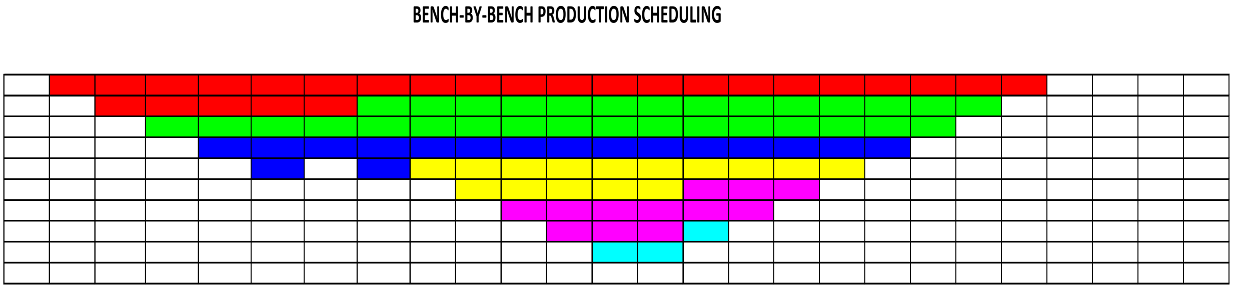


Figure 27. The gold mine project: bench-by-bench production schedule.

BENCH-BY-BENCH PRODUCTION SCHEDULING

	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6
ORE	10	10	10	10	10	3
WASTE	16	21	7	5	2	0
UCF	1827.6	1562.2	729.3	2156.0	1270.7	1057.8
WACC	0.909	0.827	0.752	0.684	0.622	0.565
DCF	1661.887	1291.744	548.382	1474.199	790.063	598.094

Figure 28. The gold mine project - bench-by-bench production schedule, key project indicators (10% WACC).

As opposed to the bench-bench production schedule, the **practical production schedule** is the sequence where the mine is depleted by mining cones of material where the mill demand is achieved and the waste removal minimised. For example, it is observed in Figures 29 and 30, that if the **practical production schedule** is used to deplete the Gold project, the first production period will have a zero stripping ratio, that is no waste is necessary to be removed in order to meet the mill target. This of course will have a significant effect in the cash flow generated during the first production period generating almost 13% more than the Bench-by-bench schedule. Observe that compared with the bench-by-bench schedule, the practical production schedule defers the waste removal to later periods generating high cash flows during early production periods.

The final results of the implementing the practical production schedule in the gold mine project, as displayed in Figure 30, suggests that the project will generate $6.5 million on average.

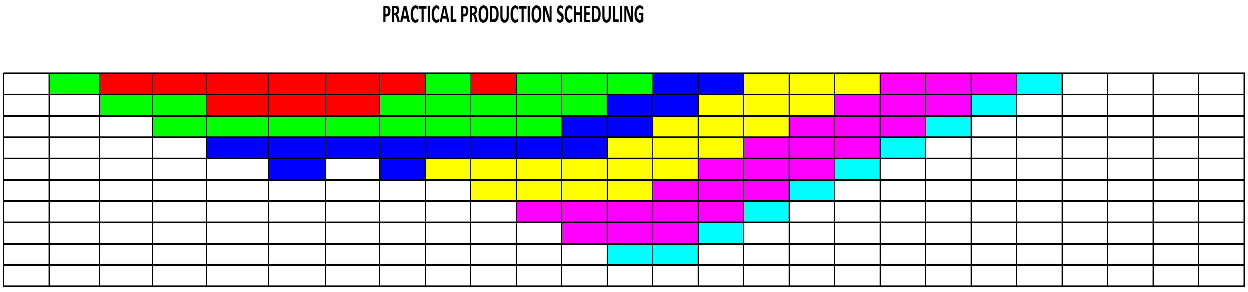


Figure 29. Gold mine project: practical production schedule.

PRACTICAL PRODUCTION SCHEDULING

	Periodo1	Periodo2	Periodo3	Periodo4	Periodo5	Periodo6
ORE	10	10	10	10	10	3
WASTE	0	10	6	12	16	7
UCF	2046.6	1783.9	757.3	2033.6	1040.3	941.9
WACC	0.9093	0.8269	0.7519	0.6838	0.6218	0.5654
DCF	1861.0735	1475.0698	569.43595	1390.471763	646.839207	532.53721

Figure 30 The gold mine project- practical production schedule, key project indicators (@10% WACC).

Figures 31 and 32 clearly indicate the variation in waste removal and cash flow generation for both the practical and bench-by-bench production schedules. As observed in the figures, the practical schedule generates higher non-cumulative cash flows during the first and second production periods due to the low stripping ratio, and lower ones during the last two periods, due to the high stripping activity.

As the time value of money affects the cash flows generated throughout the LOM, it is observed in Figure 32 that even though the practical schedule generates lower cash flows than the bench-by-bench schedule during the final periods, it still gives the maximum value when discounted to the present time.

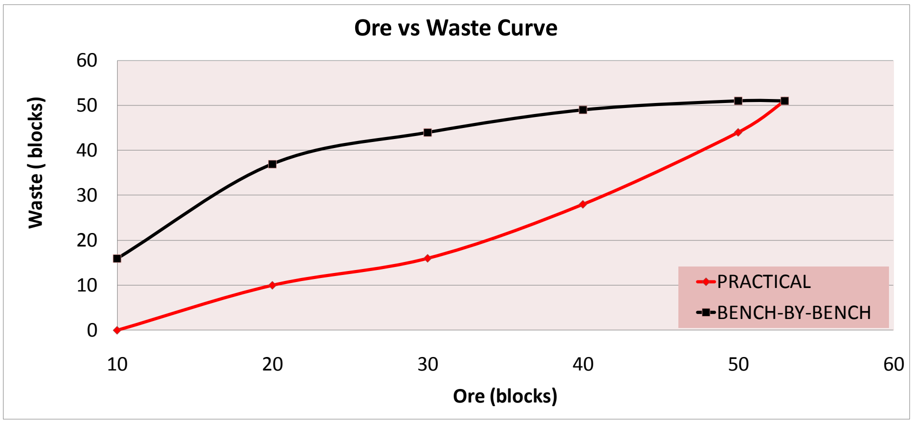


Figure 31 Ore vs. Waste curve for the practical and bench-by-bench production schedules.

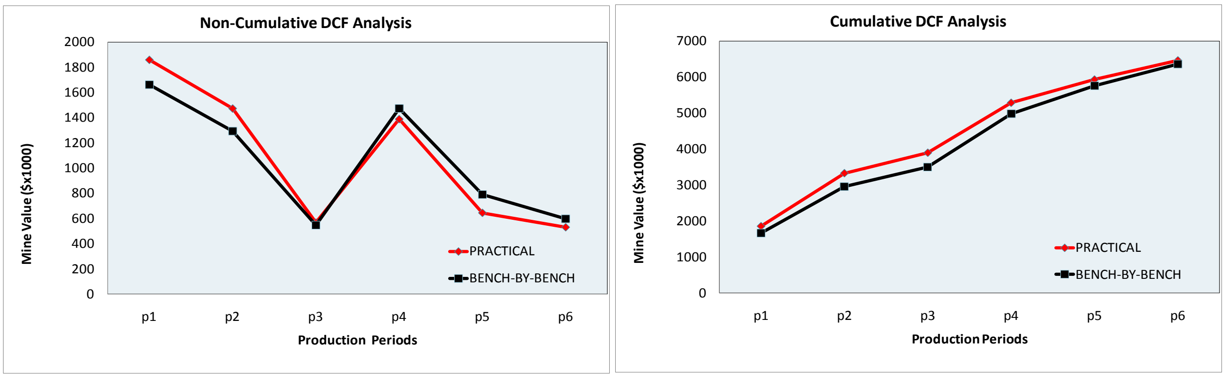


Figure 32. Non-cumulative and cumulative Discounted Cash Flow (DCF) analyses for the gold mine project considering the practical and bench-by-bench production schedules.

3.4 Conclusion

As observed throughout this chapter, the building of the base-case open pit mine plan and design is not an easy process. It requires the implementation of different mining engineering and DCF analyses to render the expected final mine project value.

Since the base-case mine design is the long-term plan of the mine project, it is very important to make sure that the base-case mine plan and design is built considering the best estimates of the technical and economic input variables as well as the best engineering procedures for open pit mine plan and design. The reason for this is that the base-case mine plan will serve as an initial platform for further analysis and also for identifying cash flow drivers, which can be considered for further analysis when implementing a complete mine evaluation process using the IVOF.

Chapter 4

The Integrated Valuation Optimisation Framework (IVOF)

4.1 Introduction

The objective of this chapter is to introduce a novel mine evaluation framework, named IVOF, as an alternative technology for mine project evaluation where uncertainty and risk are seen as allies of the mine analyst when evaluating an open pit mine project. To achieve this, Section 3.2 defines the concept of the "flaw of averages in mine project evaluation", which shows what problems can arise when single estimated values are substituted for a distribution of values when evaluating an open pit mine project in the face of uncertainty. Section 3.3 introduces the IVOF process as an alternative technology for mine project evaluation. In this section, the foundations of the IVOF are given in a comprehensive fashion, while the model framework is formally defined in Section 3.4. Conclusions and comments are given in Section 3.5.

4.2 The *"flaw of averages"* in mine project evaluation

Traditionally, mine organisations use various types of quantitative methods to estimate profit and loss associated with a proposed mine project. Among all these measures of profitability, the Net Present Value (NPV), which is based on the Discounted Cash Flow (DCF) technique (see for example Benninga, 2000), is the most widely used in the mining industry. This is because it recognises the time value of money, and accounts for risk *via* a risk adjusted discount rate, R (see Equation 1.1), giving the mine analyst a tool for making financial investment and dividend decisions. More formally, the NPV technique consists of subtracting the capital investment, $CapInv$, incurred at the beginning of the mining project (assumed to be period t_0), from the sum of the present value of the expected net cash flows ($E\{CF_t\}$) generated throughout the operating life ($t=1,2,..,T$) of the open pit mine project:

$$NPV = \sum_{t=1}^{T} \frac{E\left\{CF_t\right\}}{(1+R)^t} - CapInv \qquad (1.1)$$

In practice, the expected cash flows generated at each production period, $t = 1, 2, .., T$ are estimated using expected values for the underlying variables such as the metal price, S_t, production costs, $Cost_t$, and metal quantity, q_t, produced, that is,

$$E\left\{CF_t\right\} = E\left\{q_t\right\} \times \left(E\left\{S_t\right\} - E\left\{Cost_t\right\}\right). \qquad (1.2)$$

One consequence of using expected values when estimating cash flows is that the resulting NPV value is also assumed to be an expected value, which, as it is shown later on, may not reflect the real project's value, thereby leading to incorrect decisions.

Although some variations of the Discounted Cash Flow (DCF) technique, such as scenario analysis, have been developed to give mine analysts the flexibility of including different scenarios in the mine evaluation process, they still suffer the same problem of the DCF, that is, instead of working with the uncertain variables, these techniques work with a single estimated value[26] for each scenario, relying on the adjusted discount rate, R, to account for risk and uncertainty in the entire mine project.

The problem with evaluation techniques based on the DCF is that in cases involving uncertainty and non-linear processes, in our case the mine optimisation/evaluation process, single estimate values are often of little use because of their lack of accuracy in describing an uncertain process. In other words, as it is shown in Figure 33, serious trouble can arise when a single number is substituted for a distribution of probabilities. That is if the expected value, $E\left\{X\right\}$, of the uncertainty variable, X, is input into the non-linear process $F(\cdot)$, the resulting output, $F\left(E\{X\}\right)$, will not be the same as the expected value of the resulting outputs, $E\left\{F(X)\right\}$, generated by inputting the entire distribution of values; that is, $F\left(E\{X\}\right) \neq E\left\{F(X)\right\}$.

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[26] Observe that a single estimate in the context of a financial and engineering statement is a single number, often an average or expected value, used to represent the value of an uncertain quantity such as the average metal grade of the deposit, the price of the metal, and future mine revenues or expenses, among others. An uncertain quantity is normally represented by a probability distribution, or a bar graph, which represents the relative likelihood of various outcomes

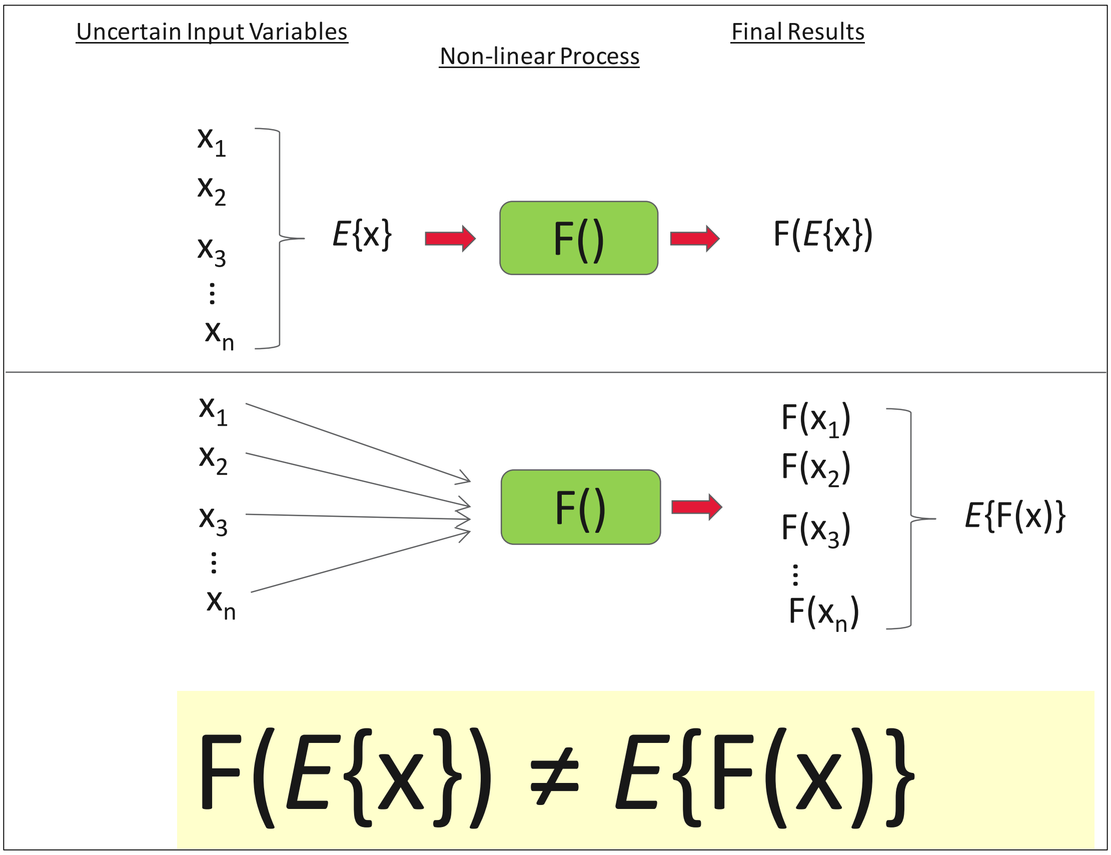

Figure 33. Scheme showing that "average inputs do not always yield average outputs" when dealing with uncertainty and non-linear processes.

Professor Savage from Stanford University refers to this problem as "the *Flaw of Averages*[27] (Savage, 2002a; Savage, 2003; Savage, 2002b)", which states that plugging average values of uncertain inputs into a non-linear process **does not** result in the average value of the process; that is, $F(E\{X\}) \neq E\{F(X)\}$. He explains this concept with the following example (see Figure 34).

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[27] Also known in finance as Jensen's inequality, which states that because the value of a project, ***x***, is a random variable and the option value, ***OV***, on the project is a convex function of the project value, then, $OV\left(E\{X\}\right) \neq E\left\{OV(X)\right\}$.

"Consider the state of a drunk, wandering back and forth on a busy highway. His average position is the centreline of the highway. Therefore the state of the drunk at his average position is alive. However, it is clear that the average state of the drunk is **dead**".

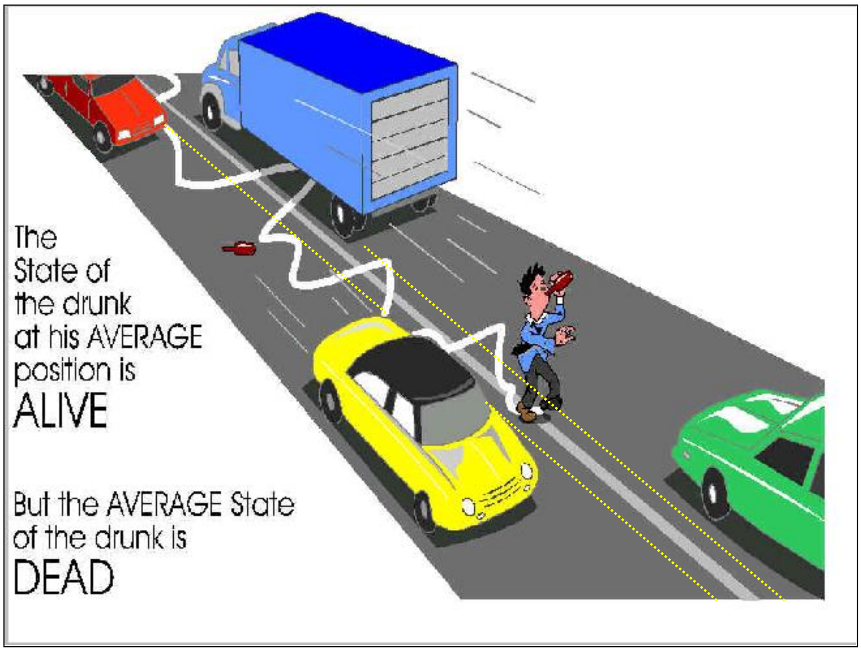
The State of the drunk at his AVERAGE position is
ALIVE
But the AVERAGE State of the drunk is
DEAD

Figure 34. A sobering example of the Flaw of Averages (from Savage, 2003).

An analogous situation happens when evaluating a mine project using traditional mine evaluation techniques that are based on the DCF. That is when evaluating a mine project it is common to use expected single values for representing all the mine variables[28] that are input into the non-linear mine optimisation process (Martinez, 2003; Dimitrakopoulos, 1998). The final output of this practice is a single estimated value for each of the project indicators, such as projected revenues and expenses, grades, metal quantities, and mining and processing costs, among others, which are assumed to be the average values to be obtained. Although it is common to perform a sensitivity analysis that uses spider and tornado diagrams to obtain a sort of interval of confidence for final mine revenues, traditional mine evaluation techniques ignore any possible realistic fluctuations in revenue or expense due to the existing uncertainty of the different input variables over time, and corresponds to the assumption that the drunk guy will always be walking on the centre line (see Figure 34).

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[28] Examples of these input variables are: the orebody model, metal prices, costs, and recoveries.

The problem is that even though sensitivity analysis is supposed to account for variations in the different input variables, it assumes that these changes will happen in a linear fashion, that is the same change will occur at each production period, which is not true. See for example Figure 34, where the yellow dashed lines represent the ± 10% confidence interval that is supposed to account for the drunk's trajectory deviation from the central line. As observed in the figure, this confidence interval does not give a realistic representation of the drunk's trajectory. Another limitation of sensitivity analysis is that it ignores the dependence structure between the underlying variables that take part in a mine evaluation process, performing changes in an isolated fashion, that is changes to a specific variable are performed keeping the other ones constant.

In the case of a mining project, metal grade variations will occur at different locations of the orebody model, but following a specific correlation structure, that is, changes at different locations will be generated following a specific correlation structure, which is a non-linear process. Similarly, metal prices will also vary at each production period but at different rates. Thus, it is important to be cautious when making decisions based on a sensitivity analysis, since it could lead to a spurious description of the current financial situation of the mining project.

One of the techniques that has been widely accepted as a unified approach to dealing with uncertainty is the **Monte Carlo Simulation** technique (Glasserman, 2004; Chan and Wong, 2006). This is because instead of taking a single best "estimate" this technique quantifies uncertainty by sampling the probability distribution of the uncertain variable while tracking the resulting outputs.

However, despite its benefits when dealing with uncertainty, the application of the Monte Carlo technique to the mine evaluation problem is not straightforward. The reason for this is that the mine optimisation process is a 3D complex, non-linear procedure where the uncertainties of the input variables are of different natures. For example, the uncertainty of the orebody model could be classified as static (Martinez, 2007) since it depends on the geology of the deposit that is uncertain; not because it changes over time, but because of the limited data (e.g., drill-hole data) obtained for its quantification[29]. On the other hand, the uncertainty of future metal prices can be classified as dynamic because it depends on the international metal market, which is affected by different mechanisms such as offer, demand and speculation, and varies over time.

Furthermore, besides the nature of the sources of technical and economic uncertainties, the mine evaluation problem is not only concerned with the opportunities and risk that can occur throughout the operating life of the mine project, but also with the planning and designing of the actual mine operation, which often exhibits complex path-dependency/interdependency that standard options theory cannot deal with.

4.3 A road to improve: Developing a novel integrated mine evaluation framework (IVOF) that accounts for uncertainty and risk

As it was mentioned in Chapter 2, different techniques have been developed to overcome the complexity of the mine evaluation problem. Although some of them have been shown to be very efficient in dealing with a specific part of the problem, none of them has been able to solve the complete problem of considering all sources of uncertainty appropriately. The reason that current techniques cannot appropriately solve the mine problem is that these techniques have been developed in isolation. That is mine evaluation techniques, such as the Upside/Downside Potential (Martinez, 2003; Dimitrakopoulos, Martinez and Ramazan, 2004), developed to deal with technical uncertainties such as the orebody model, do not account for the uncertainty of economic variables. Similarly, mine valuation techniques such as real options that deal with economic uncertainties such as metal prices do not appropriately account for technical uncertainty. So, it is logical to assume that if both technologies are combined, the final result will be a more accurate mine evaluation technique where both technical and managerial flexibility are integrated.

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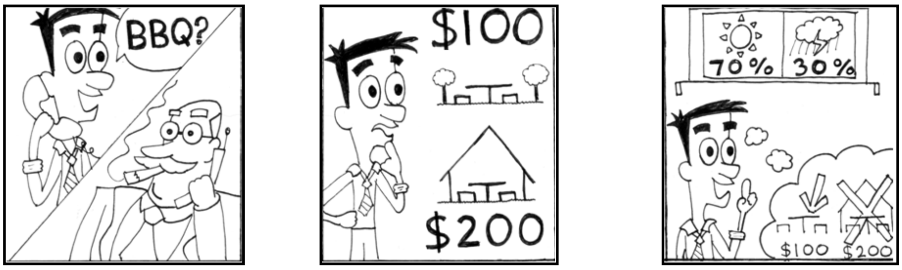
[29] It is possible to minimise the uncertainty of an orebody model by taking samples on a very small grid. However, this procedure will result in a non-profitable project because of the high cost incurred in the data collection.

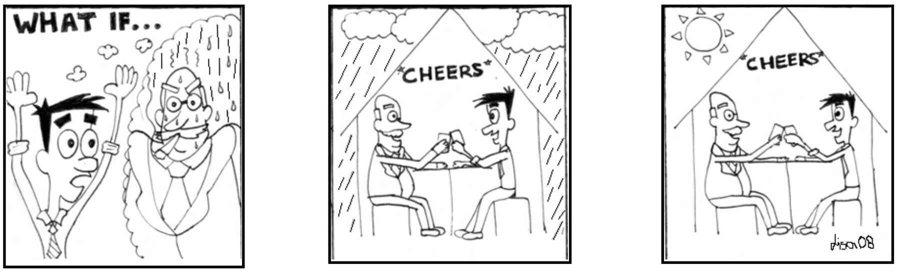
The Integrated Valuation/Optimisation Framework (IVOF) is a novel mine evaluation framework, which is not only able to account for uncertainty and risk when evaluating a mine project, but also it is able to account for mine project design flexibility. That is the IVOF breaks down the wall that isolates mine optimisation from mine valuation techniques and integrates the best of both technologies in a generalised new framework.

Derived from the financial options theory, the concept of the IVOF can be easily visualised with the following example (see Figure 35).

Imagine that tomorrow you arrange a meeting with an important client to write a **PROFITABLE CONTRACT**. It happens that the client likes BBQs but hates rain. So you arrange the meeting in a BBQ area. When booking the BBQ area you realise that there are two areas, one with a roof that costs $200 and other one without a roof that costs $100. So you see yourself with the **OPTION** of either booking an expensive BBQ area with a roof or a cheap one without a roof. Before booking, you check the weather forecast for the next day and you notice that the probabilities for raining and not raining are 30% and 70%, respectively, and the weather forecaster's expectation is a sunny day ideal for a BBQ outside. So, based on the weather forecaster's expectation, and the fact that the probability of not raining is high, you make the decision to book the cheap BBQ area without a roof for $100. Back at your house, you think about your decision and ask yourself the question **"What if it rains?"** The answer is that if it rains, you will lose the revenues that can be generated from the **PROFITABLE CONTRACT,** which you realise is a much higher cost to pay than the extra $100 needed to book the BBQ with a roof. Consequently, you realise that the best strategy to follow is to return and book the area with a roof for $200 because of the following possible outcomes: i) if it does not rain you will lose only $100, but get a **PROFITABLE CONTRACT** that will generate substantial revenues; and ii) if it does rain you still will get the **PROFITABLE CONTRACT**. This is precisely the basis of the RO technique, which is a contract that gives you the right but not the obligation of exercising it.



The opportunity to write a profitable project Humm..The *cheap* or the expensive one ? The expected weather is a sunny day – the cheap one



But,..WHAT IF IT RAINS – You lose the project Better book the expensive BBQ are with a roof: If it is either a rainy or sunny day you still have your Profitable Project.

Figure 35. A simple explanation of the real options concept: The profitable contract signed on a BBQ area.

Analogous to the example shown in Figure 35, the IVOF sees the mine evaluation problem as a multi-stage solution, rather than a holistic process, where the problem is broken down into a set of simple building blocks:

- *Base case mine plan & design model*. Here the main long-term operational and economic targets of the mine project are set up. It is at this stage where the main cash flow drivers and their respective uncertainties are identified. This is achieved by using static analysis such as sensitivity analysis that uses spider and tornado diagrams. (In Figure 35 this stage corresponds to the option to write a profitable contract and the cost of hiring a suitable BBQ area.)

- *Profit and Loss* $\left(P\,\&\,L\right)$ *model- Upside/Downside potential*. Risk analysis is a powerful tool that helps mine planners and mine managers to understand the following aspects of a mine project (see for example McNeil, Frey and Embrechts, 2005): i): what the main sources of uncertainty are and, consequently, the causes of risk and opportunity; ii) how the risk and opportunity should be measured; and iii) what the effects are of the existing risks and opportunities on the final project value. (In Figure 35 this stage corresponds to the analysis of the weather forecast and the *What if..?* question that arises due to the existing uncertainty about the weather forecast.)

- *Real options "in" mine projects model*. Here the mine analyst identifies the different technical/managerial and mine plan and design flexibilities (strategies) that will maximise project value in the face of uncertainty; and

- *Decision-making model*. Here the mine analyst makes rational final decisions based on the previous stages. (In Figure 35 this stage corresponds to the decision made about hiring the BBQ area with a roof.)

Accordingly, the IVOF is seen as a new technology that not only extends current state of the art mine evaluation techniques to incorporate uncertainty and risk, but also as a technology where the planning, design and valuation of a mine project is performed using the existing uncertainty and risk. That is the IVOF technology sees uncertainty and risk as allies instead of enemies.

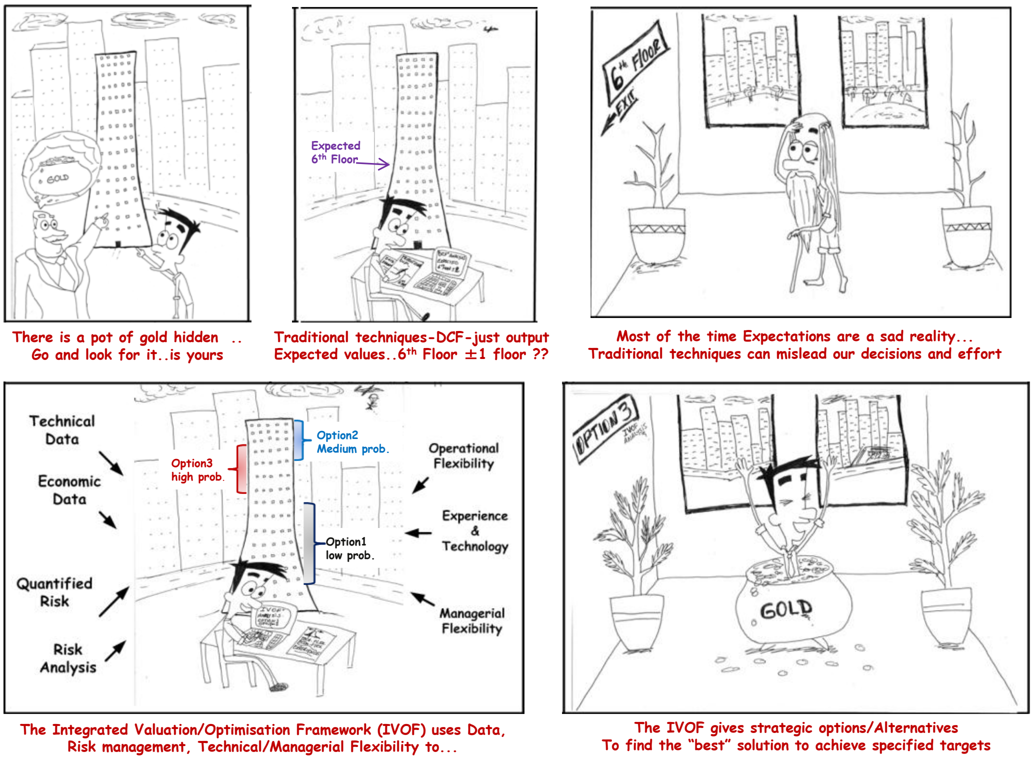
But, in layman's terms, what is the difference between traditional mine evaluation techniques and the proposed IVOF?

The main difference between traditional techniques and the IVOF, which accounts for uncertainty and risk in the mine evaluation process, resides in that the latter gives a series of alternatives with their respective likelihood of where to look for final targets while the former gives a single value with a static interval of confidence.

Imagine for example, as displayed in Figure 36, that a rich man tells us that he has hidden a pot of gold in one of the floors of a very high skyscraper. He also tells us that if we can find the gold in a specific time period we can take the gold with us. If we use traditional techniques that are based on expected values, such as the DCF and sensitivity analysis, we could conclude that the expected floor containing the gold is, let's say, floor six with floor seven and five as interval of confidence. However, practical experience has shown that most of the time expectations are a sad reality and can mislead our final decisions and efforts.

Conversely, the proposed IVOF process creates a series of options, with their respective likelihoods and outcomes, from where we could start looking for the pot of gold. minimising in this case our effort and giving us a more accurate result than traditional techniques. This is because an IVOF process not only uses the available data also considers uncertainty and risk as well as operational and economic flexibility, which are based on many factors such as experience and knowledge of the problem at hand.

Consequently, if accurate final decisions are to be made when evaluating a mine project using the IVOF structure, it is importantly firstly to quantify the main sources of existing uncertainty such as the orebody model, costs, and metal prices. Secondly, potential risks and opportunities that can arise due to uncertainty need to be identified; and thirdly the best strategies to implement in the face of a risky or opportunity event need to be identified.


GOLD
Expected 6th Floor
6th FLOOR
EXIT
Technical Data
Economic Data
Quantified Risk
Risk Analysis
Option3 high prob.
Option2 Medium prob.
Option1 low prob.
Operational Flexibility
Experience & Technology
Managerial Flexibility
OPTION 3
GOLD

There is a pot of gold hidden ..
Go and look for it..is yours

Traditional techniques-DCF-just output
Expected values..6th Floor ±1 floor ??

Most of the time Expectations are a sad reality...
Traditional techniques can mislead our decisions and effort

The Integrated Valuation/Optimisation Framework (IVOF) uses Data,
Risk management, Technical/Managerial Flexibility to...

The IVOF gives strategic options/Alternatives
To find the "best" solution to achieve specified targets

Figure 36. Graph showing the main difference between traditional mine evaluation techniques and the IVOF process.

4.4 The Novel Integrated Valuation/Optimisation Framework

In the IVOF context, a mine project, in this case an open pit mine project, is composed of two main components: the open pit mine project plan and design, ζ_t^j; and the corresponding mine project's value V_t^j; where $j = 1, 2, .., J$ indicates the mine design number.

Then, the open pit mine project evaluation problem consists of finding the best open pit mine plan and design, ζ^*, that generates a maximum current value , $V_{t_0}^*$, for the mine project, that is,

$$\left(\zeta^*, V_{t_0}^*\right) = \max_{V_{t_0}} \left\{ \left(\zeta_1, V_{t_0}^1\right), \left(\zeta_2, V_{t_0}^2\right) .., \left(\zeta_j, V_{t_0}^j\right) \right\}. \tag{1.3}$$

Equation 1.3 is, indeed, a very complex problem where the open pit mine engineering is integrated with mathematical, and economic and financial processes.

For instance, the open pit mine project plan and design variable, ζ_t^j, can be defined as

$$\zeta_{t_i}^j = \bigcup_{k=(i+1)}^{T} \xi_{t_k}^j, \tag{1.4}$$

where $\xi_t^j = \left\{ \upsilon_{x_i}^t \mid \omega_t \right\}_{i=1,2,..W}$ is the set of all blocks, $\upsilon_{x_i}^t\left(BlkVal, G, ton\right)$, which in turn are functions of block value (*BlkVal*), grade (*G*) and tonnes (*ton*), extracted at production period $t = \left\{ t_1 = \Delta t, .., t_T = T\Delta t \right\}$, given an initial set of technical and operational constraints, ω_t. Examples of these technical constraints are the open pit slope constraints and mining and processing capacities, among others.

Similarly, the value, $V_{t_0}^j$, of an open pit mine project, given a mine plan and design ζ_j, is defined as the total profit generated by the project cash flows, $CF_t^{\zeta_t^j}$, generated throughout the operating life of the open pit mine, that is, $t = \left\{ t_1 = \Delta t, .., t_T = T\Delta t \right\}$.

In open pit mine operations parlance, the variables ζ_j and ξ_t^j define the ultimate pit and the long-term production scheduling limits of the open pit mine project respectively (see Figure 37). As observed in Figure 37, each block, $\upsilon_{x_i}^t\left(BlkVal,G,ton\right)$, inside volume ς_t^j, which represents the mine production at period t, is characterised by its spatial 3-dimensional location, $x_i=\left(x_1^i,x_2^i,x_3^i\right)$, the period t where it is extracted, its block tonnage, ton[30], the metal grade, G_{x_i}, and its economic value, $BlkVal_{\upsilon_{x_i}^t}$.

The economic value of a block, $BlkVal_{\upsilon_{x_i}^t}$, is defined as the maximum between the value obtained if the block is sent to the mill, $BlkVal_{milled}^{\upsilon_{x_i}^t}$ or the value of mining it and sending it to the waste dump, $BlkVal_{dumped}^{\upsilon_{x_i}^t}$, that is,

$$BlkVal_{\upsilon_{x_i}^t}=max\left\{BlkVal_{milled}^{\upsilon_{x_i}^t},BlkVal_{dumped}^{\upsilon_{x_i}^t}\right\}. \tag{1.5}$$

Then, if the block, $\upsilon_{x_i}^t$, is mined and sent to the mill, its value is

$$BlkVal_{milled}^{\upsilon_{x_i}^t}=ton\cdot G_{x_i}\cdot y\cdot\left(S_t-c_{refining}\right)-ton\cdot\left(c_{mining}+c_{milling}\right), \tag{1.6}$$

where, c_{mining}, $c_{milling}$, $c_{refining}$, are the mining, milling and refining unit costs (\$/tonne) respectively; ton is the tonnage of the material inside block $\upsilon_{x_i}^t$; G_{x_i} is the metal grade of the block (units/tonne); y is the milling yield or milling recovery (%), and S_t is the metal price at period t.

On the contrary, if the block $\upsilon_{x_i}^t$ is sent to the waste dump, then, its value is just the cost (negative value) of mining the block,

---

[30] Observe that, although; one characteristic of block models is that each block inside the model has the same dimensions, the density and consequently the tonnes of material may or may not be the same. The reason for this is the occurrence of different rock-type material with different densities in the mineral deposit.

$$BlkVal_{dumped}^{\upsilon_{x_i}^t} = -ton \cdot c_{mining} \,. \tag{1.7}$$

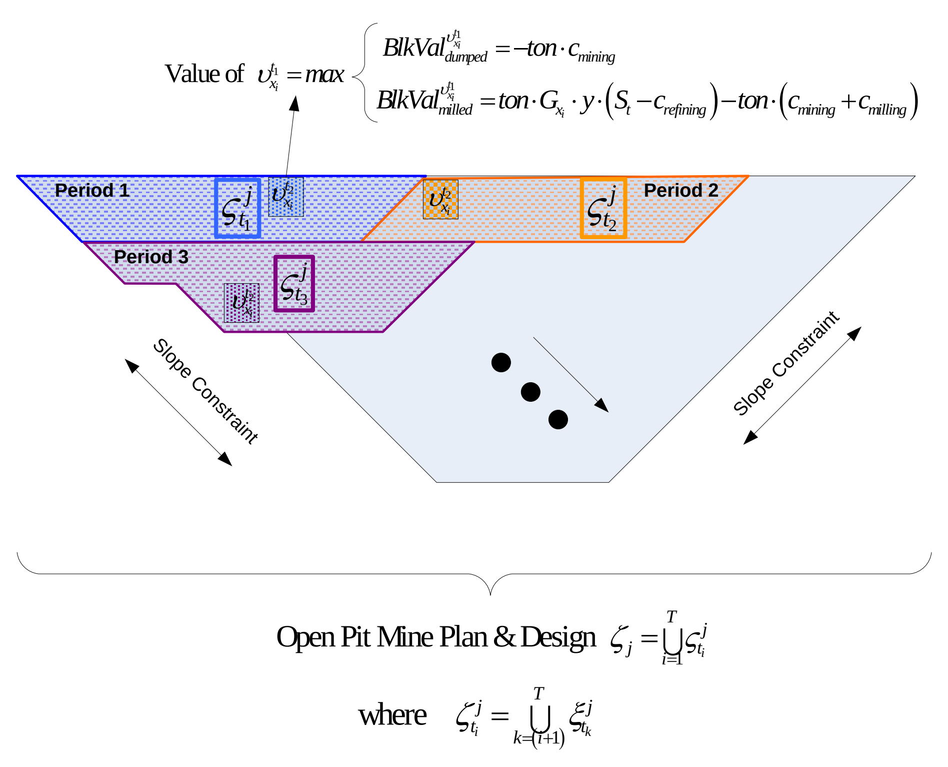

Value of $\upsilon_{x_i}^{t_1} = max \begin{cases} BlkVal_{dumped}^{\upsilon_{x_i}^{t_1}} = -ton \cdot c_{mining} \\ BlkVal_{milled}^{\upsilon_{x_i}^{t_1}} = ton \cdot G_{x_i} \cdot y \cdot (S_t - c_{refining}) - ton \cdot (c_{mining} + c_{milling}) \end{cases}$
Period 1
$\varsigma_{t_1}^j$
$\upsilon_{x_i}^{t_1}$
$\upsilon_{x_i}^{t_2}$
$\varsigma_{t_2}^j$
Period 2
Period 3
$\varsigma_{t_3}^j$
$\upsilon_{x_i}^{t_3}$
Slope Constraint
Slope Constraint
Open Pit Mine Plan & Design $\zeta_j = \bigcup_{i=1}^{T} \varsigma_{t_i}^j$
where $\varsigma_{t_i}^j = \bigcup_{k=(i+1)}^{T} \xi_{t_k}^j$

Figure 37 Schematic showing the j plan and design, ζ_j, of an open pit mine project. As observed, ζ_j is composed of the union of the production scheduling plan and designs, ς_t^j $(t = \{t_0 = 0, t_1 = \Delta t, ..., t_T = T\Delta t\}$. Also observe that the shape of the production scheduling plan and design is given by the blocks, $\upsilon_{x_i}^t$, to be extracted at each production period t and which obey technical constraints such as slope angle. The value of each block is given by Equations 3.4-3.6.

Also observe in Figure 37 that each production period plan and design, ς_t^j, will generate a cash flow, $CF_t^{\varsigma_t^j}$, defined as

$$CF_t^{\varsigma_t^j} = \sum_{i=1}^{N} BlkVal_{\upsilon_{x_i}^t} - c_{fixed} \,, \tag{1.8}$$

where N is the number of total blocks mined (that is, blocks milled + blocks dumped) at production period t, and c_{fixed} is the total fixed cost incurred in the mining process (to be assumed constant for each production period).

Then, replacing Equations 1.4-1.7 in Equation 1.8 renders the following expression for the cash flow, given a mine plan and design ζ_j:

$$CF_t^{\varsigma_t^j} = q_t \cdot y \cdot \overline{G}_t \cdot S_t - q_t \cdot y \cdot \overline{G}_t \cdot c_{refining} + q_t \cdot c_{milling} + q_t \cdot (1+sr) \cdot c_{mining} + c_{fixed} \,, \quad (1.9)$$

where,

c_{mining}, $c_{milling}$, y, and S_t, are the same as defined in Equation 1.6;

q_t: is the total tonnes of the blocks mined and sent to the mill at production period t, formally defined as *ore*;

sr: is the ratio between the tonnes of blocks sent to the dump (*waste*) and the ones sent to the mill (*ore*), formally defined as *stripping ratio*;

$\overline{G}_t$: is the average metal grade of the ore material mined at period t; and

c_{fixed}: is the fixed costs incurred at each production period t.

Then, given that the average metal grade, $\overline{G}_t \geq 0$, follows a specific probability distribution (see Chapter 2 for more details), $F_{\overline{G}_t}\left(\overline{g_t}\right)$, and using the fact that the metal grade of the orebody is independent from the metal price, the probability distribution of the expected cash flows, $CF_t^{\varsigma_t^j}$, generated at each production period, given mine design, ζ_j, and metal price $S_t = s$ can be estimated as follows:

$$\left\{CF_t^{\varsigma_t^j} \mid S_t = s\right\} = q_t \cdot y \cdot \overline{G}_t \cdot s - q_t \cdot y \cdot \overline{G}_t \cdot c_{refining} + q_t \cdot c_{milling} + q_t \cdot (1+sr) \cdot c_{mining} + c_{fixed}$$

$$\left\{CF_t^{\varsigma_t^j} \mid S_t = s\right\} = q_t \cdot y \cdot \overline{G}_t \cdot \left(s - c_{refining}\right) + q_t \cdot c_{milling} + q_t \cdot (1+sr) \cdot c_{mining} + c_{fixed}$$

then,

$$\mathrm{P}\left(CF_t^{\varsigma_t^j} \mid S_t = s \le cf\right) = \mathrm{P}\left(q_t \cdot y \cdot \overline{G}_t \cdot \left(s - c_{refining}\right) + q_t \cdot c_{milling} + q_t \cdot (1+sr) \cdot c_{mining} + c_{fixed} \le cf\right)$$

$$\mathrm{P}\left(CF_t^{\varsigma_t^j} \mid S_t = s \le cf\right) = \mathrm{P}\left(q_t \cdot y \cdot \overline{G}_t \cdot \left(s - c_{refining}\right) \le cf - q_t \cdot c_{milling} - q_t \cdot (1+sr) \cdot c_{mining} - c_{fixed}\right)$$

$$\mathrm{P}\left(CF_t^{\varsigma_t^j} \mid S_t = s \le cf\right) = \mathrm{P}\left(\overline{G}_t \le \frac{cf - q_t \cdot c_{milling} - q_t \cdot (1+sr) \cdot c_{mining} - c_{fixed}}{q_t \cdot y \cdot \left(s - c_{refining}\right)}\right)$$

$$\mathrm{P}\left(CF_t^{\varsigma_t^j} \mid S_t = s \le cf\right) = F_{\overline{G}_t}\left(\frac{cf - q_t \cdot c_{milling} - q_t \cdot (1+sr) \cdot c_{mining} - c_{fixed}}{q_t \cdot y \cdot \left(s - c_{refining}\right)}\right)$$

$$F_{CF_t^{\varsigma_t^j}}\left(cf\right) = F_{\overline{G}_t}\left(\frac{cf - q_t \cdot c_{milling} - q_t \cdot (1+sr) \cdot c_{mining} - c_{fixed}}{q_t \cdot y \cdot \left(s - c_{refining}\right)}\right),$$

where,

$$E\left\{CF_t^{\varsigma_t^j} \mid S_t = s\right\} = \int_{\alpha}^{\infty} cf \times F_{CF_t^{\varsigma_t^j}}\left(cf\right) d\left(cf\right);$$

with

$$\alpha = q_t \cdot c_{milling} - q_t \cdot (1+sr) \cdot c_{mining} - c_{fixed};$$

Now, defining the *metal quantity,* M_t, produced at each production period, t, as

$$M_t = q_t \cdot \overline{G}_t \cdot y, \tag{1.10}$$

and the *total production costs,* $TotCost_t$, incurred in the metal production process at production period, t, as

$$TotCost_t = q_t \cdot y \cdot \overline{G}_t \cdot c_{refining} + q_t \cdot c_{milling} + q_t \cdot (1+sr) \cdot c_{mining} + c_{fixed} \tag{1.11}$$

The cash flow, $CF_t^{\varsigma_t^j}$, defined in Equation 1.9, can be expressed as

$$CF_t^{\varsigma_t^j} = M_t \cdot S_t - TotCost_t. \tag{1.12}$$

Observe in Equation 1.12 that because the mine project is evaluated in the face of *in situ* metal grade and metal price uncertainties, the cash flow, $CF_t^{\varsigma_t^j}$, generated at each production period is seen as a random variable. The reason for this is that the average metal grade G_t, the metal quantity M_t, and the metal price S_t, are random variables that follow a specific stochastic process. Furthermore, even though the production rate q_t, mill recovery y, and mining, processing, refining and fixed costs, are seen as deterministic variables that can change over time, the total production cost, $TotCost_t$, is also seen as a random variable. The reason for this is that, as observed in Equation 1.11, $TotCost_t$ is a function of the average metal grade, G_t, and consequently a function of metal production, M_t, which are random variables.

In this context, at each production period $t = \{t_0 = 0, t_1 = \Delta t, ..., t_T = T\Delta t\}$, the value of the open pit mine project V_t^j, given mine plan and design ζ_j $(j = 1, 2, .., J)$, is defined as the sum of the maximum cash flow, $CF_t^{\varsigma_t^j}$, generated during the production period t, and the maximum between the continuation value, $\Phi_{t+\Delta t}$, and stopping value, $\Pi_{t+\Delta t}$, that is,

$$\begin{aligned} & V_t^j (Q_t, S_t) = \left(\max_{g_t^*} \left[E \left\{ CF_t^{\varsigma_t^j} (q_t : g_t^*) \mid S_t \right\} \right] + \left\{ max \left[\Phi_t, \Pi_t \right] \right\} \right); \\ & where: \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.13) \\ & if \quad V_t^j \left(Q_t = 0, S_t \right), \text{ then } \quad CF_t^{\varsigma_t^j} = 0; \text{ and } \Phi_t = 0; \end{aligned}$$

where Q_t is the available ore reserves at the beginning of period t, and q_t and S_t are the ore production rate and the metal price, respectively.

Observe in Equation 1.13 that the maximisation of the cash flow, $CF_t^{\varsigma_t^j}$, is performed in respect to an optimum cut-off metal grade g_t^*, defined as the minimum average metal grade, G_t, that maximises expected cash flow $CF_t^{\varsigma_t^j}$. Also observe in Equation 1.13 that if the available resources at period t are zero, that is $Q_t = 0$, then both the cash flow and continuation value are zero, that is, $CF_t^{\varsigma_t^j} = 0$ and $\Phi_t = 0$, respectively. The reason for this is that at period $t + \Delta t$ there is no production and consequently no cash flow generation.

One important feature of Equation 1.13 is that the cash flow generated at production period t is the maximum expected cash flow value conditional to the given metal price $S_t = s$. This is precisely the main different between current evaluation techniques and the IVOF technology. In this case, Equation 1.13 uses the Jensen Inequality (see Figure 33 and footnote 27) to estimate the value of the open pit mine project at each production period, given metal price $S_t = s$.

The ***continuation value***, $\Phi_{t+\Delta t}$, is defined as the expected[31] value of the mine operation if continuing (producing) operating throughout period $\left(t+\Delta t\right)$, conditional on S_t:

$$\Phi_t = \frac{1}{\left(1+r\right)^{\Delta t}} E^* \left\{ V^j_{t+\Delta t}\left(Q_{t+\Delta t}, S_{t+\Delta t}\right) \mid S_t \right\}. \tag{1.14}$$

The ***stopping value***, $\Pi_{t+\Delta t}$, is defined as the sum of the salvage value, $SVal$, of the mine operation obtained from selling all real assets, such as equipment and mine or processing facilities, and the closure costs, Clc, incurred during period $\left(t+\Delta t\right)$, that is,

$$\Pi_t = \frac{1}{\left(1+r\right)^{\Delta t}} E^* \left\{ \left(SVal + Clc\right)_{t+\Delta t} \mid S_t \right\}. \tag{1.15}$$

In this case, the decision rule for closing the mine project at any production period or continuing mining is given by the following expression

$$\Phi_{t+\Delta t} \leq \Pi_{t+\Delta t}. \tag{1.16}$$

Then, given Equations 1.13 – 1.16, the value of an open pit mine project can be defined as follows:

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[31] In this paper, $E^*_t\{.\}$ is the conditional expectation with respect to the unique risk neutral probability, which can also be denoted by $E^*_t\{.\} = E^*\{\cdot \mid I_t\}$, where I_t is the available information at period t.

- **At last production period T :** At the last production period the value of the open pit mine project is defined as the sum of the maximum cash flow, $CF_t^{\varsigma_t^j}$, and the salvage value, $SVal$, minus the cost for closing the mine, Clc, incurred for closing the mine. Then, the value of an open pit mine project at its last production period is defined as

$$V_T^j\left(Q_T, S_T\right) = \left(\max_{g_T^*} \left[E\left\{ CF_T^{\varsigma_t^j}(q_T : g_T^*) \mid S_T \right\} \right] + \Pi_T \right). \tag{1.17}$$

- ***At production period* $t_0 < t < T$ *:*** As defined previously, the value of an open pit mine project at production period $t_0 < t < T$ is defined as the sum of the maximum cash flow and the maximum of the stopping or continuation values, that is,

$$V_t^j\left(Q_t, S_t\right) = \left(\max_{g_t^*} \left[E\left\{ CF_t^{\varsigma_t^j}(q_t : g_t^*) \mid S_t \right\} \right] + \left\{ max\left[\Phi_t, \Pi_t\right] \right\} \right). \tag{1.18}$$

- ***At production period* t_0*:*** At production period t_0 the value of the open pit mine project is defined as the maximum of the continuation or stopping value incurred at period $t + \Delta t$. Then, the value of an open pit mine project at the beginning of the mine operation is defined as

$$V_{t_0}^j\left(Q_{t_0}, S_{t_0}\right) = \left\{ max\left[\Phi_{t_0}, \Pi_{t_0}\right] \right\}. \tag{1.19}$$

The final result obtained from the IVOF process is an adjusted or extended net present value, which is defined as the difference between the current mine project value, $V_{t_0}^j$, and the initial capital investment, $Investment$. That is

$$ENPV_{t_0} = V_{t_0}^j\left(Q_{t_0}, S_{t_0}\right) - Investment\,. \tag{1.20}$$

Since the current project value, $V_{t_0}^j$, is a random variable, the extended net present value of the project, $ENPV$, is also a random variable.

The extended net present value of the project, $ENPV$, can be seen as being composed of the simple net present value of the project, NPV, calculated with respect to the riskless rate of return, of certainty equivalent cash flows; and the added value, $FVal$, resulting from all possible operational and managerial flexibility, such as optimising cut-off grades and closing the mine in the face of future adverse conditions. That is

$$ENPV = \begin{cases} NPV + FVal; & FVal > 0 \\ NPV; & FVal = 0 \end{cases}. \qquad (1.21)$$

Observe in Equation 1.21 that due to the value of flexibility being defined as the value of the option to implement a specific operational/managerial strategy, the flexibility value, $FVal$, is always greater or equal than zero – that is, by definition of option pricing, if the strategy to be implemented does not add value to the project it is better not to implement it.

One important consequence of Equation 1.17 is that the minimum value that can be obtained from an IVOF process is the simple net present value of the project, that is, NPV, which is the value that owners and stakeholders of a mine project expect if the project is evaluated without flexibility.

4.4.1 Algorithm for estimating the open pit project value using the IVOF technology

As discussed in Chapter 2, the solution of Equation 1.13 is not an easy process. As shown in Equations 2.9-2.12 the solution of Equation 1.13 has to be solved by using advanced differential equations and many assumptions to relax boundaries.

Conversely with current (e)valuation processes, the IVOF process uses a discrete approach to solve Equation 1.13. In this case the IVOF uses the fact that metal grades and metal prices are independent and that the open pit mine design and plan is selected by a series of iterations, where at each iteration a given mine plan and design is evaluated and optimised under in situ metal grade and future metal price and total production cost uncertainties.

In its essence, the IVOF is composed of five general stages (see Figure 41). Chapters 5, and 6 give more details of each process. The IVOF stages are:

i) ***Base-case mine design***, which is built using current mine design techniques and used to identify main cash flow drivers. Here, an open pit design and plan,

ζ_j, is generated using traditional mine evaluation techniques and given initial expected parameters (see Chapter 3 for more details).

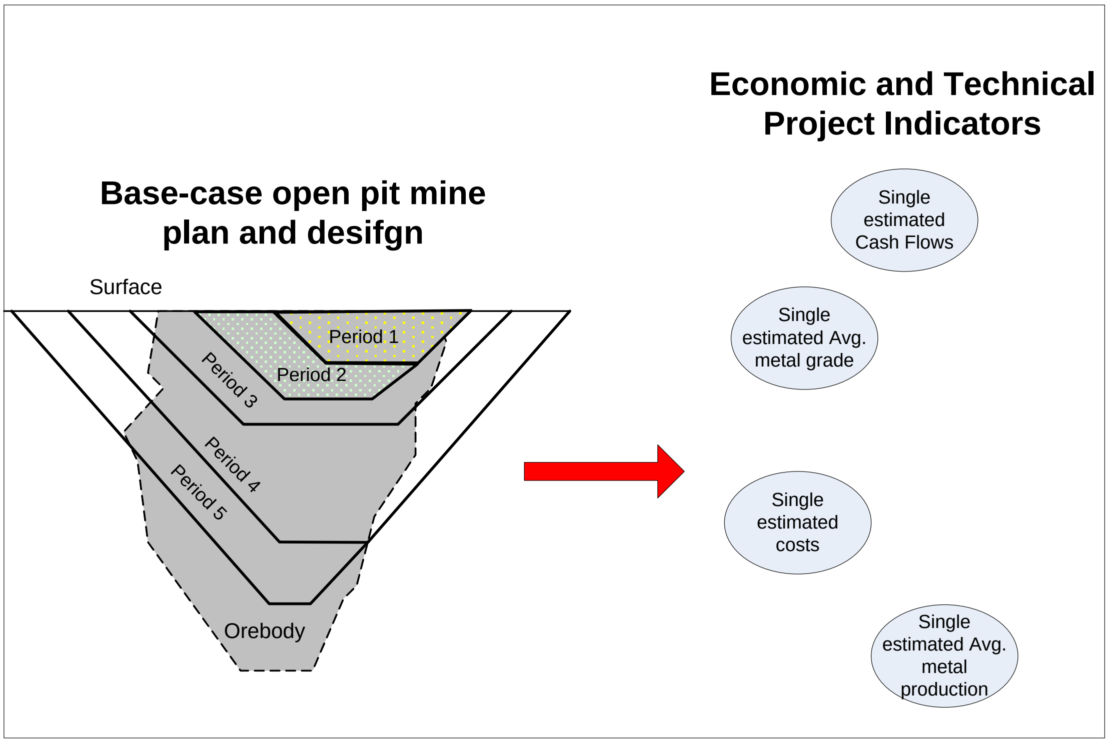

Base-case open pit mine plan and desifgn
Surface
Period 1
Period 2
Period 3
Period 4
Period 5
Orebody
Economic and Technical Project Indicators
Single estimated Cash Flows
Single estimated Avg. metal grade
Single estimated costs
Single estimated Avg. metal production

Figure 38. Diagram showing the base-case open pit mine plan and design, ζ_j*, composed of an ultimate pit and a production scheduling (see Figure 1 for more details). Observe that the final project economic and technical indicators are single estimates.*

ii) ***Incorporating the effect of geological uncertainty*** on given base-case mine design. This stage is performed using the Upside/downside potential technique (Martinez, 2003). At this stage, the base-case open pit mine plan and design, ζ_j, is assessed in the face of in situ metal grade uncertainty (see Chapter 5). The result of this stage is an open pit mine plan and design whose economic and technical indicators are characterised by a distribution of probabilities. Observe that up to this point, the in situ metal grade is the only uncertainty considered in the evaluation process.

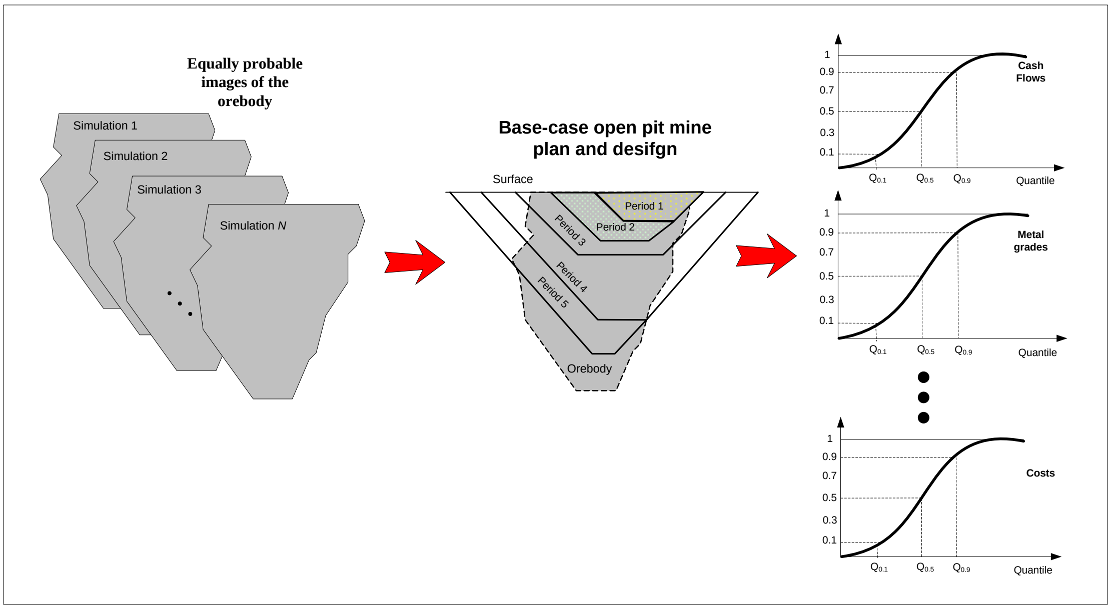

Equally probable images of the orebody
Simulation 1
Simulation 2
Simulation 3
Simulation N
Base-case open pit mine plan and desifgn
Surface
Period 1
Period 2
Period 3
Period 4
Period 5
Orebody
1
0.9
0.7
0.5
0.3
0.1
Cash Flows
Q0.1
Q0.5
Q0.9
Quantile
1
0.9
0.7
0.5
0.3
0.1
Metal grades
Q0.1
Q0.5
Q0.9
Quantile
1
0.9
0.7
0.5
0.3
0.1
Costs
Q0.1
Q0.5
Q0.9
Quantile

Figure 39. Diagram showing IVOF stage 2 where the base-case open pit mine plan and design is assessed under in situ metal grade uncertainty. Observe that in this case, contrary to Stage 1, the final project indicators are characterised by a distribution of values.

iii) ***Incorporating the effect of future metal prices*** in current mine plan and design and estimating the current market value of the open pit mine project using advanced financial techniques such as the Least-Squares Monte Carlo Simulation (Longstaff and Schwartz, 2001) and the Binomial lattice (Cox and Ross, 1976; Cox, Ross and Rubinstein, 1979). As observed in Figure 40, at this stage the base-case open pit mine plan and design obtained in the previous stage is assessed in the face of future metal price(s) uncertainty (see Chapter 5 for more details). Then, based on the results, the current value of the open pit mine is estimated. This stage is composed of two sub-stages: metal price forecasting (Figure 40-above) and cash flow generation to render the current mine value (figure 40-below). Observe that the schematic representation of the IVOF's stage 3 shown in Figure 40 uses the Binomial lattice to forecast metal price and to estimate the cash flows of the open pit mine project. Also observe that at each node of the binomial tree, the expected cash flows of the project are estimated as conditional to the given metal price, that is, $E\left\{CF_t \mid S_t\right\}$. Furthermore, to estimate the open pit mine project value, different options are considered at each production period, such as continuing production or closing the mine in adverse conditions.

iv) Selecting the best (practical) mine plan and design based on uncertainty and risk (see Figure 41). At this stage, different open pit mine designs, $\left\{\zeta_{j}, \zeta_{j+1}, ...\right\}$ are built and passed throughout stages 1-3, which results in the estimation of the current open pit mine given a specific design, $\left\{V_{t_0}^{j}, V_{t_0}^{j+1}, ...\right\}$; and

v) Making final decisions based on operational/managerial flexibility and risk analysis. At this stage, the best open pit mine design is selected based on its current value, that is, selected open pit mine design = $max\left\{V_{t_0}^{j}, V_{t_0}^{j+1}, ...\right\}$.

As can be seen from the previous description, the difficulty of implementing the IVOF process when evaluating a mine project is that the mine planner must have an in-depth understanding of four domains: i) open pit mining and specially pit optimisation and production scheduling; ii) geostatistics and in particular conditional simulation; iii) stochastic processes for modelling commodity prices, and option pricing theory and the numerical methods for evaluating financial options; and iv)real options valuation which extend the methods for evaluating financial options to real-world projects such as mines and oil fields.

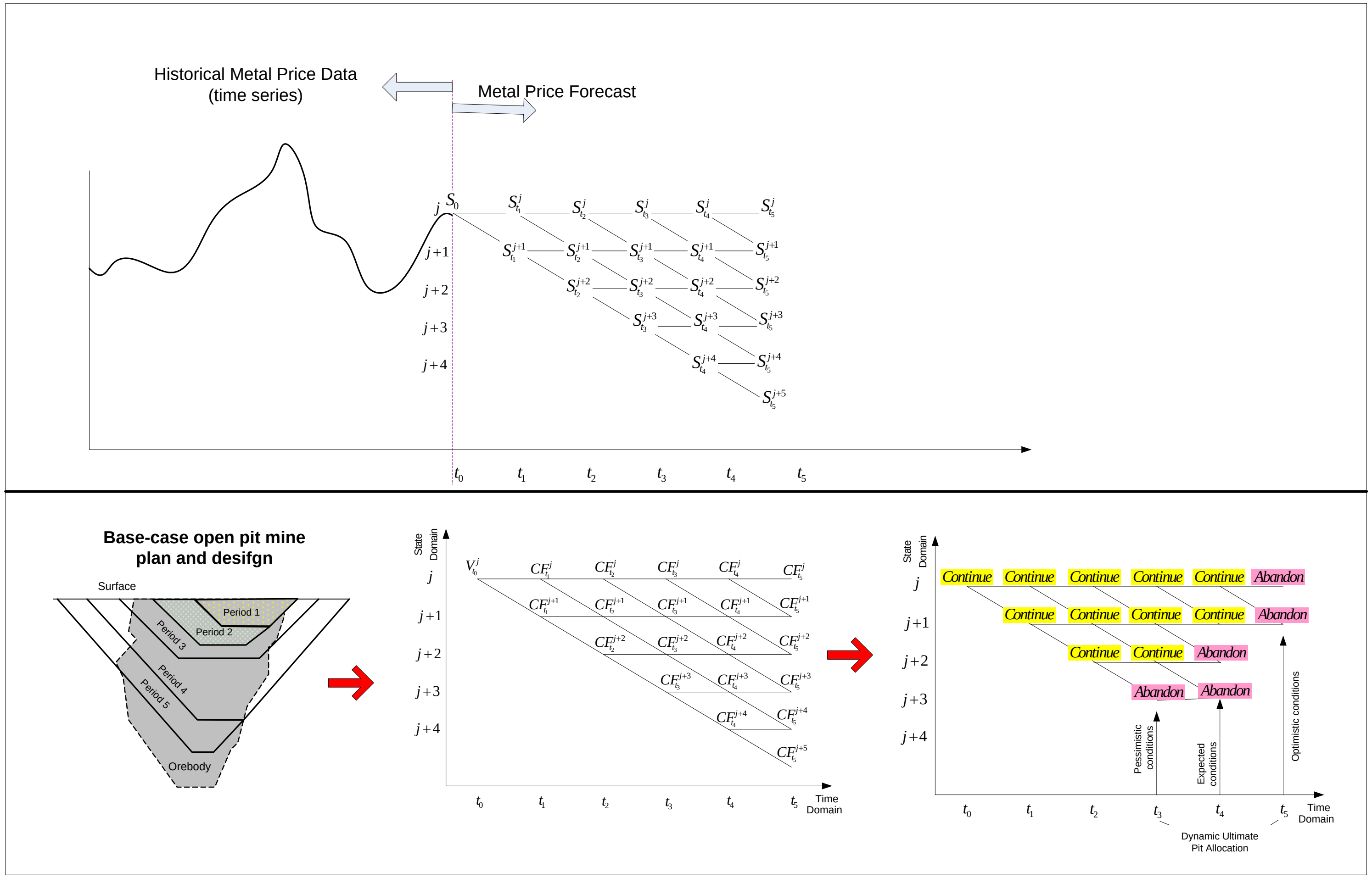

Historical Metal Price Data
(time series)
Metal Price Forecast
S_0
$S_{t_1}^j$ $S_{t_2}^j$ $S_{t_3}^j$ $S_{t_4}^j$ $S_{t_5}^j$
j
$j+1$
$j+2$
$j+3$
$j+4$
$S_{t_5}^{j+5}$
t_0 t_1 t_2 t_3 t_4 t_5
Base-case open pit mine plan and desifgn
Surface
Period 1
Period 2
Period 3
Period 4
Period 5
Orebody
State Domain
$V_{t_0}^j$
$CF_{t_1}^j$ $CF_{t_2}^j$ $CF_{t_3}^j$ $CF_{t_4}^j$ $CF_{t_5}^j$
$CF_{t_5}^{j+5}$
Time Domain
Continue
Abandon
Pessimistic conditions
Expected conditions
Optimistic conditions
Dynamic Ultimate Pit Allocation

Figure 40. Diagram showing IVOF Stage 3: Incorporating the effect of future metal prices (upper diagram) in the current mine plan and design and estimated current market value of the open pit mine project (lower diagram).

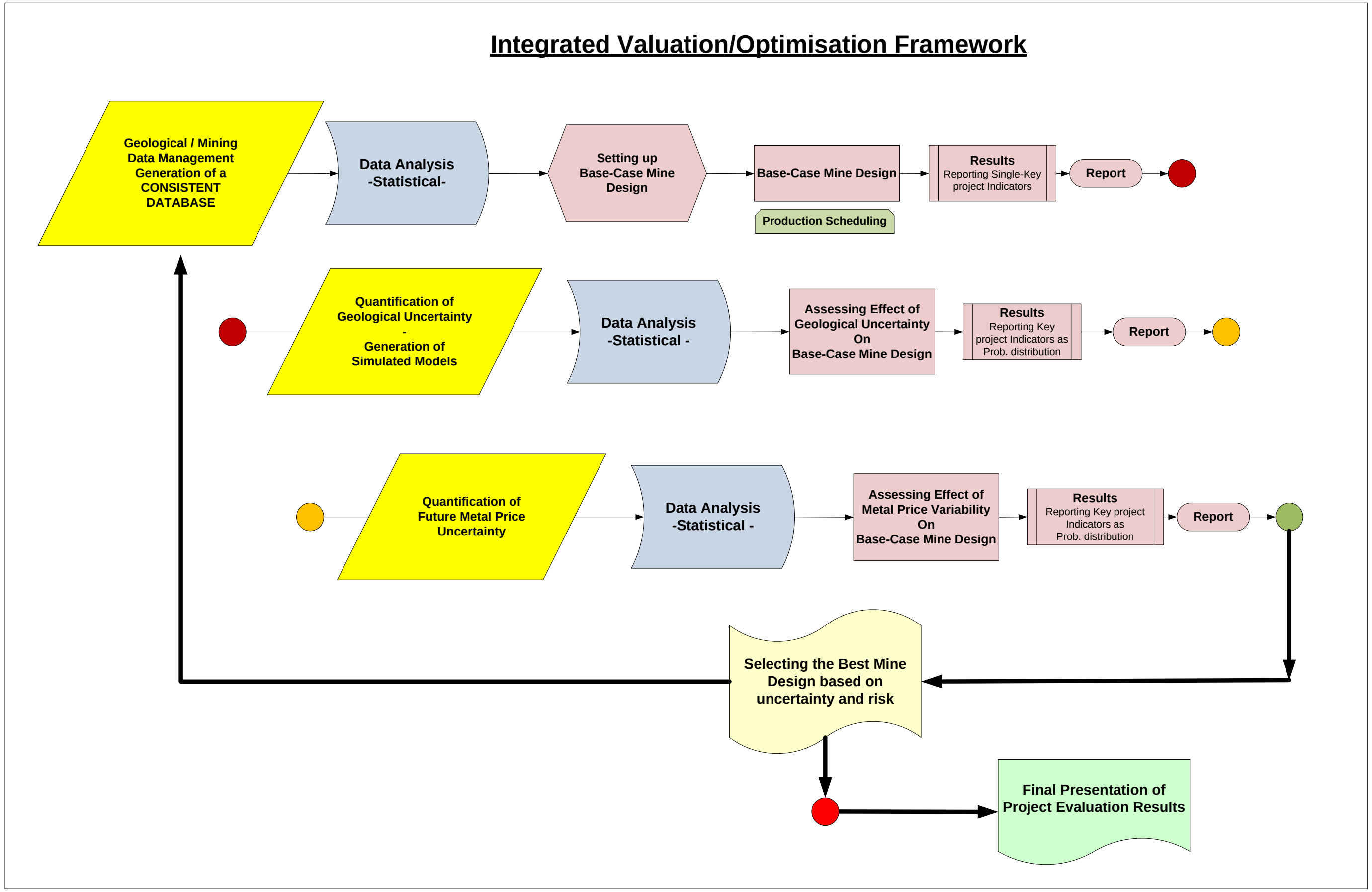

Integrated Valuation/Optimisation Framework
Geological / Mining Data Management Generation of a CONSISTENT DATABASE
Data Analysis -Statistical-
Setting up Base-Case Mine Design
Base-Case Mine Design
Production Scheduling
Results Reporting Single-Key project Indicators
Report
Quantification of Geological Uncertainty - Generation of Simulated Models
Data Analysis -Statistical -
Assessing Effect of Geological Uncertainty On Base-Case Mine Design
Results Reporting Key project Indicators as Prob. distribution
Report
Quantification of Future Metal Price Uncertainty
Data Analysis -Statistical -
Assessing Effect of Metal Price Variability On Base-Case Mine Design
Results Reporting Key project Indicators as Prob. distribution
Report
Selecting the Best Mine Design based on uncertainty and risk
Final Presentation of Project Evaluation Results

Figure 41 Diagram summarising the Integrated Valuation/Optimisation Framework.

Observe that Equations 1.14 and 1.15 use the riskless discount rate, r, instead of the aggregated risk adjusted discount rate $R > r$, which is normally used when using the DCF technique. This feature is indeed one important property of the IVOF technology, which consists of accounting for risk at the source of uncertainty rather than at the cash flow stage.

This property can be better visualised in Figures 42-45. As observed in Figure 42, traditional techniques based on the DCF rely on an aggregated risk-adjusted discount rate, R, to account for the entire uncertainty and risk of a mine project. In other words, the aggregated discount rate is the "*Atlas*" holding the entire mine project's performance. Since the estimation of an aggregate discount rate that accounts for the entire uncertainty and risk of a mine project is not an easy task, its estimation could result either in an over-estimated discount rate that will reduce the project value (see Figure 43) or an under-estimated discount rate that will not be able to support the heavy load of the project risk, leading to its collapse (see Figure 44).

Contrary to the DCF technique, as shown in Figure 45 the IVOF includes the technical and economic risk at the source of uncertainty—that is, at the project stage—allowing the mine analyst to avoid the difficult task of estimating an aggregated discount rate that accounts for these uncertainties. In this case a smaller discount rate, $r_{free} \leq r < R$, that accounts for time value of money, represented by the risk-free discount rate, r_{free}, and other external uncertainties such as politic risk, is used instead. Observe that if the uncertainty of the project is resolved in its totality, then the discount rate, r, will equal the risk-free discount rate that only accounts for the value of money over time, that is, $r = r_{free}$.

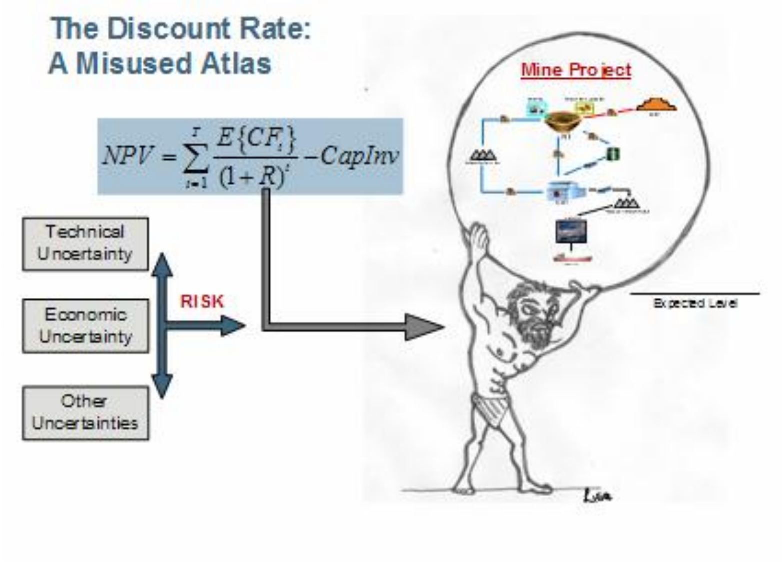

The Discount Rate:
A Misused Atlas
NPV = Σ_{t=1}^{T} E{CF_t}/(1+R)^t − CapInv
Technical Uncertainty
Economic Uncertainty
Other Uncertainties
RISK
Mine Project
Expected Level
Lisa

Figure 42 Traditional mine evaluation process where the aggregate discount rate is considered as the Atlas holding the entire risk and uncertainty of a mine project.

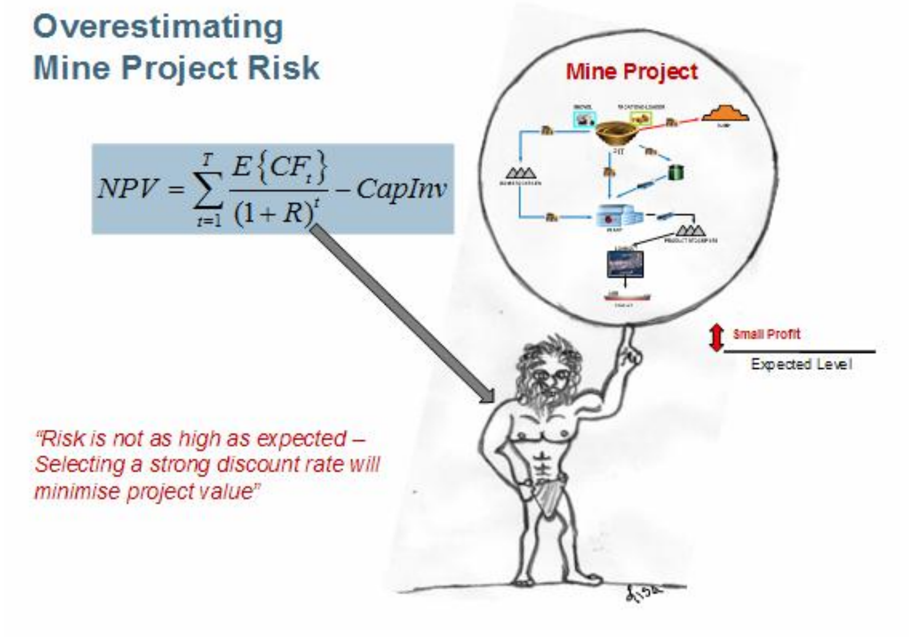

Overestimating
Mine Project Risk
NPV = Σ_{t=1}^{T} E{CF_t}/(1+R)^t − CapInv
Mine Project
Small Profit
Expected Level
"Risk is not as high as expected – Selecting a strong discount rate will minimise project value"
Lisa

Figure 43 Traditional mine evaluation techniques can over-estimate the discount rate which will minimise the mine project's value.

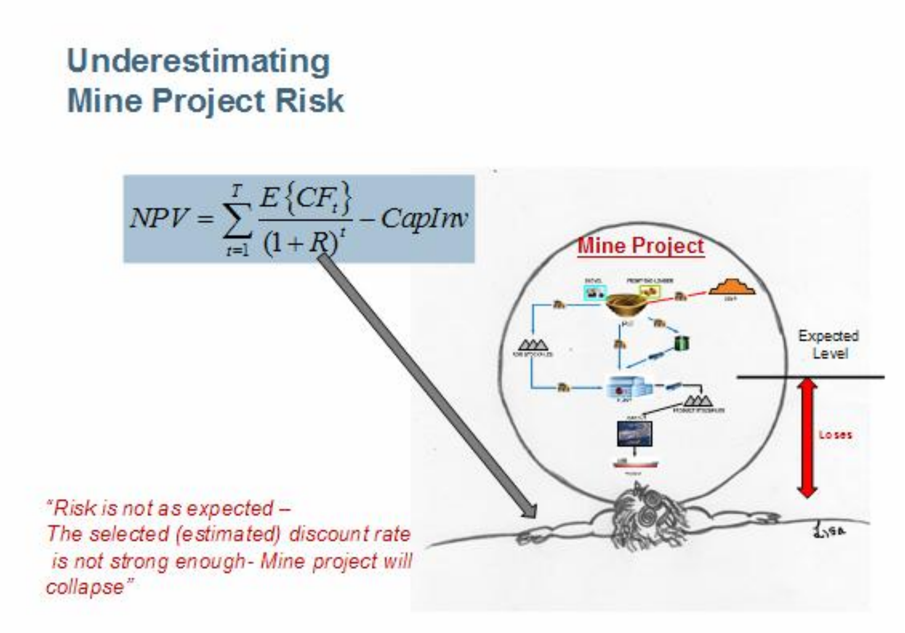

Underestimating
Mine Project Risk
$NPV = \sum_{t=1}^{T} \frac{E\{CF_t\}}{(1+R)^t} - CapInv$
Mine Project
Expected Level
Loses
"Risk is not as expected –
The selected (estimated) discount rate
is not strong enough- Mine project will
collapse"
Lysa

Figure 44 Traditional mine evaluation techniques can under-estimate the discount rate which will not be able to support the heavy load of risk of a mine project, leading to its collapse.

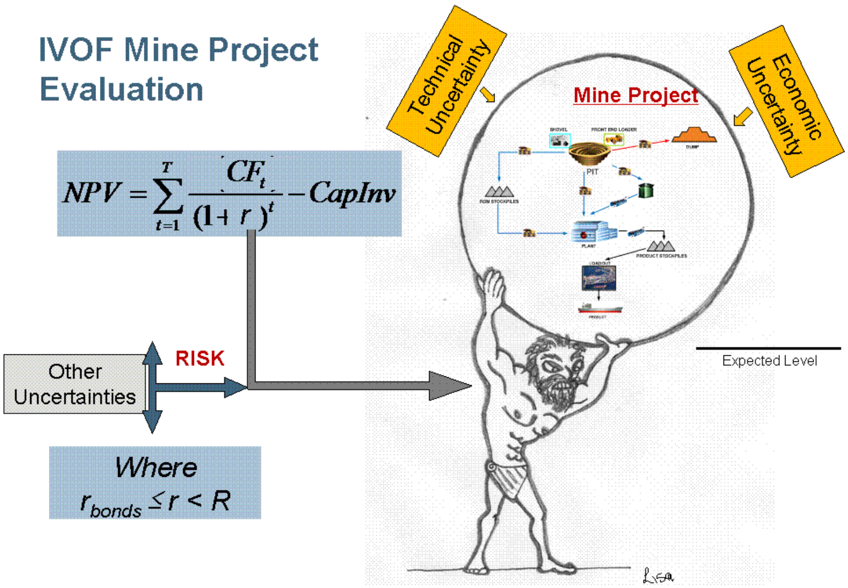

IVOF Mine Project
Evaluation
$NPV = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} - CapInv$
Technical Uncertainty
Economic Uncertainty
Mine Project
RISK
Other Uncertainties
Where
$r_{bonds} \leq r < R$
Expected Level
Lysa

Figure 45 IVOF perspective of mine project evaluation under uncertainty and risk. In this case the estimation of the discount rate, r, (represented here as the Atlas holding the entire mine project risk) is not a difficult task since it does not account for technical and economic risk, as they are accounted for at their source of uncertainty.

4.5 Conclusions and comments

Throughout this chapter, it has been seen that the mine evaluation problem is not easy to solve, but certainly not impossible to achieve. Furthermore, this chapter has shown that when dealing with projects that carry uncertainty, the use of single estimates values could mislead final decision making, giving a favourable rate of return to a project that is otherwise doubtful or an unfavourable return to a project that is otherwise profitable. Quantified uncertainty and risk analysis and management were shown to be powerful tools for the mine planner or analyst when evaluating a mine project in the face of uncertainty. The reason for this is that quantified uncertainty and risk analysis give the mine planner or analyst a realistic range of the final outcomes, giving them the flexibility to react to either adverse or favourable conditions and to implement the best operational and managerial strategies that are able to take advantage of the opportunities while mitigating the risk for losses.

In this context, the proposed Integrated Valuation/Optimisation Framework (IVOF) is seen as a technique suitable for mine project evaluation since it integrates uncertainty and risk in the mine evaluation process. One important characteristic of the IVOF is that it uses existing state of the art mine valuation/optimisation techniques to solve the mine evaluation problem in a tractable and practical fashion. One advantage of the IVOF is that it is a generic framework, that is it was not designed to be used for any specific software or open pit mine project. In fact, the IVOF can be applied to any mine project, and any software that has the relevant procedures that the IVOF needs can be used for its implementation. Another advantage of the IVOF is that because of its tractability it can allow any optimisation process to be included in the evaluation process.

In summary, as observed in Figures 38-41, when applied correctly the IVOF is able to output the following.

- An estimated value with its respective risk profile given by its probability distribution
- A map of the technical and economic mine project indicator performance where data deficient and high risk areas will be identified (e.g., high grade variability, limited geotechnical design confidence, etc.).

- The ability to assess the risk and benefits of key project interventions such as different equipment fleet, different plan design and steeper slope angles.

- The ability to identify and assess realistic and practical strategic operational and managerial decisions of mine closure, expansion or temporal stop, among others.

- The benefits of the previous result are that mining corporations will be provided with realistic ranges (as seen in Figure 46) of final operational and economic outcomes, such as average metal quantities and grades, and mining and processing costs, instead of being forced into the quarterly charade of "earnings management" with all of its flawed estimates. Furthermore, the IVOF results will also benefit investors, who would at last be provided with a realistic view of the true uncertainties of their investments.

Chapters 3, 5 and 6 will show how to build the base-case open pit mine design and plan, how to model both the orebody uncertainty, represented by the *in situ* metal grade variability, and future metal price modelling, respectively.

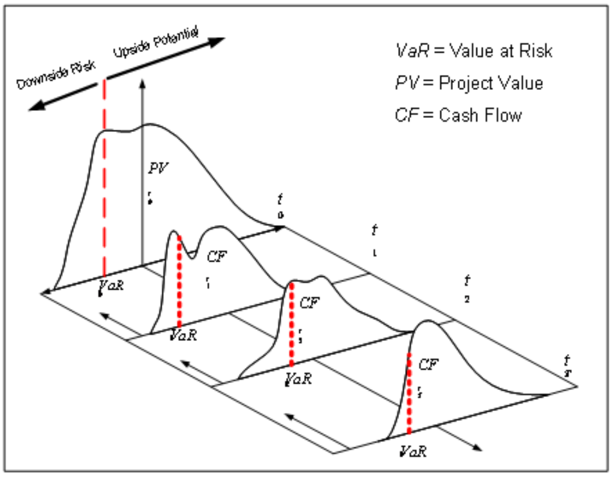


Figure 46. When applied correctly, the IVOF is able to output a map of the technical and economic mine project indicator performance, in this case the cash flow distribution generated at each production period, where high risk areas can be identified via a risk analysis, such as Value@Risk (VaR) and the Upside/downside potential.

Chapter 5

Simulation–Based Geological Risk Assessment: The Foundations of IVOF

5.1 Assessing the risk in an open pit mine project under geological uncertainty

We have previously seen that simulation techniques are of great help to the geologist when modelling an orebody. In this chapter it is shown how simulation techniques assist the mine planner in assessing the risk of a mine project and in making final strategic decisions in order to improve the potential of the project while minimising the risk of future losses. Understanding these techniques is essential for the development of IVOF.

To achieve this, firstly the topic of orebody model selection for mine design and valuation is discussed from a practical viewpoint, and secondly a small, disseminated, low grade, epithermal gold deposit is used as a case study in which an open pit mine operation, named the base-case design, is built based on an (OK) estimated orebody model, and an assessment of the risk of achieving future production and economical targets is performed. This example is a recompilation from Martinez (2003).

5.1.1 Selecting a suitable simulation orebody model(s) for mine design and valuation

Similarly to orebody modelling, one common question that is normally formulated when designing and valuing a mine project under geological uncertainty is: which simulation model(s) is the best for mine design and valuation procedures?

In practice, it is common to select one orebody model, usually estimated by linear techniques such as kriging, to design the entire mine operation. This practice results in one single value of the mine project that needs to be accepted as true when making final decisions, such as to invest in the project if the value of the mine or Net Present Value (NPV) is positive and not to invest in the project if the NPV is negative. Sensitivity analysis, which is based on spider diagrams, is frequently used in these cases to assess the variability of the mine project due to the variation (normally ±10%) of the orebody characteristics (uncertainty of the orebody), such as ore tonnes, metal grade, and metal quantities, among others (see Figure 47).

However, due to the fact that the initial source (the estimated orebody model) input in the mine optimisation process, seen here as a non-linear transfer function (see Appendix A for more details about transfer functions), is uncertain, many authors (e.g. Ravenscroft, 1992; Dimitrakopoulos et al, 2002; Rossi and Van Brunt, 1997; Farrely, 2002; Martinez, 2003) support the idea that sensitivity analyses, when applied to mine design and production scheduling, are fairly coarse and are of a potentially misleading nature. The reason for this is that sensitivity analysis considers, incorrectly, that the variability of the geological attributes such as metal grade throughout the orebody model is of a linear character. Consequently, a ±10% variation in the value of one geological attribute, say metal grade, is not a realistic representation of the variability of this attribute across the orebody model.

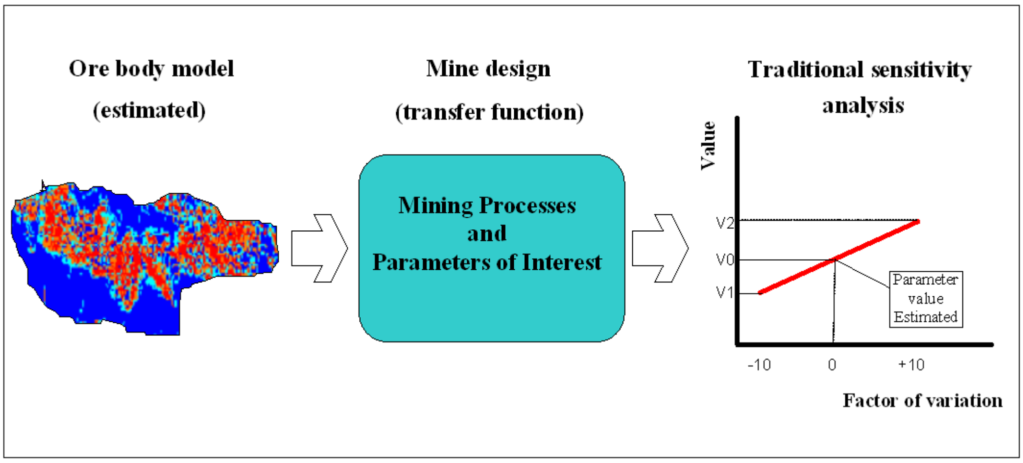

Ore body model
(estimated)
Mine design
(transfer function)
Mining Processes
and
Parameters of Interest
Traditional sensitivity
analysis
Value
V2
V0
V1
Parameter
value
Estimated
-10
0
+10
Factor of variation

Figure 47. Schematic representation of the traditional sensitivity analysis framework in which the value of a mine project is assessed against the variation (±10%) of a specific project indicator, such as the metal grade. (Reproduced from Martinez, 2003.)

Another common practice in mining projects consists of selecting three orebody models from a set of simulated models instead of a single estimated one. In this case, the selected orebody models are those with the highest, average, and lowest ore tonnes content, which are normally extracted from the cumulative distribution function (cdf) of ore tonnes that is estimated for a specific cut-off metal grade. The reason for doing this is to generate three mine project scenarios—the average, maximum and minimum project scenarios—which can give the mine analyst a type of interval of confidence for the value of the mine project. In this case, the best mine design is the one built on the best orebody model and that is expected to generate the best NPV; the average mine design is the one built on the average orebody model and that is expected to generate an average NPV; and the worst mine design is the one built on the worst orebody model and that is expected to generate the worst NPV.

Even though this makes some sense in terms of obtaining a type of interval of confidence for the NPV of the mine project, in practice it is seen that both the interpretation and implementation of this idea are not correctly executed because of the poor understanding of the following concepts:

Concept 1. The distribution of the geological attributes, such as the ore tonnes above a given cut-off grade, throughout the orebody model is nonlinear. Consequently, if two different simulated models of an orebody, such as $Sim1$ and $Sim2$, with Q_{Sim1} and Q_{Sim2} denoting their respective total ore tonnes, above a cut-off grade, g_{cutoff}, and $q_{Sim1} < Q_{Sim1}$, and $q_{Sim2} < Q_{Sim2}$, denoting the ore tonnes, above the same cut-off grade g_{cutoff}, of the zone **A** inside the orebody (see Figure 48), then the following events may happen:

$$\begin{array}{lllll} \left(Q_{Sim1} > Q_{Sim2}\right) \; AND \; \left(q_{Sim1} > q_{Sim2}\right), & & \left(Q_{Sim1} < Q_{Sim2}\right) \; AND \; \left(q_{Sim1} < q_{Sim2}\right), \\ \left(Q_{Sim1} > Q_{Sim2}\right) \; AND \; \left(q_{Sim1} \approx q_{Sim2}\right), & and & \left(Q_{Sim1} < Q_{Sim2}\right) \; AND \; \left(q_{Sim1} \approx q_{Sim2}\right), \\ \left(Q_{Sim1} > Q_{Sim2}\right) \; AND \; \left(q_{Sim1} < q_{Sim2}\right), & & \left(Q_{Sim1} < Q_{Sim2}\right) \; AND \; \left(q_{Sim1} > q_{Sim2}\right). \end{array}$$

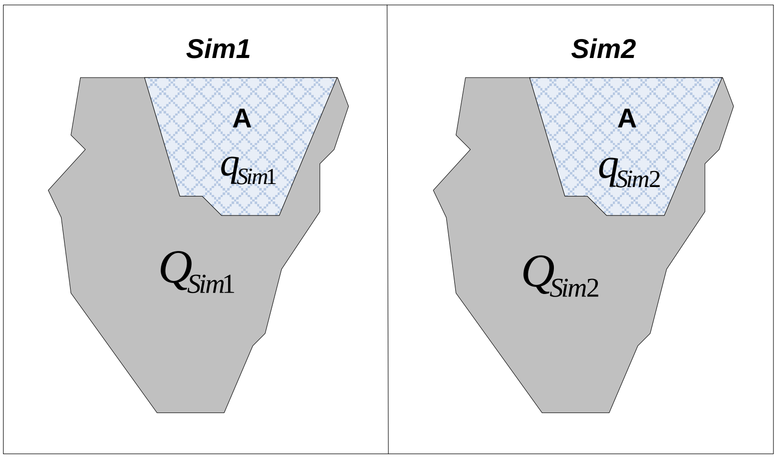


Figure 48. Diagram showing two simulations of an orebody with a common inner zone A. In the figure, Q_{Sim1} *and* Q_{Sim2} *denote the total ore tonnes of each simulated model while* q_{Sim1}*, and* q_{Sim2} *denote the ore tonnes inside the common zone A.*

Concept 2. If the differences in metal content and metal grades, among other geological characteristics, of two or more simulated models of an orebody are significant, then different (physical) mine design limits can be generated for each of them. Otherwise, one common mine design can occur for them all. Observe that for this concept to be true, the input of all other economic and technical variables in the mine design process, such as costs, metal prices, slopes, and mill capacities, need to remain constant for all simulated orebody models.

Concept 3. A mine design is a 3D volume on the ground in which the material inside its limits is seen to be economical to extract while the material outside its limits is non-economical and is left in the ground. Consequently, if two or more physical mine designs are generated on two or more simulated orebody models, then each of them will carry the uncertainty of achieving estimated initial targets, such as ore and waste tonnes, metal quantities, inside its limits.

Then, based on the previous concepts, it is clear that:

i) The selection of the best, average, and worst orebody models extracted from the ore tonnes distribution curve, or other geological attribute, will not necessarily generate the best, average and worst mine project value (see *Concept 1* above).

ii) The result of using three orebody models for mine design and valuation may result in the generation of one, two, or three pit designs which have different physical limits (see *Concept 2* above). One consequence of this result is that the mine project values obtained from the best and worst pit design scenarios cannot be used as an upper or lower limit for the mine project value obtained from the average mine design scenario (see next result).

iii) If two or more (physical) pit design scenarios are generated for a mine project, each of them is a candidate to be implemented in the project. However, each of them will be affected by the uncertainty of the orebody, that is, each of them will have the uncertainty in future productions of achieving the initial plan targets, such as ore and waste tonnes, metal quantities and metal grades.

Consequently, if two or more pit design scenarios are generated, the pit design that has the minimum risk in achieving future economic and technical targets, such as mill capacity and minimum acceptable cash flow, among others, is the best for the project.

But, how do we know which pit design is the one with the minimum risk of achieving future economic and technical targets?

One solution to the previous question could be to apply sensitivity analysis to each of the generated pit design scenarios and select the design which is less sensitive to variations in the economic and technical parameters (see Figure 49). However, as mentioned previously, sensitivity analysis is not an accurate tool for assessing the risk of a mining project because it does not consider the uncertainty of the orebody model. Consequently, a decision made based on a sensitivity analysis may bring devastating consequences to the owners of the mine and its stockholders since it could mislead the real value of the mine project.

An alternative process to quantify the mine value uncertainty, and consequently to select the best pit design based on a risk assessment of future profits and losses, is the Upside/Downside valuation framework (2002). This approach belongs to the general Monte Carlo simulation and risk assessment framework (Halton, 1970; Journel and Huijbregts, 1978; David, 1988; Dimitrakopoulos, 1998; Armstrong and Dowd, 1994; and others), and is founded on the definition of two components. The first component includes the performance of the key project indicators and the second component includes the decision-making criteria.

The Upside/Downside valuation framework uses conditional simulation techniques to provide multiple equally probable models of the orebody that reproduce the statistics and *in situ* variability of the deposit. By integrating these simulated models into a transfer function (in this case the mine design process), the orebody uncertainty is then transformed into a probability distribution of final outcomes or responses, such as the key project indicators (see Figure 50). The resulting probability distribution of the key project indictors is then used to assess the sensitivity of the overall mine economics, long-term mine planning and production scheduling to grade uncertainty. To achieve this, real decision-making criteria such as the minimum ore tonnes, the minimum amount of metal, and the minimum acceptable cash flow, produced in a specific production period, or the minimum acceptable NPV, are used as references from where the downside-risk/upside-potential of generating future values below/above these criteria are established. This process makes use of a minimum acceptable risk tolerance which is established by the analyst of the project. The final decision of selecting the best pit design for the project among two or more pit design scenarios is made by comparing their respective downside-risk and upside-potential indicators (see Figure 49).

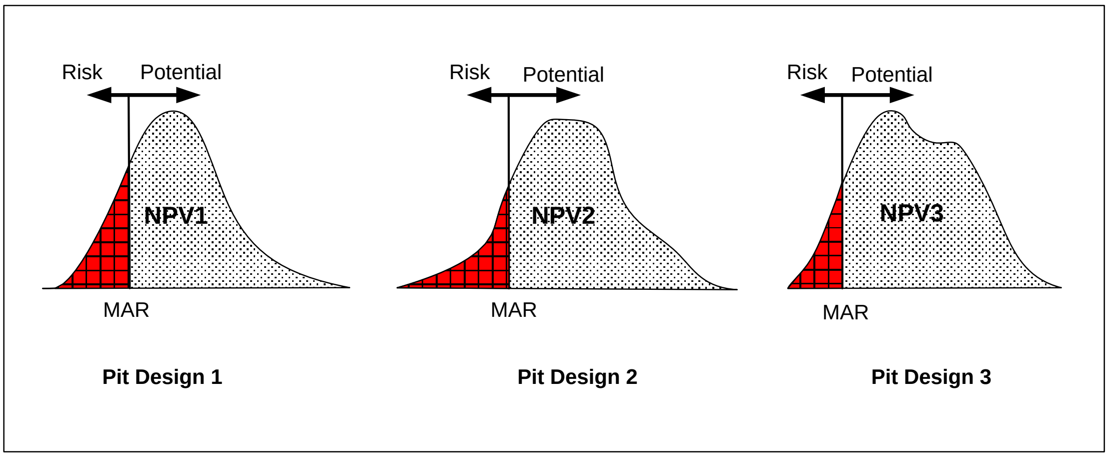


Figure 49. Diagram showing the process of comparing the probability distribution of the resulting NPVs of three pit designs. Observe that this comparison is based on the downside-risk and upside-potential of obtaining values below or above a minimum amount return (MAR).

5.1.2 Risk assessment of a gold deposit using the uncertainty of the orebody

This section demonstrates how simulation techniques help the mine analyst to assess the risk of a mine project and make final strategic decisions based on the risk assessment. To achieve this, the small disseminated gold deposit (see Figure 50) used in Martinez (2003) is used here as a case study. Table 4 displays all the pertinent economic and technical parameters to be used in the process.

The first step consists of generating the base-case open pit mine of the project, which includes the ultimate pit and production periods (cutbacks) limits. The steps for generating the base-case open pit design of the small gold deposit are as follows.

1. Estimate an orebody model of the deposit using ordinary kriging. This is done based on the information from the drill holes.

2. Pass the estimated orebody model through a pit optimisation process and find the ultimate pit limits based on a pit value analysis.

3. Based on initial considerations such as mill demand and using a mine schedule process, design the cutbacks or sequence of extraction. Observe that if mill demand is already given it could that an existing mill facility already exists and no further expansion or modification is desirable.

4. Interpret and summarise the values of the project indicators, such as DCF, ore and waste tonnage and metal (gold) quantity, in a table.

Figures 50 and 51 display the results of the previous steps for generating the base-case open pit design for the small disseminated gold deposit used as a case study. Further, Figure 50 is a table with the single project indicator’s values for each cutback or production period.

As observed in the figure, the mine project will have an operating production life of three years. At each year or cutback, the target of the mill demand (1M tonnage of ore) will be achieved. The total production of gold will be around four million grams, and the total estimated discounted cash flow (DCF) of the mine will be $22M.

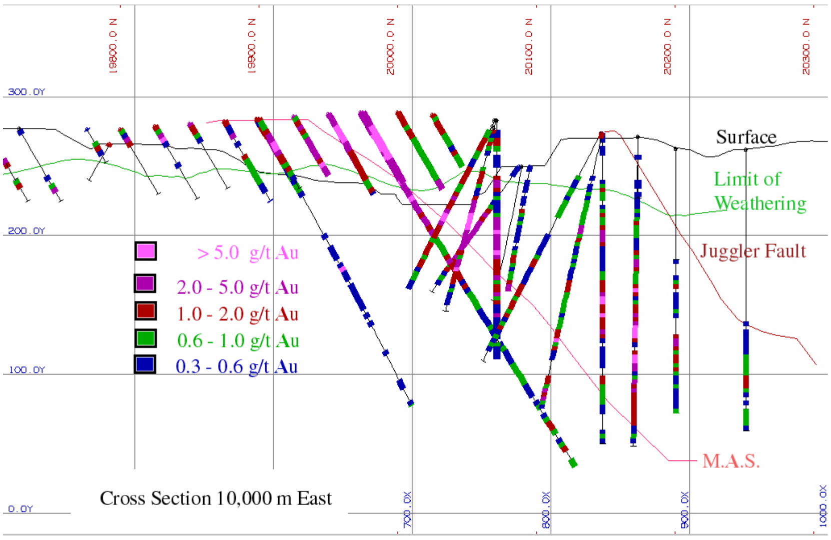

19800.0 N
19900.0 N
20000.0 N
20100.0 N
20200.0 N
20300.0 N
300.0Y
200.0Y
100.0Y
0.0Y
Surface
Limit of Weathering
Juggler Fault
M.A.S.
> 5.0 g/t Au
2.0 - 5.0 g/t Au
1.0 - 2.0 g/t Au
0.6 - 1.0 g/t Au
0.3 - 0.6 g/t Au
Cross Section 10,000 m East
700.0X
800.0X
900.0X
1000.0X

Figure 50. East-west cross-section of the gold deposit used to explain the process of risk analysis under orebody uncertainty. In the figure, the drill holes used to collect information about the deposit, with their respective 3m composites, are shown. Note that the deposit was previously mined and the objective of this study is to find out if it is possible to start a new profitable mine operation with the remaining reserves.

Mine Parameters	
Pit slope	54°
Mining cost	$1.0 per ton
Processing cost for oxide	$8.195 per ton
Processing cost for fresh	$16.86 per ton
Mill recovery for oxide	90%
Mill recovery for fresh	84%
Discounted rate	8% per year
Gold Price	Au$600/oz

Table 4. Technical and economic parameters considered when performing the process of pit design on the small disseminated gold deposit.

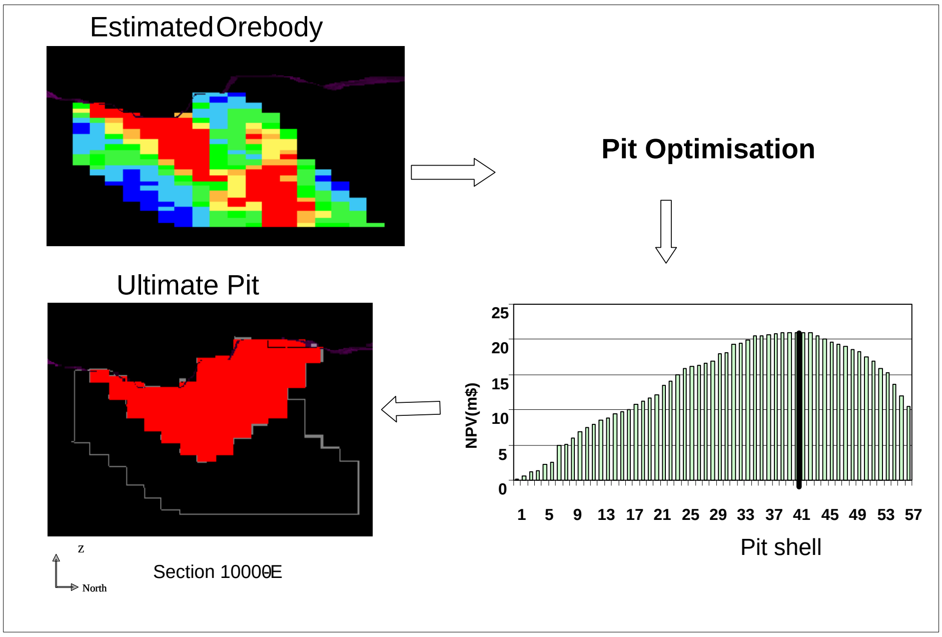
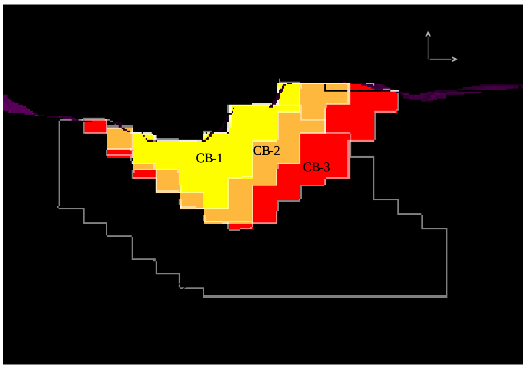

Estimated Orebody
Pit Optimisation
Ultimate Pit
NPV(m$)
25
20
15
10
5
0
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57
Pit shell
z
North
Section 10000-E

Figure 51. Schematic diagram showing the traditional pit design process: estimation of an orebody model; pit optimisation process; DCF analysis by pit shell; finding the ultimate pit (pit shell 41).


CB-1
CB-2
CB-3
Section: 10000-E

Cutback	DCF (m$)	Ore (Mt)	Waste (Mt)	Gold (Mgr)
1	14.2	1.0	0.36	1.78
2	19.0	2.0	1.4	2.9
3	22.2	3.0	2.6	4.1

Figure 52. Schematic diagram showing the traditional cutback design, and the single value for each of the project indicators considered in the analysis.

Once the base-case mine design of the project has been generated, the mandatory question to be formulated is: what is the risk of achieving the estimated project indicator values in the face of the orebody uncertainty? In other words, if the base-case mine design, shown in Figure 52, is adopted for the project, there is interest in knowing the following:

1. What is the probability that the mine will produce 1M tonnes of ore each year, for three years?
2. What is the probability that the mine will generate a total NPV of \$22M with \$14M and \$19M of cash flows during the first and second years, respectively? (In this case the investment capital is assumed to be zero).

Observe that in this case we are not interested in changing the base-case mine design limits, but in assessing the effect that the uncertainty of the geological attributes, in this case metal grade of the blocks inside base-case mine limits, has on the final value of the project indicators. Remember that the base-case mine design was built considering just one estimated orebody model (see Figure 52) in which each block was characterised by a single value of the pertinent geological attribute, that is, the metal grade.

To assess the risk of the base-case mine design under geological uncertainty the following steps are performed (after Martinez 2003).

1. Generate N simulations of the orebody of the deposit using Conditional Simulation and Monte Carlo techniques. In this case, $N = 13$ simulations of the orebody model were generated and used to assess the risk on the base-case design. It is important to mention that the 13 simulations were extracted from the NPV cumulative density function (cdf) generated by Farrelly (2002) on the same ore deposit. In this case, 10 orebody models—those whose NPV corresponded to the 10% percentiles of the cdf—plus three models selected randomly, were selected for performing the risk analysis. The idea of selecting the 10% percentile orebody models was initially proposed by Ballin and Journel (1992) in their work on reservoir characterisation.

2. Overlay the physical base-case mine design (pit limits) on each simulated orebody model and take note of the value of the selected project indicators, such as metal quantity, ore tonnes, and cash flows, among others. Note that in this step the entire physical design of the base-case mine design, which is determined by the ultimate pit and production period limits, as well as all the initial mining parameters are kept fixed so that the only parameter to vary is the orebody model. In simple words, for each of the 13 simulated orebody models we take note of the values of ore tonnes, metal quantity, and cash flows, among other, inside the limits of the base-case mine design.

3. Quantify and assess the risk for each selected key project indicator.

4. Compare and analyse the results obtained in steps 2 and 3 with the ones estimated originally for the base-case design and make final conclusions.

5. End the process.

The results of the previous analysis are depicted in Figure 53-A, B, C, D in which the distribution of the non-cumulative ore tonnes, gold quantity, waste tonnes, and discounted cash flows of the base-case mine design are shown for each production period or cutback. In the figures, the values estimated for the base-case design without considering the uncertainty of the orebody models are displayed in a dark colour bar.

As observed in the figure, the estimated project indicator values (dark colours) do not give enough information about their respective uncertainty. As a matter of fact, the figures clearly show that the base-case project indicators overestimate the quantity of ore tonnes, metal quantity and cash flow, and underestimate the generation of waste tonnes at each production period, respectively. For example, the value of the gold project, using the base-case design, was expected to generate a total DCF around $20M (see Figure 53).

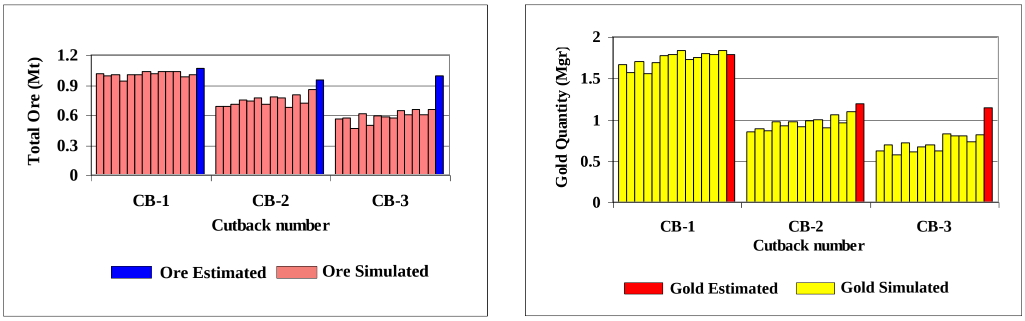
However, a simple risk analysis performed on the cumulative DCF distribution (see Figure 53) shows that on average the base-case design will generate around $16M; in fact, the likelihood that the base-case design will generate the estimated value of $20M is very low. This result is crucial when evaluating the mine project. The difference of estimating the cost of capital based on $16M or $20M for a small project like the case study (3 years), could be the key to the success or failure of the mining project. The summary of the quantification of risk on the selected key project indicators are displayed in Table 5; for the sake of simplicity, the maximum, minimum, and median values of each project indicator are displayed.

Because the result of the risk analysis performed on the base-case mine design gives a better view of the uncertainty in achieving planned targets throughout the operating life of the project, it is now possible to make strategic decisions to reduce or mitigate future risk and to add value to the project.

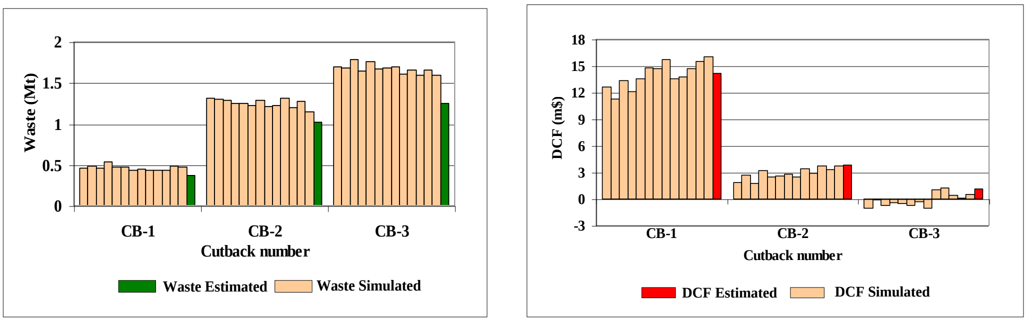
For example, Figure 53-A shows that the second year of production will be characterised by a shortfall in ore tonnes (especially at the end of the year). To avoid having problems with ore supply in that year (if this happens to be true), it could be worthwhile to plan the implementation of alternative strategies during that year. One option (strategy) could be to start a stock-piling campaign at the beginning of the second year to overcome the lack of ore tonnes at the end of the year. Another strategy could consist of starting an exploration campaign looking for new mineralised zones during the beginning of the second year in order to supply enough ore material to the mill at the end of the year.

Figure 53 also shows that the likelihood of achieving planned targets during the last production period is very low. In fact, a risk analysis indicates that this period will be characterised by low ore production and high waste removal, which will result in the generation of negative cash flow (see Figure 53-D). In this case, the strategies to follow could be: i) invest in an aggressive exploration campaign at the end of the second year to find new mineralised zones; ii) sell the property so the company that buys it is the one making the investment in exploration activities; or iii) close the mine during the last period, and recover the salvage value (the sale of mill and machinery among others).

Something interesting that can be observed from Figure 53 is that, in this case, the results estimated for the base-case mine design, is that, when using the kriging orebody model, overestimated the values of ore tonnes and metal quantity, and underestimated the values of waste material, especially during the second and third year. The reason for this is that the cut-off grade established for designing the base-case mine was larger than the overall mean of the unknown deposit. This effect can be visualised in Figure 54 where two distribution curves of metal grade are shown. Here the smoothed curve represents the kriging-smoothed effect while the unsmoothed curve represents the true distribution of the deposit. From the figure, it is observed that if a cut-off, Z_0, greater than the overall mean is considered for mine planning and design, the smoothed curve generates less metal quantity and ore tonnes and more waste tonnes than the non-smoothed curve; conversely with this, if a cut-off, Z_1, smaller than the overall mean is considered, then the smoothed curve generates more metal quantity.



(A) Ore tonnes Distributions (B) Gold Quantity Distributions



(C) Waste tonnes Distributions (D) Discounted Cash Flows Distributions

Figure 53. Base-case mine design dynamic distribution of ore tonnes, gold quantity, waste tonnes and discounted cash flows for each production period or cutback (after Martinez, 2003). Observe that the bars with different colours indicate the result obtained for the base-case mine design without considering the uncertainty of the orebody.

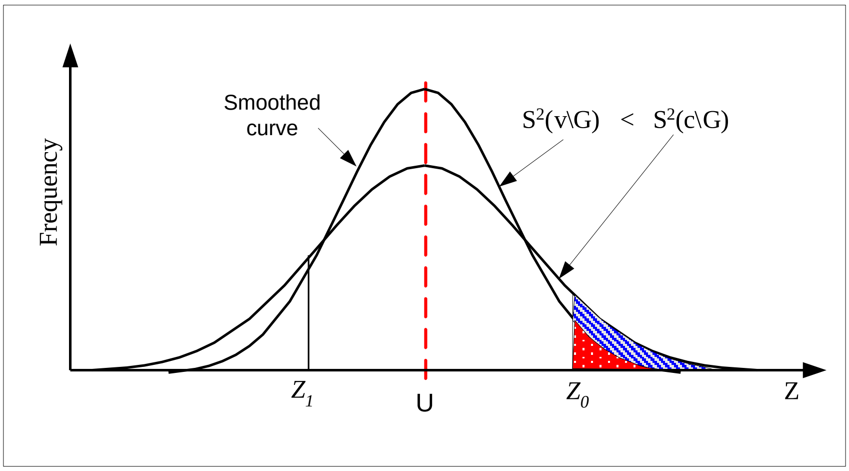


Figure 54. Scheme showing Kriging's property of overestimated or underestimated ore tonnes (adapted from Journel and Huibregts, 1978).

5.2 Comments and conclusions

The study conducted in this chapter investigates the merit of using either linear techniques or techniques based on Monte Carlo simulation frameworks for both the modelling of an orebody and the assessment of risk in a mining project, which are not accounted for when using traditional evaluation techniques. It is shown that linear techniques, such as ordinary kriging, remain as popular benchmarks for both orebody modelling and mine planning and design. However, it is also shown that linear techniques do not allow the quantification and assessment of the uncertainty of key project indicators because of their restrictive property of minimising variance. That is they fail to give a realistic representation of the orebody and, consequently, the value of a mining project.

Further, it is shown that estimation techniques based on the Monte Carlo simulation framework give a better representation of the variability of the orebody characteristic, giving mine planners more information about the uncertainty of the project value, and a tool for making final decisions. In fact, these techniques give a range or distribution of values where the true unknown value may be allocated. In this way, in the face of uncertainty the mine planner has more options from which to select an optimal design based on their experience and company policies. This is not the case when using just one estimated orebody model.

One important conclusion obtained from this study is that because the mining process can be modelled as a 3D non-linear transfer function, the use of an (OK) expected orebody model does not ensure expected values of final outcomes such as the key project indicators (see Figures 53). Take for instance the results obtained in Figure 53-D, where the risk analysis of non-cumulative discounted cash flow is displayed. As observed in the figure, the expected values of the cash flows generated at each production period are not generated by the ordinary kriging orebody model, which is the expected orebody model, but by other models representing different percentiles. This is an important result since it corroborates and supports the use of simulation techniques together with estimation techniques for open pit mine planning and design even when a risk-neutral approach is assumed in the mine modelling, planning and designing process. In simple words, to select a project value based on an average DCF it is necessary to have the probability distribution of DCFs.

This work can be expanded to approach other problems such as the selection of cutbacks in a given ultimate pit design, or the selection of the entire mine project design, which include the estimation of the optimum ultimate pit and cutback limits, based on the uncertainty of the orebody (see Martinez, 2003). However, care needs to be taken when modelling an orebody based on the traditional techniques discussed in this chapter. The reason for this is that the assumption of the properties of the random function $Z(x_i)$ as a stationary, ergodic RF, is not always true in real life, and the implementation of these assumptions in the modelling process could cause spurious results about the characteristics of the orebody and, consequently, the general performance of the mine project.

There is currently no technology that is able to perform a complete mine project valuation/optimisation process in which the geological and other underlying variables' uncertainties are included in the process.

Key project indicators	**Description**	**CB-1**	**CB-2**	**CB-3**
DCF ($ million)	Estimated	14.2	18.0	19.2
	Maximum	16.0	19.7	20.2
	Median	13.8	16.9	17.1
	Minimum	11.3	14.0	13.5
Ore (Mt)	Estimated	1.0	2.0	3.0
	Maximum	1.0	1.8	2.5
	Median	1.0	1.7	2.3
	Minimum	0.9	1.6	2.2
Waste (Mt)	Estimated	0.36	1.4	2.6
	Maximum	0.5	1.8	3.5
	Median	0.46	1.7	3.4
	Minimum	0.4	1.6	3.2
Gold quantity (Mgr)	Estimated	1.78	2.9	4.1
	Maximum	1.8	2.9	3.7
	Median	1.76	2.68	3.4
	Minimum	1.6	2.4	3.2

Table 5. Results of the risk analysis performed by cutback for key project indicators for the base-case design of the gold deposit under study (after Martinez, 2003).

Chapter 6

Understanding and Modelling Future Metal Prices in Open Pit Mine Project Evaluation

6.1 Introduction

In this chapter we demonstrate how our novel IVOF method integrates with classical price forecasting.

Metal prices are one of the most important sources of uncertainty in the evaluation process of a mining venture[33] . In fact, any variation from the expected metal prices may considerably modify the results of the entire mine project evaluation. One reason for this is that past and present price movements shape expectations about future prices and consequently the profits obtained from the mine operation. For example an overestimated metal price may result in a favourable rate of return for a project which is otherwise doubtful and, conversely, an underestimated metal price may result in an unfavourable return for a project, which is otherwise profitable. Another reason is that metal prices have a significant influence on the cost of capital for mine project investments, because the option to defer project investment is based on the available information about future metal prices. Accordingly, making future forecasts of metal prices helps decision managers in both the reduction of the uncertainty of future events and in taking full advantage of the future opportunities available. In fact, no rational decisions can be made without at least implicitly taking some view of the future.

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[33] The characteristics of the orebody model and costs, among others, are also important sources of uncertainty.

Formally defined, the price of metal is the real cash settlement that represents the equilibrium or disequilibrium of the metal market. This market is based on demand, supply, and other factors such as speculations, news events, and dividend payouts (Case and Fair, 1989; Taylor, Moosa and Cowling, 2000; Fanning and Parekh, 2004). As defined by MacAvoy (1988), uncertainty about future metal prices arises because of two reasons: the lack of exact knowledge of those factors leading to the increase or decrease in metal supply and demand; and the practices that producers or consumers perform in the face of powerful speculative and political motives.

Even though the perception of future metal price fluctuations is well known by mine planners and mine analysts, one common practice in the mining industry is to ignore it, adopting a single estimated (expected) value to perform the evaluation of the mine project throughout its entire operating life. The argument commonly used to defend this assumption is that past and present prices cannot give information about future prices.

However, as it is shown in this chapter, the assumption of a single estimated metal price value over time is not accurate in capturing changes in metal price trends and, consequently, it may over/underestimate the value of a mine project in times where low/high trends occur. Furthermore, it will be shown that the single price value forecast assumption is also based on past values, which uses a very simple forecasting rule that is based on the arithmetic average.

In comparison to traditional techniques, modern project valuation procedures that are based on risk and uncertainty model future spot metal prices as stochastic (diffusion) processes that evolve over time. Then, Monte Carlo simulation is used to quantify the effect that the uncertainty of future spot metal prices have on the mine project value. But:

What are risk-neutral metal prices; and when and why do we need to use them?

This chapter explains the answers to these questions from a practical viewpoint. To achieve this, the chapter is organised as follows. Section 6.2 discusses the traditional way for modelling metal prices in mine project evaluation. Section 6.3 discusses the modern way of modelling future spot metal prices as stochastic (diffusion) processes. Section 6.4 introduces and discusses the concept of risk (neutral) adjusted metal prices and explains its difference from spot prices in the context of mine project evaluation. Section 6.5 presents an illustrative example of the effect of metal prices on mine project evaluation where a small gold mine project is evaluated using traditional, modern and risk-neutral metal prices. Section 6.6 gives general comments and conclusions.

6.2 Traditional ways of modelling metal prices in the mining industry

When valuing a mining project it is common practice to use just one expected value for the price of the economic metal(s) present in the mineral deposit. This price is normally assumed to be either the current spot price given in the market, or the average spot price over the last three years (Rendu, 2006)[34].

One of the reasons for using just a single estimated value is the ease of comparison between mining companies, preventing the use of excessively optimistic prices. Another reason is its simplicity when estimating the mine project cash flows and, consequently, its final market value. The common statement used to defend the use of a single estimated metal price(s) in mine project evaluations, is that past and current prices do not give information about future prices.

It is important to understand, however, that the use of a single metal price is also a very simple forecast of the future metal price behaviour, and it could be of a misleading nature when evaluating mining ventures. See for example Figure 55 where the starting point of a virtual copper mine project with a large operating life (say greater than 10 years), is plotted on different periods over the historical copper (spot) price for the periods April 1986 - April 2006. As observed in the figure, if the copper mine project is started in 1986 (in the figure indicated by point A) the assumption of a single metal price, given in this case by the April 1986 copper price, will underestimate the mine project's value throughout its operating life. The reason for this is the low copper price assumed for evaluating the mine operation. Conversely, if the copper mine project is started in 1989 (in the figure indicated by point B), it is seen that the assumption of a single metal price, given by the current April 1989 copper price, will overestimate the project's value for years 1989-2004, where copper prices were lower than the assumed initial expected copper price (point B in the figure); and underestimate the project's value for years 2005 and 2006, where copper prices presented an up-jump following an increasing trend. Observe in the previous example that the metal price is obtained as the average of the last three years (in the figure indicated by the grey dashed lines - Average 1) also fails to describe rises and falls in copper prices over time and, consequently, does not give a realistic representation of the mine project's value.

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[34] See final comments and conclusions, in Section 6, for cases where the assumption of a single expected metal price could be corrected adopted when evaluating a mine project.

But, how do mining companies protect themselves against changes in future metal prices when using a single metal price value in the evaluation of their mine projects?

When performing the economic evaluation of a mine project it is common practice to use the Discounted Cash Flow (DCF) technique (Mun, 2006; Kodukula and Papadescu, 2006), which uses a global risk-adjusted discount rate, R_{WACC}, normally given by the company's Weighted Average Cost of Capital (WACC)[35] and using the Capital Asset Pricing Model (CAPM) (Benninga, 2000; Kodukula and Papadescu, 2006; Mun, 2006).

To deal with the uncertainty associated with the mine project payoff, the DCF discounts the cash flows at a specific rate, R_{WACC}, that is composed of the risk-free rate, r, that accounts for the time value of money, and a risk premium, $r_{premium}$; that is,

$$R_{WACC} = r + r_{premium} \tag{6.1}$$

In this context, the risk premium, $r_{premium}$, is an aggregated risk factor that is supposed to account for the cash flow uncertainty that also includes the uncertainty of future metal prices. Among others reasons (see for example pp84-85 of Mun, 2006), one of the main problems of using the WACC risk-adjusted rate is that the firm’s capital structure policy may have specific long-term targets and weights that do not agree with the current metal market structure. Another problem is that because it is a global, or aggregated, static indicator of risk it does not consider the dynamism of the economy that changes over time; in fact, the estimation of a dynamic risk-adjusted discount factor that agrees with a dynamic market is not an easy task to perform. Due to the fact that different businesses may involve different degrees of risk, the proponents of the “DCF-value-based planning method (Kiechel, 1981)”, typically recommend the use of risk-adjusted hurdle rates. However, many authors (see for example Reimann, 1990) argue that businesses would be better off not adjusting divisional or business unit hurdle rates for differences in risk are applied. Instead, it is suggested that the cash flows themselves should be adjusted for their relative uncertainty obtaining their certainty equivalent. Observe that the previous statement is basically the main principle of the real options valuation technique.

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[35] The risk adjusted discount rate R_{WACC} is defined as the weighted average of the cost of debt and the cost of equity (assuming that the company has equities and debts only).

In summary, the consequence of using a single constant metal price when evaluating a mine project is the lack of perception about:

i) the risk and opportunities that can be generated if low/high prices occur in the future; and

ii) the best technical strategies that can be implemented in the evaluation process to mitigate the risk for losses if low prices occur, while taking advantage of opportunities if high prices occur.

This lack of perception about future project opportunities and risks due to changes in metal prices will lead the owners and stakeholders of the mine to make final decisions, such as investing or not investing in the project, based on a single *estimated* mine value, implying tacitly that the resulting value is the best estimate of performance. Clearly, these final decisions will be based on a variety of technical and economic systematic errors because this estimate is obtained by ignoring the impact of metal price variations[36].

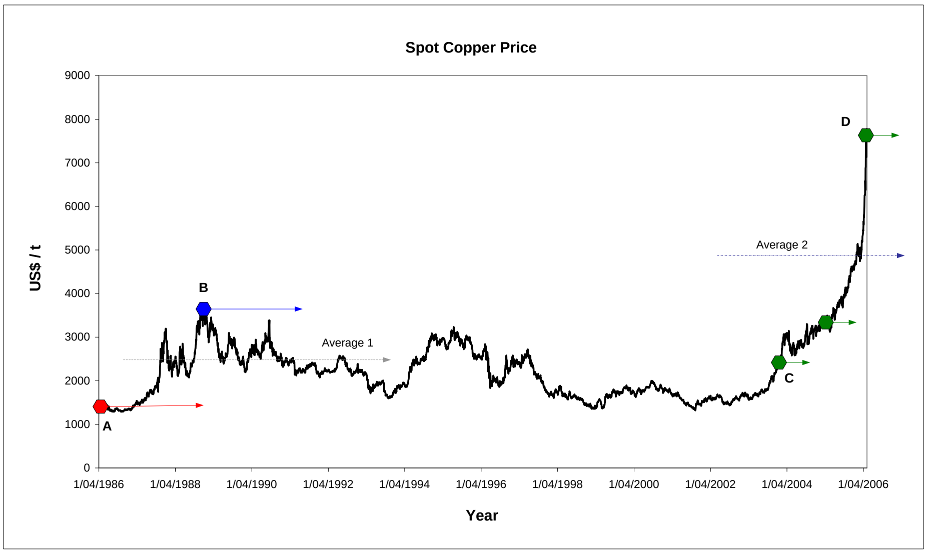

Figure 55 Chart showing the different copper price values that a copper mine will assume when evaluating the mine project using traditional metal price forecasting (in the figure, the historical copper spot price are for the period 1-04-1986 – 1-04-2006). As observed in the figure, mine projects started in1986 (point A) will underestimate its market value, while mine projects started in 1989 will overestimate its value for periods 1989-2004 and underestimate its value for periods 2004-2006.

6.3 Modelling future metal prices as stochastic processes

As opposed to the traditional way of modelling metal prices, current modern mine valuation procedures use advanced financial and economic techniques to model commodity metal prices as random variables that follow stochastic (diffusion) processes over time. This is done in order to capture the complexities of future marketing and metal production as well as other non-measurable factors such as speculation. It is important to observe that the aim of using stochastic processes for modelling future metal prices is to generate a series of possible price paths (this is done using Monte Carlo simulations) instead of a single estimated value. The purpose of doing this is to capture the unknown future behaviour of the metal price rather than estimating its exact value. This is because the process of forecasting metal prices for long time periods, which is common for mine operations, is not accurate (Davis, 1996).

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[36] In practice, mine managers prefer to work with a “number” rather than a distribution of numbers Peirson, G., et al.,2001. Business Finance, p.556-560,The McGraw-Hill Companies, Inc. , Ross, et al.,2003. Fundamentals of corporate finance, p.354-355,Irwin / McGraw-Hill .

In the literature there are many techniques available for metal price forecasting: the range goes from simple trend analysis to complex time series and econometrics models. However, the selection of an appropriate forecasting model plays a critical part in a mine project valuation process (see for example Schwartz, 1997; Schwartz, 1998). For example, classical microeconomic theory highlights that the price of investment-type assets, such as financial assets and gold, do not exhibit any kind of price reversion and, consequently, the most-used model to characterise the future behaviour of these types of assets is the Geometric Brownian Motion. Conversely, investment-type assets, and industrial-type commodities, such as copper and oil, are seen to be linked to their marginal production cost in the long term. In this way, expectations of future prices tend to fluctuate around or revert back to the marginal production costs and level of demand. Consequently, the common model used to model the future behaviour of industrial-type assets is the mean reverting model (MR).

The Geometric Brownian Motion Model (Chan and Wong, 2006; Dixit and Pyndick, 1994; Laughton and Jacoby, 1993; Brandimarte, 2006) is the most commonly used model to describe changes to commodity prices over time, S_t, in terms of the expected rate of growth, μ, and a random deviation from the expected rate written as the product of a volatility parameter, σ, and the standard Brownian motion, $dW_t = \varepsilon_t \sqrt{dt}$; where ε_t is a stochastic process following a standard normal distribution with mean equal zero and variance equal one. It is defined as follows[37]

$$dS_t = \mu S_t dt + \sigma S_t dW_t \,. \tag{6.2}$$

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[37] Observe that the model given in Equation 5.1 is simple in that the expected rate of growth, μ, and the volatility parameter, σ, are assumed to remain constant over time. A more complete model is that one in which both μ and σ varies over time.

Figure 56 shows 50 GBM paths generated by Monte Carlo simulation. The parameters for the GBM process are: $S_0 = 50$, $\mu = 10\%$, and $\sigma = 30\%$. From the figure it is observed that the GBM process never generates negative values. This is because the GBM is a lognormal process, that is, the natural logarithm of the commodity price under study is normally distributed (see for example Brandimarte pp431-433). This consistent generation of positive values is one of the reasons for which the GBM is widely used to model commodity prices since they are never negative. Note, however, that care needs to be taken when using the GBM for modelling a project's value over time. The reason for this is that the value of the project may become negative, which would not be considered if using the GBM. Another characteristic of the GBM is that the variance of the process increases proportionally with time. Consequently, projects with large production life spans following GBM behaviour will present greater uncertainty as time passes.

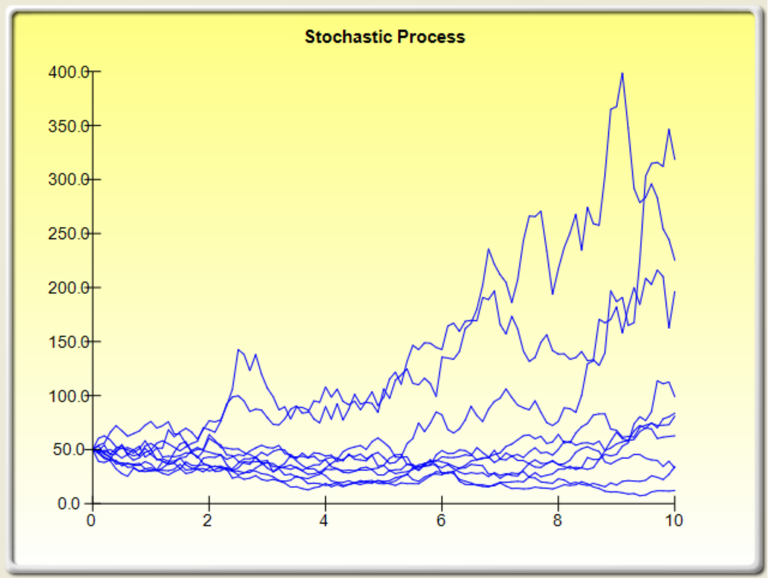


Figure 56. Twenty sample paths generated by a Monte Carlo simulation of a geometric Brownian process. The parameters for the GBM process are: $S_0 = 50$, $\mu = 10\%$, *and* $\sigma = 30\%$.

The Mean Reverting Model (Dixit and Pyndick, 1994; Laughton and Jacoby, 1993; Brandimarte, 2006) is another common model used in financial applications to describe the behaviour of future commodity (metal) prices.

To make it simple, the concept of mean reversion can be established from the theory of supply and demand: if the demand of a commodity is high/low and the supply is low/high, then the commodity price will go up/down until reaching a stable level where supply and demand are in equilibrium. In this case, the commodity price is said to follow a mean reverting process and the stable level price is the one at which supply and demand are in equilibrium. Other common definitions for mean reversion are (extracted from Exley, Mehta and Smith, 2004):

i) An asset model is mean reverting if returns are negatively correlated; and

ii) An asset model is mean reverting if interest rate (and volatilities) yields or growth rates are stationary.

Formally defined, the MR model can be expressed as

$$dS_t = \eta S_t \left(M - S_t \right) dt + \sigma dW_t \, , \tag{6.3}$$

where σ and $dW_t = \varepsilon_t \sqrt{dt}$ are the same as for the GBM, η is the speed of reversion, or mean-reversion rate, and M is the long-run equilibrium level. As observed in Equations 6.2 and 6.3, the main difference between the GBM and the MR is the drift term. Indeed, as shown in Figures 56 and 57 (where the mean reverting process given in Equation 6.3 is displayed), the MR process is a stationary process around the equilibrium level M while the GBM is a non-stationary process where the variance increases with time. In fact, the MR variance increases rapidly until reaching a stable level that remains constant over time. In this case, the MR drift is positive if the current price level S_t is lower than the equilibrium level M and negative if $S_t > M$. The speed of reversion, η, is related to the concept of half-life of the variable asset S_t. More formally, the *half-life*, H, of a variable S_t is defined as the time for which its expected value, $E\left\{S_t\right\}$, reaches the intermediate (middle) value between the current value, S_t, and the long run mean, M; that is, it is a measure of persistence. The *half-life*, H, is defined as $H = \ln(2)/\eta$ (see http://www.puc-rio.br/marco.ind/half-life.html for a more detailed analysis of speed of reversion and half-life estimation).

Figure 57 shows 100 paths generated by Monte Carlo simulation of the MR process formulated in Equation 6.2, with parameters $S_0 = 50$, $\sigma = 30\%$, $M = 35$, and $\eta = 0.8605$.

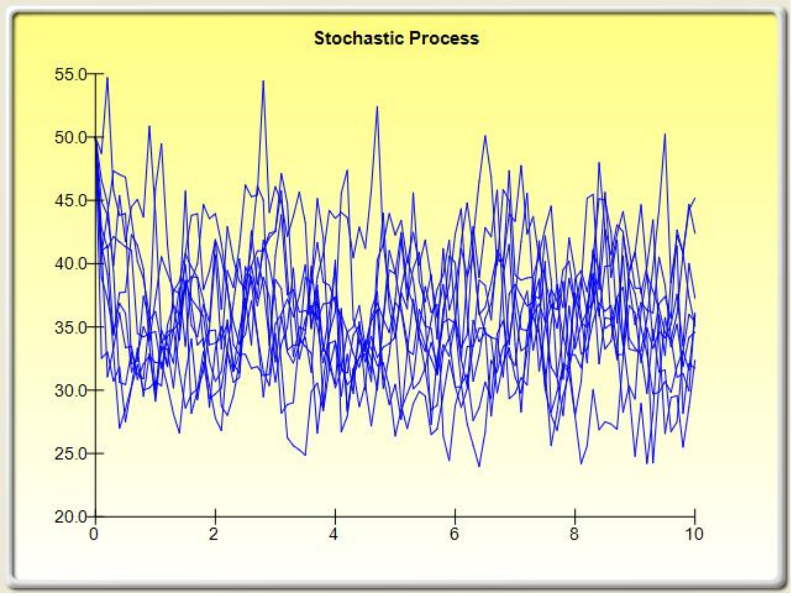


Figure 57. Fifty sample paths generated by Monte Carlo simulation of a Mean Reverting process. The MR parameters are $S_0 = 50$, $\sigma = 30\%$, $M = 35$, *and* $\eta = 0.8605$. *Observe that the Mean Reverting process tends to revert to the long-term value of* $M = 35$.

An extension of the previous MR model, given in Equation 6.3, is the adaptation to include stochastic ***jumps*** in its structure model (see for example Ch.10 of Shreve, 2004), that is,

$$\frac{dS_t}{S_t} = \eta\left(M - S_t\right)dt + \sigma dW_t - \left(\lambda k dt - dZ_t\right). \tag{6.4}$$

In Equation 6.4, the new term on the right hand side, $\left(-\lambda k dt + dZ_t\right)$, is the jump term:

$$dZ_t = \begin{Bmatrix} 0 & \text{with probability} & 1\text{-}\lambda \text{dt} \\ \Phi - 1 & \text{with probability} & \lambda \text{dt} \end{Bmatrix}, \qquad (6.5)$$
$$\textit{and}$$
$$k = E(\Phi - 1),$$

where λ is the Poisson arrival parameter, that is there is a probability λdt that a discrete jump will occur, and Φ is the size of the jump. Observe that the "minus one" appears as a matter of convention since the probability density function of Φ is defined only over positive values and then translated to jump size.

One clear example of these jumps is the recent behavior of copper spot prices shown in Figure 58. As observed in the figure, from 1986 to 2005 the copper price is seen to vary between \$1000/t and \$4000/t. However, in 2006 a jump in the price occurred, raising the copper price until it reached prices close to \$8000/t for the first time.

Figure 59 shows the 100 paths generated by Monte Carlo simulation of the MR process shown in Figure 57, with parameters $S_0 = 50$, $\sigma = 30\%$, $M = 35$, and $\eta = 0.8605$, and 10% jump rate with sizes equal to 1.5 (This is done following Equation 6.4). As observed in Figures 58 and 59, when the MR process presents stochastic jumps, the equilibrium price ($M = 35$) is reached at later stages; the reason for this is the presence of positive jumps (in this case there were more positive jumps than negative ones).

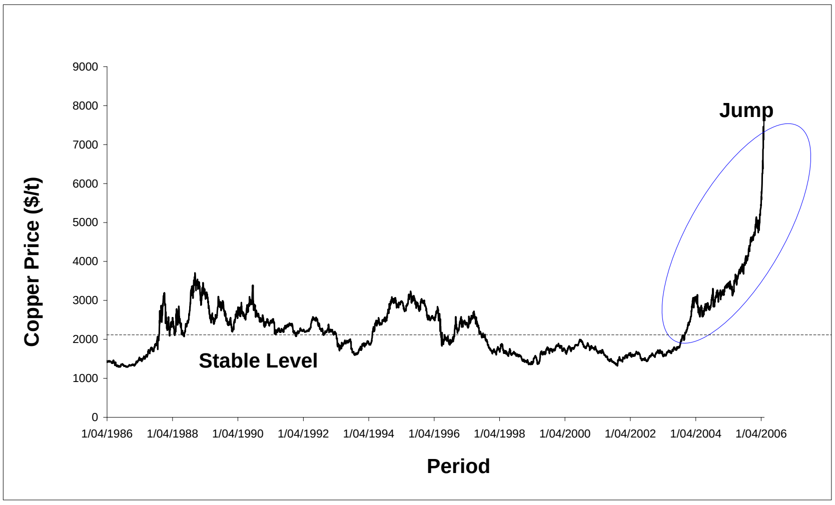


Figure 58 Chart indicating a jump in the spot price of copper in 2006.

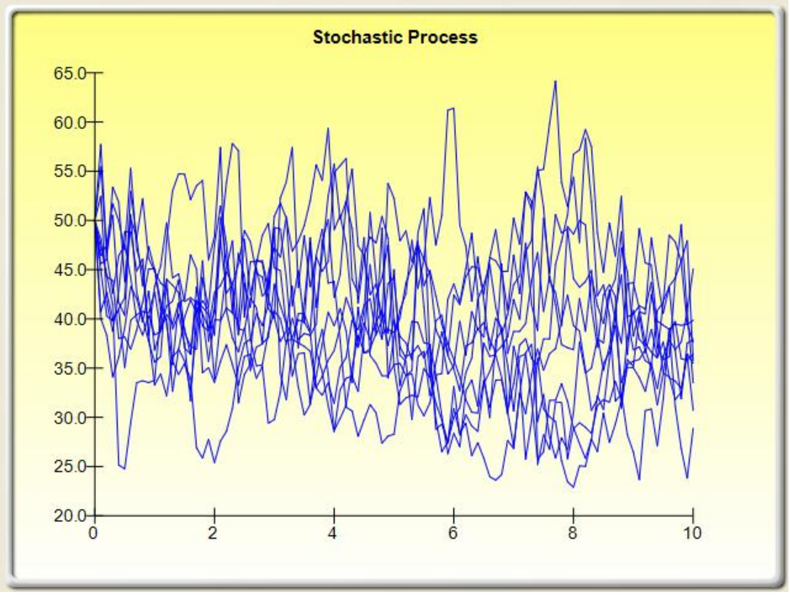


Figure 59 Fifty sample paths generated by Monte Carlo simulation of a Mean Reverting with jump process. The MR with jumps model's parameters are $S_0 = 50$, $\sigma = 30\%$, $M = 35$, *and* $\eta = 0.8605$, *and 10% jump rate with sizes equal to 1.5.*

6.4 Modelling future metal prices as risk-neutral stochastic processes

Hedging is a strategy that is commonly used by mining companies to reduce or mitigate their future risk and possible locks in profit (see pp. 3 of Billingsley, 2006). In this case, a mining company will write a forward contract to deliver a certain quantity of their final product(s), such as metal, at a specific future period and at a specified price.

When dealing with the metal market, however, the prices used to write contracts are not the current spot prices but the forward/future prices[38]. The reason for this is that the market is fair for both the buyer and the seller of the forward-contract and uses forward or *risk-neutral* prices, $F_{t_0}^{T}$. Formally defined, the risk-neutral forward price

---

[38] Observe that forward and future contracts can be considered the same if both have the same maturity time and the risk-free interest rate is constant (see for example Hull (2000) pp.60-61 for more details).

$F_{t_0}^T$ is the expected value of the future spot price, $S_{T>t_0}$, observed at the current time[39] t_0, that is, $F_{t_0}^T = E_{t_0}\left\{S_T\right\}$ (Hull, 1989). In other words, future prices are modeled in a *risk-neutral world*. It is important to note that the definition of "fair-market" is based on the concept of arbitrage. Arbitrage is the process of buying assets in one market and selling them in another to profit from unjustifiable price differences. In simple words, arbitrage is the action of taking advantage of current market conditions in order to generate a profit in the future, free of risk.

In the metal market, if the commodity is an investment asset[40] that has no storage cost and produces no income, such as gold and silver, then the relationship between forward and spot prices is:

$$F_{t_0}^T = S_{t_0}\left(1+r\right)^{(T-t_0)}, \tag{6.6}$$

where r is the risk-free discount rate and $T > t_0$. On the other hand, if storage costs are considered then the relationship between forward and spot prices is

$$F_{t_0}^T = S_{t_0}\left(1+r+u\right)^{(T-t_0)}, \tag{6.7}$$

where u is the storage cost per annum as a proportion of the spot price.

If the commodity is a consumption asset[41], such as copper and oil, Equation 6.7 may not hold since arbitrage strategies could be adopted. For example, if it is foreseen that $F_{t_0}^T > S_{t_0}\left(1+r+u\right)^{(T-t_0)}$, then an arbitrageur could borrow at the risk-free rate an amount equal to the spot value and the storage cost to buy one unit of the commodity and to pay the storage cost, and short (sell) a futures contract on one unit of the commodity (see Hull (1989) pp. 73 for other examples). Consequently, to avoid arbitrage opportunities when pricing consumption assets, the relationship between forwards and spot prices is:

---

[39] Note that if $T = t_0$, then $F_{t_0}^T = S_{t_0}$.

[40] Investment assets are held primarily for investment purposes.

[41] Consumption assets can be kept in inventory, incurring storage costs, waiting for better prices.

$$F_{t_0}^{T} = S_{t_0}\left(1 + r + u - y\right)^{(T - t_0)}, \tag{6.8}$$

where the convenience yield y provided by the asset simply measures the extent to which the left-hand side is less than the right-hand side in Equation 6.8. In fact, the convenience yield reflects the market's expectations concerning the future availability of the commodity.

Another way of interpreting the risk-adjusted, or risk-neutral, world when pricing options on financial and real assets, is as a world in which the market price of risk, λ_S, of the underlying variable (in this case the asset) is zero. The market price of risk, λ_S, associated with the price, S, is a variable that determines the growth rate of the asset. It shows that the excess return (premium) over the risk-free interest rate earned by any derivative that depends only on the asset and time is linearly related to the market price of risk of the stochastic variable underlying the derivative. The market price of risk, λ_S, is defined as:

$$\mu = r + \sigma\lambda_S, \tag{6.9}$$

where μ and σ are the discount rate and the volatility of the derivative, respectively, and r is the risk-free discount rate.

A common misconception in the mining industry, when evaluating a mining project using advanced financial techniques such as real options, is that no adjustment for risk is necessary when generating future metal price paths using a Monte Carlo simulation process (see Section 6.3). Indeed, it could be argued that generating multiple price paths accounts for variations and therefore risk, which is true under certain conditions[42]. However, it is interesting to observe the similitude between Equations 6.1 and 6.9 where the product $\sigma\lambda_S$ is a risk premium added to the risk-free rate, r, whose sum yields the risk-adjusted discount rate μ. Consequently, from equations 6.1 and 6.9 it is seen that if future metal prices are modelled as risk-neutral processes, their risk is already included in the stochastic process and does not need to be added in the adjusted discount rate used for cash flow estimation (see Section 6.2).That is the risk-free rate can be used for cash flow analysis when using risk-neutral prices. The advantage of avoiding a risk premium for metal prices in cash flow generation by modelling future metal prices as risk-neutral processe, is one of the key elements that advanced valuation techniques based on the real options concept (Amram and Kulatilaka, 1999; Armstrong and Galli, 1997; Barraquand and Martineau, 1995; Brennan and Schwartz, 2001; Cortazar, Schwartz and Casassus, 2001; Gravet, 2003; Kodukula and Papadescu, 2006; Mun, 2006) use for evaluating mining projects. In this case, real options valuation techniques value a mining project using risk-neutral metal prices and use the risk-free, r, as the discount rate instead of an adjusted discount rate[43], R_{WACC} (see Equation 6.1).

---

[42] This condition is related to the assumption that the adjusted-discount rate used in the cash flow analysis truly accounts for risk in future metal prices, which is not totally true (see general comments and conclusions for more details).

The Girsanov theorem (or Cameron-Martin-Girsanov theorem) (Cherubini, Luciano and Vecchiato, 2004b; Glasserman, 2004) is of great assistance when modeling metal prices as risk-neutral processes. The main idea is that given a Wiener process $w(t)$ defined under the filtration $(\Omega, \xi_t, \mathrm{P})$ it is possible to construct another process $w^*(t)$ which is a Wiener process under another probability space $(\Omega, \xi_t, \mathrm{Q})$. Of course, the latter will have a drift under the original measure P.

Consequently, using both the concept of the market price of risk, λ_S, and the Girsanov theorem, the risk-neutral, or risk-adjusted, GBM is defined as:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dS_t &= (r + \lambda_S \sigma) S_t dt + \sigma S_t dW_t, \\ dS_t &= r S_t dt + \sigma S_t (\lambda_S dt + dW_t), \\ dS_t &= r S_t dt + \sigma S_t dW^*_t. \end{aligned} \qquad (6.10)$$

---

[43] It is important to highlight that here we are assuming that future metal prices are the only source of uncertainty when evaluating a mining project. However, in practice there are other sources of uncertainty, such as geological and cost sources, that need to be considered and modelled before using the risk-free discount rate. But this is the topic of another paper.

Observe that the difference between Equation 6.2 and 6.10 is the drift and the random process. In the risk-neutral model the drift is replaced by the risk-free rate of return, r, and the random process is replaced by $\lambda_S dt + dW_t$. Also observe that when pricing financial assets, such as gold, because the market price of risk is zero, then $dW_t^* = dW_t$; and it means that the risk-neutral process of a financial asset following a GBM is the same as the real world model with a risk-free rate of return.

In a similar fashion, the risk-neutral model of both the MR given in Equation 6.3 and the MR with jumps given in Equation 6.4 can be expressed as (see http://www.puc-rio.br/marco.ind/half-life.html for a more detailed analysis):

Mean reversion:

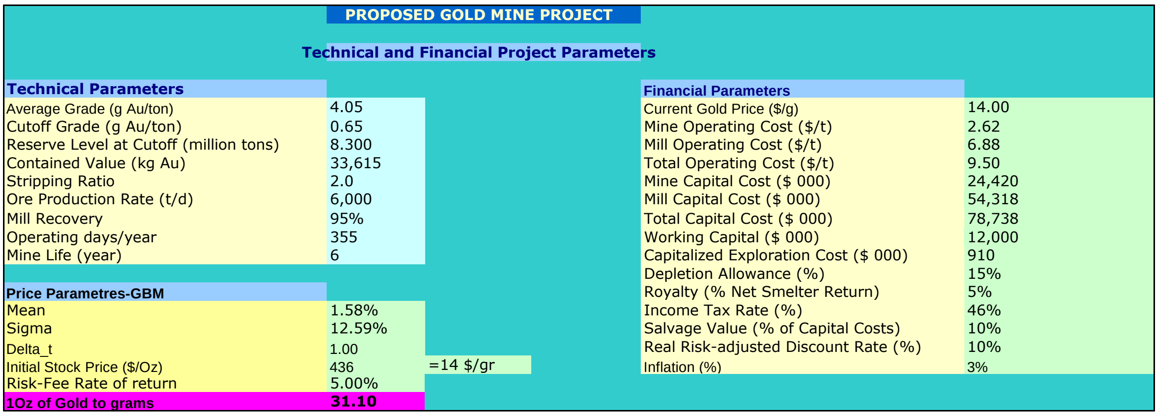
$$\begin{aligned} S_t = \exp\{&\left[\ln\left(S_{t-1}\right)\exp\left[-\eta\Delta t\right]\right] + \left[\left[\ln\left(\bar{S}\right) - \frac{(\mu - r)}{\eta}\right]\left(1 - \exp[-\eta\Delta t]\right)\right] \\ &- \left[\left(1 - \exp\left(-2\eta t\right)\right)\frac{\sigma^2}{4\eta}\right] + \sigma\sqrt{\frac{1 - \exp\left[-2\eta\Delta t\right]}{2\eta}} N\left(0,1\right)\}. \end{aligned} \tag{6.11}$$

Mean reversion with jumps:

$$\begin{aligned} S_t = \exp\{&\left[\ln\left(S_{t-1}\right)\exp\left[-\eta\Delta t\right]\right] + \left[\left[\ln\left(\bar{S}\right) - \frac{(\mu - r)}{\eta}\right]\left(1 - \exp[-\eta\Delta t]\right)\right] \\ &- \left[\left(1 - \exp\left(-2\eta t\right)\right)\frac{\sigma^2}{4\eta}\right] + \sigma\sqrt{\frac{1 - \exp\left[-2\eta\Delta t\right]}{2\eta}} N\left(0,1\right) + jumps\}. \end{aligned} \tag{6.12}$$

6.5 A simple example: the gold mine project evaluation problem

In this problem[44], a mining company is evaluating a small underground gold mining project containing an estimated one million ounces of gold. Corporate management must decide whether to invest in the gold mine venture now or wait and see how gold prices will develop[45]. The property has already been explored, but there is still some uncertainty about the total tons of ore. This translates into an uncertain project life. Management also assume uncertain capital costs, mining costs, milling costs, working capital, production rates, gold prices, ore grades, and ore recoveries. Figure 60 shows all the relevant technical and financial parameters used to evaluate the gold project.



PROPOSED GOLD MINE PROJECT

Technical and Financial Project Parameters

Technical Parameters	
Average Grade (g Au/ton)	4.05
Cutoff Grade (g Au/ton)	0.65
Reserve Level at Cutoff (million tons)	8.300
Contained Value (kg Au)	33,615
Stripping Ratio	2.0
Ore Production Rate (t/d)	6,000
Mill Recovery	95%
Operating days/year	355
Mine Life (year)	6

Price Parametres-GBM		
Mean	1.58%	
Sigma	12.59%	
Delta_t	1.00	
Initial Stock Price ($/Oz)	436	=14 $/gr
Risk-Fee Rate of return	5.00%	
1Oz of Gold to grams	**31.10**	

Financial Parameters	
Current Gold Price ($/g)	14.00
Mine Operating Cost ($/t)	2.62
Mill Operating Cost ($/t)	6.88
Total Operating Cost ($/t)	9.50
Mine Capital Cost ($ 000)	24,420
Mill Capital Cost ($ 000)	54,318
Total Capital Cost ($ 000)	78,738
Working Capital ($ 000)	12,000
Capitalized Exploration Cost ($ 000)	910
Depletion Allowance (%)	15%
Royalty (% Net Smelter Return)	5%
Income Tax Rate (%)	46%
Salvage Value (% of Capital Costs)	10%
Real Risk-adjusted Discount Rate (%)	10%
Inflation (%)	3%

Figure 60. Technical and financial parameters used to evaluate the gold mine project (see Chapter 4 for definitions of the different mining parameters).

As observed in the figure, the mine is expected to produce 6000t of ore per day with an estimated mill recovery of 95% yielding a life of mine of six years. The current (30-06-2005) gold price is at $435.50/Oz, but it is expected to change in the future.

---

[44] This model and exercise is an adaptation of the model created by Alpay Sergi, Visiting Scholar, and Graham A. Davis, Associate Professor, Division of Economics and Business, Colorado School of Mines, Golden, CO 80401, May 2002. Email gdavis@mines.edu.

[45] Observe that the decision to develop the mine is irreversible, in that after development management cannot disinvest and recover the expenditure.

Furthermore, the yearly risk-free discount rate is assumed to be $r = 5\%$ while the real annual risk-adjusted discount rate is $R_{WACC} = 10.0\%$[46].

To solve this problem, corporate management has decided to evaluate the mine using the following procedures: i) the DCF that uses an expected gold price; ii) the DCF that uses the GBM for modelling gold prices; and iii) the real option technique that uses risk-neutral GBM for modelling gold prices. The Binomial lattice technique (Kodukula and Papadescu, 2006; Mun, 2006) is used for solving the problem of investing now or one year later.

It is important to mention that for the Monte Carlo evaluation (using both the GBM and the risk-neutral GBM) specific distributions are assumed for the variables that also present uncertainty. Some of the distributions used in the analysis are: i) the truncated-normally distribution that characterises the ore production uncertainty; ii) triangular distribution used to characterise the working capital uncertainty; iii) uniform distribution used to characterise the mill recovery uncertainty; and iv) the lognormal distribution to characterise mine operating costs (Crystal ball software was used for the process). The results of the evaluation process are shown in Figures 61, 62, 63, 64 and 65 (observe that the results are obtained using the Crystal Ball software).

As observed in Figure 61, the expected after-tax NPV based on a static analysis using expected values of the uncertain variables is $73.3 million, with a payback period of 3.1 years and an IRR of 42%, which indicates that the project should proceed. The sensitivity analysis displayed in Figure 62 shows that if gold prices vary from 9$/gr to 20$/gr, the value of the mine varies linearly from $12.3M to $173.5M.

To evaluate the gold project using the GBM (see Equation 6.1) the historical gold prices and returns are first analysed. Figure 63 shows the historical monthly gold price (top) and return (bottom) between the periods 28-06-1985 to 30-06-2005. The result of a (simple) analysis shows that the annual average and standard deviation of the gold price are $\mu = 1.58\%$ and $\sigma = 12.59\%$, respectively[47].

---

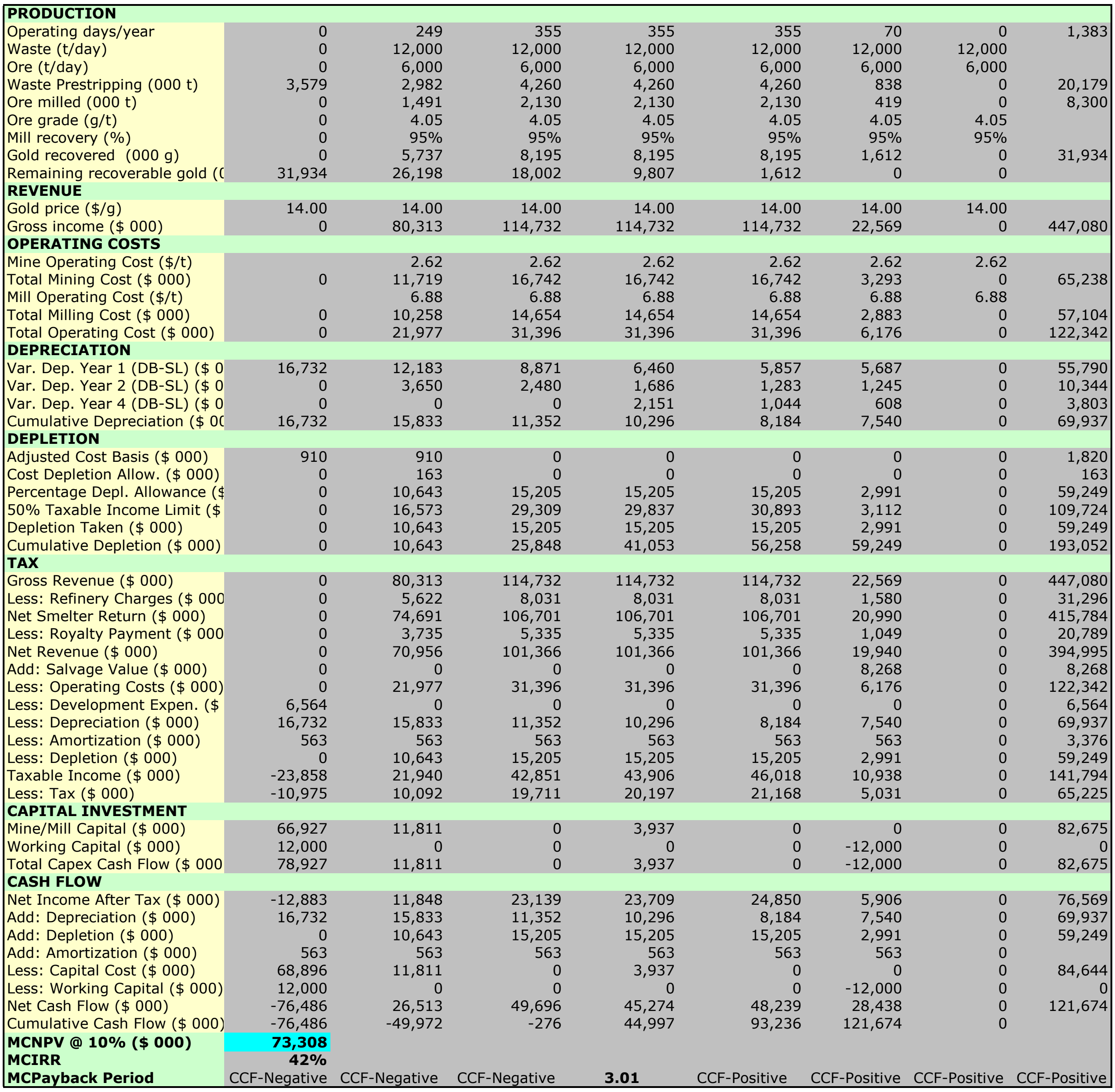
[46] The annual risk-adjusted discount rate is estimated as the Weighted Average Cost of Capital (WACC) of the mining company.

The expected NPV from the Monte Carlo analysis (showed in figure 64 - top), where metal prices are modelled as GBM (see Section 6.3), is \$87.204 million, showing that the static analysis (see Figure 61) underestimates the mean NPV by \$14 million; in fact, the likelihood of obtaining the expected static NPV of \$73.3 millions is 0.59. Figure 64 (bottom) also shows that the project has a mean IRR of 46% with a minimum of 20%, which indicates to proceed with the project (because it is greater than the adjusted discount rate of 10%). Furthermore, the Monte Carlo analysis indicates that the payback period is around 3.0 years which is similar to that obtained from the static analysis.

The expected NPV from the risk-neutral Monte Carlo analysis (showed in figure 65-top) where metal prices are modelled as risk-neutral GBM (see Section 6.4) is \$107.368 million, showing that both the static analysis (see Figure 61) and the simple MC analysis (see Figure 64) underestimate the expected NPV by around \$20 million; Figure 65 indicates that the likelihood of obtaining the expected static NPV of \$73.3 millions is 0.74. Furthermore, Figure 65 (bottom) shows that the mine project has a mean IRR of 52% with a minimum of 20%, which indicates that the project should proceed (because it is greater than the adjusted discount rate of 10%).

---

[47] Observe that other advanced techniques based on parametric and non-parametric time series analysis can be used to estimate both the mean and variance of the metal price return. However, for the sake of simplicity and practicality, a simple analysis based on moving average is used in this paper.



PRODUCTION								
Operating days/year	0	249	355	355	355	70	0	1,383
Waste (t/day)	0	12,000	12,000	12,000	12,000	12,000	12,000	
Ore (t/day)	0	6,000	6,000	6,000	6,000	6,000	6,000	
Waste Prestripping (000 t)	3,579	2,982	4,260	4,260	4,260	838	0	20,179
Ore milled (000 t)	0	1,491	2,130	2,130	2,130	419	0	8,300
Ore grade (g/t)	0	4.05	4.05	4.05	4.05	4.05	4.05	
Mill recovery (%)	0	95%	95%	95%	95%	95%	95%	
Gold recovered (000 g)	0	5,737	8,195	8,195	8,195	1,612	0	31,934
Remaining recoverable gold ((	31,934	26,198	18,002	9,807	1,612	0	0	
REVENUE								
Gold price ($/g)	14.00	14.00	14.00	14.00	14.00	14.00	14.00	
Gross income ($ 000)	0	80,313	114,732	114,732	114,732	22,569	0	447,080
OPERATING COSTS								
Mine Operating Cost ($/t)		2.62	2.62	2.62	2.62	2.62	2.62	
Total Mining Cost ($ 000)	0	11,719	16,742	16,742	16,742	3,293	0	65,238
Mill Operating Cost ($/t)		6.88	6.88	6.88	6.88	6.88	6.88	
Total Milling Cost ($ 000)	0	10,258	14,654	14,654	14,654	2,883	0	57,104
Total Operating Cost ($ 000)	0	21,977	31,396	31,396	31,396	6,176	0	122,342
DEPRECIATION								
Var. Dep. Year 1 (DB-SL) ($ 0	16,732	12,183	8,871	6,460	5,857	5,687	0	55,790
Var. Dep. Year 2 (DB-SL) ($ 0	0	3,650	2,480	1,686	1,283	1,245	0	10,344
Var. Dep. Year 4 (DB-SL) ($ 0	0	0	0	2,151	1,044	608	0	3,803
Cumulative Depreciation ($ 0(	16,732	15,833	11,352	10,296	8,184	7,540	0	69,937
DEPLETION								
Adjusted Cost Basis ($ 000)	910	910	0	0	0	0	0	1,820
Cost Depletion Allow. ($ 000)	0	163	0	0	0	0	0	163
Percentage Depl. Allowance ($	0	10,643	15,205	15,205	15,205	2,991	0	59,249
50% Taxable Income Limit ($	0	16,573	29,309	29,837	30,893	3,112	0	109,724
Depletion Taken ($ 000)	0	10,643	15,205	15,205	15,205	2,991	0	59,249
Cumulative Depletion ($ 000)	0	10,643	25,848	41,053	56,258	59,249	0	193,052
TAX								
Gross Revenue ($ 000)	0	80,313	114,732	114,732	114,732	22,569	0	447,080
Less: Refinery Charges ($ 000	0	5,622	8,031	8,031	8,031	1,580	0	31,296
Net Smelter Return ($ 000)	0	74,691	106,701	106,701	106,701	20,990	0	415,784
Less: Royalty Payment ($ 000	0	3,735	5,335	5,335	5,335	1,049	0	20,789
Net Revenue ($ 000)	0	70,956	101,366	101,366	101,366	19,940	0	394,995
Add: Salvage Value ($ 000)	0	0	0	0	0	8,268	0	8,268
Less: Operating Costs ($ 000)	0	21,977	31,396	31,396	31,396	6,176	0	122,342
Less: Development Expen. ($	6,564	0	0	0	0	0	0	6,564
Less: Depreciation ($ 000)	16,732	15,833	11,352	10,296	8,184	7,540	0	69,937
Less: Amortization ($ 000)	563	563	563	563	563	563	0	3,376
Less: Depletion ($ 000)	0	10,643	15,205	15,205	15,205	2,991	0	59,249
Taxable Income ($ 000)	-23,858	21,940	42,851	43,906	46,018	10,938	0	141,794
Less: Tax ($ 000)	-10,975	10,092	19,711	20,197	21,168	5,031	0	65,225
CAPITAL INVESTMENT								
Mine/Mill Capital ($ 000)	66,927	11,811	0	3,937	0	0	0	82,675
Working Capital ($ 000)	12,000	0	0	0	0	-12,000	0	0
Total Capex Cash Flow ($ 000	78,927	11,811	0	3,937	0	-12,000	0	82,675
CASH FLOW								
Net Income After Tax ($ 000)	-12,883	11,848	23,139	23,709	24,850	5,906	0	76,569
Add: Depreciation ($ 000)	16,732	15,833	11,352	10,296	8,184	7,540	0	69,937
Add: Depletion ($ 000)	0	10,643	15,205	15,205	15,205	2,991	0	59,249
Add: Amortization ($ 000)	563	563	563	563	563	563	0	
Less: Capital Cost ($ 000)	68,896	11,811	0	3,937	0	0	0	84,644
Less: Working Capital ($ 000)	12,000	0	0	0	0	-12,000	0	0
Net Cash Flow ($ 000)	-76,486	26,513	49,696	45,274	48,239	28,438	0	121,674
Cumulative Cash Flow ($ 000)	-76,486	-49,972	-276	44,997	93,236	121,674	0	
MCNPV @ 10% ($ 000)	**73,308**							
MCIRR	**42%**							
MCPayback Period	CCF-Negative	CCF-Negative	CCF-Negative	**3.01**	CCF-Positive	CCF-Positive	CCF-Positive	CCF-Positive

Figure 61. Static discounted cash flow analysis of the underground gold mining project. This is shown as the traditional DCF spreadsheet analysis performed when evaluating a mine project.

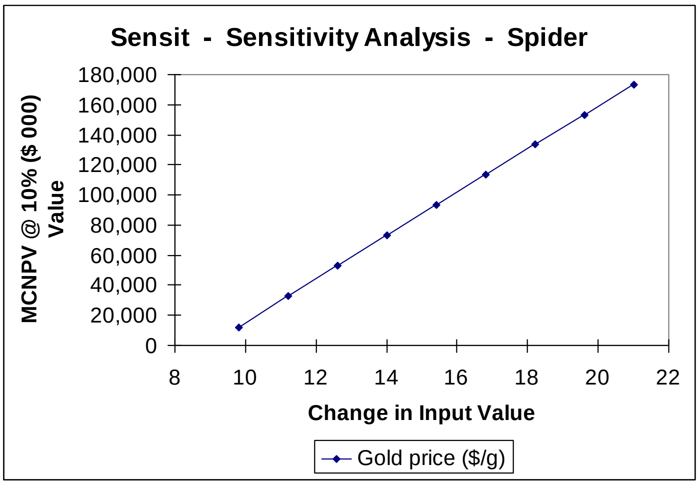


Figure 62. Sensitivity analysis, with respect to gold price variation, of the underground gold mining project.

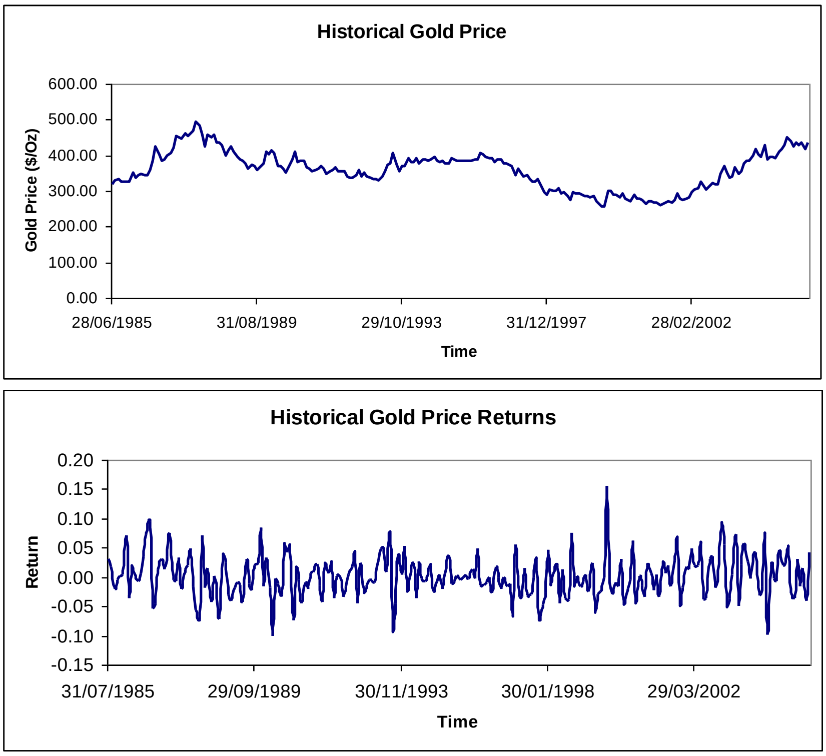


Figure 63. Historical gold price (top) and gold price return (bottom) for the period 28-06-1985 to 30-06-2005.

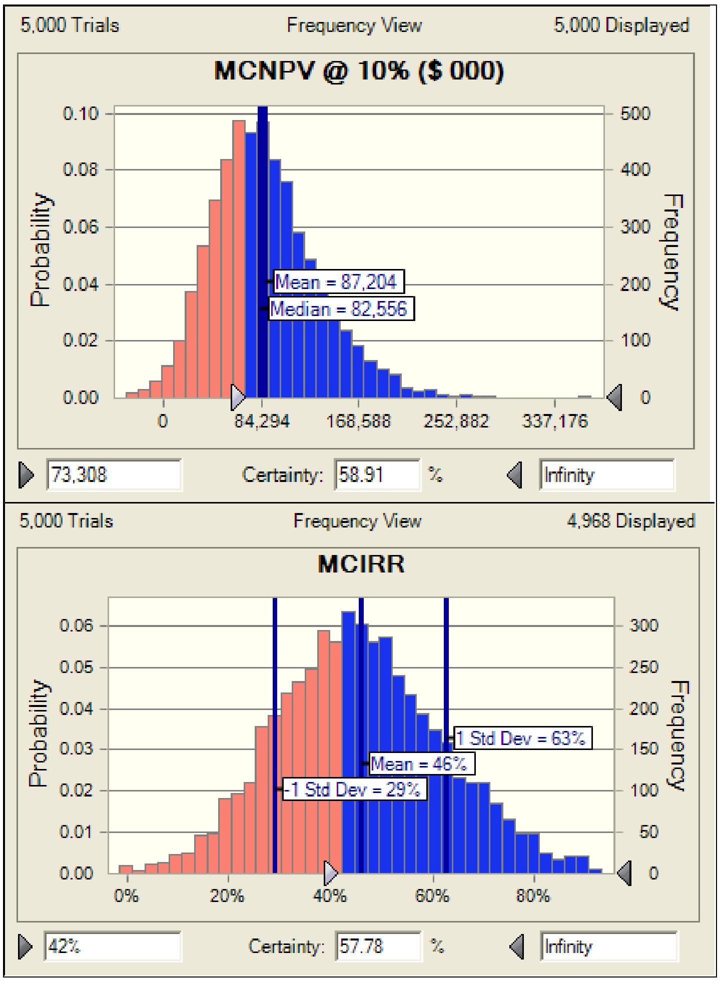

5,000 Trials
Frequency View
5,000 Displayed
MCNPV @ 10% ($ 000)
Probability
Frequency
0.10
0.08
0.06
0.04
0.02
0.00
500
400
300
200
100
0
Mean = 87,204
Median = 82,556
0
84,294
168,588
252,882
337,176
73,308
Certainty:
58.91
%
Infinity
5,000 Trials
Frequency View
4,968 Displayed
MCIRR
Probability
Frequency
0.06
0.05
0.04
0.03
0.02
0.01
0.00
300
250
200
150
100
50
0
1 Std Dev = 63%
Mean = 46%
-1 Std Dev = 29%
0%
20%
40%
60%
80%
42%
Certainty:
57.78
%
Infinity

Figure 64. Probability distribution of NPV (top) and IRR (bottom) of the underground gold mining project using the simple MC (GBM) analysis (after 5000 simulations). The results were obtained using the Crystall Ball software.

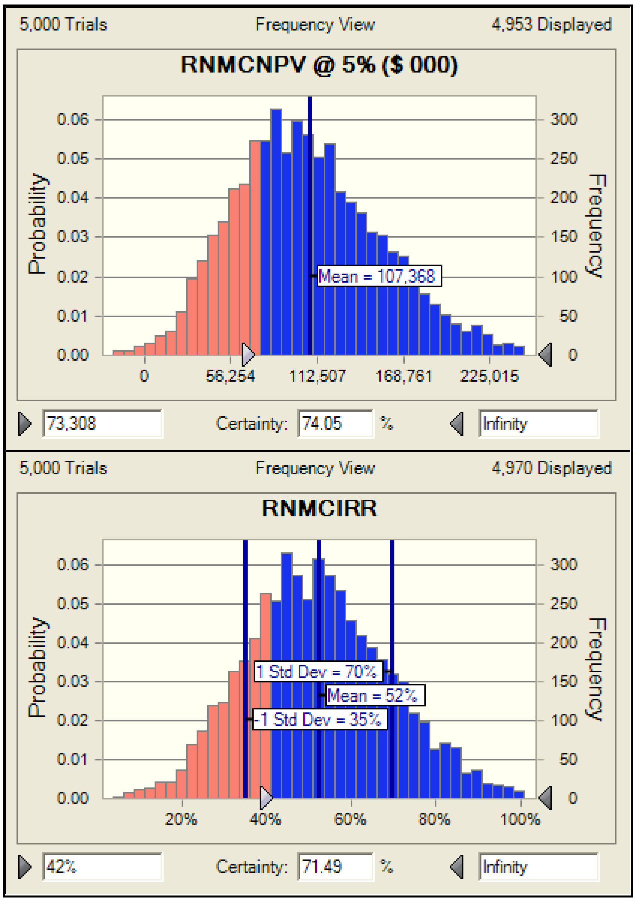


Figure 65. Probability distribution of NPV (top) and IRR (bottom) of the underground gold mining project using the risk-neutral MC (GBM) analysis (after 5000 simulations). The results were obtained using the Crystall Ball software.

6.6 General comments and conclusions

From the previous analysis, at first sight it appears that the risk-neutral analysis overestimates the expected NPV of the mine project because it uses the risk-free discount rate, which is smaller than the risk-adjusted discount rate[48]. In fact, a common analysis of the results will suggest that the difference between expected NPVs generated by the static, simple Monte Carlo technique and risk-neutral processes comes from the reduction in the discount rate used in the cash flow analysis; that is, the risk-neutral analysis uses the risk-free discount rate of 5% instead of the adjusted discount rate of 10% used by both the static and simple Monte Carlo analyses. In other words, in the risk-neutral process the cash flow stream is discounted over time with a smaller factor than in the static or simple Monte Carlo analysis.

---

[48] In fact, this was the first answer I received from the engineers working at mine sites that were in charge of the project evaluation process.

A more detailed analysis, however, will reveal that the use of a small discount rate (as is the case when using the risk-free instead of the adjusted discount rate) does not only increase the metal price value over time[49], but also costs. Indeed, since costs are now affected by a small discount rate their values over time are not as small as those obtained when using a high discount rate. Although it is not the case in this example, mine projects with high operating costs will realise that a risk-neutral cash flow analysis will suggest a smaller expected value than those obtained from a static or standard Monte Carlo analysis[50].

But, how can the mine project manager be sure that the result obtained from the risk-neutral Monte Carlo analysis is a more accurate representation of the mine project's value than the value obtained from the simple Monte Carlo analysis?

There is no direct answer to the previous question. The reason for this is that the value of a mine project will never be revealed until it is depleted and finished, meaning that at the evaluation stage it is not possible to assess which result is more accurate because there is not a true mine value to compare it to. Another reason as to why the previous question is not easy to explain is that it depends on many factors and decisions made at different levels, such as the corporate and business levels.

One important point that has to be highlighted is that the simple Monte Carlo analysis still uses a risk-adjusted discount rate for project evaluation while the risk-neutral Monte Carlo process uses a smaller discount rate equal to or bigger than the risk-free discount rate. As shown in Section 6.4 - Equation 6.8, the future cash flows of a mining project can be adjusted directly for relative risk to produce their certainty equivalents, which need only be discounted at the risk-free rate to account for the time value of money. Since this chapter focuses only on the metal price uncertainty, risk-neutral stochastic modelling is seen as an effective technique for modelling future metal prices and, consequently, accurate cash flow projections (that is, returns and growth), which in practice has been recognised by many experienced practitioners as having a higher priority for a mining project’s value than the discount rate.

---

[49] Observe that in this case the increment in price value is purely due to the effect of a small discount factor when discounting future cash flows.

[50] In some cases the risk-neutral analysis will suggest that the project is unfeasible under current conditions.

Also observe that when dealing with mine projects with two or more economic metals, the forecasting process has to be done considering the correlation between the metal prices, that is, as a multivariate process. Although in a multi-metal mine project the mine evaluation process is more complicated, the result of approaching the process as a multivariate one will give a better output than if it were considered as a mutually exclusive or independent one.

In summary, this chapter has highlighted not only the difference between forecasting metal prices as either spot or forward (risk neutral), but also its importance when selecting the appropriated discount rate for cash flow analysis, which most of the times is confused in practice. That is, the real options method uses forward prices and consequently discounts cash flows at the risk-free discount rate while simple risk analysis uses forecasted spot prices and uses the risk-adjusted discount rate.

Chapter 7

Case study: Valuing a Gold Mine Project Using the Proposed IVOF

7.1 Introduction

In this problem the corporate office of a mining company is evaluating one of their mining projects as part of their portfolio of mine projects' optimisation policy in order to decide whether or not going ahead with the venture. The mine project consists of a small disseminated high-grade gold deposit which lies very close to the surface, (so the initial waste stripping is not a significant issue). Corporate management must decide whether to invest in the gold mine venture or abandon it and focus on other alternative projects. In this case, because the project is expected to have a life of mine no bigger than 6 years (the initial guess of the corporate office), the gold price and the orebody model (in situ metal grade uncertainty) are the only sources of uncertainty considered in the evaluation process[51].

The geological data available for this project consists only of drill-holes composite data (see Figure 66) that contains information about gold grades (in grams), rock types of oxide, mixed-transitional, primary and wall rock considered as sterile (see Figure 67), and geotechnical zones (composed of 8 zones) (see Figure 68). The relevant financial and technical parameters available for the evaluation process are shown in Table 6. As observed in the table, the production period is considered to be one year each, the mill capacity is 2Mt per year and mining capacity is 9Mt of rock per year. The expected gold price is 27.5Au$/gr. Slope angles are specified for each geotechnical zone and processing costs and recoveries are specified for each rock type. Furthermore, corporate uses a yearly risk-free discount rate of $r = 5\%$ while the real annual (WACC) adjusted discount rate is $R_{WACC} = 12\%$[52]. Mining costs and processing costs are included in the orebody model as varying for each production bench and rock type, respectively. Additional information indicated that no stock piles are considered in the analysis and that the required mining width for mining machinery is 40m. At the end of the mining operation, the corporate office has estimated a salvage value of Au$60M.

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[51] Other variables such as capital costs, mining costs, milling costs, working capital, labour costs, and recoveries, among others, could also be considered in the evaluation analysis, but it would require a more detailed and complex analysis.

To solve this problem, corporate management has decided to evaluate the mine using the IVOF technique with the assumption that the gold price follows a Geometric Brownian Model (GBM)[53] (Armstrong and Dowd, 1994). The Binomial lattice technique is used for making the final mine evaluation decision and the GEMCOM suite of tools, including Surpac and Whittle[54], are used for orebody modelling and mine production scheduling design and optimisation, respectively.

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[52] The annual risk-adjusted discount rate is estimated as the Weighted Average Cost of Capital (WACC) of the mining company owner of the gold project.

[53] This was done in the light of the fact that GBM is commonly used for forecasting gold prices.

[54] These are standard analytical tools for orebody modeling and open pit production scheduling. See http://www.gemcomsoftware.com/ for details.

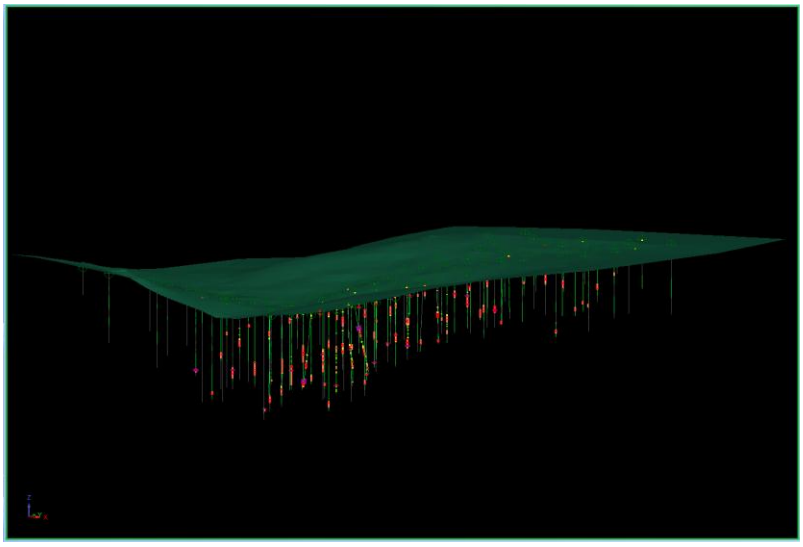

Figure 66. East view of the gold mine project drill hole composite data set showing the topography of the deposit (developed in SURPAC).

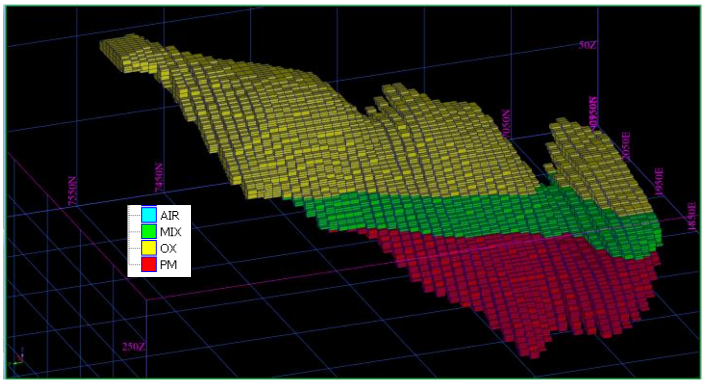
50Z
7550N
7450N
1950E
1850E
250Z
AIR
MIX
OX
PM

Figure 67. East view of the gold mine deposit showing the oxide, transitional and primary rock types (developed in SURPAC).

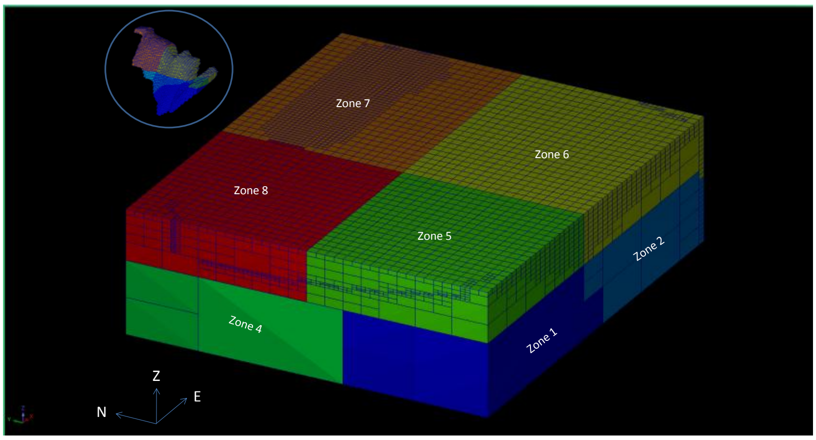

Figure 68. North-east view of the gold mine deposit showing the eight geotechnical zones (developed in SURPAC).

Parameter Description	**Value**
Assumed Production Period (years)	1
Expected Gold Price (Au$/gr)	27.5
Expected selling cost (Au$/gr)	0.6875
WACC discount rate	12%
Time cost / period (millions of Au$)	1.5
Mill capacity / period (Mt)	2
Maximum mine capacity / period (Mt)	9
Procesing cost Oxide rock (Au$/t)	15.39
Procesing cost Transitional rock (Au$/t)	16.66
Procesing cost Primary rock (Au$/t)	16.9
Recovery-Oxide rock	94%
Recovery-Transitional rock	94%
Recovery-primary rock	90%
Slope angle Zone 1 (degrees)	52
Slope angle Zone 2 (degrees)	57
Slope angle Zone 3 (degrees)	50
Slope angle Zone 4 (degrees)	48
Slope angle Zone 5 (degrees)	45
Slope angle Zone 6 (degrees)	50
Slope angle Zone 7 Degrees)	55
Slope angle Zone 8 (Degrees)	60

Table 6. Table showing some of the financial and technical parameter values used to evaluate the gold mine project.

7.2 Stage 1: The base-case mine design

The first stage in the IVOF process is to use current state-of the art mine evaluation techniques to build an initial open pit mine, named the base-case mine design. The objective of this stage is to use the base-case mine design as a benchmark to compare with the final results, and to identify important technical and economic cash flow drivers. Figure 69 shows the variogram map used to build an estimated orebody model using the ordinary kriging technique, see Figure 70. The Gemcom-Whittle software is then used for generating the base-case open pit mine design. As observed in figures 71 and 72, the base-case mine design of the gold mine project is composed of three cutbacks (see Figure 71) and the long-term production scheduling (see Figure 72). The values of the base-case mine project indicators are summarised in Figure 73. As observed in the Figure, the gold mine project is expected to have an operating life of around 3.5 years, where the mill target will be achieved during the first 3.5 years. The mine is expected to produce around 5.5M, 6M, 5M and 3.5M of grams of gold during the first, second, third and fourth year, with expected average grades of 2.9, 3.2, 2.6 and 3.4 gr./t, respectively. Figure 73 also shows that mining costs and selling costs are expected to be high during the first three years (around Au$30M), with mining costs reaching a peak during the second year (Au$35M). Processing costs are seen to remain constant, around Au$30M, throughout the operating life of the mine.

Furthermore, Figure 73 also suggests that the gold mine project will generate expected cash flows of around Au$75M, Au$65M, Au$50M, and Au$40M, for the first, second, third and fourth production periods respectively, giving a total (current) expected value of around Au$230M. A sensitivity analysis further indicates that a ±10% variation in metal prices will generate variation in the mine value to Au$300M (+10%) and Au$200M (-10%).

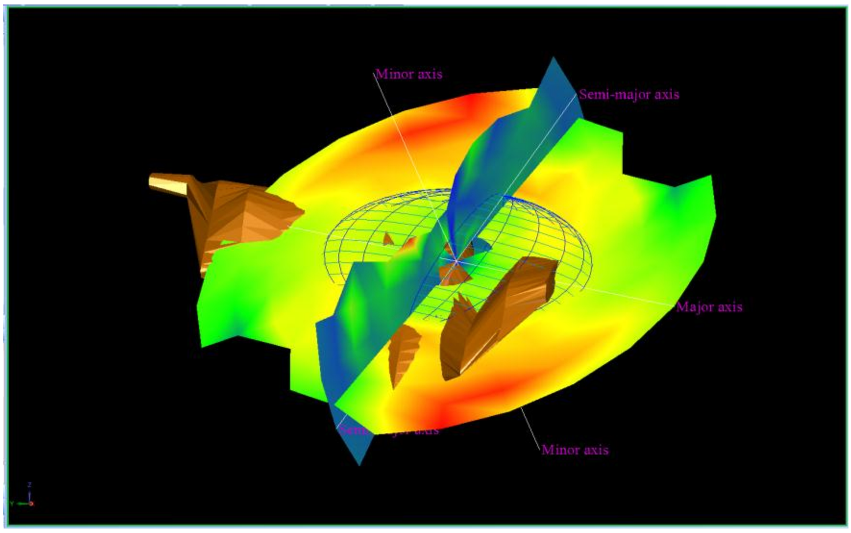
Minor axis
Semi-major axis
Major axis
Minor axis

Figure 69. Variogram analysis performed on the gold mine deposit (SURPAC-geostatistical tools).

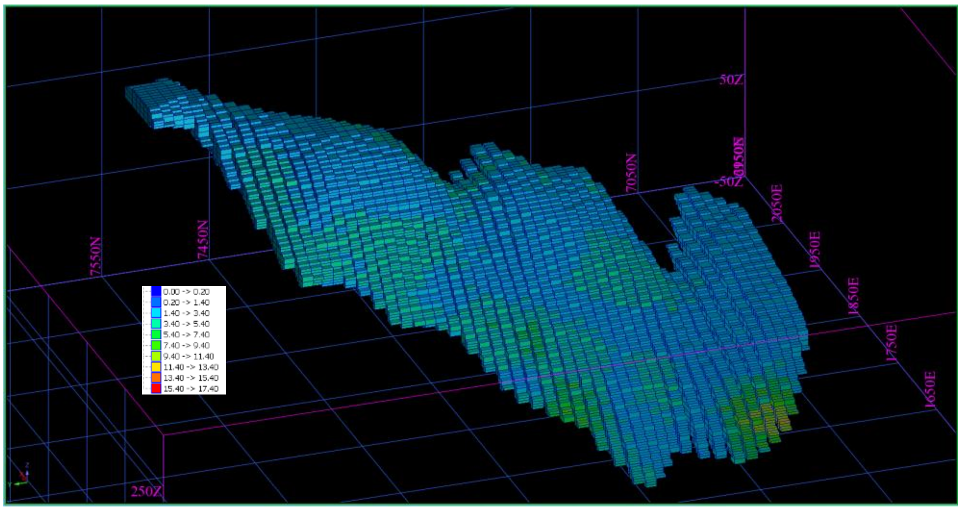
50Z
6950N
-50Z
7050N
2050E
1950E
1850E
1750E
1650E
7550N
7450N
250Z
0.00 -> 0.20
0.20 -> 1.40
1.40 -> 3.40
3.40 -> 5.40
5.40 -> 7.40
7.40 -> 9.40
9.40 -> 11.40
11.40 -> 13.40
13.40 -> 15.40
15.40 -> 17.40

Figure 70. South-east view of the estimated gold mine orebody model. In this case the ordinary Kriging technique (SURPAC-geostatistical tools) was used for generating this model.

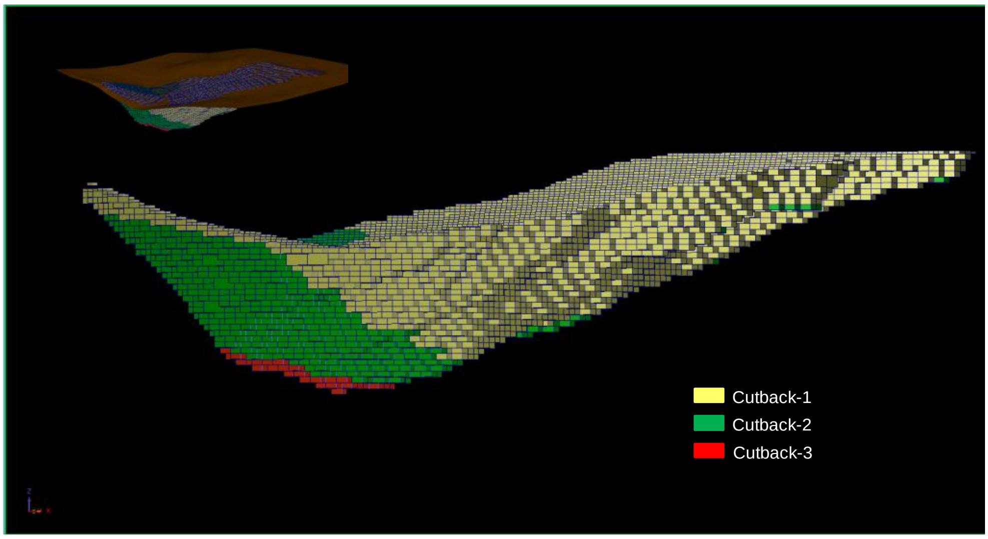
Cutback-1
Cutback-2
Cutback-3

Figure 71. North view of the gold mine Project base-case open pit mine design showing the selected cutbacks.

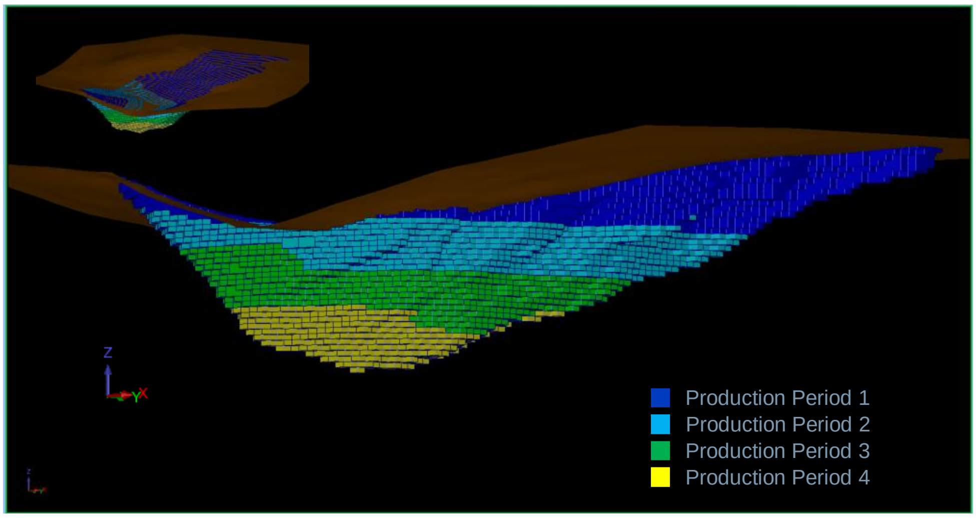
Z
Y
X
Production Period 1
Production Period 2
Production Period 3
Production Period 4

Figure 72. North view of the gold mine project base-case open pit mine design showing the long-term production scheduling.

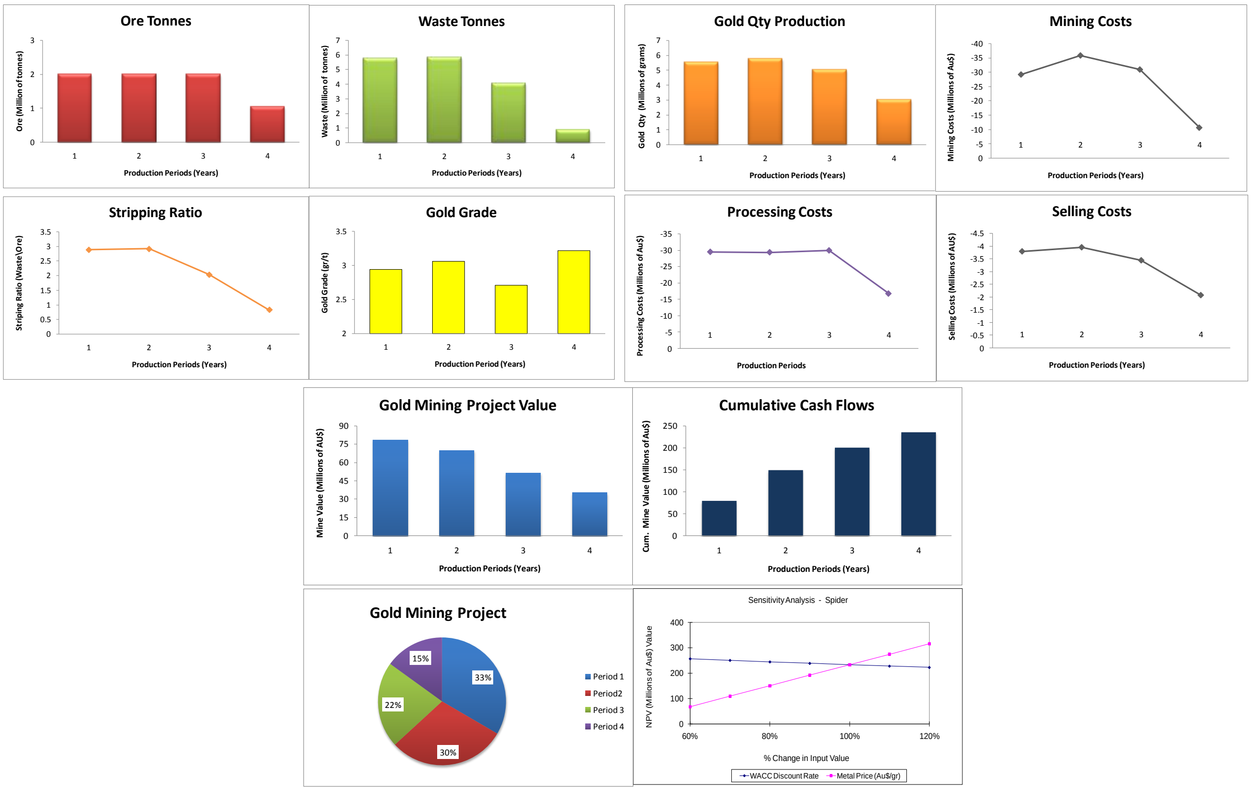

Figure 73. Summary of the gold mine project indicators.

7.3 Stage 2: What if metal grades vary? Assessing the risk of the base-case mine design indicators due to metal grade variation: An Upside/Downside-potential analysis

At this stage, the IVOF performs a risk analysis of the resulting base-case mine project indicators in the face of in situ metal grade variation. As observed in Figure 74, to find out what the effect is of metal grade uncertainty on key project indicators and mine value, the IVOF performs the following steps: i) quantify the metal grade uncertainty throughout the generation of several simulation models of the orebody. This is done using a suitable technique[55], such as the Sequential Gaussian Simulation (SGS) (Armstrong and Dowd, 1994; Chilés and Delfiner, 1999; Journel and Kyriakidis, 2004). In this case, for the sake of simplicity, only 25 simulations of the gold mine project orebody model were generated[56]; ii) overlay the base-case (physical) mine design and production scheduling on each of the simulated orebody models and record the new project indicator values; and iii) perform an Upside/Downside risk analysis (Martinez, 2003). The results of this process are displayed in Figure 75.

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[55] Observe that other more sophisticated techniques could be used for the simulation process; however, in this case the SGCS was used because it is a well-understood technique for the sake of practicality.

As observed in Figure 75, on one hand, the in situ metal grade uncertainty does not have a significant effect on ore and waste tonnes and, consequently, on the stripping ratio. The reason for this is that the production scheduling was already (physically) defined, that is, the ore and waste blocks were already defined when designing the base-case mine design (observe that the production scheduling is not being changed, rather the risk is being assessed of the existing one). One consequence of this is that, in this particular case, mining costs and processing costs do not have much variation due to the fact that no more or less ore and waste are mined or processed.

On the other hand, the results displayed in Figure 75 indicate that (as expected) the in situ metal grade uncertainty has a significant effect on the average gold grade as well as the metal quantity produced at each production period. The high uncertainty in average gold grade and gold production is also reflected in both the selling costs and the final cash flows generated at each production period. Indeed, as observed in the figure (starting from left and top, second-last line for selling costs and third line for cash flows) although the potential for having higher selling costs than the expected during the first, second and third production periods is high, it did not affect the high potential for generating high cash flows during these periods. In fact, the risk of generating cash flows below the expected levels during the first, second and third production periods is not significant, while the last period is seen to have a high risk of generating a cash flow below the expected one.

In summary, the risk analysis indicates that the gold mine project has high potential to generate values greater than the expected value (around Au$230M) with a maximum of Au$350M and a minimum of Au$ 200M.

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[56] For a better explanation of the appropriate number of simulations to be used when evaluating a mining project see Section 5.4.1.

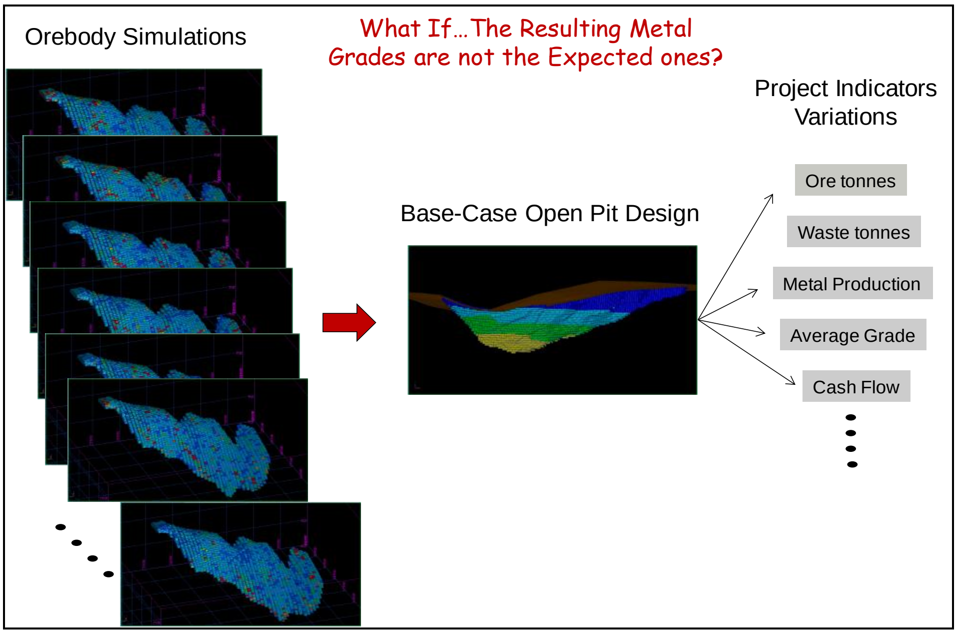

Orebody Simulations
What If…The Resulting Metal Grades are not the Expected ones?
Project Indicators Variations
Base-Case Open Pit Design
Ore tonnes
Waste tonnes
Metal Production
Average Grade
Cash Flow

Figure 74. Assessing the risk of the gold mine project base-case mine project indicators in the face of in situ metal grade uncertainty.

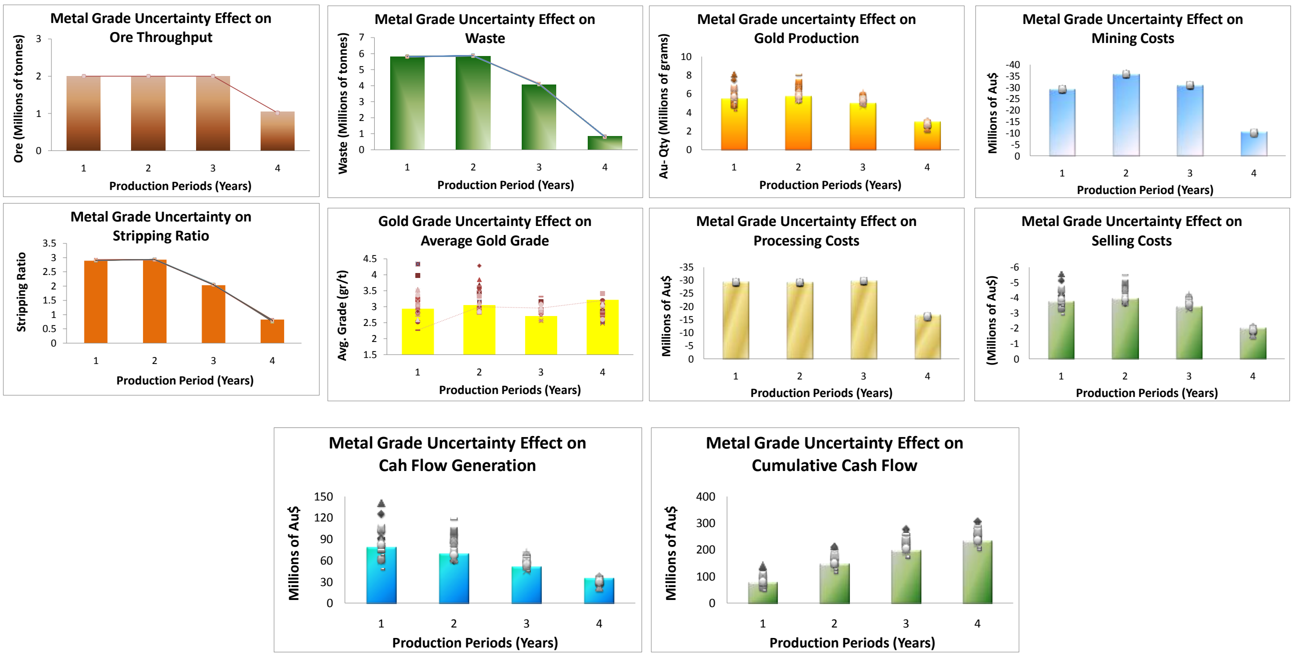
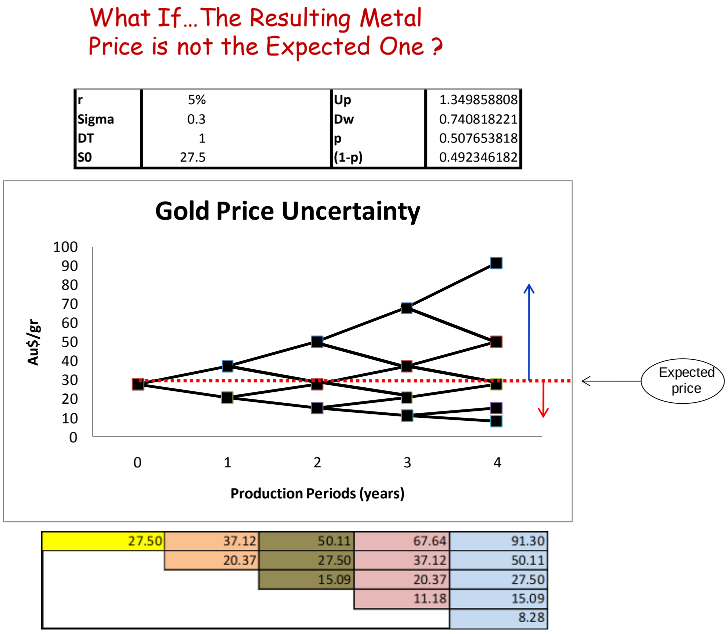


Figure 75. Chart showing the effect of metal grade uncertainty on the gold mine project key project indicators.

7.4 Stage 3: What if metal price varies? Assessing the risk of the base-case mine design indicators due to both metal grade and metal price variation

At Stage 2 it was seen how the IVOF performs an Upside/Downside potential analysis on the base-case mine design indicators due to in situ metal grade variation. In this section the IVOF includes the uncertainty of future metal price into the evaluation process and assesses the risk and potential of the base-case design project indicators. To achieve this, the future metal price uncertainty is first quantified over time via a binomial lattice (see Figure 76). Observe that in this case the gold price is assumed to follow a Geometric Brownian Model (GBM) with an annual volatility of 30%, a discount rate of 5% and a risk-neutral probability of $p=0.507$. (This is based on a simple estimation of market conditions at the time of writing.) As observed in Figure 76, the gold price is expected to go up to Au$91.30/gr. and down to Au$8.28/gr. during the last production period (4th year).

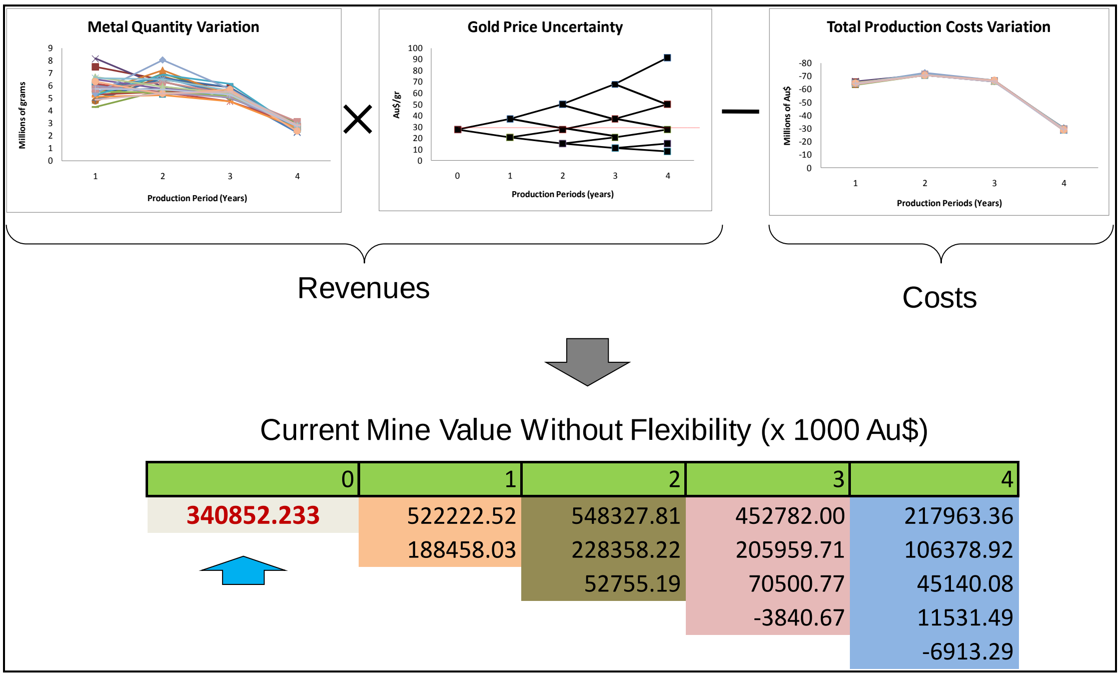
Once the future metal price uncertainty is quantified, a lattice of the cash flows generated at each production period is built considering both the "Jensen's Inequality" (see Figure 1 and footnote 3) and the existing quantified metal grade and metal price uncertainties. It is important to observe that at this stage the IVOF makes use of the fact that the metal grade and metal price uncertainties are independent and uses the metal price lattice (see Figure 76) as the base for the cash flow generation (see Figure 77). Observe that each node of the cash flow lattice contains the expected cash flow resulting from considering the uncertainties of metal quantity, cost uncertainties and the corresponding metal price. Figure 77 indicates that the expected value of the base-case mine design (in the face of metal grade and metal price uncertainty) is around Au$ 340.85M, which is significantly higher than the expected mine value obtained during Stages 1 and 2. The figure also suggests that the mine project could generate negative cash flows during productions periods 3 and 4 if gold prices fall to Au$11.8/gr. and Au$8.28/gr., respectively. *Can we take advantage of this information?*



r	5%	Up	1.349858808
Sigma	0.3	Dw	0.740818221
DT	1	p	0.507653818
S0	27.5	(1-p)	0.492346182

27.50	37.12	50.11	67.64	91.30
	20.37	27.50	37.12	50.11
		15.09	20.37	27.50
			11.18	15.09
				8.28

Figure 76. Quantification of future gold prices via a binomial lattice. In this case the price model used to describe the price behaviour is the Geometric Brownian Model (GBM).


Metal Quantity Variation
Millions of grams
9
8
7
6
5
4
3
2
1
0
1
2
3
4
Production Period (Years)
×
Gold Price Uncertainty
Au$/gr
100
90
80
70
60
50
40
30
20
10
0
0
1
2
3
4
Production Periods (years)
−
Total Production Costs Variation
Millions of Au$
-80
-70
-60
-50
-40
-30
-20
-10
0
1
2
3
4
Production Periods (Years)
Revenues
Costs
Current Mine Value Without Flexibility (x 1000 Au$)

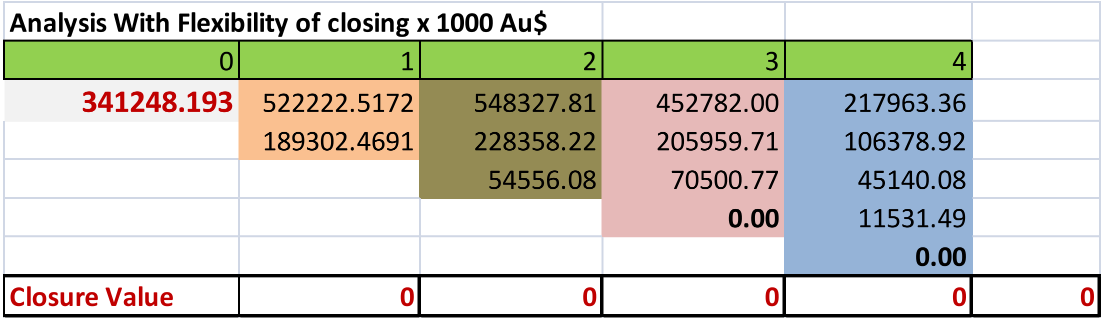
0	1	2	3	4
340852.233	522222.52	548327.81	452782.00	217963.36
	188458.03	228358.22	205959.71	106378.92
		52755.19	70500.77	45140.08
			-3840.67	11531.49
				-6913.29

Figure 77. Estimation of the Gold Mine Project value in the face of metal grade and metal price uncertainties using the IVOF technique.

7.5 Stage 4: Considering managerial flexibility: closing the mine in the presence of adverse technical and economic conditions.

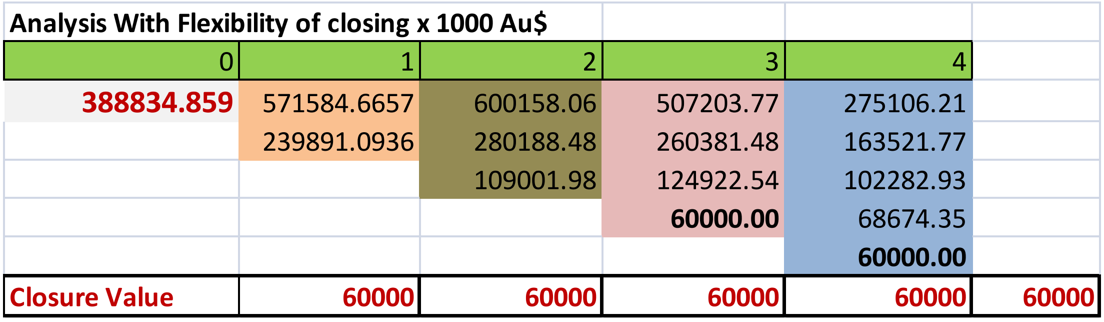
At Stages 2 and 3 it was seen how the value of the base-case design was estimated in the face of in situ metal grade and metal price uncertainties (see Figures 69-77). It was also observed that the mine is likely to generate negative cash flows in periods three and four if gold prices fall to Au$11.8/gr. and Au$8.28/gr., respectively. At this stage, the IVOF improves the value of the existing mine project by integrating managerial flexibility in the evaluation process; in this case the flexibility of closing the mine operation if future economic and technical conditions are adverse. Figures 78 and 79 show the final results of this process. As observed in the figures, the evaluation process considers two cases for closing the mine: i) the option to close the mine without a salvage value (Figure 78), and ii) the option to close the mine with a salvage value of Au$60M (Figure 79). Note that the analysis considers one year as the required time for closure and selling of mining assets.

The expected mine values are seen to be Au$341.25M and Au$388.834M, respectively (see the caption of Figures 78 and 79 for a detailed description). A quick comparison between Figures 77 and Figures 78 and 79 show that the value of the flexibility of closing the mine, if future technical and economic conditions are adverse, is either around Au$0.4M if the salvage value is zero, or around Au$47M considering a salvage value of Au$60M. This is good news for the owners and stakeholders of the mine. Furthermore, and most importantly for the project’s mine design, is that the ultimate pit limits will vary if the mine is closed either in period three or four. For example, if the mine is closed in period three it means that the ultimate pit limits will be the limits of the third production period (see Figure 72).



Analysis With Flexibility of closing x 1000 Au$					
0	1	2	3	4	
341248.193	522222.5172	548327.81	452782.00	217963.36	
	189302.4691	228358.22	205959.71	106378.92	
		54556.08	70500.77	45140.08	
			0.00	11531.49	
				0.00	
Closure Value	**0**	**0**	**0**	**0**	**0**

Figure 78. IVOF analysis of the Gold Mine Project base-case design with the option to close the mine - without salvage value. As observed in the figure, the (put) option to close the mine, with zero value, is exercised in periods 3 and 4, when gold prices fall to equal to or lower than Au$11.18/gr. and Au$8.28/gr., respectively. This is because the option value, zero, is greater than the negative cash flows that could be generated otherwise, that is, max (0, negative CF).



Analysis With Flexibility of closing x 1000 Au$					
0	1	2	3	4	
388834.859	571584.6657	600158.06	507203.77	275106.21	
	239891.0936	280188.48	260381.48	163521.77	
		109001.98	124922.54	102282.93	
			60000.00	68674.35	
				60000.00	
Closure Value	**60000**	**60000**	**60000**	**60000**	**60000**

Figure 79. IVOF analysis of the Gold Mine Project base-case design with the option to close the mine - with a salvage value of Au$60M. It is interesting to observe that in this case, the option to close the mine in the 4th production period and receive the Au$60M salvage value is exercised for gold prices equal to or lower than Au$15.09/gr. The figure also suggests that the option to close the mine and receive the salvage value in year three is exercised for gold prices equal to or lower than Au$11.18/gr.

7.6 Stage 5: Building a risk-robust open pit mine design based on uncertainty and risk analysis.

Stages 1 to 4 gave the steps for optimising and maximising the value of the base-case mine design of the Gold Mine Project, which was based on an estimated orebody model (see Stage1).

However, when quantifying the in situ metal grade uncertainty (see Stage 2), 25 simulations of the orebody were generated (see Figure 74). Then, it is logical to think that if each of these 25 orebody models are passed through Stages 1 to 4, there will be more than one mine design[57]. Although there are some techniques that claim they can automate the process of finding an optimum mine plan and design using stochastic programming techniques, this is a long and slow process which is also a black box; this is because the pit design and optimisation process is not visually analysed, and does not ensure an optimum mine design (more research needs to be done). In fact, the mine optimisation process is a discrete event where the optimisation process is achieved using not only mathematical procedures but also engineering and financial procedures as well as the experience of the mine planner or analyst, which are able to incorporate technical and operational strategies.

In this context, the IVOF uses a more practical approach where selected mine designs, such as the designs corresponding to the first ten percentiles of the mine value distribution are used for further analysis[58]. As observed in Figure 80, for the purposes of the Gold Mining Project and for the sake of practicality, the mine designs corresponding to the maximum, average, and minimum mine value selected from the mine value distribution obtained at Stage 2 are selected for further analysis, for passing them throughout Stages 1 to 4. The final results obtained from the analysis of each selected pit design are displayed for comparative purposes in Figure 81.

Figure 81 also indicates that for this specific project, the difference between physical mine designs is around Au$1M, which although not a significant increment results in a higher mine value. Some of the reasons for not obtaining a significant difference between mine values could be attributed to: i) the layout of the cutbacks and production scheduling generated for each of the selected mine designs; ii) the size of the orebody, which is a small disseminated deposit, and its proximity to the surface; iii) the continuity in metal grade variation throughout the deposit.

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[57] Observe that the use of 25 orebody models does not necessarily mean the generation of 25 mine designs. The reason for this is that one pit design may be suitable for two or more orebody models; it will depend on the metal grade variation between them (see for example Martinez and Wolff, 2007 for a more detailed explanation).

[58] Observe that because the IVOF is a generic framework it can also use advanced stochastic procedures for designing the best mine design, but this topic still needs more research.

Furthermore, an analysis of the results displayed in Figure 81 indicates that for the gold mine project case the initial base-case mine design (designed on the kriging orebody model) is the mine design that is expected to generate the highest expected mine value, in the face of in situ metal grade variation, metal price uncertainty, and managerial flexibility. Consequently, it is selected as the mine design to be implemented in the Gold Mine project.

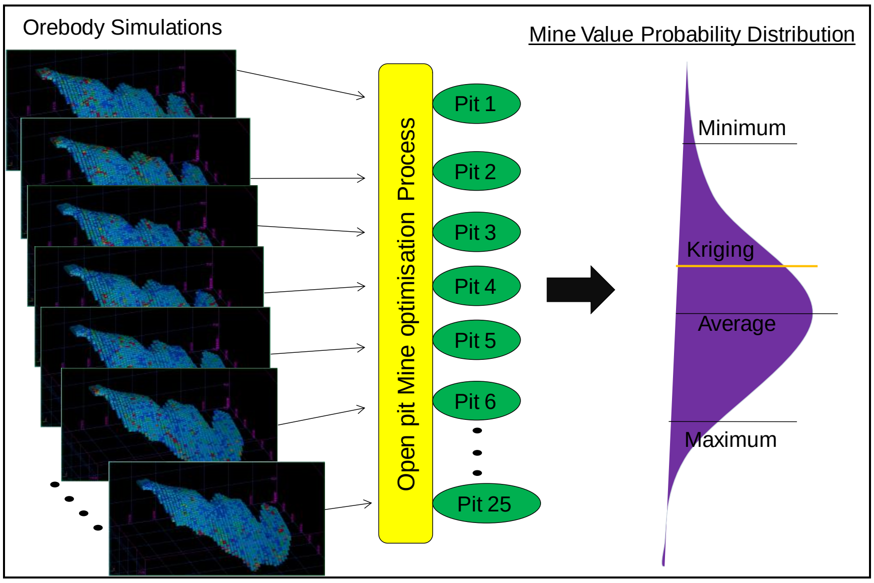
Orebody Simulations
Mine Value Probability Distribution
Open pit Mine optimisation Process
Pit 1
Pit 2
Pit 3
Pit 4
Pit 5
Pit 6
Pit 25
Minimum
Kriging
Average
Maximum

Figure 80. Chart summarising the IVOF's Stage 5: Selecting open pit mine design candidates based on the in situ metal grade uncertainty.

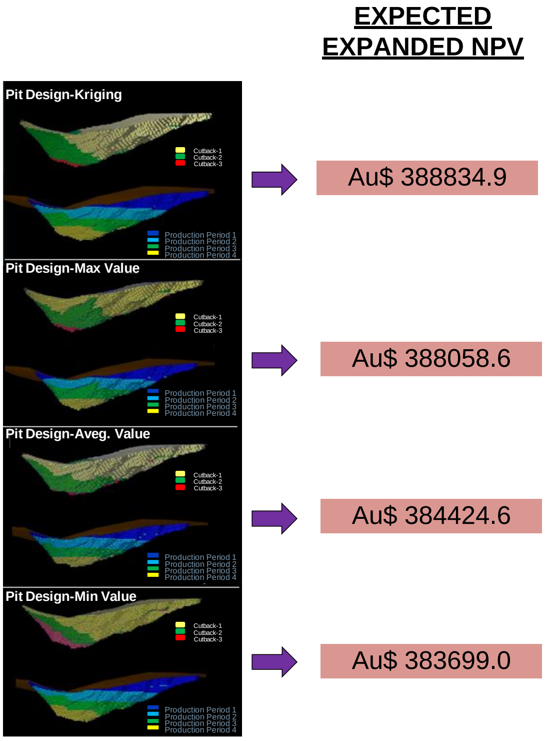
EXPECTED
EXPANDED NPV
Pit Design-Kriging
Cutback-1
Cutback-2
Cutback-3
Production Period 1
Production Period 2
Production Period 3
Production Period 4
Au$ 388834.9
Pit Design-Max Value
Cutback-1
Cutback-2
Cutback-3
Production Period 1
Production Period 2
Production Period 3
Production Period 4
Au$ 388058.6
Pit Design-Aveg. Value
Cutback-1
Cutback-2
Cutback-3
Production Period 1
Production Period 2
Production Period 3
Production Period 4
Au$ 384424.6
Pit Design-Min Value
Cutback-1
Cutback-2
Cutback-3
Production Period 1
Production Period 2
Production Period 3
Production Period 4
Au$ 383699.0

Figure 81. Chart summarising the IVOF's Stage 5: Selecting the best open pit mine design based on in situ metal grade and metal price uncertainties. The results suggest that the initial base-case mine design generates the highest expected mine value in the face of uncertainty.

7.7 General comments and conclusions

Throughout this chapter we have discussed the importance of including uncertainty and flexibilities in a mine project evaluation. As seen in Figures 77-79, the value of a mining project can significantly increase due to managing uncertainty and risk. It was seen that the inclusion of both gold grade and gold price uncertainties increased the gold mine value (Figures 75-77) when compared with the base-case scenario using a traditional DCF approach (Figure 73). The reason for this is that traditional DCF just accounts for risk through the discount risk without considering the upside potential, which is accounted for by the IVOF process. Furthermore, it was also seen that the flexibility of closing the mine project in the face of future adverse conditions added additional value to the project (Figures 77-78), regardless of future economic conditions (i.e., under high/low gold prices). Normally the decision to close a mine project is considered to happen at the end of the production life of the mine which does not account for operational or economic conditions. However, the proposed

IVOF process considers the option to close the mine as a Bermudan put option which is exercised only if the value from closing the mine project is greater than producing and closing. Observe that an accurate real options analysis would include all mine project sources of uncertainty as well as other types of managerial flexibility such as delaying mine production, among others. However this will turn the evaluation process into a more complex analysis (multi-dimensional risk analysis).

Chapter 8

Conclusions and future directions

8.1 Introduction

This thesis has given the fundamentals of a novel mine evaluation framework, IVOF, that incorporates uncertainty and risk in the evaluation process. It has shown that rather than being an enemy, uncertainty and risk are important tools for making final investment decisions.

While many authors have considered techniques similar to those used in this thesis, the major contribution of this research is demonstrating the use of binomial lattices as complements of stochastic simulation to model price uncertainty. Our methodology is easy to implement, is easy to present decision-makers in the mining industry, and gives a measure of the upside potential and downside risk involved in a mining project. While our case study consider only one type of operational flexibility, our framework is nevertheless sufficiently general to admit other types.

Some of the significant contributions of the IVOF technology are:

- It integrates current cutting-edge mine valuation/optimisation technologies into the mine evaluation process.
- It plans and designs under uncertainty.
- It allows integration of the variability of the ultimate pit into the mine evaluation process.
- Although the IVOF is based on advanced financial and technical theories, it is a practical tool that can be implemented with the collaboration of a project evaluation team.
- It is a generic framework that can be adapted to any mine project, for example, metals, coal, and underground projects.

Some of the benefits that the IVOF technology delivers when implemented are:

- The financial analysis will result in a more robust NPV with an associated measure of confidence.
- The results will include technical and operational risks and opportunities.
- It allows the identification of data deficiency and high risk areas for each production period.
- The ability to assess the risk and financial benefits of key interventions such as a different equipment fleet, different plant design and steeper slope angles.
- The ability to identify and assess realistic and practical strategic operational and managerial decisions of mine closure, expansion or temporary stopping.
- It is a sophisticated tool that improves the capability for forecasting overall risk of Greenfield/Brownfield mine projects in which vast upfront irreversible investments are required.

One important benefit that the IVOF technology brings to the mine evaluation process is that it allows the mine planner to see how the ultimate pit limits can vary over time, depending on favourable or adverse conditions. Indeed, as observed in Figure 81, depending on when the mine is closed, in the face of adverse technical and economic conditions, the limits of the ultimate pit will vary between the third or four cutbacks. Observe that the mine planner still does not know whether the third or four cutback will be the ultimate pit, but they know that it will depend on future conditions, and they will be able to include the uncertainty of allocating the ultimate pit within the mine evaluation process. Traditional evaluation techniques are not able to include the ultimate pit allocation uncertainty within the mine evaluation process.

One of the main conclusions of this thesis is that final decision making based on a simple DCF analysis could be of a misleading nature. Consequently, it is recommended that a more detailed approach using the IVOF technology is necessary to make an accurate decision in the face of uncertainty.

As a final comment, the IVOF is a structure that can be used as a platform for performing mine project evaluations integrating any type of uncertainty. More research needs to be done to ascertain what types of operational and financial uncertainties can be incorporated into the IVOF and to see how this affects the final mine value. In addition, the IVOF technology can serve as platform for more detailed analysis of the production scheduling where the plan and design of the (tactical) short-term production scheduling are subject to uncertainty.

8.2 Future directions

It is important to observe that, even though the proposed IVOF covers all different operational and managerial options, the case-study presented in Chapter 7 does not cover all different operational/managerial options, such as the deferral of the mine investment and others, due to the lack of data available and the availability of suitable mining software that can be used to perform a complete application of the IVOF technology. In fact, to date there is no specific software able to perform a complete IVOF process. The reason for this is that the IVOF requires multidisciplinary algorithms that have to be integrated to be able to deal with metal price forecasting, orebody modelling, open pit mine design, cut-off grade optimisation and decision-making based on operational and managerial flexibility. Due to time constraints this thesis could not address it.

Consequently, to show how the IVOF can assist the mine analyst in evaluating an open pit mine project, different softwares used in mining, geostatistics, economics and finance and mathematics, would be required.

We acknowledge that our evaluation framework was not set against a theoretical model from economics, and not did we explicitly address classical economic assumptions. For instance, approaches to DCF and NPV are often in the context of complete markets and we tacitly assumed this for practical purposes. It would be of interest to test the robustness of our approach against departures of this assumption, such as the existence of arbitrage opportunities in metal markets.

The objective of this thesis was not to investigate probability distributions which comprise the components of describing uncertainty in pit designs. As discussed we used standard probability distributions in our simulations and case study. A worthwhile future study would be to assess 1) efficient estimation techniques for such distributions; and 2) the robustness of the IVOF technique to choice of distributions.

We have described uncertainty as it relates to ore grade and price, and subsequently brought this uncertainty into the IVOF model. In a classical financial portfolio value is computed based on the financial attributes of the portfolio components, or *risk factors*. If it were possible to apply portfolio theory to evaluate a mine project then we would have to identify the risk factors within that project which comprise the portfolio. This study would require serious economic insight and the topic is beyond the scope of this thesis.

We note that our simulations and case study produced point estimates of value. Since the IVOF technique provides a rational, objective method for capturing uncertainty it reduces to a (possibly very sophisticated) statistical exercise to produce posterior distributions of key project indicators. Further, we could retain distributional information at each of the nodes in the binomial lattice rather than taking the mean, as an example of such as statistical extension.

The research in this thesis has raised a number of questions of related statistical and econometric methodology. Investigations of these carried out over the course of the thesis are included in the appendices, and contain further novel contributions.

Finally, there could be an interesting connection with corporate finance. Financial institutions which bankroll mine projects must clearly have their own valuation and risk methodologies. Comparing them with our IVOF method could be a useful benchmarking exercise, in as far as determining what features of risk are captured and modelled appropriately by one or the other.

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Appendix A

Independence and Conditional Expectation

Two random variables X and Y are independent if the two events $\{X \le a\}$ and $\{Y \le b\}$ are independent, that is,

$$F(a,b) = P\{X \le a, Y \le b\} = P\{X \le a\} P\{Y \le b\} = F_X(a) F_Y(b).$$

Then, it is shown that $E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$.

If the random variables X and Y are not independent, then knowing something about the value taken by one of them can give us valuable information about the other one. This leads to the concept of conditional expectation. For discrete random variables we have:

$$E\{X \mid Y = y_j\} = \sum_i x_i P\{X = x_i \mid Y = y_j\} = \frac{\sum_i x_i P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}}.$$

Similar for continuous random variables:

$$E\{X \mid Y = y\} = \frac{\int x f(x,y)\,dx}{\int f(x,y)\,dx}.$$

One fundamental property is the following:

$$E\{X\} = E\left[E\{X \mid Y\}\right].$$

Appendix B

Financial options analysis - the basics

An option on a stock is a security or contract that gives the holder the right to buy or to sell one share of the stock on or before a particular date for a predetermined price. A call/put option gives the holder the right to buy/sell a share of stock. American financial (call/put) options are contracts that give its holder the right, but not the obligation, to (buy/sell) one unit of an underlying asset for a predetermined *strike price* K at any time before the option's *expiration date* T (Fouque, Papanicolaou and Sincar, 2000). If S_t is the price of the underlying asset at time $t \le T$, then the value of this contract at any time $t \le T$, that is its *payoff* ($\wp$), is

$$\wp_{call}^{A}\left(S_t\right) = \max\left\{S_t - K, 0\right\}; \quad \text{call option},$$

$$\wp_{put}^{A}\left(S_t\right) = \max\left\{K - S_t, 0\right\}; \quad \text{put option}.$$

If the option is exercised at period $t = \tau \le T$, then τ is called the stopping time of the contract. Observe that at the expiration date T a decision needs to be made about exercising the option or not. If the option is not exercised, the holder of the option contract loses the value of purchasing the American option, f^A, paid at the beginning, that is, at time $t = 0$.

In contrast with American options, European financial (call/put) options give its holder the right, but not the obligation, to (buy/sell) one unit of an underlying asset for a predetermined *strike price* K at the option's *expiration date* T . The payoff of an European option is:

$$\wp_{call}^{E}\left(S_T\right) = \max\left\{S_T - K, 0\right\}; \quad \text{call option},$$

$$\wp_{put}^{E}\left(S_T\right) = \max\left\{K - S_T, 0\right\}; \quad \text{put option}.$$

Consequently, it can be observed that the difference between American options and European options is that the former is a dynamic process in which at any period, before and including the expiration date, the holder of the option can make the decision of exercising or not exercising their option, while the latter is a static process in which the holder of a European option has the right of exercising only at the expiration date.

To understand how an option contract can be used as a hedging strategy to minimise future risks, assume that we want to buy M tonnes of copper for delivery in one year's time (in this case the expiration date T is one year), at a specific copper (strike) price of $\$K$/tonne. We have two ways of doing this transaction: 1) We could speculate about the future copper price (1 year in the future) and write a contract for the total value, that is, $\$MK$, and see at the end of the year if we made a good purchase, that is, the copper price $S_T > K$ making the profit of $\$M\left(S_T - K\right)$; or a bad buy, that is, the copper price $S_T < K$ generating losses of $\$M\left(S_T - K\right) < 0$; and 2) We could purchase an American/European call option, paying $\$f^A/\$f^E$, to have the right of exercising the contract at any time during the year (if an American option) or at the end of the year (if a European option), respectively, making profits equivalent to $\$M\left(S_{t\le T} - K\right)$ (if an American option) or $\$M\left(S_T - K\right)$ (if a European option), or not exercising the option and losing, in this case, just the value of the option, that is, either $\$f^A$ or $\$f^E$.

The problem consists in finding out the fair value, $\$f^A/\$f^E$, of the option contract at the present time, that is, $t = 0$.

Normally, the intuitive way of pricing an American option at time $t = 0$ (today) is to maximise the expected value of its discounted payoff ($\wp$) over all the stopping times $0 \le \tau \le T$:

$$f^A\left(t = 0, S_{t=\tau}\right) = \sup_{\tau \le T}\left(E^*\left\{\wp\left(S_\tau\right)\left(1+r\right)^{-\tau}\right\}\right),$$

where r is the risk-free interest rate, normally issued by government bonds, and $E^*\{\bullet\}$ is the expected value function under risk neutral behaviour. In the case of a European option its price is defined as:

$$f^{E}\left(t=0,S_{t=T}\right)=E^{*}\left\{\wp\left(S_{T}\right)\left(1+r\right)^{-T}\right\}.$$

Observe in the previous equations that to ensure a fair market, that is, free of arbitrage opportunities, the option pricing process is performed under the risk neutral probability world. That is, the probability law governing the return of the underlying security, in our previous example the price of copper, S_t, has an expected return equal to the risk-free rate (Duan, 2002). In this way, both the buyer and the seller dealing in the market are not able to take advantage of each other, such as the buyer purchasing at a low price and selling at a higher price than the purchased price thereby making future revenues free of risk, and vice versa.

It is also clear that the option price depends mainly on the behaviour of the underlying security price, S_t. Other parameters affecting the value of an option are the expiration time T and the exercise price K. One important characteristic of the option value is that it can never be negative, that is, an option is a limited liability. For more properties about the value of an option, the reader is directed to books related to option pricing theories such as the one written by Rubinstein (Rubinstein, 1999), Hull (Hull, 1989), and Bookstaber (Bookstaber, 1987).

Appendix C

Selected open pit valuation techniques

C.1 The CA/BDH/Smith real options model

The CA/BDH/Smith RO model is a practical technique that uses Binomial lattices for solving the RO problem. One characteristic of this technique is that it views the mining project under evaluation as an American option that pays dividends at each production period. The steps of the project valuation process are as follows.

Step 1. The expected present value of the project at time zero, $E\left\{V_0\right\}$, is calculated using the RO method without flexibility (risk-neutral model) in which the certainty equivalent cash flows are generated using expected metal productions, forward-looking metal prices, and expected costs, and without considering any managerial flexibility. This step requires the use of the risk-free rate of return. The cash flows are then discounted at this risk-free discount rate, r, to obtain the expected present value of the project, $E\left(V_t\right)$, at each period $t = 0,1,..,T$ as:

$$E\left(V_t\right) = \sum_{i=t}^{T} \frac{CF_i^{CE}}{\left(1+r\right)^{i-t}} .$$

In this case, the present value of the project without options is taken as its current market price, as if the project were a traded asset. Assuming that markets are efficient, purchasing the project at this price guarantees a zero NPV, and the expected return of the project will be exactly the same as its risk-adjusted discount rate.

Step 2. The standard deviation of the returns, or volatility of the project, is estimated from a Monte Carlo simulation (2003) of the project's return. In this context, if V_1 and V_0 are the present value of the project at period one and zero, respectively, the project return, z, can be defined as:

$$z = \ln\left(\frac{V_1}{E\left\{V_0\right\}}\right).$$

To obtain the distribution of the project's return, z, several simulations of the project value at the end of the first period, v_1, are obtained by providing a new set of future cash flows cf_i ($i = 1,..,n$) and:

$$v_1 = \sum_{i=1}^{n} \frac{cf_i}{\left(1+r\right)^{i-1}} .$$

Observe that the present value of the project at period zero does not change. Once the distribution of the project's return, z, has been estimated, the standard deviation of this distribution, s^z, is used to define the annualised project's volatility, σ^V, as $\sigma^V = \frac{s^V}{\sqrt{\Delta t}}$, where Δt is the length of the period in years used in the cash flow generation.

Step 3. With the project volatility determined as previously indicated, and given the initial expected project value at time zero, $E\{V_0\}$, a binomial lattice can be constructed as follows. Define the cash flow payout rate, $\delta_t = E\{CF_t\} / E\{V_t\}$, to calculate the cash flows that are paid out at the end of each time period as a function of the project value. It is assumed here that the cash flows will vary over time, but that they will remain a constant fraction of the residual value of the project in each time period. The advantage of the previous assumption, that is, that the cash flows over time remain a constant fraction of the residual project value, is that the resulting binomial lattice is recombining (see Appendix A). The result of the previous process is that the generated cash flows, at period t and node (state) j[59] will therefore be a function of the project value and the stochastic process that drives the binomial model. To obtain the cash flows, the binomial lattice of pre-cash flow payout values is built. These values are calculated according to the following equations, where $u = e^{\sigma_V \sqrt{t}}$ and $d = u^{-1}$ are the factors that generate an up or down movement of the project value:

$$V_t^{up} = V_{t-1}\left(1-\delta_{t-1}\right)u$$
$$V_t^{down} = V_{t-1}\left(1-\delta_{t-1}\right)d.$$

The logic of this relationship should be transparent. V_{t-1} is the value of the project in the previous state, and $CF_{t-1} = V_{t-1}\delta_{t-1}$ is the cash flow paid out at the end of the period, which reduces the project value in the subsequent states. Since there is no cash flow in the initial period, that is, at $t=0$, $\delta_0 = 0$, then for $t=1$, $V_1 = V_0 u$. For all subsequent periods, the cash flow payout rate is assumed to be constant across states in each period but variable in time, so the cash flows in each period are a fixed proportion of the value of the project in that period and state. That is:

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[59] In this case j represents the upper branch of the binomial lattice at period t, while $j+1$, $j+2$, and so on, represent the immediate lower branch in the lattice.

$$\delta_t = \frac{E\{CF_t\}}{E\{V_t\}} = \frac{CF_t^j}{V_t^j}, \quad \forall j.$$

Step 4. Once the binomial lattice is built, the evaluation analysis follows a rollback analysis using dynamic programming to determine the optimal strategies at each production period. Since the binomial lattice that uses the risk neutral probabilities is used in this approach, the risk-free rate is used to arrive at the present value of the project. When there are no options, the value at each production period is given by:

$$V_t^j = CF_t^j + \frac{1}{(1+r)}\left\{pV_{t+1}^j + (1-p)V_{t+1}^{j+1}\right\},$$

so the present value of the project at period t in state j is the cash flow received in that period plus the discounted expected value in the next period. To incorporate options, we replace Equation 2.21 with an expanded version that reflects the options available in a given period as:

$$V_t^j = max\left\{CF_t^j + \frac{1}{(1+r)}\left[pV_{t+1}^j + (1-p)V_{t+1}^{j+1}\right], option1, option2, .., 0\right\}.$$

Although the CA/BDH/Smith technique is seen to be suitable for solving real options problems, it has the limitation of approximating the value of a mining project with a GBM process[60], which may mislead the real stochastic process of the value of a mine project. One point that may support rejection of the assumption that the value of a mining project follows a GBM is that it implies that the mine's value never becomes negative, which is not totally correct: the value of a mine can be negative if economic and technical conditions are unfavourable. For this reason many authors, such as Longstaff and Schwartz (2001), Smith (2005), Tsekrekos et al. and Sabour and Poulin , among others, have suggested the use of Monte Carlo techniques for solving American-type options, such as the Least-Square Monte Carlo method (henceforth LSMC). For the sake of completeness, the LSMC technique is briefly described next.

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[60] BDH also mentions the possibility of using, for example, arithmetic Brownian motion or mean-reverting processes.

C.2 The Longstaff-Schwartz Least Square Monte Carlo

Similar to the CA/BDH/Smith, the LSMC method consists of three basic steps.

1. Simulate N trajectories for all the relevant stochastic state variables that determine the price of the real option (Bermudan type) up to and including the option maturity[61].

2. Work backwards from the option maturity and generate the option payoff matrix. The option payoff matrix is a matrix detailing the time (if any) at which the option is optimally exercised along each path. To accomplish this task, that is, the estimation of the payoff matrix, least-squares regression techniques are applied to the cross-sectional data of in-the-money paths. This is done to approximate the early exercise boundary at each point in time. Thus, letting $g(x) = E\{y \mid x\}$ where y is the payoff from continuation and x represents the current state, an approximation of the conditional expectation of continuation can be used to determine the optimal exercise strategy (see for example Longstaff and Schwartz (2001) for more details). Note that the situation when the option's strike price is below (call option) or above (put option) the current market price is called an in-the-money state.

3. Finally, the option price is determined by discounting and averaging the relevant option payoffs across all matrix entries.

To illustrate how the LSMC model works, consider pricing an American put option as shown in Figure 7 (after Stentoft, 2004). In the figure, the price, P, of the American put option can be written as

$$P \equiv P\left(S(0), T\right) = \max_{\tau \le T} E\left\{e^{-r\tau} G\left(X, S(\tau), \tau\right)\right\} \tag{6.13}$$

---

[61] Observe that if there are more than two variables, the simulation of the variables needs to be done in conjunction, that is,, as a bivariate process.

where $S(t)$ is the stock price at time $t=1,2$, $S(0)$ is the spot price , r is the risk free rate of return, X is the strike price (which in this example is the same as the initial spot price), T is the expiration time (in this case it is equal to two periods), and G is the payoff function, which is a function of the strike price, stock price and time.

For demonstration purposes, in this example only six virtual potential simulated trajectories of the underlying asset (stock price) are generated, that is, $\left(\left\{S_1(t_i)\right\},\left\{S_2(t_i)\right\},..,\left\{S_6(t_i)\right\}\right)_{i=0,1,2}$. Observe that in the figure the initial price is the same for all path price, that is, $S_j(t_0)=0, \quad j=1,...,6$.

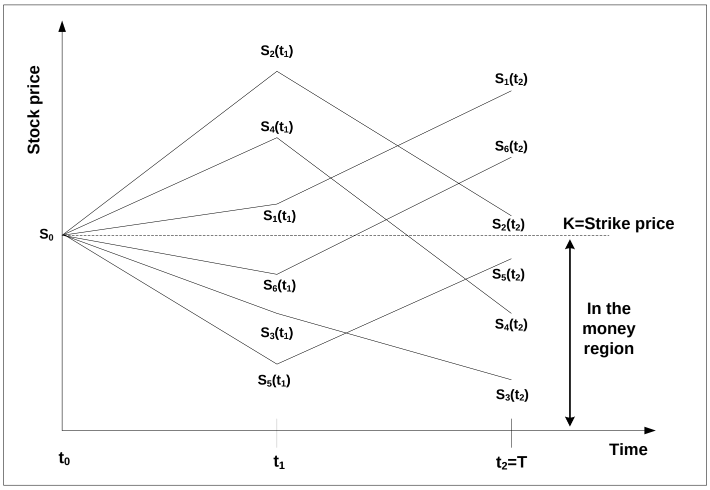
Stock price
S₀
S₂(t₁)
S₄(t₁)
S₁(t₁)
S₆(t₁)
S₃(t₁)
S₅(t₁)
S₁(t₂)
S₆(t₂)
S₂(t₂)
S₅(t₂)
S₄(t₂)
S₃(t₂)
K=Strike price
In the money region
t₀
t₁
t₂=T
Time

Figure 82. Diagram showing six potential simulated stock price paths. Note that at different stages or periods different prices are at the in-the-money region (for a put option). For example, at time t_1, prices S_6, S_3, and S_5 are at the in-the-money region.

Based on the previous example, the payoff function, G, can be written as:

$$G\left(X=S(0),S(\tau),\tau\right)=\max\left[X-S(\tau),0\right],$$

where $\tau \in [0, T]$. Observe in Equation 2.24 that the maximisation is over stopping times $0 \le \tau \le T$ adapted to the filtration generated by the relevant stock price $S(t)$, that is, the stock price is used to estimate the value of the payoff function.

The problem with the model given in Equation 2.24 is that, at any possible exercise time, the holder of the American option should compare the payoff from immediate exercise to the expected payoff from keeping the option alive. Then, the optimal decision is to exercise if the value of immediate exercise is positive and larger than the expected payoff from continuation.

To select a suitable exercise time, the process begins the analysis from the last period $t_2 = T$, where it is always optimal to exercise the option since it is worth nothing at any later time. Thus the optimal exercise strategy along each path, $j = 1, 2, \cdots, 6$, is $\tau_j = T$, and the value from following this strategy is known since this is just $max\{X - S_j(T), 0\}, \quad j = 1, 2, \cdots, 6$. At this stage, that is, at time $t_2 = T$, the option price can be obtained using:

$$P_{LSMC} = \frac{1}{j} \sum_{i=1}^{j} e^{-rT} \max\{X - S_i(T), 0\}.$$

Observe that the previous equation represents the value of the option as if it were a European option.

To calculate the optimal early exercise at period t_1, it is necessary to approximate the conditional expectation of continuing to hold the option until $t_2 = T$. To achieve this, the regression

$$\min_{(a_k)_0^K} \sum_{i=1}^{j} \left(a_0 \phi_0(x_i) + a_1 \phi_1(x_i) + \cdots + a_K \phi_K(x_i) - y_i \right)^2,$$

in which the explanatory variables $\phi_k(x_i)$ are known transformations of $S_j(t_1), \quad j = 1, 2, .., 6$, and the y s are the present value of holding the option, that is, $y_i = e^{-r(t_2 - t_1)} \max\{X - S_i(T), 0\}$.

Once the estimated values of the parameters, $\left(\hat{a}_k\right)_0^K$, are found, the values of the conditional expectation

$$\hat{g}_j\left(x\right)=E\left\{y_j\mid x_j\right\}=\sum_{k=0}^{K}\hat{a}_k\,\phi_k\left(x_j\right),$$

are estimated for each path $j=1,2,..,6$. If the value of exercising is higher than the conditional expectation, at each selected path, then $\tau_m=t_1$ is set, otherwise $\tau_m=T$.

If there is more than one exercise date prior to expiration this procedure is repeated recursively backwards through the simulation until time $t_0=0$. Then, using the approximation of the optimal early exercise strategy determined along each path $j=1,2,..,6$, the estimate of the price of an American put option can be calculated as

$$P_{LSMC}=\frac{1}{j}\sum_{i=1}^{j}\left[e^{-r\tau_i}max\left\{X-S_i\left(\tau_i\right),0\right\}\right].$$

In general terms, the LSMC framework can be adapted to deal with the problem of valuing an open pit mining project.

C.3 The Upside/Downside potential mine optimisation

The Upside/Downside potential technique developed by Martinez (2003) is an open pit evaluation framework that accounts for both the orebody uncertainty and the optimisation of the design and planning of a mining project. One key factor of this technique is that it evaluates an open pit mine project based on the assessment of the sensitivity of the overall pit economics, long-term mine planning and production scheduling to grade uncertainty. This is achieved by using conditional simulation and Monte Carlo simulation techniques. Another key factor of this technique is that it uses technical constraints, such as mill capacity and minimum cash flow generated at each production period, to assess the risk on key project indicators, and consequently to make strategic operational decisions such as the allocation of the cutbacks and ultimate pit of the mine. That is, this technique uses the uncertainty of the orebody not only to assess the risk of the project but to build an optimal open pit design that is robust to risk.

The process for performing open pit optimisation is as follows.

Step 1. Using the drill-hole data set generate several conditional simulations of the orebody. Denote each simulated model as $Sim-i$ ($i = 1, 2, ..., N \gg 50$). Observe that the simulated orebody models quantify the uncertainty of the orebody.

Step 2. Perform an analysis of resources by generating the cumulative distribution of ore tonnes for different cut-off grades.

Step 3. From the resource analysis, select $m < N$ simulated models which can preserve all the properties of the orebody. Normally, the simulated models representing the 10th, 20th, …, and 90th percentiles of the cumulative distribution of ore tonnes[62] are selected for further analysis (see, for example, Ballin for an explanation of these process).

Step 4. For each of the $m < N$ conditional simulated orebody models, generate a set of nested pit shells. Let $NPShell-i$ ($i = 1, 2, .., m$) denote the nested pit shell physical frame generated on each of the $m < N$ conditional simulated models. Observe that each pit shell will have technical information about the quantity of ore and waste, metal grade and metal quantity inside its limits.

Step 5. Overlay each of the $NPShell-i$ frames on each of the $(N-1)$ remaining simulated models and tabulate the technical information about ore tonnes and waste tonnes, metal quantity and metal grade. Observe that, as a result of this process, the technical information of each pit shell, inside each $NPShell-i$, will be characterised by a distribution of N values rather than a single estimated one.

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[62] In this case, the cut-off grade used for generating the cumulative distribution of ore tonnes is supposed to be known.

Step 6. Run a risk analysis on each $NPShell-i$ considering the mill capacity constraint, and select the pit shells that are more likely to achieve the specified mill capacity target as the cutbacks of the mine project[63].

Step 7. Run a cumulative discounted cash flow (DCF) risk analysis on each $NPShell-i$ and select the pit shell that is more likely to generate the highest cumulative DCF as the ultimate pit of the project. Observe that, as a result of this process, each $NPShell-i$ will have a pit design that includes cutbacks and ultimate pit limits. Let $PD-i$ denote the pit design generated on each $NPShell-i$.

Step 8. Run an upside/downside risk analysis on each $PD-i$ and compare the results.

Step 9. Select the pit design, $PD-i$, that has higher upside potential and minimum downside risk as the pit design for the project.

Step 10. Summarise results and end the process.

As mentioned in Section 2.3, one drawback of this model is that it is a time-consuming process. The reason for this is that a block model is normally constituted by thousands or millions of blocks (depending on the size of the deposit). Then, in order to estimate the pit limits, the pit design algorithm (in this case the Lerchs-Grossmann algorithm) needs to visit each node to assess its value while obeying technical constraints. Now, if this process is repeated using a large number of orebody models and more than one pit design, the process becomes cumbersome.

C.4 Lane’s cut-off grade optimisation

Lane’s theory of cut-off grade optimisation (1988) is an iterative model that determines an optimal cut-off grade that maximises the NPV of the project based on the production capacities of the mining, milling, and refining stages. In this case, Lane defines three economic cut-off grades: mine economic cut-off grade (g_m), treatment economic cut-off grade (g_h) and the refining economic cut-off grade (g_k), and three balanced cut-off grades: mine/treatment (g_{mh}) cut-off grade, treatment/refining (g_{hk}) cut-off grade and the mine/refining (g_{mk}) cut-off grade, from where to choose the optimal cut-off grade for the project.

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[63] Observe that in this case the probability used to make final decisions is determined by the mine analyst.

In this context, the present value, V, of a mining project at any specific period t can be expressed as:

$$V = V\left(t, Q, \Omega\right),$$

where Q is the available resource at time t, and Ω is the whole strategy that defines the setting of the variables affecting V. There is an optimum strategy Ω^* for which V is maximised. This maximum present value of the mine can be denoted as $V^*\left(t, Q\right)$ and is no longer a function of the strategy Ω, that is,

$$\max_{\Omega}\left\{V\left(t, Q, \Omega\right)\right\} = V^*\left(t, Q\right),$$

For a small decrement q in Q, consider ω as the strategy to extract that decrement, with Δt being the required time to mine q and c the cash flow per unit of resource. Both time Δt and cash flow c are functions of the fraction, q, of resource Q as well as of the adopted strategy ω for that decrement. Observe that after extracting the fraction of resource q, the remaining resource will be $Q - q$, the new time will be $t + \Delta t$, and the maximum value of the mine project will be $V = V\left(t + \Delta t, Q - q, \Omega^{'}\right)$, where $\Omega^{'}$ is the adopted strategy from that time onwards. The present value of the mine project at the beginning of period t can be expressed as the cash flow generated during the period $\tau = \left(t + \Delta t\right) - t$ plus the remaining present value, that is,

$$V\left(t, Q, \Omega\right) = qc + \left(\frac{1}{\left(1 + R^{WACC}\right)^{\Delta t = t}}\right) V\left(t + \Delta t, Q - q, \Omega^{'}\right).$$

If both sides of the previous equation are maximised with respect to the strategy ω, then,

$$V^*(t,Q) = \max_{\omega}\left\{ qc + \left(\frac{1}{\left(1+R^{WACC}\right)^{\Delta t=t}} \right) V^*(t+\Delta t, Q-q) \right\}.$$

Observe that the strategy ω is the set of all variables, such as metal prices and cut-off grade, that affect the generation of cash flow. In this context, the optimum strategy ω^* is the one that maximises the generated cash flow.

An optimal strategy can then be determined extremely efficiently by taking advantage of the sequential characteristic of Equation 6.38 which can be solved using dynamic programming as follows.

Step 1. Determine the optimum way of mining the last resource increment, that is, at period T . Equation 2.32 is then reduced to maximising the cash flow and the terminal value at the same strategy.

Step 2. Repeat ***Step 1*** at each period $T-j$ ($j=1,2,..,T$) in which the cash flow is maximised under an optimal strategy and the terminal project value is already known from the previous step.

Appendix D

Papers published from this dissertation

Besides the published paper listed above, two more unpublished papers were produced while developing this thesis. The papers are:

- The Block-Fourier Bootstrap: a new time series re-sampling technique
- The Bivariate Adaptive kernel-based nearest neighbor bootstrap

For the sake of completeness, the complete papers are given next.

The Block-Fourier Bootstrap: a new time series re-sampling technique

Luis Martínez and Rodney Wolff

Introduction

Time series bootstrapping techniques are used for re-sampling time series in order to generate surrogate or simulated series which resemble some features of the original. This procedure is performed in order to conduct inferences on the behaviour of the random variable under analysis using the data available, that is, the given time series.

Initially introduced by Efron (Efron, 1979), the bootstrap was developed for independent and identically distributed (IID) random variables, which, unfortunately, is not applicable to time series, since in most situations the assumption of IID is violated, that is, most of the models (and data) of current interest have complex forms of dependence.

In order to overcome the problem of dependence in time series, many variations of the original bootstrap technique have been developed (Jeong and Maddala, 1993). However, in the literature, most of the current time series bootstrap techniques have been developed for second order stationary time series, that is, where the mean and variance are constant over time. Consequently, when these current bootstrap techniques are applied to non-stationary time series, they generate surrogate series which do not preserve the inherent properties of the original, such as the mean, variance and higher moments, resulting in spurious outcomes regarding the behaviour of the random variable under study.

Since in real life it is common to deal with non-stationary time series, we propose a novel bootstrapping technique for time series. Basically, this technique deals with the problem of non-stationarity by means of adaptive analysis. That is the original non-stationary time series is segmented into blocks where the resulting piecewise time series are seen to behave approximately as stationary in respect to both their means and variances. Further, this technique uses a novel heuristic segmentation algorithm which is a variation of the technique developed by Bernaola-Galvan, et al. (2001) and Fukuda, et al. (2004).

The paper is organised as follows. Section 1 reviews some theory about time series analysis in the frequency domain. Section 2 introduces the algorithm used for segmenting non-stationary time series. Section 3 introduces the Block-Fourier time series bootstrap technique in detail. Section 4 presents some numerical experiments in which the proposed technique is tested. Finally Section 5 discusses conclusions, limitations and some topics for further research.

1. Discrete time series analysis: the frequency domain

Economic variables, such as flows and prices, are generally represented as series of T observations in time as a discrete time series. A discrete time series consists of a set of observations, on a specific variable Y_t, normally taken at equally spaced intervals over time. Then, a series with T observations, denoted by $y_1,\ldots y_T$, will contain information about the fluctuation or variation of the variable Y_t over time. The aim of time series analysis is to understand the process that generates the series, and thus to make inferences about its future behaviour. In fact, one of the main reasons for modelling a time series is to enable the forecast of future values (Harvey, 1988; Barnett, 2002). A time series can be represented in two ways: i) as a linear function of its own past and current values, and the past and current values of some noise process, which can be interpreted as the innovations to the system; and ii) as a non-linear function of its own past and current values and the sequences of innovations. Even though non-linear time series have a more complex structure than a linear series, their study is important since there is evidence that this type of series occurs frequently in real life. Note that existing linear statistical methods can only approximate non-linear series, and even then not entirely adequately.

A discrete time series $y_1,\ldots y_T$ is said to be weakly stationary of second order if it has a mean $E\{Y_t\}$ and a variance $Var\{Y_t\}$ independent of time. Conversely, a discrete time series is said to be non-stationary if its characteristics such as mean and variance change over time.

One alternative way of representing a time series is the Fourier model (or harmonic process) where the random variable Y_t is expressed as the sum of weighted cosine functions plus some stochastic function ε_t, where $\{\varepsilon_t\}$ are assumed to be a sequence of independent and identically distributed (i.i.d.) variables. This model is given by

$$Y_t = \sum_{j=0}^{\infty} R_j \cos\left(\omega_j t + \phi_j\right) + \varepsilon_t\,, \quad t = 1, 2\ldots, \tag{1.1}$$

where ω_j is the angular frequency, ϕ_j is the phase, and R_j is the amplitude at the jth frequency. Note that if $E\{R_j\} = 0$ and $E\{R_j R_s\} = 0$ for all $j \neq s$ then the series $\{Y_t\}$ follows a second-order stationary process, otherwise it may follow a non-stationary process. The autocovariance function of the series is defined as:

$$\gamma(k) = \operatorname{cov}(Y_t, Y_{t+k}) = E\left\{\left(Y_t - E\{Y_t\}\right)\left(Y_{t+k} - E\{Y_{t+k}\}\right)\right\}, \tag{1.2}$$

which can also be expressed as:

$$\gamma(k) = \int_{-\pi}^{\pi} e^{i\omega_j k} f\left(\omega_j\right) d\omega_j\,, \quad k = \ldots, 1, 0, 1, \ldots., \tag{1.3}$$

where $f\left(\omega_j\right)$ is the spectral representation (spectrum) of the process. The spectrum of a time series $\{Y_t\}$ is then defined as the complex Fourier transform of its autocovariance function (see Equation 1.4 below). This function breaks up the observed series Y_t into its component frequencies and their relative contribution to power in the series.

For the case of a non-stationary process, this spectrum becomes an evolutionary process, that is a spectral function which is time-dependent and admits a physical interpretation as local energy distribution (Priestley, 1965; Priestley, 1988). An evolutionary spectrum can be regarded, in practical terms, as being composed of *N* non-evolutionary spectra (Rao, 1970; Rao and Yu, 1986; Rao and Shapiro, 1970).

For the case of a stationary time series $\{Y_t\}$, the spectrum is defined as

$$f\left(\omega_j\right) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-i\omega k}\,, \quad -\pi \leq \omega_j \leq \pi\,. \tag{1.4}$$

As seen in Equation 1.4, the spectrum $f\left(\omega_j\right)$ is mathematically equivalent to the autocovariance function $\gamma(k)$. Consequently, the spectrum inherits symmetry from the autocovariance function so that $f\left(\omega_j\right)=f\left(-\omega_j\right)$ and it is also a positive definite function. However, one advantage of the spectrum is that it is subject to a single restriction, namely that it is a non-negative valued function on $(0,\pi)$. The modified periodogram $I\left(T,\omega_j\right)$, which is an estimator of the spectrum $f\left(\omega_j\right)$, is defined as

$$I\left(T,\omega_j\right)=\frac{1}{2\pi}\sum_{k=-(T-1)}^{(T-1)}\hat{\gamma}(k)\cos\left(k\omega_j\right) \tag{1.5}$$

where the *kth* lag uncentred sample autocovariance $\hat{\gamma}(k)$ is defined as

$$\hat{\gamma}(k)=\frac{1}{T}\sum_{t=1}^{T-|k|}\left(Y_t-\bar{Y}\right)\left(Y_{t+|k|}-\bar{Y}\right) \tag{1.6}$$

where $\bar{Y}=\frac{1}{T}\sum_{t=1}^{T}Y_t$. Basically, the periodogram $I\left(\omega_j\right)$ indicates the contribution at ω_j to the total variance in the series, and, consequently, the total variance of the stationary series is the area under the periodogram curve. The periodogram can also be generated using:

$$I\left(\omega_j\right)=\frac{2}{T}\left|\psi\left(\omega_j\right)\right|^2, \tag{1.7}$$

where $\psi\left(\omega_j\right)$ is the discrete Fourier transform (DFT) of the data, which is defined by

$$\psi\left(\omega_j\right)=\frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T}Y_t e^{-i\omega t}, \quad \left(-\pi\le\omega_j\le\pi\right). \tag{1.8}$$

Frequently, the discrete Fourier transform is computed at a set of equally distributed angular frequencies $\omega_j=2\pi j/T$.

From the previous analysis (Equations 1.5, 1.7 and 1.8) it is observed that Y_t can be recovered from the discrete Fourier transform as:

- if *T* is odd

$$Y_t = E\left(Y_t\right) + \sqrt{\frac{2\pi}{T}} \sum_{j=1}^{(T-1)/2} 2\sqrt{I\left(\omega_j\right)} \cos\left(t\omega_j + \theta_j\right), \tag{1.9}$$

- if *T* is even

$$Y_t = E\left(Y_t\right) + \sqrt{\frac{2\pi}{T}} \sum_{j=1}^{(T-2)/2} 2\sqrt{I\left(\omega_j\right)} \cos\left(t\omega_j + \theta_j\right) + \sqrt{\frac{2\pi}{T} I\left(T, \omega_{T/2}\right)} \cos\left(\pi t + \theta_{T/2}\right) \tag{1.10}$$

where θ_j is the phase.

2. Segmenting a non-stationary time series

In the previous section it was established that the analysis of a time series in the frequency domain gives some insight about the process generator of the random variable over time. Further, it was also established that for the case of a non-stationary time series, the spectrum becomes an evolutionary process in which the frequency distribution changes in the long-term. However, it has been shown that in a short-term interval these changes are very small (Priestley, 1965). Hence, the frequency distribution of a non-stationary time series can be seen, in practical terms, as being composed of *M* time segments in which the resulting piecewise time series is approximately stationary (Chen, Hardle and Jeong, 2005; Bernaola-Galvan et al., 2001; Fukuda, Stanley and Amaral, 2004; Adak, 1998).

In this section an innovative methodology for the segmentation of a non-stationary time series is presented and explained in detail. Note that we determine non-stationarity of a time series in forms of the variation of its variance over time assuming that its mean remains constant and equal to zero. One example of such type of time series, which is used commonly in economics and finance, is the so-called return series $\{R_t\}$ defined as:

$$R_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}},$$

where $\{Y_t\}$ is a given time series . Further, $E\{R_t\} \simeq 0$, and its volatility σ_t varies over time. This type of time series is often modelled as:

$$R_t = \sigma_t \varepsilon_t, \tag{2.1}$$

where ε_t are innovations that are assumed to IID but not necessarily normally distributed (Chen, Hardle and Jeong, 2005) with $E\{\varepsilon_t\} = 0$ and $Var\{\varepsilon_t\} = 1$.

In essence, this technique comprises two parts. The first part deals with the selection of the limits of the candidate segments or blocks, say B_i ($i = 1, 2, ..$), in which the variance of the piecewise time series, inside each block, is seen to be approximately constant and different from the variance of the piecewise series inside the contiguous blocks B_{i-1} and B_{i+1}. In this way it is ensured that no two contiguous blocks have the same mean and variance (see Section 2.1 for more details). The result of this process is the subdivision of the original series into P piecewise series (blocks) which are approximately stationary.

The second part is the verification that the piecewise series of two contiguous blocks, let's say B_i and B_{i+1}, selected by the heuristic segmentation algorithm, have different properties, such as variance and higher order statistics. To achieve this, two non-parametric tests, named the Kolmogorov-Smirnov (KS) and the Bartlett-Kolmogorov-Smirnov (BSK) two-sample statistics tests, are used to verify if both piecewise series have the same statistical properties, such as mean, variance, or in general if both piecewise series have equal marginal distributions.

Finding the candidate segments or blocks limits

To find the limits of the candidate segments (blocks), in a given non-stationary time series $\{R_t\}$ of length l, a sliding pointer moving from left to right along the squared series $\{Y_t\} = \{R_t^2\}$ is introduced. Note that here the squared series $\{Z_t\} = \{R_t^2\}$ is used instead of the series $\{R_t\}$. This is because, since the mean of $\{R_t\}$ is approximately zero, the variation in the variance (in this case, the cause of the non-stationary behaviour) can be better captured by the squared series $\{R_t^2\}$.

Then, at each position of the pointer, both the mean of the subset of the time series to the left of the pointer $\bar{Z}_{lefh}$ and to the right $\bar{Z}_{right}$ are computed and compared. Normally, when the mean of two independent Normal distributed populations need to be compared, the statistic d (see Equation 2.2) is used to test the difference of their means:

$$d = \left| \frac{\bar{Z}_{left} - \bar{Z}_{right}}{SD} \right|. \tag{2.2}$$

As a matter of fact, the statistic d, given in Equation 2.2, is the Student's *t*-test statistic (Mason, Lind and Marchal, 1999; Larsen and Marx, 2001) and SD is the pooled variance defined as

$$SD = \sqrt{\left[\frac{\left(N_{left} - 1\right) s_{left}^2 + \left(N_{right} - 1\right) s_{right}^2}{N_{left} + N_{right} - 2} \right] \left[\frac{1}{N_{left}} + \frac{1}{N_{right}} \right]}, \tag{2.3}$$

where N_{left}, N_{right}, s_{left}^2, and s_{right}^2 are the number of observations and the variances of the observations to the left and to the right of the pointer respectively. Observe that the number of observations N_{left} and N_{right} play an important role in determining the distribution of the statistic d; that is, for large values of N_{left} and N_{right} the statistic d is approximately normally distributed. If N_{left} and N_{right} are small, then d is approximately distributed as Student's *t* variable (Guttman and Wilks, 1965).

However, since in this research the time series under study are not assumed to be normally distributed nor independent, the statistic d might not be a good indicator for differentiating two adjacent means. Nevertheless, large values of d mean that the values of the means $\bar{Z}_{lefh}$ and $\bar{Z}_{right}$ are likely to be significantly different, making the point $d_{\max}$ a good candidate as a cut point.

Then, by moving the pointer along the time series, from left to right, the statistic d is calculated as a function of the position in the time series, and the point in time where the maximum value $d_{\max}$ occurs is marked and used to divide the time series into two segments, such as l_{left} and l_{right}. Next, repeat the previous process on the l_{left} series until a minimum length $l_{\min}^{1} \geq l_{o}$ is reached; l_{o} is the minimum length required in a selected segment. Then, $l_{\min}^{1}$ becomes the first segment candidate for segmenting the given non-stationary series and is marked and saved. Now, the entire process is repeated on the remaining $l - l_{\min}^{1}$ series until another minimum segment $l_{\min}^{2}$ is found and so on. The result of this process is a vector $v = \left[l_{\min}^{1}, \ldots, l_{\min}^{P} \right]$ containing the superior limits of the candidate segments or blocks to be used for segmenting the given non-stationary time series. Normally, the selection of the minimum length, that is, l_{o}, is based on the minimum number of observations required in a given segment (Politis, 2003), which will depend on the general behaviour of the non-stationary time series. Non-stationary time series with long intervals in which the variance is moderately constant will require a large l_{o}, while non-stationary time series with short intervals in which the variance is moderately constant will require a small l_{o}, so the variability in the variance can be captured appropriately. Note that there is not a specific technique to be used for selecting or choosing l_{o} automatically; in particular, Politis and Romano (Politis and White, 2004; Politis and Romano, 1995) and Politis and Romano (1995) suggest a rule of thumb in which l_{o} could be selected as twice the smallest integer, say $\hat{m}$, after which the correlogram appears not significantly different from zero. However, this rule of thumb cannot be generalised since this technique does not account for the memory of the series (either long or short memory) giving in some cases a large initial l_{o} when the series is seen to have a long memory, generating in this way false subdivisions of the original series. Consequently, one way of selecting l_{o} is based on a visual inspection of the original time series under study. In this research, it is assumed that the length of the original non-stationary time series is long enough to be subdivided in blocks containing at least 50 observations, which is considered as the initial length (l_o) for performing the split of the time series. The reason for considering the value l_o=50 is that we want the segments to be long enough for the resulting periodograms to be sufficiently informative, while at the same keeping the segments sufficiently short to ensure that the local behaviour is appropriately captured (Rosen, Stoffer and Wood, 2002). Note, however, that values of 60, 70 and 100 were also considered for splitting the time series giving similar results; that is the reason why we use the value of 50 as initial length. Also, it is important to remark that the Kolmogorov-Smirnov two-sample test is more flexible with respect to the value of the minimum length l_o; it can accept values greater than or equal to 35 observations (Gibbons and Chakraborti, 1992), which are necessary to use the critical values of the test statistic (see Equation 2.4), tabulated by Smirnov (1948).

Selecting blocks

Once the limits of the candidate segments have been obtained, a process of selection is run in order to select the blocks in which the piecewise time series are second order stationary.

The traditional test for measuring differences between means of two normal random samples is the so-called Student *t*-test; however, this test is not applicable to this case since the distributions of the values in the selected blocks are not necessarily normal. As mentioned previously, to overcome this problem, the proposed technique uses two non-parametric techniques to measure the significance of the difference between two contiguous blocks: i) the non-parametric Kolmogorov-Smirnov two-sample test statistic (SKS) (Gibbons and Chakraborti, 1992); and ii) a variation of the Kolmogorov-Smirnov test named the Bartlett Kolmogorov Smirnov (BKS) (Priestley, 1981; Bartlett, 1966; Diggle and Fisher, 1991). The proposed splitting technique can use either test in order to select the blocks that divide the non-stationary time series into segments where it behaves as a stationary one.

The Kolmogorov-Smirnov two sample test statistic (SKS)

This technique is used here as a goodness of fit criterion between the empirical distribution of two contiguous piecewise time series, or blocks, to identify differences in their means and variances. Note that the lengths of each piecewise series do not need to be the same. Then, if the empirical distributions of each piecewise time series are denoted as, $S^{m}_{left}(x)$ and $S^{r}_{rigth}(x)$ (of lengths equal to m and r respectively), they are assumed to be reasonable estimates of their respective population distributions $F_{left}(x)$ and $F_{rigth}(x)$ respectively. If the null hypothesis $H_0 : F_{left}(x) = F_{right}(x)$ (for all x) is true, then the populations are identical and $S^{m}_{left}(x)$ and $S^{r}_{rigth}(x)$ have equal distribution functions. Consequently, allowing for sampling variation, under H_0 there should be a reasonable agreement between the two-sample empirical distributions. Then, the two-sample test criterion, denoted by $D_{m,r}$, is the maximum absolute difference between the two empirical distributions

$$D_{m,n} = \max_{x} \left| S^{m}_{left}(x) - S^{r}_{rigth}(x) \right|. \tag{2.4}$$

Note that in this test only the magnitudes, and not the directions, of the deviations are considered. For a more detailed analysis of this test, the reader is referred to Gibbons and Chakraborti (Gibbons and Chakraborti, 1992) for instance.

The Bartlett-Kolmogorov-Smirnov (BKS)

This technique is a variation of the simple Kolmogorov-Smirnov two-sample test statistic which estimates the distance

$$D_p = \sup_{\omega_p} \left| U_x(\omega_p) - U_y(\omega_p) \right|, \tag{2.5}$$

between the normalised cumulative periodograms of two time series (in this case between two contiguous piecewise time series). The cumulative normalised periodogram of a time series is given by

$$U\left(\omega_p\right) = \frac{\sum_{q=1}^{p} I\left(\omega_q\right)}{\sum_{q=1}^{m} I\left(\omega_q\right)}, \tag{2.6}$$

where $m = l/2$, if l is even, or $m = \left(l+1\right)/2$ if l is odd, $I\left(\omega_p\right)$ is the periodogram at frequency ω_p ($p = 1, 2, \ldots, m$). The null hypothesis H_0 is that both periodograms $I_x\left(\omega_p\right)$ and $I_y\left(\omega_p\right)$ are the same, for $p = 1, 2, \ldots, m$. In practice it is not feasible to calculate the distribution of D_p, but it can be approximated very adequately by calculating $D_2, \ldots, D_s$ for some large number $s-1$ of frequencies ω_s, and calculating the significance probability of the observed D_p-value.

This task is not an easy process since the available data are just two consecutive piecewise time series, which do not necessarily have the same length (making the process of comparing spectra difficult), and in some cases are as small as the minimum length l_0. This is the reason why the minimum length needs to be selected carefully. There have been some attempts to calculate the distribution of D_p (Diggle and Fisher, 1991) for some specific time series models with specific lengths. However, it is not always true that these distributions will work when time series with different lengths and models to the ones specified in these attempts are analysed.

In order to overcome the previous problem, that is, finding the distribution of D_p, in this paper a bootstrap approach is used as follows.

Given two consecutive piecewise stationary time series, let's say $\left\{Y_t^{left}\right\}$ and $\left\{Y_t^{right}\right\}$ do: 1) split $\left\{Y_t^{left}\right\}$ in two sub-series, let's say $\left\{a\right\}$ and $\left\{b\right\}$; 2) estimate the cumulative spectra (see Equation 2.6) of $\left\{a\right\}$ and $\left\{b\right\}$, that is, $U_a\left(\omega_p\right)$ and $U_b\left(\omega_p\right)$, respectively. Because $\left\{a\right\}$ and $\left\{b\right\}$ come from the same stationary series, in this case $\left\{Y_t^{left}\right\}$, they should be the same; 3) Calculate the distance $d_{\max}$ between $U_a\left(\omega_p\right)$ and $U_b\left(\omega_p\right)$, respectively. This is done for different frequencies ω_p (for

$p = 1, 2, .., k$). Note that for a large k, it is necessary to taper both sub-series ($\{a\}$ and $\{b\}$); 4) Once the distance $d_{\max}$ has been obtained for different frequencies ω_p (for $p = 1, 2, .., k$), use bootstrap techniques to find the distribution of $d_{\max}$ and estimate critical values for different significance levels, e.g., 0.05, 0.01, and 0.1; 5) use the estimated critical values, generated in the previous step, to evaluate the significance of D_p between $\{Y_t^{left}\}$ and $\{Y_t^{right}\}$; that is with a certain level of significance, the probability that D_p will exceed $d_{\max}$ ($1 - P\left(D_p \leq d_{\max}\right)$) is used for the test.

Note that this process needs to be repeated at each point of comparison since the values of the distribution of $d_{\max}$ change for different lengths of the time series under analysis (multiple comparisons). Consequently, the process of estimating the distribution of D_p along the original time series is a dynamic process.

3. The Block-Fourier Bootstrap

The Block-Fourier bootstrap is a technique that re-samples a given non-stationary time series $\{R_t\}$ to generate surrogate series $\{R_t^*\}$. If this operation is repeated N times, then N surrogate series $\left(\{R_t^*\}_1, \{R_t^*\}_2, \ldots, \{R_t^*\}_N\right)$, each of them keeping the first and second moment of the original non-stationary time series $\{R_t\}$ are generated. In essence, the Block-Fourier bootstrap uses the algorithm defined in the previous section to segment the non-stationary time series into M blocks where the piecewise time series are approximately stationary. After that, it uses a bootstrap procedure called the Fourier bootstrap or phase scrambling bootstrap (Davison and Hinkley, 1998) in each of the M selected blocks in order to generate surrogate series of the piecewise time series inside the selected block. In this way a surrogate series of the original series is generated by connecting the M piecewise surrogate series. One limitation of this technique, however, is that the pseudo-data generated by concatenating re-sampled blocks of data will not preserve the dependence structure of the original data near block end-points (Kunsch and Carlstein, 1990).

The *Fourier bootstrap* or *phase randomisation* method is a technique that takes the discrete Fourier transform of a given time series $\{X_t\}$ (see equations 1.9 and 1.10) then randomising the phase θ_j at each frequency uniformly on $(0, 2\pi)$ and back-transforming (taking the inverse Fourier transform) renders a so-called surrogate series. In fact, this technique matches the periodogram of the series to a closer degree than any other technique (Barnett, 2002).

Note that since this technique does not change either the amplitude of the Fourier transform nor the expected value of the random variable, both the power spectrum (second order structure) and the mean (first order structure) of the original series are preserved in the surrogated series X_t^* within blocks. Furthermore, it has been shown by Nur, Wolff and Mengersen (Nur, Wolff and Mengersen, 2001) that the Fourier bootstrap, under certain conditions, also keeps the third order structure or bispectrum of the time series. These conditions are related to the type of time series process under study. For example, the authors shown that when the Fourier method is applied to non-linear and non-stationary processes, the higher order moments are preserved.

One important feature of the proposed technique is that it generates more information about the behaviour of the random variable at each point in time. In fact, the proposed technique allows us to recover the (empirical) distribution function $F_{R_t}^t(r) = P(R_t \le r)$ of the random variable R_t over time. Since the distribution function of a random variable carries the whole information about the behaviour of it, then any statistic of the random variable can be estimated, such as the estimation of the volatility at each point in time among others. In particular, the estimation of the distribution is of great importance when performing risk evaluation, such as value at risk (VaR) models to quantify portfolio risk, where it is required to estimate or predict the entire distribution of probabilities of the economic indicator under study. In other words, by applying the Block-Fourier bootstrap to a given non-stationary series it is possible to recover the evolutionary probability distribution function of the process, which is not necessarily constant over time (see Figures 92, 93, and 94).

More formally, the Block–Fourier bootstrap process consists of the following steps:

- Split the given non-stationary time series $\{R_t\}$ into small segments or blocks, let's say M, $\left(\{B_t\}_1, \{B_t\}_2, \ldots, \{B_t\}_M\right)$ where the piecewise time series $\{B_t\}$ are approximately stationary. This process is performed using the technique introduced in Section 2.

- Apply the Fourier bootstrap to each block $\left(\{B_t\}_1, \{B_t\}_2, \ldots, \{B_t\}_M\right)$ and generate a surrogate series for each of them $\left(\{B_t^*\}_1, \{B_t^*\}_2, \ldots, \{B_t^*\}_M\right)$.

- Connect the generated surrogate series of each of the blocks $\left\{\{B_t^*\}_1, \{B_t^*\}_2, \ldots, \{B_t^*\}_M\right\}$ and generate, in this way, a surrogate series $\{R_t^*\}_1$ of the original series $\{R_t\}$.

- Repeat the previous process N times and generate N surrogate series of the original time series, that is $\left(\{R_t^*\}_1, \{R_t^*\}_2, \ldots, \{R_t^*\}_N\right)$.

- End the process.

4. Application of the Block-Fourier bootstrap to simulated data

In this section, the Block-Fourier bootstrap performance is tested on three simulated non-stationary time series, that is $\{X_t\}$, $\{Y_t\}$, and $\{Z_t\}$. In this case, time series $\{X_t\}$ and $\{Y_t\}$ are simulated in such a way so that they present two jumps in their volatilities (see Equations 2.7 and 2.8 respectively), while time series $\{Z_t\}$ is generated in a way that presents a continuous change in its volatility (see Equation 2.9). As a reference, the time series $\{X_t\}$ was extracted from Chen et al. (2005) pp.13. $\{Y_t\}$ is just a variation of $\{X_t\}$, and $\{Z_t\}$ was extracted from Adak (1998).

Simulation 1: time series $\{X_t\} = \sigma_t^X \varepsilon_t$, where $\varepsilon_t \sim N(0,1)$ and

$$\sigma_t^X = \begin{Bmatrix} 0.01 & : & 1 \le t \le 400 \\ 0.05 & : & 400 < t \le 750 \\ 0.10 & : & 750 < t \le 1000 \end{Bmatrix}. \tag{2.7}$$

Simulation 2: time series $\{Y_t\} = \sigma_t^Y \varepsilon_t$, where $\varepsilon_t \sim N(0,1)$ and

$$\sigma_t^Y = \begin{Bmatrix} 0.01 & : & 1 \le t \le 400 \\ 0.05 & : & 400 < t \le 750 \\ 0.01 & : & 750 < t \le 1000 \end{Bmatrix} \tag{2.8}$$

Simulation 3: modulated time series $\{Z_t\}$, where $\varepsilon_t \sim N(0,100^2)$ with initial values for $Z_1^0 = 0, \quad Z_2^0 = 0$, and

$$\begin{aligned} &Z_{t,N} = \exp\left[\left(\frac{-25}{2}\right)\left(\frac{t}{N} - \frac{1}{3}\right)^2\right] Z_t^0 \\ &Z_t^0 = 0.8 Z_{t-1}^0 - 0.4 Z_{t-2}^0 + \varepsilon_t \\ &t = 3, 4, ..1000. \end{aligned}$$

Note that the volatility of $\{Z_t\}$ is time dependent and is defined as

$$Var(Z_t) = \left[\exp\left[\left(\frac{-25}{2}\right)\left(\frac{t}{N} - \frac{1}{3}\right)^2\right]\right]^2 Var(Z_t^0), \tag{2.9}$$

Note that Z_t^0 follows an AR(2) process and its variance is given by

$$Var(Z_t^0) = \left(\frac{1+0.4}{1-0.4}\right) \frac{Var(\varepsilon_t)}{\left[(1+0.4)^2 - 0.8^2\right]} = 17676.5. \tag{2.10}$$

Note that 1000 points each were generated for the time series, $\{X_t\}$, $\{Y_t\}$, and $\{Z_t\}$.

The time series plots for the simulated data, that is, $\{X_t\}$, $\{Y_t\}$, and $\{Z_t\}$, as well as their respective time-varying volatilities, that is, $\{\sigma_t^X\}$, $\{\sigma_t^Y\}$, and $\{\sigma_t^Z\}$,are displayed in Figures 83, 84, and 85 respectively. As observed from the figures, $\{X_t\}$

and $\{Y_t\}$ have constant volatility between points 0-400, 400- 750, and 750-1000; while $\{Z_t\}$ has a continuously varying volatility which increases and then decreases over time.

The objectives of this exercise are as follows.

- Apply the Block-Fourier bootstrap to each simulated time series, that is, $\{X_t\}$, $\{Y_t\}$, and $\{Z_t\}$, and generate 50 surrogate series of each of them. Note that there is not a specific reason for generating only 50 surrogate series. As a matter of fact, if more accuracy is required, then 200 surrogates will give good results (Efron and Tibshirani, 1993). In this paper, the value of 50 was adopted for the sake of simplicity.

- In order to analyse the sensitivity of the surrogate series, the volatility series for each simulated time series is then estimated and compared with their respective theoretical models. Observe that this step is now possible since, as a result of the previous step, there are 51 values (the original plus 50 surrogates) of each random variable, at each point in time. The volatility at each point in time is estimated by the sample cross sectional volatility, e.g.,

$$\hat{\sigma}_t = \sqrt{\frac{1}{51}\sum_{i=1}^{N+1} X_{t,i}^2} \; . \tag{2.11}$$

- Compare the estimated time varying volatility obtained from the surrogate data and the theoretical models.

Figures 86 and 87 show the results of applying the Block-Fourier bootstrap to the first time series $\{X_t\}$ using both the SKS and BKS techniques respectively. Figures 86-a (left) and Figure 87-a (left) show the original time series (in black) and 50 surrogate series (in light colour) while Figures 86-b (right) and 87-b (right) display the estimated adaptive volatility $\hat{\sigma}_{X,t}$ (in light colour) and the true volatility (in black). The minimum length when selecting the blocks was 50.

As observed in the figures, the surrogate series honour very well the changes in variance in the original time series, and consequently, changes in its volatility over time (this is shown in Figures 86-b and 87-b respectively). Note, however, that a comparison of Figures 86-b and 87-b suggest that the surrogate series generated by the SKS technique (Figure 86-a) performed better than those generated by the BKS technique (Figure 87-a) when estimating the true volatility of $\{X_t\}$. This is because, in this case, the BKS technique split the original time series with one extra block than the SKS technique, which means that the BKS is more sensitive to small changes in the spectrum, probably due to the noise rather than changes in the volatility, than the SKS technique. However, in general, terms both techniques performed well in capturing the variation of the variance and volatility of the original time series $\{X_t\}$. For this case, the SKS technique split the non-stationary time series $\{X_t\}$ at points $t = 348$ and 749, which were very close to the true values, that is $t_b = 400, 750$; on the other hand, the BKS technique split the series at points $t = 398, 749, 827$ (one more than the SKS technique). The same results were obtained in several simulations where different minimum lengths, such as 60, 70 and 80, were used.

Similar analyses were performed for the series $\{Y_t\}$ and $\{Z_t\}$. Again both techniques, that is, SKS and BKS, were used to split the non-stationary time series and were compared. These results are shown in Figures 88, 89 (for $\{Y_t\}$), and 90 and 91 (for $\{Z_t\}$) respectively. The results obtained from the analysis of time series $\{Y_t\}$ were similar to those obtained for $\{X_t\}$, that is both techniques (SKS and BKS) performed well when splitting the non-stationary time series into blocks where the piecewise series behaves as though stationary. Also it is observed that the SKS technique performs better when the estimated adaptive volatility is compared with the true model.

For the case of the modulated series $\{Z_t\}$, which has a continuous variation in its volatility, the results were different from the previous cases. In this case, as observed in Figures 90 and 91, both techniques, that is, the BKS and SKS, performed fairly well in detecting the continuous changes in volatility. Both techniques generated surrogate series which honour very well the changes in variance in the original time series (this is shown in Figures 90-a and 91-a respectively). However, it is clear, as seen in Figures 90-b and 91-b respectively, that neither technique is able to detect continuous changes in the volatility. This is not a surprise since the segmenting technique used in this paper is highly discrete. Consequently, since in this case the changes in volatility occur continuously and very slowly, which is different from the previous cases where the changes occurs abruptly in specific points on time, the differences between the sample distributions and the spectrums of the two piecewise time series also occurs so slowly that both the SKS and BKS are not able to detect these changes. One solution might be to smooth the estimated volatility in order to have more continuity in the results.

Further, it is observed in Figures 90-b and 91-b that the SKS technique performs better than the BKS when detecting changes in volatility. The reason for this is that the SKS technique detected more points in time where there were significant changes in the volatility than the BKS technique. Consequently, the BKS split the original time series with more blocks than the BKS technique. In actual fact, the SKS technique detected changes in volatility at epochs $t = 76, 148, 409, 537, 709, 831, 893$, while the BKS technique detected changes in volatility at epochs $t = 76, 222, 617, 831, 893$. Also, note that the SKS technique did performed better in detecting changes in volatility throughout the original series than the BKS technique, which was not able to detect changes in volatility at early points (see Figure 91-b).

From the previous result it is concluded that, in general, the SKS technique is more sensitive to changes in volatility than the BKS. It is not a surprise since the SKS technique compares distribution functions to see changes in volatility, while the BKS technique just compares spectrums. Then, the SKS technique is seen to be very useful when dealing with time series with both discrete and continuous changes in volatility, while the BKS technique performs well with series where the changes in the variance occur less frequently (discrete changes).

Another important result obtained from the application of the block-Fourier bootstrap to the given series is that it allows the recovery of the adaptive empirical distribution function over time of the random variable under study. In our example these results are shown in Figures 92, 93, and 94 where the adaptive probability distribution function of X_t, Y_t (obtained from the SKS technique surrogates) and Z_t (obtained using the BKS technique) for the points $t = 1, 100, 250, 600, 800, 1000$ (for the case of Z_t, the last point was allocated at $t = 950$) are shown. As observed in the figures, the distribution functions do not remain constant over time, but change, which means that the entire process is dynamic rather than static.

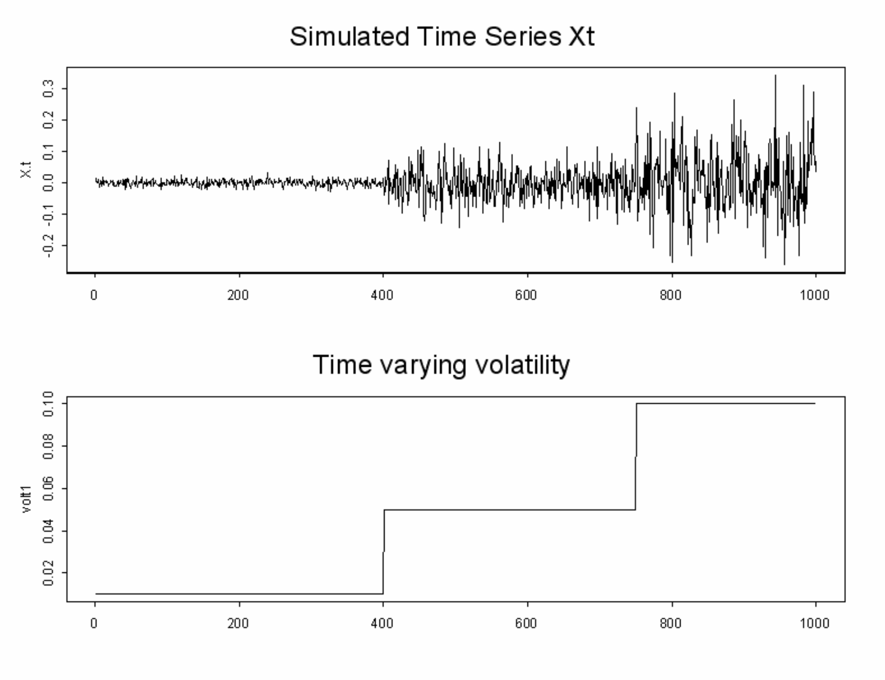

Figure 83. Original time series $\{X_t\}$ and its respective time-dependent volatility.

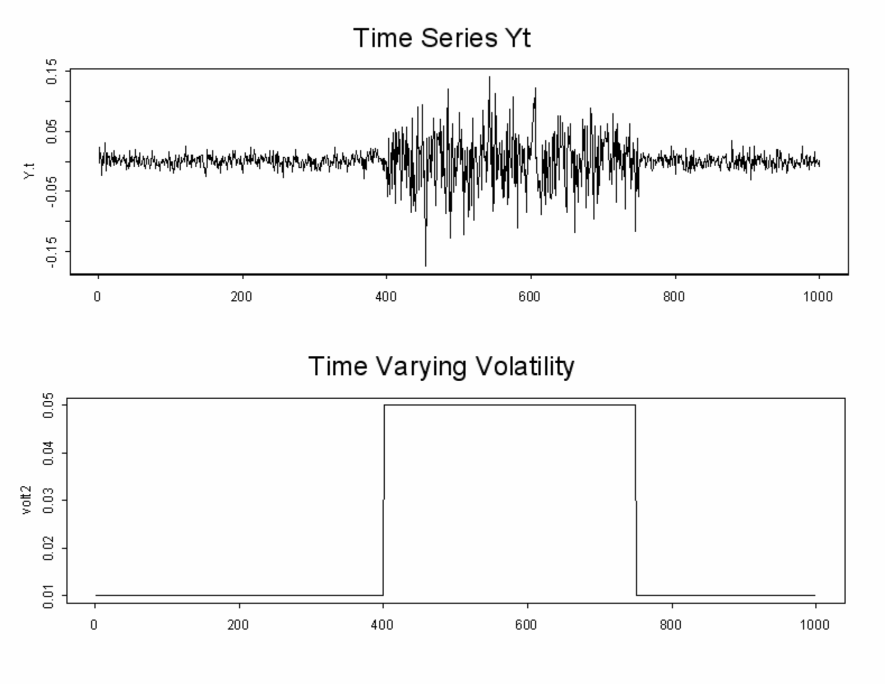

Figure 84. Original time series $\{Y_t\}$ *and its respective time-dependent volatility.*

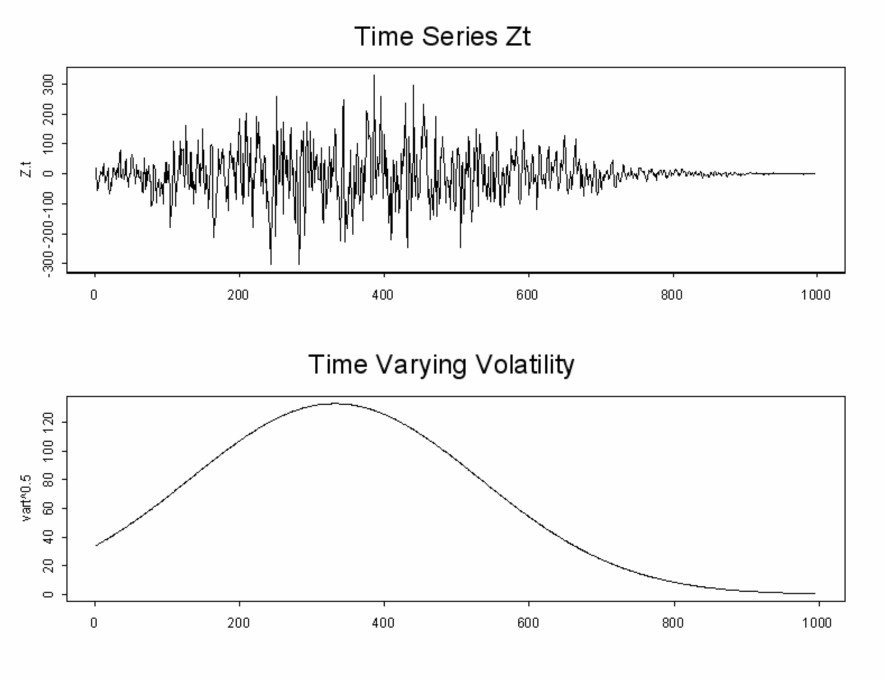

Figure 85. Original time series $\{Z_t\}$ *and its respective time-dependent volatility.*

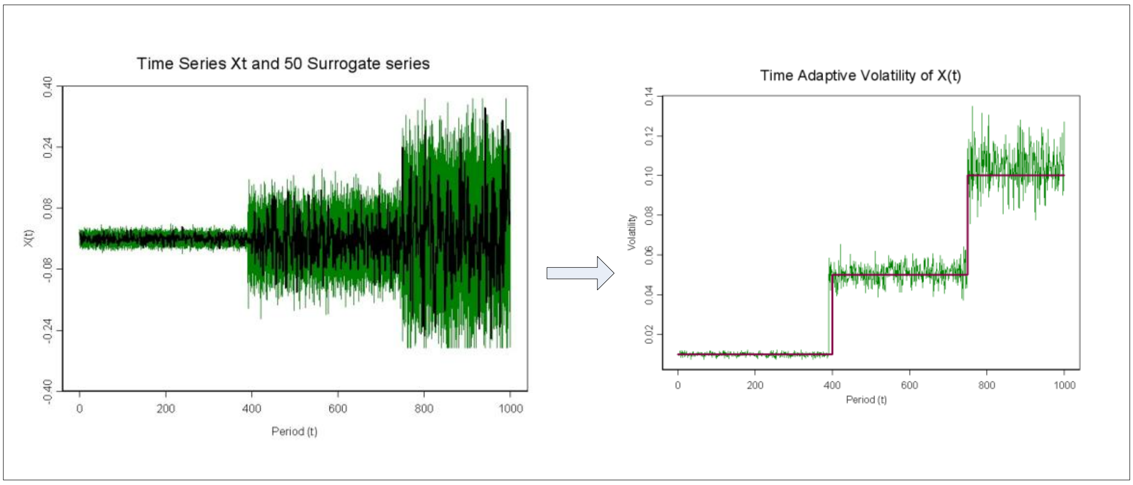

Figure 86. Analysis of surrogate series and adaptive volatility of time series $\{X_t\}$ *using the SKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).*

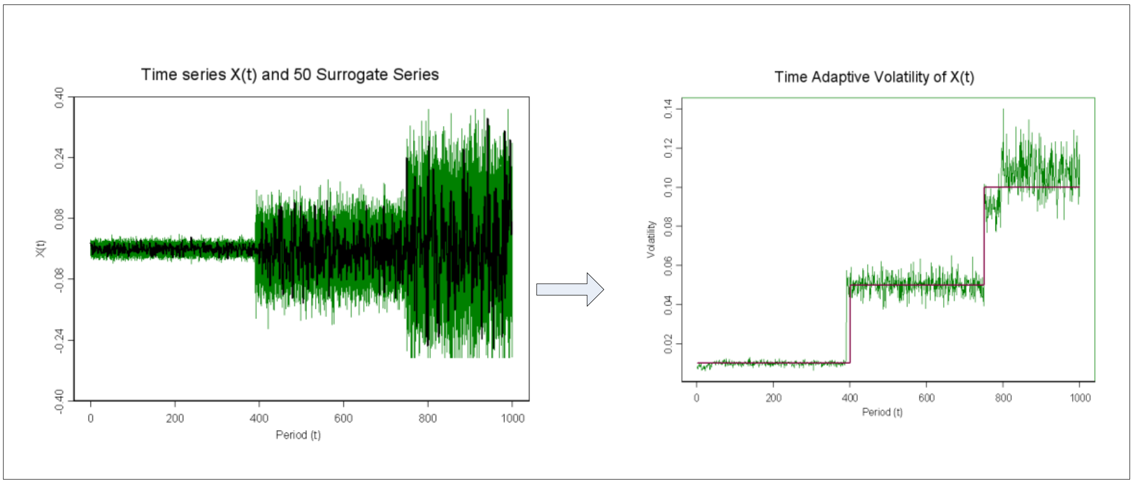

Figure 87. Analysis of surrogate series and adaptive volatility of time series $\{X_t\}$ *using the BKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).*

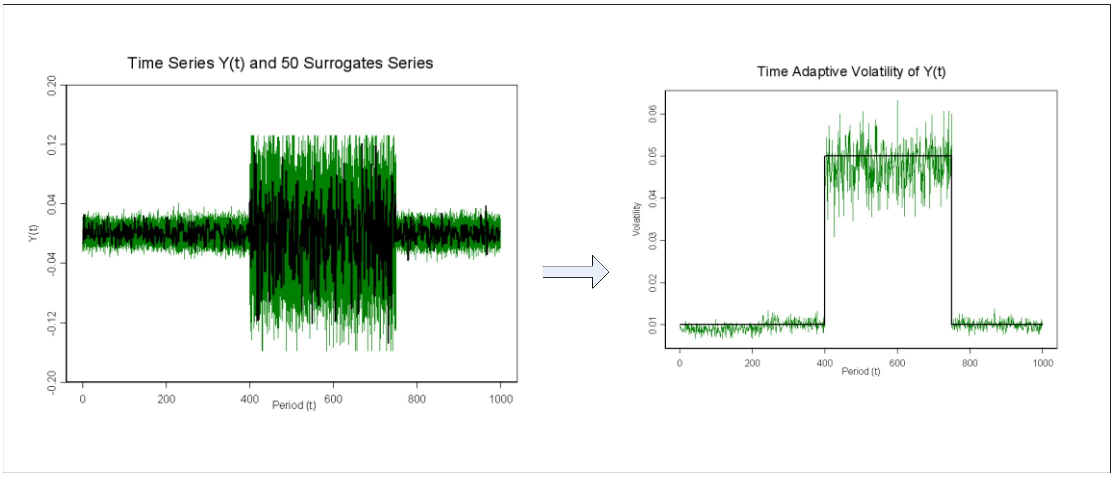

Figure 88. Analysis of surrogate series and adaptive volatility of time series $\{Y_t\}$ *using the SKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).*

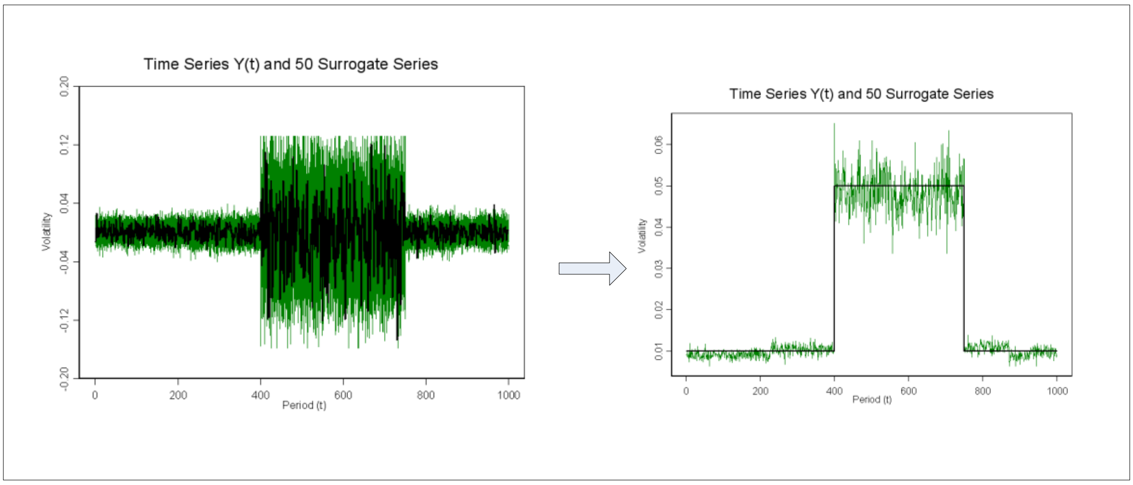

Figure 89. Analysis of surrogate series and adaptive volatility of time series $\{Y_t\}$ *using the BKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).*

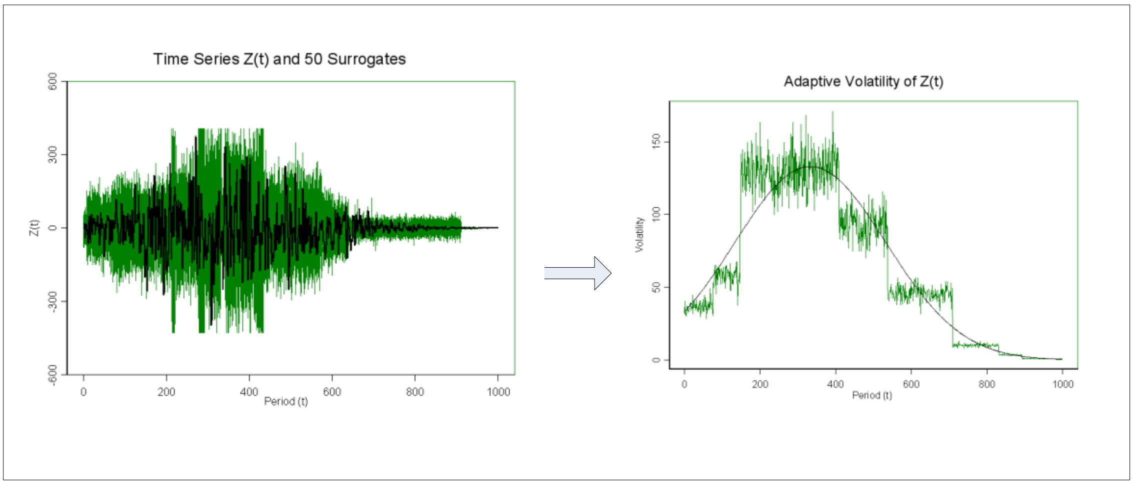

Figure 90. Analysis of surrogate series and adaptive volatility of time series $\{Z_t\}$ *using the SKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).*

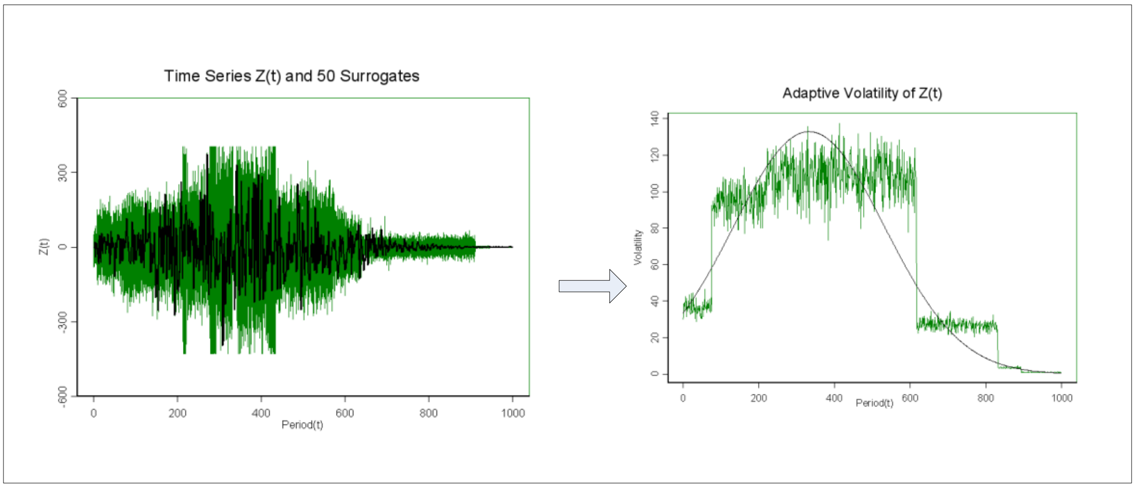

Figure 91. Analysis of surrogate series and adaptive volatility of time series $\{Z_t\}$ *using the BKS technique. Left figure: Original time series (black colour) and 50 surrogate series (light colour). Right figure: Adaptive estimated volatility (light colour) and true adaptive volatility (dark line).*

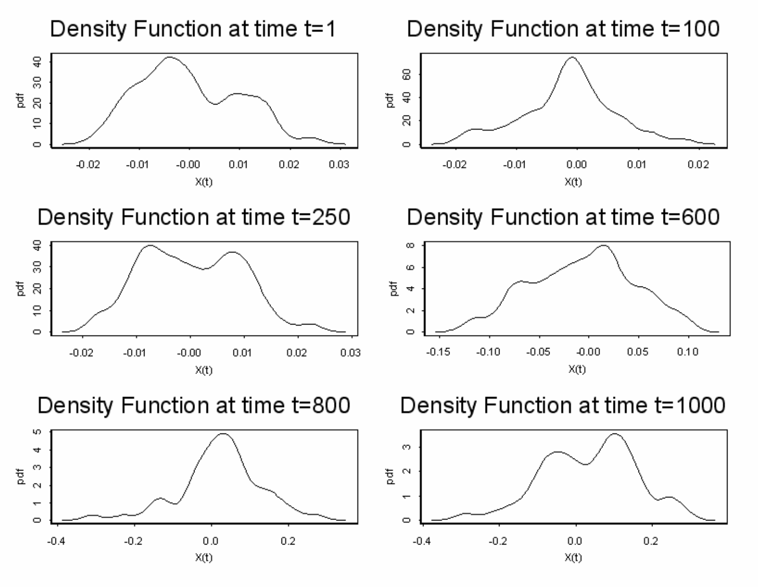

Figure 92. Adaptive distribution function of the time series $\{X_t\}$ *for points* $t = 1, 100, 250, 600, 800, 1000$ *(SKS technique).*

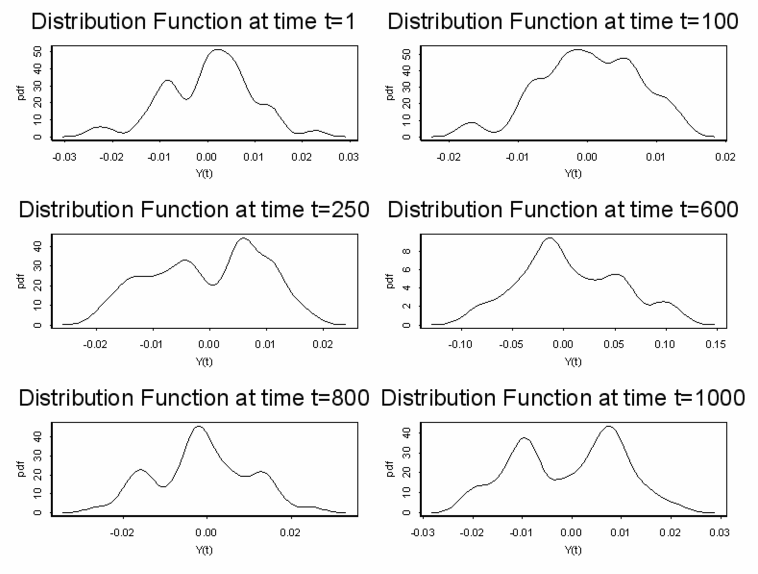

Figure 93. Adaptive distribution function of the time series $\{Y_t\}$ *for points* $t = 1, 100, 250, 600, 800, 1000$ *(SKS technique).*

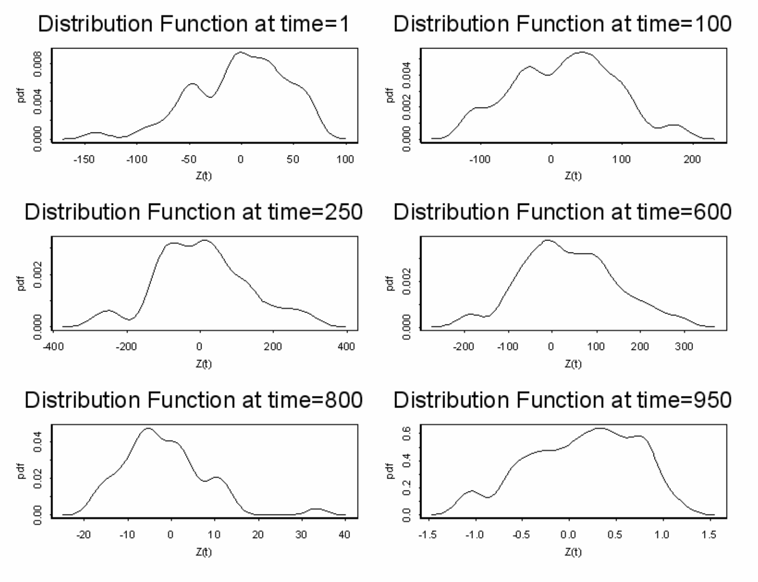

Figure 94. Adaptive distribution function of the time series $\{Y_t\}$ *for points* $t = 1,100,250,600,800,$ *(BKS technique).*

5. Summary, conclusions and further considerations

In this paper, we have proposed a novel time series re-sampling technique, the Block-Fourier bootstrap, which deals with zero-mean non-stationary time series. As explained throughout the paper, this technique is based on the assumption that a non-stationary time series can be seen as being composed of M blocks where each piecewise series is approximately stationary. Consequently, this technique has its fundamentals in both the adaptive volatility estimation (SKS), the adaptive spectrum estimation (BKS) and the scrambling or Fourier bootstrap.

The principal conclusions of this study are as follows.

- Techniques used for splitting a non-stationary time series based on adaptive spectrum and adaptive volatility can be used together with the phase scrambling technique (or other stationary time series bootstrap techniques) to resample a non-stationary time series.

- Techniques based on the change of volatility, such as the SKS, have been shown to perform very well in detecting changes in variance of a non-stationary time series, while techniques based on the adaptive spectrum, such as BKS, are more sensitive to abrupt changes and perform well with series where the changes in the variance occur in frequently.

- The probability density function is not static over time but dynamic. This last result could be of little help if the process is analysed from a linear viewpoint, because the changes observed in the data-generating process do not follow a linear pattern but a non-linear one. Hence, if an analysis of pattern identification needs to be done, for example, toforecast future values, non-linear techniques need to be used, such as the nearest neighbour techniques (NN).

- A novel non-parametric technique for res-ampling zero-mean non-stationary time series has been developed.

As mentioned previously, one limitation of this technique is that since it re-samples the time series in a block by block fashion, information about the exiting correlation or dependence (if there is one) across the boundaries of the blocks are lost.

Another limitation of this technique is that it deals just with zero-mean non-stationary time series. Consequently, an extension of this research is to generalise the Block-Fourier bootstrap so it is capable of dealing with a time series which is non-stationary in respect of both its mean and variance. Further, it would be very interesting to include higher order moments in the frequency domain, such as the bispectrum in the process of resampling so the third order statistics or skewness can be preserved by the resulting surrogates.

Even though throughout this research it has been seen that the block-Fourier bootstrap performs very well in generating surrogate series from an original, this is an initial idea which can be improved by, for example, using techniques based on wavelets for estimating the periodogram, and consequently have more precision when splitting the time series into blocks which are seen as stationary. Also, it would be logical to think about generating a hybrid technique which could be able to combine both the SKS and BKS techniques. Another improvement can be made in using another technique, such as the Cressie-Whitford (Reed III, 2005) techniques, which uses the skewness and kurtosis for detecting changes in volatility, when selecting candidates blocks. These limitations and improvements are undoubtedly the topics of future research.

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The bivariate adaptive kernel-based nearest neighbour bootstrap technique: a new bivariate time series re-sampling technique

Luis Martinez and Rodney Wolff

Introduction

In a previous unpublished paper (see the Block Fourier Bootstrap paper above), Martinez and Wolff (Martinez and Wolff, 2005a) developed a new technique, called the block-Fourier bootstrap, for re-sampling a zero-mean non-stationary time series. In their paper, the authors show how the block-Fourier bootstrap is able to generate surrogate series from a zero-mean non-stationary time series by means of adaptive analysis; that is the original non-stationary time series is segmented into blocks where the resulting piecewise series is seen to behave approximately as stationary, and using the Fourier bootstrap (Davison and Hinkley, 1998) on each block, a surrogate or simulation, of the original time series, is generated by concatenating the surrogates generated for each block. The previous technique, however, was limited just to univariate non-stationary time series.

In this paper, a new technique for jointly resampling two non-stationary time series (a bivariate process), named the adaptive kernel-based nearest neighbour bootstrap (BAKNNB), is developed and explained as an extension of the block-Fourier bootstrap technique. Note that, similarly to the block-Fourier bootstrap, this technique deals with time series which are non-stationary due to the fluctuation in variance over time (heteroskedasticity) assuming that their means remain constant and equal to zero.

One example of such type of time series, which is a common transformation in economics and finance, is the so-called return series $\{R_t\}$ defined as

$$R_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}},$$

where $\{Y_t\}$ is a given time series, such as flows and prices . Further, $E\{R_t\} \cong 0$, and its volatility σ_t varies over time. This type of time series can be modelled as

$$R_t = \sigma_t \varepsilon_t ,$$

where ε_t are innovations (residual series) that are assumed to be independent identically distributed (IID) but not necessarily normally distributed (Chen, Hardle and Jeong, 2005) with $E\{\varepsilon_t\} = 0$ and $Var\{\varepsilon_t\} = 1$.

Basically, as it will be explained with more detail later, the strategy of this technique is to estimate the dynamic dependence structure between two time series, estimate their respective dynamic joint distribution, and sample it in order to render a surrogate series for each of the given (original) time series. One important feature of this technique is that it generates bivariate surrogate series which honour both their marginal statistical properties, of each of the original time series, as well as their joint properties, such as their linear correlation (cross correlation) over time. To achieve this, the proposed bootstrap technique makes use of the following concepts.

1 The *k*-nearest neighbour (*k*-NN) (Rajagopalan and Lall, 1999) and the local bootstrap (Paparoditis and Politis, 2002) as tools for capturing the joint behaviour, over time, of the two risky assets or random variables under study;

2 the non-parametric Kendall's tau coefficient of correlation (Kendall and Ord, 1990; Nelsen et al., 2003) and copulas (Cherubini, Luciano and Vecchiato, 2004a; Caillault and Guegan, 2003; Patton, 2002; Sklar, 1959; Van den Goorbergh, Genest and Werker, 2003; Matteis, 2001; Genest and MacKay, 1986; Nelsen et al., 2003) as tools for both measuring dependence and for modelling the joint distribution of two random variables without assuming either elliptical processes, such as Gaussian or t-distribution processes, nor linear dependence over time. It is important to

mention that both the estimation of the correlation and the estimation of the joint distribution of the two risky assets are performed dynamically, or in other words, at each point over time, and only in a linear fashion.

The paper is organised as follows. In Section 1, the concepts of Kendall's tau correlation and bivariate copulas are reviewed as a means of measuring and modelling dependence and bivariate distributions, respectively. Section 2 introduces a new bivariate bootstrap concept called the bivariate adaptive kernel-based nearest neighbour bootstrap (BAKNNB). In Section 3 an example of the use of the new technique applied to two assets, daily log-returns of FFR/USD versus DEM/USD exchange rates, is presented with results and comments. Section 4 gives conclusions, limitations and some topics for further research.

1. The Kendall's tau coefficient of correlation and copulae functions

The Kendall's tau correlation is a non-parametric statistic that measures the dependence between two random variables based on their rank statistics. More formally, given the observation of two random variables X and Y ($\left\{(x_1, y_1), (x_2, y_2), .., (x_n, y_n)\right\}$, the Kendall's sample correlation is defined as

$$\bar{\rho}_\tau = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{n} sign\left[\left(x_i - x_j\right)\left(y_i - y_j\right) \right] \right), \tag{1.1}$$

where

$$sign(x) = \begin{pmatrix} 1 & if & x > 0; \\ 0 & if & x = 0; \\ -1 & if & x < 0; \end{pmatrix}.$$

Further, as it will be shown in the next section, the Kendall's tau coefficient for the random variables X and Y with copula C is defined as

$$\rho_\tau = 4 \iint_{I^2} C(u,v) dC(u,v) - 1, \tag{1.2}$$

where, I is the unit interval. Hence, there is a close relationship between the Kendall's tau correlation coefficient and the bivariate copula (especially the Archimedean copula) of X and Y.

A two dimensional (bivariate) copula is nothing else but the cumulative bivariate distribution of two random variables, which depends on the marginal distributions of each random variable. More formally, given two random variables X and Y, with cumulative marginal distributions $F_X(x)$ and $G_Y(y)$ respectively, the bivariate copula is defined as

$$C_{UV}(u,v) = H_{XY}(x,y), \tag{1.3}$$

where $U = F_X(x)$ and $V = G_Y(y)$ are the cumulative marginal distributions of X and Y respectively. Note that the marginal distributions of X and Y, that is, $F_X(x)$ and $G_Y(y)$, and the copula $C(u,v)$ are standard uniform random variables. If $F_X(x)$ and $G_Y(y)$ are continuous, then there exists a unique copula $C(u,v)$; however, if they are not continuous, then the copula is not unique. In fact, every copula that has the same range as $Range(F_X) \times Range(G_Y)$ is a valid copula.

One important property of the copula is that it allows the modelling of the bivariate distribution function to be performed in two steps: i) the estimation of the marginal distributions; and ii) the estimation of the dependence structure between the two random variables. Indeed, these properties can be visualised from the following analysis

$$C_{UV}(u,v) = H_{XY}(x,y), \tag{1.4}$$

$$\frac{\partial^2 C_{UV}(u,v)}{\partial x \partial y} = \frac{\partial^2 H_{XY}(x,y)}{\partial x \partial y}, \tag{1.5}$$

$$f_X(x) g_Y(y) c\big(F(x), G(y)\big) = h_{XY}(x,y), \tag{1.6}$$

where $c(F(x),G(y)) = \dfrac{\partial^2 C(F(x),G(y))}{\partial F_X \partial G_Y}$, is the coupling function of X and Y, which expresses the dependence structure between these two random variables, and both $f_X(x)$ and $g_Y(y)$ are the density functions of X and Y respectively. Note that here we have made use of the following equivalences:

$$U = F_X(x) \rightarrow \frac{\partial U}{\partial x} = \frac{\partial F(x)}{\partial x} = f_X(x), \tag{1.7}$$

and

$$V = G_Y(y) \rightarrow \frac{\partial V}{\partial y} = \frac{\partial G(y)}{\partial y} = g_Y(y). \tag{1.8}$$

Note that Equation 1.6 is very important when modelling bivariate distributions since it implies that the combination of any two marginal distribution functions, for X and Y respectively, with a given dependence structure $c(U,V)$ (expressed by the copula) results in a valid bivariate distribution function (Sklar, 1959).

The class of *one-parameter Archimedean copulae* is a special family of parametric copulae which may be constructed using a function $\varphi_\alpha : I \rightarrow \Re^{*+}$ (the non-negative extended real line), continuous, decreasing, convex and such that $\varphi_\alpha(1) = 0$; this function is called the generator function of the copula, and α is the parameter of the Archimedean copula.

As mentioned in the previous section, one important characteristic of Archimedean copulas is their close relationship with measures of association, such as the Kendall's tau. In fact, Genest and Mackay (1986) demonstrated that the Kendall's tau correlation of two random variables X and Y, is given by

$$\rho_\tau = 4\int_I \frac{\varphi_\alpha(\omega)}{\varphi_\alpha^{'}(\omega)} d\omega + 1, \tag{1.9}$$

where the first derivative of the generator function ($\varphi_\alpha^{'}(\omega)$) exists almost everywhere (other than in a set of Lebesgue measure zero).

Fitting an Archimedean copula to a bivariate process

In this subsection, the Genest and Rivest non-parametric technique (Genest and Rivest, 1993; Matteis, 2001) is used to estimate and fit an Archimedean copula to a bivariate process $\left(X_t, Y_t\right)$ (for $t = 1, 2, \cdots, n$). The technique consists of the following steps.

Given the observations of a bivariate process $\left\{\left(x_1, y_1\right), \left(x_2, y_2\right), \ldots, \left(x_n, y_n\right)\right\}$, compute the following

1. Determine the non-parametric distribution function $\hat{K}_C$ for each pair $\left(X_t, Y_t\right)$. Let $\left\{\left(x_1, y_1\right), \left(x_2, y_2\right), \ldots, \left(x_n, y_n\right)\right\}$ be observations of the bivariate distribution $H\left(x, y\right)$, with continuous marginal $F\left(x\right)$ and $G\left(y\right)$, and Archimedean copula

$$C_\alpha\left(u,v\right) = C_\alpha\left(F\left(x\right), G\left(y\right)\right) = \varphi^{-1}\left(\varphi\left(u\right) + \varphi\left(v\right)\right).$$

An estimate of the univariate distribution function of $C_\alpha\left(U,V\right)$ is defined, on the interval $\left[0,1\right]$, as:

$$K\left(w\right) = \Pr\left[C\left(U,V\right) \le w\right] = \Pr\left[H\left(X,Y\right) \le w\right]. \tag{1.10}$$

To find this, the random variable W_i is defined as:

$$W_i = \hat{H}\left(X_i, Y_i\right) = \frac{\#\left\{\left(X_j, Y_j\right) : X_j < X_i, \quad Y_j < Y_i\right\}}{\left(n-1\right)}. \tag{1.11}$$

Thus, $\hat{H}$ is an empirical estimate of the bivariate distribution function H. Then, a non-parametric estimator of $K_C\left(w\right)$ is given by:

$$\hat{K}_C\left(w\right) = \frac{\#\left\{i : \quad 1 \le i \le n, \quad W_i \le w\right\}}{n}. \tag{1.12}$$

For an examination of the properties and a better explanation of $\hat{K}_C\left(w\right)$, refer to Matteis (2001, p41) and Genest and Rivest (1993).

2. Determine the parametric distribution function of K_C for a set of selected Archimedean copulas (see appendix A). Once an estimate $\hat{K}_C(w)$ of the distribution $C_\alpha(U,V)$ has been computed through Equation 1.12, it is used as a tool to help in identifying the parametric family of Archimedean copulas that provides the best possible fit to the data. Because the fact that every known Archimedean copula has its specific generator φ_α, it is possible to determine through $\hat{\alpha}$ (the estimated of the copula's parameter) the parametric estimate of the distribution function $\hat{K}_C(w)$ which takes the well known form given by:

$$K_C(t) = t - \frac{\varphi(t)}{\varphi'(t)}, \tag{1.13}$$

where $K_C(t)$ is the distribution of $C_\alpha(U,V)$ (for the proof refer to Nelsen, 1999, p130; or Matteis, 2001 p29).

Note that the generator functions ($\varphi(t)$) of some important Archimedean copulas are already given in tables. See for example, Nelsen (1999) pp94-96; Joe (1997) pp138-149; and Matteis (2001) pp. 72-74. One important characteristic that some of these tables display is the Kendall's tau coefficient as a function of the parameter of the Archimedean copula α (see for example Matteis, 2001).

Then considering the Kendall's tau coefficients estimated previously, and the generator functions for the different Archimedean copulas, a set of candidate copulas can be selected. These selection are based on the domain (range of α) of each generator function.

As an example (taken from Matteis, 2001), consider the generator function of the Clayton family expressed as a function of α by $\varphi_\alpha(w) = \frac{1}{\alpha}\left(w^{-\alpha} - 1\right)$, then by solving $\hat{\rho}_\tau = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt$, the Kendall's tau coefficient as a function of the parameter of the Archimedean copula is obtained as $\hat{\rho}_\tau = \frac{\alpha}{\alpha+2}$. Then if we have a pair of random variables (X,Y) with an estimated Kendall's tau, say $\hat{\rho}=0.4634316$, the value of the estimated parameter of the copula is $\hat{\alpha}=1.7274$. Note that since the Clayton copula's parameter has the interval $\alpha \in [-1,\infty)\setminus\{0\}$ as domain (Appendix A), and $\hat{\alpha}=1.7274$, the Clayton copulas could be selected as a candidate to be the copula of (X,Y).

3. Compare the non-parametric distribution function $\hat{K}_C(w)$ with the selected parametric distribution functions $K_C(w)$. The idea is that the estimator $\hat{\alpha}$ is adequate if the parametric distribution function $K_C(w)$ is similar to the non-parametric $\hat{K}_C(w)$. One way of comparison is through model fitting. In this research, the Kolmogorov-Smirnov test statistic (Hogg and Klugman, 1984) is used to perform the comparison between the two distribution functions. Then, the parametric model that obtains the lowest Kolmogorov-Smirnov value is selected as the ideal copula for the data under study. We put aside the issue of significance of the Kolmogorov-Smirnov for the time being.

2. The bivariate adaptive kernel-based nearest neighbour bootstrap

The bivariate adaptive kernel-based nearest neighbour bootstrap technique (BAKNNB) is a new resampling technique for a bivariate zero-mean non-stationary time series. As mentioned in the introduction section, the strategy of this technique is to estimate the dynamic joint distribution function of the two random variables under study, and sample it in order to render a surrogate series for each of the given time series; each surrogate series must honour the statistical properties of each of the original time series, as well as their linear correlation (cross correlation) over time.

One important feature of this technique is that it works with the stochastic processes ε_t and μ_t (see Equation 1.15), which are assumed to be strictly stationary and independent and identically distributed (i.i.d) (see for example Frank, Kreiss and Mammen (Franke, Kreiss and Mammen, 2002)). Consequently, given two zero-mean non-stationary time series, let's say:

$$R_t^x = \sigma_t^x \varepsilon_t$$
$$R_t^y = \sigma_t^y \mu_t, \tag{1.14}$$

the estimated devolatised series $\hat{\varepsilon}_t$ and $\hat{\mu}_t$ can be modelled as

$$\hat{\varepsilon}_t = R_t^x \big/ \hat{\sigma}_t^x$$
$$\hat{\mu}_t = R_t^y \big/ \hat{\sigma}_t^y, \tag{1.15}$$

where $\hat{\sigma}_t^x$ and $\hat{\sigma}_t^y$ are the values representing the estimated volatility of each process (which are estimated using the block-Fourier bootstrap). Note that $\hat{\sigma}_t^x$ and $\hat{\sigma}_t^y$ are not constant over time but dynamic (they change at each point in time). Then, i.i.d. bootstraps can be performed on the jointly stochastic process $\left(\hat{\varepsilon}_t, \hat{\mu}_t\right)$, using the proposed technique. Then, using the recursion (1.14) with $\hat{\sigma}_t^x$ and $\hat{\sigma}_t^y$ in place of σ_t^x and σ_t^y - a bivariate surrogate series $\left\{R_t^{x*}\right\}$ and $\left\{R_t^{y*}\right\}$ of the original series can be generated; from here, any target statistic, such as their cross correlation, can be computed. The block-Fourier bootstrap (Martinez and Wolff, 2005a) is used here to estimate the dynamic volatility $\hat{\sigma}_t^x$ and $\hat{\sigma}_t^y$ of $\left\{R_t^x\right\}$ and $\left\{R_t^y\right\}$ respectively.

To characterise the joint behaviour of the two random variables, that is, ε_t and μ_t, the BAKNNB uses a variation of the *k*-nearest neighbour technique for finding suitable (joint) neighbours. It is important to observe that this procedure is performed in a dynamic fashion in order to capture the existing dependence structure (if there is one) between the two random variables ε_t and μ_t at each point in time. More formally, given two zero-mean non-stationary discrete time series, let's say $\left\{R_t^X\right\} = \left\{r_1^x, r_2^x, \cdots, r_{n-1}^x, r_n^x\right\}$, and $\left\{R_t^Y\right\} = \left\{r_1^y, r_2^y, \cdots, r_{n-1}^y, r_n^y\right\}$, the steps to generate joint surrogate series using the BAKNNB are as follows.

Step1. Apply the block-Fourier bootstrap to each time series, and estimate their dynamic volatility, that is, $\hat{\sigma}_t^x$ and $\hat{\sigma}_t^y$, respectively.

Step 2. Estimate the stochastic processes $\hat{\varepsilon}_t$ and $\hat{\mu}_t$ using Equation 1.15.

Step3. Estimate the joint neighbourhood region of the current values, that is e_k and u_k (for $k = 1, 2, \ldots, n$) (see subsection 2.1 for more details about the joint neighbourhood).

Step4. Find the nearest neighbours of e_k and u_k (for $k = 1, 2, \ldots, n$) given by the pairs that fall inside their joint neighbourhood.

Step5. Estimate their joint distribution expressed by the (Archimedean) copula $C\left(F_{\varepsilon_k}(e), G_{\mu_k}(u)\right)$, where $F_{\varepsilon_k}(e)$ and $G_{\mu_k}(u)$ are the cumulative marginal distribution of $\hat{\varepsilon}_k$ and $\hat{\mu}_k$ (for $k = 1, 2, \ldots, n$) (see subsection 1.1).

Step 6. Sample the estimated copula (see appendix B) obtaining bootstrap values ε_k^* and μ_k^* for $\hat{\varepsilon}_k$ and $\hat{\mu}_k$ respectively (for $k = 1, 2, \ldots, n$). Note that because these two bootstrap values are sampled from their copula they are guaranteed to keep their respective dependence structure.

Step 7. Use Equation (1.14) to generate surrogate values, that is, R_k^{x*} and R_k^{y*} (for $k = 1, 2, \ldots, n$), of R_k^x and R_k^y respectively;

Step 8. Repeat steps 2 to 7 N times in order to generate N bivariate surrogates series of the original bivariate process.

Step 9. End the process.

Finding a suitable neighborhood for a bivariate process

In this research, the kernel bandwidth estimator is the method used to determine the size of the neighborhoods, h_x and h_y, for each current value, that is, x_k and y_k (for

$k = 1, 2, \ldots, n$). In essence, this method uses a strip of constant width or bandwidth such that the points that fall inside are considered to be nearest neighbors of the current value (point in time under study), which is located at the centre of this strip by definition. More generally, one can think of a bandwidth, h_x and h_y, as simply a parameter used to determine the size of the neighborhood around a specific point, in this case the current value of the time series (DiNardo and Tobias, 2001). In this paper we use the Silverman's rule of thumb bandwidth generator

$$h = 1.06 \times \min\left\{\hat{\sigma}, IQR/1.34\right\} \times n^{-1/5}, \tag{1.16}$$

where $\hat{\sigma}$ is the estimated sample standard deviation of the data set, IQR is the interquartile range, and n is the length of the time series. Further, since the rule of thumb (naïve bandwidth) assumes that the unknown data generator belongs to the normal family (which it is not the case in this research), the bandwidth h is adjusted to

$$\begin{aligned}\hat{h} &= 2.6226\left[1.06 \times \min\left\{\hat{\sigma}, IQR/1.34\right\} \times n^{-1/5}\right] \\ \hat{h} &= 2.78 \times \min\left\{\hat{\sigma}, IQR/1.34\right\} \times n^{-1/5}\end{aligned}, \tag{1.17}$$

using the canonical bandwidth property. For details of the bandwidth selections, the reader is referred to (Silverman, 1986; Wand and Jones, 1995; Hardle, 1990; Härdle, 1990; Härdle et al., 2004). This process can be better visualised in Figure 95 below where the current value x_k (black circle) of a time series, the strip (rectangle) indicating the size of its neighbourhood, as well as its respective set of nearest neighbours (hollow circles) $\left\{x_j, x_p, \cdots, x_l, x_n\right\}$ are shown.

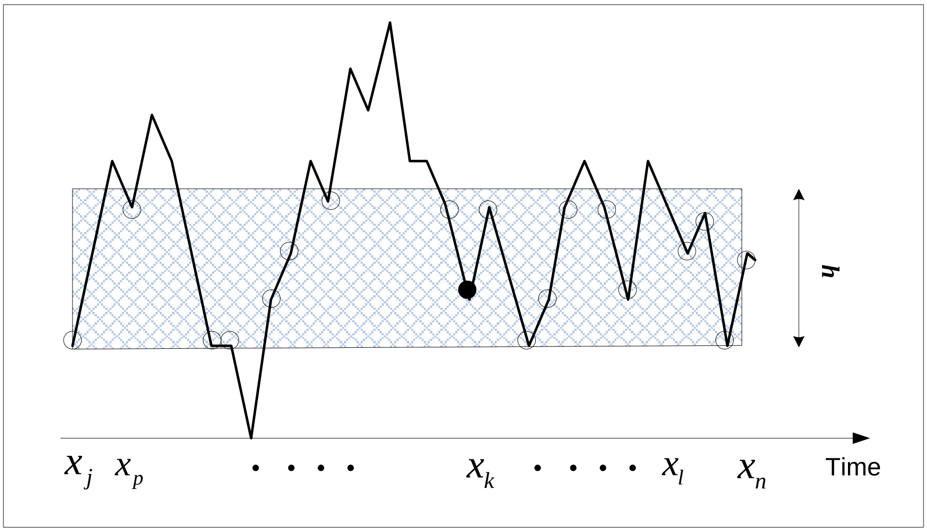


Figure 95. Diagram showing the current value x_k *(black circle), its neighbourhood* h *(rectangle), and its nearest neighbours* $\left\{x_j, x_p, \cdots, x_l, x_n\right\}$ *(hollow circles).*

Note, however, that since in this case because the process under study is bivariate, the neighbours of each random variable need to be selected jointly at the same time or period; so, in this way, the joint behaviour or dependence structure (if it exists) is captured. Basically, in this paper the joint neighbour of the current values x_t and y_t, at a specific point in time $t = k$, is the region inside the ellipsoid which has as main axis the bandwidths h_x and h_y, estimated using equation 1.17 for each random variable, respectively. Then the joint neighbourhood of x_t and y_t is defined as the ellipsoid

$$\frac{\left(x_k - x\right)^2}{\left({}^{h_x}\!/\!_2\right)^2} + \frac{\left(y_k - y\right)^2}{\left({}^{h_y}\!/\!_2\right)^2} = 1. \tag{1.18}$$

Consequently, the nearest neighbours of x_t and y_t will be the pair of points $\left(x_j, y_j\right)$ that falls inside the ellipsoid defined in (1.18). More formally, the nearest neighbours of x_t and y_t are the pair of points $\left(x_j, y_j\right)$ such as

$$\left\{\left(x_j, y_j\right); \frac{\left(x_k - x_j\right)^2}{\left({}^{h_x}\!/\!_2\right)^2} + \frac{\left(y_k - y_j\right)^2}{\left({}^{h_y}\!/\!_2\right)^2} \le 1 \qquad j = 1, 2, .., n\right\}. \tag{1.19}$$

This procedure is better visualised in Figure 96 where the joint neighbourhood of x_t and y_t is shown. In the figure, it is observed that the ellipsoid is centred at point (x_k, y_k), and has as its main axis the bandwidths h_x and h_y, of x_t and y_t respectively, and a neighbour (x_j, y_j) which falls inside its limits. Observe that this procedure, that is, selecting joint nearest neighbours, is performed at each point in time.

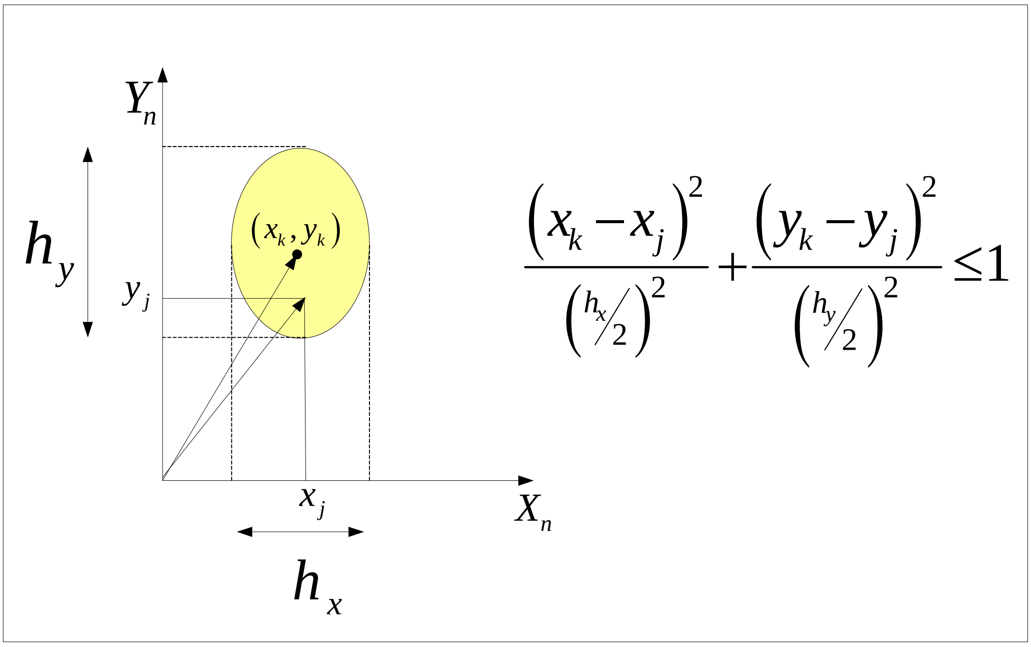

Y_n
h_y
(x_k, y_k)
y_j
x_j
X_n
h_x
$\frac{(x_k - x_j)^2}{(h_x/2)^2} + \frac{(y_k - y_j)^2}{(h_y/2)^2} \le 1$

Figure 96. Figure showing the ellipse or joint neighbourhood of the current value x_k *and* y_k *of a bivariate process.*

3. Application of the proposed bootstrap technique

In the previous section, a novel bivariate bootstrap technique which deals with bivariate zero mean non-stationary time series, named the bivariate adaptive kernel-based nearest neighbour bootstrap technique (BAKNNB), was introduced and explained in detail. In this section, this technique is tested by meaning of generating jointly surrogate series of two log-return time series, that is, the daily log-returns of FFR/USD versus DEM/USD exchange rates. Note that the data sets contain just weekday spot prices (weekends and holidays are not considered); furthermore, the database consists of 2.2 years of daily data (from 1st May 1982 to 17th August 1984) and is composed of 600 daily data points.

A time series analysis indicated that both time series have expected values approximately equal to zero, and a standard deviation of 0.0069, for the FFR/USD, and 0.0058, for the DEM/USD, respectively. A cross analysis indicated that both time series have a linear correlation (Pearson's correlation) equal to 0.83, which indicates that both return series are apparently highly linear correlated over time. Figure 97 shows the time series for each of the exchange rates return time series, and Table 7 indicates a summary statistics for both processes.

The target of this application is to apply the BAKNNB to the two exchange rates time series, that is FFR/USD and DEM/USD, and generate $N = 100$ surrogate series of each of them in order to have more information about their joint behaviour over time. For the sake of simplicity, this process will be explained in a step-by-step fashion.

Let $\left\{R_t^x\right\}$ and $\left\{R_t^y\right\}$ represent the time series of FFR/USD and DEM/USD exchange rates respectively, then each of them can be modelled as (see Equation 1.14)

$$\begin{aligned} R_t^x &= \sigma_t^x \varepsilon_t \\ R_t^y &= \sigma_t^y \mu_t, \end{aligned}$$

where σ_t^x and σ_t^y are their respective volatilities, and, ε_t and μ_t, are their respective stochastic processes which can be estimated as (see Equation 1.15)

$$\begin{aligned} \hat{\varepsilon}_t &= R_t^x \big/ \sigma_t^x \\ \hat{\mu}_t &= R_t^y \big/ \sigma_t^y . \end{aligned}$$

Steps 1 and 2. Apply the block-Fourier bootstrap to each exchange rate series and estimate both their respective dynamic volatility $\hat{\sigma}_t^x$ and $\hat{\sigma}_t^y$, at each point in time, and their respective processes $\hat{\varepsilon}_t$ and $\hat{\mu}_t$, respectively. The results of these steps are shown in Figures 98 and 99 where both the dynamic volatility and the stochastic process (de-volatised series) for the FFR/USD and DEM/USD exchange rates are shown, respectively. As shown in the Figure 98, the FFR/USD exchange rate is seen to have one marked change in its volatility, which occured at the 3rd quarter of 1983, while, as shown in Figure 99, the DEM/USD is seen to have 4 changes in volatility occurring at the end of 1982, 1st and 3rd quarters of 1983, and at the beginning of 1984.

It is important to acknowledge that in this paper the processes $\overset{\wedge}{\varepsilon_t}$ and $\overset{\wedge}{\mu_t}$ are assumed to be strictly stationary (see Section1). However, it could happen that they are not totally stationary; one example of the previous problem is visualised in Figure 98 where the de-volatised series (bottom figure) of the FFR/USD exchange rates series is seen to be, apparently, non-stationary with respect to the variance; in fact, there are three spikes at the end of year 1982, beginning and middle of 1983, respectively, which make it difficult to accept the stationarity of the process. A visual comparison of the original and de-volatised FFR/USD exchange rates series (Figures 98-top and 99-top) will indicate that the spikes founded in the de-volatised series are due to isolated extreme values in the original data series. Because the BFB (Martinez and Wolff, 2005b) technique is used in this step for estimating the adaptive volatility of the FFR/USD exchange rates series, it could happen that this technique (BFB) was not able to capture the change in volatility in these isolated extreme values, giving a smooth value, which results in extreme values in the de-volatised series. One solution could be to improve the BFB in a way that is able to capture volatility in isolated extreme values or normalise the original data avoiding these extreme cases.

Despite the previous problem, it is important to mention that the proposed technique does not need to work with a strictly stationary process, since it uses, as stated in Steps 3 and 4, a nearest neighbour approach to capture the existing dependence between the two given time series which has no restrictions about the behaviour of the series (see the final results and comments in Section 4) ; of course, working with a strictly stationary process (i.i.d) will result in a more accurate analysis since the selection of neighbourhood (kernel bandwidth) would be based on a stationary distribution function. Consequently, in this paper, the normalisation of the de-volatised series is not considered in detail.

Steps 3, 4 and 5. Estimate the joint neighbourhood region of the current values, that is x_k and y_k (for $k = 1, 2, \ldots, n$) under study, estimate their dynamic cross-correlation given by the Kendall's tau coefficient, and their joint distribution or dynamic Archimedean copula function. The results of these steps are shown in Figures 100 and 101 where the dynamic Kendall's tau (*k-tau*) coefficient of correlation, the dynamic Archimedean copula function, and the dynamic Archimedean parameter (alpha), of both exchange rates series, are shown. As seen in Figure 100, both exchange rates have, most of the time, a positive rank correlation varying between 0 and 1, which indicates that these two exchange rates are positively correlated most of the time; however, it is also observed that there are some points where these two exchange rates have negative and zero correlation which indicates that in some periods these two FX rates are negative correlated or that they could be independent (not necessarily). The figure also shows that the joint distribution function of these two exchange rates, determined by their copula function (bottom of Figure 6), is not static over time but dynamic; it means that both the copula structure and its respective parameter, α, change over time (see Figure 101). In other words, at different points in time, different copula functions are seen to describe the joint behaviour of these two exchange rates. Further, it is observed in Figure 100 (bottom) that the joint distribution function (copula function) does not follow a linear pattern over time which is a more realistic result, indicating that the entire bivariate process is nonlinear.

The previous analysis is in itself of great importance since it shows the evolution of the process over time, indicating that the entire process is not static but changes over time (which is more realistic in terms of real processes). Further, this last result indicates that traditional assumptions of assuming a constant Gaussian distribution function (with constant parameters) over time may not be correct, and could cause spurious results about the joint behaviour of these two random variables over time.

Steps 6, 7 and 8. Sample the estimated copula obtaining bootstrap values $\overset{\wedge}{\varepsilon}_t^*$ and $\overset{\wedge}{\mu}_t^*$, for $\overset{\wedge}{\varepsilon}_t$ and $\overset{\wedge}{\mu}_t$ (for $k = 1, 2, \ldots, n$), respectively, and use equation (1.14) to generate surrogates of $\left\{R_t^x\right\}$ and $\left\{R_t^y\right\}$. Note here that the volatilities of each log-return series were already estimated in Step 1 (see Figures 98 and 99 respectively).

The results of these steps are shown in Figures 102 and 103 respectively, where the original time series (top) and 100 surrogate series (bottom) of each exchange rate series are plotted. As seen from the figures, the proposed technique, that is, the BAKNNB, generates bivariate surrogate series which honour both the univariate (marginal) and bivariate (jointly) characteristics of the original time series. The previous results can be better visualised in Figures 104 and 105, where the empirical marginal distribution functions (S+ was used to generate these distributions) of the original and three surrogate series, of each exchange rates, are shown. As observed in the figures, the empirical marginal distribution of the three surrogate series (in this case surrogates 10, 50, and 90) honour very well the characteristics of the empirical distribution of the original series, such as mean, variance and skewness, and kurtosis. In fact it is difficult to differentiate the empirical distribution of the surrogate series from the empirical distribution of the original series, respectively. Also, observe in Figure 106 that the linear Pearson's cross-correlation of the 100 surrogate bivariate series (points 1 to 100) honour very well the cross-correlation of the original bivariate series (point zero), which is equal to 0.083 (see Table 7). This last result is very important since it indicates that the proposed technique is able to generate bivariate surrogate series which also honours the existing linear correlation between the original exchange rate series. Note that the re-sampling method does not force this last result. Conversely, the proposed technique uses a data driven re-sampling methodology which is able to capture the whole information contained in the original data. In other words, no assumptions were done about both the marginal and joint distribution of the original data sets.

4 Summary, conclusions and further recommendations

In this paper, we have proposed a novel bivariate time series resampling technique, named the bivariate adaptive kernel-based nearest neighbour bootstrap technique (BAKNNB), which deals with non-stationary bivariate time series of the type given in Equation 1.14. As explained throughout the paper, this technique uses the Block-Fourier bootstrap to estimate the dynamic (adaptive) volatility of each given time series, and works with the resulting stochastic time series (see Equation 1.15). Also, the proposed technique uses a variation of the *k*-nearest neighbour technique to capture the existing dynamic dependence between the two given time series: this is done using the Kendall's tau correlation and the (Archimedean) copula function. In this way, the BAKNNB is able to generate bivariate surrogate series which honour both their marginal characteristics, such as mean and variance, and their joint characteristics, such as their cross correlation.

The principal conclusions of this research are as follows.

- The Pearson's linear correlation and Gaussian distribution are not good indicators of dependence structure and data generator of a bivariate process. The reason for this is that, as shown throughout this paper, the bivariate data generator process (joint probability density function) is not static over time but dynamic.

- Kendall's tau correlation and copula functions are good alternatives to the traditional linear Pearson's correlation and elliptical distributions for estimating dependence structure and estimating the joint distribution function of a bivariate process.

- A novel non-parametric technique for re-sampling zero-mean bivariate non-stationary time series has been developed. In this context, it is important to acknowledge that in this paper a zero-mean non-stationary time series implies a time series which has a variance that fluctuates over time assuming that orders higher than two remain stationary.

One limitation of this technique, however, is that it depends on the size of the neighbourhood used to estimate the joint behaviour (over time) of the two random variables under study; consequently, the selection of this neighbourhood needs to be carefully selected. In this case the selection was based on the rule of thumb given by Silverman (1986) and Hardle et al. (2004).

Another limitation of this technique is that it just deals with zero-mean bivariate non-stationary time series; consequently, an extension of this research is to generalise this technique so it would be capable of dealing with a bivariate time series which is non-stationary in respect of both its mean and variance.

As further recommendations, it is important to observe that other non-parametric techniques based either on rank correlations, such as the Spearman's correlation

(DiNardo and Tobias, 2001; Gibbons and Chakraborti, 1992; Nelsen, 1999), or based on the use of higher order moments and non-parametric characterisation of more general dependence structure (Wolff, 2005), could be used for capturing the dependence structure and the selecting of an appropriate copula.

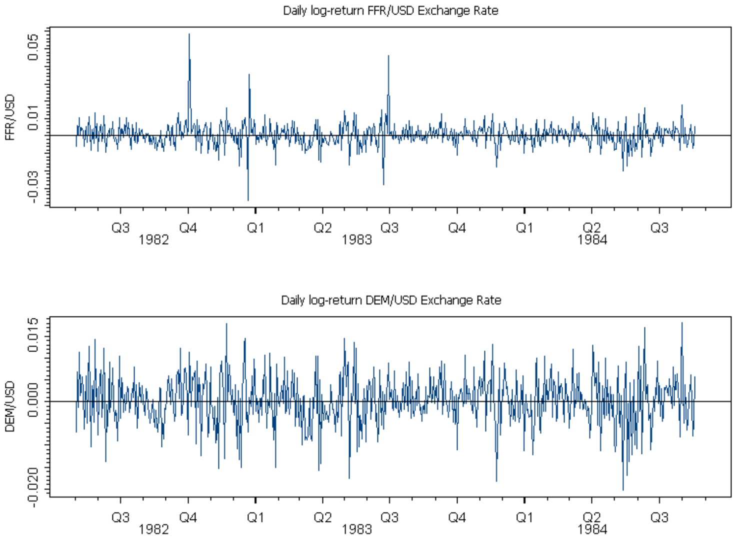
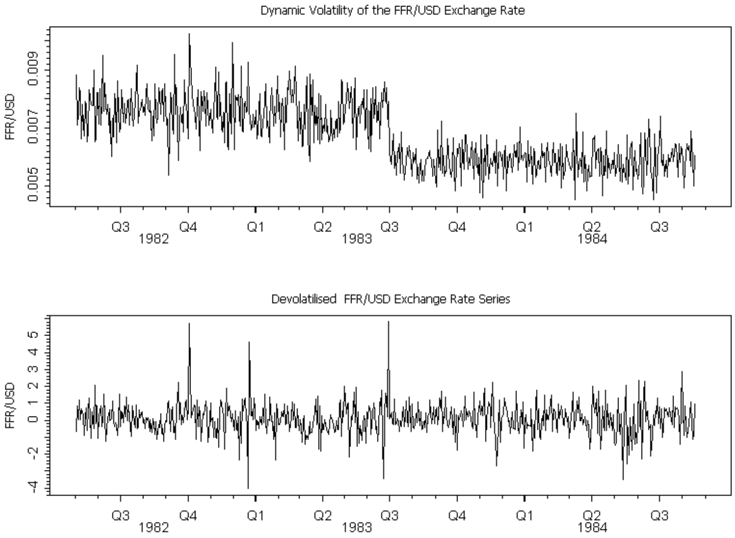


Figure 97. Time series of the daily log-returns of both FFR/USD and DEM/USD exchange rates, respectively, for the period 01/05/82 – 17/08/84. The data comprises 600 daily values.

Statistics	FFR/USD	DEM/FFR
Mean	0.000	0.000
Median	0.00045	0.00048
Std dev	0.0068	0.0058
Skewness	1.159	-0.19
Kurtosis	0.136	0.602

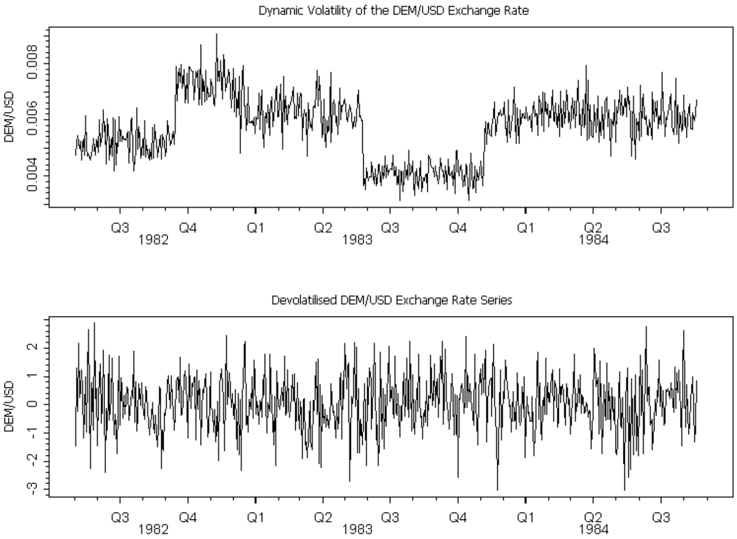
Table 7. Descriptive statistics for the FFR/USD and DEM/USD FX indexes.



Dynamic Volatility of the FFR/USD Exchange Rate

Devolatilised FFR/USD Exchange Rate Series

Figure 98. Plot showing the dynamic volatility and de-volatised series of the FFR/USD exchange rates for period 01/05/82 – 17/08/84.



Dynamic Volatility of the DEM/USD Exchange Rate

Devolatilised DEM/USD Exchange Rate Series

Figure 99. Plot showing the dynamic volatility and de-volatised series of the DEM/USD exchange rates for period 01/05/82 – 17/08/84.

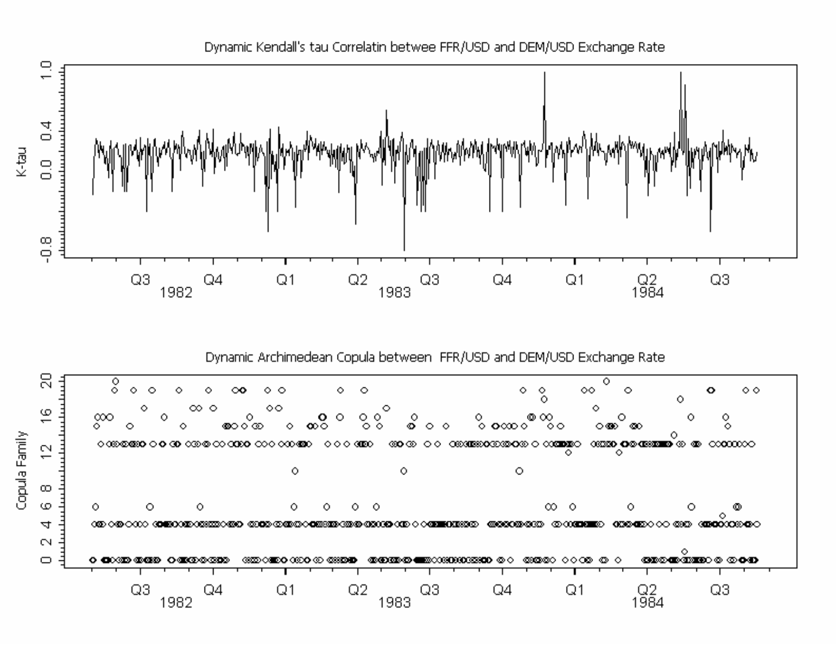


Figure 100. Dynamic Kendall’s tau coefficient of correlation (above) and dynamic Archimedean copula (below) between FFR/USD and DEM/USD exchange rates.

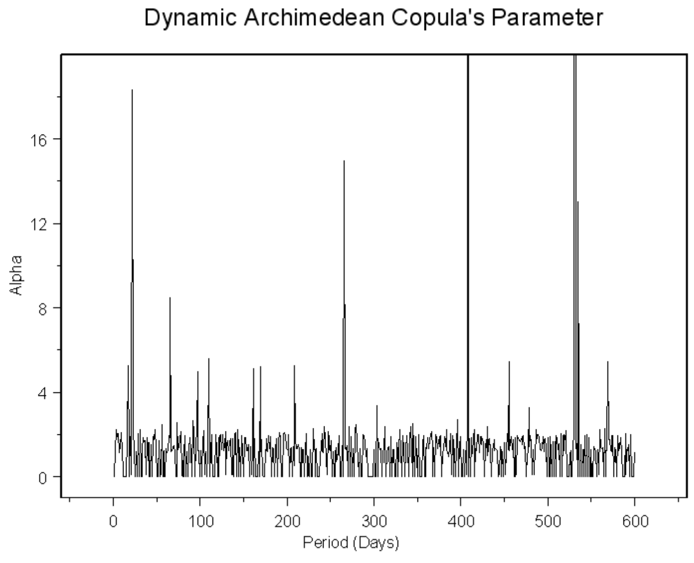


Figure 101. Dynamic Archimedean copula parameter.

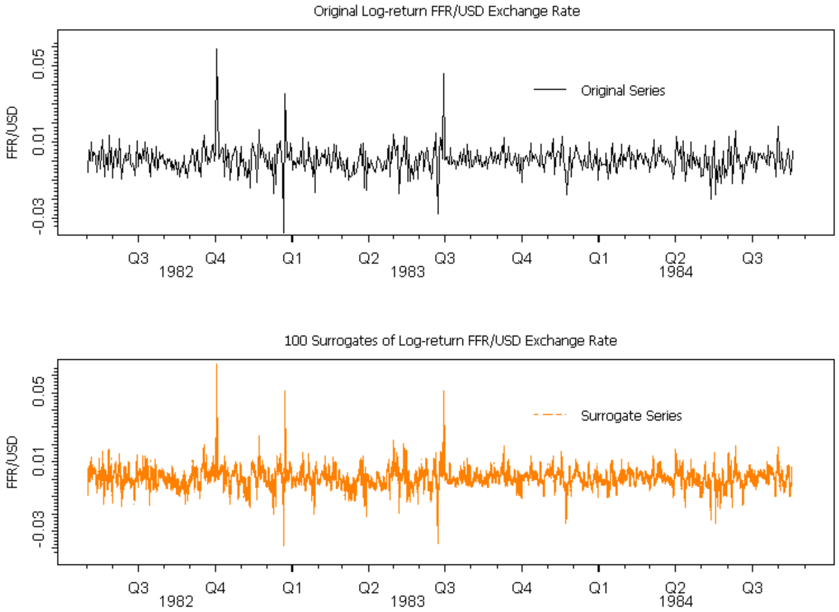


Figure 102. Original time series (top) and 100 surrogate series (bottom) of the FFR/USD exchange rate.

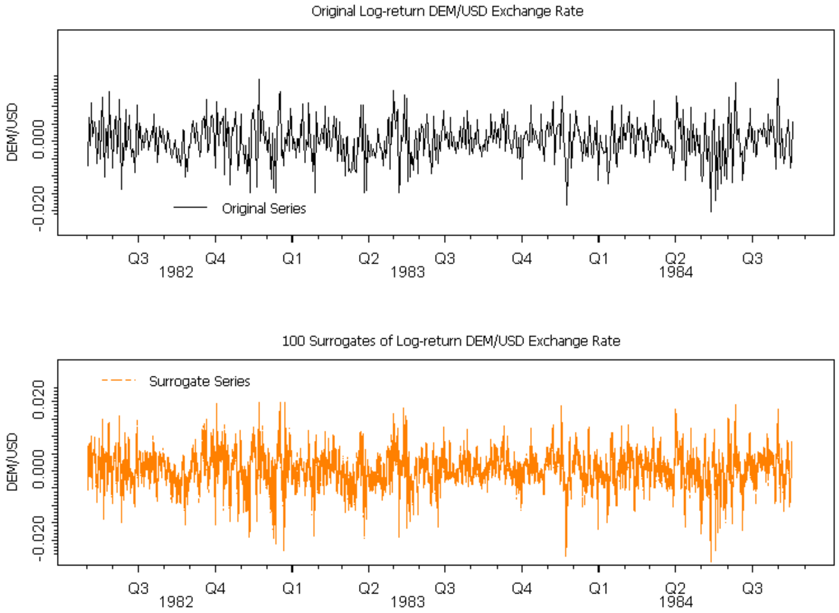


Figure 103. Original time series (top) and 100 surrogate series (bottom) of the DEM/USD exchange rate.

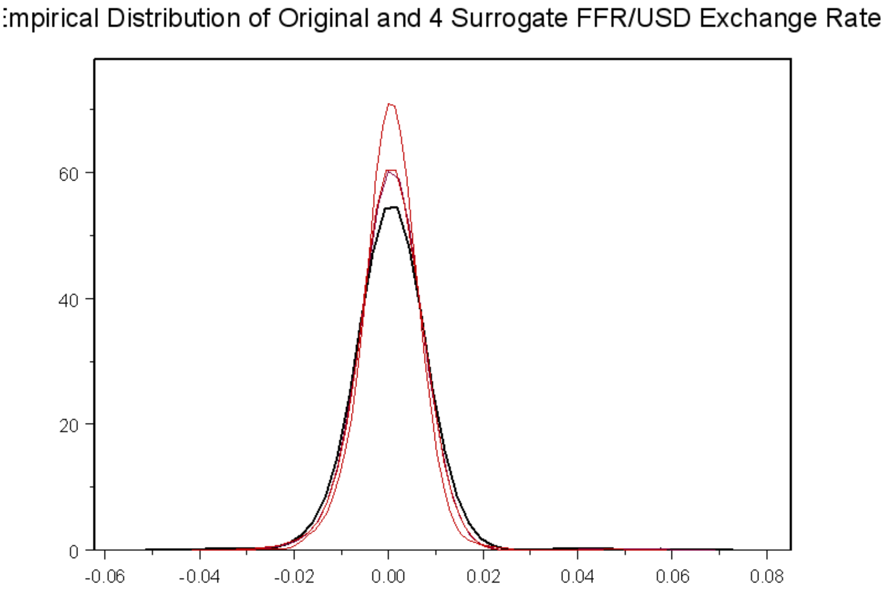


Figure 104. Empirical distribution function of original (black) and 4 surrogate series (red) of FFR/USD exchange rate.

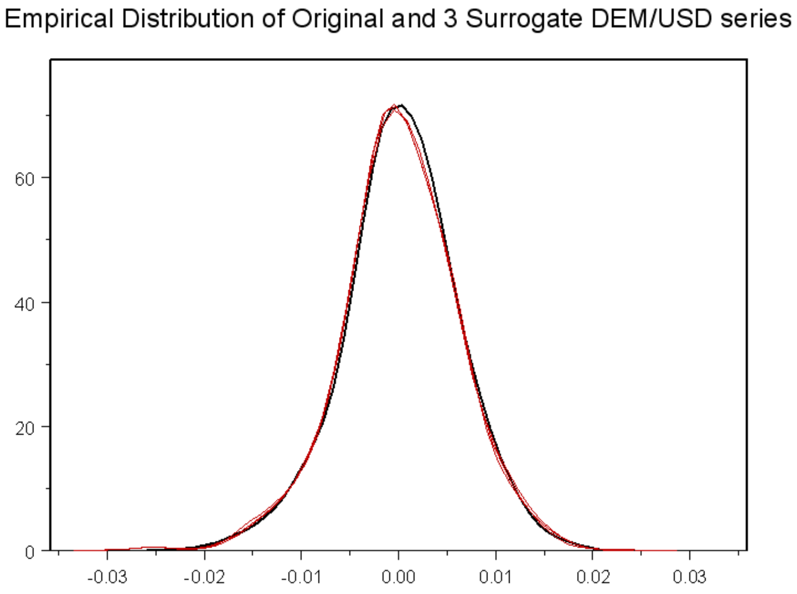


Figure 105. Empirical distribution function of original (black) and 4 surrogate series (red) of DEM/USD exchange rate.

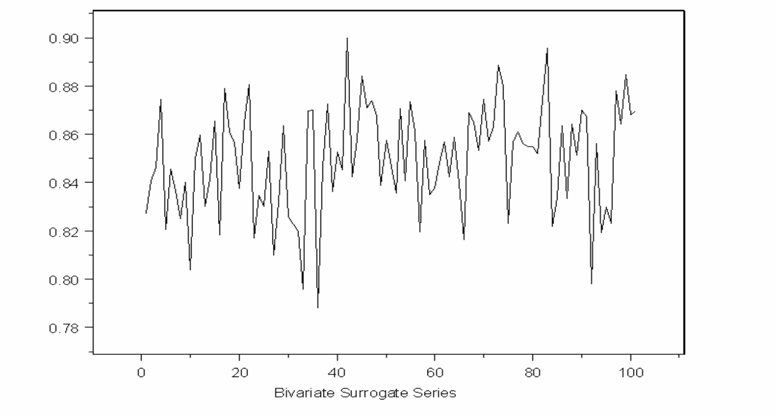


Figure 106. Linear cross-correlation between original bivariate process and 100 bivariate surrogate series of FFR/USD and DEM/USD exchange rates.

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Appendix A: List of Archimedean copula functions

In this research, 20 Archimedean copulas were used to select an appropriate one when performing dynamic analysis. The table indicating the structure of these 20 family of copulas are similar to the ones shown in Nelsen (Nelsen, 1999) and Matteis (Matteis, 2001), so the reader is directed to these two authors for a more detailed analysis.

Appendix B: Sampling a bivariate copula function - Conditional copula

The task of this section is that one of the generating pairs (u,v) of observations of uniformly distributed, in $[0,1]^2$, random variables U and V whose joint distribution function is the Archimedean copula $C_\alpha(u,v)$ (see Appendix A). In this case, the method of conditional copula (Rank, 2000; Nelsen, 1999; Matteis, 2001; Joe, 1997; DiNardo and Tobias, 2001) will be used.

Let C_U^α be the parametric conditional Archimedean copula, with parameter α, for the random variable V at a given value $U = u$ defined as

$$C_U^\alpha(v) = P(V = v \mid U = u) = \frac{\partial C_\alpha(u,v)}{\partial u},$$

where $C_\alpha(u,v)$ is a given parametric Archimedean copula. Observe also that because the properties of the copula (see Nelsen for a more detailed analysis) the following theorem is true.

Theorem 1 Let $C_\alpha(u,v)$ be a copula function. For every $v \in [0,1]$, the partial derivative $\partial C_\alpha(u,v) \big/ \partial u$ exist for almost all (in the sense of Lebesgue measure) $u \in [0,1]$. For such u and v one has

$$0 \le \frac{\partial C_\alpha(u,v)}{\partial u} \le 1 .$$

To generate a pair (u,v) from the given copula $C_\alpha(u,v)$ the following steps need to be performed.

- Generate two independent uniform pseudo random numbers $u, w \in [0,1]$; u is already the first number we are looking for.
- Compute the inverse function of C_U^α. In general, it will depend on the parameters of the copula, in this case α, and on the value u, which can be seen, in this context, as an additional parameter of C_U^α. Set $v = Inverse\left(C_U^\alpha(w)\right)$ to obtain the second value v.

Note, however, it may happen that the inverse function of the copula, that is, $Inverse\left(C_U^\alpha(w)\right)$, cannot be calculated analytically. In this case, numerical algorithms need to be used to determine the value of v.