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AN EXPLORATION OF YOUNG STUDENTS' ABILITY TO GENERALISE FUNCTION TASKS

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The *Early Years Generalising Project* involves Australian students, Years 1–4 (age 5–9), and explores how the students grasp and express generalisations. This paper focuses on the data collected from clinical interviews with Year 3 and 4 cohorts in an investigative study focussing on the identification, prediction and justification of function rules. It reports on students attempts to generalise from function machine contexts, describing the various ways students express generalisation and highlighting the different levels of justification given by students. Finally, we conjecture there are a set of stages in the expression and justification of generalisations that assist students to reach generality within tasks.

The *Early Years Generalising Project* (EYGP)[1] is a series of cross-sectional studies of cohorts of students from Year 1 to Year 4 (age 5 to 9) that aims to build theories regarding young students' ability to grasp and express generalisations, the two components of the act of generalisation in terms of Radford, 2006). Each cross-sectional study covers a particular context and form of generalisation (e.g., growing patterns and pattern rules, equivalence and equation principles, operations and arithmetic processes and structures). Each study has two stages: (a) exploration—an initial stage where a small sample of students (n=5) from each Year level participate in one-on-one clinical interviews; and (b) validation—a final stage where, as a result of these interviews, conjectures were posed and tested in one-on-one semi-structured interviews conducted with a further cohort of 20 students from each Year level, selected to represent a wide range of academic abilities and cultures.

This paper presents a single aspect of the project; an exploration of how Year 3 and 4 students (age 7 to 9) express and justify generalisations for the context of input-output changes using function machines and the form function rules. It covers two year levels of the initial stage of the cross-sectional study on function machines.

Context

For EYGP, mathematics consists of relating and transforming things (numbers, shapes, variables) with relationships and transformations being two ways of looking at the same idea (Scandura, 1971), and the power of mathematics being the way relationships and

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transformations on their own or together give rise to generalisations (Warren, 2005). Functional thinking emerges from the transformational perspective but can be understood in relationship terms, and is the ability to identify the rules that relate two or more varying quantities (Smith, 2008).

There are some studies that suggest that young students can think functionally and generalise in functional situations. For example, Blanton and Kaput (2005) found that students can engage in co-variational thinking as early as Kindergarten and use t-charts and express rules in Years 3 to 5, while Cooper and Warren (2008) found that Years 3 and 4 students can generalise rules for function contexts. However, there is still little known about how young students' identify and generalise function rules. Most studies of functional thinking have focused on middle years' students and explored functions represented as growing patterns (e.g., Lannin, 2005, Radford, 2006). These studies require students to coordinate two variables where one is explicitly represented (e.g., the visual representation of the growing pattern) and the other variable is more abstract (e.g., the position of each term). By focusing on function machines and input-output changes, this paper explores the question that, if we represent both variables and the function action more explicitly, does this assist students to reach more explicit generalisations?

Studies with older students with the focus on growing patterns have identified the different approaches students use when completing generalisation tasks. Harel (2001) identified two approaches: (a) results generalisation where a generality is developed from a few examples usually involving trial and error; and (b) process generalisation where a generality is developed and justified when considering progression across many steps. This classification is supported by Radford (2006) who has labelled the two approaches as naive induction and generalisation and Lannin (2005) who has labelled them non-explicit and explicit. To investigate this classification in younger students, this paper also explores the extent to which young children can justify their generalisations.

Theoretical framework

Underpinning this research project is the theoretical perspective of semiotics. Mathematics has been depicted as an intrinsic symbolic activity which is achieved through communicating using oral, bodily, written and other signs (Radford, 2006). The discipline of semiotics is based on perceivable signs that assist understanding of the mathematics processes of thought, symbolisation and communication. Of particular importance to this paper is the use of body and language, seen best through the physical activity of students as they interact with artefacts (Sabena, 2008). Additionally, studies have noted that cognition is strongly related to the use of the body (Lakoff & Núñez, 2000). It was this framework that drove the construction of the activities and framed the data analysis.

Method

Ten students from Years 3 and 4 (4 males and 6 females with an average age of 8.5 years) were selected to be interviewed in the initial exploration stage of this study. The students were from a middle socio-economic school in the outer suburbs of a major city and had a range of academic abilities and cultural backgrounds. The interviews consisted of 6 tasks; two having a language focus, one having a geometry focus, and three having a number focus. The aim of the tasks was to probe students' understanding of functions. The interview was video recorded and was of approximately 20 minutes duration. The students were presented with activities involving concrete materials and whole body movement starting from unnumbered situations and moving to numbered situations. Table 1 presents the six tasks, each tasks function rule, and an example of the input and output values for each rule.

Table 1. Example of tasks given to students.

	Unnumbered situations situations				Numbered		Numbered situations					
Task	Language (1)		Language (2)		Shape (3)		Number (4)		Number (5)		Number (6)	
Rule	Add 'ip'		Add 'ap'		Make it thinner and smaller		Add two		Subtract three		Double	
Example	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
	T	Tip	M	Map	Red, large, thick triangle	Red, large, thin triangle	5	7	10	7	4	8

Initially, students were introduced to a cardboard box function machine called Rosie. The input and output values were presented on cards or as physical shapes. The interview began with the first language task—Language (1). Each student was shown a letter and asked to place it into Rosie's ear (input) and then the researcher produced the output card from the opposite ear (output). This occurred for three input numbers. Then they were asked to predict the output value for given input values. Each student was then asked to identify the rule.

The questions posed were contingent on the responses given by the student. After the first question, depending on their responses, students were either given further examples or were asked to predict output values for given input values. They were asked then to predict input values for given output values and to identify the reverse rule. The researcher asked students to justify their answers and express the rule and its inverse in general terms. This process was repeated for each task. In practice, the process mirrored an "acting out" of input-output tables (t-tables) and identifying the relationship between the corresponding pairs of values in the table. From a semiotic perspective, the signs were the cards and kinaesthetic movement.

All video recordings were transcribed with attention paid to both the students' verbal responses and their manipulation of the concrete materials, in particular how students engaged with the signs and interpreted these signs as they identified the function. The data was analysed by two researchers and member checks were performed. Semiotics has been used throughout the research project to analyse the data. Within this particular study, the data sets have emerged out of the semiotic analysis conducted. The interpretation of actions are not included in this paper, but if interested please refer to Warren, Miller, and Cooper (2011).

Results and discussion

The data associated with each task is organised into four sections, namely, the student's ability to correctly predict: (a) output values from given input values, (b) the function rule, (c) input values from given output values, and (d) the inverse function rule.

Table 2 presents the tasks together with the frequency of students who were successful in each section.

Table 2. Frequency of student's correct responses to six tasks.

Tasks	Language tasks		Shape task	Number tasks		
	(1)	(2)	(3)	(4)	(5)	(6)
Rule	Add 'ip'	Add 'ap'	Make it smaller & thinner	Add 2	Subtract 3	Doubling
Predict output	8	8	6	10	9	8
Identify output rule	8	8	8	9	8	8
Predict Input	10	8	7	8	8	8
Identify Input (inverse) rule	7	7	6	8	7	4

The results indicate that students could predict the output card for given input cards when asked. The shape task was the only task in which students appeared to have difficulty and this pertained to their inability to describe the attributes of the particular shape (colour, size, thickness).

At least 80% of students could identify the rule Rosie was using to create the output value. However, students were not always able to identify the input rule (inverse rule). This was particularly so for the last number task (doubling) as students did not appear to have the mathematical language to describe the action of halving or dividing by two.

The students were then asked three questions to explore their ability to generalise the three number tasks.

- First, they were asked to pick the largest number they knew as an input value and identify the corresponding value that would come out of the function machine. For the purposes of this study this has been labelled a quasi generalisation (Cooper & Warren, 2008, adapted from Fujii and Stephens', 2001, notion of quasi-variable).
- Second, the students were given a fictitious number (e.g., finky) as the input value and asked to predict the output value.
- Third, their ability to inverse the process was also probed by asking them if 'finky' came out what value would they put in the machine.

These questions were included to determine if the student could generalise the rule beyond the use of numbers and move to a more abstract understanding that entailed the use of variables.

Table 3, below, identifies students' responses to each of these three questions for each of the tasks. The tick indicates that their quasi-generalisation was correct and the written text identifies the rules they predicted for the fictitious number 'finky'.

Table 3. Student's success in quasi generalisation and generalising the number tasks.

Student	Plus 2 (number task 4)			Subtract 3 (number task 5)			Double (number task 6)		
	Quasi	Finky in	Finky out	Quasi	Finky in	Finky out	Quasi	Finky in	Finky out
S1	✓	Take out the inky	nr	✓	nr	nr	✓	nr	nr
S2	✓	2Finky	Finky	✓	00Finky	3finky	✓	2finky	nr
S3	✓	2Finky	-2finky	✓	-3finky	3finky	✓	Finky2	nr
S4	✓	Finky2	Finky-2	✓	finky-3	finky3	✓	Double finky	Half finky
S5	nr	nr	nr	nr	nr	nr	nr	nr	nr
S6	✓	It will turn into a 2	Finky-ky	✓	nr	nr	✓	nr	nr
S7	✓	Finky add 2 letters	Finky take 2 letters	nr	finky -3 letters	Finky plus 3 letters	nr	nr	nr
S8	✓	K + 2 = 7 therefore k=5	p –2 = 4 therefore p=6	✓	n – 3 = 16 therefore n = 19	N + 3 =19 therefore n = 16	✓	Double q	Halve q
S9	✓	nr	nr	✓	nr	nr	nr	nr	nr
S10	✓	Finky + 2	Finky has to go down by 2	✓	Finky – 3	Frisky + 3	✓	Finky x 2	Finky divided in half

Note: nr – no response

Of the students who were asked to generalise the 'add two rule' using the word 'finky', 2 students were successful in expressing the generalisation. The other students would either talk about the generalisation in regard to adding two letters or express it as 'finky2' without using the mathematical operation involved with the function. S8 required a value for the variable and therefore he used expressions that incorporated single letters.

Nine students' generalisations aligned with Harel's (2001) process generalisation (showing generalisation across a number of steps) or Lannin's (2005) explicit generalisation (linking the dependent variable with the independent variable). The different levels of process/explicit generalisation tended to be related to misunderstandings of the notation system used to represent variables and expressions involving operations. Many of these misunderstanding reflected the categories identified by Küchemann (1981): particularly *Letter as object*, *Letter as specific unknown*, *Letter as generalised number*, and *Letter as variable*. It did not seem that the students were engaging in 'guess and check' either in the initial stages of identifying the rule or in "whole-object" strategies as identified in past research involving growing patterns (e.g., Lannin, 2005; Radford, 2006).

Table 4 presents the levels of expressions for generalisation together with examples of each descriptor for each level. Statements such as 'finky2' were accompanied by utterances such as "You add 2, it is finky2", which aligns with adding two to 50 and obtaining 52. In all there were 60 responses related to describing the generalisation (6 per student).

Table 4. Levels of expression for explicit generalisations together with frequency of student usage.

Level	Descriptor	Example (+2 rule)	Frequency
1	No expression	R: What if I had a made up number like finky and put that into Rosie. What would come out? S: A donkey. R: What do you have to do with it? What does the machine do to it? S: I don't know.	25
2	Letter as object	R: So what do you think would happen to finky? S: Finky add two letters	7
3	Letter as specific unknown	S: It is a K so K plus 2 is & so K is 5.	4
4	Letter as generalised number or variable	R: What do you think would happen if I put in a number called finky? S: 2 Finky you add two.	12
		R: What if I put in a number like finky? What would come out? S: Finky add 2.	12

As indicated in the results, 40% of the students' responses (n=24) incorporated the use of letters as generalised numbers or as variables. This was accompanied with students reiterating that 'finky' meant any number. Most of the responses that were considered as Level 1 responses were proffered by three students, S1, S5, and S9. From the results, stages of expression of justification were hypothesised. These stages relate to the use of numbers and unknowns in the students' general statements, and reflect the stages proffered by past research (e.g., Lannin, 2005). Table 5 presents the three stages with the associated exit points of each student.

Table 5. Stages of expression of justification.

Stage		Descriptor	Exit point
1	Numeric evidence (countable numbers)	Used small countable numbers to justify the rule	S5, S7, S9
2	Quasi - generic evidence (uncountable numbers)	Used quasi-variables to justify the rule	S1, S6, S8
3	Generic evidence (algebraic expression)	Used letter notation to justify the rule	S2, S3, S4, S8, S10

This research makes the distinction between using large numbers to justify generalisations and using algebraic notation. This reflects the distinction that Fujii and Stephens (2001) make with regard to the use of variables and the quasi-generalisation of

Cooper and Warren (2008). We conjecture that for young students, moving from familiar numeric situations to using large uncountable numbers represents a leap in understanding. It shows that students are moving from a simple computational situation to evidencing an understanding of the applicability of that computation across the number system.

Conclusion and implications

This research presents three main tentative conclusions. First, young students can engage in activities that require them to express and justify generalisations. This result suggests that there is a need for young students to experience functional thinking activities within the classroom to develop higher levels of mathematical understanding. It would be suggested that kinaesthetic activities that link directly to the learning context of the student would be beneficial. The level of thinking they exhibited mirrors that shown in past research in growing patterns with older students. In this instance though there is one distinctive difference in these students' responses which is the absence of Lannin's (2005) terms of non-explicit generalisations or recursive thinking, building on the previous term or terms in the sequence to determine subsequent terms. We suggest that this is a result of how the activity was constructed where the signs for the input and output were explicit (represented as input and output cards) and the linking of the data sets was accompanied by physical movement. In addition the input numbers were randomly selected thus ensuring that there was no implicit relationship in one data set (e.g., the input or output numbers).

Second, we conjecture that young students' ability to reach generalisations was assisted by the types of activities that were selected and the way they were presented to the students. The crux of problems involving functional situations is the need to coordinate two data sets, the independent and dependent variables and identify the relationship between these sets. The activities for this research were deliberately chosen so that this relationship was transparent. From a semiotic perspective the signs for each were visible and required the students to be actively involved in their creation. Blanton and Kaput (2005) also chose tasks where the variables were explicitly related, for example, the number of eyes and tails on puppy dogs, and hence the students demonstrated success in this task. In addition, the EPGP study, the function or change process was represented kinaesthetically by gesturing with hands across the front of the function machine. This assisted students to focus on the underpinning concept embedded in all of these activities, which is co-variational thinking.

Third, we also conjecture that the context for growing patterns in previous studies is restrictive and abstract. The position of each term as one of the variables is not transparent and we conjecture this contributes to the use of guess and check and recursive strategies. Additionally, in past research students have been asked to engage in the exploration of functional problem solving situations with little prior experience in co-variational thinking. This adds to their difficulties. Our research suggests that young students can deal with co-variational situations as long as both variables are explicitly represented and the rule is clear for students. The tasks presented in this study focus on the relationships within the function, that is, it is not obscured by other aspects as it is in patterning. When using examples such as patterning sequences, students tend to 'run along' the pattern instead of recognising the covariant relationship between pattern terms and their positions. Additionally, cards were displayed to the students in a random sequence forcing students to focus on the relationship between the input and output (horizontal relationship) rather than on the relationship of just the output cards (vertical relationships).

This paper has focused on students' attempts to generalise from function machine contexts, describing the various ways students express generalisation. Furthering the conjectures presented the *Early Years Generalising Project* is continuing to further investigate functional thinking with larger cohorts of students.

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