Practical Stability with Respect to Model Mismatch of Approximate Discrete-Time Output Feedback Control

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***Abstract*— This paper establishes asymptotic stability results for stochastic discrete-time output feedback control problems involving mismatch between the exact system to be stabilised and the approximating system used to design the controller. The stability results established are in the sense of an asymptotic bound on the expected error bias induced by the model approximation. Importantly, the stability results established here do not require the approximating system to be of the same model type as the exact system. An example is presented to illustrate the nature of the presented results.**

I. INTRODUCTION

In most realistic situations, control design is performed on the basis of system models that are not known with complete certainty or a true system that is too complex to be directly considered in the design process. Sometimes the uncertain nature of the system description is incorporated via robust design techniques that include explicit description of the uncertainty [1]. However, in many other situations, controllers are designed on the basis of a model that closely approximates the true system in some sense. This is the situation considered in this paper and we investigate practical stability of output feedback controller designed on the basis of an approximating model.

Output feedback stabilisation problems have previously been considered by many control researchers, for example [7]–[11]. Unfortunately, much of the previous work has assumed that knowledge of the exact system is available for controller design, and has not considered system stability under approximate output feedback control.

More recently, control design problems involving mismatch between true and design models have been investigated in a number of important situations [2]–[6]. In [2], practical stability of approximating state-feedback controller is established for sampled-data control problem. In [5], [6], this type of practical stability result is extended to infinite horizon, finite horizon and receding horizon control designs. There has also been some preliminary extension of these results to output feedback control (but under assumptions of neither measurement nor process noises being present) [12]. In recent work, these deterministic practical stability results have been extended to a stochastic setting to establish stability results for approximate filtering approaches [19].

In this paper, we consider a more general model-mismatch output feedback control problem by investigating practical stability of the combined filtering and stochastic control dynamics with respect to modelling errors. Under mild conditions, we establish asymptotic bounds on the expected performance error when the true stochastic system is under approximate output feedback control. Practical stability results are also established (in a manner similar to the approach of [2]).

This paper is structured as follows. In Section II, our nominal dynamics, our information state concepts, and our modelling approximations are presented. In Section III, we establish the asymptotically bounded error and practical stability results for stochastic output feedback control. Section IV presents an example mismatch output feedback control problem. Conclusions are then presented in Section V.

II. PROBLEM FORMULATION

A. Dynamics

Consider a nonlinear discrete-time system described by a state process $x_k \in \mathbb{R}^n$ and a measurement process $y_k \in \mathbb{R}^m$, for time-step $k > 0$,

$$\begin{aligned} x_k &= f^e(x_{k-1}, u_{k-1}) + v_k \\ y_k &= c^e(x_k) + w_k \end{aligned} \tag{1}$$

where x_0 has *a priori* distribution σ_0, $u_k \in U \subset \mathbb{R}^d$, $f^e(\cdot) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$, and $c^e(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$. Here, $v_k \in \mathbb{R}^n$ and $w_k \in \mathbb{R}^m$ are sequences of independent and identically distributed *i.i.d* random variables with densities $\phi_v(\cdot)$ and $\phi_w(\cdot)$, respectively. We will assume that v_k, w_k, and x_0 are mutually independent for all k. In this paper, we will use $|x_k|_1$ to denote 1-norm of x_k. Throughout this paper, we also use the shorthand $x_{[\ell,m]}$ to denote the state sequences $\{x_\ell, \dots, x_m\}$. We likewise define $y_{[\ell,m]}$, $v_{[\ell,m]}$, and $w_{[\ell,m]}$.

We consider all processes to be defined on a probability space $(\Omega, \mathcal{F}, P)$ where Ω is defined to consist of all infinite sequences $\{x_0, \dots, x_k, \dots; y_1, \dots, y_k, \dots\}$ (with elements $\omega \in \Omega$), $\mathcal{F}$ is defined to be a σ-algebra generated by these sequences, and P will be a probability measure given by Kolmogorov extension theorem applied to these sequences [15]. We will let $\mathcal{Y}_{[\ell,m]}$ denote the complete filtration generated by the sequence $y_{[\ell,m]}$, see [13, p. 18].

B. Normalised Information State

We now introduce some information state concepts that describe our estimation operations. Consider the space $L^\infty(\mathbb{R}^n)$ and its dual $L^{\infty *}(\mathbb{R}^n)$ which includes $L^1(\mathbb{R}^n)$ (see [14] for an introduction into vector space concepts). We will introduce the $\langle \cdot, \cdot \rangle$ notation to denote the operation of $\xi(\cdot) \in L^1(\mathbb{R}^n)$ and $\psi(\cdot) \in L^\infty(\mathbb{R}^n)$ as $\langle \xi, \psi \rangle \triangleq \int_{\mathbb{R}^n} \xi(x)\psi(x)dx$.

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We will also introduce the L_1 norm on functions $\xi(\cdot) \in L^1(\mathbb{R}^n)$ as $||\xi(\cdot)||_1 \triangleq \int_{x\in\mathbb{R}^n} |\xi(x)|dx$ [14].

Let $\bar{L}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ denote functions in $L^1(\mathbb{R}^n)$ that have L_1 norm equal to 1 in the sense that $\bar{L}^1(\mathbb{R}^n) \triangleq \{\xi(\cdot) : \xi(\cdot) \in L^1(\mathbb{R}^n) \text{ and } ||\xi(\cdot)||_1 = 1\}$. We can now define a normalised information state process $\sigma_k^e(\cdot) \in \bar{L}^1(\mathbb{R}^n) : \mathbb{R}^n \to \mathbb{R}$, based on the exact model, by

$$\langle \sigma_k^e, \psi \rangle = E\left[\psi(x_k) \middle| \mathcal{Y}_{[1,k]}, \sigma_0\right] \tag{2}$$

for all $k > 0$, and all test functions $\psi(\cdot) \in L^\infty(\mathbb{R}^n)$, where $\sigma_0 \in \bar{L}^1(\mathbb{R}^n)$ is the *a priori* distribution of x_0. This definition highlights that the normalised information state $\sigma_k^e(\cdot)$ can be interpreted as a conditional probability distribution function of x_k given measurement sequences $y_{[1,k]}$ and *a priori* distribution σ_0. The evolution of this normalised information state process σ_k^e is given by

$$\sigma_{k+1}^e = \frac{1}{N_k^e} \Sigma^e(u_k, y_{k+1})\sigma_k^e \tag{3}$$

where $\sigma_0^e \in L^1(\mathbb{R}^n) = \sigma_0$ and $\Sigma^e : L^{\infty *}(\mathbb{R}^n) \to L^{\infty *}(\mathbb{R}^n)$ is the linear operator defined as

$$\Sigma^e(u,y)\sigma^e(x) = \frac{\phi_w^e(y - c^e(x))}{\phi_w^e(y)} \int_{\mathbb{R}^n} \phi_v^e(x - f^e(z,u))\sigma^e(z)dz \tag{4}$$

Here, $N_k^e = ||\Sigma^e(u_k, y_{k+1})\sigma_k^e||_1$ is the normalisation factor. Throughout this paper, we will write $\sigma^e_{k|[\ell,k],\sigma_{\ell-1}}(\cdot)$ to denote the normalised information state $\sigma_k^e(\cdot)$ after evolution by measurements $y_{[\ell,k]}$ from initial distribution $\sigma_{\ell-1}$ at time $k = \ell - 1$. Also, we define $\sigma^e_{0|[1,0],\sigma_0} \triangleq \sigma_0$.

C. Parameterised Class of Approximating Model

Let $h > 0$ parameterises a class of approximating models (for example, h might be spatial discretisation size). For each h, let us now consider the following approximating model of x_k and y_k:

$$\begin{aligned} x_k &= f^h(x_{k-1}, u_{k-1}) + v_k^h \\ y_k &= c^h(x_k) + w_k^h \end{aligned} \tag{5}$$

for $k > 0$, where x_0 has *a priori* distribution σ_0^h, $u_k \in U \subset \mathbb{R}^d$, $f^h(\cdot) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$, and $c^h(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$. Here, $v_k^h \in \mathbb{R}^n$ and $w_k^h \in \mathbb{R}^m$ are *i.i.d.* random variables with densities $\phi_v^h(\cdot)$ and $\phi_w^h(\cdot)$, respectively. The random variables v_k^h, w_k^h, and x_0 are assumed to be mutually independent for all k. Using the previously introduced sample space Ω, with elements ω, we can use the approximating model (5) to infer new probability distribution functions and then use the Kolmogorov extension theorem to generate a measure P^h corresponding to the approximating model, see [15] for more details. This allows us to define an appropriate expectation operation $E^h[\cdot]$.

Throughout this paper, we will assume that P^h is absolutely continuous with respect to P, see [15, p. 413]. We will write $P^h \gg P$ to denote this absolute continuity condition.

Similar to the true system (1), we define a normalised information state process $\sigma_k^h(\cdot) \in \bar{L}^1(\mathbb{R}^n) : \mathbb{R}^n \to \mathbb{R}$, for the approximating model, as

$$\langle \sigma_k^h, \psi \rangle = E^h\left[\psi(x_k) \middle| \mathcal{Y}_{[1,k]}, \sigma_0^h\right] \tag{6}$$

for all $k > 0$, and all test function $\psi(\cdot) \in L^\infty(\mathbb{R}^n)$, where $\sigma_0^h \in \bar{L}^1(\mathbb{R}^n)$ is the *a priori* distribution of x_0. Likewise, the evolution of this normalised information state process is given by $\sigma_{k+1}^h = \frac{1}{N_k^h}\Sigma^h(u_k, y_{k+1})\sigma_k^h$ where $\sigma_0^h \in L^1(\mathbb{R}^n)$. Here, $\Sigma^h : L^{\infty *}(\mathbb{R}^n) \to L^{\infty *}(\mathbb{R}^n)$ is the bounded linear operator which is defined as $\Sigma^h(u,y)\sigma^h(x) = \frac{\phi_w^h(y-c^h(x))}{\phi_w^h(y)} \int_{\mathbb{R}^n} \phi_v^h(x - f^h(z,u))\sigma^h(z)dz$, and $N_k^h = ||\Sigma^h(u_k, y_{k+1})\sigma_k^h||_1$ is the normalisation factor. We will call $\Sigma^h(u,y)$ the *approximating information state filter*.

In the following, we will write $\sigma^h_{k|[\ell,k],\sigma^h_{\ell-1}}(\cdot)$ to denote the normalised information state $\sigma_k^h(\cdot)$ after evolution by measurements $y_{[\ell,k]}$ from initial distribution $\sigma_{\ell-1}^h$ at time $k = \ell - 1$. Also, we will define $\sigma^h_{0|[1,0],\sigma_0^h} \triangleq \sigma_0^h$.

D. Output Feedback Control Design

In this paper, we are interested in the control problem where the exact model (1) is unknown. We will assume that exact system is under an output feedback control solution designed on the basis of the approximating model (5). For this purpose, we will define a control process, u_k, associated with an approximating information state, σ_k^h, as

$$u_k = g^h(\sigma_k^h) \tag{7}$$

where $g^h(\cdot) : \bar{L}^1(\mathbb{R}^n) \to \mathbb{R}^d$ is the control law designed using the approximating system (5). We will call this the *approximating output feedback controller* and we will assume that this is always based on the approximating information state, $\sigma_k^h(\cdot)$.

We will now introduce some notation that helps our presentation. Let $x^e_{m|[\ell,m]}\left(x_{\ell-1}, \sigma_{\ell-1}, \Sigma^h, g^h, v_{[\ell,m]}, w_{[\ell,m]}\right)$ denote the state of the true system (1) at time m starting from $x_{\ell-1}$ with the initial distribution $\sigma_{\ell-1}$, the approximating output feedback controller $g^h(\cdot)$, the approximating information state filter $\Sigma^h(\cdot)$, and the noise processes $v_{[\ell,m]}$ and $w_{[\ell,m]}$ (these noise processes generate the measurement process $y_{[\ell,m]}$). This notation highlights that output feedback control designed on the approximating system (5) is used for output control of the true system model (1). In this paper, we examine the behaviour of such a closed loop system.

To assist with our analysis, we also introduce notation to describe the approximating system under a similar output control solution. We will also write $x^h_{m|[\ell,m]}\left(x_{\ell-1}, \sigma^h_{\ell-1}, \Sigma^h, g^h, v^h_{[\ell,m]}, w^h_{[\ell,m]}\right)$ to denote the state of the approximating system (5) at time m starting from $x_{\ell-1}$ with the initial distribution $\sigma^h_{\ell-1}$, the approximating output feedback controller $g^h(\cdot)$, the approximating information state filter $\Sigma^h(\cdot)$, the noise processes $v^h_{[\ell,m]}$ and $w^h_{[\ell,m]}$ (for the measurement process $y_{[\ell,m]}$).

E. Mixed State

Let $\mathscr{X} \subset \mathbb{R}^n$ be a compact set. To help characterise the behaviour of our closed loop dynamics, we will introduce the following mixed description of the dynamics

$$X_k\left(x_k, \sigma_k, \sigma_k^h\right) = \begin{bmatrix} x_k \\ \sigma_k(\cdot) - \sigma_k^h(\cdot) \end{bmatrix}$$

where $x_k \in \mathscr{X}$ and $\sigma_k, \sigma_k^h \in \bar{L}^1(\mathbb{R}^n)$. This *mixed state* describes both the state dynamics and the relative dynamics of the true and approximating information states. We will also let $\mathbb{X} \in \mathscr{X} \times \bar{L}^1(\mathbb{R}^n)$ denote the set of possible mixed state values in the sense that $X_k(x_k, \sigma_k, \sigma_k^h) \in \mathbb{X}$. For this mixed state we will define a mixed $\mathbb{1}$-norm of the mixed state process as $\left|X_k\left(x_k, \sigma_k, \sigma_k^h\right)\right|_{\mathbb{1}} \triangleq |x_k|_1 + ||\sigma_k(\cdot) - \sigma_k^h(\cdot)||_1$. We highlight that we consider the control problems where the control objective is regulation of $|x_k|_1$ to zero (however, other control objectives can be considered by simple redefinition of the mixed state X_k).

For the purposes of describing the behaviour of the true system under approximate output feedback control, we will define the mixed state $X^e_{m|[\ell,m]}(x_{\ell-1}, \sigma_{\ell-1}, \sigma^h_{\ell-1}, v_{[\ell,m]}, w_{[\ell,m]})$ as

$$\begin{bmatrix} x^e_{m|[\ell,m]}\left(x_{\ell-1}, \sigma^h_{\ell-1}, \Sigma^h, g^h, v_{[\ell,m]}, w_{[\ell,m]}\right) \\ \sigma^e_{m|[\ell,m],\sigma_{\ell-1}} - \sigma^h_{m|[\ell,m],\sigma^h_{\ell-1}} \end{bmatrix}.$$

This mixed state describes the evolution of the true system under the control of information from the approximating system and also describes the relative dynamics of the true and approximating information state (based on measurement from the true model). For this reason, this mixed state describes all the information important in understanding our system under control.

For clarity of presentation, from this point forward, we will suppress the dependency on the noise processes $v_{[\ell,m]}$ and $w_{[\ell,m]}$ and write $X^e_{m|[\ell,m]}(x_{\ell-1}, \sigma_{\ell-1}, \sigma^h_{\ell-1})$ as shorthand. We likewise use the shorthand notation $X^h_{m|[\ell,m]}(x_{\ell-1}, \sigma_{\ell-1}, \sigma^h_{\ell-1}, v^h_{[\ell,m]}, w^h_{[\ell,m]})$ to denote the mixed state

$$\begin{bmatrix} x^h_{m|[\ell,m]}\left(x_{\ell-1}, \sigma^h_{\ell-1}, \Sigma^h, g^h, v^h_{[\ell,m]}, w^h_{[\ell,m]}\right) \\ \sigma^h_{m|[\ell,m],\sigma_{\ell-1}} - \sigma^h_{m|[\ell,m],\sigma^h_{\ell-1}} \end{bmatrix}$$

describing the evolution of the approximating system under approximating output feedback control (in this case, output feedback control designed with knowledge of the system under control). Again, we suppress the dependency of the mixed state on the noise processes $v^h_{[\ell,m]}$ and $w^h_{[\ell,m]}$ and write $X^h_{m|[\ell,m]}(x_{\ell-1}, \sigma_{\ell-1}, \sigma^h_{\ell-1})$ to denote the mixed state described above.

Finally, we write $X^e_{k|[k]}(x_{k-1}, \sigma_{k-1}, \sigma^h_{k-1})$ as shorthand for $X^e_{k|[k,k]}(x_{k-1}, \sigma_{k-1}, \sigma^h_{k-1})$. Similarly, we write $\sigma^e_{k|[k],\sigma_k-1}$, $x^e_{k|[k]}(x_{k-1}, \sigma^h_{k-1}, \Sigma^h, g^h, v_k, w_k)$, and all corresponding quantities with h superscripts.

III. PRACTICAL STABILITY OF OUTPUT FEEDBACK CONTROL

In this section, we establish our main results which describe under what situations both the true system controlled by the approximating output feedback controller and the approximating information state filter will exhibit desirable behaviours.

We will first introduce some important definitions and some required assumptions before we end the section by presenting the main results of this paper.

A. Definitions of Asymptotically Bounded and Practical Stability

Let us say that a function $\gamma(\cdot)$ is of class-$\mathscr{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$. A function of class-$\mathscr{K}$ is also of class-$\mathscr{K}_\infty$ if it is unbounded. Moreover, a function $\beta(\cdot,\cdot)$ is of class-$\mathscr{KL}$ if $\beta(\cdot, t) \in \mathscr{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$ (see [18, Ch.4] for descriptions of system stability involving such functions). We will now introduce definitions of asymptotically bounded error and practical stability.

Definition 1: ((β, N_x, N_σ)-asymptotically bounded in a stochastic sense) Let $N_x \subset \mathbb{R}^n$ and $N_\sigma \subset \mathbb{R}^+$ be open sets containing the origin. A system (1) with the approximating information state filter $\Sigma^h(\cdot)$ and the approximating output feedback controller $g^h(\cdot)$ (some fixed h) is said to be (β, N_x, N_σ)-asymptotically bounded in a stochastic sense if, there exists a $\beta(\cdot,\cdot) \in \mathscr{KL}$ and $\gamma_v(\cdot), \gamma_x(\cdot) \in \mathscr{K}_\infty$ such that, for all $x_0 \in N_x$, all $||\sigma_0 - \sigma_0^h||_1 \in N_\sigma$, and all $k \geq 0$, we have that

$$\begin{aligned} &E_{[1,k]}\left[\left|X^e_{k+1|[1,k+1]}(x_0, \sigma_0, \sigma_0^h)\right|_{\mathbb{1}}\right] \\ &\leq \beta\left(\left|X_0(x_0, \sigma_0, \sigma_0^h)\right|_{\mathbb{1}}, k+1\right) + R \\ &\quad + E_{[1,k]}\left[\gamma_v(|v_{k+1}|_1) + \gamma_x\left(\left|\Delta\hat{x}^h_{k|[1,k],\sigma_0^h}\right|_1\right)\right] \ P\text{-a.s.} \end{aligned} \tag{8}$$

where $E_{[1,k]}[\cdot] \triangleq E[\cdot|x_0, y_{[1,k+1]}, \sigma_0, \sigma_0^h]$ is a P-measure conditional expectation and $\Delta\hat{x}^h_{k|[1,k],\sigma_0^h} \triangleq \hat{x}^h_{k|[1,k],\sigma_0^h} - x_k$, for all $x \in \mathbb{R}^n$, is the previous estimation error. Here, $\hat{x}^h_{k|[1,k],\sigma_0^h} = \int_{\mathbb{R}^n} \sigma^h_{k|[1,k],\sigma_0^h}(x) x dx$. Note that the $\gamma_x\left(\left|\Delta\hat{x}^h_{k|[1,k],\sigma_0^h}\right|_1\right)$ term in (8) seems to be a natural bias term in many problems of interest (but could be replaced by a general expression involving the information state, if required).

We note that the above definition of (β, N_x, N_σ)-asymptotically bounded is related to the following definition of Lyapunov asymptotically bounded.

Definition 2: (Lyapunov asymptotically bounded in a stochastic sense) A system (1) with the approximating information state filter $\Sigma^h(\cdot)$ and the approximating output feedback controller $g^h(\cdot)$ (some fixed h) is said to be Lyapunov asymptotically bounded in a stochastic sense if, there exists a Lyapunov function $V(\cdot) : \mathbb{R}^n \times \bar{L}^1(\mathbb{R}^n) \to \mathbb{R}$, $\gamma_v(\cdot), \gamma_x(\cdot) \in \mathscr{K}_\infty$, a compact set $\mathscr{X} \subset \mathbb{R}^n$, and finite constants $a_1, a_2, a_3 > 0$ such that, for all initial conditions $x_k \in \mathscr{X}$, all $\sigma_k, \sigma_k^h \in \bar{L}^1(\mathbb{R}^n)$, and all $k \geq 0$, we have that

$$a_1\left|X_k(x_k,\sigma_k,\sigma_k^h)\right|_{\mathbb{1}} \leq V\left(X_k(x_k,\sigma_k,\sigma_k^h)\right) \leq a_2\left|X_k(x_k,\sigma_k,\sigma_k^h)\right|_{\mathbb{1}} \quad P\text{-a.s.} \tag{9}$$

and

$$\begin{aligned} &E_k\left[V\left(X^e_{k+1|[k+1]}(x_k, \sigma_k, \sigma_k^h)\right) - V\left(X_k(x_k, \sigma_k, \sigma_k^h)\right)\right] \\ &\leq -a_3\left|X_k(x_k, \sigma_k, \sigma_k^h)\right|_{\mathbb{1}} + R \\ &\quad + E_k\left[\gamma_v(|v_{k+1}|_1)\right] + \gamma_x(\left|\Delta\hat{x}_k^h\right|_1) \qquad P\text{-a.s..} \end{aligned} \tag{10}$$

where $E_k[\cdot] \triangleq E[\cdot|x_k, y_{k+1}, \sigma_k, \sigma_k^h]$ is a P-measure conditional expectation, and $\Delta\hat{x}_k^h \triangleq \hat{x}_k^h - x_k$, for all $x \in \mathbb{R}^n$, is the previous estimation error. Here, $\hat{x}_k^h = \int_{\mathbb{R}^n} \sigma_k^h(x) x dx$.

We highlight that Lyapunov asymptotically bounded in a stochastic sense implies that the mixed state is attracted to the set $E\left[\left|X^e_{k+1|[1,k+1]}(x_0, \sigma_0, \sigma_0^h)\right|_{\mathbb{1}}\right] \leq \frac{1}{a_3}\left(R + E\left[\gamma_v(|v_{k+1}|_1) + \gamma_x\left(\left|\Delta\hat{x}^h_{k|[1,k],\sigma_0^h}\right|_1\right)\right]\right)$ as $k \to \infty$. This is the type of asymptotically bounded behaviour that we will establish in later results.

We highlight that when additional conditions hold, it will be possible to establish practical stability. Consider the following definition.

Definition 3: (Lyapunov practically stable in a stochastic sense) A system (1) with the class of approximating information state filters $\Sigma^h(\cdot)$ and the class of approximating output feedback controllers $g^h(\cdot)$ is said to be Lyapunov practically stable in a stochastic sense if, for any $R > 0$, there exists a $H > 0$, a Lyapunov function $V(\cdot) : \mathbb{R}^n \times \bar{L}^1(\mathbb{R}^n) \to \mathbb{R}$, $\gamma_v(\cdot), \gamma_x(\cdot) \in \mathscr{K}_\infty$ a compact set $\mathscr{X} \subset \mathbb{R}^n$, and finite constants $a_1, a_2, a_3 > 0$ such that, for all $h \in (0, H]$, all initial conditions $x_k \in \mathscr{X}$, all $\sigma_k, \sigma_k^h \in \bar{L}^1(\mathbb{R}^n)$, and all $k \geq 0$, we have that (9) and (10) hold.

B. Assumptions

In the following, we will establish our asymptotically bounded and practical stability results in a stochastic sense for output feedback control in the presence of modelling errors. Our results will be established using some finite error growth properties between the true and the approximating systems. We will now introduce these important definitions.

Definition 4: (Finite filter error over one step) An approximating filter $\sigma^h_{k|[1,k],\sigma_{k-1}}(\cdot)$ (some fixed h) is said to have finite error over one timestep with respect to the true filter $\sigma^e_{k|[1,k],\sigma_{k-1}}(\cdot)$ if, for all initial conditions $\sigma_{k-1} \in \bar{L}^1(\mathbb{R}^n)$ and all $k > 0$, we have that

$$E_{k-1}\left[\left|\left|\sigma^e_{k|[k],\sigma_{k-1}} - \sigma^h_{k|[k],\sigma_{k-1}}\right|\right|_1\right] \leq \rho \qquad P\text{-a.s.} \quad (11)$$

where $\rho > 0$ is finite.

Definition 5: (Finite model error over one step) An approximating model (5) with the approximating information state filter $\Sigma^h(\cdot)$ and the approximating output feedback controller $g^h(\cdot)$ (some fixed h) is said to have finite error over one timestep with respect to the true model (1) with the approximating information state filter $\Sigma^h(\cdot)$ and the approximate output feedback controller $g^h(\cdot)$ if, there exists a compact set $\mathscr{X} \subset \mathbb{R}^n$ such that, for all initial state $x_{k-1} \in \mathscr{X}$, all initial conditions $\sigma_{k-1} \in \bar{L}^1(\mathbb{R}^n)$, and all $k > 0$, we have that

$$E_{k-1}\left[\left|x^e_{k|[k]}(x_{k-1}, \sigma_{k-1}, \Sigma^h, g^h, v_k, w_k) - x^h_{k|[k]}(x_{k-1}, \sigma_{k-1}, \Sigma^h, g^h, v_k^h, w_k^h)\right|_1\right] \leq \alpha \quad P\text{-a.s.} \ (12)$$

where $\alpha > 0$ is finite.

We will now introduce a definition on our approximating model (5) that will be used to establish our main asymptotically bounded and practical stability results.

Definition 6: (Lyapunov asymptotically stable in a stochastic sense with respect to initial conditions) An approximating model (5) with the approximating information state filters $\Sigma^h(\cdot)$ and the approximating output feedback controllers $g^h(\cdot)$ (some fixed h) is said to be Lyapunov asymptotically stable in a stochastic sense with respect to initial conditions if, there exists a Lyapunov function $V(\cdot) : \mathbb{R}^n \times \bar{L}^1(\mathbb{R}^n) \to \mathbb{R}$, $\gamma_v(\cdot), \gamma_x(\cdot) \in \mathscr{K}_\infty$, a compact set $\mathscr{X} \in \mathbb{R}^n$, and a finite constant $a_3 > 0$ such that, for all $x_k \in \mathscr{X}$, all $\sigma_k, \sigma_k^h \in \bar{L}^1(\mathbb{R}^n)$, and all $k \geq 0$, we have that

$$E_k\left[V\left(X^h_{k+1|[k+1]}(x_k, \sigma_k, \sigma_k^h)\right) - V\left(X_k(x_k, \sigma_k, \sigma_k^h)\right)\right] \leq -a_3\left|X_k(x_k, \sigma_k, \sigma_k^h)\right|_{\mathbb{1}} + E_k\left[\gamma_v(\left|v^h_{k+1}\right|_1)\right] + \gamma_x(\left|\Delta\hat{x}_k^h\right|_1) \quad P\text{-a.s.} \quad (13)$$

and the Lyapunov function $V(\cdot)$ also satisfies the following properties: there exists $a_1, a_2 > 0$ such that, for all $X_k(x_k, \sigma_k, \sigma_k^h) \in \mathbb{X}$ and all $k \geq 0$,

$$a_1\left|X_k(x_k,\sigma_k,\sigma_k^h)\right|_{\mathbb{1}} \leq V\left(X_k(x_k,\sigma_k,\sigma_k^h)\right) \leq a_2\left|X_k(x_k,\sigma_k,\sigma_k^h)\right|_{\mathbb{1}} \quad P\text{-a.s.} \quad (14)$$

and there exists a $L > 0$ such that, for all $X_k(x_k, \sigma_k, \sigma_k^h), \bar{X}_k(\bar{x}_k, \bar{\sigma}_k, \bar{\sigma}_k^h) \in \mathbb{X}$ and all $k \geq 0$,

$$V(X_k(x_k, \sigma_k, \sigma_k^h)) - V(\bar{X}_k(\bar{x}_k, \bar{\sigma}_k, \bar{\sigma}_k^h)) \leq L\left|X_k(x_k, \sigma_k, \sigma_k^h) - \bar{X}_k(\bar{x}_k, \bar{\sigma}_k, \bar{\sigma}_k^h)\right|_{\mathbb{1}} \quad P\text{-a.s.}. \quad (15)$$

We highlight that this definition relates to the behaviour of the approximating system under output feedback control (designed with the full knowledge of the system under control). It seems reasonable to require the output feedback control to (at least) stabilise the system that the controller was designed for. Note that condition (13) is expressed under P measure rather than using, as might be expected, the P^h measure defined for the approximate processes (but please see later example of how (13) can be established).

Definition 7: (Finite stochastic mismatch) Consider the $\gamma_v(\cdot) \in \mathscr{K}_\infty$ holding in Definition 6. Let $\Delta\gamma^{h|v}_{k+1} \triangleq \gamma_v(\left|v^h_{k+1}\right|_1) - \gamma_v(|v_{k+1}|_1)$, be a measure of stochastic mismatch between models. We will say there is finite stochastic mismatch between true and approximate models when $E_k\left[\left|\Delta\gamma^{h|v}_{k+1}\right|\right] \leq \epsilon$ where $\epsilon > 0$ is finite.

C. Main Results

We now state an important error bound result.

Theorem 1: Consider a state process x_k and a measurement process y_k generated by the true system (1). Also consider an approximating system (5) with approximate information state filter $\Sigma^h(\cdot)$ and approximating output feedback controller $g^h(\cdot)$ (some fixed h). Assume Definitions 4, 5, 6 and 7 hold. Then the output feedback control solution satisfies (9) and (10). That is, the true system under approximating output feedback control is Lyapunov asymptotically bounded in a stochastic sense (Definition 2), with the bound $R = L(\alpha + \rho) + \epsilon$.

Proof: Let $\mathscr{X}$ be the compact set holding in Definition 6. For all $x_k \in \mathscr{X}$ and all $\sigma_k, \sigma_k^h \in \bar{L}^1(\mathbb{R}^n)$, and all $k \geq 0$,

$$
\begin{aligned}
&E_k\left[V\big(X^e_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)-V\big(X_k(x_k,\sigma_k,\sigma^h_k)\big)\right]\\
&=E_k\left[V\big(X^e_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)-V\big(X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)+V\big(X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)-V\big(X_k(x_k,\sigma_k,\sigma^h_k)\big)\right]\\
&=E_k\Big[V\big(X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)-V\big(X_k(x_k,\sigma_k,\sigma^h_k)\big)\Big]+E_k\Big[\Big|V\big(X^e_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)-V\big(X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)\Big|\Big]\\
&\le -a_3\left|X_k(x_k,\sigma_k,\sigma^h_k)\right|_{\mathbb{1}}+E_k\left[\gamma_v(\left|v^h_{k+1}\right|_1)\right]+\gamma_x(\left|\Delta\hat{x}^h_k\right|_1)\\
&\quad+E_k\left[L\left|X^e_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)-X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\right|_{\mathbb{1}}\right] \qquad P\text{-a.s.}\\
&=-a_3\left|X_k(x_k,\sigma_k,\sigma^h_k)\right|_{\mathbb{1}}+E_k\left[\gamma_v(\left|v^h_{k+1}\right|_1)\right]+\gamma_x(\left|\Delta\hat{x}^h_k\right|_1)+LE_k\left[\left|x^e_{k+1|[k+1]}(x_k,\sigma^h_k,\Sigma^h,g^h,v_{k+1},w_{k+1})\right.\right.\\
&\quad\left.\left.-x^h_{k+1|[k+1]}(x_k,\sigma^h_k,\Sigma^h,g^h,v^h_{k+1},w^h_{k+1})\right|_1+\left|\left|\sigma^e_{k+1|[k+1],\sigma_k}-\sigma^h_{k+1|[k+1],\sigma_k}\right|\right|_1\right] \qquad P\text{-a.s.}\\
&\le -a_3\left|X_k(x_k,\sigma_k,\sigma^h_k)\right|_{\mathbb{1}}+L(\alpha+\rho)+E_k\left[\gamma_v(\left|v^h_{k+1}\right|_1)\right]+\gamma_x(\left|\Delta\hat{x}^h_k\right|_1) \qquad P\text{-a.s.}\\
&\le -a_3\left|X_k(x_k,\sigma_k,\sigma^h_k)\right|_{\mathbb{1}}+L(\alpha+\rho)+E_k\left[\gamma_v(|v_{k+1}|_1)\right]+E_k\left[\left|\Delta\gamma^{h|v}_{k+1}\right|\right]+\gamma_x(\left|\Delta\hat{x}^h_k\right|_1) \qquad P\text{-a.s..}
\end{aligned} \tag{16}
$$

consider the steps in (16). In the 2nd step of (16), we have used that

$$
\begin{aligned}
&V\big(X^e_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)-V\big(X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)\\
&\le\Big|V\big(X^e_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)-V\big(X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\big)\Big|.
\end{aligned}
$$

In the 3rd step, we have applied (13) and (15). The 4th step follows from the definition of the mixed $\mathbb{1}$-norm. In the 2nd last step, we have used (11) and (12). In the last step, we have introduced $\Delta\gamma^{h|v}_{k+1}$ and overbounded this term by its absolute value. The result then holds by setting $R=L(\alpha+\rho)+\epsilon$ and considering the definition of Lyapunov asymptotically bounded in a stochastic sense. ■

Theorem 2: Consider a state process x_k and a measurement process y_k generated by the true system (1). Also consider a class of approximating systems (5) with approximate information state filter $\Sigma^h(\cdot)$ and approximating output feedback controller $g^h(\cdot)$. Assume that, for any $\rho>0$, $\alpha>0$, and $\epsilon>0$, there exists a $H>0$ such that, for all $h\in(0,H]$, Definitions 4, 5, 6 and 7 hold. Then the true system (1) with the class of approximating information state filters $\Sigma^h(\cdot)$ and the class of approximating output feedback controllers $g^h(\cdot)$ is Lyapunov practically stable in a stochastic sense (Definition 3).

Proof: Under the theorem assumptions, for any selected R, there are suitable $\rho>0$, $\alpha>0$, $\epsilon>0$, and $H>0$, so that Theorem 1 can be applied for all $h\in(0,H]$. The theorem statement then follows. ■

IV. Example

In this section, we illustrate the nature of our asymptotically bounded result in an example output feedback control problem where both the true model and the approximating model are linear systems under control by a linear quadratic regulator (LQR) controller.

For $k>0$, consider a stable system described by a state process $x_k\in\mathbb{R}$ and measurement process $y_k\in\mathbb{R}$,

$$
\begin{aligned}
x_k &= 0.9x_{k-1}+u_{k-1}+v_k\\
y_k &= x_k+w_k
\end{aligned} \tag{17}
$$

with the initial state value $x_0=5$. Here, $v_k\in\mathbb{R}$ and $w_k\in\mathbb{R}$ are zero-mean Gaussian noise processes with variances of 0.01 and 1, respectively.

We will consider approximation of the above system by the following:

$$
\begin{aligned}
x_k &= 0.5x_{k-1}+u_{k-1}+v^h_k\\
y_k &= x_k+w^h_k.
\end{aligned} \tag{18}
$$

where $v^h_k\in\mathbb{R}$ and $w^h_k\in\mathbb{R}$ have the same densities as v_k and w_k above.

Our approximating output feedback control is designed on the basis of an approximating Kalman filter and an infinite-horizon LQR controller designed for the approximating model (18) (see [16], [17] for details). The Kalman filter assumed an initial estimate $\hat{x}_0=10$ and initial covariance $P_0=100$, and the LQR control was designed on the basis of the cost function $J=\sum_{k=0}^{\infty}3x_k^2+u_k^2$. The achieved LQR controller $g^h(\cdot)$ is

$$
g^h(\sigma^h_k)=-0.375\hat{x}^h_k \tag{19}
$$

where $\hat{x}^h_k$ is the Kalman filter estimate. Note that the Kalman filter mean and variance defined a probability density function that serves as our approximate information state, $\sigma^h_k(\cdot)$, in this example.

To show that the conditions of Theorem 1 hold, we consider the Lyapunov function $V(X_k(x_k,\sigma_k,\sigma^h_k))=\left|X_k(x_k,\sigma_k,\sigma^h_k)\right|_{\mathbb{1}}$. For this choice of Lyapunov function, we can then show (using Minkowski's inequality) that Definition 4 holds with $\rho=2$ (at least), and that Definition 5 holds because both dynamics are linear (ie. finite difference over one time step). Properties (14) and (15) of Definition 6 hold immediately from our selected Lyapunov function. We then note that

$$
\begin{aligned}
&E_k\Big[V\left(X^h_{k+1|[k+1]}(x_k,\sigma_k,\sigma^h_k)\right)-V\left(X_k(x_k,\sigma_k,\sigma^h_k)\right)\Big]\\
&=E_k\Big[\Big|x^h_{k+1|[k+1]}(x_k,\sigma^h_k,\Sigma^h,g^h,v^h_{k+1},w^h_{k+1})\Big|_1-|x_k|_1\Big]\\
&\quad+E_k\Big[\Big|\Big|\sigma^h_{k+1|[k+1],\sigma_k}-\sigma^h_{k+1|[k+1],\sigma^h_k}\Big|\Big|_1-\big|\big|\sigma_k-\sigma^h_k\big|\big|_1\Big].
\end{aligned} \tag{20}
$$

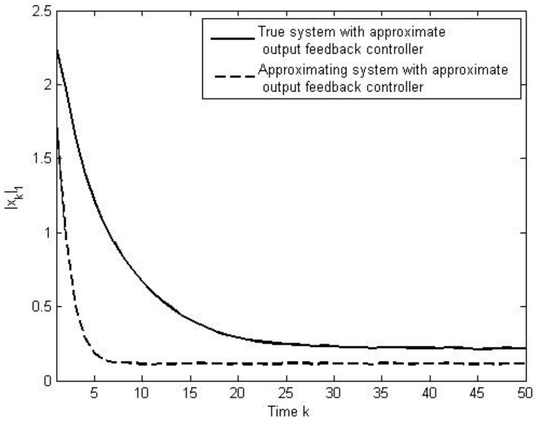

Fig. 1. Comparison of the true state under the control of true output feedback controller and true state under the control of approximate output feedback controller

From the exponential forgetting property of Kalman filters [16, pp. 76-82], it can be shown that

$$\begin{aligned}&\left|\left|\sigma^h_{k+1|[k+1],\sigma_k} - \sigma^h_{k+1|[k+1],\sigma^h_k}\right|\right|_1 - \left|\left|\sigma_k - \sigma^h_k\right|\right|_1 \\ &\leq -a^h_3 \left|\left|\sigma_k - \sigma^h_k\right|\right|_1 \qquad P^h\text{-a.s.}\end{aligned} \tag{21}$$

where $a^h_3 > 0$ is a finite constant. We can also show by substitution in approximate dynamics (18) that

$$\begin{aligned}&\left|x^h_{k+1|[k+1]}(x_k, \sigma^h_k, \Sigma^h, g^h, v_{k+1}, w_{k+1})\right|_1 - |x_k|_1 \\ &\leq -0.875|x_k| + |v^h_{k+1}| + 0.375|\hat{x}^h_k - x_k| \quad P^h\text{-a.s..}\end{aligned} \tag{22}$$

Under the assumption that $P^h \gg P$, (21) and (22) also hold P-a.s.; hence taking the expectation operation and applying (20) establishes that Definition 6 holds, where $\gamma_v(v) = v$ and $\gamma_x(x) = Bx$ for some finite $B > 0$. Definition 7 holds because the stochastic error can be overbound by a linear function of the state (eg. $\Delta\gamma^{h|v}_k \leq 0.4|x_k|$) and the expected value of the state is bounded. Theorem 1 can then be applied to establish that the expected mixed-state error is asymptotically bounded.

To illustrate the properties described by Theorem 1, we conducted a small simulation study. Figure 1 shows a closed-loop trajectory of the true system (17) under approximate output feedback control (solid line). For comparison purposes, this figure also shows a closed-loop trajectory of the design system (18) under approximate output feedback control (dashed line). This figure illustrates that that state regulation error is asymptotically bounded in this example, and that there is only a moderate loss in performance due to the model approximations involved.

V. CONCLUSION

In this paper, we investigated an output feedback stabilisation problem for stochastic discrete-time nonlinear systems when there is a mismatch between the true system and the approximating system used to design the controller. Under mild conditions, this paper establishes asymptotic bounds on the expected control and filtering errors when the true stochastic system is under approximate output feedback control. Practical stability is also established under stronger conditions.

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