Vibration of L-shaped plates under a deterministic force or moment excitation: a case of Statistical Energy Analysis application

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Abstract

Analytical and closed form solutions are presented in this paper for the vibration response of an L-shaped plate under a point force or a moment excitation. Inter-relationships between wave components of the source and the receiving plates are clearly defined. Explicit expressions are given for the quadratic quantities such as input power, energy flow and kinetic energy distributions of the L-shaped plate. Applications of statistical energy analysis (SEA) formulation in the prediction of the vibration response of finite coupled plate structures under a single deterministic forcing are examined and quantified. It is found that the SEA method can be employed to predict the frequency averaged vibration response and energy flow of coupled plate structures under a deterministic force or moment excitation when the structural system satisfies the following conditions: (1) the coupling loss factors of the coupled subsystems are known; (2) the source location is more than a quarter of the plate bending wavelength away from the source plate edges in the point force excitation case, or is more than a quarter wavelength away from the pair of source plate edges perpendicular to the moment axis in the moment excitation case due to the directional characteristic of moment excitations. SEA overestimates the response of the L-shaped plate when the source location is less than a quarter bending wavelength away from the respective plate edges owing to wave coherence effect at the plate boundary.

Keywords: vibration, L-shaped plate, SEA, energy flow, input power

1. Introduction

Statistical Energy Analysis (SEA) is often employed to estimate the vibration energy flow between coupled structures such as a ship structure when only band-averaged quantities are the main concern. However, the application of SEA is questionable when the excitation frequency is in the low or medium frequency ranges where modal overlap is low. SEA also assumes broadband random excitations with uncorrelated statistically independent forces. In contrast, excitation sources in practical applications often appear in deterministic forms (e.g., engine excitations on ship structures through engine mounts). Although analytical solutions can be obtained for the vibration response of simple regular shaped coupled plate structures with well defined boundary conditions and excited by a deterministic forcing, coupled structures in practical applications such as a ship hull or an aircraft fuselage often appear in complex forms with uncertain boundary conditions. For such structures, an analytical solution is difficult to obtain. The limitation thus motivates this study to examine the extension of SEA in the prediction of vibration response of finite coupled plate structures under a deterministic force or moment excitation, at low and medium frequencies.

Plate/plate coupled structures are often encountered in engineering applications where plates are coupled to form an integral engineering structure, such as a ship hull or an aircraft fuselage. Guyader and co-workers [1, 2] studied the vibration response of finite plates coupled at L, T and cross junctions by employing a propagation wave approach. Wave coefficients of the propagation wave solution were determined by matrix inversions in their work. Cuschieri [3] obtained a closed form solution in predicting the power transmission of an L-shaped plate using the mobility power flow approach. By including in-plane waves in their investigation, Cuschieri and McCollum [4] expanded the analysis further to vibration response of an L-shaped plate in both thin and thick plate models. They found that in-plane waves could be neglected at low frequencies when the product of the plate bending wavenumber and the plate thickness is less than 0.1. Kessissoglou [5] studied the in-plane contribution to the energy flow of a coupled plate structure at low and high frequency ranges. It was observed that contributions of the in-plane wave to the energy flow between coupled plates are significant at high frequencies where in-plane waves can act as efficient transmitters of flexural energy through plate/plate junctions. By employing a modal receptance formulation, Farag and Pan [6] studied the vibration response of two finite plates coupled at arbitrary angles where flexural, shear and in-plane wave components were included in a single matrix formulation. They observed that the coupling between two plates is dominated by moment couplings at frequencies up to the cut-off frequency of the first in-plane mode, and is controlled by both out-of-plane and in-plane vibration at frequencies above this cut-off frequency.

Wester and Mace [7] studied the validity of SEA in a wave guide system comprising two coupled simply supported rectangular plates by employing a propagation wave approach. Two wave parameters (reflectance and transmission coefficients) were defined to quantify the coupling strength between plates as weak, strong and very strong. They illustrated that the coupling loss factor estimated from the classical SEA assumption of semi-infinite subsystems and diffuse field holds only for the finite coupled system when the coupling strength is weak. It was noted that the coupling strength and the accuracy of coupling loss factor between two finite coupled plates could not be indicated by the modal overlap of the system. Dimitriadis and Pierce [8] obtained a closed form solution for the vibration response of two rectangular plates coupled at right angles by utilizing a modal solution where all plate edges including the coupled plate/plate junction were assumed to be simply supported. The solution was utilized to extract the 'apparent' quantities corresponding to the coupling loss factor and modal density ratio used in SEA. By considering both in-plane and flexural vibration, Rebillard and Guyader [9] derived an analytical solution to predict the response of a plate structure comprising a number of finite plates coupled at arbitrary angles. They showed that the effect of angular defects to vibration response of the coupled plate structure depends on the coupling angle. A small angle defect can lead to a large variation of response when the angle between two coupled plates is small. In contrast, large modification of the angle has little effect on the response when the angle of two coupled plates is large.

Typical SEA parameters of coupled plate structures have been evaluated and discussed in the literature [10-14]. Skeen [12] investigated wave transmission coefficients of finite and semi-infinite coupled plate structures, and found that the random incidence wave transmission coefficient of infinite plate structures would not provide a good estimation for that of a finite L-shaped plate under a single point force excitation. As a result, 100 random point excitations were applied to calculate the averaged total transmission coefficient of a finite coupled plate structure in his work. Park et al [13] studied the influence of modal behaviors to wave transmission between a finite and a semi-finite plate using a combined wave model and dynamic stiffness approach. They showed that finite boundary conditions have strong influence on wave transmission coefficients of the coupled plate structure. Skeen and Kessissoglou [14] examined the typical SEA parameter of the modal and total wave transmission coefficient of finite and semi-infinite coupled plate structures under a point force excitation. They found that the transmission coefficient between two coupled finite or semi-infinite plates due to a point force excitation is largely affected by excitation locations, particularly when the excitation location is close to the plate/plate junction and the coupled plate is a finite structure. However, they did not provide an adequate explanation in their study to quantify the condition of how close to the junction the transmission coefficient would be largely affected by the excitation locations. This condition will be examined and quantified in this work. The work also aims to extend the application of SEA formulation in the prediction of the frequency averaged response of coupled plate structures under a deterministic force or moment excitation at low and medium frequencies. Although in-plane waves are often included in the vibration analysis of plate/plate coupled structures, it was found that the contribution of in-plane vibration to the power transmission of an L-shaped plate is only significant at higher frequencies [4-6]. In-plane waves were also neglected by Nilsson [15] in the study of wave propagation in a scaled ship model. They found that vibration response of the ship model predicted by considering only flexural vibration agrees well with experimental results. Since this work concerns only the vibration response and energy flow of coupled plate structures at low frequencies, only flexural waves are considered in the analysis.

For simplicity, a simply supported L-shaped plate model is chosen in the study. Analytical and closed form solutions for the vibration response of the L-shaped plate under a deterministic point force or moment excitation are presented in Section 2. SEA formulation for the energy flow between two plates coupled at right angles is also briefly discussed in the section. In Section 3, conditions for the application of SEA in the prediction of the frequency averaged vibration response and kinetic energy distribution of L-shaped plates under a deterministic force/moment excitation, at low and medium frequency ranges, are examined and quantified. The main findings are summarized in Section 4.

2. Formulation

2.1 An analytical solution by matrix inversion

The L-shaped plate and the associated coordinate system are shown in Fig. 1. It is assumed that all plate edges other than the coupled plate/plate junction are simply supported. Using a thin plate vibration model, the governing equations for flexural displacements of the source (W) and the receiving (U) plates excited by an external source (Q) are given as:

$$\nabla^4 W + k_{p1}^4 W = \frac{Q}{D_1}, \tag{1}$$

and

$$\nabla^4 U + k_{p2}^4 U = 0, \tag{2}$$

where k_{p1} and k_{p2} are the bending wavenumber of the source and the receiving plates respectively and D_1 is the bending rigidity of the source plate.

Equations (1) and (2) can be solved by employing a propagation wave approach similar to that described by Lin et al [16]. However, when the calculation domain of the source plate is large (i.e. a large L_x), the exponential terms associated with the traveling wave solution of the source plate can have extremely large or small values, which could lead to numerical overflow [9]. Adopting the approach of Ref. [9], the source plate is partitioned into three sub-domains in this study to avoid such occurrence. After the partition, the source location becomes (x_0^a, y_0) in the local coordinate system of the source section as shown in Fig. 2(b).

The traveling wave solution of flexural displacements for the three sub-domains of the source plate gives:

$$W_1 = \sum_n [A_{1n} e^{-k_{n1}(L_{x1} - x_I)} + A_{2n} e^{-k_{n1} x_I} + A_{3n} e^{-k_{n2}(L_{x1} - x_I)} + A_{4n} e^{-k_{n2} x_I}] \phi_n(y), \qquad 0 \le x_I \le L_{x1}, \tag{3}$$

$$W_{2I} = \sum_n [A_{5n} e^{-k_{n1}(L_{x2} - x_{II})} + A_{6n} e^{-k_{n1} x_{II}} + A_{7n} e^{-k_{n2}(L_{x2} - x_{II})} + A_{8n} e^{-k_{n2} x_{II}}] \phi_n(y), \quad 0 \le x_{II} \le x_0^a, \tag{4}$$

$$W_{2II} = \sum_n [A_{9n} e^{-k_{n1}(L_{x2} - x_{II})} + A_{10n} e^{-k_{n1} x_{II}} + A_{11n} e^{-k_{n2}(L_{x2} - x_{II})} + A_{12n} e^{-k_{n2} x_{II}}] \phi_n(y), \; x_0^a \le x_{II} \le L_{x2}, \tag{5}$$

and

$$W_3 = \sum_n [A_{13n} e^{-k_{n1}(L_{x3} - x_{III})} + A_{14n} e^{-k_{n1} x_{III}} + A_{15n} e^{-k_{n2}(L_{x3} - x_{III})} + A_{16n} e^{-k_{n2} x_{III}}] \phi_n(y), \; 0 \le x_{III} \le L_{x3}, \tag{6}$$

where $\phi_n(y) = \sin(k_n y)$, $k_n = n\pi / L_y$, $k_{n1} = \sqrt{k_n^2 + k_{p1}^2}$ and $k_{n2} = \sqrt{k_n^2 - k_{p1}^2}$.

The traveling wave solution for the flexural displacement of the receiving plate gives:

$$U = \sum_n [A_{17n} e^{-k_{n3}(L_z - z)} + A_{18n} e^{-k_{n3} z} + A_{19n} e^{-k_{n4}(L_z - z)} + A_{20n} e^{-k_{n4} z}] \phi_n(y), \qquad 0 \le z \le L_z, \tag{7}$$

where $k_{n3} = \sqrt{k_n^2 + k_{p2}^2}$ and $k_{n4} = \sqrt{k_n^2 - k_{p2}^2}$.

The unknown wave coefficients in Eqs. (3) – (7) can be determined from the boundary and continuity conditions of the coupled plate structure. The simply supported boundary conditions of the two plate edges at $x = L_x$ and $z = L_z$ are well known, which are given by:

$$W\big|_{x=L_x} = (\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2})\bigg|_{x=L_x} = U\big|_{z=L_z} = (\frac{\partial^2 U}{\partial z^2} + \nu \frac{\partial^2 U}{\partial y^2})\bigg|_{z=L_z} = 0, \quad (8)$$

where ν is Poisson's ratio.

The compatibility conditions at the coupled plate/plate junction are:

$$W\big|_{x=0} = U\big|_{z=0} = 0, \ \frac{\partial W}{\partial x}\bigg|_{x=0} = \frac{\partial U}{\partial z}\bigg|_{z=0}, \ (\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2})\bigg|_{x=0} = -r(\frac{\partial^2 U}{\partial z^2} + \nu \frac{\partial^2 U}{\partial y^2})\bigg|_{z=0}, \quad (9)$$

where $r = \frac{D_2}{D_1}$ is the rigidity ratio of the receiver and source plates.

The continuity conditions at the artificial boundaries of the sub-domains as shown in Fig. 2 are:

$$W_1\big|_{x_I=L_{x1}} = W_{2I}\big|_{x_{II}=0}, \ \frac{\partial W_1}{\partial x_I}\bigg|_{x_I=L_{x1}} = \frac{\partial W_{2I}}{\partial x_{II}}\bigg|_{x_{II}=0}, \ \frac{\partial^2 W_1}{\partial x_I^2}\bigg|_{x_I=L_{x1}} = \frac{\partial^2 W_{2I}}{\partial x_{II}^2}\bigg|_{x_{II}=0},$$

$$\frac{\partial^3 W_1}{\partial x_I^3}\bigg|_{x_I=L_{x1}} = \frac{\partial^3 W_{2I}}{\partial x_{II}^3}\bigg|_{x_{II}=0}, \quad (10)$$

and

$$W_{2II}\big|_{x_{II}=L_{x2}} = W_3\big|_{x_{III}=0}, \ \frac{\partial W_{2II}}{\partial x_{II}}\bigg|_{x_{II}=L_{x2}} = \frac{\partial W_3}{\partial x_{III}}\bigg|_{x_{III}=0}, \ \frac{\partial^2 W_{2II}}{\partial x_{II}^2}\bigg|_{x_{II}=L_{x2}} = \frac{\partial^2 W_3}{\partial x_{III}^2}\bigg|_{x_{III}=0},$$

$$\frac{\partial^3 W_{2II}}{\partial x_{II}^3}\bigg|_{x_{II}=L_{x2}} = \frac{\partial^3 W_3}{\partial x_{III}^3}\bigg|_{x_{III}=0}. \quad (11)$$

The continuity conditions at the source location for a point force excitation source, $Q = F_0 \delta(x - x_0)\delta(y - y_0)$, are [9]:

$$W_{2I}\big|_{x_{II}=x_0^a} = W_{2II}\big|_{x_{II}=x_0^a}, \frac{\partial W_{2I}}{\partial x_{II}}\bigg|_{x_{II}=x_0^a} = \frac{W_{2II}}{\partial x_{II}}\bigg|_{x_{II}=x_0^a}, \frac{\partial^2 W_{2I}}{\partial x_{II}^2}\bigg|_{x_{II}=x_0^a} = \frac{\partial^2 W_{2II}}{\partial x_{II}^2}\bigg|_{x_{II}=x_0^a},$$

$$\left.\frac{\partial^3 W_{2II}}{\partial x_{II}^3}\right|_{x_{II}=x_0^a} - \left.\frac{\partial^3 W_{2I}}{\partial x_{II}^3}\right|_{x_{II}=x_0^a} = \frac{F_0}{D_1}\delta(x-x_0)\delta(y-y_0)\,. \tag{12}$$

For a moment excitation source, $Q = M_0 \sin\varphi \dfrac{\partial\delta(x-x_0, y-y_0)}{\partial x} - M_0 \cos\varphi \dfrac{\partial\delta(x-x_0, y-y_0)}{\partial y}$,

where M_0 is the amplitude of the external moment source and φ is the angle of moment axis with respect to the x-axis of the coordinate system [17] as shown in Fig. 1. Under moment excitation, the continuity conditions at the source locations become:

$$W_{2I}\big|_{x_{II}=x_0^a} = W_{2II}\big|_{x_{II}=x_0^a}, \left.\frac{\partial W_{2I}}{\partial x_{II}}\right|_{x_{II}=x_0^a} = \left.\frac{\partial W_{2II}}{\partial x_{II}}\right|_{x_{II}=x_0^a},$$

$$\left.\frac{\partial^2 W_{2II}}{\partial x_{II}^2}\right|_{x_{II}=x_0^a} - \left.\frac{\partial^2 W_{2I}}{\partial x_{II}^2}\right|_{x_{II}=x_0^a} = \frac{M_0 \sin\varphi}{D_1}\delta'(x_{II}-x_0)\delta(y-y_0)\,,$$

$$\left.\frac{\partial^3 W_{2II}}{\partial x_{II}^3}\right|_{x_{II}=x_0^a} - \left.\frac{\partial^3 W_{2I}}{\partial x_{II}^3}\right|_{x_{II}=x_0^a} = -\frac{M_0 \cos\varphi}{D_1}\delta(x_{II}-x_0)\delta'(y-y_0)\,, \tag{13}$$

where the prime (') indicates a spatial derivative.

Traditionally, wave coefficients of the traveling wave solution are determined by matrix inversion [5, 12-14], which is given by:

$$\mathbf{A_n} = \mathbf{B_n}^{-1}\mathbf{F_n}\,, \tag{14}$$

where $\mathbf{A_n} = \lfloor A_{1n} \quad A_{2n} \quad \ldots\ldots \quad A_{19n} \quad A_{20n} \rfloor^T$ is the unknown wave coefficient vector to be determined, the superscript T indicates a vector transpose. $\mathbf{B_n}$ is a 20×20 square matrix, which is given in Appendix A. The vector $\mathbf{F_n}$ is the external force vector. For a point force excitation source, the force vector is given by:

$$\mathbf{F_n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{F_0\phi_n(y_0)}{D_1\Lambda_n k_{n1}^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T . \qquad (15)$$

where $\Lambda_n = \dfrac{L_y}{2}$ is a modal constant.

If the external excitation source is a moment source, the force vector becomes:

$$\mathbf{F_n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{M_0 \sin\varphi\phi_n(y_0)}{D_1\Lambda_n k_{n1}^2} & -\dfrac{M_0 \cos\varphi\phi_n'(y_0)}{D_1\Lambda_n k_{n1}^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T .$$

(16)

Eq. (14) provides a straight forward solution for the vibration response of an L-shaped plate. However, dependency of vibration response of the L-shaped plate, such as input power, energy flow and kinetic energy distribution, on structural properties of the plate is not explicitly demonstrated in the solution. For a better understanding of the mechanism controlling the energy flow from an excitation source to the coupled plate structure, and from the source plate to the receiving plate, vibration response of the L-shaped plate is given explicitly in terms of structural properties of the L-shaped plate in the following section.

2.2 A closed form solution for a point force excitation case

In this paper, the receiving plate is replaced by a coupled boundary condition to obtain a closed form solution of the L-shaped plate. By doing so, the simply supported boundary conditions at $z = L_z$ corresponding to $\left.(\dfrac{\partial^2 U}{\partial z^2} + \nu \dfrac{\partial^2 U}{\partial y^2})\right|_{z=L_z} = 0$ and $U\big|_{z=L_z} = 0$, and the zero flexural displacement ($U\big|_{z=0} = 0$) at the coupled plate/plate junction ($z = 0$) are utilized first to express the first three wave coefficients in Eq. (7) in terms of the fourth wave coefficient of the receiving plate as:

$$A_{17n} = c_{21}\beta A_{20n}\,,\; A_{18n} = -\beta A_{20n}\,,\; A_{19n} = -c_{22}A_{20n}\,, \tag{17}$$

where $\beta = \dfrac{1-c_{22}^2}{1-c_{21}^2}$, $c_{21} = e^{-k_{n3}L_z}$ and $c_{22} = e^{-k_{n4}L_z}$ are exponential wave constants for a given frequency and a trace wavenumber k_n .

Secondly, the two compatibility conditions at the plate/plate junction ($\left.\dfrac{\partial W_1}{\partial x_I}\right|_{x_I=0} = \left.\dfrac{\partial U}{\partial z}\right|_{z=0}$, $\left.(\dfrac{\partial^2 W_1}{\partial x_I^2} + \nu \dfrac{\partial^2 W_1}{\partial y^2})\right|_{x_I=0} = -r\left.(\dfrac{\partial^2 U}{\partial z^2} + \nu \dfrac{\partial^2 U}{\partial y^2})\right|_{z=0}$) are utilized in conjunction with Eq. (17) to form a coupled boundary condition in terms of wave coefficients of the source plate only as:

$$(\Delta_2 - \Delta_1)A_{1n}c_1 - (\Delta_2 + \Delta_1)A_{2n} + (\Delta_2 H_1 - \Delta_1 H_1^2)A_{3n}c_2 - (\Delta_2 H_1 + \Delta_1 H_1^2)A_{4n} = 0\,, \tag{18}$$

where $\Delta_1 = [H_2\beta(1+c_{21}^2) - H_3(1+c_{22}^2)]$, $\Delta_2 = r(H_2^2 + H_3^2)(1-c_{22}^2)$. $c_1 = e^{-k_{n1}L_{x1}}$ and $c_2 = e^{-k_{n2}L_{x1}}$ are wave constants; $H_1 = \dfrac{k_{n2}}{k_{n1}}$, $H_2 = \dfrac{k_{n3}}{k_{n1}}$ and $H_3 = \dfrac{k_{n4}}{k_{n1}}$ are wavenumber ratios.

Eq. (18) together with the zero flexural displacement condition of the source plate at $x_I = 0$ ($\left.W_1\right|_{x_I=0} = 0$) provide the two necessary boundary conditions at the plate/plate junction for a closed form solution. Combining Eq. (18) with other boundary conditions of the source plate lead to analytical expressions for the wave coefficients of the L-shaped plate, which are given in Appendix B.

(1) Input mobility

Once the wave coefficients are determined, the point force input mobility of the L-shaped plate can be calculated from Eq. (4) as:

$$Y_{in}^{F} = j\omega \frac{W|_{(x_0,\,y_0)}}{F_0} = \frac{j\omega}{F_0}\sum_n (A_{5n}c_5 + A_{6n}c_6 + A_{7n}c_7 + A_{8n}c_8)\phi_n(y_0)\,, \tag{19}$$

where $c_5 = e^{-k_{n1}(L_{x2}-x_0)}$, $c_6 = e^{-k_{n1}x_0}$, $c_7 = e^{-k_{n2}(L_{x2}-x_0)}$ and $c_8 = e^{-k_{n2}x_0}$ are wave constants.

Alternatively, the input mobility of the L-shaped plate can be expressed in terms of structural properties of the plate by substituting the analytical expressions of wave coefficients into Eq. (19) as:

$$Y_{in}^{F} = \frac{j\omega}{4D_1\Lambda_n k_{p1}^2}\sum_n \frac{\phi_n^2(y_0)}{k_{n1}}\left(\frac{\alpha_2\gamma_2 - \alpha_3\gamma_1}{\alpha_2 - \alpha_1\alpha_3}R_1 + \frac{\gamma_1 - \alpha_1\gamma_2}{\alpha_2 - \alpha_1\alpha_3}R_2\right), \tag{20}$$

where $\alpha_1 = \Theta_1 + c_1c_3^2c_9^2\Theta_3$, $\alpha_2 = \Theta_2 + c_1c_3^2c_9^2\Theta_4$, $\alpha_3 = c_2c_4^2c_{10}^2$, $\gamma_1 = \dfrac{c_6^2 - c_3^2c_9^2}{c_6}$,$\gamma_2 = \dfrac{c_8^2 - c_4^2c_{10}^2}{c_8H_1}$,
$R_1 = \dfrac{1}{c_8} - (\dfrac{1}{c_6}\Theta_1 + c_1c_6\Theta_3)$, $R_2 = c_2c_8 - (\dfrac{1}{c_6}\Theta_2 + c_1c_6\Theta_4)$, $c_3 = e^{-k_{n1}L_{x2}}$, $c_4 = e^{-k_{n2}L_{x2}}$,$c_9 = e^{-k_{n1}L_{x3}}$ and $c_{10} = e^{-k_{n2}L_{x3}}$.

$\Theta_1 - \Theta_4$ are terms relating to the coupling of the receiving plate which are defined by:

$$\Theta_1 = \frac{(1+H_1)[\Delta_2 + \Delta_1(1-H_1)]\,c_2}{2\Delta_2 c_1},\ \Theta_2 = \frac{(1-H_1)[\Delta_2 + \Delta_1(1+H_1)]}{2\Delta_2 c_1},$$

$$\Theta_3 = \frac{(1-H_1)[\Delta_2 - \Delta_1(1+H_1)]\,c_2}{2\Delta_2},\ \ \Theta_4 = \frac{(1+H_1)[\Delta_2 - \Delta_1(1-H_1)]}{2\Delta_2}. \tag{21}$$

(2) Vibration energy flow

The power injection to the structure by the point force can now be written as:

$$\left\langle \overline{P}_{in} \right\rangle = \frac{\left|F_0\right|^2}{2}\mathrm{Re}(Y_{in}^F) = -\frac{\omega\left|F_0\right|^2}{8\Lambda_n}\mathrm{Im}[\frac{1}{D_1 k_{p1}^2}\sum_n \frac{\phi_n^2(y_0)}{k_{n1}}(\frac{\alpha_2\gamma_2-\alpha_3\gamma_1}{\alpha_2-\alpha_1\alpha_3}R_1+\frac{\gamma_1-\alpha_1\gamma_2}{\alpha_2-\alpha_1\alpha_3}R_2)]\,. \tag{22}$$

Eq. (22) illustrates that the input power of the L-shaped plate is largely controlled by the structural properties of the source plate, particularly at frequencies away from the resonant response of the receiving plate. This will be discussed further in Section 3.

The time averaged, steady state energy flow from the source plate to the receiving plate can be calculated by:

$$\left\langle \overline{P}_{12} \right\rangle = \frac{1}{2}\mathrm{Re}\int_0^{Ly} M_y\big|_{(0,\,y)}\,\dot{\theta}_y^*\big|_{(0,\,y)}\,dy\,, \tag{23}$$

where the asterisk sign (*) indicates a complex conjugate, M_y and $\dot{\theta}_y$ are respectively, the moment and angular velocity distribution along the plate/plate junction. The moment distribution at the plate/plate junction is evaluated by:

$$\begin{aligned} M_y\big|_{(0,\,y)} &= D_1(\frac{\partial^2 W_1}{\partial x_I^2}+\nu\frac{\partial^2 W_1}{\partial y^2})\Big|_{x_I=0} \\ &= -\frac{F_0}{2\Lambda_n}\sum_n\frac{\phi_n(y_0)}{k_{n1}}(\frac{1-c_2\alpha_3}{\alpha_2-\alpha_1\alpha_3}\gamma_1+\frac{c_2\alpha_2-\alpha_1}{\alpha_2-\alpha_1\alpha_3}\gamma_2)\phi_n(y) \end{aligned}\,. \tag{24}$$

The angular velocity distribution at the plate/plate junction is given by:

$$\begin{aligned} \dot{\theta}_y\big|_{(0,\,y)} &= j\omega\sum_n[k_{n1}(A_{1n}c_1-A_{2n})+k_{n2}(A_{3n}c_2-A_{4n})]\phi_n(y) \\ &= -\frac{j\omega F_0}{2D_1\Lambda_n}\sum_n\frac{\phi_n(y_0)}{k_{n1}^2}(\frac{1-c_2\alpha_3}{\alpha_2-\alpha_1\alpha_3}\gamma_1+\frac{c_2\alpha_2-\alpha_1}{\alpha_2-\alpha_1\alpha_3}\gamma_2)\frac{\Delta_1}{\Delta_2}\phi_n(y) \end{aligned}\,. \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (23) gives:

$$\langle \overline{P}_{12} \rangle = -\frac{\omega |F_0|^2}{8\Lambda_n} \mathrm{Im}[\frac{1}{D_1^*} \sum_n \frac{\phi_n^2(y_0)}{|k_{n1}|^2 k_{n1}^*} \left| \frac{(1 - c_2\alpha_3)\gamma_1 + (c_2\alpha_2 - \alpha_1)\gamma_2}{\alpha_2 - \alpha_1\alpha_3} \right|^2 (\frac{\Delta_1}{\Delta_2})^*] . \tag{26}$$

Examination of Eq. (26) reveals that the energy flow from the source plate to the receiving plate is not only governed by the source plate stiffness D_1 but is also controlled by the receiving plate stiffness D_2 via the stiffness ratio in the coupling term $\frac{\Delta_1}{\Delta_2}$. Furthermore, the energy flow from the source plate to the receiving plate is also affected by the wavenumber ratios between the two plates.

2.3 A closed form solution for a moment excitation case

The inter-relationships between wave coefficients in the moment excitation case are the same as in the point force excitation case except for those directly associated with the moment source, corresponding to $A_{5n} - A_{12n}$. The inter-relationships of these wave coefficients ($A_{5n} - A_{12n}$) and analytical expressions of wave coefficients A_{3n} and A_{4n} in the moment excitation case are given by Eqs. (B.18) to (B.23) in Appendix B.

(1) Input mobility

There are two moment input mobility components, each associated with one of the two angular velocity components at the source location. These two input mobility components are given by:

$$\begin{aligned} Y_{in}^x = \frac{j\omega}{M_0} \frac{\partial W}{\partial y}\bigg|_{(x_0,\ y_0)} &= \frac{j\omega}{4D_1\Lambda_n k_{p1}^2} \sum_n [\frac{\phi_n'^2(y_0)\cos\varphi}{k_{n1}} (\frac{\alpha_2\gamma_2 - \alpha_3\gamma_1}{\alpha_2 - \alpha_1\alpha_3} R_1 + \frac{\gamma_1 - \alpha_1\gamma_2}{\alpha_2 - \alpha_1\alpha_3} R_2) \\ &\quad - \phi_n(y_0)\phi_n'(y_0)\sin\varphi (\frac{\alpha_2\gamma_4 - \alpha_3\gamma_3}{\alpha_2 - \alpha_1\alpha_3} R_1 + \frac{\gamma_3 - \alpha_1\gamma_4}{\alpha_2 - \alpha_1\alpha_3} R_2)] \end{aligned} , \tag{27}$$

and

$$Y_{in}^{y}=\frac{j\omega}{M_0}\frac{\partial W}{\partial x}\bigg|_{(x_0,\ y_0)}=\frac{j\omega}{4D_1\Lambda_n k_{p1}^2}\sum_n[\frac{\phi_n(y_0)\phi_n'(y_0)\cos\varphi}{k_{n1}}(\frac{\alpha_2\gamma_2-\alpha_3\gamma_1}{\alpha_2-\alpha_1\alpha_3}R_3+\frac{\gamma_1-\alpha_1\gamma_2}{\alpha_2-\alpha_1\alpha_3}R_4)$$
$$-\phi_n^2(y_0)\sin\varphi(\frac{\alpha_2\gamma_4-\alpha_3\gamma_3}{\alpha_2-\alpha_1\alpha_3}R_3+\frac{\gamma_3-\alpha_1\gamma_4}{\alpha_2-\alpha_1\alpha_3}R_4)], \quad (28)$$

where $R_3=k_{n1}(c_1c_6\Theta_3-\frac{1}{c_6}\Theta_1)+\frac{1}{c_8}k_{n2}$, $R_4=k_{n1}(c_1c_6\Theta_4-\frac{1}{c_6}\Theta_2)-c_2c_8k_{n2}$, $\gamma_3=\frac{c_6^2+c_3^2c_9^2}{c_6}$,

$\gamma_4=\frac{c_8^2+c_4^2c_{10}^2}{c_8}$.

The input power of a moment excitation thus has two components in association with the two input mobility components, which is given by:

$$\langle\overline{P}_{in}\rangle=\langle\overline{P}_{in}\rangle^x+\langle\overline{P}_{in}\rangle^y=\frac{|M_0|^2}{2}\cos^2\varphi\,\mathrm{Re}(Y_{in}^x)+\frac{|M_0|^2}{2}\sin^2\varphi\,\mathrm{Re}(Y_{in}^y). \quad (29)$$

(2) Vibration energy flow

The moment distribution at the plate/plate junction for this excitation case is given by:

$$M_y\big|_{(0,\ y)}=-\frac{M_0}{2\Lambda_n}\sum_n[\phi_n(y_0)\sin\varphi(\sigma_1\gamma_3+\sigma_2\gamma_4)-\frac{\phi_n'(y_0)\cos\varphi}{k_{n1}}(\sigma_1\gamma_1+\sigma_2\gamma_2)]\phi_n(y), \quad (30)$$

where $\sigma_1=\frac{1-\alpha_3c_2}{\alpha_2-\alpha_1\alpha_3}$ and $\sigma_2=\frac{\alpha_2c_2-\alpha_1}{\alpha_2-\alpha_1\alpha_3}$.

The angular velocity distribution at the coupled junction is determined by:

$$\dot{\theta}_y\big|_{(0,\ y)}=-\frac{j\omega M_0}{2D_1\Lambda_n}\sum_n[\frac{\phi_n(y_0)\sin\varphi}{k_{n1}}(\sigma_1\gamma_3+\sigma_2\gamma_4)-\frac{\phi_n'(y_0)\cos\varphi}{k_{n1}^2}(\sigma_1\gamma_1+\sigma_2\gamma_2)]\frac{\Delta_1}{\Delta_2}\phi_n(y). \quad (31)$$

Substituting Eqs. (30) and (31) into Eq. (23) gives:

$$\langle\overline{P}_{12}\rangle=-\frac{\omega|M_0|^2}{8\Lambda_n}\mathrm{Im}[\frac{1}{D_1^*}\sum_n\frac{1}{k_{n1}^*}\left|\phi_n(y_0)\sin\varphi(\sigma_1\gamma_3+\sigma_2\gamma_4)-\frac{\phi_n'(y_0)\cos\varphi}{k_{n1}}(\sigma_1\gamma_1+\sigma_2\gamma_2)\right|^2(\frac{\Delta_1}{\Delta_2})^*]. \quad (32)$$

2.4 Kinetic energy distribution

For both excitation cases, the time averaged, steady state kinetic energy distribution of the source plate is given by:

$$\langle \overline{T}_1 \rangle = \frac{1}{2}\int_{S_1} \rho_1 \dot{W}\dot{W}^* dS = \frac{\rho_1 \Lambda_n \omega^2}{2}\sum_{i=1}^{4} T_i \, , \tag{33}$$

where S_1 and ρ_1 are respectively, the surface area and material density of the source plate. T_i is the energy component obtained by integrating over the surface area of i^{th} sub-domain of the source plate. Analytical expressions for these energy components of the source plate are given by Eqs. (B.24) – (B.27) in Appendix B.

The time averaged, steady state kinetic energy distribution of the receiving plate is given by:

$$\begin{aligned}\langle \overline{T}_2 \rangle = \frac{\rho_2 \Lambda_n \omega^2}{2}\sum_n \{ & \frac{|A_{17n}|^2 + |A_{18n}|^2}{2\mathrm{Re}(k_{n3})}(1-|c_{21}|^2) + \frac{|A_{19n}|^2 + |A_{20n}|^2}{2\mathrm{Re}(k_{n4})}(1-|c_{22}|^2) \\ & - \frac{2\mathrm{Re}(A_{17n}^* A_{18n})\mathrm{Im}(c_{21})}{\mathrm{Im}(k_{n3})} - \frac{2\mathrm{Re}(A_{19n}^* A_{20n})\mathrm{Im}(c_{22})}{\mathrm{Im}(k_{n4})} \\ & + 2\mathrm{Re}[\frac{A_{17n}^* A_{20n} + A_{18n}^* A_{19n}}{k_{n3}^* - k_{n4}}(c_{22} - c_{21}^*) + \frac{A_{17n}^* A_{19n} + A_{18n}^* A_{20n}}{k_{n3}^* + k_{n4}}(1 - c_{21}^* c_{22})]\} \end{aligned} . \tag{34}$$

2.5 SEA formulation for vibration energy flow of an L-shaped plate

The steady state energy balance equation of a two-subsystem SEA model can be written as [10]:

$$\mathbf{T} = \frac{1}{\omega_c}\begin{bmatrix} \eta_1 + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta_2 + \eta_{21} \end{bmatrix}^{-1} \mathbf{P} \, , \tag{35}$$

where $\mathbf{T} = \lfloor \langle \overline{T}_1 \rangle \quad \langle \overline{T}_2 \rangle \rfloor^T$ is the kinetic energy vector . $\langle \overline{T}_1 \rangle$ and $\langle \overline{T}_2 \rangle$ are respectively, the time averaged, steady state kinetic energy of subsystems 1 and 2. η_1 and η_2 are the internal loss factor of subsystems 1 and 2, η_{12} and η_{21} are the coupling loss factors between the two subsystems, ω_c is the centre frequency of a band. $\mathbf{P} = \lfloor \langle \overline{P}_1 \rangle \quad \langle \overline{P}_2 \rangle \rfloor^T$ is the input power vector in which $\langle \overline{P}_1 \rangle$ and $\langle \overline{P}_2 \rangle$ are the time averaged, steady state power injection to subsystems 1 and 2 by external sources. Only power injection into the source plate is considered in the numerical simulation presented in the next section, thus, $\langle \overline{P}_2 \rangle = 0$.

Eq. (35) could be utilized to estimate the vibration energy distribution of a finite L-shaped plate under a deterministic force excitation if the frequency averaged power injection to the L-shaped plate can be approximated by that of an infinite plate. For a point force excitation, the power injection to an infinite plate is given by [18]:

$$\langle \overline{P}_1 \rangle = \frac{|F_0|^2}{2} \mathrm{Re}(Y_{\mathrm{inf}}^{F_0}) = \frac{|F_0|^2}{8} \mathrm{Re}(\frac{1}{\sqrt{D_1 \rho_{s1}}}), \tag{36}$$

where ρ_{s1} is the surface mass of the source plate. For a moment excitation, the power injection is given by [18]:

$$\langle \overline{P}_1 \rangle = \frac{|M_0|^2}{2} \mathrm{Re}(Y_{\mathrm{inf}}^{M_0}) = \frac{\omega |M_0|^2}{16 D_1} \mathrm{Re}[1 - j\frac{4}{\pi} \ln(1.781 k_{p1} a)], \tag{37}$$

where *j* is the imaginary number, and *a* is the dipole distance that measures the distance between the forces forming a dipole moment.

The coupling loss factors between the component plates of an L-shaped plate needs to be determined. For two rectangular plates coupled along a common edge, the coupling loss factor between the two component plates of an L-shaped plate is given in terms of the wave transmission coefficient as [10, 18, 19]:

$$\eta_{12} = \frac{2c_{b1}L_y\tau_{12}}{\pi\omega_c S_1}, \tag{38}$$

where c_{b1} is the bending wave speed of component Plate 1, τ_{12} is the wave transmission coefficient from Plate 1 to Plate 2 across the line junction. The coupling loss factor between Plate 2 and Plate 1 can be evaluated similarly.

Furthermore, the random incidence wave transmission coefficient τ_{12} is approximated in terms of the normal incidence transmission coefficient τ_0 by [20]:

$$\tau_{12} = \tau_0 \frac{2.754R}{1+3.24R}, \tag{39}$$

where $R = \frac{t_1}{t_2}$ is the ratio of plate thicknesses, and the normal incidence transmission coefficient τ_0 is given by:

$$\tau_0 = 2\left(\frac{\rho_1^{1/2}c_{L1}^{3/4}t_1^{5/4}}{\rho_2^{1/2}c_{L2}^{3/4}t_2^{5/4}} + \frac{\rho_2^{1/2}c_{L2}^{3/4}t_2^{5/4}}{\rho_1^{1/2}c_{L1}^{3/4}t_1^{5/4}}\right)^{-2}, \tag{40}$$

where c_{L1} and c_{L2} are the longitudinal wave speeds of Plates 1 and 2.

3. Results and discussion

3.1 Accuracy evaluation

The plates used in the numerical simulation are assumed to be made of aluminum panels with the material properties $E_0 = 7.1\times10^{10}\,\mathrm{N/m^2}$, $\rho = 2660\,\mathrm{Kg/m^3}$ and $\nu = 0.3$, and are assumed to have constant internal loss factors $\eta_1 = \eta_2 = 0.01$. 1Hz frequency resolution is used for all simulations in the study except for the simulation results presented in Fig. 5 where 0.1Hz frequency resolution is used instead. Four plate configurations are chosen for the L-shaped plate in the simulation: Model A, the source and the receiving plates both have relatively high modal density; Model B, the receiving plate has higher modal density than the source plate; Model C, the receiving plate has lower modal density than the source plate; and Model D, the source and the receiving plates both have relatively low modal density. Physical dimensions of the four L-shaped plate models and the value of a quarter bending wavelength of the source plate at 100Hz and 1000Hz are given in Table 1. The plate bending wavelength of the source plate is calculated by:

$$\lambda = \frac{2\pi \left(D_1 / \rho_{s1} \right)^{1/4}}{\sqrt{\omega}} . \tag{41}$$

Model B is used in the initial accuracy evaluation. The real part of input mobility of the L-shaped plate due to a point force excitation applied at the source plate location $(x_0, y_0) = (0.32\text{m}, 0.32\text{m})$ is calculated by using the closed form solution (Eq. (20)), the matrix inversion method (Eq. (14)) and by finite element analysis using the modal frequency response analysis module provided by MSC/NASTRAN. The source and the receiving plates are meshed by 150×50 and 75×50 mesh grids respectively in the FEA model. The results are shown in Fig. 3 for comparison. Good agreement is found between the results. Furthermore, it is shown in Fig. 3(b) that the averaged response of the L-shaped plate can be represented by that of the infinite plate at frequencies above the first few peak responses. This corresponds to frequencies where the source location is more than a quarter of the plate bending wavelength away from the source plate boundary, that is, at frequencies greater than 12Hz. This interesting observation will be discussed further in the next section.

Fig. 4 shows the input mobility of the L-shaped plate due to a moment excitation ($\varphi = 0^o$) applied at the same location as in the point force case together with that of the corresponding infinite plate for comparison. It is shown that the frequency averaged principal component of the moment input mobility (the component in the direction of the moment excitation) can be represented by that of the corresponding infinite plate at frequencies above the first few peak responses of the L-shaped plate but not the secondary component. The real part of the secondary moment input mobility component (the component perpendicular to the moment source) is not always positive and its frequency averaged value is substantially less than that of the infinite plate.

For better understanding of the mechanism controlling the input power of L-shaped plates, the input mobility of the four L-shaped plate models described in Table 1 together with those of the corresponding uncoupled component plates are calculated and shown in Fig. 5. It is shown that the coupling of the receiving plate leads to a slightly increased modal stiffness of the corresponding modes of the uncoupled source plate (i.e., an increased modal frequency and a decreased modal amplitude). Peaks corresponding to the uncoupled modes of the receiving plate are also presented in the response, which typically have much smaller amplitude than peaks due to the source plate, particularly when the receiving plate has much smaller bending stiffness than the source plate such as Model C in Fig. 5. The overall input mobility of L-shaped plates is largely controlled by the bending stiffness of the source plate as indicated by Eq. (22).

3.2 Applications of SEA in a deterministic source excitation case

In this simulation, vibration response due to a unit point force/moment excitation applied at various plate locations are calculated for the four plate models described in Table 1. This is to examine the validity of SEA application in the prediction of vibration response and energy flow between coupled plate structures due to a deterministic force/moment excitation. The excitation source is applied at $(x_0, y_0) = (0.11, 0.32)$, $(0.035, 0.32)$ and $(0.01, 0.32)$ in three separate simulations for models A and B, and is at $(x_0, y_0) = (0.22, 0.32)$, $(0.07, 0.32)$ and $(0.01, 0.32)$ for models C and D. The first two excitation locations in each model are chosen such that the distance between the source and the plate/plate coupling edge is corresponding to a quarter plate bending wavelength at 100Hz and 1000Hz as shown in Table 1. The third excitation location is chosen such that the distance between the source and the coupling edge is far less than a quarter plate bending wavelength in the whole frequency range of consideration.

(1) Point force excitations

The energy distribution of the source and the receiving plates of the four plate models are shown in Figs. 6 – 9, together with those predicted by SEA using Eq. (35). In the SEA approach, the input power of the L-shaped plates is calculated by that of the infinite plate using Eq. (36). The energy distribution of the source and receiving plates is evaluated at discrete frequencies (i.e., $\omega_c = \omega$) in Eq. (35) rather than band averaged values.

It is shown that the frequency averaged energy distribution of the L-shaped plates due to a deterministic point force excitation at low and medium frequencies can be predicted by using

SEA formulation when the source location is more than a quarter of the plate bending wavelength away from the coupling plate edge, for instance, at frequencies above 100Hz when the force excitation is applied at $(x_0, y_0) = (0.11, 0.32)$ for plate models A and B, and at $(x_0, y_0) = (0.22, 0.32)$ for plate models C and D. Similar results are found when the source location is more than a quarter bending wavelength away from other edges of the source plate. In contrast, SEA over predicts the vibration response of the L-shaped plates when the source location is less than a quarter bending wavelength away from the respective source plate edge. This can be explained by the wave coherence effect at the plate boundary or by the effect of the plate boundary stiffness (infinite in this case) on the input mobility of the plate.

The excitation is not affected by wave coherence at the plate boundary when the source location is more than a quarter of the plate bending wavelength away from the source plate edges. The boundary stiffness of the plate has little effect on the point force input mobility and therefore the input power of the L-shaped plate at this condition. It starts to affect and contribute to the stiffness term of the input mobility when the source location is less than a quarter bending wavelength from the boundaries. The plate boundary stiffness effect increases when the source location is moved closer to the plate edges or diminishes when the source location is moved away from the edges. At the latter condition, the input mobility is mainly controlled by the bending stiffness of the source plate so that the frequency averaged input mobility can be represented by that of the corresponding infinite plate. Fig. 10 shows the input mobility of the L-shaped plate (Model A) due to the three force excitation conditions. It is shown that the frequency averaged input mobility of the L-shaped plate can be represented approximately by that of the infinite plate at frequencies above 100Hz when the source location is 0.11m (corresponding to a quarter plate bending wavelength at this frequency) away from the coupling edge, and above 1000Hz when the source is 0.035m away from the coupling edge. However, the vibration response of the plate model is substantially less than that of the infinite plate in the whole frequency range of interest when the source location is 0.01m away from the coupling edge attributing to the strong wave coherent effect from the plate boundary.

(2) Moment excitations

The input mobilities of the L-shaped plate (Model A) due to a bending moment excitation ($\varphi = 0^o$) applied at the same locations as in the point force excitation case are shown in Fig. 11. Additionally, the input mobilities of the L-shaped plate due to the same moment excitation but applied at three other locations are shown in Fig. 12 for comparison. In contrast to that of point force excitation cases, it is found that only the pair of source plate edges perpendicular to the moment axis can affect the input power of the moment excitation. The amplitude of the moment input mobility is largely affected by the infinite boundary stiffness of this pair of plate edges when the source location is less than a quarter bending wavelength away from these edges, but is not affected by the other pair of plate edges. This interesting result is attributed to the directional characteristic of moment excitations. Similar to that observed in point force cases, effects of boundary stiffness on the moment input mobility diminish when the source location is greater than a quarter of the plate bending wavelength away from the pair of source plate edges perpendicular to the moment axis. Under this condition, the frequency averaged vibration response and the kinetic energy flow of the L-shaped plate can be predicted by SEA as shown in Fig. 13. Similar results are also found when the L-shaped plate is excited by a torsional moment ($\varphi = 90^o$).

4. Conclusions

Analytical and closed form solutions are presented in this paper by utilizing a traveling wave solution to predict the vibration response of an L-shaped plate under a point force or a moment excitation. Explicit expressions are given for wave coefficients of the traveling wave solution where no matrix inversion is required. The quadratic quantities of the plate response such as input mobilities, energy flow and kinetic energy distribution of the L-shaped plate are also expressed explicitly in terms of structural properties of the plate. The closed form solution not only provides a direct understanding of the mechanism controlling the power injection and power flow between two coupled plates under a deterministic force/moment excitation, it also has computational benefit than traditional matrix inversion approaches since no matrix inversion is required in the solution. Discussions are given by comparing the input mobility of the L-shaped plate and that of the corresponding infinite plate. It is shown that the frequency averaged input mobility of the L-shaped plate can be represented by that of the corresponding infinite plate when the source location is more than a quarter of the plate bending wavelength away from the source plate edges in the point force excitation case, or is more than a quarter wavelength away from the pair of plate edges perpendicular to the moment axis in moment excitation cases due to the directional characteristic of moment excitations.

This paper also shows that SEA formulation could be utilized to predict the frequency averaged vibration response of an L-shaped plate due to a single deterministic force/moment excitation at low and medium frequencies when the source location is more than a quarter of the plate bending wavelength away from the plate boundary. Although only results of simple rectangular plate structures are used in this work, results obtained from this study indicate that SEA technique can also be employed to predict the frequency averaged vibration response of coupled plate structures with different shapes providing that the coupling loss factors are known and the excitation source is more than a quarter plate bending wavelength away from the source plate boundary. It is also noted that the findings presented in this paper are not affected by internal loss factors of the coupled subsystems.

Appendix A

The matrix $\mathbf{B_n}$ in Eq. (14) is given by:

$$
\mathbf{B_n} = \left[\begin{array}{cccccccccc}
c_1 & 1 & c_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & c_1 & 1 & c_2 & -c_3 & -1 & -c_4 & -1 & 0 & 0 \\
1 & -c_1 & H_1 & -H_1 c_2 & -c_3 & 1 & -H_1 c_4 & H_1 & 0 & 0 \\
1 & c_1 & H_1^2 & H_1^2 c_2 & -c_3 & -1 & -H_1^2 c_4 & -H_1^2 & 0 & 0 \\
1 & -c_1 & H_1^3 & -H_1^3 c_2 & -c_3 & 1 & -H_1^3 c_4 & H_1^3 & 0 & 0 \\
c_1 & -1 & H_1 c_2 & -H_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_1 & 1 & H_1^2 c_2 & H_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_5 & c_6 & c_7 & c_8 & -c_5 & -c_6 \\
0 & 0 & 0 & 0 & c_5 & -c_6 & H_1 c_7 & -H_1 c_8 & -c_5 & c_6 \\
0 & 0 & 0 & 0 & c_5 & c_6 & H_1^2 c_7 & H_1^2 c_8 & -c_5 & -c_6 \\
0 & 0 & 0 & 0 & -c_5 & c_6 & -H_1^3 c_7 & H_1^3 c_8 & c_5 & -c_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.
$$

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -H_2c_{21} & H_2 & -H_3c_{22} & H_3 \\
0 & 0 & 0 & 0 & 0 & 0 & rH_2^2c_{21} & rH_2^2 & rH_3^2c_{22} & rH_3^2 \\
-c_7 & -c_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-H_1c_7 & H_1c_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-H_1^2c_7 & -H_1^2c_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H_1^3c_7 & -H_1^3c_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & c_4 & -c_9 & -1 & -c_{10} & -1 & 0 & 0 & 0 & 0 \\
H_1 & -H_1c_4 & -c_9 & 1 & -H_1c_{10} & H_1 & 0 & 0 & 0 & 0 \\
H_1^2 & H_1^2c_4 & -c_9 & -1 & -H_1^2c_{10} & -H_1^2 & 0 & 0 & 0 & 0 \\
H_1^3 & -H_1^3c_4 & -c_9 & 1 & -H_1^3c_{10} & H_1^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & c_9 & 1 & c_{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & c_9 & H_1^2 & H_1^2c_{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{21} & 1 & c_{22} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & c_{21} & 1 & c_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & c_{21} & H_4^2 & H_4^2c_{22}
\end{matrix}\right]
$$

where $H_4 = \dfrac{k_{n4}}{k_{n3}}$.

Appendix B

1. Inter-relationships of wave coefficients in point force excitation cases

Combining Eq. (18) with other boundary and continuity conditions of the source plate, one has:

$$A_{3n} = \frac{\alpha_2\gamma_2 - \alpha_3\gamma_1}{\alpha_2 - \alpha_1\alpha_3} F_n \,, \tag{B. 1}$$

and

$$A_{4n} = \frac{\gamma_1 - \alpha_1\gamma_2}{\alpha_2 - \alpha_1\alpha_3} F_n \,, \tag{B. 2}$$

where $F_n = \dfrac{F_0\phi_n(y_0)}{4D_1\Lambda_n k_{n1}k_{p1}^2}$.

Analytical expressions of other wave coefficients can also be obtained from the following inter-relationships between wave coefficients determined from the compatibility conditions at the interfaces and the boundary conditions of the plate structure. For instance, Eq. (12) together with the boundary condition $W_1\big|_{x_I=0} = 0$ at the plate/plate junction gives:

$$A_{1n} = -\Theta_1 A_{3n} - \Theta_2 A_{4n} \,, \tag{B.3}$$

and

$$A_{2n} = -\Theta_3 A_{3n} - \Theta_4 A_{4n} \,. \tag{B.4}$$

The continuity and compatibility conditions at the virtual interface of the first and the second sub-domains yield:

$$A_{5n} = \frac{A_{1n}}{c_3} \,, \tag{B.5}$$

$$A_{6n} = A_{2n}c_1, \tag{B.6}$$

$$A_{7n} = \frac{A_{3n}}{c_4}, \tag{B.7}$$

and

$$A_{8n} = A_{4n}c_2. \tag{B.8}$$

The compatibility and continuity conditions at the source location give:

$$A_{9n} = A_{5n} + \frac{F_n}{c_5}, \tag{B.9}$$

$$A_{10n} = A_{6n} - \frac{F_n}{c_6}, \tag{B.10}$$

$$A_{11n} = A_{7n} - \frac{F_n}{c_7 H_1}, \tag{B.11}$$

and

$$A_{12n} = A_{8n} + \frac{F_n}{c_8 H_1}. \tag{B.12}$$

The compatibility and continuity conditions at the virtual interface of the second and the third sub-domains in conjunction with the simply supported boundary conditions of the source plate edge at $x = L_x$ provide:

$$A_{14n} = A_{10n}c_3, \tag{B.13}$$

$$A_{13n} = -A_{14n}c_9, \tag{B.14}$$

$$A_{15n} = \frac{A_{11n}}{c_{10}}, \tag{B.15}$$

and

$$A_{16n} = A_{12n} c_4 . \tag{B.16}$$

Combining Eq. (18) and one of the two coupled boundary conditions at the plate/plate junction gives:

$$A_{20n} = \frac{A_{1n} c_1 - A_{2n} + A_{3n} H_1 c_2 - A_{4n} H_1}{\Delta_1} . \tag{B.17}$$

Eq. (17) provides the inter-relationships of other wave coefficients of the receiving plate in terms of A_{20n} .

2. Inter-relationships of wave coefficients in moment excitation cases

For the moment excitation case, the compatibility and continuity conditions at the source location yield:

$$A_{9n} = A_{5n} - \frac{M_n^x - M_n^y}{c_5} , \tag{B.18}$$

$$A_{10n} = A_{6n} - \frac{M_n^x + M_n^y}{c_6} , \tag{B.19}$$

$$A_{11n} = A_{7n} + \frac{M_n^x - M_n^y / H_1}{c_7} , \tag{B.20}$$

and

$$A_{12n} = A_{8n} + \frac{M_n^x + M_n^y / H_1}{c_8} . \tag{B.21}$$

Following the same procedure as in the point force case, one has:

$$A_{3n} = \frac{\alpha_2 \gamma_4 - \alpha_3 \gamma_3}{\alpha_2 - \alpha_1 \alpha_3} M_n^y + \frac{\alpha_2 \gamma_2 - \alpha_3 \gamma_1}{\alpha_2 - \alpha_1 \alpha_3} M_n^x , \tag{B.22}$$

and

$$A_{4n}=\frac{\gamma_3-\alpha_1\gamma_4}{\alpha_2-\alpha_1\alpha_3}M_n^y+\frac{\gamma_1-\alpha_1\gamma_2}{\alpha_2-\alpha_1\alpha_3}M_n^x, \tag{B.23}$$

where $M_n^y=\dfrac{M_0\sin\varphi\phi_n(y_0)}{4D_1\Lambda_n k_{p1}^2}$, $M_n^x=\dfrac{M_0\cos\varphi\phi_n'(y_0)}{4D_1\Lambda_n k_{p1}^2 k_{n1}}$.

Inter-relationships of other wave coefficients in the moment excitation case are the same as in the point force excitation case, which can again be used to determine an analytical expression for each wave coefficient in this excitation case.

3. Energy components

The four energy components (T_1 - T_4) in Eq. (30) are given by:

$$\begin{aligned}T_1=\int_0^{L_{x1}}\sum_n|w_{1n}|^2dx=\sum_n\{&\frac{|A_{1n}|^2+|A_{2n}|^2}{2\operatorname{Re}(k_{n1})}(1-|c_1|^2)+\frac{|A_{3n}|^2+|A_{4n}|^2}{2\operatorname{Re}(k_{n2})}(1-|c_2|^2)\\&-\frac{2\operatorname{Re}(A_{1n}^*A_{2n})\operatorname{Im}(c_1)}{\operatorname{Im}(k_{n1})}-\frac{2\operatorname{Re}(A_{3n}^*A_{4n})\operatorname{Im}(c_2)}{\operatorname{Im}(k_{n2})}\\&+2\operatorname{Re}[\frac{A_{1n}^*A_{4n}+A_{2n}^*A_{3n}}{k_{n1}^*-k_{n2}}(c_2-c_1^*)+\frac{A_{1n}^*A_{3n}+A_{2n}^*A_{4n}}{k_{n1}^*+k_{n2}}(1-c_1^*c_2)]\}\end{aligned}, \tag{B.24}$$

$$\begin{aligned}T_2=\int_0^{x_0}\sum_n|w_{2na}|^2dx=\sum_n\{&\frac{|A_{5n}|^2(|c_5|^2-|c_3|^2)+|A_{6n}|^2(1-|c_6|^2)}{2\operatorname{Re}(k_{n1})}\\&+\frac{|A_{7n}|^2(|c_7|^2-|c_4|^2)+|A_{8n}|^2(1-|c_8|^2)}{2\operatorname{Re}(k_{n2})}-\frac{\operatorname{Im}[A_{5n}^*A_{6n}(c_3e^{-2j\arg(c_5)}-c_3^*)]}{\operatorname{Im}(k_{n1})}\\&-\frac{\operatorname{Im}[A_{7n}^*A_{8n}(c_4e^{-2j\arg(c_7)}-c_4^*)]}{\operatorname{Im}(k_{n2})}+2\operatorname{Re}[\frac{A_{5n}^*A_{8n}(c_5^*c_8-c_3^*)+A_{6n}^*A_{7n}(c_4-c_6^*c_7)}{k_{n1}^*-k_{n2}}\\&+\frac{A_{5n}^*A_{7n}(c_5^*c_7-c_3^*c_4)+A_{6n}^*A_{8n}(1-c_6^*c_8)}{k_{n1}^*+k_{n2}}]\}\end{aligned}, \tag{B.25}$$

$$\begin{aligned}T_3 = \int_{x_0}^{L_{x2}} \sum_n |w_{2nb}|^2 dx = \sum_n \{ &\frac{|A_{9n}|^2(1-|c_5|^2) + |A_{10n}|^2(|c_6|^2 - |c_3|^2)}{2\,\mathrm{Re}(k_{n1})} \\ &+ \frac{|A_{11n}|^2(1-|c_7|^2) + |A_{12n}|^2(|c_8|^2 - |c_4|^2)}{2\,\mathrm{Re}(k_{n2})} - \frac{\mathrm{Im}[A_{9n}^* A_{10n} c_3 (1 - e^{-2j\arg(c_5)})]}{\mathrm{Im}(k_{n1})} \\ &- \frac{\mathrm{Im}[A_{11n}^* A_{12n} c_4 (1 - e^{-2j\arg(c_7)})]}{\mathrm{Im}(k_{n2})} + 2\,\mathrm{Re}[\frac{A_{9n}^* A_{12n}(c_4 - c_5^* c_8) + A_{10n}^* A_{11n}(c_6^* c_7 - c_3^*)}{k_{n1}^* - k_{n2}} \\ &+ \frac{A_{9n}^* A_{11n}(1 - c_5^* c_7) + A_{10n}^* A_{12n}(c_6^* c_8 - c_3^* c_4)}{k_{n1}^* + k_{n2}}]\}\end{aligned}, \quad \text{(B.26)}$$

and

$$\begin{aligned}T_4 = \int_{0}^{L_{x3}} \sum_n |w_{3n}|^2 dx = \sum_n \{ &\frac{|A_{13n}|^2 + |A_{14n}|^2}{2\,\mathrm{Re}(k_{n1})}(1-|c_9|^2) + \frac{|A_{15n}|^2 + |A_{16n}|^2}{2\,\mathrm{Re}(k_{n2})}(1-|c_{10}|^2) \\ &- \frac{2\,\mathrm{Re}(A_{13n}^* A_{14n})\,\mathrm{Im}(c_9)}{\mathrm{Im}(k_{n1})} - \frac{2\,\mathrm{Re}(A_{15n}^* A_{16n})\,\mathrm{Im}(c_{10})}{\mathrm{Im}(k_{n2})} \\ &+ 2\,\mathrm{Re}[\frac{A_{13n}^* A_{16n} + A_{14n}^* A_{15n}}{k_{n1}^* - k_{n2}}(c_{10} - c_9^*) + \frac{A_{13n}^* A_{15n} + A_{14n}^* A_{16n}}{k_{n1}^* + k_{n2}}(1 - c_9^* c_{10})]\}\end{aligned}. \quad \text{(B.27)}$$

References

[1] J. L. Guyader, C. Boisson and C. Lesueur, Energy transmission in finite coupled plates, Part I: Theory. *Journal of Sound and Vibration*, 81 (1982), 81-92.

[2] C. Boisson, J. L. Guyader, P. Millot and C. Lesueur, Energy transmission in finite coupled plates, Part II: Application to an L shaped structure. *Journal of Sound and Vibration* 81 (1982), 93-105.

[3] J. M. Cuschieri, Structural power-flow analysis using a mobility approach of an L-shaped plate. *Journal of the Acoustical Society of America* 87 (1990), 1159-1165.

[4] J. M. Cuschieri and M. D. McCollum, In-plane and out-of-plane waves' power transmission through an L-plate junction using the mobility power flow approach. *Journal of the Acoustical Society of America* 100 (1996), 857-870.

[5] N. J. Kessissoglou, Power transmission in L-shaped plates including flexural and in-plane vibration. *Journal of the Acoustical Society of America* 115 (2004), 1157-1169.

[6] N. H. Farag and J. Pan, On the free and forced vibration of single and coupled rectangular plates. *Journal of the Acoustical Society of America* 104 (1998), 204-216.

[7] E. C. N. Wester and B. R. Mace, Statistical Energy Analysis of two edge-coupled rectangular plates: Ensemble averages. *Journal of Sound and Vibration* 193 (1996), 793-822.

[8] E. K. Dimitriadis and A. D. Pierce, Analytical solution for the power exchange between strongly coupled plates under random excitation: A test of Statistical Energy Analysis concepts. *Journal of Sound and Vibration* 123 (1988), 397-412.

[9] E. Rebillard and J. L. Guyader, Vibrational behaviour of a population of coupled plates: Hypersensitivity to the connexion angle. *Journal of Sound and Vibration* 188 (1995), 435-454.

[10] R. H. Lyon, *Statistical Energy Analysis of Dynamical Systems: Theory and Applications,* MIT Press, Cambridge, MA, 1975.

[11] J. Tratch Jnr, *Vibration transmission through machinery foundation and ship bottom structure*, MSc Thesis, MIT, Massachusetts, 1985.

[12] M. B. Skeen, *An investigation of energy flow through coupled plate structures*, PhD dissertation, University of New South Wales, Sydney, 2008

[13] W. H. Park, D. J. Thompson and N. S. Ferguson, The influence of modal behaviour on the energy transmission between two coupled plates, *Journal of Sound and Vibration* 276 (2004), 1019-1041

[14] M. B. Skeen and N. J. Kessissoglou, An investigation of transmission coefficients for finite and semi-infinite coupled plate structures, *Journal of the Acoustical Society of America* 122 (2007), 814 - 822.

[15] A. C. Nilsson, Attenuation of structure-borne sound in superstructures on ships. *Journal of Sound and Vibration* 55 (1977), 71-91.

[16] T. R. Lin, J. Pan and C. Mechefske, The effect of off-neutral axis excitation on the vibration response of finite ribbed plates. *Journal of Vibration and Acoustics, Transactions of ASME* 131 (2009), 011011-1 - 011011-11.

[17] T. R. Lin and J. Pan, A closed form solution for the vibration response of finite ribbed plates. *Journal of the Acoustical Society of America* 119 (2006), 917 - 925.

[18] L. Cremer, M. Heckl and B. A. T. Petersson, *Structure-Borne Sound,* 3rd Ed. Springer, Berlin, 2005.

[19] M. P. Norton, *Fundamentals of Noise and Vibration Analysis for Engineers,* Cambridge University Press, Cambridge UK, 1989.

[20] D. A. Bies and S. Hamid, In-situ determination of loss and coupling loss factors by the power injection method. *Journal of Sound and Vibration* 70 (1980), 187 - 204.

Table 1. Physical dimensions of the four L-shaped plate models used in the study

	Source Plate				Receiving Plate		
Model	Surface area (m^2)	Thickness (mm)	Modal density n(*f*)	¼ bending wavelength (m)	Surface area (m^2)	Thickness (mm)	Modal density n(*f*)
A	3.0×1.0	2	0.48	0.11 at 100Hz 0.035 at 1000Hz	3.6×1.0	2	0.58
B	3.0×1.0	2	0.48		1.5×1.0	8	0.06
C	1.2×1.0	8	0.048	0.22 at 100Hz 0.07 at 1000Hz	3.6×1.0	2	0.58
D	1.2×1.0	8	0.048		1.5×1.0	8	0.06

Figure Captions

Fig. 1. Model description and the coordinate system of the L-shaped plate.

Fig. 2. Local coordinate systems of the three sub-domains of the source plate, (a) sub-domain I; (b) sub-domain II; and (c) sub-domain III.

Fig. 3. Point force input mobility of the L-shaped plate.

Fig. 4. Moment input mobility of the L-shaped plate due to a moment excitation ($\varphi = 0^0$), (a) x component; (b) y component.

Fig. 5. Point force input mobility of the four plate models listed in Table 1 and those of the corresponding uncoupled component plates.

Fig. 6. Kinetic energy distribution of the L-shaped plate (Model A) due to a point force excitation applied at various plate locations away from the coupled edge.

Fig. 7. Kinetic energy distributions of the L-shaped plate (Model B) due to a point force excitations applied at various plate locations.

Fig. 8. Kinetic energy distributions of the L-shaped plate (Model C) due to a point force excitations applied at various plate locations.

Fig. 9. Kinetic energy distributions of the L-shaped plate (Model D) due to a point force excitations applied at various plate locations.

Fig. 10. Input mobility of the L-shaped plate (Model A) due to a point force excitation applied at various locations on the plate.

Fig. 11. Input mobility of the L-shaped plate (Model A) due to a moment excitation ($\varphi = 0^o$) applied at various locations away from the coupled edge.

Fig. 12. Input mobility of the L-shaped plate (Model A) due to the same moment excitations ($\varphi = 0^o$) but applied at various locations away from a plate edge parallels the moment axis.

Fig. 13. Kinetic energy distribution of the L-shaped plate (Model A) due to a moment excitations ($\varphi = 0^o$) applied at various locations away from the coupled edge.