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Lie group symmetry analysis for granular media stress equations

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Abstract

The Airy stress function, although frequently employed in classical linear elasticity, does not receive similar usage for granular media problems. For plane strain quasi-static deformations of a cohesionless Coulomb-Mohr granular solid, a single nonlinear partial differential equation is formulated for the Airy stress function by combining the equilibrium equations with the yield condition. This has certain advantages from the usual approach, in which two stress invariants and a stress angle are introduced, and a system of two partial differential equations is needed to describe the flow. In the present study, the symmetry analysis of differential equations is utilised for our single partial differential equation, and by computing an optimal system of one-dimensional Lie algebras, a complete set of group-invariant solutions is derived. By this it is meant that any group-invariant solution of the governing partial differential equation (provided it can be derived via the *classical* symmetries method) may be obtained as a member of this set by a suitable group transformation. For general values of the parameters (angle of internal friction ϕ and gravity g) it is found there are three distinct classes of solutions which correspond to granular flows considered previously in the literature. For the two limiting cases of high angle of internal friction and zero gravity, the governing partial differential equation admit larger families of Lie point symmetries, and from these symmetries, further solutions are derived, many of which are new. Furthermore, the majority of these solutions are exact, which is rare for granular flow, especially in the case of gravity driven flows.

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1 Introduction

The presence of granular or powder-like materials and their usage arises in many aspects of human life, varying from day-to-day activities in the home and the workplace, through to important industrial applications. The economies of countries such as Australia and South Africa heavily depend upon agricultural and mining export industries, and therefore there is considerable interest from many industries in the handling of bulk solids and more generally in the transport, storage and flow properties of granular materials (see Roberts [17], Arnold and Wypych [4], and Spencer and Hill [21]).

Granular materials constitute an intermediate state between solids and fluids and can behave like either, depending on the bulk density of the solid particles ρ. Many formulations for the equations which govern the deformation and the flow of these materials have been proposed, however there is no single model which is generally accepted to describe and predict the behaviour of all real materials under all practical or experimental conditions. At present there are mathematical models arising from continuum mechanics [23], statistical mechanics [12, 16], molecular dynamics modelling [1] and cellular automata modelling [18]. It is likely that different models will be required, not only for different materials, but also for the same material under different conditions.

Here, we follow the continuum mechanical approach, for which it is well accepted that the Coulomb-Mohr yield condition provides a reasonable basis for the determination of the stress profiles for quasi-static steady flow of granular materials (see for example, Spencer [23, 24]). The Coulomb-Mohr yield condition for a frictional and cohesive granular material postulates that slip occurs on the surface element with unit normal $\mathbf{n}$ if the magnitude of the shear component of traction τ attains the critical value

$$|\tau| = c - \sigma \tan \phi, \tag{1.1}$$

where ϕ is the angle of internal friction, c is the cohesion, and σ denotes the normal component of traction, here taken positive in tension. This sign convention, which means that positive forces produce positive stretches, is the one normally adopted in continuum mechanics. However, we observe that in the context of granular materials, the majority of stress distributions are compressive and therefore the stresses are negative. The two mechanical properties of the material are the cohesion c and the angle of internal friction ϕ. The special case $\phi = 0$ corresponds to a frictionless or purely cohesive material and gives rise to the Tresca yield condition of metal plasticity (see R. Hill [8]), while the limiting case $\phi = \pi/2$ corresponds to an idealised 'highly frictional' material (see Section 4). For the present study we restrict ourselves to cohesionless materials ($c = 0$), such as dry powders, but note that in the highly frictional limit ($\phi = \pi/2$) the governing equations are the same, regardless of the value of c.

In plane strain linear elasticity, the Airy stress function, which satisfies the bi-harmonic equation, is commonly employed to formulate governing equations to determine stresses. However, in the mechanics of granular materials, the usual convention is to introduce two stress invariants and a stress angle, the latter of which describes the direction of the maximum principal stress. This representation arises from the Mohr diagram and constitutes the fundamental preferred method of analysis adopted by the engineering profession. However, from a mathematical perspective the use of a stress angle, necessarily involving trigonometric functions, tends to introduce unnecessary complexities, which is the prime motivation for this study. Here we exploit the classical Airy stress function, which we denote by $\psi(x, y)$, to formulate the single nonlinear second order partial differential equation

$$\psi_{xy}^2 = \psi_{xx}\psi_{yy} + \rho g y \psi_{yy} - \frac{1}{4}(\psi_{xx} + \psi_{yy} + \rho g y)^2 \cos^2 \phi, \tag{1.2}$$

which governs the quasi-static gravity flow of a cohesionless Coulomb-Mohr granular material in two dimensions, under the influence of gravity. The quantities ρ and g are the bulk density and the acceleration due to gravity, respectively.

By considering the Lie point symmetries admitted by (1.2), we are able to derive a complete set (or optimal system) of group-invariant solutions for quasi-static flow. This means that apart from solutions which may arise from non-classical symmetries, we may exclude the possibility of there being group-invariant solutions to (1.2) which are not equivalent to ones presented here. The motivation is therefore to find and classify all group-invariant solutions for quasi-static flow, identify solutions which have been considered before in the literature, and highlight any new solutions. To achieve this end we use the Airy stress function, since this approach yields a single governing partial differential equation (1.2) which is more amenable to computer algebraic symmetry methods. In contrast, the traditional approach commonly used employs the stress invariants and the stress angle, and results in a system of two highly coupled nonlinear first order partial differential equations for which the derivation of all Lie point symmetries is much more difficult, because it involves trigonometric nonlinearities. Thus, although the two formulations are evidently equivalent, the Airy stress function approach results in a much simpler partial differential equation. We note that although the flow of granular materials occurs in many practical circumstances, only a limited number of exact analytical solutions are known, especially those incorporating gravity. Accordingly, any exact solutions to the governing equations have many potential applications.

The plan of the paper is as follows. In the following section, we summarise the basic continuum mechanics equations for two-dimensional flow of granular materials and derive the governing partial differential equation (1.2) for the Airy stress function. In Section 3 we use the Lie point symmetries admitted by this equation to derive an optimal system of group-invariant solutions. We find there are three equivalence classes of solutions, examples from which can be used to describe flow down an include, flow between contracting vertical walls, and flow through a converging wedge. The limiting case of $\phi = \pi/2$ is considered in Section 4. Here equation (1.2) admits further Lie point symmetries, and the optimal system contains nine equivalence classes of group-invariant solutions. All of these solutions can be derived exactly by solving the corresponding nonlinear ordinary differential equations. While the solutions presented in Section 3 have been considered before in the literature, many of those derived in Section 4 are new. Sections 5 is concerned with granular flows for which the effects of gravity may be ignored. Here we consider the group transformations admitted by (1.2) (with $g = 0$), and again derive the optimal system of group-invariant solutions. Finally, Section 6 contains a brief discussion.

2 Basic equations of granular continuum mechanics

In this section, we briefly summarise the basic equations that describe plane strain quasi-static steady flow of a granular material which conforms to the Coulomb-Mohr yield condition.

We consider here the two-dimensional state of stress using the usual rectangular Cartesian coordinates (x, y). The in-plane physical stress components are denoted by σ_{xx}, σ_{xy} and σ_{yy}. These components of the stress tensor satisfy the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho g, \tag{2.1}$$

where, as noted in the Introduction, ρ denotes the bulk density, which we assume to be constant, and g is the acceleration due to gravity. For cohesive granular materials, the stress relations are completed with the assumption of the Coulomb-Mohr yield condition

$$\{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\}^{1/2} = 2c\cos\phi - (\sigma_{xx} + \sigma_{yy})\sin\phi, \tag{2.2}$$

where ϕ denotes the angle of internal friction which we assume to be constant, and c denotes the coefficient of cohesion. The above equations are generally accepted as a reasonable basis for the determination of the stress components, although, as stated in the introduction, other more complicated theories exist.

We now introduce the Airy stress function $\psi = \psi(x, y)$, which is defined by the relations

$$\sigma_{xx} = \psi_{yy}, \quad \sigma_{xy} = -\psi_{xy}, \quad \sigma_{yy} = \psi_{xx} + \rho g y, \tag{2.3}$$

where the subscripts associated with the stress function ψ denote partial derivatives. The equilibrium equations (2.1) are now automatically satisfied, and the yield condition (2.2) becomes

$$\begin{aligned} (\psi_{xx} + \psi_{yy} + \rho g y)^2 \quad &+ \quad 4(\psi_{xy}^2 - \psi_{xx}\psi_{yy} - \rho g y \psi_{yy})\sec^2\phi = 4c^2 \\ &- \quad 4c(\psi_{xx} + \psi_{yy} + \rho g y)\tan\phi. \end{aligned} \tag{2.4}$$

The case $\phi = 0$ corresponds to metal plasticity, and equation (2.4) with $g = 0$ is given in Ames [2]. We do not consider this case here.

From this point onwards we consider only cohesionless granular materials, and thus assume $c = 0$. With this assumption, equation (2.4) reduces to (1.2), which is a non-linear second-order partial differential equation with variable coefficients. In the following four sections we provide a Lie symmetry analysis and deduce solutions for various special cases of this equation.

3 Group-invariant solutions for $\phi \neq \pi/2$ and $g \neq 0$

In this section, we seek group-invariant solutions of equation (1.2). Using the symbolic manipulator package DIMSYM [19], we find (1.2) admits a 6-parameter Lie group of transformations, with the associated group operators

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial \psi}, \quad \Gamma_3 = x\frac{\partial}{\partial \psi}, \quad \Gamma_4 = y\frac{\partial}{\partial \psi},$$

$$\Gamma_5 = \frac{\partial}{\partial y} - \tfrac{1}{2}\rho g x^2 \frac{\partial}{\partial \psi}, \quad \Gamma_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3\psi\frac{\partial}{\partial \psi}. \tag{3.1}$$

These operators form a basis for the corresponding Lie algebra, and the list of commutators

$$[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i \tag{3.2}$$

is presented in Table 1.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	Γ_2	0	$-\rho g\Gamma_3$	Γ_1
Γ_2	0	0	0	0	0	$3\Gamma_2$
Γ_3	$-\Gamma_2$	0	0	0	0	$2\Gamma_3$
Γ_4	0	0	0	0	$-\Gamma_2$	$2\Gamma_4$
Γ_5	$\rho g\Gamma_3$	0	0	Γ_2	0	Γ_5
Γ_6	$-\Gamma_1$	$-3\Gamma_2$	$-2\Gamma_3$	$-2\Gamma_4$	$-\Gamma_5$	0

Table 1: Table of commutators of operators (3.2) for equation (1.2)

Since the partial differential equation (1.2) involves two independent variables only, we may reduce it to an ordinary differential equation in the usual way by considering any one-parameter subgroup of the symmetry group (see [9], [14], for example). There are, of course, infinitely many of these subgroups (any linear combination of the Lie operators given in (3.1) corresponds to a one-parameter subgroup), so it is instructive to seek an "optimal system" of one-parameter subgroups, which will lead to an optimal system of group-invariant solutions, thus avoiding any redundancy and unnecessary computations.

We say that for a given differential equation, an optimal system of group-invariant solutions is a set of such solutions with the property that any other group-invariant solution is related to exactly one of these solutions by a group transformation admitted by the differential equation. That is, the set of all invariant solutions splits into equivalence classes, and an optimal system is one which contains an invariant solution from each class. In a similar way, an optimal system of subalgebras is defined to be a system for which every other subalgebra is equivalent to a member of this system under some element of the adjoint representation of the group. It can be shown that the problem of finding an optimal system of either one-parameter subgroups or group-invariant solutions is equivalent to finding an optimal system of one-parameter subalgebras. For details the reader is referred to the texts [14] and [15].

To compute the adjoint representation of the Lie algebra spanned by (3.2) we use the formula

$$\mathrm{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j = \Gamma_j - \epsilon[\Gamma_i, \Gamma_j] + \frac{1}{2}\epsilon^2[\Gamma_i, [\Gamma_i, \Gamma_j]] - \ldots,$$

where $[\Gamma_i, \Gamma_j]$ is the usual commutator, defined in (3.2). The resulting operators are given in Table 2. The optimal system of subalgebras is found by taking a general linear combination of the basis vectors (3.2) and simplifying it as much as possible by subjecting it to carefully chosen adjoint transformations. Again, the reader is referred to Olver [14] for details. The result is that an optimal system of one-dimensional subalgebras for this example is provided by those generated by each of the following basis vectors,

$$\{\Gamma_1 + \alpha\Gamma_5 + \beta\Gamma_4, \quad \Gamma_5 + \alpha\Gamma_4, \quad \Gamma_3 + \alpha\Gamma_4, \quad \Gamma_2, \quad \Gamma_4, \quad \Gamma_6\} \tag{3.3}$$

where α and β are arbitrary constants. We now consider each of the corresponding group-invariant solutions separately.

Example 3.1 By considering the set of operators $\Gamma_1 + \alpha\Gamma_5 + \beta\Gamma_4$, we may obtain the functional form

$$\psi(x, y) = \beta(xy - \tfrac{1}{2}\alpha x^2) - \tfrac{1}{6}\rho g\alpha x^3 + f(y - \alpha x),$$

where f is an arbitrary function of $z = y - \alpha x$. On substitution of this equation into (1.2)

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_2$	Γ_4	$\Gamma_5 + \rho g\epsilon\Gamma_3 - \frac{1}{2}\rho g\epsilon^2\Gamma_2$	$\Gamma_6 - \epsilon\Gamma_1$
Γ_2	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	$\Gamma_6 - 3\epsilon\Gamma_2$
Γ_3	$\Gamma_1 + \epsilon\Gamma_2$	Γ_2	Γ_3	Γ_4	Γ_5	$\Gamma_6 - 2\epsilon\Gamma_3$
Γ_4	Γ_1	Γ_2	Γ_3	Γ_4	$\Gamma_5 + \epsilon\Gamma_2$	$\Gamma_6 - 2\epsilon\Gamma_4$
Γ_5	$\Gamma_1 - \rho g\epsilon\Gamma_3$	Γ_2	Γ_3	$\Gamma_4 - \epsilon\Gamma_2$	Γ_5	$\Gamma_6 - \epsilon\Gamma_5$
Γ_6	$e^{\epsilon}\Gamma_1$	$e^{3\epsilon}\Gamma_2$	$e^{2\epsilon}\Gamma_3$	$e^{2\epsilon}\Gamma_4$	$e^{\epsilon}\Gamma_5$	Γ_6

Table 2: Table of adjoint operators. The (i,j)th entry is $\mathrm{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j$

we have

$$(1+\alpha^2)^2 f''^2 + 2[\rho g(1+\alpha^2-2\sec^2\phi)z - \alpha\beta(1+\alpha^2+2\sec^2\phi)]f'' + (\rho g z - \alpha\beta)^2 + 4\beta^2\sec^2\phi = 0, \tag{3.4}$$

where the primes denote differentiation with respect to z, and (3.4) may be written as

$$\begin{aligned} f'' &= \frac{[\alpha\beta(1+\alpha^2+2\sec^2\phi) - \rho g(1+\alpha^2-2\sec^2\phi)z]}{(1+\alpha^2)} \\ &\pm \frac{2\rho g\sec\phi\sqrt{\tan^2\phi-\alpha^2}}{(1+\alpha^2)}\sqrt{(z+A)^2-B^2}, \end{aligned}$$

where the quantities A and B are given by

$$A = \frac{\alpha\beta\sec^2\phi}{\rho g(\tan^2\phi-\alpha^2)}, \quad B = \frac{\beta(1+\alpha^2)\tan\phi}{\rho g(\tan^2\phi-\alpha^2)},$$

provided that either

$$z \le -\frac{|\beta|}{\rho g}\left(\frac{\mathrm{sgn}(\beta)+\alpha\tan\phi}{\alpha-\mathrm{sgn}(\beta)\tan\phi}\right) \quad \text{or} \quad z \ge \frac{|\beta|}{\rho g}\left(\frac{\mathrm{sgn}(\beta)-\alpha\tan\phi}{\alpha+\mathrm{sgn}(\beta)\tan\phi}\right).$$

Hence, the solution of (3.4) is given by

$$\begin{aligned} f(z) &= \frac{[3\alpha\beta(1+\alpha^2+2\sec^2\phi)z^2 - \rho g(1+\alpha^2-2\sec^2\phi)z^3]}{6(1+\alpha^2)} \\ &\pm \frac{\rho g\sec\phi\sqrt{\tan^2\phi-\alpha^2}}{(1+\alpha^2)^2}\left[\frac{1}{3}\sqrt{(z+A)^2-B^2}\{(z+A)^2+2B^2\}\right. \\ &- \left. B^2(z+A)\log\left|z+A+\sqrt{(z+A)^2-B^2}\right|\right] + C_1 z + C_2, \end{aligned} \tag{3.5}$$

where C_1 and C_2 are arbitrary constants of integration.

This solution describes quasi-static flow down an incline (Sokolovskii [20]). The parameter α corresponds to $\alpha = -\tan\delta$, where δ is the angle which the incline makes with the horizontal. In the limit $\alpha \to 0$ with $\beta \neq 0$, the functional form is $\psi(x,y) = \beta xy + f(y)$, and for this case the solution is

$$\begin{aligned} f(y) &= \tfrac{1}{6}\rho g y^3(2\sec^2\phi - 1) \pm \rho g \sec\phi\tan\phi \left[\frac{1}{3}\sqrt{y^2 - \frac{\beta^2\cot^2\phi}{\rho^2 g^2}}\left(y^2 + \frac{2\beta^2\cot^2\phi}{\rho^2 g^2}\right)\right. \\ &- \left.\frac{\beta^2\cot^2\phi}{2\rho^2 g^2} y \log\left|y + \sqrt{y^2 - \frac{\beta^2\cot^2\phi}{\rho^2 g^2}}\right|\right] + C_1 y + C_2, \qquad (3.6) \end{aligned}$$

provided that $|y| \geq \beta\cot\phi/\rho g$.

The parameter β is a constant related to the shear stress. In the limit $\beta \to 0$ with $\alpha \neq 0$, the functional form is $\psi(x,y) = -\rho g \alpha x^3/6 + f(y - \alpha x)$ and for this case the solution is

$$f(z) = -\frac{\rho g z^3}{6(1+\alpha^2)^2}[(1 + \alpha^2 - 2\sec^2\phi) \mp 2\sec\phi\sqrt{\tan^2\phi - \alpha^2}] + C_1 z + C_2, \qquad (3.7)$$

provided that $|\alpha| \leq \tan\phi$. In this limit there is a traction-free surface located at $z = 0$.

Finally, in the limit $\alpha, \beta \to 0$, the solution corresponds to horizontal flow with zero shear stress. Here the functional form becomes $\psi(x,y) = f(y)$ and for this case the solution is

$$f(y) = \tfrac{1}{6}\rho g y^3 \tan^2\left(\frac{\pi}{4} \pm \frac{\phi}{2}\right) + C_1 y + C_2. \qquad (3.8)$$

Example 3.2 The functional form $\psi(x,y) = \frac{1}{2}(\alpha y^2 - \rho g x^2 y) + f(x)$ may be obtained by considering the operator $\Gamma_5 + \alpha\Gamma_4$ of (3.3), where f is an arbitrary function of x. Upon substitution of this functional form into (1.2), we obtain the following equation

$$f''^2 + 2\alpha(1 - 2\sec^2\phi)f'' + [\alpha^2 + (2\rho g x\sec\phi)^2] = 0, \qquad (3.9)$$

which may be rewritten in the form

$$f'' = \alpha(2\sec^2\phi - 1) \pm 2\sec\phi\sqrt{\alpha^2\tan^2\phi - \rho^2 g^2 x^2},$$

where the primes denote differentiation with respect to x. Upon solving we obtain the general solution for (3.9)

$$f(x) = \frac{\alpha}{2}(2\sec^2\phi - 1) \pm \rho g \sec\phi\left[\frac{\alpha^2\tan^2\phi}{\rho^2 g^2}\left\{x \arcsin\left(\frac{\rho g x\cot\phi}{\alpha}\right) + \sqrt{\frac{\alpha^2\tan^2\phi}{\rho^2 g^2} - x^2}\right\}\right.$$

$$- \frac{1}{3}\left(\frac{\alpha^2 \tan^2\phi}{\rho^2 g^2} - x^2\right)^{1/3}\Bigg] + C_1 x + C_2, \tag{3.10}$$

provided that $\alpha \neq 0$ and $x^2 \leq \alpha^2 \tan^2\phi/(\rho g)^2$, where C_1 and C_2 are arbitrary constants of integration. For $\alpha \to 0$, the ordinary differential equation obtained by the substitution of the functional form $\psi(x, y) = -\frac{1}{2}\rho g x^2 y + f(x)$ generated by Γ_5 gives rise to complex solution which is non-physical. For $\alpha = -\rho g l \cot\phi$, where l is the positive constant and $|x| \leq l$, the solution (with positive sign) corresponds to one given by Spencer and Bradley [22], and describes material which is compressed between two vertical walls.

Example 3.3 Finally, the operator Γ_6 leads to the important functional form

$$\psi(x, y) = x^3 f\left(\frac{y}{x}\right). \tag{3.11}$$

Here the function f satisfies the nonlinear ordinary differential equation

$$(1+\xi^2)^2 f''^2 - 8\xi(1+\xi^2)f'f'' + 12(1+\xi^2 - 2\sec^2\phi)ff'' + 16(\xi^2 + \sec^2\phi)f'^2$$

$$-48\xi f f' + 36 f^2 + \rho^2 g^2 \xi^2 + 2\rho g\xi(6f - 4\xi f' + (1+\xi^2)f'') = 0,$$

where the primes denote differentiation with respect to $\xi = y/x$. The similarity solution (3.11) corresponds to the well-known so-called 'radial stress field', first considered by Jenike [10] and Sokolovskii [20]. It can be used to model the flow of granular materials near the outlet of a wedged-shaped hopper. Unfortunately we are unable to solve this ordinary differential equation analytically.

We note that the operators $\Gamma_3 + \alpha\Gamma_4$, Γ_4 and Γ_2 have no group-invariant solutions, so that all group-invariant solutions which may be obtained via classical means fall into either one of these three equivalence classes, and can be obtained by one of the solutions presented in Examples 3.1-3.3 by an appropriate group transformation.

4 Group-invariant solutions for $\phi = \pi/2$ and $g \neq 0$

Materials for which $\phi = \pi/2$ may be referred to as 'highly frictional', as this is the limiting value for the angle of internal friction. In this limit, the Coulomb-Mohr yield condition states that $\sigma_{xy}^2 = \sigma_{xx}\sigma_{yy}$, which implies the maximum principal stress (smallest in magnitude) tends to zero at every point in the material. In practise, the largest measured angle of internal friction is about 70° (see Sture [25], for example), however it is still instructive to proceed for the following reasons. Firstly, there are materials for which $\cos^2\phi$ is small, and hence this idealised theory can describe a bound, or limit, for physically meaningful materials. Secondly, the governing equation for $\phi = \pi/2$ can be used to describe the leading order term of a regular perturbation for ψ, with correction terms (not considered here) being of order $(1-\sin\phi)$. Thirdly, it so happens that for the limit $\phi = \pi/2$, we are able to construct a number of exact solutions to the nonlinear partial differential equation (1.2), which is an exception in the theory of granular materials, and hence worth pursuing. Finally, these exact solutions may be used to validate numerical schemes which are devised for the more general case of $\phi < \pi/2$.

With the value $\phi = \pi/2$, equation (1.2) reduces to

$$\psi_{xy}^2 = \psi_{xx}\psi_{yy} + \rho g y \psi_{yy}, \tag{4.1}$$

from which the following Lie-point symmetries are obtained

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial \psi}, \quad \Gamma_3 = x\frac{\partial}{\partial \psi}, \quad \Gamma_4 = y\frac{\partial}{\partial \psi}, \quad \Gamma_5 = \frac{\partial}{\partial y} - \tfrac{1}{2}\rho g x^2\frac{\partial}{\partial \psi},$$
$$\Gamma_6 = y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}, \quad \Gamma_7 = x\frac{\partial}{\partial y} - \tfrac{1}{6}\rho g x^3\frac{\partial}{\partial \psi}, \quad \Gamma_8 = x\frac{\partial}{\partial x} + 2\psi\frac{\partial}{\partial \psi}.$$

We compute an optimal system of subalgebras in the same way as that described in the previous section, however for brevity we omit the details. The result is that an optimal system is spanned by each of the operators

$$\{\Gamma_8 - \alpha\Gamma_6, \quad \Gamma_8 \pm \Gamma_5, \quad \Gamma_8 + \Gamma_6 \pm \Gamma_7, \quad \Gamma_8 - 2\Gamma_6 \pm \Gamma_2, \quad \Gamma_8 - \Gamma_6 \pm \Gamma_3, \quad \Gamma_6 + \alpha\Gamma_1 + \beta\Gamma_4,$$
$$\Gamma_1 + \alpha\Gamma_4 + \beta\Gamma_7, \quad \Gamma_7 \pm \Gamma_4, \quad \Gamma_7 \pm \Gamma_2, \quad \Gamma_5 \pm \Gamma_4, \quad \Gamma_7, \quad \Gamma_5, \quad \Gamma_4, \quad \Gamma_3, \quad \Gamma_2\}, \tag{4.2}$$

where α and β are arbitrary constants. We now present the corresponding functional forms, and solve for the group-invariant solutions exactly.

Example 4.1 The operator $\Gamma_8 - \alpha\Gamma_6$ corresponds to interesting similarity solutions of the form

$$\psi(x, y) = x^{2-\alpha} f(x^\alpha y),$$

which on substitution into (4.1) yields

$$4f'^2 = f''[\rho g\xi + (\alpha-1)(\alpha-2)f - \alpha(\alpha+1)\xi f'], \tag{4.3}$$

where the primes denote differentiation with respect to $\xi = x^\alpha y$. We are able to solve (4.3) exactly for four different values of α.

For $\alpha = -1$, the similarity solution is of the form is $\psi = x^3 f(y/x)$. This is the same functional form as that considered in Example 3.3, and hence is a special case of the solution which describes the radial stress field. In contrast to Example 3.3, we are able to solve for f exactly. From (4.3) the appropriate differential equation is

$$(6f + \rho g\xi)f'' = 4f'^2, \tag{4.4}$$

which, with the use of the transformation $u(\xi) = 6f + \rho g\xi$, can be reduced to

$$3uu'' = 2(u' - \rho g)^2.$$

After making the substitution $u'(\xi) = p(u)$ and solving the resultant equation, we obtain

$$6f + \rho g\xi = C_1(p - \rho g)^{3/2} \exp\left\{-\frac{3\rho g}{2(p-\rho g)}\right\}. \tag{4.5}$$

A further integration gives

$$\xi = \frac{6f}{p-\rho g} + \frac{pC_1}{p-\rho g}\int^p \frac{1}{t^2}(t-\rho g)^{3/2}\exp\left\{-\frac{3\rho g}{2(t-\rho g)}\right\} dt + \frac{pC_2}{p-\rho g}, \tag{4.6}$$

where C_1 and C_2 are constants of integration, and therefore, the parametric solution of (4.4) is given by equations (4.5)-(4.6). A solution equivalent to this has been presented by Hill and Cox [7].

For $\alpha = 0$, the resulting functional form is $\psi = x^2 f(y)$. From (4.3), we find that

$$(2f + \rho g y)f'' = 4f'^2, \tag{4.7}$$

which can be reduced to

$$uu'' = 2(u' - \rho g)^2. \tag{4.8}$$

by making the substitution $u(y) = 2f + \rho g y$. Furthermore, we transform (4.8) into $upp' = 2(p - \rho g)^2$ by using $u'(y) = p(u)$, and upon solving this equation, we obtain

$$2f + \rho g y = C_1(p - \rho g)^{1/2} \exp\left\{-\frac{\rho g}{2(p-\rho g)}\right\}. \tag{4.9}$$

A further integration yields

$$y = \frac{2f}{p-\rho g} + \frac{pC_1}{p-\rho g}\int^p \frac{1}{t^2}(t-\rho g)^{1/2}\exp\left\{-\frac{\rho g}{2(t-\rho g)}\right\} dt + \frac{pC_2}{p-\rho g}, \tag{4.10}$$

where C_1 and C_2 are constants of integration, and therefore, the parametric solution of (4.7) is given by (4.9)-(4.10). This solution is equivalent to one presented by Thamwattana and Hill [26].

For $\alpha = 1$, the functional form is $\psi = xf(xy)$. From (4.3), we deduce

$$\xi f''(\rho g - 2f') = 4f'^2, \tag{4.11}$$

where the primes denote differentiation with respect to $\xi = xy$. This equation may be reduced to

$$\xi(\rho g - 2p)p' = 4p^2, \tag{4.12}$$

following the transformation $f'(\xi) = p(\xi)$, and integration of (4.12) yields

$$\xi = C_1 p^{-1/2}\exp\left\{-\frac{\rho g}{4p}\right\}, \tag{4.13}$$

where C_1 denotes an arbitrary constant of integration. A further integration gives

$$f(\xi) = p\xi - C_1\int^p t^{-1/2}e^{-\rho g/4t}dt + C_2, \tag{4.14}$$

where C_2 is constant of integration, and therefore, the parametric solution of (4.11) is given by (4.13)-(4.14), where p is the parameter.

For $\alpha = 2$, the functional form is $\psi = f(x^2y)$ and equation (4.3) becomes

$$\xi f''(\rho g - 6f') = 4f'^2. \tag{4.15}$$

Upon making the substitution $f'(\xi) = p(\xi)$ into (4.15) we obtain

$$\xi p'(\rho g - 6p) = 4p^2,$$

and on solving this equation, we find

$$\xi = C_1 p^{-3/2}\exp\left\{\frac{-\rho g}{4p}\right\}, \tag{4.16}$$

where C_1 denotes an arbitrary constant of integration. A further integration gives

$$f(\xi) = p\xi - C_1\int^p t^{-3/2}\exp\left\{\frac{-\rho g}{4t}\right\} dt + C_2, \tag{4.17}$$

where C_2 is a further arbitrary constant. Equations (4.16)-(4.17) constitute the general parametric solution of (4.15).

We note that while solutions equivalent to those for $\alpha = -1$ and $\alpha = 0$ have been derived previously in the literature, the ones for $\alpha = 1$ and $\alpha = 2$ are new. Unless indicated otherwise, the remainder of the solutions given in this section are new.

Example 4.2 The functional form $\psi(x,y) = -\frac{1}{2}\rho g x^2 \log x + x^2 f(y - \log x)$ may be obtained by considering the operator $\Gamma_8 + \Gamma_5$ ($\Gamma_8 - \Gamma_5$ may be obtained from $\Gamma_8 + \Gamma_5$ by discrete symmetry) of (4.2). Upon substitution of this functional form into (4.1), we obtain

$$2f'f'' - 8f'^2 + [4f + \rho g(2z - 3)]f'' = 0, \tag{4.18}$$

where the primes denote differentiation with respect to $z = y - \log x$. The transformation $u(z) = 4f + \rho g(2z - 3)$ reduces (4.18) to

$$u''(u' - 2\rho g + 2u) = 4(u' - 2\rho g)^2, \tag{4.19}$$

and upon making the substitution $p(u) = u'(z) - 2\rho g$ into (4.19) we obtain

$$\frac{dq}{du}(p + 2\rho g)(p + 2u) = 4p^2.$$

On rewriting this equation in the following form

$$\frac{du}{dp} - \frac{(p + 2\rho g)}{2p^2}u = \frac{p + 2\rho g}{4p}$$

and solving it, we obtain

$$u = (1/4)p^{1/2}e^{-\rho g/p}[I(p) + C_1], \tag{4.20}$$

where C_1 denotes an arbitrary constant and the integral $I(p)$ is given by

$$I(p) = \int^p (t + 2\rho g)t^{-3/2}e^{\rho g/t}dt.$$

A further integration gives

$$z = \frac{4f - 3\rho g}{p} + \frac{p + 2\rho g}{4p}\int^p \frac{\sqrt{t}}{(t + \rho g)^2}e^{-\rho g/t}[I(t) + C_1]\,dt + \frac{C_2(p + 2\rho g)}{p}, \tag{4.21}$$

where C_2 is a further arbitrary constant. Equations (4.20)-(4.21) constitute the general parametric solution of (4.19), from which we may deduce the general parametric solution of the original equation (4.18).

Example 4.3 The functional form $\psi(x,y) = -\frac{1}{6}\rho g x^3 \log x + x^3 f(y/x - \log x)$ corresponds to the operator $\Gamma_8 + \Gamma_6 + \Gamma_7$ of (4.2) ($\Gamma_8 + \Gamma_6 - \Gamma_7$ may be obtained from $\Gamma_8 + \Gamma_6 + \Gamma_7$ by the discrete symmetry). Here, the function f satisfies the differential equation

$$4f'^2 = [6f - f' + \rho g(z - 5/6)]f'', \tag{4.22}$$

where the primes denote differentiation with respect to $z = y/x - \log x$. Equation (4.22) may be solved in a similar manner as described in Example 4.2 and we merely state the parametric solutions of (4.22):

$$\begin{aligned} 6f &+ \rho g(z - 5/6) = \frac{1}{4}(C_1 - I(p))p^{3/2}e^{-3\rho g/2p}, \\ z &= \frac{36f - 5\rho g}{6p} + \frac{p + 2\rho g}{4p}\int^p \frac{1}{4}\frac{[C_1 - I(t)]t^{3/2}e^{-3\rho g/2t}}{(t+\rho g)^2}\,dt + \frac{C_2(p + 2\rho g)}{p}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants of integration, and the integral $I(p)$ is given by

$$I(p) = \int^p (t + \rho g)t^{-5/6}e^{3\rho g/2t}\,dt.$$

Example 4.4 We may obtain the functional form $\psi(x,y) = \log x + f(yx^2)$ from $\Gamma_8 - 2\Gamma_6 + \Gamma_2$ of (4.2) ($\Gamma_8 - 2\Gamma_6 - \Gamma_2$ may be obtained from $\Gamma_8 - 2\Gamma_6 + \Gamma_2$ by the discrete symmetry). On substitution of this functional form into (4.1), we find the equation

$$4f'^2 = [\rho g z - 1 - 6zf']f'', \tag{4.23}$$

where the primes denote differentiation with respect to $z = x^2 y$. Upon solving the equation (4.23), we obtain the parametric solution:

$$\begin{aligned} z &= -\frac{1}{4}[I(p) + C_1]p^{-3/2}e^{-\rho g/4p}, \\ f(z) &= pz + \frac{1}{4}\int^p [I(t) + C_1]t^{-3/2}e^{-\rho g/4t}\,dt + C_2, \end{aligned}$$

where C_1 and C_2 are arbitrary constants of integration, and the integral $I(w)$ is given by

$$I(p) = \int^p t^{-1/2}e^{-\rho g/4t}\,dt.$$

Example 4.5 The operator $\Gamma_8 - \Gamma_6 + \Gamma_3$ of (4.2) corresponds to the group-invariant solution $\psi(x,y) = x\log x + xf(xy)$, with f satisfying

$$4f'^2 = [\rho g z + 1 - 2zf']f''. \tag{4.24}$$

This equation is similar to one solved in the previous Example 4.4; we simply provide the parametric solution

$$\begin{aligned} z &= \frac{1}{4}[I(p)+C_1]p^{-1/2}e^{-\rho g/4p}, \\ f(z) &= pz - \frac{1}{4}\int^p [I(t)+C_1]t^{-1/2}e^{-\rho g/4t}\,dt + C_2, \end{aligned}$$

where C_1 and C_2 are arbitrary constants of integration, and the integral $I(p)$ is given by

$$I(p) = \int^p t^{-3/2}e^{-\rho g/4t}\,dt.$$

Example 4.6 The functional form $\psi(x,y) = \beta y \log y + y f(x - \alpha \log y)$ is found by considering the operator $\Gamma_6 + \alpha\Gamma_1 + \beta\Gamma_4$ from (4.2). Upon substitution of this functional form into (4.1) we obtain the equation

$$f'^2 + \rho g(\alpha f' - \beta) = [\beta + \rho g\alpha^2 + \alpha f']f'',$$

where the primes denote differentiation with respect to $z = x - \alpha \log y$. For $\alpha, \beta \neq 0$, we obtain the parametric solution

$$\begin{aligned} z &= \frac{\alpha}{2}\log|p^2 + \rho g(\alpha p - \beta)| + \frac{2\beta + \rho g\alpha}{2\sqrt{\rho g(4\beta + \rho g\alpha^2)}}\log\left|\frac{p + \frac{1}{2}\rho g\alpha - \beta}{p + \frac{1}{2}\rho g\alpha + \beta}\right| + C_1, \\ f(z) &= p(z+\alpha) - \frac{[2p(2\beta + \rho g\alpha) - (\alpha + 4\beta)(\rho g\alpha)^2]}{4\sqrt{\rho g(4\beta + \rho g\alpha^2)}}\log\left|\frac{p + \frac{1}{2}\rho g\alpha - \beta}{p + \frac{1}{2}\rho g\alpha + \beta}\right| \\ &- \frac{\alpha}{4}(2p + \rho g\alpha)\log|p^2 + \rho g(\alpha p - \beta)| - \frac{1}{2}[(2\beta - \rho g\alpha) + \log|p + \tfrac{1}{2}\rho g\alpha - \beta| \\ &+ (2\beta + \rho g\alpha)\log|p + \tfrac{1}{2}\rho g\alpha + \beta|] - C_1 p + C_2, \end{aligned}$$

where C_1 and C_2 are arbitrary constants of integration.

For $\alpha, \beta \to 0$, the functional form is $\psi(x,y) = yf(x)$, which yields the trivial solution $f(x) =$constant.

For $\alpha \to 0$ and $\beta \neq 0$, the functional form becomes $\psi(x,y) = \beta y \log y + y f(x)$ and its corresponding ordinary differential equation is

$$f'^2 - \rho g\beta = \beta f''.$$

The solution to this equation is

$$f(x) = \sqrt{\rho g\beta}x - \beta\log|1 - C_1 e^{2x\sqrt{\rho g/\beta}}| + C_2.$$

For $\beta \to 0$ and $\alpha \neq 0$, the functional form is $\psi(x,y) = yf(x - \alpha \log y)$, so that f must satisfy equation is

$$(f' - f'')(f' + \rho g \alpha) = 0,$$

where the primes denote differentiation with respect to $z = x - \alpha \log y$. It follows that there are two possible solutions

$$f(z) = -\rho g \alpha z + C_1, \quad f(z) = \alpha C_2 e^{z/\alpha} + C_3,$$

where C_3 is a further arbitrary constant.

Example 4.7 The functional form $\psi(x,y) = \alpha(xy - \frac{1}{3}\beta x^3) - \frac{1}{24}\rho g \beta x^4 + f(y - \frac{1}{2}\beta x^2)$ may be found from the operator $\Gamma_1 + \alpha\Gamma_4 + \beta\Gamma_7$ of (4.2). Upon substitution of this functional form into (4.1), we obtain the equation

$$[\rho g z - \beta f']f'' = \alpha^2, \tag{4.25}$$

where the primes denote differentiation with respect to $z = y - \frac{1}{2}\beta x^2$. Upon solving (4.25) we find the parametric solution for $\alpha, \beta \neq 0$

$$\begin{aligned} z &= \frac{\beta}{\rho g}\left(p + \frac{\alpha^2}{\rho g}\right) + C_1 e^{\rho g p/\alpha^2}, \\ f(z) &= pz - \frac{\beta}{2\rho g}\left(p + \frac{\alpha^2}{\rho g}\right)^2 - \frac{\alpha^2 C_1}{\rho g} e^{\rho g p/\alpha^2} + C_2, \end{aligned} \tag{4.26}$$

where C_1 and C_2 are arbitrary constants of integration.

For $\alpha, \beta \to 0$, the functional form is $\psi(x,y) = f(y)$, which gives rise to trivial solution in y.

For $\alpha \to 0$ and $\beta \neq 0$, the functional form is $\psi(x,y) = -\frac{1}{24}\rho g \beta x^4 + f(y - \frac{1}{2}\beta x^2)$ and the corresponding differential equation is

$$[\rho g z - \beta f']f'' = 0.$$

There are two possible solutions

$$f(z) = C_1 z + C_2, \quad f(z) = \frac{\rho g}{2\beta} z^2 + C_3,$$

where C_3 is an further arbitrary constant of integration.

For $\beta \to 0$ and $\alpha \neq 0$, the functional form is $\psi(x,y) = \alpha xy + f(y)$ and the corresponding differential equation is $f'' = \alpha^2/(\rho g y)$, with solution

$$f(y) = \frac{\alpha^2}{\rho g}(y \log y + y) + C_1 y + C_2.$$

This latter solution and the following two examples are less interesting, and have been considered before by Thamwattana and Hill [26].

Example 4.8 By considering the operator $\Gamma_7 + \Gamma_4$ of (4.2), we may derive group invariant solutions of the form $\psi(x,y) = y^2/2x - \frac{1}{6}\rho g x^2 y + f(x)$ ($\Gamma_7 - \Gamma_4$ may be obtained from $\Gamma_7 + \Gamma_4$ by the discrete symmetry). On substitution of this functional form into (4.1), we obtain

$$f''(x) = \left(\frac{\rho g}{3}\right)^2 x^3,$$

which gives rise to the solution

$$f(x) = \left(\frac{\rho g}{3}\right)^2 \frac{x^5}{20} + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants of integration.

Example 4.9 For $\Gamma_5 + \Gamma_4$, we obtain the functional form $\psi(x,y) = \frac{1}{2}(y^2 - \rho g x^2 y) + f(x)$, which leads to the ordinary differential equation $f''(x) = (\rho g x)^2$. Integration reveals the solution

$$\psi(x,y) = \tfrac{1}{12}(\rho g)^2 x^4 + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants of integration.

It is noted that the operators $\Gamma_7 \pm \Gamma_2$, Γ_7, Γ_5, Γ_4, Γ_3 and Γ_2 have no group-invariant solutions. Thus, for the limiting case $\phi = \pi/2$, all group-invariant solutions of (4.1) which may be obtained via classical Lie symmetries are equivalent to exactly one of the Examples 4.1-4.9.

5 Group-invariant solutions for $\phi \neq \pi/2$ and $g = 0$

In this section we examine an optimal system of group-invariant solutions of equation (1.2) for the case in which the effects of gravity may be ignored. We see that on neglecting gravity, equation (1.2) becomes

$$\psi_{xy}^2 = \psi_{xx}\psi_{yy} - \frac{1}{4}(\psi_{xx} + \psi_{yy})^2 \cos^2\phi, \tag{5.1}$$

which is presented in Ames [2]. With the use of DIMSYM [19], we find this equation admits the 8-parameter Lie group of operators

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial y}, \quad \Gamma_3 = \frac{\partial}{\partial \psi}, \quad \Gamma_4 = x\frac{\partial}{\partial \psi}, \quad \Gamma_5 = y\frac{\partial}{\partial \psi},$$
$$\Gamma_6 = \psi\frac{\partial}{\partial \psi}, \quad \Gamma_7 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \Gamma_8 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}. \tag{5.2}$$

An optimal system of one-dimensional subalgebras of (5.1) is derived in the same fashion as in Sections 3 and 4. For the sake of brevity, we simply state the result

$$\{\Gamma_7 + \alpha\Gamma_4 + \beta\Gamma_6 - \gamma\Gamma_8, \quad \Gamma_7 + \alpha\Gamma_4 + \beta\Gamma_6 - \gamma\Gamma_8 \pm \Gamma_1, \quad \Gamma_7 + \alpha(\Gamma_6 - \Gamma_8) \pm \Gamma_3,$$
$$\Gamma_8 + \alpha\Gamma_6, \quad \Gamma_8 \pm \Gamma_4, \quad \Gamma_8 - \Gamma_6 \pm \Gamma_3, \quad \Gamma_6, \quad \Gamma_6 \pm \Gamma_1, \quad \Gamma_1 + \alpha\Gamma_4, \quad \Gamma_4, \quad \Gamma_3\}, \tag{5.3}$$

where α, β, and γ are arbitrary constants. Now we provide a complete list of functional forms, and solve for the group-invariant solutions.

Example 5.1 From the first operator $\Gamma_7 + \alpha\Gamma_4 + \beta\Gamma_6 - \gamma\Gamma_8$ of (5.3), we obtain the functional form

$$\psi(r, \theta) = \frac{\alpha r}{1 + \beta^2}(\sin\theta - \beta\cos\theta) + e^{(\beta-\gamma)\theta} f(re^{\gamma\theta}), \tag{5.4}$$

where $r = (x^2 + y^2)^{1/2}$ and θ is given by $\theta = \arctan(y/x)$. Since the functional form (5.4) is in r and θ, we reduce the equation (5.1) to

$$[r^2\psi_{rr} + \psi_{\theta\theta} + r\psi_r]^2 + 4\sec^2\phi[r^2\psi_{r\theta}^2 + \psi_\theta^2 - r^2\psi_{rr}\psi_{\theta\theta} - r^3\psi_r\psi_{rr} - 2r\psi_\theta\psi_{r\theta}] = 0 \tag{5.5}$$

by the transformations $x = r\cos\theta$ and $y = r\sin\theta$. From (5.4)-(5.5), we obtain the non-linear ordinary differential equation

$$4\sec^2\phi[(1+\gamma^2)\xi^3 f' f'' + (\beta^2 - \gamma^2)\xi^2 f f'' - (\beta-\gamma)^2\xi^2 f'^2 + 2(\beta-\gamma)^2\xi f f' - (\beta-\gamma)^2 f^2]$$
$$-[(1+\gamma^2)\xi^2 f'' + (2\beta\gamma - \gamma^2 + 1)\xi f' + (\beta-\gamma)^2 f]^2 = 0, \tag{5.6}$$

where the primes denote differentiation with respect to $\xi = re^{\gamma\theta}$. Equation (5.6) may be reduced to

$$4\sec^2\phi[(1+\gamma^2)u'u'' + (\beta^2-\gamma^2)uu'' - ((\beta-\gamma)^2+1+\gamma^2)u'^2 + (\beta-\gamma)(\beta-3\gamma)uu'$$
$$-(\beta-\gamma)^2u^2] - [(1+\gamma^2)u'' + 2\gamma(\beta-\gamma)u' + (\beta-\gamma)^2u]^2 = 0 \qquad (5.7)$$

by the transformation $f(\xi) = u(z)$ with $z = \log\xi$, where here the primes denote differentiation with respect to z. It is noted that the first term in (5.4) is itself an exact solution to (5.5), so without loss of generality, we set $\alpha = 0$ in what follows. We are unable to solve equation (5.7) for general β and γ, however we shall solve (5.7) for two particular cases.

Firstly, if we look for solutions of (5.7) of the form $u(z) = Ae^{mz}$ for certain constants A and m, we find that A is arbitrary while m satisfies the quartic equation

$$[(1+\gamma^2)m^2+2\gamma(\beta-\gamma)m+(\beta-\gamma)^2]^2 = 4\sec^2\phi(m-1)[(1+\gamma^2)m^2+2\gamma(\beta-\gamma)m+(\beta-\gamma)^2].$$

Accordingly, m may be determined as a root of one of the following quadratic equations

$$(1+\gamma^2)m^2 + 2\gamma(\beta-\gamma)m + (\beta-\gamma)^2 = 0, \qquad (5.8)$$

or

$$(1+\gamma^2)m^2 + 2\gamma(\beta-\gamma)m + (\beta-\gamma)^2 = 4(m-1)\sec^2\phi, \qquad (5.9)$$

and since the roots of (5.8) are necessarily complex, we conclude that (5.7) admits two real solutions of the form $u(z) = Ae^{mz}$, where A is an arbitrary constant and $m = m_1, m_2$ are the two (assumed real) roots of (5.9).

Secondly, $\beta = \gamma$, the functional form (5.4) is $\psi(r,\theta) = f(re^{\gamma\theta})$ and equation (5.7) becomes

$$\tfrac{1}{4}(1+\gamma^2)\cos^2\phi\, u''^2 - u'u'' + u'^2 = 0. \qquad (5.10)$$

On solving the equation (5.10), we find that

$$u(z) = \frac{1}{2}C_1\cos\phi(\sec\phi \mp \sqrt{\tan^2\phi-\gamma^2})\exp\left\{\frac{2z}{1+\gamma^2}(\sec\phi \pm \sqrt{\tan^2\phi-\gamma^2})\sec\phi\right\} + C_2,$$

provided that $|\gamma| \leq \tan\phi$ and here C_1 and C_2 are arbitrary constants of integration. Thus, for this case the solution of (5.6) is

$$f(\xi) = \frac{1}{2}C_1\cos\phi(\sec\phi \mp \sqrt{\tan^2\phi-\gamma^2})\xi^{\frac{2}{1+\gamma^2}(\sec\phi\pm\sqrt{\tan^2\phi-\gamma^2})\sec\phi} + C_2, \qquad (5.11)$$

Example 5.2 By considering the operator $\Gamma_7+\alpha\Gamma_4+\beta\Gamma_6-\gamma\Gamma_8+\Gamma_1$ of (5.3) (the discrete symmetry $(x,y,\psi)\mapsto(x,y,-\psi)$ will map $\Gamma_7+\alpha\Gamma_4+\beta\Gamma_6-\gamma\Gamma_8-\Gamma_1$ to $\Gamma_7+\alpha\Gamma_4+\beta\Gamma_6-\gamma\Gamma_8+\Gamma_1$), we obtain the functional form

$$\begin{aligned}\psi(R,\Theta) &= -\frac{\alpha\gamma}{(1+\gamma^2)(\beta-\gamma)}+\frac{\alpha R}{1+\beta^2}(\sin\Theta-\beta\cos\Theta)+e^{(\beta-\gamma)\Theta}f(Re^{\gamma\Theta}), \quad \text{if} \quad \beta\neq\gamma,\\ \psi(R,\Theta) &= \frac{\alpha\gamma\Theta}{(1+\gamma^2)}+\frac{\alpha R}{1+\gamma^2}(\sin\Theta-\gamma\cos\Theta)+f(Re^{\gamma\Theta}), \quad \text{if} \quad \beta=\gamma, \end{aligned} \tag{5.12}$$

where R and Θ are defined by

$$R=\left\{\left(x-\frac{\gamma}{1+\gamma^2}\right)^2+\left(y-\frac{1}{1+\gamma^2}\right)^2\right\}^{1/2}, \quad \text{and} \quad \Theta=\arctan\left(\frac{(1+\gamma^2)y-1}{(1+\gamma^2)x-\gamma}\right).$$

Since the functional form is in R and Θ, we transform the equation (5.1) into

$$\begin{aligned}&4\sec^2\phi[R^2\psi_{R\Theta}^2+\psi_\Theta^2-R^2\psi_{RR}\psi_{\Theta\Theta}-R^3\psi_R\psi_{RR}-2R\psi_\Theta\psi_{R\Theta}]\\ &+[R^2\psi_{RR}+\psi_{\Theta\Theta}+R\psi_R]^2=0\end{aligned} \tag{5.13}$$

by the transformations $x-\gamma/(1+\gamma^2)=R\cos\Theta$ and $y-1/(1+\gamma^2)=R\sin\Theta$. For $\beta\neq\gamma$, substitution of $(5.12)_1$ into (5.13), we obtain

$$\begin{aligned}&4\sec^2\phi[(1+\gamma^2)\xi^3f'f''+(\beta^2-\gamma^2)\xi^2ff''-(\beta-\gamma)^2\xi^2f'^2+2(\beta-\gamma)^2\xi ff'-(\beta-\gamma)^2f^2]\\ &-[(1+\gamma^2)\xi^2f''+(2\beta\gamma-\gamma^2+1)\xi f'+(\beta-\gamma)^2f]^2=0,\end{aligned} \tag{5.14}$$

which is identical to equation (5.6), where here the primes denote differentiation with respect to $\xi=Re^{\gamma\Theta}$. Accordingly, equation (5.14) may be further reduced to (5.7) by the transformation $f(\xi)=u(z)$ with $z=\log\xi$, where here the primes denote differentiation with respect to z, and the first particular case listed above also applies here.

For $\beta=\gamma$, by substituting $(5.12)_2$ into (5.13), we arrive at

$$(1+\gamma^2)^4(\xi f'+\xi^2f'')^2=4[(1+\gamma^2)^3\xi^3f'f''-\alpha^2\gamma^2]\sec^2\phi, \tag{5.15}$$

where the primes denote differentiation with respect to $\xi=Re^{\gamma\Theta}$. In a similar fashion, upon making the transformations $f(\xi)=u(z)$ with $z=\log\xi$, equation (5.15) may be reduced to

$$\frac{1}{4}(1+\gamma^2)^4\cos^2\phi\, u''^2-(1+\gamma^2)^3u'u''+(1+\gamma^2)^3u'^2+\alpha^2\gamma^2=0, \tag{5.16}$$

where the primes denote differentiation with respect to z. By the transformation $u'(z) = p(z)$, equation (5.16) may be reduced to

$$\frac{1}{4}(1+\gamma^2)^4\cos^2\phi\, p'^2 - (1+\gamma^2)^3 pp' + (1+\gamma^2)^3 p'^2 + \alpha^2\gamma^2 = 0,$$

from which we find that

$$\begin{aligned} z &= \frac{1}{4}\log|(1+\gamma^2)^3 p^2 + \alpha^2\gamma^2| \\ &\quad \pm\frac{1}{2}(1+\gamma^2)^2\cos\phi \int^p \frac{\sqrt{(1+\gamma^2)^2(\tan^2\phi-\gamma^2)t^2-\alpha^2\gamma^2}}{(1+\gamma^2)^3t^2+\alpha^2\gamma^2}\,dt + C_1, \\ u(z) &= \frac{1}{2}\left[p - \frac{\alpha\gamma}{(1+\gamma^2)^{1/3}}\arctan\left(\frac{(1+\gamma^2)^{1/3}}{\alpha\gamma}p\right)\right] \\ &\quad \pm\frac{1}{2}(1+\gamma^2)^2\cos\phi \int^p \frac{t\sqrt{(1+\gamma^2)^2(\tan^2\phi-\gamma^2)t^2-\alpha^2\gamma^2}}{(1+\gamma^2)^3t^2+\alpha^2\gamma^2}\,dt + C_2, \end{aligned} \tag{5.17}$$

where C_1 and C_2 are arbitrary constants of integration. Equations (5.17) constitute the general parametric solutions of (5.16), from which the general parametric solutions of the original equation (5.15) can be deduced.

Example 5.3 By considering the generator $\Gamma_7 + \alpha(\Gamma_6 - \Gamma_8) + \Gamma_3$ of (5.3) (the discrete symmetry $(x, y, \psi) \mapsto (x, y, -\psi)$ will map $\Gamma_7 + \alpha(\Gamma_6 - \Gamma_8) - \Gamma_3$ to $\Gamma_7 + \alpha(\Gamma_6 - \Gamma_8) + \Gamma_3$), we find the functional form $\psi(x, y) = \theta + f(re^{\alpha\theta})$. Substitution of this functional form into (5.5), leads to the non-linear second-order ordinary differential equation

$$(1+\alpha^2)^2(\xi^2 f'' + \xi f')^2 = 4\sec^2\phi[(1+\alpha^2)\xi^3 f'f'' + 2\alpha\xi^2 f'' - 1], \tag{5.18}$$

where the primes denote differentiation with respect to $\xi = re^{\alpha\theta}$. Equation (5.18) may be transformed into

$$\frac{(1+\alpha^2)^2}{4}\cos^2\phi\, u''^2 - (1+\alpha^2)u'u'' - 2\alpha u'' + (1+\alpha^2)u'^2 + 2\alpha u' + 1 = 0 \tag{5.19}$$

by making the transformation $f(\xi) = u(z)$ with $z = \log\xi$, where this time the primes denote differentiation with respect to z. Equation (5.19) may be reduced to the non-linear first-order ordinary differential equation

$$\frac{(1+\alpha^2)^2}{4}\cos^2\phi\, p'^2 - [(1+\alpha^2)p + 2\alpha]p' + [(1+\alpha^2)p^2 + 2\alpha p + 1] = 0, \tag{5.20}$$

by the substitution $u'(z) = p(z)$. Upon solving (5.20), we find that

$$\begin{aligned} z &= \frac{1}{4}\log|(1+\alpha^2)p^2+2\alpha p+1| + \frac{\alpha}{2}\tan^{-1}\{(1+\alpha^2)p+\alpha\} \\ &\pm \frac{(1+\alpha^2)\cos\phi\sqrt{\tan^2\phi-\alpha^2}}{2}\int^p \frac{\sqrt{(t-A)^2-B^2}}{(1+\alpha^2)t^2+2\alpha t+1}\,dt + C_1, \end{aligned} \tag{5.21}$$

provided that $|\alpha| \leq \tan\phi$, where C_1 is an arbitrary constant of integration and the constants A and B are given by

$$A = \frac{\alpha(1+\alpha^2-2\sec^2\phi)}{(1+\alpha^2)(\tan^2\phi-\alpha^2)}, \quad B = \frac{\tan\phi}{\tan^2\phi-\alpha^2}.$$

A further integration yields

$$\begin{aligned} u(z) &= \frac{1}{2}[p-\tan^{-1}\{(1+\alpha^2)p+\alpha\}] \\ &\pm \frac{(1+\alpha^2)\cos\phi\sqrt{\tan^2\phi-\alpha^2}}{2}\int^p \frac{t\sqrt{(t-A)^2-B^2}}{(1+\alpha^2)t^2+2\alpha t+1}\,dt + C_2, \end{aligned} \tag{5.22}$$

where C_2 is another arbitrary constant. Equations (5.21)-(5.22) constitute the general parametric solution of (5.19), from which we may deduce the the general parametric solution of original equation (5.18).

For $\alpha = 0$, the functional form becomes $\psi(x,y) = \theta + f(r)$. In this case we can evaluate the integral in (5.22), with the result

$$\begin{aligned} z &= \frac{1}{4}\log|p^2+1| \pm \sin\phi\int^p \frac{\sqrt{t^2-\cot^2\phi}}{t^2+1}\,dt + C_1, \\ u(z) &= \frac{1}{2}(p-\tan^{-1}(p)) \pm \frac{1}{2}[\sin\phi\sqrt{p^2-\cot^2\phi} - \tan^{-1}(\sin\phi\sqrt{p^2-\cot^2\phi})] + C_2, \end{aligned} \tag{5.23}$$

where C_1 and C_2 are constants of integration.

Example 5.4 The consideration of $\Gamma_8 + \alpha\Gamma_6$ of (5.3) leads to the functional form $\psi(x,y) = x^{\alpha+1}f(y/x)$. Upon substitution of this functional form into (5.1) we obtain the highly non-linear ordinary differential equation

$$[(1+\xi^2)f'' - 2\alpha\xi f' + \alpha(1+\alpha)f]^2 = 4\alpha\sec^2\phi[(1+\alpha)ff'' - \alpha f'^2], \tag{5.24}$$

where the primes denote differentiation with respect to $\xi = y/x$. For $\alpha = -1$, a complex solution may be obtained, while for $\alpha = 0$ corresponds to the trivial solution $f(\xi) = C_1\xi + C_2$, where C_1 and C_2 are arbitrary constants.

Highly non-linear second-order ordinary differential equations of type (5.24) were first studied by Appell [3] in the late 1880's. He derived the condition, later known as Appell's condition, to determine non-singular solutions. Curtiss [5] obtained the same condition using a different argument. Recently, Chalkley [6] established that the non-linear second-order ordinary differential equations of type (5.24) satisfy Appell's condition if and only if equations of type (5.24) satisfy the conditions stated in Chalkley [6]. Using these conditions, the value of α for which the non-trivial non-singular solution (singular solution as well) may be found for equation (5.24) is $\alpha = 1$, and these solutions are presented by Johnpillai and Hill [11].

Example 5.5 The operator $\Gamma_8+\Gamma_4$ of (5.3) (the discrete symmetry $(x, y, \psi) \mapsto (-x, y, \psi)$ will map $\Gamma_8 - \Gamma_4$ to $\Gamma_8 + \Gamma_4$) leads to the functional form

$$\psi(x, y) = x \log x + x f(y/x). \tag{5.25}$$

where the function f satisfies the non-linear ordinary differential equation

$$[(1+\xi^2) f'' + 1]^2 = 4 f'' \sec^2 \phi \tag{5.26}$$

with the primes here denoting differentiation with respect to $\xi = y/x$. We may solve the quadratic (5.26) to give

$$f''(\xi) = -\frac{1}{(1+\xi^2)^2}[1 + \xi^2 - 2\sec^2\phi \mp 2\sec\phi\sqrt{\tan^2\phi - \xi^2}],$$

provided that $|\xi| \leq \tan\phi$. By integrating twice, we obtain the solution

$$\begin{aligned} f(\xi) \; = \; & \left[\xi \tan^2\phi \tan^{-1}\xi + \frac{1}{2}\log(1+\xi^2)\right] \\ & \pm \left[\sec\phi\sqrt{\tan^2\phi - \xi^2} + \xi\tan^2\phi \arcsin\left(\frac{\xi\csc\phi}{\sqrt{1+\xi^2}}\right)\right. \\ & \left. + \frac{1}{2}\log\left(\frac{\sqrt{\tan^2\phi - \xi^2} - \sec\phi}{\sqrt{\tan^2\phi - \xi^2} + \sec\phi}\right)\right] + C_1\xi + C_2, \end{aligned} \tag{5.27}$$

where C_1 and C_2 are constants of integration. Thus, a group-invariant solution of (5.1) is given by (5.25), and $f(\xi)$ is given by (5.27).

Example 5.6 From the operator $\Gamma_8-\Gamma_6+\Gamma_3$ of (5.3) (the discrete symmetry $(x,y,\psi)\mapsto(x,y,-\psi)$ will map $\Gamma_8-\Gamma_6-\Gamma_3$ to $\Gamma_8-\Gamma_6+\Gamma_3$), we find the functional form $\psi(x,y)=\log x+f(y/x)$. The following non-linear ordinary differential equation

$$[(1+\xi^2)f''+2\xi f'-1]^2=-4\sec^2\phi(f''+f'^2) \tag{5.28}$$

may be obtained by substituting this functional form into (5.1), where the primes denote differentiation with respect to $\xi=y/x$. Equation (5.28) may be reduced to the first-order non-linear ordinary differential equation

$$(1+\xi^2)^2p'^2-2[1+\xi^2-2\sec^2\phi-2\xi(1+\xi^2)p]p'+4(\xi^2+\sec^2\phi)p^2-4\xi p+1=0$$

by making a transformation $f'(\xi)=p(\xi)$. Unfortunately, we are unable to make any further progress solving this equation analytically.

Example 5.7 We may obtain the functional form $\psi(x,y)=e^xf(y)$ by considering the operator $\Gamma_6+\Gamma_1$ of (5.3) (the discrete symmetry $(x,y,\psi)\mapsto(-x,y,\psi)$ will map $\Gamma_6-\Gamma_1$ to $\Gamma_6+\Gamma_1$). From (5.1) we find the function f satisfies the non-linear second-order ordinary differential equation

$$f''^2+2(1-2\sec^2\phi)ff''+4\sec^2\phi f'^2+f^2=0. \tag{5.29}$$

On making the transformation $u(y)=f'/f$, equation (5.29) reduces to

$$u'^2+2[u^2+(1-2\sec^2\phi)]u'+(u^2+1)^2=0. \tag{5.30}$$

Upon solving (5.30), we obtain

$$y=G(u,C_1), \tag{5.31}$$

where C_1 is a constant of integration and G is given by

$$\begin{aligned} G &= \frac{u\sec^2\phi}{u^2+1}+\tan^2\phi\tan^{-1}(u) \\ &\mp \left[\tan^2\phi\arcsin\left(\frac{u\csc\phi}{\sqrt{u^2+1}}\right)+\frac{u\sec\phi}{u^2+1}\sqrt{\tan^2\phi-u^2}\right]+C_1. \end{aligned} \tag{5.32}$$

From the transformation $u(y)=f'/f$ and (5.31), it follows that

$$\log f=yu-\int G(u,C_1)\,du+C_2,$$

where C_2 is a further arbitrary constant. This equation gives on integration

$$\begin{aligned}\log f &= yu - \log\sqrt{u^2+1} - u\tan^2\phi\tan^{-1}(u) \\ &\pm \left[u\tan^2\phi\arcsin\left(\frac{u\csc\phi}{\sqrt{u^2+1}}\right) + \sec\phi\sqrt{\tan^2\phi - u^2}\right. \\ &- \left.\tan^2\phi\log\left(\frac{\sec\phi + \sqrt{\tan^2\phi - u^2}}{\sqrt{u^2+1}}\right)\right] + C_1u + C_2,\end{aligned}$$

which together with (5.31) and (5.32) constitutes the general parametric solution of (5.29) provided that $|u| \leq \tan\phi$ (see Murphy [13]).

Example 5.8 The operator $\Gamma_1 + \alpha\Gamma_4$ of (5.3) corresponds to the group-invariant solution $\psi(x,y) = \alpha\frac{1}{2}x^2 + f(y)$, where f satisfies the ordinary differential equation

$$f''^2 + 2\alpha(1 - 2\sec^2\phi)f'' + \alpha^2 = 0.$$

By solving this quadratic and integrating, we obtain the solution

$$f(y) = \frac{\alpha}{2}\left[y\tan\left(\frac{\pi}{4} \pm \frac{\phi}{2}\right)\right]^2 + C_1y + C_2,$$

where C_1 and C_2 denote arbitrary constants of integration.

It is noted that the generators Γ_6, Γ_4, and Γ_3 of (5.3) have no group-invariant solutions. Thus the above solutions provided in Examples 5.1-5.8 constitute an optimal system of group-invariant solutions to the equation (5.1) whereby any other group-invariant solution can be found by transforming one of these solutions by a suitable group element.

6 Discussion

In the study of quasi-static plane flow of a Coulomb-Mohr granular material, the usual approach is to introduce the stress invariants p and q and the stress angle $\bar{\psi}$, defined by

$$p = -\tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}), \quad q = \tfrac{1}{2}\left\{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right\}^{1/2}, \quad \tan 2\bar{\psi} = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}. \qquad (6.1)$$

As mentioned in the Introduction, with the use of (6.1), the equilibrium equations (2.1) and the yield condition (2.2) can be combined to give two highly nonlinear coupled first order partial differential equations for q and $\bar{\psi}$, and the introduction of trigonometric functions make the equations difficult to treat analytically. Physically, p is the average hydrostatic pressure, q is the maximum shear stress, and $\bar{\psi}$ is the angle which the maximum principal stress makes with the x-axis. These quantities are usually used graphically in conjunction with Mohr diagrams, and are especially favoured by engineers in the analysis of problems. Again, the reader is referred to Spencer [23] for details on this formulation.

In contrast, here we have made use of the Airy stress function ψ, which is borrowed from the study of plane strain linear elasticity. This function is a potential function for the stresses, and once ψ is determined, we must differentiate twice to recover the stresses. However, this approach has the important advantage in that the equilibrium equations (2.1) and the yield condition (2.2) combine to give a single nonlinear partial differential equation (1.2), which is far more amenable to classical Lie symmetry methods. We have therefore been able to derive a complete set of group-invariant solutions for this problem, and in particular, derive new exact solutions for the special limiting case of the angle of internal friction $\phi = \pi/2$.

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