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REUNITISING HUNDREDTHS: PROTOTYPIC AND NONPROTOTYPIC REPRESENTATIONS

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This paper reports on a study in which 29 Year 6 students (selected from the top 30% of 176 Year 6 students) were individually interviewed to explore their ability to reunitise hundredths as tenths (Behr, Harel, Post & Lesh, 1992) when represented by prototypic (PRO) and nonprototypic (NPRO) models. The results showed that 55.2% of the students were able to unitise both models and that reunitising was more successful with the PRO model. The interviews revealed that many of these students had incomplete, fragmented or non-existent structural knowledge of the reunitising process and often relied on syntactic clues to complete the tasks. The implication for teaching is that instruction should not be limited to PRO representations of the part/whole notion of fraction and that the basic structures (equal parts, link between name and number of equal parts) of the part/whole notion needs to be revisited often.

The notion of a unit underlies the decimal number system. However, Steffe (1986) has identified four different ways of thinking about a unit, namely, *counting (*or *singleton) units*, *composite units*, *unit-of-units* and *measure unit,* with each type apparently representing an increasing level of abstraction. When considering whole numbers, *singleton units, composite units* and *unit-of-units* need to be considered (see Figure 1) whereas with decimal fractions, the *measure unit* needs to be invoked (Behr, Harel, Post, & Lesh, 1992). (See Figure 2.) There is a consensus in the literature (Behr et al. 1992, Harel & Confrey, 1994; Hiebert & Behr, 1988, Lamon, 1996) that the cognitive complexity involved in connecting referents, symbols and operations can be attributed mainly to the changes in the nature of the unit.

Partitioning, unitising and *reunitising* are important to the development of rational number concepts but are often the source of young students' conceptual and perceptual difficulties in interpreting rational-number representations (Baturo, 1996; Behr et al, 1992; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983). In particular, reunitising, the ability to change one's perception of the unit, requires a flexibility of thinking that may be beyond young children. This has importance for hundredths which need to be thought of as a number of hundredths sometimes and as a number of tenths at other times. Similarly, tenths need to be thought of as a number of tenths or as a number of hundredths.

The cognitive complexity required to process the unit-of-units notion has major implications for acquiring an understanding of the decimal number system. For example, each place needs to be reunitised in terms of the unit/one for a complete understanding of the place-value relationships to be known. Figure 1 shows the ways in which 5 tens (represented by 5 base-10 blocks) can be unitised in terms of singleton and composite units and composite unit-of-units.

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Figure 1. Various notions of a unit applied to tens and ones.

Figure 2 shows that similar thinking is required to process a number such as 0.20. However, the extra dimension of the *unit measure* needs to be invoked (Behr et al., 1992) to relate the part to the whole. To transform the units in the different ways and to keep track of these transformations with respect to the shaded parts requires a great deal of flexible thinking and would most likely place a strain on cognitive loading.

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Figure 2. Units-of-units notion applied to tenths and hundredths.

When a whole is partitioned into tenths only, students need only unitise once (i.e., the 10 × 1-unit is unitised as 1 × 10-unit) and therefore there is only one measure unit to be invoked. Similarly, if hundredths only are to be considered. However, when hundredths need to be perceived as both tenths and hundredths, as they are for recording purposes and for renaming from one place to the other (equivalence), then the cognition required becomes much more complex.

THE STUDY

One hundred and seventy-six students from two schools (low-middle and middle-high socioeconomic backgrounds) were administered a diagnostic instrument that was developed to assess the students' understanding of the numeration processes (i.e., number identification, place value, regrouping, ordering, and estimating) related to tenths and hundredths. The students were classified in terms of their overall mean for the test and 29 students were selected from the top 30% for interviewing. This group of students comprised 12 high-performing students (HP – ≥ 90%), 11 medium-performing students (MP – 80-90%) and 8 low performing students (LP – 70-80%).

Semistructured individual interviews were undertaken and incorporated a set of tasks (presented in the same order) designed to probe the students' structural knowledge with respect to reunitisng hundredths for both PRO and NPRO area representations. Figure 3 shows the two tasks on which this paper reports. The full study was reported in Baturo (1996).

TASK 1 (prototypic)	TASK 2 (nonprototypic)
Shade 0.6 of the shape below.	Shade 0.2 of the shape below.
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Figure 3. The reunitising tasks.

The interviews were conducted at the students' schools and took approximately 30 minutes to complete. They were video-taped, transcribed into protocols and then analysed for commonalities in achievement and strategy use within and between the performance categories (HP, MP, LP).

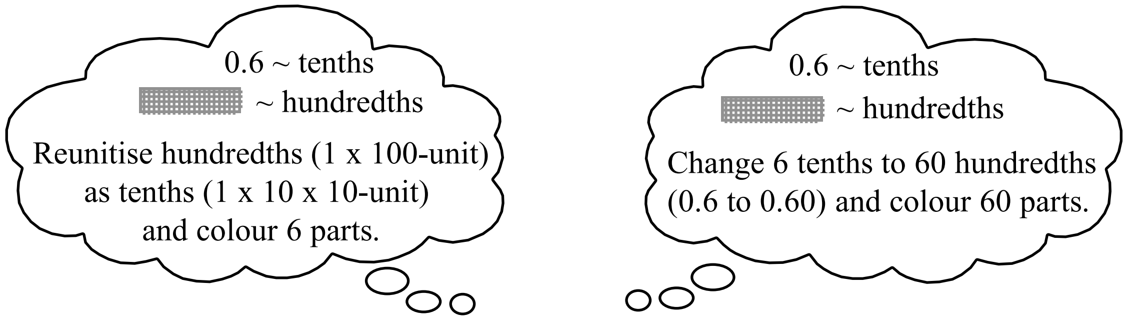
RESULTS

Task 1

Twenty-one (10 HP, 8 MP, 3 LP) of the 29 students were correct, shading either 6 rows or 6 columns. The remaining 8 students(2 HP, 3 MP, 3 LP) all coloured 6 hundredths. No student mentioned that they counted the number of parts in order to unitise the shape as 1 × 100-units; rather, they seemed to have the expectation that there were 100 equal parts, an expectation that could be attributed to the overuse of the PRO model. When asked to read how much had to be shaded, 4 of the 8 incorrect students (1 HP, 1 MP, 2 LP) immediately realised their error (e.g., *I should have shaded 6 strips* – MP8) and shaded the correct amount. Three of the remaining 4 students (1 HP, 2 MP) were able to identify and rectify their incorrect response only after they had been focused on unitising the shape. The remaining student (LP4), whose protocol is provided, appeared to be so bewildered by her original answer that she seemed to lose all ability to unitise.

LP4 [I: How much did you have to shade here?] *A six – I don't know really*. [I: What's this number (pointing to the 0.6 again because she seemed to be looking at what she had coloured)?] *Six* (after a pause). [I: Six what?] *Is it one sixth*? [I: That's (writing $^1/_6$) 1 sixth. What's this number (the 0.2 she had read correctly in an earlier task)?] *One second or something.*

Two different strategies could be identified from the students' responses to the question*: How did you work out how much to shade*? These were classified as *reunitising* (RU) in which the 1 × 100-unit of the given diagram was reunitised as 1 × 10 × 10-units (either rows or columns) or as *equivalence* (EQ) in which the number, 0.6, was reunitised as 0.60, and 60 hundredths were shaded. Figure 4 shows the difference in thinking required by the reunitisation and equivalence strategies.



A. Reunitisation strategy B. Equivalence strategy

Figure 4. Cognitive differences in reunitisation and equivalence.

Both strategies required an understanding of equivalence between tenths and hundredths (i.e., 10 h = 1 t) in order to be applied successfully and this notion was often explicated by students. A third category, *prototypic* was suspected because some students referred to tenths as "strips" or "lines" which may have been the result of prototypic thinking and not as a consequence of having equivalence. That is, the 10 × 10 PRO model always has tenths arranged in rows or columns and therefore they can be <u>perceived</u> without requiring the cognition of equivalence (10 h = 1 t) or reunitisation (1 × 100-unit can be reunitised as 1 × 10 × 10-units). However, this strategy was too subtle to distinguish from the reunitisation strategy so students who were suspected of employing a prototypic strategy were given the benefit of the doubt and classified as using the reunitisation strategy.

The EQ strategy appeared to be used by 10 students (4 HP, 5 MP, 1 LP) and was identified in protocols such as the following. (No student shaded 60 hundredths at random; rather, each student shaded groups of 10.)

HP3: *Because 6 tenths is the same as 60 hundredths and it* (indicating the diagram) *was divided into hundredths so I just shaded 60.* [I: Show me the 6 tenths parts.] *The whole rows* (indicating).

HP10: I *just see these* (hundredths) *as ones and so I colour 60.*

MP12: *It* (diagram) *was divided up into hundredths so you had to colour 60.* [I: Did you change that (0.6) in your mind to 60 hundredths?] *Yes.*

LP2: *Six tenths is the same as 60 hundredths so I thought of zero on the end* (of 0.6) *and just coloured 60.*

Nineteen students (8 HP, 6 MP, 5 LP) appeared to use the RU strategy as they made reference to restructuring the hundredths in the diagram. The following protocols show the variety of thinking that was used in reunitising hundredths as tenths.

HP4: *Cos 60 hundredths also makes 6 tenths, what I did I thought that these* (his shaded columns) *could also be these* (indicating the tenths in an earlier task in which the PRO model had been partitioned into 10 equal columns) *and shaded 6.*

HP6: *There were 100 pieces and if 10 were 1 tenth then I'd need to colour in 6* (indicating her shaded columns). [I: So can you see that (the whole shape) as 100 little parts and as 10 of something else?] *Yes.* [I: When you divide it in your mind in 10 parts, what does that 10 part look like?] *Like that* (indicating a tenth in an earlier task). *Or if I had a 100 of those little cube things* (possibly referring to MAB ones), *I could divide them into 10 groups evenly* (indicating separate groups with her hands).

MP1: *I shaded just one – I guess I took them – the vertical ones* (partitions) *– out of my mind and just shaded it in* (his shaded 6 rows). [I: You blocked the little bits from your mind so you could see these rows going across?] *Yes* [I: So you saw them as 10 rows of 10 then?] *Yes.*

MP5: *I just did 6* (indicating the shaded columns) *because there's 6 there* (0.6) *and forgot about the boxes.*

MP7: *Well, I saw the little squares and there* (0.6) *it says to show 6 tenths in hundredths so I coloured 6 of these* (indicating the rows).

The following protocols provide examples of what was suspected of being prototypic reasoning.

HP11: *Well you just – you know that six take away ten is four so you miss four columns and you just colour in the rest*. [I: So how did you see the tenths? Do the tenths just go across?] *Well, you just know that that's tenths* (pointing to the rows).

MP8: *I should have coloured strips*. (She had shaded 6 hundredths.)

Task 2

Nineteen (8 HP, 7 MP, 4 LP) of the 29 students correctly shaded 1 row, 2 half-rows or 4 columns of the NPRO shape. Of the 10 incorrect students, 1 (LP6) had not attempted the task, 1 (MP12) had shaded half the shape whilst the remaining 8 students had shaded 2 hundredths, 2 rows or 2 columns. Shading 2 parts was thought to be the most naive strategy because no attempt had been made to ratify the numerical amount with the pictorial representation. Shading 2 rows or columns was thought to be less naive because an attempt to ratify the symbolic and pictorial representations had been made but prototypic reasoning (strips, rows, columns) had been used to reunitise the hundredths as tenths.

With respect to unitising, no student mentioned counting the parts, in Task 1, in order to unitise the model as 1 × 100-unit and this behaviour had been attributed to the expectation of 100 equal parts that is generated by the overuse of the PRO pictorial representation of hundredths. In this task, 8 students (6 HP, 1 MP, 1 LP), all of whom shaded the correct amount, mentioned counting the parts to establish how may there were in order to unitise the shape as 1 × 100-unit. However, when asked to read the number and then say whether the shape represented tenths all but one student (MP7) immediately recognised their error and made the appropriate changes. MP7 (who had shaded 2 columns of 5) revealed that he had a problem in unitising the shape as hundredths as shown by his protocol.

I: *Now how do we know whether that's* (his shading) *right or wrong*?

S: *Count up here* (top row) *and see how many altogether. Well, there's 20 in each row* (after counting) *so 20, 40 60, 80, 100* (pointing to the end of each row as he counted). [I: So what would 1 tenth of that be?] *It would be just one of these* (indicating a small square). [I: No, that's 1 hundredth. What about 1 tenth?] (No response) [I: You said before that that (indicating the first column he had shaded) was 1 tenth. Do you still think that's 1 tenth of the whole thing?] *Yes.*

With respect to reunitising, the protocols revealed the same types of strategies that were revealed in Task 1, namely, the RU strategy (used by 21 students – 9 HP, 7 LP, 5 LP) and the EQ strategy (used by 7 students – 3 HP, 3 MP, 1 LP).

Results across the tasks

Table 1 provides the students' initial and amended solutions for both reunitisation tasks and shows that 5 students (2 HP, 2 MP, 1 LP) who had shaded the correct amount in Task 1 did not shade the correct amount in Task 2.

This behaviour supports the belief that reunitisation is not established until it can be applied to both PRO and NPRO representations.

Table 1 also shows that 5 (2 HP, 2 MP, 1 LP) of the 8 students (2 HP, 3 MP, 3 LP) who were incorrect in Task 1 were also incorrect for Task 2 and, with the exception of the LP student who was unable to provide a solution, made the same error, namely, coloured the numbers given (i.e., 6 and 2) irrespective of the pictorial representation. The behaviour (i.e., incorrect in the first task but correct in the second task) of the remaining 3 students (1 MP, 2 LP) could probably be attributed to the NPRO model. For example, the model was different from the model usually given to represent hundredths and therefore this oddity acted as a metacognitive “trigger”, alerting the students to examine the task more closely.

The 8 students who self-corrected their response revealed that they had the appropriate reunitising knowledge available but had not accessed it at the time of the test. Failure to access the knowledge could have been due to external environmental factors (one student said she couldn’t think because the teacher was talking), to internal personal factors such as tiredness, illness, early closure, or to task novelty clashing with task expectations (for example, being asked to shade hundredths only when the diagram represents hundredths and to shade tenths only when the diagram is partitioned into tenths). On the other hand, the interview probably had had some teaching effects because of the probes regarding the whole, the equality of the parts and the number of equal parts that comprise the whole.

Table 1

Students’ responses and solution strategies accessed in the reunitisation tasks.

	Task 1		**Task 2**			**Task 1**		**Task 2**	
	Shading	Strategy	Shading	Strategy		Shading	Strategy	Shading	Strategy
HP1	6 C	RU	4 C	RU	MP1	6 R	RU	4 C	RU
HP2	6 C	RU	4 C	RU	MP2	6 C	EQ	4 C	RU
HP3	6 R	EQ	1 R	RU	MP3	6 h; 6 R	EQ	2 h; 4 C	EQ
HP4	6 C	RU	4 C	RU	MP4	6 h; 6 R	RU	1 R	EQ
HP5	6 C	RU	1 R	RU	MP5	6 C	EQ	2 h; 4 C	RU
HP6	6 C	RU	4 C	RU	MP6	6 C	RU	2 × ½ R	RU
HP7	6 h; 6R	RU	2 h; 1 R	EQ	MP7	6 R	RU	2 C	—
HP8	6 C	EQ	4 C	EQ	MP8	6 h; 6 R	RU	2 h; 2 × ½ R	RU
HP9	6 C	RU	4 C	RU	MP9	6 R	EQ	4 C	RU
HP10	6 C	EQ	2 R; 2 × ½ R	EQ	MP10	6 R	RU	4 C	RU
HP11	6 R	RU	$^1/_2$; 4 C	RU	MP11	6 C	EQ	4 C	EQ
HP12	6 h; 6 R	EQ	2 h; 1 R	RU					
LP1	6 h; 6 R	RU	1 R	RU					
LP2	6 C	EQ	4 C	RU					
LP3	6 C	RU	4 C	RU					
LP4	6 h	RU	4 C	EQ					
LP5	6 C	RU	2 h; 1 R	RU					

LP6	6 h; 6 R	RU	—; 1 R	RU

Table 1 also reveals that 9 students (3 HP, 4 MP, 2 LP) did not maintain their strategy across the two tasks. Six students (2 HP, 3 MP, 1 LP) changed from the EQ to the RU strategy whilst 3 students (1 HP, 1 MP, 1 LP) changed from the RU to the EQ strategy.

CONCLUSIONS

Table 2 provides the correct solutions (based on initial responses) in terms of the performance categories. It shows that, with respect to performance overall, the students were able to reunitise the PRO representation (Task 1) more easily than the NPRO representation (Task 2).

Tale 2

Correct initial responses to both tasks in terms of the performance categories.

	Performance categories			**Overall**
	HP (*n* = 12)	MP (*n* = 11)	LP (*n* = 6)	All (n = 29)
Task 1	10 (83.3%)	8 (72.7%)	3 (50.0%_	21 (72.4%)
Task 2	8 (66.7%)	7 (63.7%)	4 (66.7%)	19 (65.5%)
Both correct	8 (66.7%)	6 (54.5%)	2 (33.3%)	16 (55.2%

With respect to the performance categories, Table 2 shows that differential exists between the categories in Task 1 but not in Task 2. Within the categories, differential between tasks was exhibited by the LP group. The deviant behaviour of the LP students on Task 2 was attributed to the teaching effects of the interview in Task 1.

With respect to identifying students who understand tenths and hundredths, this study revealed that performance alone is not a sound indicator. However, it also revealed that, even when the student's strategy is probed, it is sometimes difficult to know whether syntactic features are used as a crutch or whether they are the end-product of structural knowledge which has been integrated and simplified. The interviews also revealed that high-performing students are not necessarily sound in all aspects of fraction knowledge. For example, some may have a sound understanding of the notion of fraction but cannot reunitise tenths as hundredths whilst others exhibit a sound understanding of the concept and the unitising, reunitising and partitioning processes when PRO representations are provided but cannot extend this understanding to NPRO representations. Moreover, some LP students who had performed poorly on the test performed quite well in the interview, indicating that they had the available knowledge but could not access this knowledge at the time of the test.

There seems to be evidence, however, that: (a) the fraction concept and the unitising, reunitising and partitioning processes are essential for performing in decimal fractions with competence; (b) each of these components needs to be connected if a student is to be labelled as having an understanding of decimal fractions; and (c) instruction must include PRO and NPRO representations.

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The complexity would appear to be increased if a number with fewer than 10 hundredths is to be recorded (e.g., 0.03). This hypothesis is based on the fact that students need to invoke the composite unit (10 × 10-unit), unitise it as 1 × 10 × 10-unit, invoke the measure unit and then find that the cupboard is bare!

This paper report on a study which explored Year 6 students' reunitising strategies for PRO and NPRO representations of hundredths.

HP6 – p. 5

[I: So it doesn't matter to you whether those little lines (horizontal partitions) are in there or not, you can see that (column) as 10 hundredths or 1 tenth?] *Yes.*