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CONSTRUCTION OF MULTIPLICATIVE ABSTRACT SCHEMA FOR DECIMAL-NUMBER NUMERATION

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This paper reports on an intervention study planned to help Year 6 students construct the multiplicative structure underlying decimal-number numeration. Three types of intervention were designed from a numeration model developed from a large study of 173 Year 6 students' decimal-number knowledge. The study found that students could acquire multiplicative structure as an abstract schema if instruction took account of prior knowledge as informed by the model.

Baturo (1997) explored students' acquisition of, and access to, the cognitions required to function competently with decimal numbers. One hundred and seventy-three Year 6 students from two schools (different socioeconomic backgrounds) were tested with a pencil-and-paper instrument that included items designed to assess number identification, place value, counting, regrouping, comparing, ordering, approximating and estimating for tenths and hundredths. As a result of analyses of the students' performances and of the cognitions embedded in decimal-number numeration processes, Baturo developed the numeration model shown in Figure 1 to show these cognitions and how they may be connected.

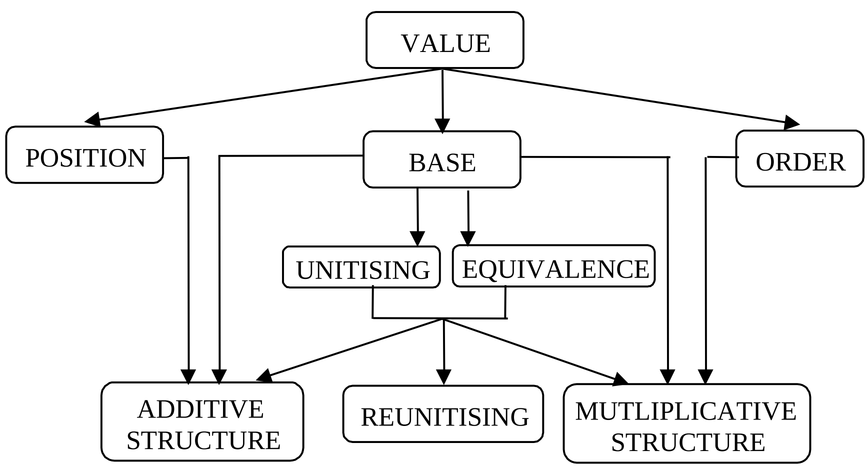

Figure 1. Cognitions and their connections embedded in the decimal number system (Baturo, 1997).

The model depicts decimal-number numeration as having three levels of knowledge that are hierarchical in nature and therefore represent a sequence of cognitive complexity. *Level 1 knowledge* is the baseline knowledge associated with *position*, *base* and *order*, without which students cannot function with understanding in numeration tasks. Baseline knowledge is unary in nature comprising static memory-objects (Derry, 1996) from which all decimal-number numeration knowledge is derived. *Level 2* *knowledge* is the "linking" knowledge associated with *unitisation* (Behr, Harel, Post & Lesh, 1994; Lamon, 1996) and *equivalence,* both of which are derived from the notion of base. It is binary in nature and therefore represents relational mappings (Halford, 1993). *Level 3 knowledge* is the structural knowledge that provides the superstructure for integrating all levels and is associated with *reunitisation*, *additive structure* and *multiplicative structure*. It incorporates ternary relations that are the basis of system mappings (Halford, 1993).

Within the model, *multiplicative structure* relates *position* and *base* into an exponential system (Behr, Harel, Post, & Lesh, 1994; Smith & Confrey, 1994) to give *value* and *order*. It is continuous and bi-directional and, for binary relationships, relates all adjacent places to the left through multiplication by 10 and to the right through division by 10. (For ternary relationships, it relates all adjacent-but-one places to the left through multiplication by 100 and to the right through division by 100.) It is the knowledge structure that underlies the concept of place value, the development of which is a major teaching focus in the primary school. Thus, an understanding of multiplicative structure is crucial and, as argued by Baturo (1997), if not explicated for whole numbers, denies students one of the major conceptual underpinnings of decimal numbers. It is also an excellent example of an abstract schema (Ohlsson, 1993) as shown in Figure 2.

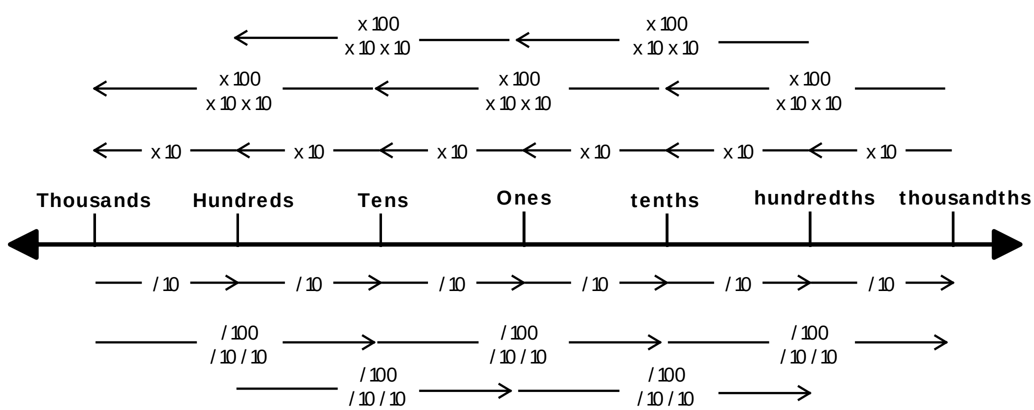

Figure 2. Place-value relationships embedded in the decimal number system.

The model was used by Baturo (1997) to develop interviews designed to probe students' understanding of Levels 1, 2 and 3 knowledge with respect to decimal numbers to hundredths. These interviews were administered to all students whose test performance was very high (≥90%), high (80-90%), and medium (60-80%). Thus, the interview selection comprised 16 very-higher performers (VHP), 16 high performers (HP) and 13 medium performers (MP). Responses to the interviews (and the tests) showed that a majority of the students did not have multiplicative structure to Level 3; in fact, a significant proportion of the medium students did not have multiplicative structure at Level 1 (knowledge of position and order). Therefore, intervention was undertaken, individually, with 17 of the 45 interview students to help them construct multiplicative abstract schema for decimal-number numeration.

The intervention study

Three types of intervention were given to the 17 students (1 VHP, 7 HP, 9 MP). Type 1 intervention was employed if the student had indicated evidence of procedural knowledge for interview tasks such as "0.3 × 10 = __". This intervention aimed to connect the student's procedural knowledge to the appropriate structural knowledge through focusing on reverse tasks such as "change 7 tenths to 7 ones using a calculator".

Students were given Type 2 intervention if Type 1 failed or if procedural knowledge was weak or unavailable. In Type 2 intervention used a large place value chart (PVC) and digit cards in conjunction with the calculator. The students were asked to model a binary relationship in one direction (×) by showing 7 tenths on the PVC, making a change to the 7 tenths to show 7 ones, and mirroring this process with the calculator. If students were successful on this task, they were then asked to model a ternary relationship (e.g., 8 ones to 8 hundredths) in the opposite direction (÷). The decimal point was represented with the students' choice of small adhesive stickers of hearts, geometric shapes, flowers or small animals. This was done to: (a) make the students aware that the decimal point, like all mathematical symbols, is a cultural artifact; (b) to add some excitement and motivation to an otherwise fairly dull task, and (c) to make the symbol more meaningful by allowing the students to choose their own representation. In this stage, the language used was vital in helping the students connect the concrete/iconic place value procedures to the symbolic calculator procedures.

Type 3 intervention was given to those students who, in Type 2 intervention, had shown an understanding of the bi-directional operations (×, ÷) that would effect the direction of the shift but who did not understand the role of the base in binary (adjacent places) relationships (and, therefore, ternary relationships). Students were shown the sets of whole-number statements below and asked which statement in each set was correct. The statements were chosen to be within the students' syntactic understanding for multiplying and dividing by 10 (i.e., to be solvable by invoking rules such as "add/take off a zero"). This task was used in conjunction with the PVC and digit cards to represent the multiplicative structure of whole numbers in order to transfer this knowledge to decimal numbers.

Set 1: $60 \times 10 = 600$, $60 \div 10 = 600$, $60 + 10 = 600$, $60 - 10 = 600$;

Set 2: $800 \times 10 = 80$, $800 \div 10 = 80$, $800 + 10 = 80$, $800 - 10 = 80$.

Results and discussion

Type 1 intervention. This intervention involved Claire (VHP) and Kylie (HP) who had exhibited robust procedural knowledge in the interview. They were encouraged to make the connection between place change and operation. Claire was asked to change 7 tenths to 7 ones using the calculator. She entered 7 tenths correctly, but her finger hovered over the + key and then over the 0 key. She finally shook her head and said: *I can't do it*. However, she very quickly made the connection when directed to examine her correct answers to the procedural tasks (e.g., $0.3 \times 10 = 3$), as the following protocol indicates. (Students' responses are in square brackets; I: = interviewer; S: = student.) *I: Here, we had 3 tenths multiplied by 10 equals 3 (pointing to each component of the procedural item). [S: Ohhh (immediately reaching for the calculator and entering × 10).]* On Kylie's first attempt, she entered "+ 7"; for her second attempt, she entered "7.00". She made one more attempt but then realised that that didn't work either. At this stage, she was given intervention similar to Claire with the same success.

Type 2 intervention. Each of the remaining 15 students were asked to show *7 tenths* on the place value chart and then to move the digit to show *7 ones*. They were then asked in which direction (right or left) they had moved the digit and whether the digit had become larger or smaller than it was before. Thus the students' kinaesthetic knowledge of position change was developed though moving the digit card whilst the associated language linked direction with size. The students were then asked to show this change on their calculator. This process was repeated for other adjacent places so that the relationship between the operation (× 10) and the leftwards direction was consolidated. Once the leftwards direction was associated with an increase in value, the students were asked to predict which way they would have to move the digit to effect a decrease in value and then asked to use their calculator to show the operation that would make the digit shift one place to the right. Again, the relationship between the operation (÷ 10) and the rightwards shift was consolidated with other adjacent places. The same process was repeated to establish the relationship between the operation, the direction of the shift and the number of places shifted with ternary relationships. However, although the continuous and bi-directional properties could be simulated and promoted through the place value chart (PVC) activity, the exponential property could not. So for those students who did not have an understanding of the role of the base, this activity was not effective. However, for those who did have the notion of the role of the base, this intervention seemed to have an immediate positive effect on making the appropriate connection between procedural and structural knowledge.

Once the students had moved the digits themselves (both directions) and then mirrored the processes required for both binary and ternary relationships, their new-found understanding was consolidated through activities where the interviewer moved the digit card (random direction and relationship but limited to ternary) whilst they mirrored the shifts on their calculator. Himansu's (MP) protocol exemplifies the language used throughout Type 2 intervention. *I: Show me 7 tenths on the place value chart. [He did so.] Now show me where you want to get it to show 7 ones. [He slid the digit card from the tenths place to the ones place.] Have you made the 7 larger or smaller in value? [S: Bigger] How many times bigger? [S: Ten times bigger.] Now enter 7 tenths on the calculator. [He did so.] What will you do to make the digit shift from the tenths place to the ones place? [He entered × 10 and was delighted to see that the operation produced the required shift.] Now, how do you think you could change the 7 ones back to 7 tenths? [He entered ÷ 10.] Well done.*

This stage of the intervention was repeated until he had shown a connection between ternary shifts to the left with multiplication (× 100) and to the right with division (÷ 100). The next stage of the intervention was then undertaken. *I: Now I'm going to move the digit (PVC) from there (7 tens) to there (7 hundreds). How can you do that on the calculator? [S: Multiply by 10.] So you make it one place bigger when you multiply by 10. How do you think we could get the 7 hundreds back to 7 ones (showing on the PVC)? Is it getting larger or smaller in value? [S: Smaller. (He entered − 100 and had 600.) No, that's wrong.] What undoes multiplication? [S: Divide.] Well, leave your 6 hundreds and make it into 6 ones (showing on the PVC). [He divided by 100.] Now, watch carefully because I'm going to try to catch you (shifting the PVC digit from 6 ones to 6 hundredths). [He divided by 100.] Excellent. How did you know to divide by 100? [S: Because 10 times 10 is a hundred.] Well done! And did you get larger or smaller when you went from there to there (indicating ones to hundredths on PVC)? [S: Smaller.] One more go but I'm not going to say anything so you have to watch what I do (placing the digit, 3, in the tenths place and moving it to show 3 tens). [He entered × 100, looking very pleased with himself.] What a champ!* The success experienced by Himansu was particularly gratifying as he had been totally unsuccessful on the interview tasks related to position and order . His body language during the intervention changed from what apparent nervousness to confidence whilst his smiles indicated that this intervention had boosted his self-esteem.

This stage of intervention was very successful for 8 of the 15 students. Of the remaining 7 students, 5 eventually associated the leftwards shift with multiplication for both binary and ternary relationships but continued to associate the rightwards shift with subtraction. These students knew the equivalence relationship of 10 between adjacent places and the relationship of 100 between adjacent-but-one places but were unable to connect the relationship to multiplicative operations. Kirsty's protocol exemplifies the difficulties in eliciting the connection between equivalence and the required operation. *I: Show me 7 tenths on the place value chart. [She did so.] Now show me where you want to get it to show 7 ones. [She slid the digit from the tenths place to the ones place.]*

Have you made the 7 larger or smaller in value? [S: Larger] How many times larger? [S: 10] Show me on your calculator how to change 7 tenths to 7 ones. [She entered 0.7 and then entered 10; she was bewildered when she saw the result, 0.71.] How many times larger than 7 tenths is 7 ones? [S: Ten times larger. (Her finger hovered over the + key but she didn't press it.] What can you do to tenths to get ones? [She entered + 10 and again was bewildered by the result, 10.7.] What else could you do? [No response] You made the 7 tenths 10 times bigger here [PVC], didn't you? [S: Yes] So what else could you do apart from adding 10 to shift 7 tenths to 7 ones? [S: Times by 10?] Try it. [She entered × 10 and looked very pleased with herself when she saw the result.] Now, I'm going to shift the 7 ones back to 7 tenths (showing on PVC). How can you make the calculator do that? [She entered − 10!]

For all students who could not connect equivalence with the multiplicative operations, the following questions usually elicited the given responses. *How many tens equal a hundred? [10] How many times larger than tens are hundreds? [10 or 10 times larger] What can you do to tens to get hundreds? [Add 10; add 90; × 10 (not often)] What can you do to hundreds to get tens? [Subtract 10; subtract 90; ÷ 10 (not often)]* Thus, the first two questions elicited the base (10) but not the operation, whilst the last two questions elicited an operation which, for most lower-performing students, will be addition and subtraction (additive structure) or multiplication and subtraction (conflict between multiplicative and additive structure). The latter response, giving the multiplication operation but not the division operation, may have been the result of the word "times" in the previous question. The students with this type of problem were not provided with the third type of intervention because they already had an awareness of the base. However, although the consolidation activities helped these students, it was thought that they would require other, more intensive, remediation to establish the connection between equivalence and multiplicative operations and to develop the notion of division as the inverse of multiplication.

The remaining 2 students, Dean and Sarah (both HP) revealed that they had associated the appropriate operations with the bi-directional shifts but they were not aware of the role of the base in binary and ternary relationships. These two students were given Type 3 intervention.

Type 3 intervention. This intervention initially focused on the binary patterns in Set 1 of the mathematical statement (i.e., 60 × 10 = 600, 60 ÷ 10 = 600, 60 + 10 = 600, 60 − 10 = 600). Instruction followed this sequence of steps: (a) the students' attention was drawn to the similarities between the starting and finishing numbers (i.e., 60 and 600); (b) they were asked to show 6 tens on the PVC and then shift the 6 to its finishing position; (c) they were asked to select the operation from the list of statements that would make that shift; (d) they were asked to show the shift from 6 tens to 6 hundreds on the calculator; (e) they were asked to show similar binary multiplication shifts for other adjacent places (e.g., 7 tenths to 7 ones; 5 hundredths to 5 tenths); and (f) they were asked to use the calculator to show ternary multiplication shifts that were shown on the PVC (e.g., 8 tens to 8 thousands). These steps were followed for the second set of statements to extend the role of the base in binary and ternary relationships to division. This intervention, in combination with Type 2 intervention, was successful for both Dean and Sarah.

IMPLICATIONS

Teaching

This study showed that students need the following "levels" of knowledge to understand and access multiplicative structure (see Baturo, 1997, for a discussion of this): (a) Level 1 – the "baseline" knowledge of position and order (seen in the syntactic features of Figure 1), without which students cannot hope to function with any understanding in numeration tasks; (b) Level 2 – the "linking" of base and equivalence that connects and provides meaning to Level 1 knowledge; and (c) Level 3 – the "structural" knowledge of the continuous, bi-directional, and exponential nature of the relationship between positions (see Figure 2) that provides the superstructure for integrating all levels. The interview responses showed that a majority of the students did not have multiplicative structure to Level 3; in fact, a significant proportion of the students did not have multiplicative structure at Level 1 (knowledge of position and order). The students whose responses indicated understanding of, and access to, multiplicative structure came predominantly from the HP students. This means that teachers need to facilitate knowledge and integration of the three levels of knowledge.

Bi-directional teaching. With respect to Level 3 knowledge, many students seemed to be aware of the continuous nature of the relationships between positions. However, only 53.3% of the students (46% of whom were HP students) revealed an understanding of the bi-directional nature of multiplicative structure whilst 11.1% had a unidirectional understanding (multiplication, but not division). The remaining 35.6% of the students had no understanding of the bi-directional nature of multiplicative structure. The implication for teaching, therefore, is that instruction focusing on this bi-directional relationship has to be a priority and must start with whole numbers and be extended to decimal numbers.

The study showed that students were able to apply the exponential relationship more successfully: (a) within the domains of whole numbers or decimal numbers (e.g., tens and hundreds, ones and tenths) than across these domains (to nonprototypic examples such as tens and tenths); and (b) between adjacent positions (e.g., tens and hundreds) than between non-adjacent positions (e.g. tens and thousands). Therefore, teaching must include examples across domains and between non-adjacent positions.

The introduction of new whole-number positions requires a focus on grouping (multiplication) by ten whereas the introduction of new decimal positions requires a focus on partitioning (division) by ten. Whilst, it is necessary to introduce new positions in a unidirectional manner, the implication of this study is that such instruction should be extended to include *reverse* activities, that is, partitioning for whole numbers and grouping for decimal numbers.

Materials. Students need to experience material usage that reinforces size and bi-directional relationships. Therefore, grouping material such as MAB should be used to show the size of a ten in comparison to a hundred and to show the bi-directional relationship (i.e., a ten is 1 tenth of a hundred; a hundred is equivalent to 10 tens). Partitioning material such as 10×10 grids (a square divided into 100 smaller squares in 10 rows of 10) should be used to show the size of 1 tenth compared to 1 hundredth and the bi-directional relationship (1 tenth is 10 hundredths; 1 hundredth is a tenth of a tenth). Activities of this type stress the role of base 10 and equivalence between places.

The place value chart is an invaluable aid in showing position and order of positions. However, it does not show size in a concrete way and, by itself, does not show the exponential relationships nor the effect of applying such relationships. Calculators, on the other hand, can show the effect of applying exponential relationships. Therefore, these two aids need to be used in tandem for full understanding to occur. The intervention episodes in the study showed their effectiveness in promoting bi-directional exponential relationships. Furthermore, the actions of the students in follow-up activities revealed that they had internalised the place value chart as an exponential model rather than as a simple positional model. For example, some students nodded their heads twice as they mentally moved from tenths to tens (for example) while others indicated with their fingers that they were moving across two places. Therefore, activities which require the students to physically move digits from one place to another on the place value appear to develop the kinaesthetic aspect of the exponential relationship whilst the calculator verifies the operation that effects the shift in position and together, they provide a connection from external representations to internal representations.

Future research

This study has given rise to several issues that have implications for teaching and has provided suggestions as to how teachers could take advantage of its findings. However, what was not addressed was whether students in the middle school are developmentally ready to accommodate the suggested structural knowledge architecture for decimal-number numeration (see Figure 1). Therefore, one direction for future research would be to examine the development of structural knowledge within the domain of whole numbers to determine how students can be helped to establish this knowledge and then to examine the best practices for transferring this knowledge to the decimal-fraction domain.

Thus, a longitudinal or cross-sectional study of students in Years 2 to 7 should be undertaken to determine whether/how these students develop structural knowledge of the whole numbers. Some research questions that may be answered from these types of research would be: (a) Is the construction of an abstract schema for whole-number numeration a naturally occurring phenomenon and, if so, at what year level is the abstract schema most likely to be constructed? (b) At what year level do students understand multiplicative structure, particularly its bi-directional and exponential properties? (c) How and when do students translate "10 times smaller" to "÷ 10"? (d) Does having an abstract schema for whole-number numeration guarantee the successful accommodation of tenths, then hundredths, then thousandths, and so on?

The interviews in this study revealed that most students knew the relationship between adjacent places (binary) and adjacent-but-one places (ternary). That is, for binary relationships, they knew that the place on the left of any given place was 10 times larger than the given place and they knew the converse, that is, the place on the immediate right of the given place was 10 times smaller. These notions were also known with respect to ternary relations (i.e., 100 times smaller/larger). As well, the students in this study knew that you need 10 of one place to make the place on its immediate left (i.e., 10 tenths = 1 one; 10 ones = 1 ten; and so on). However, whilst their responses indicated a knowledge of base and equivalence, the students in this study appeared to be unaware of the role of base and equivalence in linking decimal number places. Moreover, their knowledge appeared to be available in static conditions only (connecting two given places) and was generally not translated to dynamic conditions in which "10 times larger" needed to be associated with a shift one place to the left and "10 times smaller" needed to be associated with a shift one place to the right. During the interviews, it was necessary to ask the questions, "What do you do to tens to get hundreds?" and "What do you do to hundreds to get tens?" to elicit the multiplicative operations of ×, ÷. For several students, these questions elicited partial multiplicativity and partial additivity (always ×, −; never +, ÷) or full additivity (+, −). Therefore, further research is required to tease out these behaviours and to determine: (a) how some students know when it is appropriate to access multiplicativity and when it is appropriate to access additivity; and (b) how "10 times larger" is connected with "× 10" and how "10 times smaller" is connected with "÷ 10". Because many teachers use the word "times" synonymously with "multiplication", it is tempting to think that the word "times" in "10 times larger" would then be translated to multiplication. However, this raises the question of why "times" in "10 times smaller" is not associated with multiplication but is often translated to subtraction. Accessing the appropriate structure appears to be a behaviour that is restricted to very high-performing students, a behaviour that warrants further research if teaching practices are to be enhanced.

However, because having knowledge of multiplicative structure and knowing when to access this knowledge were found to be the major factors differentiating very high-performing students from other high-performing students and from low-performing students, research related to developing and consolidating knowledge of multiplicative structure would appear to be a priority in enabling students to process decimal numbers with understanding. This study revealed that many students had not constructed appropriate mental models to accommodate the continuous, bi-directional and exponential properties of multiplicative structure (the exponential model) and the relationship between the whole-number and decimal-fraction place names (the symmetry model).

The interventions that were undertaken to help the students construct these models were both efficient in terms of time and effective in promoting the appropriate knowledge. However, further research is required to determine: (a) the long-term effects of the intervention episodes in terms of maintenance of the mental models over time; and (b) whether the construction of the mental models has a positive effect on students' ability to process decimal numbers with understanding.

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