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Mathematical models for describing the shape of the *in-vitro* unstretched human crystalline lens

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Abstract

We developed orthogonal least-squares techniques for fitting crystalline lens shapes, and used the bootstrap method to determine uncertainties associated with the estimated vertex radii of curvature and asphericities of five different models. Three existing models were investigated including one that uses two separate conics for the anterior and posterior surfaces, and two whole lens models based on a modulated hyperbolic cosine function and on a generalized conic function. Two new models were proposed including one that uses two interdependent conics and a polynomial based whole lens model. The models were used to describe the *in-vitro* shape for a data set of twenty human lenses with ages 7 to 82 years. The two-conic-surface model (7 mm zone diameter) and the interdependent surfaces model had significantly lower merit functions than the other three models for the data set, indicating that most likely they can describe human lens shape over a wide age range better than the other models (although with the two-surfaces-conic model being unable to describe the lens equatorial region). Considerable differences were found between some models regarding estimates of radii of curvature and surface asphericities. The hyperbolic cosine model and the new polynomial based whole lens model had the best precision in determining the radii of curvature and surface asphericities across the five considered models. Most models found significant increase in anterior, but not posterior, radius of curvature with age. Most models found a wide scatter of asphericities, but with the asphericities usually being positive and not significantly related to age. As the interdependent surfaces model had lower merit function than three whole lens models, there is further scope to develop an accurate model of the complete shape of human lenses of all ages. The results highlight the continued difficulty in selecting an appropriate model for the crystalline lens shape.

Key Words: asphericity, in-vitro, lens, merit function, surface shape

1. Introduction

If we are to better understand the optical properties of the human crystalline lens, we must be able to model it accurately. Its two critical properties are shape and internal distribution of the gradient refractive index. In this paper, we explore different ways we can describe the lens shape mathematically and apply this to a set of real lens measurements. A summary of previous *in-vivo* and *in-vitro* investigations of changes in the surface parameters as a function of age is provided in Table 1 (Brown, 1974; Smith, Pierscionek, & Atchison, 1991; Pierscionek, 1993; Pierscionek, 1995; Glasser & Campbell, 1999; Dubbelman & Van der Heijde, 2001; Koretz, Cook, & Kaufman, 2001; Koretz, Cook, & Kaufman 2002; Koretz, Strenk, Strenk, & Semmlow, 2004; Manns, Fernandez, Zipper, Sandadi, Hamaoui, Ho, & Parel, 2004; Schachar, 2004; Strenk, Strenk, Semmlow, & DeMarco, 2004; Dubbelman, Van der Heijde, & Weeber, 2005; Jones, Atchison, Meder, & Pope, 2005; Rosales, Dubbelman, Marcos, & van der Heijde, 2006; Rosen, Denham, Fernandez, Borja, Ho, Manns, Parel, & Augusteyn, 2006; Urs, Manns, Ho, Borja, Amelinckx, Smith, Jain, Augusteyn, & Parel, 2009). It includes regression equations in which surfaces have been fitted as conics (see equation (1)). Where no age dependence was found, mean values are shown.

In-vivo investigations of lens surface shape have involved phakometry, Scheimpflug photography, and magnetic resonance imaging (MRI). While the first two types of studies are subject to optical distortions that must be taken into account, MRI is free of optical distortions and can provide reliable information on lens shape provided that proper precautions are taken (Atchison, Jones, Schmid, Pritchard, Pope, Strugnell, & Riley, 2004). However, resolution is limited by pixel size and by eye movements during measurement. The studies show that the vertex radii of curvature of unaccommodated lens surfaces decrease with age, but at a greater rate for anterior than for the posterior surfaces (Brown, 1974; Dubbelman & Van der Heijde, 2001; Koretz et al., 2001). There is a wide range of estimates of surface asphericity Q when surfaces are modeled as conics (Brown, 1974; Dubbelman & Van der Heijde, 2001; Koretz et al., 2001).

For the *in-vivo* studies, the radii of curvature decrease with accommodation, but more so for the anterior than for the posterior surface (Brown, 1974; Koretz et al., 2002; Dubbelman et al., 2005; Rosales et al., 2006). Dubbelman et al. (Dubbelman et al., 2005) found changes in anterior surface asphericity as a function of accommodation without any age dependence, but did not determine posterior surface changes in asphericity because of poor reliability.

In-vitro investigations of surface shape have involved photography, corneal topographers, shadow photogrammetry and MRI. Except for the study of Pierscionek (Pierscionek, 1993; Pierscionek, 1995) the studies in Table 1 involved unstretched lenses. These lenses were expected to be near the maximum state of accommodation as changes in lens power with age and differences in power between unstretched and fully stretched lenses approximately match the decline in accommomodation amplitude with age (Fisher, 1973; Glasser & Campbell, 1998; Jones et al., 2005). There are much wider ranges of radii of curvature and asphericities in these studies than those occurring for the *in-vivo* lens studies. Only a few of the *in-vitro* studies found significant age trends, mainly with the surfaces flattening with increase in age (Glasser & Campbell, 1999; Rosen et al., 2006) although one study found slight steepening of the anterior surface (Schachar, 2004)(Fig. 1a). Although not shown in Table 1, stretching lenses decreases the power of lenses young enough to be capable of changing shape (Glasser & Campbell, 1998) and hence increases the radii of curvature of surfaces. Two studies have reported asphericities of unstretched lenses, finding very different mean values, but with no age dependence (Manns et al., 2004; Rosen et al., 2006)(Fig. 1b).

While there is general agreement on trends, at least for the *in-vivo* investigations, different studies often gave very different numerical values for radii of curvature and asphericities. These differences may be real and occur because of limited sample sizes taken from a population with large variations in values. However, the differences may not be real but arise from other sources such as inadequate modeling. Lens surfaces are not likely to be exact conics, and so the type of equation used to describe the surface shape and the amount of surface used will affect curve fitting.

Differences between various studies can also arise from measurement error, for which there may be many sources. Not one of the above studies provided a detailed error analysis of their procedures and corresponding measurement uncertainties on individual values. Both inter-subject variation effects on small samples and data analysis procedures (e.g., area of lens used) could have more influence on surface asphericity than on radius of curvature.

We have access to the digitized edges of 20 *in-vitro* lenses from a nuclear magnetic resonance imaging (MRI) study of lens shape and refractive index distribution (Jones et al., 2005) achieving a pixel size of 0.08 mm *in-vitro*. The ages of these lenses ranged from 7 to 82 years of age. Because they are *in-vitro*, they were in accommodated states.

As mentioned above, MRI edge profiles are not distorted by any intervening optics as in the case of techniques such as Scheimpflug photography. The main sources of error in MRI imaging are the geometric linearity of the imaging process (which is determined by the homogeneity of the static magnetic field and the linearity of the magnetic field gradients used for spatial resolution), and the error in identifying the lens edge, which may be slightly fuzzy because of low contrast, low signal-to-noise ratio, and the finite size of the sampling voxels. We use a statistical bootstrap technique to estimate the uncertainties arising from these errors.

Using the lens data, we explore lens shape by two approaches. The first approach is to describe the two lens surfaces by separate equations. These may be independent or dependent, with the former having been used in previous studies. The second approach is to describe lens shape by single equations.

We present a number of equations. Whichever is the best equation depends upon application. Accuracy of fit is an important criterion, but if two or more similar equations have similar accuracy, other criteria must be considered. If we want to estimate lens power, we choose an equation that readily provides vertex radii of curvature. If we want to estimate the aberration contribution, we need an equation that also provides a surface asphericity or allows conventional ray tracing. Another criterion is to primarily give a good overall anatomical shape and give the optics secondary importance.

2. Crystalline lens shape modeling

2.1 Initial analysis of surface data

We used edge co-ordinates obtained by Jones et al. (Jones et al., 2005) from MRI derived two-dimensional refractive index maps for a set of 20 lenses. Lenses were removed within 24hrs post-mortem after which the lenses were stored in AAH with an indicator to monitor lactate and the medium was changed regularly to ensure the lenses remained in good condition. MRI measurements were performed between 2 and 5 days post-mortem. The use of the refractive index maps (rather than the raw grey scale MR images) eliminated image shading (e.g., due to static and RF field inhomogeneities), facilitating the use of a simple thresholding method (written in MATLAB) for determining the lens edges(Jones et al., 2005). The surfaces were uniformly sampled in the Y-direction at about 80 points. We used a simple routine to eliminate tilt. The assembly of surface points was least-squares fitted to a straight line. If a lens is correctly orientated (and symmetric about the optical axis), the slope of the line is zero. The computed slope angle was taken as the angle of tilt and the lens edges rotated appropriately.

2.2 Merit function and bootstrapping

We optimized surface fits by minimizing the sum of squares of the distances between each data point and the curve, measured along the normal to the curve (orthogonal least squares) (Ahn, Rauh, & Warnecke, 2001) The optimization used to estimate the merit function, $MF = SSE_{Ort}$ (sum of squared errors), is the Conjugate Direction method. (Press, Flannery, Teukolsky, & Vetterling, 1989) To evaluate the statistical performance of the estimated radii of curvature and asphericities, a non-parametric bootstrap method was used (Efron & Tibshirani, 1993; Zoubir & Iskander, 2004). In the method the residuals, that are normal to the surface between the given estimated lens shape and the original data, were first calculated. The residuals were then detrended (Efron & Tibshirani, 1993; Zoubir & Iskander, 2004). A set of bootstrapped residuals was then generated by resampling with replacement from the original set of residuals, assumed to be independent and identically distributed, by putting a probability mass function 1/*N*, *N* being the sample length, at each observation. A new lens shape was then created by adding the resampled residuals to the originally estimated lens shape function. The orthogonality of the residuals to the lens shape function was maintained. A number of $B = 100$ bootstrap replications was chosen as this is a sufficient number to calculate standard errors (Tibshirani, 1988). The bootstrap procedure simply simulates multiple acquisitions of the original lens data.

2.3 Lens shape in terms of two independent conic equations (two-conic-surface model)

The most commonly used shape for describing lens (and corneal) shapes is the conic, described in the Y-Z section by the equation

$$Y^2 + (1+Q)Z^2 - 2RZ = 0 \tag{1}$$

where Y is the radial distance from the surface vertex along the vertex plane, Z is the surface sagitta, R is the vertex radius of curvature, and Q is the conic asphericity. An alternate form of expressing the conic is

$$Z = \frac{Y^2}{R + \sqrt{R^2 - (1+Q)Y^2}} \tag{1a}$$

The radius of curvature R_T at point (Y, Z) is given by

$$R_T = \left(R^2 - QY^2\right)^{3/2} / R^2 \tag{1b}$$

The simple equations (1) and (1a) provide vertex radii of curvature and asphericity, and predict power and aberrations. As an example, the more negative or less positive the asphericity, the more negative or less positive is the spherical aberration.

However, conics are not ideal for two reasons. Firstly, Q is often more negative than −1 (Table 1), in which case the lens surface cannot smoothly join at the equator. Secondly, because the real surfaces are not exact conics the estimates of R and Q depend on the diameter over which the data are fitted.

In a preliminary study to estimate the influence of zone diameter, we fitted equation (1) to each surface of the 20 *in-vitro* lenses over central zones of 3, 4, 5, 6, 7 and 8 mm diameter. In general, the merit functions increased with increase in zone diameter, indicating fitting was becoming poorer (Fig. 2). However, the uncertainties for the radii and asphericities for the smaller zone diameter were high, indicating that the parameters were sensitive to edge digitization noise. At zone diameters of 6 mm and higher, the mean ratio of uncertainty to radius reduced to less than 10% (Fig. 3). To balance the best merit function, favored by small zones, and uncertainties, smaller at larger zones, we compromised at a 7 mm zone diameter where the uncertainties in radius have settled to less than 5%. Fig. 4 shows the fits for one lens and Table 2 shows the merit functions, radii of curvature and asphericities of the lenses.

We investigated fitting the 20 lenses to either two connected equations or to single equations that fit the whole lens, rather than just a central zone. In all cases, we have a predicted vertex radius R, but obtaining an asphericity Q is not always possible. In all cases, we compare the predicted radii of curvature with those from the 7 mm zone values shown in Table 2.

2.4. Lens shape in terms of two interdependent equations

We can improve the fit by modifying the conic. In optical design, a spherical or conic surface is often modified, in order to change its aberrations, by adding "figuring" terms of the form $f_n Y^{2n}$, for $n = 2,4,\ldots$ to equation (1a) where f_n are the figuring coefficients. However, for describing the whole lens shape, modifying equation (1) by adding figuring terms is not the best approach because at the equator slopes must be zero. This cannot occur with a figured form of equation (1a), unless there is an infinite number of figuring coefficients. A better approach is as follows. First, we express equation (1) in the form

$$Y^2 = 2RZ - (1+Q)Z^2 \tag{2}$$

Second, we add extra figuring higher order terms for *Z* to give

$$Y^2 = 2RZ - (1+Q)Z^2 + v_1 Z^3 + v_2 Z^4 + v_3 Z^5 + \ldots \tag{3}$$

With more figuring coefficients the fit accuracy improves. Of practical importance is the convergence of the process, that is, how many figuring coefficients we need to make a suitably accurate fit. This depends upon how well the actual lens surface resembles a conic. The better the fit, the fewer terms will be required. For the present, we will stop at the Z^5 term. At the anterior and posterior surfaces we have

$$Y^2 = 2R_1 Z - (1+Q_1)Z^2 + v_{11}Z^3 + v_{12}Z^4 + v_{13}Z^5 \tag{4a}$$

$$Y^2 = 2R_2(d-Z) - (1+Q_2)(d-Z)^2 + v_{21}(d-Z)^3 + v_{22}(d-Z)^4 + v_{23}(d-Z)^5 \tag{4b}$$

where the subscripts 1 and 2 denote the anterior and posterior surfaces, respectively, and *d* is the lens thickness. We must satisfy two conditions. First, the surfaces must have the same value of *Y* at the equator. Setting (*Z*, *Y*) at the equator to (a, ρ) we have

$$\rho^2 = 2R_1 a - (1+Q_1)a^2 + v_{11}a^3 + v_{12}a^4 + v_{13}a^5 \tag{5a}$$

$$\rho^2 = 2R_2(d-a) - (1+Q_2)(d-a)^2 + v_{21}(d-a)^3 + v_{22}(d-a)^4 + v_{23}(d-a)^5 \tag{5b}$$

Second, we need a smooth joint at the equator. Differentiating equations (4) and setting $dY/dZ = 0$ when $Z = a$, gives

$$0 = 2R_1 - 2(1+Q_1)a + 3\nu_{11}a^2 + 4\nu_{12}a^3 + 5\nu_{13}a^4 \tag{6a}$$

$$0 = -2R_2 - 2(1+Q_2)(d-a) - 3\nu_{21}(d-a)^2 - 4\nu_{22}(d-a)^3 - 5\nu_{23}(d-a)^4 \tag{6b}$$

However, satisfying the two conditions does *not* ensure complete smoothness of the join. This can be done only by requiring all derivative orders to be the same for both surfaces at the equatorial joint. Although the first derivative must be zero, the higher derivates may have non-zero values. If we consider the second derivative d^2Y/dZ^2, from equations (4a and 4b) we have

$$(dY/dZ)^2 + 2Y\,d^2Y/dZ^2 = -2(1+Q_1) + 6\nu_{11}Z + 12\nu_{12}Z^2 + 20\nu_{13}Z^3$$

$$(dY/dZ)^2 + 2Y\,d^2Y/dZ^2 = -2(1+Q_2) + 6\nu_{21}(d-Z) + 12\nu_{22}(d-Z)^2 + 20\nu_{23}(d-Z)^3$$

At the equator $dY/dZ = 0$, $Z = a$ and $Y = \rho$ and these equations reduce to

$$\rho\,d^2Y/dZ^2 = -(1+Q_1) + 3\nu_{11}a + 6\nu_{12}a^2 + 10\nu_{13}a^3 \tag{7a}$$

$$\rho\,d^2Y/dZ^2 = -(1+Q_2) + 3\nu_{21}(d-a) + 6\nu_{22}(d-a)^2 + 10\nu_{23}(d-a)^3 \tag{7b}$$

These two second order differentials must be equal, so now we have

$$\begin{aligned} &-(1+Q_1) + 3\nu_{11}a + 6\nu_{12}a^2 + 10\nu_{13}a^3 = \\ &\qquad -(1+Q_2) + 3\nu_{21}(d-a) + 6\nu_{22}(d-a)^2 + 10\nu_{23}(d-a)^3 \end{aligned} \tag{8}$$

We have ten unknowns and only four equations. To solve the problem, we first find values for the two pairs of R and Q, by least squares fitting, over the central 7 mm of the surface. This will leave us with six unknowns and four equations. For the moment, neglecting the values of ν_{13} and ν_{23}, we have four equations and four unknowns and we can separate the four equations into two independent pairs. For the anterior surface, equations (5a) and (6a) are

$$\nu_{11}a^3 + \nu_{12}a^4 = \rho^2 - 2R_1a + (1+Q_1)a^2 \tag{9a}$$

$$3\nu_{11}a^2 + 4\nu_{12}a^3 = -2R_1 + 2(1+Q_1)a \tag{9b}$$

For the posterior surface, the equivalent equations are, from equations (5b) and (6b)

$$\nu_{21}(d-a)^3+\nu_{22}(d-a)^4=\rho^2-2R_2(d-a)+(1+Q_2)(d-a)^2 \qquad (10a)$$

$$-3\nu_{21}(d-a)^2-4\nu_{22}(d-a)^3=2R_2-2(1+Q_2)(d-a) \qquad (10b)$$

Solutions to these equations give only four figuring coefficients (two per surface), which limits the potential accuracy of fit. Initially we introduced a third figuring coefficient and then neglected it, because at the time, there did not appear to be a way of finding its value. To find the two values of third figuring coefficients (one for each surface), we introduce the optimization procedure (described earlier) which refines the values of R, Q, v_1 and v_2. The merit function for the optimization procedure includes adherence to equations (8), (9) and (10). Table 3 gives the results of the first lens shown in Table 2.

2.5. Whole lens shape in terms of a single equation

To the best of our knowledge there have been only two publications presenting single equations for describing the whole lens surface, those of Kasprzak (Kasprzak, 2000) and Kasprzak and Iskander. (Kasprzak & Iskander, 2006) Recently, Urs et al. proposed a one curve lens model simultaneously describing halves of anterior and posterior surfaces but this model is not considered here as it is essentially a polynomial approximation to the model of Kasprzak (Kasprzak, 2000).

2.5.1. Modulated hyperbolic cosine whole lens model

The whole lens equation has the form (Kasprzak, 2000)

$$\rho(\phi) = \rho_A(\phi) + \rho_P(\phi) - d/2 \qquad (11)$$

where $\rho(\phi)$ is the distance from the lens centre in the direction ϕ and has anterior and posterior contributions given by

$\rho_A(\phi) = (a_A/2)\{\cosh[(\pi - \phi)b_A] - 1\}\{1 - \tanh[m(s_A - \phi)]\} + d/2$ (11a)

$\rho_P(\phi) = (a_P/2)\{\cosh[\phi b_P] - 1\}\{\tanh[m(s_P - \phi)] + 1\} + d/2$ (11b)

The parameters include d the lens thickness, m a dampening factor so that Eq.s (11a) and (11b) dominate at the anterior and posterior surfaces, respectively, and a_A and a_P, b_A and b_P, and s_A and s_P. The function has eight independent variables: d, m, a_A, a_P, b_A, b_P, s_A and s_P. Equation (11) is effectively two equations, one for the anterior and one for the posterior surface, that are combined by the damping factor m.

The anterior and posterior radii of curvature are given by Kasprzak's Eq. 8:

$R_A = (d/2)^2/(d/2 - a_A b_A^2)$, $R_P = (d/2)^2/(d/2 - a_P b_P^2)$ (12)

Kasprzak did not give an equation for the asphericity Q in terms of the above parameters and it appears that no exact equation is possible. All that can be done is an approximation by matching the conic with the hyperbolic cosine function. As Kasprzak did not suggest a method of finding the values of the above eight parameters from a set of digitized lens edge data, we have resorted to optimization. The results for lens 1 are given in Table 4.

2.5.2. Generalized conic whole lens model

Kasprzak and Iskander (Kasprzak & Iskander, 2006) offered the generalized conic equation (their Eq. 10)

$$F(Y,Z) = Z^4 + 2A^2Z^2 + 2A^2CY^2Z^2 - A^4(B - C^2)Y^4 = 0 \quad (13a)$$

which we have simplified to

$$F(Y,Z) = Z^4 + c_1Z^2 + c_2Y^2Z^2 + c_3Y^4 = 0 \quad (13b)$$

They gave the following for the anterior vertex radius of curvature (their Eqs. (5) and (8))

$$R = 1\big/\left[A\sqrt{2(B - C^2)}\right] \quad (14a)$$

and in terms of the c coefficients, we have

$$R = \sqrt{-c_1/c_3}\,/\,2 \tag{14b}$$

They did not give an equation for the asphericity *Q*, but we derived one in Appendix 1 as

$$Q = 4c_2\sqrt{-c_3/c_1^3}\,R^3 - 1 \tag{14c}$$

They also did not provide any solutions for the posterior surface, but the corresponding values of *R* and *Q* can be found by turning the lens around and reanalyzing the data. This equation can be solved using linear least squares. However, we used this method to find an initial solution and then used the optimization procedure for orthogonal least squares described earlier. The results for lens 1 are given in Table 5. Two merit functions are determined, one with the lens reversed to obtain the posterior surface parameters. These are generally similar and we have used their average in the Table and in further analysis of the lenses.

2.5.3. Another solution: polynomial based whole lens

We start with the premise that to a first approximation, in two dimensions the lens has a similar shape to the ellipse

$$(Z-a)^2/a^2 + Y^2/b^2 = 1 \tag{15}$$

where *a* and *b* are semi-diameters along the *Z* and *Y* directions, respectively. We assume that we can make this ellipse asymmetric by a non-linear stretch in the *Z* direction with the transformation

$$Z \Rightarrow Z + \alpha Z^2$$

so that equation (15) becomes

$$(Z + \alpha Z^2 - a)^2/a^2 + Y^2/b^2 = 1 \tag{16}$$

If we allow a similar stretch in the vertical direction, we have

$$(Z + \alpha Z^2 - a)^2/a^2 + (Y + \beta Y^2)^2/b^2 = 1 \tag{17}$$

We expand the brackets, dropping the odd power of *Y* because the lens model is symmetrical about the optical (*Z*) axis, to give

$$(-2/a)Z + [(1-2a\alpha)/a^2]Z^2 + (2\alpha/a^2)Z^3 + (\alpha/a)^2 Z^4 +$$
$$(1/b^2)Y^2 + (2\beta/b^2)Y^3 + (\beta/b)^2 Y^4 = 0$$

Because this model is simple, we take a more general solution by allowing the coefficients to be independent, rather than connected as above. We can express the final result in the more general form

$$F(Y,Z) = Y^2 + \nu_0 Y^4 + \nu_1 Z + \nu_2 Z^2 + \nu_3 Z^3 + \nu_4 Z^4 = 0 \qquad \text{`} \qquad (18)$$

where we have divided through by the coefficient of Y^2. Equations for the vertex radii of curvature and asphericity of the anterior surface are

$$R = -\nu_1 / 2 \qquad (19a)$$

$$Q = \nu_2 + \nu_0 \nu_1^2 - 1 \qquad (19b)$$

To find corresponding values of the posterior surface, the *v* coefficients are first altered using

$$\nu_k{}' = (-1)^k \sum_{n-k}^{n=4} {}^n C_k \nu_n d^{n-k}, \; k = 1,2,3,4 \qquad (20)$$

where ${}^n C_k$ is the combination symbol and the vertex radii of curvature and asphericity of the posterior surface are obtained as for the anterior surface, but with *v*' coefficients from Eq. (20) replacing the *v* coefficients in Eq. (19). As with the preceding Kasprzak and Iskander equation, the *v* coefficients can be found by a linear least squares solution. But once again, this was regarded as an initial solution and an optimization process was used to determine final values in an orthogonal least squares sense. The results for lens 1 are given in Table 6.

2.6. Further generalisation

The above models can be generalised by assuming that the lenses are not aligned with the optical axis. In such a case, three additional parameters in terms of the lateral shift (*x*,*y*) and the rotation angle of the lens surface are included in each model (Kasprzak & Iskander, 2006). Such a generalised representation would avoid the lens tilt correcting procedure described in section 2.1 and could also lead to a better fit to the data. However, at the same time, the additional three parameters could lead to a less stable optimisation procedure making it prone to stop at local minima. Hence, in our work we have invoked the principle of parsimony and pre-corrected the lens tilt to limit the number of parameters to be estimated in each of the models.

3. Group results

The five models of lens shape were used to analyze the 20 *in-vitro* lenses, referred to earlier, with comparisons between the methods and assessment of change in shape with age.

3.1. Comparison of methods

Figures 5−9 show the merit functions, vertex radii of curvature and surface asphericities as a function of lens number. For each parameter, the lenses were arranged according to the order in Table 1 (in order of increasing age).

For the merit function (Fig. 5), generally the interdependent surfaces model has the lowest values (mean 0.0019 mm^2, standard deviation 0.0007 mm^2), followed closely by the two conic surfaces model (0.0022±0.0014 mm^2), and then by the modulated hyperbolic cosine whole lens model (0.0031±0.0017 mm^2), our polynomial based whole lens model (0.0036±0.0015 mm^2) and the generalized conic whole lens model (0.0038±0.0017 mm^2). The interdependent surfaces model and the two conic surfaces model are significantly superior to the other models and the modulated hyperbolic cosine whole lens model is significantly better than the generalized conic whole lens model (paired *t*-tests, $p \leq 0.001$) at least for this data set.

As expected, the patterns for the two conic surfaces and the interdependent surfaces models are similar because the second model is an extension of the first model. Both sets are based upon conics. The patterns for the modulated hyperbolic cosine and generalized conic whole lens models are also similar despite obvious functional differences between the models.

Fig. 6 shows the anterior surface vertex radii of curvature. There are considerable differences between the different models, with the modulated hyperbolic cosine whole lens model showing the most extreme variations, with some particular high values (off-scale in the figure).

Considerable differences between models are also features of the posterior vertex radii of curvature and of the surface asphericities in Figs. 7−9. Fig. 7 shows the anterior surface asphericities; the most extreme asphericities are provided by the interdependent surface model. Fig. 8 shows the posterior surface vertex radii of curvature; the different models give similar results apart from a few large differences. Fig. 9 shows the posterior surface asphericities.

3.2. Age trends

Table 7 shows the trends of the radii of curvature and asphericity with age according to the models. For the anterior radius of curvature, four out of five models show increase with age, consistent with most of the literature and with two of three previous literature results (Fig. 1a). If the three extreme results beyond 20 mm are removed for the modulated hyperbolic cosine whole lens model, it shows a significant increase with age also, although at about twice the rate of the other models.

The anterior asphericities for most lenses are positive with all methods (means are +2 to +7 across the age range and significantly different from zero). The generalized conic whole lens model and the polynomial based whole lens model show age dependence. The two papers in the literature have mean values of +3 and −1, with the first being accompanied by age dependence (Manns et al., 2004; Rosen et al., 2006).

For the posterior radius of curvature, only the interdependent surfaces model shows significant age dependence with lenses becoming flatter with increasing age; the regressions for the other models show a similar trend with probabilities of the slopes being significantly different from zero ranging from 0.12 to 0.50. Two of the three papers in the literature find no age dependence, with Glasser and Campbell (1999) finding a small quadratic dependence and Manns et al (2004) finding an increase in radius of curvature with age (Fig. 1a).

For the posterior surface asphericity, there is no age dependence. The means range from −0.2 to +1.5, and only the generalized conic whole lens model has a mean significantly different from zero. The two papers in the literature have mean values of −2 and −1, with no age dependence (Fig. 1b).

4. Discussion and conclusions

To describe surface shapes of human lenses, in most previous studies the front and back surfaces have been fitted separately with conics across particular diameters of the surfaces and no account has been taken of uncertainty in measurements. We found that the resulting vertex radii and asphericities depended upon the diameter of the central zone fitted, indicating that the central region of the lens is not well described by conics. As a result, the size of the diameter of the zone fitted affect the results, putting some doubt on the reliability of previously published conic fittings. We could only conclude that the lens surface would be better fitted by more complex models able to describe the shape of the whole lens rather than only some optically relevant zone. To explore this issue, we fitted the lens with four additional models and compared the goodness of fit. The models included two recently proposed models by Kasprzak (Kasprzak, 2000) and by Kasprzak and Iskander (Kasprzak & Iskander, 2006), and two new models. The latter include a model in which there is interdependence between the two surfaces and a whole lens model based on a polynomial. While using these models we included estimates of uncertainty fit using the bootstrap. The quality of the fit, particularly in the regions near the equator, was aided by procedures in which fits were made along normals to the surface rather than parallel to the optical axis of the lenses.

For the set of *in-vitro* lens data to which we had access (Jones et al., 2005), the two conic surfaces model (7 mm zone diameter) and the interdependent surfaces model had considerably and significantly lower merit functions than the other three models, indicating that most likely they can describe human lens shape better than the other models (although with the two conic surfaces model not being able to describe the lens equatorial region). On the other hand, bootstrap analysis showed that the hyperbolic cosine model of Kasprzak and the new polynomial based whole lens model are characterized with the best precision in determining the crystalline lens parameters. The trade-off between minimizing the unknown bias and the estimated merit function highlights the difficulty in determining the best parametric model for the crystalline lens shape.

Considerable differences were found between some models regarding estimates of radii of curvature and surface asphericities. As the interdependent surfaces model gave the lowest merit functions, it seems that there is still scope for finding a single equation that fits the whole human lens well.

Some variation between models was found when assessing age related changes for *in-vitro* lenses. Most models showed increase in anterior surface radius of curvature with age, which is in line with two of four papers in the literature (Fig. 1a), with another paper failing to show a change and one showing a significant decrease – this latter paper is distinct in the small inter-lens variation compared with other *in-vitro* studies (Schachar, 2004). All but one model show no change in posterior radius of curvature with age, which again is in line with two of four papers in the literature (Fig. 1a). The models show a wide range of estimates of asphericity and dependences with age, with surprising the model with the lowest merit function (the interdependent surfaces model) showing the greatest fluctuations. Two out of four models show the anterior surface asphericity becoming more positive with age (and with asphericity generally being positive), while no models show age dependence for the posterior surface asphericity and with only one model finding it significantly different from zero. Two previous studies of *in-vitro* surface asphericity gave significant means, three of which were slightly negative (Fig. 1b) (Manns et al., 2004; Rosen et al., 2006).

We considered the effect that the different lens models would have on the determination of surface power. To do this, we used refractive indices for the aqueous/vitreous and the lens surface of 1.336 and 1.371, respectively (Jones et al., 2005). The average power is 9.4D, and is in line with our previous estimate that the majority of lens power resides in the gradient index (Jones et al., 2005). All models predict significant decreases in surface power with increasing age (linear regression, $p < 0.01$), varying from 2.4 D to 5.0 D.

By giving us measures of the uncertainties in parameters, the bootstrapping procedure has shown that the asphericity is much more sensitive to noise in the original data than is the vertex radius of curvature. The accuracies of the models cannot be inferred from the real data, unless some kind of an artificial lens with known geometry was imaged using MRI. The use of simulated data could indicate the robustness of a given model when the data is generated under a different model. However, such analysis would obviously be limited to the few considered models. In the case when the true model is unknown, the bootstrap analysis of the precision seems to be more appropriate.

It would be good to apply these methods to the data for in-vivo methods, but it is probably not reasonable to do this as yet because of the shortcomings of methods in which the lens is imaged by the cornea, such as the Scheimpflug method, or by the limited resolution of magnetic resonance imaging.

As mentioned earlier, the lenses are presumably in shapes similar to those near the limit of their accommodative ranges. In the unaccommodated state, the lens of the young eye contributes negative spherical aberration (Artal, Guirao, Berrio, & Williams, 2001) and this increases as the eye accommodates eg (Atchison, Collins, Wildsoet, Christensen, & Waterworth, 1995; Cheng, Barnett, Vilupuru, Marsack, Kasthurirangan, Applegate, & Roorda, 2004; Singh, Atchison, Kasthurirangan, & Guo, 2009). However here we have found some models indicating positive asphericity which would tend to give positive spherical aberration to the lens. The most likely reason for why the lenses do not contribute positive spherical aberration is the dominance of the gradient index of the lenses, not only for power but also for aberration. Smith & Atchison (2001) have provided some theoretical evidence that the gradient index distribution can simultaneously give positive power and negative spherical aberration.

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Figure captions

Fig. 1. Linear fits of a) vertex radii of curvature and b) surface asphericity as a function of age from recent *in-vitro* studies. Where regressions are not significant, means are shown. B) includes full data for the Manns et al. study (Manns et al., 2004).

Fig. 2. Mean merit functions for the set of lenses, as a function of zone diameter. Error bars indicate standard deviations.

Fig. 3. Mean absolute ratio of uncertainty of radius of curvature to the radius of curvature, as a function of zone diameter. Error bars indicate standard deviations.

Fig. 4. Cross-section digitized points for a 7 year old lens (the first lens in Table 2) with best fitting conics out to 7 mm diameter for anterior and posterior surfaces,

Fig. 5. Merit functions for the five lens models and the set of 20 lenses. The lens numbers match those of Table 2.

Fig. 6. Anterior surface vertex radii of curvature for the five lens models and the set of 20 lenses. The lens numbers match those of Table 2.

Fig. 7. Anterior surface asphericity Q for four lens models and the set of 20 lenses. The lens numbers match those of Table 2.

Fig. 8. Posterior surface vertex radii of curvature for the five lens models and the set of 20 lenses. The lens numbers match those of Table 2.

Fig. 9. Posterior surface asphericity Q for four lens models and the set of 20 lenses. The lens numbers match those of Table 2.

Appendix 1. Estimates of radius R and asphericity Q for generalized conic whole lens model

We will express Eq. (13b)

$$F(Y,Z) = Z^4 + (d + eY^2)Z^2 + fY^4 = 0 \tag{A1}$$

as a power series in Y and compare it with the equivalent power series form of Eq. (1), which is

$$Z(Y) = (1/2)(1/R)Y^2 + (1/8)(1/R^3)(1+Q)Y^4 + O(Y^6) \tag{A2}$$

where $O(Y^6)$ indicates terms in Y of the sixth order and higher.

Solving for Z^2 in Eq. (A1) we have

$$2Z^2 = (d + eY^2) \pm \sqrt{(d + eY^2)^2 - 4fY^4} \tag{A3}$$

and

$$2Z^2 = -(d + eY^2) \pm \sqrt{(d + eY^2)^2 - 4fY^4} \tag{A4}$$

for the posterior and anterior surfaces, respectively. This can be written as

$$\begin{aligned} 2Z^2 &= (d + eY^2)\{[1 - 4fY^4(d + eY^2)^{-2}]^{1/2} - 1\} = \\ &(d + eY^2)\{[1 - 4fY^4(1 + (e/d)Y^2)^{-2}/d^2]^{1/2} - 1\} \end{aligned} \tag{A5}$$

Expanding the $(\cdot)^{-2}$ expression using the binomial expansion gives

$$\begin{aligned} 2Z^2 &= (d + eY^2)[1 - 4fY^4(1 +^{-2} C_1(e/d)Y^2 +^{-2} C_2)e/d)Y^4 + O(Y^6))/d^2]^{1/2} - 1 \\ &= (d + eY^2)\{[1 - 4fY^4(1 - 2(e/d)Y^2 + 3(e/d)Y^4 + O(Y^6))/d^2]^{1/2} - 1\} \\ &= (d + eY^2)\{[1 - 4fY^4/d^2 + O(Y^6)]^{1/2} - 1\} \end{aligned} \tag{A6}$$

The next step is to similarly expand the $(\cdot)^{1/2}$ expression to give

$$2Z^2 = (d + eY^2)[1 - (1/2)4fY^4/d^2 + O(Y^6) - 1] = (d + eY^2)[-2fY^4/d^2 + O(Y^6)] \tag{A7}$$

Dividing both sides by 2 and taking the square root gives

$$Z = \sqrt{d}\sqrt{1 + (e/d)Y^2}\sqrt{-f/d^2}Y^2 \tag{A8}$$

Once again we expand the square rooted expression to get

$$Z = \sqrt{(-f/d)}Y^2[1+(1/2)(e/d)Y^2 + O(Y^4)] = \sqrt{(-f/d)}Y^2 + (1/2)e\sqrt{(-f/d^3)}Y^4 \qquad \text{(A9)}$$

On comparing coefficients of the Y orders in Eqs. (A2) and (A9), we have

$$\begin{aligned} R &= \sqrt{(-d/f)}/2 \\ Q &= 4e\sqrt{(-f/d^3)}R^3 - 1 \end{aligned} \qquad \text{(A10,14b,14c)}$$

	Anterior surface		Posterior surface		Comment
	R (mm)	Q	R (mm)	Q	
***In-vivo* – relaxed eyes**					
Brown (1974) – Scheimpflug photography	$+16.82 - 0.104x$ (100)	$-2.042 + 0.0225x$ (100)	$-8.719 + 0.015x$ (100)	$+0.855 + 0.0020x$ (100)	Regression fits determined by Smith et al. (1991). Radii of curvature used to determine Q using Eqn. (1b)
Koretz et al. (2001) – Scheimpflug photography	$+11.16 - 0.020x$ (100)	−1 (100)	$-8.27 + 0.0203x$ (100)	−1 (100)	Polynomial fit; parabola (Q -1) chosen as higher order terms not significant
Dubbelman & van der Heijde (2001) – Sch'pflug photography	$+12.9 - 0.057x$ (102)	−5 (90)	$-6.2 + 0.012x$ (65)	−4 (41)	
Koretz et al. (2004) – Scheimpflug photography	$+13.95 - 0.076x$ (65)		−6.07 (57)		
Koretz et al. (2004) – MRI	$+13.48 - 0.081x$ (25)		−5.63 (25)		Different subjects from those used for Scheimpflug photography
***In-vivo* – accommodated eyes**					
Koretz et al. (2002) – Scheimpflug photography	ΔR/D: $-0.60 + 0.009x$ (100)		ΔR/D: $+0.25 - 0.003x$ (100)		Changes in R per diopter of accommodation stimulus
Dubbelman et al. (2005) – Scheimpflug photography	ΔR/D: −0.61 (65)	ΔQ/D: −0.5 (37)	ΔR/D: +0.13 (37)		Changes in R or Q per diopter of accommodation stimulus. Q changes not analyzed for posterior surface
In-vitro - unstretched lenses					
Data of Pierscionek – presented by Smith et al. (1991)	+9.6 (11)	$+5.3 - 0.03x$ (11)	−8.0 (11)	+2.1 (11)	Whole surfaces fitted to oblate ellipsoids ($Q > 0$), so asphericities in particular should be treated with caution
Pierscionek (1993, 1995) – photography	+7.41 (7)	−1 (7)	−4.42 (7)	−1 (7)	Polynomial fit; parabola (Q -1) chosen as higher order terms not significant
Glasser & Campbell (1999) –photography	$+4.32 + 0.068x$ (13)	−1	$-3.143 - 0.0536x$ $+0.0004173x^2$ (19)	−1	Anterior surface results for < 65 years. About 40% of surfaces fitted to paraboloids. Decapsulating lenses increases absolute R
Schachar (2004) – corneal topography	+10.0±0.5 (30)		−6.8±0.9 (30)		Age relationship not explored. Results shown for 1.0 mm from vertex
Manns et al. (2004) – corneal topography	+10.15±1.39 (24)	$-2.66 + 0.077x$ (24)	$-2.313 - 0.050x$ (18)	−1.7±1.8 (18)	Lenses attached to most of eye. Lenses aged 46 – 93 years.
Data of Jones et al. (2005) – MRI photography	+11.6±4.8(20)		−7.2±2.1(20)		Original analysis done with curvatures showed significant flattening of surfaces with age
Rosen et al. (2006)[1] - shadowphotogrammetry	$+7.5 + 0.046x$ (37)	−0.8±1.7 (37)	−5.5 (37)	−1.1 ± 1.5 (37)	Resolution 0.12 µm /pixel. Asphericities determined over diameters of 8 (anterior) and 3-3.5 mm (posterior)
Urs et al. (2009) - shadowphotogrammetry					27 lenses aged 6-82 years. 10th order polynomial fits to the 10th order. Some terms changed significantly with age. R and Q not estimated.

Table 1. Radii of curvature (R) and conic asphericities (Q) of lenses from *in-vivo* and *in-vitro* studies and from two model eyes. In R and Q columns, regressions are shown on age in years, except that means are given if there is no significant age dependence. Numbers of eyes used to determine parameters are in brackets.

Lens	Age (years)	*MF* anterior (mm^2)	*R* anterior (mm)	*Q* anterior	*MF* posterior (mm^2)	*R* posterior (mm)	*Q* posterior
1	7	0.00061	6.191±0.106	+0.921±0.226	0.00073	−4.849±0.064	+0.296±0.104
2	7	0.00160	5.989±0.174	+1.044±0.365	0.00165	−5.567±0.121	+0.683±0.231
3	20	0.00161	8.505±0.213	+4.247±0.463	0.00168	−5.136±0.152	−1.625±0.370
4	20	0.00182	6.509±0.290	−1.168±0.760	0.00332	−5.992±0.305	−0.052±0.738
5	27	0.00210	9.577±0.393	+5.539±1.138	0.00346	−5.932±0.320	+0.330±0.768
6	27	0.00265	8.560±0.273	+4.366±0.612	0.00290	−5.347±0.191	+0.351±0.366
7	35	0.00151	10.816±0.417	+5.024±1.409	0.00145	−5.561±0.207	−0.462±0.473
8	35	0.00244	10.272±0.362	+7.646±0.997	0.00406	−11.196±0.706	+8.749±2.088
9	40	0.00248	15.921±0.966	+12.655±4.736	0.00101	−6.555±0.240	−0.175±0.625
10	40	0.00150	15.057±0.665	+12.832±3.470	0.00086	−6.015±0.150	−1.108±0.419
11	50	0.00118	9.222±0.278	+2.220±0.947	0.00128	−6.211±0.173	+0.440±0.464
12	50	0.00119	10.208±0.505	+3.441±1.907	0.00284	−5.235±0.234	−1.044±0.552
13	51	0.00675	8.841±0.684	+1.837±2.360	0.00274	−6.780±0.286	+0.990±0.721
14	55	0.00112	9.323±0.465	−5.540±2.181	0.00212	−7.152±0.305	−0.329±0.921
15	55	0.00136	13.332±0.503	+13.221±1.580	0.00277	−12.723±0.553	+11.138±1.840
16	63	0.00190	6.898±0.696	−3.519±2.586	0.00314	−9.743±0.349	+6.376±0.807
17	63	0.00323	9.062±0.866	−2.310±3.509	0.00643	−8.946±1.174	+4.173±3.612
18	72	0.00136	11.429±0.622	+2.321±2.683	0.00130	−5.576±0.200	−2.395±0.566
19	82	0.00153	11.147±0.980	−11.064±5.251	0.00520	−8.099±0.340	+4.196±0.700
20	82	0.00150	15.057±0.665	+12.832±3.470	0.00086	−6.015±0.150	−1.108±0.419

Table 2. Age, merit function *MF*, vertex radius of curvature *R*, surface asphericity *Q* and uncertainties (standard deviations) of *R* and *Q* for both surfaces of each of 20 lenses using the two independent equations approach

Parameter	Anterior surface	Posterior surface
R (mm)	+6.263 ± 0.261	–4.759 ± 0.161
Q	+0.718 ± 1.396	+0.054 ± 0.757
v_1 (mm^{-1})	–0.529621± 1.39084947	–0.102213± 0.71157277
v_2 (mm^{-2})	+0.191767± 0.58659049	+0.078712± 0.27960906
v_3 (mm^{-3})	–0.040665± 0.08826684	–0.029001± 0.03834987
a (mm)	2.284	
d (mm)	4.994	
ρ (mm)	4.004	
MF (mm^2)	0.00065	

Table 3. Parameters and merit function for Lens 1 derived using the interdependent surfaces model.

Parameter	Anterior surface	Posterior surface
a	0.965	0.977
b	1.186	1.061
s	1.808	1.669
R (mm)	5.431±0.055	–4.454±0.014
m	3.706	
d (mm)	5.014	
MF (mm^2)	0.00076	

Table 4. Parameters and merit function for Lens 1 derived using the modulated hyperbolic cosine whole lens model.

	Anterior surface	Posterior surface
c_1 (mm^2)	−23.758694	−24.286047
c_2	+0.927992	+0.646926
c_3	+0.088096	+0.210240
R (mm)	+7.91±0.391	−5.32±0.056
Q	+4.169±0.65	+0.571±0.0536
MF(mm^2)	0.00190	0.00161
mean MF (mm^2)	0.00176	

Table 5. Parameters and merit function for Lens 1 derived using the generalised conic whole lens model.

	Anterior surface	Posterior surface
v_0 (mm^{-2})	−0.00011 ± 0.00019	−0.0350 ± 0.0185
v_1 (mm)	−11.8147 ± 0.025	−7.647 ± 0.754
v_2	+0.9167 ± 0.0221	−1.530 ± 1.748
v_3 (mm^{-1})	+0.7345 ±0.0606	+1.605 ± 0.1689
v_4 (mm^{-2})	−0.0895 ± 0.0054	−0.0895 ± 0.0054
R (mm)	5.904 ± 0.078	−3.806 ± 0.063
Q	−0.091 ± 0.143	−2.579 ± 0.117
MF (mm^2)	0.00084	

Table 6. Parameters and merit function for Lens 1 derived using the polynomial based whole lens model.

Model	anterior surface vertex radius of curvature (mm)	anterior surface asphericity	posterior surface vertex radius of curvature (mm)	posterior surface asphericity
two conic surfaces	$+7.27 + 0.064age$	$+3.3\pm6.5$	-6.9 ± 2.2	$+1.5\pm3.6$ n.s.
Interdependent surfaces	$+7.08 + 0.085age$	$+5.9\pm12.1$	$-5.11 - 0.033age$	-0.2 ± 3.1 n.s.
Modulated hyperbolic cosine whole lens	$+14.3\pm 16.2$	-	-6.5 ± 2.0	-
Generalized conic whole lens	$+6.83 + 0.075age$	$+1.68 + 0.085age$	-6.7 ± 0.8	$+0.9\pm0.7$
Polynomial based whole lens	$+5.76 + 0.076age$	$-0.31 + 0.087age$	-6.5 ± 1.8	$+0.3\pm3.0$ n.s.

Table 7. Vertex radii of curvature and asphericities of lenses as a function of age (years) from the different models. Where linear regression slopes are not significantly different from zero ($p > 0.05$), means and standard deviations are given. The term n.s. denotes not significantly different from zero.

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