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The role of fluency in a mathematics item with an embedded graphic: Interpreting a pie chart

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Abstract

The purpose of this study was to identify the pedagogical knowledge relevant to the successful completion of a pie chart item. This purpose was achieved through the identification of the essential fluencies that 12- 13-year-olds required for the successful solution of a pie chart item. Fluency relates to ease of solution and is particularly important in mathematics because it impacts on performance. Although the majority of students were successful on this multiple choice item, there was considerable divergence in the strategies they employed. Approximately two-thirds of the students employed efficient multiplicative strategies which recognised and capitalised on the pie chart as a proportional representation. In contrast, the remaining one third of students used a less efficient additive strategy that failed to capitalise on the representation of the pie chart. The results of our investigation of students' performance on the pie chart item during individual interviews revealed five distinct fluencies were involved in the solution process: conceptual (understanding the question), linguistic (key words), retrieval (strategy selection), perceptual (orientation of a segment of the pie chart) and graphical (recognising the pie chart as a proportional representation). In addition, some students exhibited mild disfluencies corresponding to the five fluencies identified above. Three major outcomes emerged from the study. First, a model of knowledge of content and students for pie charts was developed. This model can be used to inform instruction about the pie chart and guide strategic support for students. Second, perceptual and graphical fluency were identified as two aspects of the curriculum which should receive a greater emphasis in the primary years due to their importance in interpreting pie charts. Finally, a working definition of fluency in mathematics was derived from students' responses to the pie chart item.

Keywords Fluency, Problem solving, Information graphics, Pie chart, Learning, Teaching.

1 Introduction

The early 21st century has been marked by an unremitting deluge of quantitative data about various aspects of life including weather patterns, mobile phone statistics, stock market trends, and government expenditure. To avoid being swamped or paralyzed by the volume and complexity of information, there has been a marked increase in the use of graphical displays of data (e.g., pie charts) in everyday and professional life. Data displays can be invaluable because (1) they enable data sets to be presented concisely, and (2) particular graphics can cue users to attend to patterns and relationships within the data. For example, *a pie chart* would be effective for showing the proportion of government funding on education whereas *a line graph* would be ideal for showing education spending over time. Henceforth, we will refer to these data displays as "information graphics" to highlight the informational purpose of these visual-spatial displays.

Though information graphics are useful for organizing, managing and communicating data, the creation of appropriate information graphics for presenting mathematical information is only half the solution to the "data deluge". By necessity, the other half of the solution rests with the capabilities of individuals' to interpret information graphics and extract mathematical information from them. However, the interpretation of information graphics in mathematics is paradoxical. While the use of graphics in the presentation of mathematical information *addresses the problem* of information overload for the individual, it *introduces the problem* of interpreting information graphics in mathematics. The visual-spatial quality of information graphics distinguishes them in representation and reasoning from text or symbolic languages in mathematics and the associated sequential reasoning (See Adams, 2003 for a discussion of the complexity of reading mathematical texts). However, despite their visual-spatial structure in representation, information graphics are not homogeneous. Information graphics comprise a diverse group of visual representations which vary substantially in representation and reasoning from each other (Mackinlay, 1999). For example, a pie chart, line graph, and calendar are all unique information graphics.

It is true that information graphics have been part of mathematics for countless years but what is different today is the ubiquitous nature of graphics in a data-rich society and their variety. There are over 4000 graphics in common use (Harris, 1996), and in schools alone, graphics are embedded in mathematics instruction, texts, tests, and resource materials including software. Thus, teachers of mathematics are confronted with the important and urgent task of educating their students to interpret a myriad of graphics competently. The consequences of individuals being unable to interpret information graphics are extreme and unacceptable in a democratic society (Steen, 1997): "an innumerate citizen of today is as vulnerable as the illiterate peasant in Gutenberg's time" (p. xv).

This paper takes an initial step towards improving students' interpretation of information graphics by investigating the fluency of primary students' interpretation of a pie chart. We purposefully narrowed our investigation to this topic for four reasons. First, the component of pedagogical content knowledge that appears to affect student performance is teachers' "knowledge of content and students" (KCS) (Hill, Ball, & Schilling, 2008, p. 377). Thus, to provide guidance for teachers, we need to glean information about *the content of particular graphics and students' knowledge of graphics*. Second, we examine *primary* students' performance because graphics are introduced to students early in their schooling. Third, we chose to investigate performance on a *pie chart* because it is a common graphic in everyday mathematics and used from primary school onwards. Additionally, research indicates a lack of correlation between students' performances on various information graphics even when they are informationally equivalent (Baker, Corbett, & Koedinger, 2001). Hence, we need to examine performance on particular graphics separately. Finally, we elected to focus on *fluency* of a task because it affects mathematical performance (Oppenheimer, 2008). Ultimately, we are interested in contributing to an evidence base that can guide instruction and support primary students to interpret graphics successfully on mathematics tasks. The investigation of primary students' fluency on a pie chart item is consistent with this goal.

As a background to our investigation, we describe contemporary thinking about the relationship between teacher knowledge and student performance which characterises our theoretical stance (Section 2.1). We then discuss representations and graphics in mathematics (Section 2.2), the purpose of pie charts (Section 2.3) and outline our perspective on fluency in interpreting graphics (Section 2.4). Subsequently, we present the design and methods for our investigation (Section 3), the results (Section 4) and a discussion of these results (Section 5). We conclude the paper with a preliminary model that contains knowledge of the content of a pie chart and students' fluencies in interpreting this graphic (Section 6) together with a working definition of fluency in mathematics.

2 Background

2.1 The relationship between teaching and learning

Since Shulman (1986), identified the concept of pedagogical content knowledge (PCK) over two decades ago, it has permeated thinking about mathematics education. However, Hill et al. (2008) argue that despite assumptions about the importance of PCK, there is limited evidence of the effectiveness and content of

PCK nor a large scale study of the relationship between teachers' PCK and gains in students' knowledge. They argue that to fully understand PCK, there is a need to consider "Knowledge of content and students" (KCS) as distinct from "Knowledge of content and teaching" (KCT) and "Knowledge of curriculum" (Hill et al., 2008, p. 377). Figure 1 shows Hill et al.'s conceptualisation of the components of pedagogical knowledge and subject matter knowledge and their relationship. Specifically, Hill et al. propose that KCS is a subset of PCK and requires an understanding of how students learn particular content including the errors that students make. They argue that knowledge of content and students (KCS) is integral to knowledge of content and teaching (KCT) and call for studies across various mathematics topics in mathematics to identify knowledge of content and student learning.

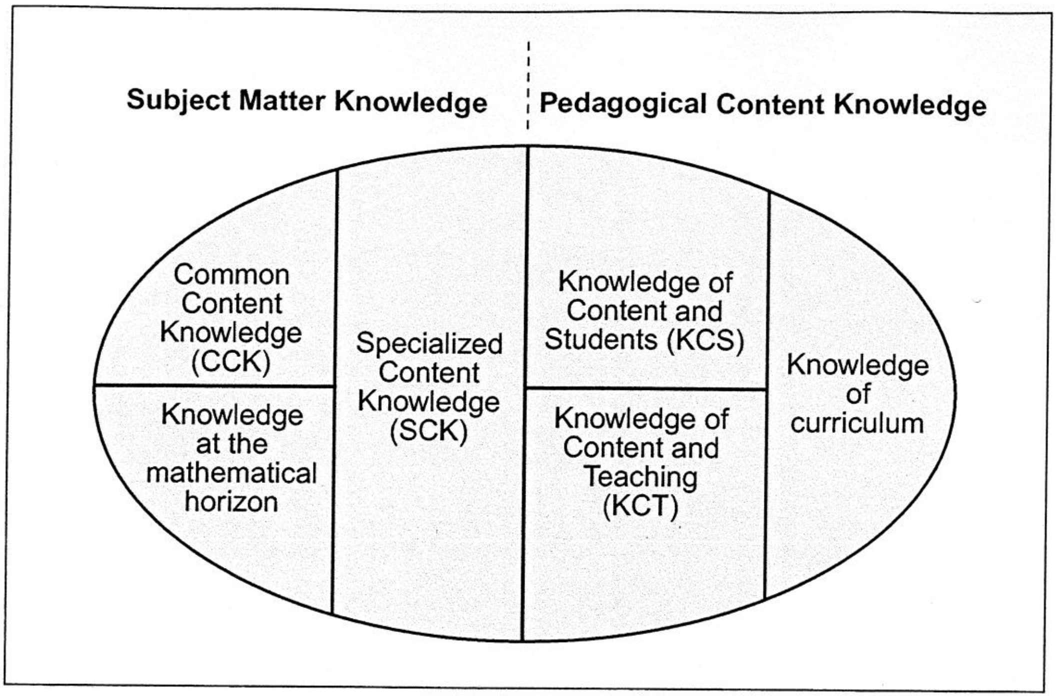


Fig. 1 Domain map for mathematical knowledge for teaching (Hill et al., 2008, p. 377).

Our selected focus on information graphics is highly relevant to the mathematics concept strand of Data because information graphics are data displays. Hence, the implementation of a curriculum that includes Data should address the range of information graphics use as data displays including pie charts. Here, we focus on the establishment of KCS of the pie chart within Data which subsequently can inform KCT of the pie chart.

Although Hill et al. (2008) appear to be the first to produce a "Domain map for mathematical knowledge for teaching" (Fig. 1), that isolate KCS and KCT within PCK others have argued the importance of knowledge of content and students albeit using different terminology. For example, proponents of cognitively guided instruction suggest that there are five facets of pedagogical content knowledge that impact related to students' performance on mathematical tasks (Carpenter, Fennema, & Franke, 1996) (see Table 1). They also argue that these facets provide an explicit guide for listening and questioning by the teacher (Carpenter et al., 1996). That is in Hill et al.'s terms, they inform knowledge of content and teaching (KCT).

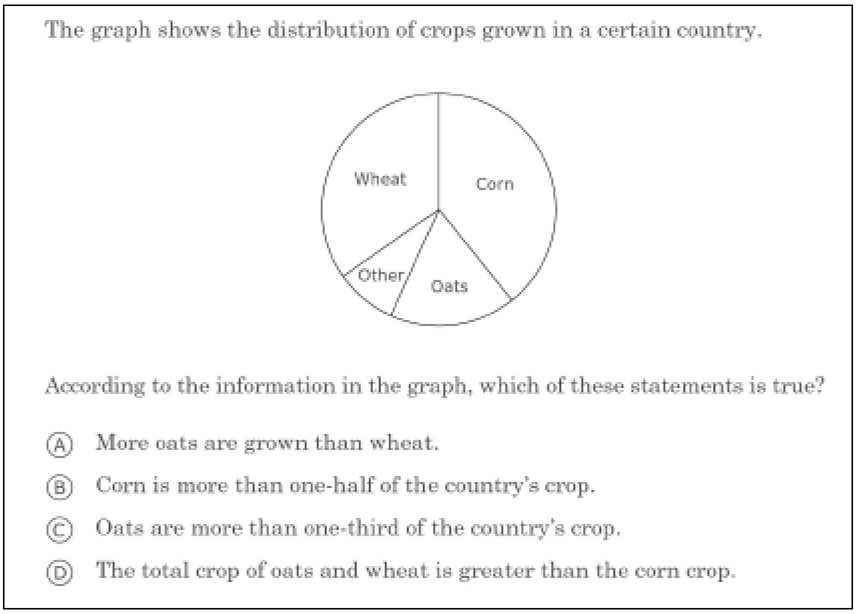
Table 1 Facets of pedagogical content knowledge (Carpenter et al., 1996)

What problems students can typically solve and how students solve them.
How students connect new ideas to existing ideas.
What is difficult and what is easy for students.
Common errors made by students.
An understanding of individual students' thinking.

We concur with Carpenter et al. (1996) and Hill et al. (2008) on the importance of knowledge of content and students in pedagogical content knowledge and have found this to be a productive line of inquiry in our research on particular graphics, such as number lines (Diezmann & Lowrie, 2006). It is within Carpenter et al.'s and Hill et al.'s frameworks of pedagogical content knowledge and KCS respectively that our investigation of fluency in the interpretation of the pie chart is undertaken. Hence, within the scope of investigating fluency we particularly focus on students' solutions and thinking about a pie chart and their errors and difficulties on this item.

2.2 Representations and graphics in mathematics

Mathematics draws on multiple representational systems to enable individuals to convey and think about mathematical ideas. These systems are oral language, text (e.g., word stories), manipulatives (e.g., multi-base arithmetic blocks), symbols (e.g., 204), and external visual representations (e.g., pictures, diagrams) (Lesh, Post, & Behr, 1987). The visual representation system includes information graphics (e.g., line graph, pie chart, map, calendar) which variously convey quantitative, ordinal and nominal information through a range of perceptual elements (Harris, 1996; Mackinlay, 1999). Students need to be able recognize and manipulate ideas within each of these systems, and to translate ideas between representational systems (Lesh et al., 1987). For example, on the following pie chart item (Fig. 2), the solver needs to assimilate information presented in the text and graphic in the solution process and select a textual response (Fig. 2).


The graph shows the distribution of crops grown in a certain country.

According to the information in the graph, which of these statements is true?

(A) More oats are grown than wheat.
(B) Corn is more than one-half of the country's crop.
(C) Oats are more than one-third of the country's crop.
(D) The total crop of oats and wheat is greater than the corn crop.

Fig. 2 Pie chart of crop distribution (International Association for the Evaluation of Educational Achievement (IAE), 2007)

Information graphics can be classified into six graphical languages, which, with the exception of the Miscellaneous graphics, have unique spatial structures based on their perceptual elements and the encoding techniques that are used to represent information (Mackinlay, 1999) (See Table 2). Miscellaneous graphics are a catchall for the remaining unclassified graphics and include the pie chart (e.g., Fig. 2). These graphics do not share conventions across the Miscellaneous languages; rather, each type of graphic has its own unique conventions of use. For example, a pie chart and a calendar, which are both Miscellaneous graphics, vary enormously from each other in their representation and conventions of use.

Table 2 Descriptions of graphical languages and associated encoding techniques

Graphical Languages	Encoding Techniques
Axis Languages (e.g., horizontal and vertical axes	A single-position encodes information by the placement of a mark on an axis.
Apposed-position Languages (e.g., line graph, bar chart, plot chart)	Information is encoded by a marked set that is positioned between two axes.
Retinal-list Languages (i.e., graphics featuring colour, shape, size, saturation, texture, orientation.)	Retinal properties are used to encode information. These marks are not dependent on position.
Map Languages (e.g., road map, topographic map)	Information is encoded through the spatial location of the marks.
Connection Languages (e.g., tree, acyclic graph, network)	Information is encoded by a set of node objects with a set of link objects.
Miscellaneous Languages (e.g., pie chart, Venn diagram)	Information is encoded with additional graphical techniques (e.g., angle, containment).

2.3 The Pie Chart

The pie chart is a commonplace information graphic that shows "the relative sizes of components to one another and to the whole" (Harris, 1996, p. 280). Pie charts are often used in everyday life, for example, in a newspaper showing the proportion of household expenditure on various goods and services. These graphics also feature in international and national mathematics assessments. For instance, the Grade 8 mathematics items in the 2003 Trends in International

Mathematics and Science Study (TIMSS) included a pie chart of crop distribution (Fig. 2) (IAE, 2007). Thus, there is an expectation that by Grade 8, students can interpret this common graphic.

Despite the need to interpret pie charts in everyday situations and in school activities, many students are unsuccessful on items that incorporate this type of graphic. For example, internationally, in the TIMSS 2003 assessment, only 71% of Grade 8 students were successful on the pie chart items (Fig. 2) (National Center for Education Statistics (NCES), n.d.). It is likely that students' understanding of the graphic embedded in this item, as well as the text, contributed to their success (or lack thereof) because the individual plays an important role as the interpreter of a representation (von Glasersfeld, 1987). Although large scale quantitative studies can provide information about the percentage of target populations' success on pie chart items (or other information graphics), what cannot be determined from these studies and the statistics reported is the knowledge of content and students (KCS) for a pie chart. Thus, to complement large scale quantitative studies which include pie charts; there is a need for in-depth qualitative studies to explicate the KCS for the pie chart. The ascertainment of this knowledge is particularly important because in turn it can inform knowledge of content and teaching (KCT). Prior to describing our approach to examining students' knowledge of a pie chart, we first explore the likely role that fluency that plays in KCS.

2.4 Fluency

Fluency is the ease of completing a mental task and success on the task (Oppenheimer, 2008). Despite the importance of fluency in cognition, there is no universal definition. To accommodate the lack of a precise definition, Oppenheimer (2008) has created a taxonomy of fluency affects. These affects include conceptual fluency, attentional fluency, retrieval fluency and encoding fluency. To complement encoding fluency, which involves creating a representation from given information, we would add the complementary *decoding fluency,* which would involve interpreting the information from a representation. Henceforth, we use the term *graphical fluency* in lieu of decoding fluency to highlight the graphical nature of this particular type of fluency. Consistent with Oppenheimer's (2008) taxonomic approach to fluency, the relevance of various fluencies in relation to ease of solution can be considered in relation to the task and the individual.

At the task level, ease relates to the fluencies required of the task and in the solution strategy. On a mathematics task that incorporates text and mathematical symbols or operations, the fluencies will include: *conceptual fluency* for an understanding of the task; *linguistic fluency* for the language used; and *mathematical fluency* for the mathematical operations required. However, when a mathematics task also includes a graphic, *perceptual and graphical fluency* will also be involved. *Perceptual fluency* relates to the employment of spatial perception skills on the spatial elements of the graphic, such as figure-ground contrast (Willems & Van der Linden, 2006). *Graphical fluency* relates to the appropriate interpretation of and reasoning with a particular graphic. Ideally, in the solution of a pie chart item, the individual will interpret the graphic as a proportional representation and use proportional reasoning in the solution process (Harris, 1996). Thus, multiple fluencies are likely to be involved in any mathematics task with an embedded graphic.

At the individual level, ease involves the cognitive demand of the task for the solver. Based on a review of studies of fluency, Oppenheimer (2008) identified three ways that fluency affects an individual's judgement, and subsequently, their success. First, fluency can affect judgement through *mental representation* (Oppenheimer, 2008). Students' conception of a task can affect judgement through retrieval fluency and strategy selection. Retrieval fluency is of particular importance because it provides the solver with possible strategies to apply. However, strategy selection during the retrieval process is affected by the frequency with which a particular strategy has been accessed previously (Hertwig, Herzog, Schooler, & Reimer, 2008). Thus, fluency involves an understanding of what strategies students can use and retrieve on a pie chart task. Second, fluency can affect judgement through *cognitive operations.* Notably, Oppenheimer (2008) argues that fluency should not be conceived as a straightforward positive cue. He argues that disfluency can be beneficial if it leads participants to use more systematic processing strategies through greater attention to the task and a slowing down in processing. Hence, the role of disfluency in interpreting a pie chart needs to be examined. Third, fluency can affect judgement through *attribution.* Objects can be recognized as fluent (or familiar) based on frequency, recency or duration of exposure (Oppenheimer, 2008). Thus, there is a need to be attuned to students' commentaries about the attribution of their decisions on a pie chart task. These three impacts on fluency and success (mental representation, cognitive operations, attribution) at the individual level suggest that fluency with a task will be mediated by an individual's own fluencies.

3 Design and Methods

A qualitative approach was adopted to enable an in-depth examination of the research question, *What aspects of fluency impact on students' performance on a pie chart item?* Three sub questions were associated with the research question.

The first two sub questions focussed on understanding how students engaged with the item with a particular focus on the selection and employment of a strategy and its appropriateness. These sub questions relate to retrieval fluency. The final sub question sought to identify additional fluencies associated with success on this particular but typical pie chart item.

- *What strategies did students use on the pie chart item?*
- *How appropriate were students' strategies for the pie chart item?*
- *What fluencies were associated with the pie chart item?*

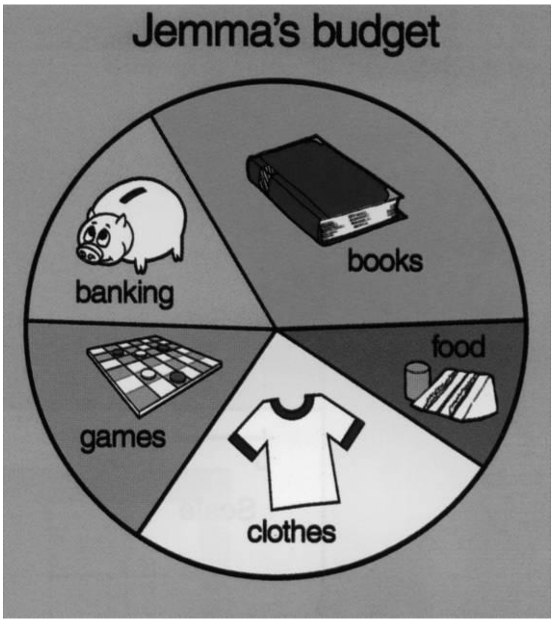
3.2 The participants

The participants comprised 15 students (M = 4; F = 11) aged between 12 and 13 years from a small intact Grade 7 class, which was ideal for an in-depth investigation. These students attended a government school in an Australian capital city and had English as their first language. Pie charts were part of the curriculum for this class. Notably, one year after our data were collected, this class would have been an eligible population for a Grade 8 TIMSS assessment which has previously included a pie chart item (e.g., IAE, 2007).

3.3 The Pie Chart item

The item selected for investigation was *Jemma's Budget* (Queensland Studies Authority (QSA), 2002) (Fig. 3). This item requires students to identify the total budget expenditure from a pie chart. This item was appropriate for our investigation for two reasons. First, the item is typical of a pie chart item used to represent data for students of this age. In addition, it was similar to the pie chart item presented in the TIMSS Grade 8 assessment (IAE, 2007) (see Fig. 2). That item was categorised as a routine problem (Mullis, Martin, & Foy, 2005). Second, the selected item was useful because the linguistic and numerical demands of the task were low. Thus, any errors or difficulties within the solution process are more likely to be related to the interpretation of the graphic than to the text or calculations.

In 2003, Jemma budgeted $30 on clothes. Approximately how much money did she get that year?



Answer

- ☐ $90
- ☐ $120
- ☐ $150
- ☐ $180

Fig. 3 A pie chart (QSA, 2002 p. 6, p. 1 of insert)

3.4 Data collection and analysis

3.3.1 Data collection

The data on the Pie Chart item were collected during individual interviews on pairs of items from three of the six graphical languages. (On another occasion, students were interviewed about three pairs of items from the other three graphical languages.) The students selected the multiple choice response for each pair of items and were then invited to explain their responses. Here, we report on the data collected on the Pie Chart item (Fig. 3) which was one of the two Miscellaneous language items in the interview (The other item was not a pie chart, and hence, is not discussed in this paper).

3.3.2 Data analysis

Students' solutions on the Pie Chart item were scored "1" for the selection of the correct multiple choice response and "0" for the selection of an incorrect response. Explanations for the selected responses were then transcribed and analysed thematically according to students' strategies and the associated fluencies.

4. Results

Fourteen of the fifteen students (93%) of students successfully selected the correct multiple choice responses for *Jemma's Budget* (Fig. 3). However, as the following results will demonstrate, students' convergence in their selection of the correct responses masked a divergence in their strategy use and fluencies on this task.

4.2.1 What strategies did students use on the pie chart item?

Collectively, the class had a repertoire of six strategies for this task (Table 3). All students who used the following five strategies were successful: *the Fraction strategy; the Diameter strategy; the Estimate quantity and add strategy; the Estimate size, quantity and add strategy;* and the *Visualize and add strategy.* The final strategy was *Guessing.* The only student to use this strategy was unsuccessful.

Table 3 Pie chart strategies

Strategies	N = 15	Percentage of Overall Success
Fraction strategy	8	53.3%
Diameter strategy	1	6.7%
Estimate quantity and add strategy	3	20%
Estimate size, quantity and add strategy	1	6.7%
Visualize and add strategy	1	6.7%
Guessing	1	0%

1. Fraction strategy

The *Fraction strategy* was a single step process that involved the combined use of a key segment of the circle as the basis for identifying annual income and the relevant textual information. That segment was "Clothes" because this was where expenditure of $30 had been specified. Amy's (A) response illustrates this combination of segment and amount as the basis for her calculation.

A: $120……..Well the clothes looked about a quarter of the whole circle….. and the clothes looked about a quarter of the circle and so if she only spent um clothes for a quarter of the circle then if you times that ($30) by 4 because there are 4 quarters in the circle then she would have had the……she would have had around 120.

This strategy was used by eight successful students including Amy and capitalised on the utility of a pie chart as a proportional representation.

2. Diameter strategy

The *Diameter strategy* was a two-step process in which a line was extended from one of the radii bounding the clothes segment (Step 1). Upon visual confirmation that this extended line formed the diameter of a circle, the Clothes segment was verified as a quarter of the circle. This was followed by the calculation of 4 times $30 as in the *Fraction strategy* (Step 2). Megan (M), the only student to employ this strategy, explains:

M: I chose $120 because um well I saw how big the clothes were … I tried to see if I finished drawing off the diameter, from one of the radius sides, from one of the lines of clothes and continued it to the other side of the circle, making a diameter. I was going to see if it would make it 1/2, and I tried that with both sides to see if it end up being a quarter which it pretty much resembled, pretty close to perfect. So then I guessed it would be $120 because 3 times 4 is 12.

This strategy involved first confirming that the Clothes segment was a quarter and then using a quarter as the basis for the solution similarly to the Fractional strategy.

3. Estimate quantity and add strategy

The *Estimate quantity and add strategy* was also a two-step process that involved an estimation of the cost of various represented items and a calculation of this total. Holly (H) began with the known segment of $30 for clothes and then estimated the other sectors (Step 1) before calculating the answer (Step 2).

H: Okay well if she spent $30 on clothes um she probably I reckon she spent about 10 or no…$15 for that.

I: Which was food?

H: Which is ($) 35 if you add them together and then books would probably be about ($) 50 … ($) 85 and um they would probably be about $40 or something

I: For banking and games?

H: Yeah. How much did I say with them all added up together?

I: ($) 85

H: Oh yeah….yeah

I: And so you've chosen ($) 120.

H: Yeah

Holly and two other students used the strategy successfully. However, the lengthiness of this strategy creates multiple points for errors to occur.

4. Estimate size, quantity and add strategy

The *Estimate size, quantity and add strategy* was a three-step process that involved a series of iterations in which the size of the segment was compared to the size of a segment for which the quantity was known (Step 1), the quantity of the new segment was identified (Step 2), and the total of all sectors was calculated (Step 3). Caleb (C), the only student to use this strategy, reported:

C: And *Jemma's Budget* and so I looked and $30 for clothes is just…books is a little bit bigger as well and I thought if that's an extra $10…so that's 70…and the food would be about 10 cause it's quite small and about three could fit into clothes so that makes 80 and then those two are about the same size and they're half as big as that so it must be 20 each…that 40…50…and then that's 120.

There are similarities between this three-step strategy and the two-step *Estimate quantity and add strategy.* In this three-step strategy, the size comparison of each segment was explicit whereas in the two-step strategy it was implicit.

5. Visualize and add strategy

The *Visualize and add strategy* was a relatively complex two-step process. The strategy commenced with the visualized comparison of one of the sectors to a known segment to establish the quantity it represented (Step 1). This information was held in memory and then the total of all sectors was calculated (Step 2). Byron, the only student to use this strategy, explained the method and its inherent difficulty.

B: $120. I didn't really understand it at first so I just looked at it and I saw the clothes. So that would be 30 **and then, in my head, I measured the games and I like put it on top of that** (emphasis added) and there was about, 'cause I counted them up and there was about 10 there, and another 10 there, and another 10 there and put that on, and that would have been $120.

I: So you put it on top of like the clothes section to see how it matched type of thing?

B: And there was like $10 left. And then I put the bank in and I did the same thing and it came up as the same as games. And then I put the books on top of it and it was bigger, so I put the clothes on top of the books and it had ($) 20. I think it had ($) 20 left over. Oh no it was ($) 10. Yes it was ($) 10 and so that was ($) 50 and then I did the food and it came up as ($) 10, yes, and then I added the clothes and it (sic) came up with ($) 120.

Hence, an essential component of this strategy was the ability to visualize and manipulate images.

6. Guessing strategy

Cathy was the only student to select the incorrect response and the only student to employ a *Guessing strategy.* Although she correctly identified the Clothes sector as a quarter, she did not make use of this information to identify the total.

C: $150…….and I just guessed…..so yeah

I: So you just guessed at the amount, was there anything that could have helped you to make that guess?

C: Um well **the clothes is a quarter** of it and yeah (emphasis added)

I: Are you still thinking?

C: No

I: Okay so the clothes is a quarter of the graph and how much did she spend on clothes?

C: **$30** (emphasis added)

I: So you're sticking with $150?

C: Yeah

Cathy identified the Clothes sector as a quarter and was clear that this sector represented $30. However, she did not attempt to use this knowledge to check the accuracy of her guess. Thus, the existence of prior knowledge alone appears to be insufficient for its use in the solution process.

4.2.2 How appropriate were students' strategies for a pie chart item?

The appropriateness of students' strategies was determined by the "fit" between the strategies for a pie chart item and the cognitive demand that resulted from the execution of these strategies. Recall, that cognitive demand is also mediated by individual capacities and preferences.

Each of the five strategies used successfully on *Jemma's Budget* (Fig. 3) had a proportional component. Two strategies had a multiplicative base and the other three strategies had an additive base.

The two multiplicative strategies were the *Fraction strategy* and the *Diameter strategy.* Both strategies involved reasoning about the proportion of the key segment (i.e., Clothes) relative to the whole circle. The *Fraction strategy* is somewhat more efficient than the *Diameter strategy* because the latter strategy involved the extra step of checking that the Clothes segment was a quarter by extending one of the radii to become the diameter of the circle

The three additive strategies were the *Estimate quantity and add strategy; the Estimate size*, *quantity and add strategy;* and the *Visualize and add strategy.* These strategies involved the comparison of a key segment of the circle with other segments of the circle rather than with the whole circle. In all cases, the proportional reasoning lacked precision in the explanations of the comparison between segments. This was evident in the students' use of qualitative rather quantitative language. For example Caleb commented "clothes is just…books (segment) is **a little bit bigger** as well" (emphasis added). A lack of precision was also evident in students' uncertainty about the quantity represented by a segment. For example, Holly (above) commented, "I reckon she spent about **10 or no…$15** for that (Food) (emphasis added)". Because proportional reasoning was less effective in the additive strategies, the multiplicative strategies are a better "fit" for the task.

The cognitive load created by the additive and multiplicative strategies differed with the former creating a higher load than the latter. Unlike the multiplicative strategies, each of the additive strategies required students to perform two cognitive processes simultaneously. The three additive strategies involved the repeated comparison of the segments of the circle against the key segment (Clothes). In tandem, the students either needed to keep a running tally of amounts mentally or hold the amounts in memory and then add them. The need for students using the additive strategy to undertake two cognitive processes simultaneously creates a higher cognitive demand than engaging in one process using a multiplicative strategy. Thus, generally a multiplicative strategy is considered to be more efficient than an additive strategy, and hence, have a lower cognitive demand.

There were also differences in the cognitive demand among the additive strategies. There is a quantitative difference between the two-step *Estimate quantity and add strategy* and the three-step *Estimate size, quantity and add strategy.* Hence, the former is less demanding than the latter. However, there is a qualitative difference between the demands of additive strategies involving estimation and the remaining additive strategy of *Visualize and add.* This latter strategy has been identified as similar in difficulty to *Estimate quantity and add* because both strategies involve two steps. However, depending on an individual's preference for visual or non visual strategies (Kulm, Campbell, Frank, & Talsma, 1981), the visualization strategy could be relatively more or less demanding.

A comparison of the relative ease of the five strategies that led to success is shown on Fig. 4. The multiplicative strategies were more efficient than the additive strategies because they capitalised on proportional reasoning. Additionally, in all cases of similar strategy use, strategies with fewer steps were considered to be more efficient than those with more steps. Such strategies also had fewer opportunities for error in the solution process. Finally, both two-step additive strategies were considered to be similarly appropriate albeit qualitatively different. This difference relates to an individual preference for visual or non visual strategies (Kulm et al., 1981).

<table>
<tr><td rowspan="2">Multiplicative Strategies</td><td colspan="2">Fraction strategy (1 step)</td></tr>
<tr><td colspan="2">Diameter strategy (2 steps)</td></tr>
<tr><td></td><td colspan="2">↑Appropriateness↓</td></tr>
<tr><td rowspan="2">Additive Strategies</td><td>Estimate quantity and add strategy (2 steps)</td><td>Visualize and add (2 steps)</td></tr>
<tr><td>Estimate size, quantity and add strategy (3 steps)</td><td>(NA)</td></tr>
</table>

Fig. 4 Relative appropriateness of strategies

4.2.3 What fluencies were associated with the pie chart item?

In addition to retrieval fluency which is implicit in strategy use, four types of fluency were identified from students' interactions about the pie chart, namely conceptual fluency, linguistic fluency, perceptual fluency and graphical fluency.

1. Conceptual fluency

Students' conceptual fluency was indicated by their understanding of the question. This understanding of the question ranged from instant comprehension to confusion. Alan (A) comprehended the question and commented on his familiarity with this type of task.

A: **Well I got virtually what they wanted me to do immediately. I've done stuff like that before** …I've seen things like that before in puzzle books and I've had to draw the other half, those sorts of things (emphasis added).

In contrast, other students had difficulty comprehending the task and difficulty explaining what they failed to comprehend. Two types of difficulty were apparent. One difficulty related to the annual budget. For example Chiara (C) commented:

C: **It wasn't quite so straight forward, we had to (pause) the question wasn't quite as clear.** (emphasis added)

I: So what was it about the question that made it a bit unclear?

C: Well you had to figure out, **it doesn't exactly say that this is one year in the diagram,** yeah so you just have to assume that yeah (emphasis added).

The other difficulty related to the segment of the circle identified as Clothes. For example, Bella (B) stated:

B: Because I sort of get mixed up. **I didn't really get the question at first** but then I realised it (Clothes segment) was ¼ (emphasis added).

Notwithstanding differences in the source of the difficulties, any conceptual difficulties have the potential to impact negatively on success.

2. Linguistic fluency

A lack of knowledge of the term "budgeted" was problematic on this item. For Elise (E), this lack of vocabulary knowledge went beyond success on the tasks to impact on her self esteem.

E: I couldn't remember what budgeted meant, **I felt really stupid.** (emphasis added)

Hence, in the short term, familiarity with key mathematical vocabulary is essential for success on a task. In addition, in the longer term, competence and confidence with mathematical vocabulary could impact on students' self esteem.

3. Perceptual fluency

The perceptual skill required in this task was to recognize a quarter of a circle in an atypical orientation. However, the orientation of the key segment (Clothes) was problematic for Helena (H). She referred to the Clothes segment as not being an "outright quarter", which could be interpreted as a quarter in a more typical orientation with the radii of the segment in horizontal and vertical positions.

H: I just found it a bit harder by putting the other things in the pie graph and not making clothes, like, **outright quarter.** (emphasis added)

Thus, being able to orient shapes underpins success on the Pie Chart item for some students.

4. Graphical fluency

Graphical fluency involves knowledge of the particular type of graphic and how to use that graphic. Megan (M) identified that she was unfamiliar with the pie chart and was also lacking the requisite tool of a protractor. Although a protractor might be essential for some pie chart tasks, it was unnecessary for this task.

M: It's quite hard to do pie graphs, if you haven't done it before, and if you don't have a protractor.

I: It's not like you hadn't done it before though.

M: I hadn't. I guessed how to do it.

Megan's lack of knowledge of the proportional representation of a pie chart was demonstrated further during the interview when she explained that the item would be easier if the representation was changed from a pie chart to a bar graph.

M: **You can probably make it easier by putting it into a standard graph** like that (indicating a bar graph). A bar graph and like have the labels near it or something, either under it or on it or something like that and then you could have like a little thing, a little scale beside it. And you'd be able to make it a lot easier especially if you didn't understand how to do a pie graph. (emphasis added)

The choice of representation, however, is the province of the creator of a graphic rather than its interpreter. Hence, Megan needs to be able to interpret a pie chart which seems to be unfamiliar to her at present.

Graphical fluency was also affected by knowing how to use the pie chart. Caleb experienced difficulty using the pie chart but was able to identify a simpler strategy during the following interaction, thus demonstrating the benefit of explaining in stimulating thinking.

C: I had to sort of measure out about which one (segment) would fit most into the other one (segment).

I: How much of the circle do you think the clothes part actually represents?

C: Um…I think it represents about maybe … well… I'd say about quarter of it yes so **another way I could have done it** is make it a quarter yeah. (emphasis added)

I: And then what would have you done?

C: What like another way of …

I: Working it out…

C: **Oh another way** yeah if I looked to see how it was a quarter…what was half…cause I got the $30 and it looks like a quarter and I could've said well a quarters is four…so four times three is 12 so it must be $120. (emphasis added)

Thus, graphical fluency is fundamental to understanding the purpose and use of a pie chart, specifically its conventions of use.

Collectively, each of these fluencies contributed to students' ease with the pie chart task. The existence of additional fluencies is not excluded; however, they were not demonstrated explicitly by this class on the pie chart item.

5 Discussion

Our analysis of students' performance on the Pie Chart item reveals three points of note. First, despite the high success of students in choosing the correct multiple choice responses on the Pie Chart item, retrieval fluency was limited for some students. These students might not have retrieved and implemented an efficient (multiplicative) strategy because they did not know one. Hence, they either resorted to their default (e.g., additive) strategy (Blöte, Van der Burg, & Klein, 2001) or they developed an innovative strategy (for them) which was less than efficient (Schwartz, Bransford, & Sears, 2005). In both these situations, a lack of knowledge of or confidence with a multiplicative strategy was problematic for some students.

Second, disfluency was problematic but not obstructive for most students. In addition to retrieval disfluency, described above, students exhibited conceptual, linguistic, perceptual, and graphical disfluency. Conceptual disfluency involved failing to appreciate the task requirements: "I didn't really get the question". Linguistic disfluency was limited to the term "budgeted" on the pie chart item. However, failing to understand this term could halt students' ability to successfully solve this problem. Perceptual disfluency involved a difficulty with orientation, which is a foundational perceptual skill for mathematics (Del Grande, 1990). Graphical disfluency relates to the inability to use a graphic appropriately. On the Pie Chart item some students demonstrated a lack of understanding of the conventions of a pie chart. Thus, our findings lend support to Oppenheimer's

(2008) claim that disfluency can be beneficial to cognition because despite their disfluencies students were able to complete the item successfully. However, we propose that if there is a serious disfluency it would be obstructive rather than beneficial. For example, a serious disfluency in English language would render the text inaccessible, and hence, with the exception of guessing, the student would be unable to complete the item (See Shorrocks-Taylor & Hargreaves, 2000 for a discussion of language demands and mathematics tests).

Finally, despite most students selecting the correct multiple choice responses on the Pie Chart item, correctness was a poor indicator of students' understanding. As demonstrated above, there, is considerable scope to improve students' strategy selection and to address various disfluencies.

6 Concluding Points

The outcomes of our investigation suggest three main points. First, our investigation has provided some insight into the knowledge of content and students (KCS) required on a pie chart item which we have represented as a model of knowledge of content and students on pie charts (Figure 5). The grey regions on this model represent two aspects of KCS. The first aspect, represented by a circular region, identifies the key content knowledge for a pie chart. That is, a pie chart is a proportional information graphic. The second aspect, represented by a hexagonal region, presents the five fluencies (conceptual, retrieval, linguistic, perceptual, graphical) that emerged as essential and/or problematic from the students' performance on the pie chart item. These fluencies relate to understanding students' solutions and thinking about a pie chart and their errors and difficulties on this item. An additional segment labelled "Other Fluencies" has been included in this hexagonal region as a reminder that it is likely that additional fluencies might impact on other pie chart items. Recall, Oppeneimer (2008) identified additional fluencies in his taxonomy of fluency effects. Clearly, the model of KCS for the pie chart needs to be tested with larger populations and on different pie chart items, and, subsequently refined to relate generically to pie charts. The white region on the model surrounding the grey regions represents Knowledge of Content and Teaching (KCT). KCT was not the focus of this study. Nevertheless, it is included in this model as a reminder that one of the purposes in identifying KCS was to provide an evidence base to inform instructional decisions and strategic support for students (KCT).

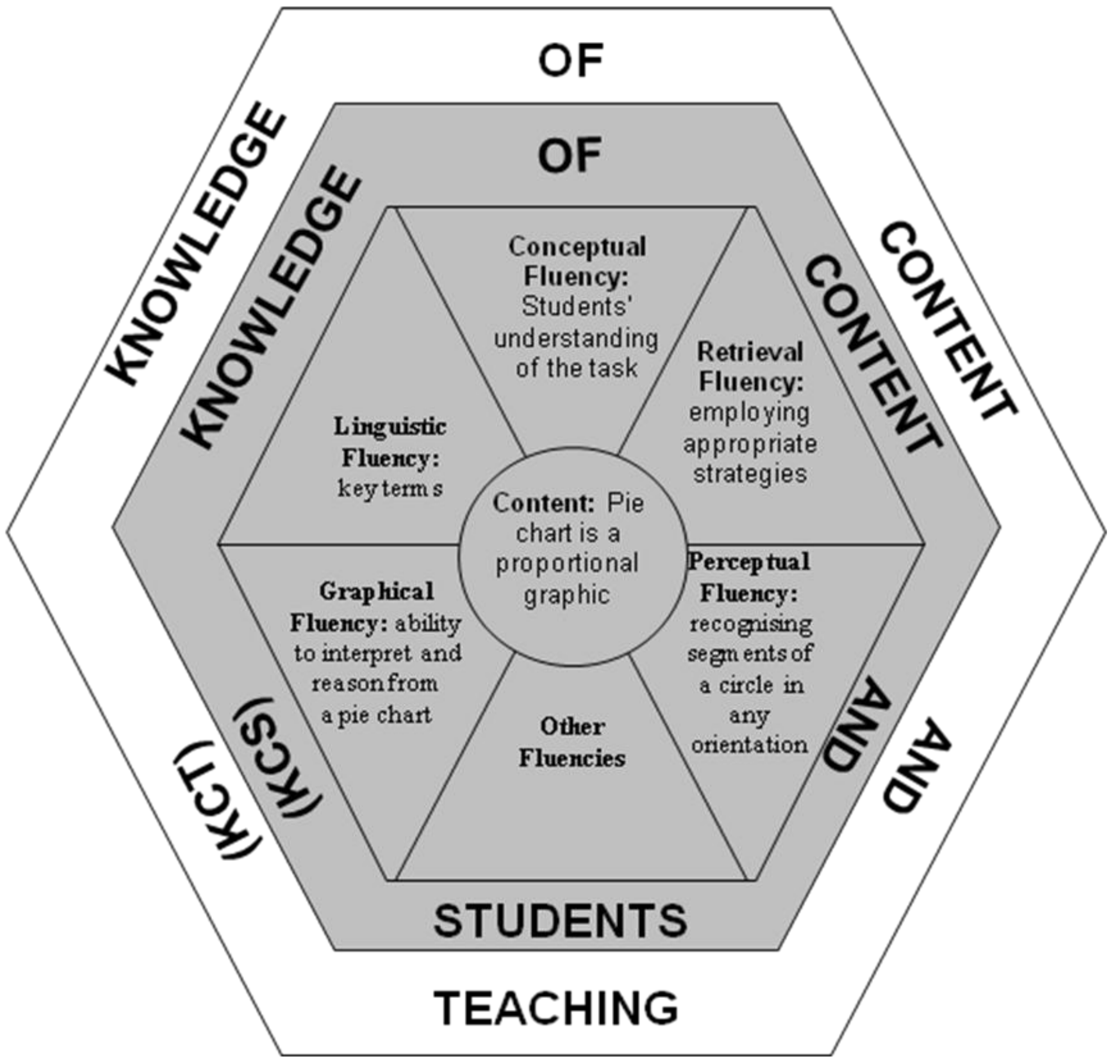


Fig. 5. A model of knowledge of content and students for pie charts

Second, our investigation of KCS for the pie chart has indicated the need to review knowledge of curriculum (Fig.1). Typically, mathematics curriculum includes attention to conceptual and linguistic fluency (i.e., key vocabulary) but generally perceptual and graphical fluency are overlooked in the middle to latter primary grades. Perceptual fluency is often the focus in the early years of schooling (Del Grande, 1990) but this focus needs to continue until students have a strong foundational fluency in perception. In interpreting a pie chart, perceptual fluency is critical because perceptual elements form the basis of this graphic (Harris, 1996). In addition, graphical fluency is as fundamental in the technological age as arithmetical fluency was in the industrial age. Hence, curriculum must include attention to graphical fluency. Moreover, more focused attention is needed towards retrieval fluency. Too often, classroom discussions of strategies relate only to alternative ways of solving a task rather than comparing and critiquing strategies in the solution process and determining which strategy is most appropriate for a particular task. As shown with the pie chart item, the single step multiplicative strategy which capitalised on proportional reasoning was superior to the other strategies (Fig. 4).

Finally, based on our investigation of students' responses to the pie chart, we propose a two-part working definition of fluency for use with mathematics items with embedded graphics, namely that:

1. *Fluency* is the ease of executing the requisite set of cognitive knowledge, skills and processes essential to the solution of a particular task; and
2. *Disfluency* is the lack of ease in executing the requisite set of cognitive knowledge, skills and processes essential to the solution of a particular task <u>or</u> the inappropriate use of this knowledge. Although mild

disfluencies can be overcome, serious disfluencies can be obstructive in the solution process.

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