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PAPER PRESENTATION #2 FOR TSG #19

METHODOLOGIES FOR INVESTIGATING RELATIONSHIPS BETWEEN CONCEPT DEVELOPMENT AND THE DEVELOPMENT OF PROBLEM SOLVING ABILITIES

Richard Lesh
Indiana University
ralesh@indiana.edu

Lyn English
Queensland University of Technology
l.english@qut.edu.au

Thomas Fennewald
Indiana University
tfennewa@indiana.edu

This paper is the second in a pair that Lesh, English, and Fennewald will be presenting at ICME TSG 19 on Problem Solving in Mathematics Education. The first paper describes three shortcomings of past research on mathematical problem solving. The first shortcoming can be seen in the fact that knowledge has not accumulated – in fact it has atrophied significantly during the past decade. Unsuccessful theories continue to be recycled and embellished. One reason for this is that researchers generally have failed to develop research tools needed to reliably observe, document, and assess the development of concepts and abilities that they claim to be important. The second shortcoming is that existing theories and research have failed to make it clear how concept development (or the development of basic skills) is related to the development of problem solving abilities – especially when attention is shifted beyond word problems found in school to the kind of problems found outside of school, where the requisite skills and even the questions to be asked might not be known in advance. The third shortcoming has to do with inherent weaknesses in observational studies and teaching experiments – and the assumption that a single grand theory should be able to describe all of the conceptual systems, instructional systems, and assessment systems that strongly molded and shaped by the same theoretical perspectives that are being used to develop them. Therefore, this paper will describe theoretical perspectives and methodological tools that are proving to be effective to combat the preceding kinds or shortcomings. We refer to our theoretical framework as *models & modeling perspectives* (MMP) on problem solving (Lesh & Doerr, 2003), learning, and teaching. One of the main methodologies of MMP is called multi-tier design studies (MTD).

Three Shortcomings of Past Problem Solving Research

1. Relationships are Unclear between Concept Development and the Development of Problem Solving Competencies. One shortcoming of past problem solving research is that it has not been clear how concept development is expected to interact with the development of relevant problem solving heuristics, beliefs, dispositions, or processes. In fact, in most curriculum standards documents (NCTM, 2000; http://www.doe.state.in.us/standards/), problem solving tends to be listed as the name of a chapter-like topic similar to algebra, geometry, or calculus. In other words, the implicit assumption is conveyed that problem solving ability is expected to increase by: (a) first, mastering relevant concepts, (b) second, mastering relevant problem solving heuristics, strategies, beliefs, dispositions, or processes, and (c) third, learning to put these concepts and processes together to solve problems. Consequently, when such assumptions are coupled with the flawed belief that students must first learn concepts and processes as abstractions before they can put them together and use them in "real life" problem solving situations, problem solving tends to end up never getting taught at all in many classrooms. So, one of the most critical challenges for future problem solving research is to clarify the nature of relationships that should exist between concept development and the development of problem solving competencies.

2. Relationships are Unclear between Competencies on Textbook Word Problems and the Levels and Types of Understandings needed to use Mathematics Concepts and Abilities Beyond School. A second shortcoming of past problem solving research is that, because of its almost exclusive emphasis on textbook word problems, mathematics educators have given relatively little attention to the kinds of problem solving competencies that are needed when mathematical thinking is required outside of school. The main exception to this rule has been in the area of elementary arithmetic concepts – where the emphasis has been on skill development rather than problem solving. This neglect of problem solving beyond school is especially significant because, in research about problem solving in fields (such as engineering) that are heavy users of mathematics and technology (Lesh, Hamilton & Kaput, 2007; Zawojewski, Diefes-dux & Bowman, 2008), it has become clear that, in a technology-based *age of information*, problem solving outside of school tends to be significantly different than problem solving in the context of *word problems* of the type emphasized in school textbooks and tests (Lesh & Caylor, 2008). Therefore, even if past research on mathematical problem solving would have been successful at explaining students' behaviors in the context of traditional kinds of textbook word problems, it is unlikely that such explanations would be useful without modification in the kinds of future-oriented problem solving situations that are emerging beyond school. Of specific relevance is that, in the world beyond school, specific questions being answered might not be known until long after the problem (dilemma, decision, discomfort) begins to be addressed. (Knorr-Cetina & Mulkay, 1982; Latour, 1987; Sawyer, 2006)

3. Research on Mathematical Problem Solving has not Accumulated. Failed or flawed concepts or conjectures have continued to be recycled or embellished – with no significant changes being made in the underlying theoretical perspectives. This is exacerbated by the fact that mathematics education researchers have generally avoided tasks that involve developing critical tools for their own use. Unlike their counterparts in more mature sciences (physics, chemistry, biology), where some of the most significant kinds of research often involve the development of tools to reliably observe, document, or measure the most important constructs that are hypothesized to be important, mathematics educators have developed very few tools for observing, documenting, or measuring most of the understandings and abilities that are believed to contribute to problem solving expertise. Furthermore, partly because operational definitions and tools have not been developed to observe, document, and assess the development of most constructs that have been claimed to be important, apparently failed or flawed concepts tend to be continually embellished or recycled. For example, the first paper in this pair describes the continuous embellishment of a theory that focuses on explicitly-learned rules. It also describes instances where "new" theoretical constructs clearly consist of nothing more than new names for previously discredited constructs. The basic problem has been that short lists of rules (e.g., heuristics, beliefs, metacognitive processes) tend to have descriptive but not prescriptive power; longer lists of prescriptive rules become so long that knowing when to use them becomes as important as knowing how to use them. Yet, introducing meta-rules (i.e., higher-level rules that operate on lower order rules) simply transfers the same basic shortcoming to a higher level.

Using new names to recycle old constructs is not the same as producing durable constructs, and continuous embellishment is not the same as theory development. As Popper (1963) emphasized, one of the most important characteristics that distinguishes a scientific theory from an ideology rests on the potential falsifiability of its assumptions and claims. In fact, according to Popper, one of the most important ways that theories develop is by rejecting hypotheses. Yet, rejected hypotheses rarely occur in the kind of research that has dominated inquiry about mathematical problem solving. So, for mathematics educators who are interested in research on problem solving, what we believe to be most needed are: (a) tools for observing, documenting, and measuring important constructs, (b) theoretical perspectives which do not encourage orthodoxy, and (c) research methodologies which encourage the consideration of diverse ways of thinking – but which also encourage selection among alternatives.

Shortcomings Associated with Theories & Research Methodologies

1. Concerning the development of useful theories. The theoretical perspective that we will emphasize in this paper traces its roots primarily to Piaget (Piaget and Beth, 1966) and to *American Pragmatists* such as Pierce, James, Holmes, Meade, and Dewey (Lesh and Doerr, 2003) who were originators of many of the most important ideas underlying modern views of situated cognition and socio-cultural factors that mold and shape knowledge development. Like *pragmatism*, the theoretical perspective that we emphasize is not so much a theory as it is a framework for developing theories. Like the pragmatists, we believe that no single theory is likely to provide solutions to the complex kind of problems that mathematics educators most need to understand and solve. One reason why this is true is because mathematics education tends to be more like engineering than physics – in the sense that the systems that we need to understand are largely products of human design or human guided development. So, the same conceptual systems that are used to make sense of these products of human development are also used to change them.

How does this challenge the idea that there can be a grand theory of education? Consider the fact that, in fields such as aerospace engineering, we will never have a fixed and final theory of things like *space shuttles*. One reason why this is true is because as soon as we understand them better, we will change them. So, every conceptual system that is developed for thinking about them is really the n^{th} in a continuing series. Furthermore, the design of such engineered artifacts usually involves trade-offs among competing perspectives and interests - because we usually want products that are low in cost but high in quality, or low in risk but high in possible gain, and so on. In fields like engineering, such observations have prompted the well known quip that: *Engineering is the science of understanding and designing things when there is not enough time, money, or other resources - and when trade-offs need to be considered which involve conflicting conceptions of success.* Education is much like this – and we will not be able to reach a final grand theory because of it.

In general, in research based on MMP, we adopt the pragmatists' point of view that what research on problem solving most needs is not another grand theory which claims to explain everything from cooking, to carpentry, to students' behaviors on textbook word problems (Schoenfeld, 2007). So, we do not expect realistic solutions to realistically complex problems to be solved by single research studies, nor even by single theories. Instead, what are most needed are models which are embodied in artifacts and tools that are designed to be powerful, sharable and reusable. Such models also need to integrate ways of thinking drawn from a variety of practical and theoretical perspectives. Once this is done, model development can lead to theory development; no single theory should be expected to provide guidance for most important problems or decision-making issues related to mathematical problem solving, rather models and theories should guide decision-making.

2. Concerning inadequacies of observational studies and teaching experiments. Two research methodologies which have dominated research of mathematical problem solving are: (a) observational studies in which researchers observe (and often analyze videotapes of) students solving problems, and (b) teaching experiments in which the researcher attempts to validate the importance of some heuristic, belief, disposition, or process by demonstrating that it can be taught.

One shortcoming of observational studies tends to be that constructs that are useful for describing past problem solving behaviors are not necessarily useful for prescribing future problem solving behaviors. A second shortcoming of observational studies is that the things that observers see are always strongly influenced by the researchers' preconceived notions. This shortcoming is especially powerful during a time when "heavy users" of mathematics claim that beyond school, significant changes have been occurring in the nature of situations where some type of mathematical thinking is needed beyond school.

One shortcoming of teaching experiments is that they simply have not been successful. Small treatments have produced small effects, and large treatments either do not get implemented sufficiently or their complexity tends to makes it impossible to draw causal inferences. So, in either case, such studies usually end up concluding that the researcher didn't try hard enough – or in the right way. But, beyond these practical considerations, studies designed to show that "it" works confront the same problems that led to the demise of aptitude-treatment interaction theories (ATIs). ATIs assumed that if students can be classified as having one set of profiles or traits as variables, then it should be possible to match the students with suitable treatments, so that particular learning goals will be reliably achieved. However, as Cronbach and Snow (1977) showed, ATI treatments were not effective assessments of student learning because complex interactions amongst the variables often mattered more than the variables themselves; to test all of the combinatorial combinations of the variables was not only computationally impractical task but also theoretically impossible.

Not only did such "treatments" prove to be combinatorically impractical, but they also proved to be theoretically impossible because of feedback loops, second-order effects, and other emergent properties associated with complex systems which have chaotic (unpredictable) outcomes. In fact, many of the most powerful actions that determine success tend to involve two-way interactions (rather than one-way actions) among students, teachers, and other relevant agents or resources (e.g, parents, programs, policy makers, curriculum materials, assessment systems, etc.).

3. Concerning the usefulness of design research methodologies. A recent *Handbook of Design Research in SMET Education* (Kelly, Lesh & Baek, 2008) describe a powerful new class of research methodologies that should be especially useful in research on mathematical problem solving. Although design research methodologies are new to learning scientists and mathematics educators, they have been used for years in many design sciences such as engineering (Zawojewski, Diefes-dux, & Bowman, 2008) – where the "thing" that needs to be understood and explained are also being designed by the relevant researchers.

Learning scientists tend to believe that design research methodologies were developed by theoreticians who were attempting to make results of their lab-based research more practical (Brown, 1992) or by software developers or educational program developers who were attempting to provide better theoretical grounding for best practices (Collins, 1992). But, in mathematics education, and in particular in our own research investigating the nature of students' developing mathematical concepts and processes, design research methodologies developed mainly out of attempts to minimize the amount of researcher guidance (Lesh, 2002). This work was modeled on Soviet-style teaching experiments (Krutetskii, 1976).

One dilemma that influenced our research was that regardless whether we focus on the development of students' conceptual systems, or the design of curriculum materials or programs, the "things" that we want to study are "things" that we ourselves are helping to develop or design. So, how can hypotheses be tested when they involve phenomena that are continually changing – and when the changes are partly driven by the conceptual systems that researchers are developing? How can we be certain that principles which are rejected or accepted today will not need to be revisited tomorrow?

A second way to describe the preceding dilemma is to recognize that when we investigate the nature of students' interpretation abilities, students' interpretations clearly are influenced by both their own structuring abilities and also by the structure of the tasks that researchers or teachers present. So, when researchers examine students interacting with activities designed by researchers, the interactions and the explanations of the students are not likely to be reducible to simple input-output rules, because researchers are a part of the system they are studying.

To deal with the preceding kinds of issues, it is useful to notice that mathematics educators are not alone! Similar dilemmas confront engineers or other design scientists when they are attempting to understand worlds that they themselves are designing – and worlds filled with feedback loops and second-order effects (where A impacts B, B impacts C, and C impacts A). The following observations about the ways that engineers confront these sorts of issues are useful to consider:

- Regardless whether engineers are designing software or space shuttles, the underlying designs tend to be important parts of the products that are designed. So, when trial products are tested, the underlying design principles also are tested.
- The design "specs" that engineers are given enable them to test products and to choose among strengths and weaknesses associated with alternative designs. This allows engineers to move in directions that are increasingly better without basing decisions on preconceived notions of what is "best."

On the one hand, such procedures cannot overcome the fact that any current model can only be the n^{th} iteration in a continually evolving series. On the other hand, sequences of such models often provide auditable trails of documentation which reveal important trends or patterns that otherwise would not be apparent. These patterns often enable generalizations to be made.

Models & Modeling Perspectives on Mathematics Problem Solving, Learning & Teaching

Foundations of MMP have been described in a number or recent publications (Lesh & Doerr, 2003; Lesh & Lehrer, 2002; Lesh & English, 2005; Lesh & Sriraman, 2005). MMP evolved primarily out of Piagetian and American Pragmatist perspectives - which also presaged many modern situated and socio-cultural views of problem solving, learning, and teaching (Lesh & Doerr, 2003, p. 519-556). Compared with other theoretical perspectives that have been used to investigate mathematical problem solving: (a) MMP emphasizes interpretation and communication aspects of understanding as much as it emphasizes procedural capabilities, (b)

MMP investigates problem solving processes developmentally – using techniques similar to those that others have used to investigate what it means to understand the development of concepts ranging from early number concepts (Steffe et al.,1983; Clements & Bright, 2003; Fuson, 1992), to rational numbers and proportional reasoning (Lesh, Post & Behr, 1985; Middleton et al., 2001) to the foundations of algebra (Driscoll et al., 2001), statistics (Konold & Lehrer, in press), or calculus (Kaput, 1997), and (c) MMP emphasizes the fact that, as we enter the 21st century, significant changes have been occurring in both the kinds of situations where some type of mathematical thinking is needed for success beyond school, and the levels and types of understandings and abilities that are needed for success in these situations (Lesh, Hamilton & Kaput, 2007). So, even if past theories of problem solving would have proven to be adequate for describing students' thinking in the context of traditional textbook word problems, MMP research entertains the notion that these theories may need to be modified significantly to describe the kind of mathematical thinking that is needed beyond school in a *technology-based age of information* (Lesh, Hamilton & Kaput, 2007).

Unlike most past theories that have been used to investigate mathematical problem solving, MMP was not developed primarily to explain problem solving per se. Instead, MMP was designed to investigate the development of mathematics concepts. Nonetheless, MMP also has used developmental studies to investigate what it means to "understand" problem solving processes. These studies have shown that concept development and the development of problem solving processes are closely and synergistically related. For example, one of the earliest questions that MMP researchers investigated was: *What is it, beyond the kind of understandings emphasized in most textbooks, tests, and classroom teaching, that enables students to use the things they have learned in real life situations beyond school?* (Lesh, Landau, & Hamilton, 1983) Results of these studies made it clear that the following two questions are significantly different.

- *What should students do when they are stuck (i.e., when they are not aware of any relevant concepts or processes)?*

- *What additional levels or types of understanding do students need to develop in order to be able to use concepts and abilities which: (a) are recognized as being relevant to a given problem solving situation, (b) are at intermediate stages of development, and (c) need to be adapted significantly to be useful in current circumstances?*

MMP investigates questions such as: (a) *What does it mean for students to "understand" relevant heuristics, strategies, beliefs, dispositions, or metacognitive processes?* (b) *What is the nature of students' early understandings of relevant heuristics, strategies, beliefs, dispositions, or metacognitive processes?* (c) *How (or in what ways) do students' early understandings develop?*

In the most general terms, models can be thought of as being systems for describing or designing other systems. As such, they are conceptual systems or interpretation systems, and because they are developed for a purpose, they are purposeful conceptual systems. Then, a distinctive characteristic of mathematical and scientific models is that they focus on systemic (or emergent[1]) properties of systems-as-a-whole.

One reason why MMP focuses on interpretation abilities is because, outside of school, in virtually every area where researchers have investigated similarities and differences between experts and novices (or between gifted versus average ability students, or between successful versus relatively unsuccessful problem solvers), results have shown that experts not only *do* things differently, but they also *see* (or interpret) things differently.

A second reason why MMP focuses on interpretation abilities is because Piaget-inspired researchers have shown that the development of most mathematics concepts depend on the development of students' abilities to make sense of situations using operational/relational systems-as-a-whole. That is, relevant concepts do not take on their appropriate mathematical meanings until students are able to think systemically. Examples of systemic properties include invariance with respect to a system of operations, transitivity with respect to a system of relations, or properties that involve minimization, optimization, or stabilization of operational-relational systems. So, what Piagetians showed is that, if the conceptual systems that students use to interpret their experiences are not yet functioning as systems-as-a-whole, then students' thinking tends to be unstable (e.g., they lose the metaphorical "forest" when their attention focuses on "trees" - or vice versa). Their thinking also tends to be characterized by: (a) centering – losing cognizance of one attribute when others are noticed, or (b) conceptual egocentrism – lacking the ability to be self-critical, or to consider alternative ways of thinking.

When we say that modeling is about interpretation and expression, this includes the fact that modeling is about the description and explanation of existing systems, and it also is about the design and development of new systems. According to MMP, the development of mathematical competence is about the development of powerful mathematical models and modeling abilities at least as much as it is about the acquisition of mathematical facts, skills, or processes. Yet, when we focus on the mathematics of description and explanation, this does not mean that the mathematics of computation and derivation are neglected. Such neglect would be as foolish in mathematics as it would be to ignore basic skills in athletics or performing arts – where equal attention also is given to scrimmages, competitions, and performance in other complex decision-making situations where the emphasis is on much more than isolated basic skills – and where knowing when to do things is as important as knowing how to do them.

Because of the preceding perspectives, *MMP defines problem solving activities to be goal-oriented activities in which problem solvers need to make significant adaptations to their current ways of thinking in order to achieve the desired goal*. Consequently, MMP focuses on problem solving situations in which model development is an important part of the product that problem solvers produce – or the underlying design is an important part of the conceptual tools that are

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[1] According to MMP, mathematics is the study of structure (Lesh and English, 2005). If we look at the undefined terms that occur in the formal axiomatic systems that are used to define mathematical concept, then every "undefined term" in these axiomatic systems is an emergent property of the systems. That is, all of its mathematical meaning comes from the systems that are used to define them. Similarly, underlying every statement of value, MMP expects there to be a system of values. Underlying every heuristic or metacognitive process, MMP expects there to be a conceptual system. And, underlying every fact or skill, MMP expects there to be a conceptual system in which the fact or skill becomes meaningful.

designed. Then, when solution development involves conceptual adaptation, at least as much as information processing, it is misleading to characterize problem solving as *getting from givens to goals when the path is not obvious*. In fact, model development tends to involve several express-test-revise cycles in which significant changes generally need to be made to initial conceptions of givens, goals, and possible solution processes. So, the development of solutions involves the adaptation of existing conceptual systems much more than it involves the search for ideas and procedures which have been misplaced. So, *the kinds of heuristics that are most useful are those that help students' ways of thinking evolve beyond current conceptions – all of which tend to be at intermediate stages of development*.

MMP-based Design Research Studies & Tools to Support Research Collaborations

According to MMP, researchers, teachers, and students – all are considered to be model developers. Students develop models in response to problems that are simulations of important new kinds of situations where important types of mathematical thinking are needed beyond school in the 21st century. Teachers develop models (and conceptual tools) for making sense of students' modeling activities. Researchers develop models of interactions between students and teachers. At all three levels, model developers express their current ways of thinking in the form of artifacts or tools which are designed explicitly to be useful for some specifically targeted purpose. Because the "design specs" make it clear that the underlying design (or conceptual system) is an important part of the artifact that is designed, when the artifact or tool is tested the important aspects of the conceptual system that it embodies is also tested. Furthermore, because the artifacts or tools that are produced need to be powerful (in the specific situations in which they were created), sharable (with other people), and reusable (in the future, and in other situations beyond the one in which they were created), they also contribute to community building and to the accumulation of knowledge. So, at all three interacting levels of model development, the products that model developers produce are expected to go beyond simply being tested for usefulness, sharability, and generalizability; they also are designed to have these attributes.

The preceding perspectives lend themselves to *multi-tier design studies* (Kelly, Lesh & Baek, 2008) which are aimed at investigating interactions among the model development activities of students, teachers, and researchers. Furthermore, *multi-tier design studies* also are specifically designed to coordinate the work of multiple researchers who are working at multiple sites and who represent a variety of practical or theoretical perspectives which may range from student development, to teacher development, to curriculum development, to theory development. At all levels of *multi-tier design studies,* many of the most important products that problem solvers produce are powerful, sharable, and reusable tools for their own use. So again, the result is to promote community building and the accumulation of knowledge.

MMP Alternatives to Past Problem Solving Research Methodologies

1. Model-Eliciting Activities - Alternatives (or Supplements) to Clinical Interviews or Videotape Analyses: MMP research now uses model-eliciting activities in many situations where we once used clinical interviews or videotape analyses – both of which tend to be very labor intensive and difficult to replicate. (Lesh & Lehrer, 2000). Unlike most of the problem solving situations that have been used in research on mathematical problem solving, MEA's were designed, first and foremost, for research purposes (Lesh & Caylor, 2008). The fact that they also have proven to be useful to support student learning (Lesh & Doerr, 2003), assessment (Lesh &

Lamon, 1992), and teacher development (Schorr & Lesh, 2003) is mostly due to inherent synergies between our views of mathematics education research and practice. Principles for designing MEA's have been explained in several resent publications (Lesh, et. al. 2000; Hjalmarson & Lesh, 2008).

As their name suggests, MEA's are activities in which students' develop a model - or an artifact or a tool which explicitly embodies an important conceptual system (explanation, interpretation, design) that the researcher wants to investigate. Therefore, because the underlying conceptual system is an important part of the designed artifact or tool, testing these products also involves testing the underlying design principles that they embody. Furthermore, because underlying conceptual systems are expressed in forms that can be examined and assessed by students, teachers, and researchers, solutions to MEA's tend to involve sequences of iterative express-test-revise cycles similar to the kind that are involved in the first-, second-, and n^{th}-drafts that are involved in the development of other kinds of written or drawn descriptions of situations. Therefore, auditable trails of documentation tend to be produced automatically, and important aspects of the evolving models can be inspected by both students and teachers (or researchers). These documentation trails often supplement the kind of information that can be obtained with time-consuming videotape analyses. Furthermore, because MEA's are designed so that significant conceptual adaptations occur during relatively brief periods of time (e.g., 60-90 minute problem solving episodes), MEA's often function something like little Petrie dishes in science laboratories. That is, important developments occur in easily observable forms during sufficiently brief periods of time so that researchers can go beyond observing successive states of knowledge to also observe processes that lead from one state to another. Furthermore, compared with the kind of information that can be gained from clinical interviews or videotape analyses, it often is possible to involve far more students using MEA's – and the results tend to be far more sharable.

When we compare information that can be gained from MEAs versus clinical interviews, it is noteworthy that a primary goal of clinical interviews is to follow students thinking – rather than simply investigating how close students can come to the researchers preconceived notions about how students should think about important mathematical concepts or processes. But, especially when the kind of thinking that is being investigated focuses on students' interpretation abilities, every interpretation that students produce is influenced by both the structure of the task and the students' structuring abilities. Therefore, each time a student develops a new interpretation, interpretation abilities tend to be impacted. And, if the interpretation involves a powerful mathematical construct, these impacts tend to be significant. So, in MMP research, we address this fundamental difficulty by trying to be as explicit as possible about how both students and teachers (or researchers) structure the tasks at hand – and about how they interact.

Whereas, in clinical interviews, researchers adapt their questioning to the thinking revealed by individual students, in MEAs, the students are able to interpret a single problem in a variety of ways and at a variety of levels of sophistication. So, MEAs tend to be self adapting.

2. <u>Local Conceptual Development Studies - Alternatives to Expert-Novice Studies</u>: MMP research often compares problem-solvers-who-are-isolated-individuals to problem-solvers-who-are-groups – in somewhat the same way that other researchers from other theoretical perspectives have compared experts and novices, or gifted problem solvers and average-ability problem solvers. Using MEA's, one result of this approach is that it has become clear that problem solvers' early interpretations of *model-eliciting activities* usually involve a collection of

partly-overlapping yet undifferentiated partial interpretations of different aspects of the relevant situations. So, regardless whether the problem solving is an individual or a group, model development tends to involve gradually sorting out, clarifying, revising, refining, and integrating the preceding kinds of gradually evolving ways of thinking. Furthermore:

(a) The evolution of these initially-unstable communities of constructs tends to resemble the evolution of complex and diverse ecological systems – far more than they resemble the movement of a point along a path (i.e., *getting from givens to goals when the path is unclear*).

(b) Heuristics that are intended to help problem solvers make productive adaptations to existing ways of thinking often are significantly different than heuristics that are intended to help problem solvers figure out what to do when they are stuck (with no apparent concepts available).

(c) Heuristics and metacognitive processes evolve in ways that are often quite similar to the dimensions of development that apply to other types of concepts or abilities that mathematics educators have studied. In particular, Vygotsky's (1978) concept of internalizing external functions often results in early understandings of heuristics that are distinctly social in character. So, instead of "looking at a similar problem" it often is useful to "look at the same problem from another point of view" (and to be aware of the fact that one's current point of view is not the only possible point of view.

(d) In the context of MEA's, heuristics and metacognitive processes generally function tacitly rather than as explicitly executed rules; in MEAs, heuristics and metacognitive processes have far less to do with helping students know what to do next, and have far more to do with helping them interpret the situation (including alternative ways of thinking about givens, goals, personal competencies, and "where they are" in solution processes). For example, when athletes or performing artists analyze videotapes of their own performances (or those of others), it is useful for them to develop a language for describing these past performances. But, this language usually is not intended to result in prescriptive rules about what to do at some given point in future performances. Instead, the language and imagery that they develop tends to be aimed mainly at helping them make sense of things during future performances. In other words, they are aimed mainly at the development of more powerful models.

3. <u>Multi-Tier Design Research - Alternatives to Japanese Lesson Plan Studies</u>: MMP research often investigates the development of teacher knowledge by engaging teachers in the development of tools for facilitating, documenting, analyzing, or assessing the development of students' models during MEA's. For example, one class of teacher-level tools has been referred to as *ways of thinking sheets* (Berry, 2006; Carmona, 2004; Hjalmarson, 2004; Lesh, Doerr, Carmona, & Hjalmarson, 2003). *Ways of thinking sheets* are sharable and reusable tools that teachers often find it useful to develop making sense of students' work – and for recording alternative ways of thinking that students adopt (a) while they are working on solutions to MEA's, (b) when they are giving oral reports of their results for MEA's, or (c) when they submit written reports of their results for MEA's. For teachers, the purposes of these ways of thinking sheets usually is to help them give students feedback about strengths and weaknesses of their work, or to help them identify appropriate follow-up activities that harvest students' insights and addresses their needs. Or, for teacher developers, the development of *ways of thinking sheets* often functions similarly to *Japanese Lesson Plan Studies* (Driscoll et al., 2001). Teachers often work in teams of three, and tool development goes through a series of iterative express-test-revise cycles. However, because research has shown that one of the most powerful ways to

positively influence teachers' teaching practices is to help them become more insightful about the nature of their students' thinking (Zawojewski, Diefes-dux & Bowman, 2008), and because the development of *ways of thinking sheets* minimize the amount of time that teachers are taken away from teaching, they lend themselves to effective on-the-job teacher development activities.

- In real classrooms (those that are not ongoing research laboratories), the learning that has occurred through the use of MEA's has been more impressive when teacher-development and student-development go hand-in-hand.
- In a sense, *ways of thinking sheets* are MEA's for teachers. Just like MEA's for students, they tend to generate auditable trails of documentation about the development of teachers' knowledge and abilities.

4. Evolving Expert Studies - Alternatives to Ethnographic Observations or Questionnaires: MMP research often involves evolving expert studies in which researchers, and other relevant experts, are engaged in the development of models, artifacts or tools in which the underlying conceptual system is an important part of the product. For example, as discussed in the book, *Foundations for the Future in Mathematics Education* (Lesh, Hamilton & Kaput, 2007), in the book, *Models and modeling in Engineering Education: Designing experiences for all students* (Zawojewski, Diefes-dux & Bowman, 2008), and in a series of semester-long follow-up studies with professional engineers, business managers, and others who are heavy users of mathematics, we engaged three-person teams of these experts to work with teachers and learning science researchers to co-design MEA's which they believed would help clarify insightful and future-oriented responses to the following questions. *What is the nature of typical problem-solving situations where elementary-but-powerful mathematical constructs and conceptual systems are needed for success in a technology-based age of information? What kind of "mathematical thinking" is emphasized in these situations? What does it mean to "understand" the most important of these ideas and abilities? How do these competencies develop? What can be done to facilitate development? How can we document and assess the most important (deeper, higher-order, more powerful) achievements that are needed: (i) for informed citizenship, or (ii) for successful participation in wide ranges of professions that are becoming increasingly heavy users of mathematics, science, and technology? How can we identify students who have exceptional potential which are not measured on standardized tests?*

Unlike studies in which researchers observe or interview experts, the preceding studies recognized that the opinions of researchers, teachers, and other experts would be certain to evolve significantly if they participated in activities in which they repeatedly expressed their current ways of thinking in forms that were tested and revised iteratively – based on peer review, and based on trials in which their draft activities were tried out with students. In the beginning, these experts' opinions focused on traditional kinds of skill building, but by the end, there was a consistent and overwhelming consensus that future-oriented problem solving will involve: (a) designing and making sense of complex systems, (b) working in teams of diverse specialists each of whom use continually evolving tools, and (c) participating in multiple-stage projects in which relevant abilities emphasize the multiple-media communicating, collaborating, planning, monitoring, and assessing. Also, computational and multi-media modeling often replaces models that were based on single, solvable, differentiable functions.

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