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Generalising Unitary Time Evolution

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Abstract. In this third Quantum Interaction (QI) meeting it is time to examine our failures. One of the weakest elements of QI as a field, arises in its continuing lack of models displaying proper evolutionary dynamics. This paper presents an overview of the modern generalised approach to the derivation of time evolution equations in physics, showing how the notion of symmetry is essential to the extraction of operators in quantum theory. The form that symmetry might take in non-physical models is explored, with a number of viable avenues identified.

1 Quantum Interactions are not Evolving

As a field Quantum Interaction (QI) has progressed well in recent years [10, 8]. It is clear that something is to be gained from applying the quantum formalism to the description of systems not generally considered physical [1, 4, 14, 16, 23]. However, despite this initial promise, there are many elements of quantum theory that have yet to be properly applied within this framework. Perhaps most notably, it is clear that time evolution has yet to be properly implemented (i.e. derived) for any of these systems. This is a very significant weakness. Without an appreciation of how an entangled quantum-like system might come about it becomes rather difficult to justify the quantum collapse model that is very often leveraged in the quantum interaction community. This paper will explore the notion of time evolution in standard quantum theory (QT), sketching out the modern approach to extracting Hamiltonians and unitary operators. We shall then utilise this approach to suggest some interesting avenues that might be pursued in the future extraction of a fully-fledged quantum-like theory capable of evolving, entangling and then collapsing.

There is no *apriori* reason to expect that the Schrödinger equation is the only form of time evolution equation available in a quantum-like theory. This paper will discuss the reasons lying behind this, and propose ways in which the QI community might work to establish a new time dynamics, or to prove that the application of Schrödinger dynamics is appropriate. Even if some justification can be found for the application of the Schrödinger equation beyond the description of physical systems, it is highly unlikely that the common techniques used in the extraction of a quantum description will work. This is because the standard approach to constructing a quantum theory generally involves finding a description of the system of interest that bears resemblance to an existing quantum description and then making use of a perturbative approach to extract the new quantum dynamics. Given that the systems modelled within the QI community are not necessarily physical in origin we might expect that this method will prove difficult to apply in this field.

It is worth emphasising at this point the necessity of these considerations. While the problem of describing composite quantum systems is well understood, there is no reason to expect that the systems described by the QI community will behave identically to physical systems. While entanglement and measurement are commonly used by QI models, almost none of them show how a quantum-like system might evolve to the point where it could be measured. One of the most commonly used techniques in the modelling of physical systems involves showing an approximate equivalence with a system already modelled and then applying that model to the new system. This may work for some QI models, but there is a very real possibility that not all QI systems will have direct physical analogues. This paper has been written in order to show those of the QI community who do not have a background in physics how they might proceed in constructing an evolving quantum-like theory if this becomes necessary.

2 Transformations in Quantum Theory

Time evolution is well understood in the standard quantum formalism, and the choices made in creating a model generally have very compelling reasons behind them. In this section we shall sketch out the modern approach to quantization, showing how this can be used to extract Schrödinger dynamics. The full approach can be found in any good modern text on QT [5, 21].

Physics has come a long way by assuming that the laws of nature are invariant under certain space-time transformations. These can include displacements, rotations and changes between frames of reference in uniform relative motion. In quantum theory, transformations of both states $|\psi\rangle \to |\psi'\rangle$ and observables $\hat{A} \to \hat{A}'$ must be considered together, and this places restrictions on the form that any transformation can take. Specifically, if $A|\phi_n\rangle = a_n|\phi_n\rangle$, then we must have $A'|\phi'_n\rangle = a_n|\phi'_n\rangle$ after transformation. Thus the eigenvalues of observable A cannot change under a transformation since the observable cannot be changed by the way we are looking at it. It is also essential that $|\langle\phi_n|\psi\rangle|^2 = |\langle\phi'_n|\psi'\rangle|^2$, which means that the probabilities for equivalent events in two different frames of reference should be equivalent. This requirement leads to Wigner's theorem [21], which places a strong restriction on the form that such a transformation can take, with only the above very minimal assumption about the nature of the inner product. This theorem shows that any mapping of a vector space onto itself that preserves the value of the inner product must be implemented by an operator U that is either unitary and linear or anti-unitary and anti-linear [25].[3] Unitary operators are very widely used in QT, as they are the only ones

[3] A unitary transformation is one such that $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$, whereas an anti-unitary transformation satisfies $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle^*$.

that can describe continuous transformations such as translations and rotations (since every continuous transformation must have a square root [5]). However, anti-unitary transformations also play a part in the quantum formalism as they are used in the description of discrete time reversal symmetries.

Together, these very minimal requirements place strong constraints upon the form that transformation operators can take in a standard quantum theory. In the particular case of continuous transformations, we find that while states must transform according to

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle, \tag{1}$$

observables must transform according to

$$\hat{A} \rightarrow A = UAU^{-1}. \tag{2}$$

Thus, with the assumption that the symmetries in quantum-like models will be continuous, we find ourselves to be looking for unitary operators satisfying equations (1) and (2).

In order to start sketching out the general form that such operators must take, we shall consider a set of unitary matrices $U(\alpha_1, \alpha_2, \dots)$ which depend upon the continuous parameters α_j. With a good choice of parameters, we find that these matrices are in a 1–1 correspondence with a continuous *group* of transformations, $\mathcal{G}_U$. That is, we find that the matrices satisfy:

Closure: for every $U_a, U_b \in \mathcal{G}_U$, the product of the two matrices is in the group, $U_a U_b \in \mathcal{G}_U$.

Associativity: for every $U_a, U_b, U_c \in \mathcal{G}_U$, $U_a(U_b U_c) = (U_a U_b)U_c$ (note that this property is automatically satisfied by matrices).

Identity element: there exists one, and only one, identity matrix in the group. We customarily define this matrix such that $U(0, 0, \dots) = \mathbb{1}$.

Inverse element: every matrix $U_a \in \mathcal{G}_U$ has a unique inverse also in the set. That is, there exists a matrix $U(\beta_1, \beta_2, \dots) \in \mathcal{G}_U$ such that:

$$U(\alpha_1, \alpha_2, \dots)U(\beta_1, \beta_2, \dots) = \mathbb{1}$$

Any set of matrices satisfying these properties forms a symmetry group. This is a remarkably important concept in modern physics. It is essential to realise that symmetry groups can take many different forms, the one sketched above for unitary matrices relies heavily upon multiplication, but the closure criterion could be just as easily framed for addition, or even some other operator. For example the Integers form a symmetry group under addition.

2.1 What is a Symmetry?

The concept of symmetry has a very particular meaning in physics, where it applies to any physical or mathematical feature of a system that is preserved, or invariant, under some transformation. Thus, the concept is quite broad in physics, compared to the common lay usage which generally refers to properties of a more geometrical nature. Consider for example the following way in which a symmetry group can be constructed for motion in one dimension.

> *To appreciate the link between group theory and motion, imagine you are standing on a straight road that goes on forever both in front and behind you. Stand stock still; this is the identity of a group. Walk forwards a little, then a little more. But you are now where you would have been had you just walked further in the first place. So moving along a straight line exhibits the closure property. Associativity can be demonstrated by walking different distances forwards and backwards in different sequences and noting that the end result is always the same. Finally, if you walk forwards a bit then backwards to where you started you have discovered the inverse.* [24]

Thus, a symmetry is not necessarily something that looks the same along an axis of view (like a mirror reflection symmetry), it has a much broader set of connotations.

Any transformation that satisfies the above group structure is a symmetry. Symmetries that commute with time evolution correspond to a conserved quantity in physics via Noether's theorem. This important theorem amounts to a statement that for every physical system exhibiting symmetry under time evolution there is some conserved physical property of that system, and conversely that each conserved physical quantity has a corresponding symmetry. Thus, symmetries can have physical consequences in their own right.

2.2 Symmetries, Operators, and Hamiltonians

It can be shown that any unitary transformation that depends upon a single parameter α (e.g. a rotation about a fixed axis by an angle $\alpha = \theta$) can be expressed as an exponential of a Hermitian *generator*, G, that is independent of α [5, 21]:

$$U(\alpha) = e^{-i\alpha G}. \tag{3}$$

The generators of transformations corresponding to symmetry properties often have simple physical meanings (such as energy, momentum, electric charge etc. in physics). It is important to realise that these symmetries often work together, forming larger groups which describe all allowable transformations within that space. Thus are the Gallilei, and Poincaré groups formed, as well as the larger groups used in The Standard Model of modern particle physics.

The Galilei group arises in non-relativistic quantum mechanics. It consists of a 10 dimensional representation of the symmetries of classical mechanics.[4] This group describes all of the rotations, displacements and transformations that can occur between uniformly (and slowly) moving frames of reference. Thus, this group describes all transformations of the form:

$$\mathbf{x} \rightarrow \mathbf{x}' = R\mathbf{x} + \mathbf{a} + \mathbf{v}t \tag{4}$$

$$t \rightarrow t' = t + s. \tag{5}$$

[4] Maxwell's equations do not satisfy this group and their inclusion in the group structure of modern physics led to the development of the Poincaré group which includes the Lorentz transformations of special relativity.

Here, R is a rotation (which can be thought of as a 3×3 matrix acting on a 3-vector $\mathbf{x}$), $\mathbf{a}$ is a space displacement, $\mathbf{v}$ is the velocity of a moving coordinate transformation and s is a small displacement of the time t.

Here, we are interested in the time evolution of a quantum system. In physics, time evolution is a symmetry of spacetime given by

$$t \to t + s,\ x \to x,\ y \to y,\ z \to z \tag{6}$$

with a conserved quantity that corresponds to the energy of the system. A system with more energy will move faster as time passes, so this conservation law is intuitively understandable. It is possible to derive Schrödinger's equation[5]

$$\frac{d}{dt}|\psi(x,t)\rangle = -iH(x,t)|(x,t)\rangle \tag{7}$$

from considerations of the dynamics of a free particle invariant under the full Galilei group of space-time transformations [5]. To do this, we make use of the properties of the Galilei group. We start by considering two sets of transformations, τ_1 followed by τ_2, and an equivalent single transformation τ_3. The equivalence means that $\tau_1\tau_2 = \tau_3$, and since these transformations are the same transformations we must require that $U(\tau_2)U(\tau_1)|\psi\rangle$ and $U(\tau_3)|\psi\rangle$ describe the same state. They do not necessarily have to be the same vector, they can differ up to a complex phase, which gives

$$U(\tau_3) = e^{i\omega(\tau_1,\tau_2)}U(\tau_2)U(\tau_1). \tag{8}$$

So symmetries must be relatable using some complex phase factor. Indeed, corresponding to the time displacement $t \to t' = t + s$, we find that the following vector space transformation holds [5]:

$$|\psi(t)\rangle \to e^{isH}|\psi(t)\rangle, \tag{9}$$

but if we consider figure 1 we quickly see that this can be written equivalently as $|\psi(t-s)\rangle$. We use this symmetry, by setting $s = t$, which gives $|\psi(x)\rangle = e^{-itH}|\psi(x)\rangle$. Finally, we note that only an equation of form (7) can generate this solution. □

While finding the Schrödinger equation through the application of symmetry information about time translations is the main point of this article, it will most likely prove useful to the QI community to see how this technique extends further. Indeed, it can be used to extract the full commutative structure of QT. We shall not perform that analysis here, the interested reader can refer to [5].

2.3 How do Commutation Relations Relate to Symmetry?

As was mentioned above, symmetries that commute with time evolution correspond to conserved quantities via Noether's theorem. However, the commutation

[5] Here we have used natural units (which gives $\hbar = 1$).

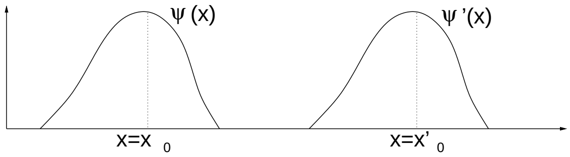

Fig. 1. A unitary time translation of the function $\psi(x)$, from a point around $x = x_0$ to a point around $x = x_0'$ is equivalent to a change in coordinate frame; there is an inverse relationhip between transformations on a function space and transformations on coordinates. Representing the change in coordinates as τ, we find that $\psi'(\tau x) = \psi(x)$, and hence that $U(\tau)\psi(x) = \psi(\tau^{-1}x)$ [5].

relations of QT have a wider set of relationships with the symmetry group of a physical theory.

In extracting the generators of a Galileian group describing a QT it is necessary to couple the symmetry structure of the transformations in the group with the unitary requirements of (1) and (2). In doing this we find that the standard commutation relationships of QT must be satisfied [5].

Thus, it is possible to fully derive the structure of QT from a consideration of symmetry and unitarity, and this is the modern approach to quantization. It is likely that this approach will prove most effective in the construction of a fully-fledged quantum-like theory.

3 Symmetry Groups for Quantum-like Theories?

There is no reason to believe that the symmetry groups of a quantum-like theory will be the same as for those of standard physics. Many of the relevant spaces considered in the field are of a very high dimension, and they do not need to satisfy the same set of physical conditions. Consider for example the very high dimensional cognitive spaces that are being modelled using QT [3, 9]. We would not immediately expect such systems to display the same symmetry behaviour as a standard QT. This raises an intriguing question; what form of symmetry could be satisfied by such models?

There are some early hints that we might explore in developing new symmetries, relevant to a much broader class of system. Some interesting avenues that we feel hold promise include:

- The use of Quantum Field Theory (QFT) in the modelling of biological systems [23, 17] and the use of symmetry breaking techniques in the modelling of dynamical emergence. This requires the identification of symmetry groups beyond those standard to physics, and it appears possible that complete groups might be identified as these theories develop; some of these might point towards a temporal symmetry that might be leveraged in deriving general time evolution equations in standard first quantized models.

- Some interesting work examining the concept of symmetry in object oriented programming languages has been performed [13, 27]. Here, the use of inheritance in the extraction of symmetry relations suggests that if a symmetry group could be found for such systems then it should share some features with any biological models that make use of intergenerational symmetries (within the same species for example).
- A concept of *superfractals* has been coined [6] to describe the mathematics of natural imagery, art, and biology. Among the mathematics developed here, it is possible to make use of iterated function systems to generate complex landscape and biological images using a computer, which *look similar* to a human observer. This conception of similarity holds promise, and the mathematical nature of the theory leaves it ideal for extension to a theory of symmetry with respect to human cognition. This idea will be explored elsewhere.
- The different senses or meanings of a word might also be developed into a group theory. Such a theory would probably leverage the intuition that even when changing word senses you still have the same token. Thus, *bat* stands for "furry flying mammal" or "sporting implement". If a group denoting this could be found then it might even fit into a larger group structure of language, after all, the set of different languages still describes the same set of senses, at least approximately.

All of these different avenues are currently under investigation, but the problem of finding proper formalisations of what are generally quite vague arguments is very difficult. We might wonder if perhaps there is a new generalised mathematics of symmetry groups waiting to be found.

3.1 Towards a New Mathematics?

Group theory as it currently stands is concerned with relatively simple structures and behaviour. It has been developed primarily for physical systems, and we might wonder if the behaviour of quantum-like systems can be described by the same sets of groups developed for physics. Some reasons to believe that this is probably not the case will be briefly discussed in this section.

Many of the systems described by the QI community display complex behaviour [18], and as such they will have features such as internal structure, hierarchical organisation, contingent dependency upon historical events, and an evolving dynamics. This would lead us to suggest that their symmetries will be far more difficult to extract, and in themselves far more complex, than those of physical systems.

4 Towards Time Evolution in Quantum-like theories

In this section we shall summarise some recently developed ideas that we feel hold sufficient promise for the future creation of a fully-fledged quantum-like theory. Both of them have been generated through attempts to develop the idea of a symmetry group of the system of interest, and in particular to find properties of that system that are conserved under time evolution and so might be used to generate some sort of quantum time evolution dynamics.

4.1 Quantum Models of Biological Development

Symmetries play a vital role in models of biological development. In fact, it is the breaking of symmetries that generates actual outcomes in terms of axial orientation, and through this eventual cell differentiation. It will be instructive to consider some of the issues involved in constructing a full description of the dynamics involved in this process.

Let us consider a perfectly spherical egg. It is symmetrical under all rotations and translations in space, and as such could be represented by the O(3) group. Differentiation of the cell starts from the moment it is impregnated by a sperm cell; a new axis of symmetry arises from the line joining the site of sperm penetration with the centre of the egg. Once this event occurs the O(3) rotational symmetry of the egg is lost, and developmental events will quickly lead to a loss of more and more symmetry. However, over time, there is a sense of conservation; the organism remains the same organism, even if it gradually becomes very different in form.

It is hoped that this idea might be leveraged in order to develop a quantum-like model of biological development. Here we would see a situation where the environment in which the fertilised egg is developing influences the eventual form of the egg itself, however, there is every reason to suppose that a QFT would prove most appropriate for such systems. This is because QFT's allow for the existence of unitarily inequivalent ground states [22, 23, 19], which allows for a model of development that sees the organism as growing through a number of different stable states. The alternative picture supplied by a first quantized theory (such as is discussed in this paper) would see the developing through a process of excitation, this is not feasible, after all, such a model would open up the possibility that a fully developed organism might de-excite back to the ground state!

Can we find a situation where a first quantized model is the most appropriate approach?

4.2 Quantum Models of Semantic Structure

Cognitive scientists have produced a collection of models which have an encouraging, and at times impressive, track record of replicating human information processing, such as word association norms. These are generally referred to as *semantic space* models. As used here, the term "semantic" derives from the intuition that the meaning of a word derives from the "company it keeps", as the linguist J.R. Firth (1890-1960) famously remarked. For example, the words "mobile" and "cellular" would exhibit a strong association in semantic space as the distribution of words with which they co-occur tends to be similar, even though the two words almost never co-occur themselves. Although the details of the various semantic space models differ, they all process a corpus of text and "learn" representations of words in a high dimensional space.

There is already an existing body of work linking semantic space theory to QT [2, 26, 7, 12, 11]. In one set of examples [12, 11], a semantic space S_w surrounding word w is constructed by collecting a corpus of traces centred around w. Such matrices are square symmetric matrices and hence self-adjoint. For a given set of words u, v and w we shall represent the corresponding matrices as S_u, S_v and S_w.

We shall now sketch how a symmetry group might be developed for a semantic space model, and perhaps eventually used to generate a model of semantic dynamics.

The product $S_u S_v$ can be interpreted as the the effect on the semantic representation of u when seen in the context of word v. That is, how much of u's semantic representation project onto that of v. This product satisfies closure, since the word itself is still in the combined semantic space. Combining semantic representations using such a product is also associative: $S_u(S_v S_w) = (S_u S_v)S_w$. The identity operator can be easily identified as the word itself, it has the same semantic representation as itself.

The question of an inverse S_w^{-1} for S_w for an arbitrary word w in not a straightforward issue. Intuitively we might expect the inverse to be something that "undoes" the projection, hence removing a word from its context. However, a word removed from its context is a highly artificial thing, and this is not necessarily the best way to proceed. Perhaps instead a notion of inverse might be developed that would produce a representation that is "orthogonal" to the meaning of w. One possible candidate is the Householder reflection:

$$S_w^{-1} = I - 2|w\rangle\langle w|$$

This formula exploits the complementary representations S_w and $|w\rangle$ noted in [7]. S_w is a matrix representation for the word w, but also the unit vector $|w\rangle$ is a prominent column vector in S_w. The above formula produces a self adjoint matrix S_w^{-1} which is a reflection in the hyperplane perpendicular to the vector $|w\rangle$. This problem of the inverse is something that will be investigated in future work.

Obviously there are significant details to be worked out in such an approach. Firstly the product $S_u S_v$ is not guaranteed to be self adjoint. This is not necessarily a problem, but it would be much cleaner if a product operation could be defined which resulted in a self adjoint matrix. In addition, the above definition of an inverse only covers the inverses corresponding to individual words, not compound representations, e.g., $S_u S_w$ which are also elements of the group. There is a whole avenue of research in relation to forming the semantic representations of compounds, indeed, there has been some speculation that concepts are entangled [3, 9] Finally, there is the question about what the interpretation of invariance should be in relation to the a semantic representation. As semantic space models are derived directly from an underlying corpus, the semantic representations of the words change accordingly. That is, the meaning of the words changes according how the company around them evolves [20]. As a consequence, the strength of semantic association between words varies as the corpus evolves. However, there will be a point where the semantic representations stabilise and semantic associations will stabilise. The stabilisation of semantic association was demonstrated recently in relation to the BEAGLE model [15]. In BEAGLE, representations are primed initially by random vectors. So each time the model is run over a given corpus the actual semantic representations of words will be different. However the strength of semantic association is largely invariant across different runs of the model.

5 Conclusions, and a Question for the Future

This article is obviously of a very exploratory nature. Here we shall ask a question in the hope that others might be interested in considering it.

The symmetry groups of modern physics are, in a number of ways, boring. The requirement to satisfy space-time symmetries is a very strong one, which leads to some very profound restrictions upon the nature of physical reality. Such restrictions do not necessarily apply in the high dimensional conceptual spaces often considered in QI. This actually makes the derivation of group structures in this field much more challenging as there are no clear restrictions to incorporate into our models. However, as we have seen in sections 3 and 4.2, there are some early intuitive ideas that might be investigated. More generally, there are many of mathematically interesting ideas that could be considered. For example, the structure of the unitary operators must be taken into account, many interesting systems take a hierarchical form, and while the Standard Model does have something of a nested structure, it was not necessary to consider any truly hierarchical behaviour in the construction of this model. However, we can ask if there might be a way of constructing a set of more general tensorial operators, ones that could incorporate the complex and interrelated hierarchical symmetries of biological systems. This is a problem that will be investigated in future work.

If QI is to truly come of age then it must start to develop complete theories. These must include both time evolution and symmetry considerations. The entanglement so often relied upon in the field must emerge from a truly evolving quantum model, not just be assumed to exist at the outset. This paper has presented some ideas about how such models might be constructed, and pointed at some of the possible avenues that might be pursued in the future. We hope that these ideas might prove fruitful to any future investigations of dynamics in the new field of quantum interaction..

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