Solutions for the Flow of a Micropolar Fluid in a Porous Channel

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2 Introduction

How various additives can increase some cardio-vascular diseases and effects of transport for albumin and glucose through permeable membranes are some important studies in biomechanics. The rolling phenomena of the leucocytes gives rise to an inflammatory reaction along a vascular wall. Initiated by Eringen [5], a micropolar fluid is a satisfactory model for flows of fluids which contain micro-constituents which can undergo rotation.

2 Problem formulation

We consider a fully developed flow of such a fluid bounded by two infinite parallel porous plates, shown in Figure 1. The details of the governing equations are found in [1, 14]. They are :

continuity

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \qquad (1)$$

momentum

$$\hat{u}.\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}.\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{1}{\rho}.\frac{\partial \hat{p}}{\partial \hat{x}} + \nu_k.\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right) + \frac{k}{\rho}.\frac{\partial \hat{N}}{\partial \hat{y}} \qquad (2)$$

$$\hat{u}.\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}.\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{1}{\rho}.\frac{\partial \hat{p}}{\partial \hat{y}} + \nu_k.\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right) - \frac{k}{\rho}.\frac{\partial \hat{N}}{\partial \hat{x}} \qquad (3)$$

moment of momentum

$$\rho.j\left(\hat{u}\frac{\partial \hat{N}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{N}}{\partial \hat{y}}\right) = -2k\hat{N} + k\left(\frac{\partial \hat{v}}{\partial \hat{x}} - \frac{\partial \hat{u}}{\partial \hat{y}}\right) + \gamma\left(\frac{\partial^2 \hat{N}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{N}}{\partial \hat{y}^2}\right) \qquad (4)$$

Here $\nu_k = \nu + k/\rho$ is a characteristic of the viscosity of the fluid and $\hat{N}$ is the component of the gyration vector normal to the $(\hat{x}, \hat{y})$ plane. For consistency with other studies [10, 11] all physical parameters are taken as independent and constant. In our geometry, the flow is determined by the height h between the two plates and the normal velocity at the walls $q \neq 0$. Several studies, for example [15, 17], show the possibility of multiple solutions when suction $q < 0$ is imposed on each boundary.

At either wall, the volume flow rate Q per unit wall width may be written as

$$Q = L.q = U.h \qquad (4)$$

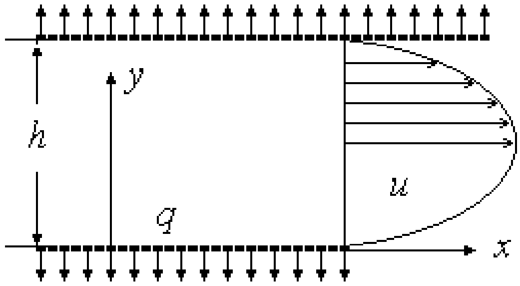

FIGURE 1. Schematic of flow.

where U and L are characteristic longitudinal velocity and length scales, respectively. Using these scales, we introduce the following dimensionless variables and parameters:

$$x = \hat{x}/L \quad ; \quad y = \hat{y}/h \quad ; \quad p = \hat{p}/\rho.q^2$$
$$u = \hat{u}/U \quad ; \quad v = \hat{v}/q \quad ; \quad \mathrm{N} = \hat{N}/(U/h) \qquad (5)$$

$$Re = \frac{q.h}{\nu} \;;\; N_1 = \frac{k}{\rho.\nu} \;;\; N_2 = \frac{\gamma}{\rho.\nu^2}.\frac{\nu}{h^2} \;;\; N_3 = \frac{j}{\nu}.\frac{\nu}{h^2}$$

Here, $Re \neq 0$ is the cross flow Reynolds number, where $Re \leq 0$ corresponds to suction, and $Re \geq 0$ to injection.

To simplify the governing equations, we generalise Berman's similarity solution [3] to include the microrotation N, by assuming

$$u = x.V' \quad ; \quad v = -V \quad ; \quad \mathrm{N} = x.N \qquad (6)$$

where V and N are functions of y only, and a prime denotes differentiation with respect to y.

Using (6), equation (1) is identically satisfied, and equation (3) becomes :

$$V.V' = -\frac{\partial p}{\partial y} - \frac{1+N_1}{Re}.V'' - \frac{N_1}{Re}.N \qquad (7)$$

Equation (7) differentiated with respect to x reduces to $\partial^2 p/\partial x.\partial y = 0$, and equation (2) differentiated with respect to y yields:

$$V'.V'' - V.V''' = \frac{1+N_1}{Re}.V'''' + \frac{N_1}{Re}.N'' \qquad (8)$$

Using (6), equation (4) becomes

$$Re.N_3(NV' - VN') = -N_1(V'' + 2N) + N_2 N'' \qquad (10)$$

and equations (8) and (10) are to be solved subject to the following conditions:

Velocity

$$\hat{u}(0)=0 \Rightarrow V'(0)=0;\ \hat{v}(0)=q \Rightarrow V(0)=-1$$
$$\hat{u}(h)=0 \Rightarrow V'(1)=0;\ \hat{v}(h)=-q \Rightarrow V(1)=1 \qquad (11)$$

microrotation

$$\hat{N}(0)=-s.\left.\frac{\partial \hat{u}}{\partial \hat{y}}\right|_{\hat{y}=0} \quad \Rightarrow \quad N(0)=-s.V''(0)$$
$$\hat{N}(h)=-s.\left.\frac{\partial \hat{u}}{\partial \hat{y}}\right|_{\hat{y}=h} \quad \Rightarrow \quad N(1)=-s.V''(1) \qquad (12)$$

For the microrotation boundary condition given in equation (12), we follow [6, 11] and assume a linear relationship between N and the surface shear stress. In this work we use the value $s=0$, corresponding to the condition $N(x,0)=0$. The latter represents the case where the particle density is sufficiently great such that microelements close to the wall are unable to rotate.

3 Perturbation development

Choosing the Reynolds number Re as the perturbing parameter, we expand the similarity functions using regular perturbation expansions

$$V=F+Re.f_{(1)}+Re^2.f_{(2)}+\ .....$$
$$N=G+Re.g_{(1)}+Re^2.g_{(2)}+\ .....$$

and substitute them into the boundary-layer equations (8) and (10). By collecting terms in equal powers of Re, for the zeroth order, we obtain :

$$(1+N_1).F''''+N_1.G''=0$$
$$-N_1.(F''+2.G)+N_2.G''=0 \qquad (13)$$

We introduce a vortex viscosity coupling coefficient $C_1=N_1/(1+N_1)$. In a recent study [12], we have shown that C_1 may only vary in the range $0 \le C_1 < 1$. In case of a newtonian fluid $C_1=0$ (and $N_2=N_3=0$), the first two terms for the velocity are deduced easily (see [4, 16]). In our case :

$$F\big|_{C_1=0} \approx -1+6.y^2-4.y^3+$$
$$\frac{Re}{35}\left(-3y^2+16y^3-35y^4+42y^5-28y^6+8y^7\right)$$
$$F'\big|_{C_1=0} \approx 12.y-12.y^2+ \qquad (14)$$
$$\frac{2yRe}{35}\left(24y-3-70y^2+105y^3-84y^4+28y^5\right)$$

For a non zero value for C_1, equation (13a) may be integrated twice. This leads to

$$F''+C_1.G=(C_1-2).(\varphi.y+\psi) \qquad (15)$$

This equation is substituted in equation (13b) which leads to :

$$G''-\frac{1}{N_2}\frac{C_1.(2-C_1)}{1-C_1}G=\frac{1}{N_2}\frac{C_1}{1-C_1}(\varphi.y+\psi) \qquad (16)$$

whose solution is :

$$G=\alpha.e^{+r.y}+\beta.e^{-r.y}+(\varphi.y+\psi) \qquad (17)$$

with $r^2=\frac{C_1}{N_2}.\frac{2-C_1}{1-C_1}$. From equation (15), we obtain

$$F'=\frac{-C_1}{r}\left(\alpha e^{ry}-\frac{\beta}{e^{ry}}\right)-2\left(\frac{\varphi.y^2}{2}+\psi.y+\delta\right)$$
$$F''=-C_1\left(\alpha e^{ry}+\frac{\beta}{e^{ry}}\right)-2(\varphi.y+\psi) \qquad (18)$$

The constants φ, ψ and δ, ε (this last constant appears in F) are expressed in terms of α and β using the conditions on the microrotation function and two conditions on the velocity function for the lower plate. The constants α and β are then found using the conditions on the upper boundary.

4 Numerical procedure

Equations (8) and (10) represent two coupled two-point non-linear differential equations The original two point boundary value problem is transformed into a first order system. An approximate solution is given by $W=\{W_k\,;k=1,6\}=\{N,N',V,V',V'',V'''\}$ and $Z=\{Z_k\,;k=1,6\}$ is an improved solution. A first order Newton's development around a former solution gives the two linear coupled equations :

$$Z_2'=\frac{1}{C_1}.\left[Z_1.(2.u_1+u_2.W_4)-Z_2.(u_2.W_3)\right.$$
$$\left.-Z_3.(u_2.W_2)+Z_4.(u_2.W_1)+Z_5.(u_1)\right] \qquad (19)$$
$$+\frac{1}{C_1}.\left[u_2.(W_2.W_3-W_1.W_4)\right]$$

and

$$Z_6'=\left[-Z_1(2u_1+u_2W_4)-Z_2u_2W_3+\right.$$
$$Z_3(u_2W_2-u_0W_6)+Z_4(u_0W_5-u_2W_1)+$$
$$\left.Z_5(u_0W_4-u_1)-Z_6u_0W_3\right] \qquad (20)$$
$$+\left[u_0(W_3W_6-W_4W_5)-u_2(W_2W_3-W_1W_4)\right]$$

with

$$u_0=Re(1-C_1)\,;\ u_1=C_1.\frac{C_1}{N_2.(1-C_1)}$$
$$u_2=C_1Re\frac{N_3}{N_2} \qquad (21)$$

Our numerical procedure is described in [2, 4]. This method may be related to those described in [10].

5 Results and Comments

5.1 Perturbation development. We begin by comparing the solutions (14) and (18) with numerical results. In Figure 2, we draw the longitudinal velocity for two cases : $C_1 = 0.6\,;N_2 = 0.05$ and $C_1 = 0.4\,;N_2 = 0.2$ using (18). These results are compared with the velocities obtained using (14) for the cases $Re = -10.0$ and $Re = 25.0$. These developments have been used indiscriminately in our numerical procedure. For all our investigations, the convergence is obtained in less than five iterations. We have used, in complement, a second convergence criterion based on a comparison of the RMS values of the sixth Z_k while the first criterion analyses the weights of different results participating to the solution (see [2, 4]).

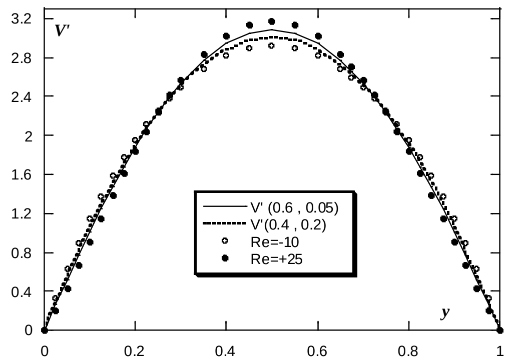

FIGURE 2. Comparison between solutions (14; black and white dots) and (18; plain and broken lines): velocity profiles.

The following graph summarizes the consequences of the formulae (14-18) in which we consider the variation of the main part of the wall shear stress $F''(0)$ as a function of the parameters N_2, C_1 and Re. For a fixed value $C_1 = 0.8$, an increasing value for N_2 increases the wall shear stress to an asymptotic value corresponding to an impermeable wall. For a fixed value of $N_2 = 0.4$, if C_1 increases, the wall shear stress decreases and an injection ($Re \geq 0$) has the same effect.

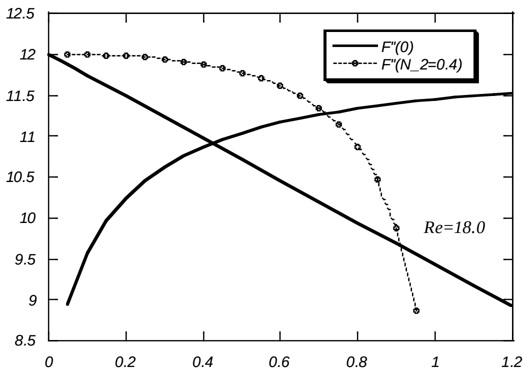

FIGURE 3. Comparison between solutions (14 ; black line from $Re = 0$ to +18.) and (18). Variation of the wall shear stress $F''(0)$.

5.2 The parameter values. A review of the literature [7, 8, 18] shows that the vortex viscosity N_1 is often taken less than 0.2 but we found values of 4.5 in [9] or 50 in [6]. Our previous study [12] indicates that computations for large values for N_2 is typically stable and relatively easy to obtain. Therefore, we mainly considered the computationally more demanding range $N_2 \leq 2.0$ to investigate the effects of parameter variations. The other physical parameters are taken in the ranges : $-30.0 \leq Re \leq +30.0$ and, in contrast with [13] we specified $N_3 \leq 0.15 \leq N_2$. In the main literature, the physical parameter corresponding to the micro inertia density is often omitted [7, 8, 18].

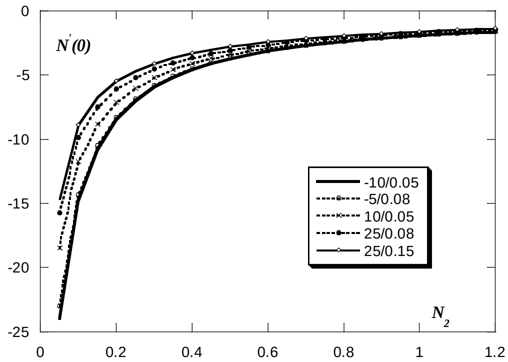

FIGURE 4. Profiles of the couple wall derivative $N'(0)$, function of N_2 for various N_3 and wall injection values.

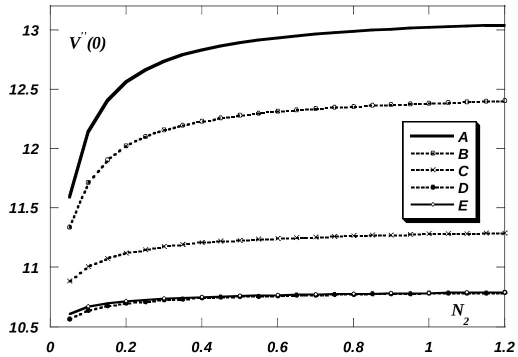

FIGURE 5. Profiles of the wall shear stress $V''(0)$, function of N_2 for various N_3 and wall injection values.

In Figures 4 and 5, we consider the influence of this parameter on the wall couple stress $N'(0)$ and the wall shear stress. With C_1 = 0.5, our values are :

	A	B	C	D	E
N_3	0.05	0.08	0.05	0.08	0.15
Re	-10.	-5.	10.	25.	25.

From Figures 4 and 5, we observe the asymptotic decreasing influence of N_2, and we do not encounter any problems in the various calculations. The influence of N_3 is negligible in comparison with the Reynolds number, which increases the absolute value of the wall shear stress.

5.3 Effect of suction. For our geometry, if the suction number is near –14.0 or less, it is possible to obtain several solutions [15, 17]. We found this phenomena in the case of a sliding pad [2]. Here, we have attempted to obtain more detailed information for the case of moderate suction. No difficulty in obtaining numerical results was found using our perturbation developments as a first estimate for the solution.

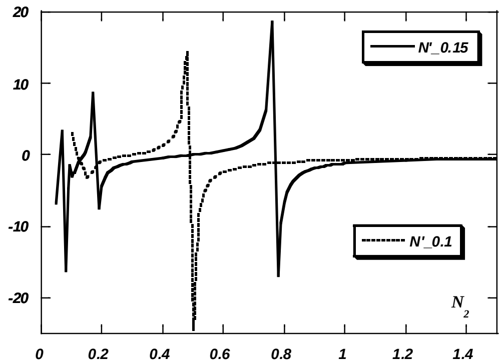

FIGURE 6. Profiles of the couple wall derivative $N'(0)$ function of N_2 for two values for N_3.

To observe the influence of N_3, we consider $C_1 = 0.2$ and suction corresponding to $Re = -20.0$. The parameter values chosen were $N_3 = 0.10\ ;\ 0.15$. From Figure 6, we see numerous discontinuities, the number of which decreases for a decrease in N_3. (In Figure 4, we did not detect any discontinuities for the case of injection). The variations are not monotone for $0.05 \le N_2 \le 0.18$, and the last discontinuity is observed for a value $N_2 \approx 0.76$.

Conclusion

1. Our numerical procedure seems to be one among the most accurate and well adapted to the problem of a micropolar fluid flow between two parallel porous plates.
2. Attempt to decrease the wall shear stress is obtained with injection or using fluids with large coupling coefficient values C_1.
3. A large ratio for the viscosities N_2 or a low value for the micro inertia density are favourable to any calculation.
4. Some discontinuities in couple wall derivative profiles $N'(0)$ can arise in case of a suction combined with moderate values for N_3.

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