A FRACTIONAL STOCHASTIC EVOLUTION EQUATION DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Abstract. This paper introduces a semilinear stochastic evolution equation which contains fractional powers of the infinitesimal generator of a strongly continuous semigroup and is driven by Hilbert space-valued fractional Brownian motion. Fractional powers of the generator induce long-range dependence in space, while fractional Brownian motion induces long-range dependence in time in the solution of the equation. An approximation of the evolution solution is then constructed by the splitting method. The existence and uniqueness of the solution and mean-square convergence of the approximation algorithm are established.

1. Introduction

In many applications such as heat conduction and fluid flow in porous media, propagation of seismic waves, diffusion and transport of macromolecules in living tissues (Levin [20], Barabasi and Stanley [6], Shlesinger *et al.* [23], Innaccone and Khokha [16], Carpintera and Mainardi [9], Hilfer [15]), the non-homogeneities of the medium may alter the laws of Markov diffusion in a fundamental way. In particular, the correlation function of the diffusion process may decay to zero at a much slower rate than the usual exponential rate of Markov diffusion, resulting in long-range dependence (LRD) (Beran [8], Anh and Heyde [3], Leonenko [19]). A class of models which are suitable for describing this phenomenon in space and time is that of evolution equations:

$$dX\left(t,x\right)=A_{x}X\left(t,x\right)dt+f\left(t,x,X\left(t,x\right)\right)dt+g\left(t\right)dB\left(t,x\right), \tag{1.1}$$

$$X\left(0,x\right)=X_{0}\left(x\right),\quad x\in\mathbb{R},$$

where A_x is a fractional (in space) differential operator and $B\left(t,x\right)$ is a fractional (in time) Brownian motion. An application of the semigroup theory (Da Prato and Zabczyk [22]), or the monotone operator theory (Krylov and Rozovskij [18], Grecksch and Tudor [14]), or the Green function theory (Manthey [21], Anh and Leonenko [4]) can be invoked to analyse Eq. (1.1). Here, we interpret A_x as the infinitesimal generator of a semigroup $S\left(t\right)$ of contractions and define the solution of (1.1) as

$$X\left(t,x\right)=S\left(t\right)X_{0}\left(x\right)+\int_{0}^{t}S\left(t-s\right)f\left(s,x,X\left(s,x\right)\right)ds+\int_{0}^{t}S\left(t-s\right)g\left(s\right)dB\left(s,x\right), \tag{1.2}$$

where the stochastic integral in (1.2) will be defined in Section 2.

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It is known that, in general, a fractional power of an infinitesimal generator is not an infinitesimal generator again (Balakrishnan [5], Ahmed [1], Bensoussan *et al.* [7]). In particular, if A is an infinitesimal generator, then $-(-A)^{\alpha/2}$ generates a strongly continuous semigroup only for $\alpha \leq 1$. Two fundamental examples are $A = \Delta$, the Laplacian operator (Feller [11]) and $A = \Delta - I$ (Grecksch and Anh [13]). The resulting operators $(-\Delta)^{\alpha/2}$ and $(I-\Delta)^{\alpha/2}$ are inverses of the Riesz potential and the Bessel potential respectively (Donoghue [10], Stein [24]). The operator $(-\Delta)^{\alpha/2}$, which defines a dynamic model of fractional Brownian motion (fBm) through the equation

$$(-\Delta)^{\alpha/2} X(t) = \varepsilon(t), \quad t \in \mathbb{R}, \tag{1.3}$$

$\varepsilon(t)$ being white noise, plays an important role in the theory of LRD processes (Anh *et al.* [2]). As noted above, $-(-\Delta)^{\alpha/2}$ generates a semigroup for $0 < \alpha \leq 1$. But a process with stationary increments defined by (1.3) displays LRD only for $1 < \alpha < 3/2$ (Anh *et al.* [2]). Hence, in order to have the semigroup solution (1.2) which possesses LRD, the operator A_x must take a different form. In this paper, we consider the composition

$$A_x = -(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}, \tag{1.4}$$

which defines fractional Riesz-Bessel motion (fRBm) through the equation

$$(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2} X(t) = \varepsilon(t), \quad t \in \mathbb{R} \tag{1.5}$$

(Anh *et al.* [2]). It is noted that the spectral density of fBm is given by

$$f_{fBm}(\lambda) = \frac{c}{|\lambda|^{2\alpha}}, \quad 1/2 < \alpha < 3/2, c > 0, \lambda \in \mathbb{R}$$

(cf. (1.3)), while that of fRBm is

$$f_{fRBm}(\lambda) = \frac{c}{|\lambda|^{2\alpha}\left(1+\lambda^2\right)^{\gamma}}, \quad 1/2 < \alpha < 3/2, \gamma \geq 0, c > 0, \lambda \in \mathbb{R}$$

(cf. (1.5)), which reduces to f_{fBm} for $\gamma = 0$ and displays LRD by definition. We will show that the operator (1.4) generates a strongly continuous semigroup (Proposition 1). The solution of (1.1) will then possess spatial LRD via the operator (1.4) and temporal LRD via fBm which drives the equation. The Hilbert space-valued fBm $B(t,x)$ of (1.1) will be defined in Section 2, together with the corresponding stochastic integral. The existence and uniqueness of (1.2) as a generalised solution over a Gelfand triple of Hilbert spaces will be established in Theorem 1.

In Section 3, a splitting method is described to approximate (1.2). The basic idea of the splitting method consists of the construction of two sequences of equations with a time discretisation. The first sequence contains equations which are defined with probability 1 and can be solved as a deterministic problem. The second sequence contains equations with stochastic integrals and can be solved as a purely stochastic problem, such as by simulation of a stochastic integral. The convergence of the approximation algorithm, in the mean square sense, is provided in Theorem 2.

2. A stochastic differential equation with fractional operators and fractional Brownian motion

We consider the stochastic partial differential equation

$$dX(t,x) = -(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2} X(t,x)\,dt + f(t,x,X(t,x))\,dt + g(t)\,dB(t,x) \tag{2.1}$$

with $X(0,x)=X_0(x)$, $x\in\mathbb{R}$. We want to define the generalised solution of (2.1) with the help of semigroup theory.

Let us denote the Fourier transform of f by $\hat{f}$:

$$\hat{f}(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)\,e^{-ixy}dy$$

for $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. We define $(-\Delta)^{\alpha/2}$ by

$$(-\Delta)^{\alpha/2} f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ixy}\,|y|^{\alpha}\,\hat{f}(y)\,dy$$

for $f\in\mathfrak{D}\left((-\Delta)^{\alpha/2}\right)=\{f\in L^p_w(\mathbb{R})\,;\,f,|y|^{\alpha}\hat{f}(y)\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{ixy}|y|^{\alpha}\hat{f}(y)\,dy\in L^p_w(\mathbb{R})\}$, where $L^p_w(\mathbb{R})$, $p>1$, is the Banach space $\{f: f$ is measurable and $\int_{\mathbb{R}}|f(x)|^p\,w(x)\,dx<\infty\}$ and $w(x)=(1+x^2)^{-\mu/2}$, $\mu>1$. For simplicity of notation, we put $m=\nu\mu$. The operator $(I-\Delta)^{\gamma/2}$ is defined by

$$(I-\Delta)^{\gamma/2} f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ixy}\left(1+y^2\right)^{\gamma/2}\hat{f}(y)\,dy$$

for $f\in\mathfrak{D}\left((I-\Delta)^{\gamma/2}\right)=\{f\in L^p_w: f,\,(1+y^2)^{\gamma/2}\hat{f}(y)\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{ixy}(1+y^2)^{\gamma/2}\hat{f}(y)\,dy\in L^p_w(\mathbb{R})\}$.

We consider the Green function of the operator $\frac{\partial}{\partial t}+(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}$:

$$p(\alpha,\gamma,\mu;t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(i\lambda x-\mu t\,|\lambda|^{\alpha}\left(1+\lambda^2\right)^{\gamma/2}\right)d\lambda,$$

$1/2<\alpha<3/2$, $\gamma\geq 0$, $t\geq 0$, $x\in\mathbb{R}$, and define

$$\begin{aligned} S(t)\,f(x) &= \int_{-\infty}^{\infty} p(\alpha,\gamma,\mu;t,x)\,f(x-y)\,dy, \\ S(0)\,f(x) &= f(x) \end{aligned}$$

for f in $L^p_w(\mathbb{R})$. It is seen that

$$|S(t)\,f(x)|\leq\int_{-\infty}^{\infty}|p(\alpha,\gamma,\mu;t,y)\,f(x-y)|\,dy$$

$$=\int_{-\infty}^{\infty}\left|p\left(\alpha,\gamma,\mu;t,t^{1/(\alpha+\gamma)}u\right)t^{1/(\alpha+\gamma)}\right|\left|f\left(x-t^{1/(\alpha+\gamma)}u\right)\right|du$$

$$\leq\left(\int_{-\infty}^{\infty}|p(\alpha,\gamma,\mu;1,u)|^{p'}\left(1+u^2\right)^{\frac{\nu p'}{2}}du\right)^{1/p'}\left(\int_{-\infty}^{\infty}\left|f\left(x-t^{1/(\alpha+\gamma)}u\right)\right|^{p}\left(1+u^2\right)^{-\frac{\nu p}{2}}du\right)^{1/p}$$

by Hölder's inequality. Thus,

$$\int_{-\infty}^{\infty}|S(t)f(x)|^{p}\left(1+x^{2}\right)^{-\frac{\nu p}{2}}dx \tag{2.2}$$
$$\leq \left(\int_{-\infty}^{\infty}|p(\alpha,\gamma,\mu;1,u)|^{p'}\left(1+u^{2}\right)^{\frac{\nu p'}{2}}du\right)^{p/p'}$$
$$\times\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|f\left(x-t^{1/(\alpha+\gamma)}u\right)\right|^{p}\left(1+u^{2}\right)^{-\nu p/2}du\right)\left(1+x^{2}\right)^{-\nu p/2}dx.$$

Lemma 1. *$\int_{|x|\geq 1}|p(\alpha,\gamma,\mu;1,x)|^{p'}|x|^{\nu p'}dx<\infty$ for $m<p/q,\ q\in(1,2]$.*

Proof. By integration by parts,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixy}\frac{d}{dy}\left(\exp\left(-\mu|y|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)dy\right)=(-ix)\,p(\alpha,\gamma,\mu;1,x).$$

Thus,

$$\int_{|x|\geq 1}|p(\alpha,\gamma,\mu;1,x)|^{p'}|x|^{\nu p'}dx$$
$$=\int_{|x|\geq 1}\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixy}\left(\frac{d}{dy}\exp\left(-\mu|y|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)\right)dy\right|^{p'}|x|^{(\nu-1)p'}dx$$
$$\leq\left(\int_{|x|\geq 1}|x|^{(\nu-1)p's'}dx\right)^{1/s'} \tag{2.3}$$
$$\times\left(\int_{|x|\geq 1}\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixy}\left(\frac{d}{dy}\exp\left(-\mu|y|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)\right)dy\right|^{p's}\right)^{1/s}.$$

By the Hausdorff-Young inequality,

$$\left(\int_{-\infty}^{\infty}\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixy}\left(\frac{d}{dy}\exp\left(-\mu|y|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)\right)dy\right|^{q'}dx\right)^{1/q'}$$
$$\leq A(q)\left(\int_{-\infty}^{\infty}\left|\frac{d}{dy}\exp\left(-\mu|y|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)\right|^{q}dy\right)^{1/q}.$$

Thus, writing $p's=q'$, inequality (2.3) becomes

$$\int_{|x|\geq 1}|p(\alpha,\gamma,\mu;1,x)|^{p'}|x|^{\nu p'}dx$$
$$\leq (A(q))^{q'/s}\left(\int_{|x|\geq 1}|x|^{(\nu-1)p's'}dx\right)^{1/s'} \tag{2.4}$$
$$\times\left(\int_{-\infty}^{\infty}\left|\frac{d}{dy}\exp\left(-\mu|y|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)\right|^{q}dy\right)^{\frac{q'}{qs}}.$$

We have

$$\int_{|x|\geq 1}|x|^{(\nu-1)p's'}dx<\infty \text{ if } (\nu-1)p's'+1<0, \tag{2.5}$$

i.e. if $m < p/q$. Now,

$$\frac{d}{dy}\exp\left(-\mu\left|y\right|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)=-\mu\left(sgn\ y\right)^{\alpha}$$

$$\times\left(\gamma y^{\alpha-1}\left(1+y^{2}\right)^{\gamma/2}+\alpha y^{\alpha+1}\left(1+y^{2}\right)^{\frac{\gamma}{2}-1}\right)\exp\left(-\mu\left|y\right|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right).$$

Thus,

$$\int_{-\infty}^{\infty}\left|\frac{d}{dy}\exp\left(-\mu\left|y\right|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)\right|dy<\infty,$$

and also

$$\int_{-\infty}^{\infty}\left|\frac{d}{dy}\exp\left(-\mu\left|y\right|^{\alpha}\left(1+y^{2}\right)^{\gamma/2}\right)\right|^{q}dy<\infty \tag{2.6}$$

for some $q\in(1,2]$. The lemma now follows from (2.4)-(2.6). ∎

Lemma 2. $\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|f\left(x-t^{1/(\alpha+\gamma)}u\right)\right|^{p}\left(1+u^{2}\right)^{-\nu p/2}du\right)\left(1+x^{2}\right)^{-\nu p/2}dx$

$\leq c\left(1+t^{(m-1)/(\alpha+\gamma)}\right)\left\|f\right\|_{w,p}^{p}$, *c being a constant.*

Proof. See Takano [25] ∎

Lemma 3. *$S(t)$ is a bounded operator on $L_{w}^{p}(\mathbb{R})$ if $(m-1)/((\alpha+\gamma)p)<1$ and $m<p/q$ for some $q\in(1,2]$.*

Proof. This result follows from (2.2) and Lemmas 1 and 2. ∎

Lemma 4. *$S(t)S(s)=S(t+s)$ for $t,s\geq 0$, and $S(t)f\to f$ as $t\to 0^{+}$ for all f in $L_{w}^{p}(\mathbb{R})$ in the L_{w}^{p}-norm.*

Proof. See Takano [25] ∎

The following result is thus obtained:

Proposition 1. *$\{S(t), 0\leq t<\infty\}$ is a strongly continuous one-parameter semigroup for $\alpha+\gamma>\frac{m-1}{p}$ and $m<\frac{p}{q}$ for some $q\in(1,2]$.*

We now choose $p=2$. Let $\left\{B_{j}^{h}(t), t\geq 0\right\}$, $j=1,2,...$, be independent centered Gaussian processes with $B_{j}^{h}(0)=0$ on a given probability space $(\Omega,\mathcal{F},P)$, where we assume

$$E\left(B_{j}^{h}(t)-B_{j}^{h}(s)\right)^{2}=\left|t-s\right|^{2h}\mu_{j},\ j=1,2,..., \tag{2.7}$$

$$\mu_{j}>0,\quad\sum_{j=1}^{\infty}\mu_{j}<\infty,\ h\in(1/2,1). \tag{2.8}$$

The processes $\left(B_{j}^{h}(t)\right)$ are independent fractional Brownian motions with Hurst index h and $E\left(B_{j}^{h}(1)\right)^{2}=\mu_{j}$, $j=1,2,...$ It follows from Kleptsyna *et al.* [17] that

$$B_{j}^{h}(t)=\left(\int_{-\infty}^{0}\left(\left|t-r\right|^{h-1/2}-\left|r\right|^{h-1/2}\right)dW_{j}(r)+\int_{0}^{t}\left|t-r\right|^{h-1/2}dW_{j}(r)\right), \tag{2.9}$$

where $\{W_{j}(t), t\geq 0\}$, $j=1,2,...$ are real independent Wiener processes with $EW_{j}^{2}(t)=\mu_{j}t$.

Let H be a separable Hilbert space with scalar product $(\cdot,\cdot)_H$ and $\{e_j,\ j=1,2,...\}$ denotes a complete orthogonal system in H. Then

$$\sum_{j=1}^{\infty} E\left\|B_j^h(t)\,e_j\right\|_H^2 = t^{2h}\sum_{j=1}^{\infty}\mu_j < \infty$$

and the following definition is appropriate:

Definition 1. *$B^h(t)=\sum_{j=1}^{\infty}B_j^h(t)\,e_j$ is called a H-valued fractional Brownian motion where the sum is defined in mean square.*

Lemma 5. *The covariance operator of $\left\{B_j^h(t),\ t\geq 0\right\}$, $j=1,2,...,$ is a positive nuclear operator $\mathcal{Q}(t,s)$ with*

$$tr\left(\mathcal{Q}(t,s)\right)=\frac{1}{2}\sum_{j=1}^{\infty}\mu_j\left(t^{2h}+s^{2h}-|t-s|^{2h}\right). \tag{2.10}$$

Proof. See Grecksch and Anh [12] ∎

Lemma 5 shows that $B^h(t)$ is a H-valued Gaussian process and the trace of the covariance operator is given by (2.10).

Let us now consider $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with

$$|f(t,x,y)|^2\leq K^2\left(1+y^2\right), \tag{2.11}$$

$$|f(t,x,y)-f(t,x,z)|^2\leq K^2|y-z|^2 \tag{2.12}$$

for all $t\in[0,T]$, $x,y,z\in\mathbb{R}$ and $K>0$ being a fixed constant. We assume that f is measurable with respect to t,x. Also, let g be a measurable function defined on $[0,T]$ with $|g(t)|\leq C$, where C is a positive constant.

Assume that the initial condition X_0 is a function in $L_w^2(\mathbb{R})$. We then define the problem (2.1) by the integral equation

(2.13)

$$X(t,x)=S(t)\,X_0(x)+\int_0^t S(t-s)\,f(s,x,X(s,x))\,ds+\int_0^t S(t-s)\,g(s)\,dB(s,x),$$

where $S(t)$ is the semigroup established in Proposition 1 and the stochastic integral is defined by

$$\int_0^t S(t-s)\,g(s)\,dB(s,x)=\sum_{j=1}^{\infty}\int_0^t S(t-s)\,g(s)\,e_j(x)\,dB_j^h(s).$$

Let $t\in[0,T]$ be fixed and $0=t_0<t_1<...<t_n=t$, $g(s)=g_i$ for $s\in[t_i,t_{i+1})$. Then

Lemma 6.

$$E\left\|\int_0^t S(t-s)\,g(s)\,dB(s,x)\right\|_{L_w^2(\mathbb{R})}^2\leq C_t^2C^2t^{2n}\sum_{j=1}^{\infty}\mu_j.$$

Proof. Let $m \geq 1$ be chosen arbitrarily. Then

$$
\begin{aligned}
& E\left\|\sum_{j=1}^{m}\int_0^t S(t-s)\,g(s)\,e_j(x)\,dB_j^h(s)\right\|^2_{L^2_w(\mathbb{R})} \\
= & \; E\left\|\sum_{j=1}^{m}\sum_{k=0}^{n-1} S(t-t_k)\,g_k e_j(x)\left(B_j^h(t_{k+1})-B_j^h(t_k)\right)\right\|^2_{L^2_w(\mathbb{R})} \\
= & \; E\left(\sum_{j=1}^{m}\sum_{k=0}^{n-1} S(t-t_k)\,e_j(x)\left(B_j^h(t_{k+1})-B_j^h(t_k)\right)g_k\right. \\
& \left.\times\sum_{j=1}^{m}\sum_{k=0}^{n-1} S(t-t_k)\,e_j(x)\left(B_j^h(t_{k+1})-B_j^h(t_k)\right)g_k\right)_{L^2_w(\mathbb{R})} \\
= & \; \sum_{j=1}^{m}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\left(S(t-t_k)\,e_j(t),S(t-t_l)\,e_j(x)\right)_{L^2_w(\mathbb{R})} g_k g_l \\
& \times E\left(\left(B_j^h(t_{k+1})-B_j^h(t_k)\right)\left(B_j^h(t_{l+1})-B_j^h(t_l)\right)\right) \\
\leq & \; C_t^2 C^2 \sum_{j=1}^{m}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} E\left(B_j^h(t_{k+1})-B_j^h(t_k)\right)\left(B_j^h(t_{l+1})-B_j^h(t_l)\right) \\
= & \; C_t^2 C^2 \sum_{j=1}^{m} E\left(\sum_{k=0}^{n-1} B_j^h(t_{k+1})-B_j^h(t_k)\right)^2 \\
= & \; C_t^2 C^2 \sum_{j=1}^{m} E\left(B_j^h(t)\right)^2 = C_t^2 C^2 t^{2h}\sum_{j=1}^{m}\mu_j
\end{aligned}
$$

for all m, since $E\left(\left(B_j^h(t_{k+1})-B_j^h(t_k)\right)\left(B_j^h(t_{l+1})-B_j^h(t_l)\right)\right) \geq 0$. Consequently,

$$
\begin{aligned}
E\left\|\int_0^t S(t-s)\,g(s)\,dB(s,x)\right\|^2_{L^2_w(\mathbb{R})} &= \sum_{j=1}^{\infty} E\left\|\int_0^t S(t-s)\,g(s)\,e_j(x)\,dB_j^h(s)\right\|^2_{L^2_w(\mathbb{R})} \\
&\leq C_t^2 C^2 t^{2n}\sum_{j=1}^{\infty}\mu_j.
\end{aligned}
$$

∎

We see that $\int_0^t S(t-s)\,e_j(x)\,g(s)\,dB_j^h(s)$ is properly defined and it is not difficult to extend this definition of stochastic integral to general bounded measurable functions $g(s)$.

Theorem 1. *There is a unique $L^2_w(\mathbb{R})$-valued solution process $\{X(t,\cdot), t\in[0,T]\}$ of (2.13) with* $\sup_{0\leq t\leq T} E\left\|X(t,\cdot)\right\|^2_{L^2_w(\mathbb{R})} < \infty$.

Proof. Let a be a real number with $a \geq 2K^2C_T^2$. We note that $\left\|X\right\| = \left(\sup_{0\leq t\leq T} e^{-at}E\left\|X(t,\cdot)\right\|^2_{L^2_w(\mathbb{R})}\right)^{1/2}$ is an equivalent norm on the Banach space of all $L^2_w(\mathbb{R})$-valued processes $\{X(t,\cdot), t\in[0,T]\}$ with $\left(\sup_{0\leq t\leq T} E\left\|X(t,\cdot)\right\|^2_{L^2_w(\mathbb{R})}\right)^{1/2} <$

∞. We denote this space by M. Then the operator $\mathcal{T}$ defined by

$$\mathcal{T}(X)(t,x) = S(t)X_0(x) + \int_0^t S(t-s) f(s,x,X(s,x))\,ds + \int_0^t S(t-s) g(s)\,dB(s,x)$$

is an M-valued operator. For $X, Y \in M$, it holds that

$$\begin{aligned}\|\mathcal{T}(X) - \mathcal{T}(Y)\|^2 &= \sup_{0\le t\le T} e^{-at} E \left\| \int_0^t S(t-s)\left[f(s,x,X(s,x)) - f(s,x,Y(s,x))\right] ds \right\|^2_{L^2_w(\mathbb{R})} \\ &\le K^2 C_T^2 \sup_{0\le t\le T} e^{-at} E \int_0^t e^{as} \sup_{0\le\tau\le T} e^{-a\tau} \|X(\tau,\cdot) - Y(\tau,\cdot)\|^2_{L^2_w(\mathbb{R})}\, ds \\ &= K^2 C_T^2 \sup_{0\le t\le T} \frac{1}{a}\left(1 - \frac{1}{e^{at}}\right) \|X - Y\|^2 \\ &\le \frac{K^2 C_T^2}{a} \|X - Y\|^2 .\end{aligned}$$

Thus, for $a \le 2K^2C_T^2$, $\mathcal{T}$ is a contraction in M. The Banach fixed-point theorem then gives the statement. ∎

3. An approximation

We consider a partition $0 = t_1^{(r)} < t_2^{(r)} < ... < t_{N_r}^{(r)} = t \in [0,T]$ with $\lim_{r\to\infty} \max_{0\le j\le N_r-1} \left(t_{j+1}^{(r)} - t_j^{(r)}\right) = 0$. For brevity of notation, we shall write $t_j^{(r)} = t_j$. We introduce stochastic processes $\left\{X_j^r(s),\ s \in [t_j, t_{j+1})\right\}$ with values in $L^2_w(\mathbb{R})$ defined by

(3.1)
$$X_j^r(s,x) = S(s-t_j) X_j^r(t_j,x) + \int_{t_j}^s S(s-u) f\left(u,x,X_j^r(u,x)\right) du,\ j = 0,...,N_r - 1,$$

where

$$X_j^r(t_j,x) = S(t_j - t_{j-1}) X_{j-1}^r(t_j - 0, x) + \int_{t_{j-1}}^{t_j} S(t_j - u) g(u)\,dB(u,x) \tag{3.2}$$

and $X_0^r(0,x) = X_0(x)$.

Analogous to (2.13) as the generalised solution of (2.1), the solution (3.1) can be defined as the generalised solution of

$$\frac{\partial}{\partial s} X_j^r(s,x) = -(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2} X_j^r(s,x) + f\left(s,x,X_j^r(s,x)\right)$$

with

$$X_j^r(t_j,x) = S(t_j - t_{j-1}) X_{j-1}^r(t_j - 0, x).$$

We can prove in a similar manner to the proof of Theorem 1 the following result:

Lemma 7. *There is a unique solution process* $\left\{X_j^r(s,x),\ s\in[t_j,t_{j+1})\right\}$ *of (3.1), (3.2) with* $\sup_{s\in[t_j,t_{j+1})} E\|X(s,\cdot)\|^2_{L^2_w(\mathbb{R})} < \infty$ *for all* $j = 0,...,N_r-1$ *and* $r = 1,2,...$.

If we introduce the process $\tilde{X}^r(s,x) = \tilde{X}_j(s,x)$ for $s \in [t_j, t_{j+1})$, $j = 0, ..., N_r - 1$, then we can write for (3.1), (3.2) in the case $s \in [t_j, t_{j+1})$

$$\begin{aligned} \tilde{X}^r(s,x) &= S(t_j) X_0(x) + \int_0^{t_j} S(t_j - u) f\left(u, x, \tilde{X}^r(u,x)\right) du \\ &\quad + \int_0^{t_j} S(t_j - u) g(u) dB(u,x) + \int_{t_j}^{s} S(s-u) f\left(u, x, \tilde{X}^r(u,x)\right) du \end{aligned} \tag{3.3}$$

for $j = 0, ..., N_r - 1$. We define

$$\begin{aligned} \tilde{X}^r(t,x) &= S(t - t_{N_r - 1}) \tilde{X}^r(t_{N_r-1}, x) + \int_{t_{N_r-1}}^{t} S(t-u) f\left(u, x, \tilde{X}^r(u,x)\right) du \\ &= S(t) X_0(x) + \int_0^t S(t-u) f\left(u, x, \tilde{X}^r(u,x)\right) du \\ &\quad + \int_0^{t_{N_r-1}} S(t - t_{N_r-1}) g(u) dB(u,x). \end{aligned} \tag{3.4}$$

We want to prove that $\tilde{X}^r(t,x)$ is an approximation of $X(t,x)$. We first prove an *a priori* estimate.

Lemma 8. *There is a positive constant $\widetilde{C}$ independent of r with $E\left\|\tilde{X}^r(s,\cdot)\right\|^2_{L^2_w(\mathbb{R})} \le \widetilde{C}$ for all r and $s \in [0,t]$.*

Proof. We choose arbitrary numbers $j \in \{0, ..., N_r - 1\}$ and $r \in \{1, 2, ...\}$. Since t is finite, it follows from the proof of Proposition 1 that

$$\|S(u)\| \le D \tag{3.5}$$

for a fixed constant $D > 0$ and all $u \in [0,t]$. Then, from Schwarz's inequality, the properties of f, g and Lemma 6, we get

$$\begin{aligned} E\left\|\tilde{X}^r(s,\cdot)\right\|^2_{L^2_w(\mathbb{R})} &\le 4D^2 E\|X_0(\cdot)\|^2_{L^2_w(\mathbb{R})} \\ &\quad + 4t_j D^2 \int_0^{t_j} \left(1 + E\left\|\tilde{X}^r(u,\cdot)\right\|^2_{L^2_w(\mathbb{R})} du\right) \\ &\quad + 4D^2 C^2 t_j^{2h} \sum_{k=1}^{\infty} \mu_k \\ &\quad + 4(s - t_j) D^2 \int_{t_j}^{s} \left(1 + E\left\|\tilde{X}^r(u,\cdot)\right\|^2_{L^2_w(\mathbb{R})}\right) du \\ &\le 4D^2 \left(E\|X_0(\cdot)\|^2_{L^2_w(\mathbb{R})} + 2t^2 + C^2 t^{2h} \sum_{k=1}^{\infty} \mu_k\right) \\ &\quad + 4tD^2 \int_0^s E\left\|\tilde{X}^r(u,\cdot)\right\|^2_{L^2_w(\mathbb{R})} du. \end{aligned}$$

An application of the Gronwall lemma then gives the statement. ∎

Theorem 2. *It holds that*

$$\lim_{r\to\infty} E\left\|\tilde{X}^r(t,\cdot) - X(t,\cdot)\right\|^2_{L^2_w(\mathbb{R})} = 0.$$

Proof. It follows from (2.13) and (3.4) that

$$\tilde{X}^r(t,x) - X(t,x) = \int_0^t S(t-u)\left[f\left(u,x,\tilde{X}^r(u,x)\right) - f(u,x,X(u,x))\right]du + \int_{t_{N_r-1}}^t S(t-t_{N_r-1})\,g(u)\,dB(u,x).$$

From (3.5), the Schwarz inequality, (2.12) and Lemma 6, we get

$$E\left\|\tilde{X}^r(t,\cdot) - X(t,\cdot)\right\|^2_{L^2_w(\mathbb{R})} \le 2D^2K^2\int_0^t E\left\|\tilde{X}^r(u,\cdot) - X(u,\cdot)\right\|^2_{L^2_w(\mathbb{R})}du + 2D^2C^2(t-t_{N_r-1})\sum_{j=1}^{\infty}\mu_j.$$

An application of the Gronwall lemma then yields the statement for $r \to \infty$. ∎

References

[1] N. U. Ahmed. *Semigroup Theory with Applications to Systems and Control.* Longman, Harlow, 1991.

[2] V. V. Anh, J. M. Angulo, and M. D. Ruiz-Medina. Possible long-range dependence in fractional random fields. *Journal of Statistical Planning and Inference*, 80(1/2):95–110, 1999.

[3] V. V. Anh and C. C. Heyde, editors. *Special Issue on Long-Range Dependence.* Journal of Statistical Planning and Inference, 80(1/2):1–292, 1999.

[4] V. V. Anh and N. N. Leonenko. Spectral analysis of fractional kinetic equations with random data. *Journal of Statistical Physics*, 104(5/6):1349–1387, 2001.

[5] A. V. Balakrishan. Fractional powers of closed operators and the semigroup generated by them. *Pacific J. Math.*, 10:419–437, 1960.

[6] A. L. Barabasi and H. E. Stanley. *Fractal Concepts in Surface Growth.* Cambridge University Press, 1995.

[7] A. Bensoussan, G. D. Prato, M. G. Delfour, and S. K. Mitter. *Representation and Control of Infinite Dimensional Systems.* Birkhauser, London, 1992.

[8] J. Beran. *Statistics for Long-Memory Processes*, volume 61 of *Monographs on Statistics and Applied Probability.* Chapman & Hall, 1994.

[9] A. Compte and R. Metzler. Generalised Cattaneo equation for the description of anomalous transport processes. *J Phys A: Math Gen*, 1997.

[10] W. J. Donoghue. *Distributions and Fourier Transforms.* Academic Press, New York, 1969.

[11] W. Feller. On a generalization of Marcel Riesz' potential and the semi-groups generated by them. In *Comm. Sém. Matém. Université de Lund*, pages 73–81. Tome suppl. dédié a M. Riesz, 1952.

[12] W. Grecksch and V. V. Anh. A parabolic stochastic differential equation with fractional Brownian motion input. *Statistical & Probability Letters*, 41:337–346, 1999.

[13] W. Grecksch and V. V. Anh. Approximation of stochastic evolution equations and application to equations with fractional power of infinitesimal operators. *Random Operators & Stochastic Equations*, 8:27–38, 2000.

[14] W. Grecksch and C. Tudor. *Stochastic Evolution Equations*, volume 85 of *Mathematical Research.* Akademie Verlag, 1995.

[15] R. Hilfer. Fractional time evolution. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 87–130. World Scientific, Singapore, 2000.

[16] P. M. Iannaccone and M. Khokha, editors. *Fractal Geometry in Biological Systems.* CRC Press, 1996.

[17] M. Kleptsyna, P. E. Kloeden, and V. V. Anh. Existence and uniqueness theorems for fBm stochastic differential equations. *Probl. Inform. Transm.*, 34(4):51–61, 1998.

[18] N. V. Krylov and B. L. Rozovskij. Itogi nanki i techniki, sevija sov remennych problemy matematiki. *VINITI, Mozkva*, 14:71–146, 1979.

[19] N. Leonenko. *Limit Theorems for Random Fields with Singular Spectrum.* Kluwer, 1999.

[20] S. A. Levin, editor. *Frontiers in Mathematical Biology*. Springer-Verlag, 1994.

[21] R. Manthey. On the Cauchy problem for reaction-diffusion equations with white noise. *Math. Nach.*, 136:209–228, 1988.

[22] G. D. Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge University Press, Cambridge, 1992.

[23] M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch, editors. *Lévy Flights and Related Topics in Physics*, volume 450 of *Lecture Notes in Physics*. Springer, 1995.

[24] E. M. Stein. *Singular Integrals and Differential Properties of Functions*. Princeton University Press, New Jersey, 1970.

[25] K. Takano. One-parameter semigroups with infinitesimal generators of fractional powers of the Laplacean on weighted L^P-spaces. *Bull. Fac. Sci. Ibaraki. Univ., Math.*, 13:45–54, 1981.

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