J. Appl. Math. & Computing Vol. x *(200y), No. z, pp.*

TIME FRACTIONAL ADVECTION-DISPERSION EQUATION

F.LIU, V.V.ANH, I.TURNER AND P.ZHUANG

Abstract. A time fractional advection-dispersion equation is obtained from the standard advection-dispersion equation by replacing the first-order derivative in time by a fractional derivative in time of order $\alpha(0 < \alpha \leq 1)$. Using variable transformation, Mellin and Laplace transforms, and properties of H-functions, we derive the complete solution of this time fractional advection-dispersion equation.

AMS Mathematics Subject Classification: 26A33, 49K20, 44A10.
Key words and phrases: time fractional advection-dispersion equation, Mellin transform, Laplace transform.

1. Introduction

A space-time fractional partial differential equation, obtained from the standard partial differential equation by replacing the second order space-derivative by a fractional derivative of order $\beta > 0$ and the first order time-derivative by a fractional derivative of order $\alpha > 0$ has been recently treated by a number of authors. Wyss (1986) considered the time fractional diffusion equation and the solution was given in closed form in terms of H-functions. Schneider and Wyss (1989) considered the time fractional diffusion and wave equations. The corresponding Green functions were obtained in closed form for arbitrary space dimensions in terms of H-functions and their properties were exhibited. Gorenflo, Iskenderov, and Luchko (2000) used the similarity method and the Laplace transform method to obtain the scale-invariant solution of the time-fractional diffusion-wave equation in terms of the Wright function. Benson, Whearcraft and Meerschaert (2000a,b) considered space fractional advection-dispersion equation. They gave an analytical solution featuring the α-stable error function. Liu, Anh and Turner (2002) presented a numerical solution of the space fractional advection-dispersion equation. Mainardi, Luchko and Pagnini (2001) considered the space-time fractional diffusion equation and provided a

Received , . Revised , .

general representation of the Green functions in the terms of Mellin-Barnes integrals in the complex plane. Anh and Leonenko (2000) proposed scaling laws for fractional diffusion-wave equations with singular data. Angulo, Ruiz-Medina, Anh and Grecksch (2000) introduced a fractional heat equation, where the diffusion operator is the composition of the Bessel and Riesz potentials. Wyss (2000) considered a fractional Black-Scholes equation and gave a complete solution of this equation. Some partial differential equations of space-time fractional order were successfully used for modelling relevant physical processes (Giona and Roman, 1992; Mainardi, 1994; Hilfer, 1995; Caputo, 1996; Benson, 2000a,b; El-Sayed and Aly, 2002; Basu and Acharya, 2002).

In this paper, we consider the time fractional advection-dispersion equation. This equation is obtained by replacing the time-derivative in the advection dispersion equation by a generalized derivative of order α with $0 < \alpha \leq 1$. We consider

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = -\nu \frac{\partial C(x,t)}{\partial x} + D\frac{\partial^2 C(x,t)}{\partial x^2}, \quad x > 0, t > 0, 0 < \alpha \leq 1 \tag{1}$$

with the initial condition

$$C(x,0) = C_0(x), \qquad x \geq 0 \tag{2}$$

where $\nu \geq 0$, $D > 0$ and $\frac{\partial^\alpha C(x,t)}{\partial t^\alpha}$ is a fractional derivative.

Properties and more details about the fractional derivatives can be found in Samko, Kilbas and Marichev (1993). Using variable transformation, the time fractional advection-dispersion equation is reduced to a more familiar form. Using Schneider and Wyss's (1989) and Wyss's (2000) techniques we derive in this paper the complete solution of this time fractional advection-dispersion equation (TFADE).

2. The reduced time fractional advection dispersion equation

Eq. (1) can be expressed by the following integral equation (Wyss, 2000; Schneider and Wyss, 1989):

$$C(x,t) = C(x,0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[-\nu\frac{\partial C(x,\tau)}{\partial x} + D\frac{\partial^2 C(x,\tau)}{\partial x^2}]d\tau \tag{3}$$

with $n-1 < \alpha \leq n$,$n = 1$. To reduce (3.1) to a more familiar form, let

$$C(x,t) = u(\xi,t)exp(\frac{\nu\xi}{2\sqrt{D}}), \qquad x = \frac{\xi}{\sqrt{D}} \tag{4}$$

with the initial condition:

$$C(x,0) = C_0(x) = u(\xi,0)exp(-\frac{\nu\xi}{2\sqrt{D}}), \qquad \xi > 0. \tag{5}$$

Let $\mu^2 = \frac{\nu^2}{4D}$, this leads to the equation

$$u(\xi,t) = u(\xi,0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2} - \mu^2 u(\xi,\tau)]d\tau \tag{6}$$

and

$$u(\xi,0) = C_0(\frac{\xi}{\sqrt{D}})exp(-\frac{\nu\xi}{2\sqrt{D}}). \tag{7}$$

3. The Green function for the reduced TFADE

According to the properties of the Laplace transform and (49), the Laplace transform of (6) with respect to t gives for $p > 0$

$$\widetilde{u}(\xi,p) = p^{-1}u(\xi,0) + p^{-\alpha}[\frac{\partial^2\widetilde{u}(\xi,p)}{\partial\xi^2} - \mu^2\widetilde{u}(\xi,p)] \tag{8}$$

or

$$\frac{\partial^2\widetilde{u}(\xi,p)}{\partial\xi^2} - [\mu^2 + p^\alpha]\widetilde{u}(\xi,p) = -p^{-1+\alpha}u(\xi,0). \tag{9}$$

Letting $w^2 = \mu^2 + p^\alpha$, we have the equation

$$\frac{\partial^2\widetilde{u}(\xi,p)}{\partial\xi^2} - w^2\widetilde{u}(\xi,p) = -p^{-1+\alpha}u(\xi,0). \tag{10}$$

According to Schneider and Wyss (1989), Eq. (10) has the solution

$$\widetilde{u}(\xi,p) = \int_0^\infty \widetilde{G}_\mu^\alpha(|\xi-y|,p)u(y,0)dy, \tag{11}$$

where

$$\widetilde{G}_\mu^\alpha(r,p) = p^{\alpha-1}\kappa(r,w) = p^{\alpha-1}\kappa(r,\sqrt{\mu^2+p^\alpha}) \tag{12}$$

and

$$\kappa(r,w) = (\frac{r}{2\pi w})^{\frac{1}{2}}K_{\frac{1}{2}}(wr). \tag{13}$$

A direct transition to the time domain (i.e., inverting the Laplace transform) does not seem to be feasible. This difficulty is circumvented by passing through the intermediate step of the Mellin transform (33), connected with the Laplace transform by (50). Thus, to invert the Laplace transform of Eq. (12) to find

Green's function $G_\mu^\alpha(r,t)$, we first compute its Mellin transform according to Eq. (50):

$$G^{*\alpha}_{\mu}(r,s) = \frac{1}{\Gamma(1-s)} \int_0^\infty p^{-s} \widetilde{G}_\mu^\alpha(r,p)dp \tag{14}$$

or

$$G^{*\alpha}_{\mu}(r,s) = (\frac{r}{2\pi})^{\frac{1}{2}} \frac{1}{\Gamma(1-s)} \int_0^\infty p^{-s+\alpha-1}[\mu^2+p^\alpha]^{-\frac{1}{4}} K_{\frac{1}{2}}(r\sqrt{\mu^2+p^\alpha})dp. \tag{15}$$

Letting $q = p^{\frac{\alpha}{2}}$, we obtain

$$G^{*\alpha}_{\mu}(r,s) = \frac{2}{\alpha}(\frac{r}{2\pi})^{\frac{1}{2}} \frac{1}{\Gamma(1-s)} \int_0^\infty q^{[2-\frac{2s}{\alpha}]-1}[\mu^2+q^2]^{-\frac{1}{4}} K_{\frac{1}{2}}(r\sqrt{\mu^2+q^2})dq. \tag{16}$$

Using formula (42) with $b=\mu, \nu=\frac{1}{2}, a=r$, we have

$$G^{*\alpha}_{\mu}(r,s) = \frac{1}{\alpha\sqrt{\pi}}(\frac{2\mu}{r})^{\frac{1}{2}}(\frac{2\mu}{r})^{-\frac{s}{\alpha}} \frac{\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-s)} K_{\frac{s}{\alpha}-\frac{1}{2}}(\mu r). \tag{17}$$

From the H-function representation (52), we have

$$H^{*1,0}_{1,1}(-s) = H^{*1,0}_{1,1}\left(z \,\middle|\, \begin{matrix}(1,1)\\(1,\frac{1}{\alpha})\end{matrix}\right)(-s) = \frac{\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-s)}. \tag{18}$$

Let $\varphi(p) = \frac{1}{2}exp[-\frac{1}{2}\mu r(p^\alpha + p^{-\alpha})]$ and $\phi(p) = p^{-\frac{\alpha}{2}}\varphi(p)$. From the properties of the Mellin transform and Eqs. (38) and (41), we have

$$\varphi^*(s) = \frac{1}{\alpha} K_{\frac{s}{\alpha}}(\mu r), \qquad \phi^*(s) = \frac{1}{\alpha} K_{\frac{s}{\alpha}-\frac{1}{2}}(\mu r). \tag{19}$$

Thus the Mellin transform $G^{*\alpha}_{\mu}(r,s)$ can be written as

$$G^{*\alpha}_{\mu}(r,s) = \frac{1}{\sqrt{\pi}}(\frac{2\mu}{r})^{\frac{1}{2}}(\frac{2\mu}{r})^{-\frac{s}{\alpha}} \phi^*(s) H^{*1,0}_{1,1}(-s). \tag{20}$$

From (38), (40) and (43), we have

$$\int_0^\infty \phi((\frac{2\mu}{r})^{\frac{1}{\alpha}}\frac{t}{z}) H^{1,0}_{1,1}(z^{-1})\frac{dz}{z} \overset{M}{\longleftrightarrow} [(\frac{2\mu}{r})^{\frac{1}{\alpha}}]^{-s} \phi^*(s) H^{*1,0}_{1,1}(-s). \tag{21}$$

Letting $\zeta = \frac{1}{z}$, we have

$$\int_0^\infty \phi((\frac{2\mu}{r})^{\frac{1}{\alpha}}\frac{t}{z}) H^{1,0}_{1,1}(z^{-1})\frac{dz}{z} = \int_0^\infty \phi((\frac{2\mu}{r})^{\frac{1}{\alpha}}\zeta t) H^{1,0}_{1,1}(\zeta)\frac{d\zeta}{\zeta}. \tag{22}$$

Therefore, the inverse Mellin transform leads to

$$G^{\alpha}_{\mu}(r,s) = \frac{1}{\sqrt{\pi}}(\frac{2\mu}{r})^{\frac{1}{2}} \int_0^\infty \phi((\frac{2\mu}{r})^{\frac{1}{\alpha}}\zeta t) H^{1,0}_{1,1}(\zeta)\frac{d\zeta}{\zeta} \tag{23}$$

or

$$G^{\alpha}_{\mu}(r,s) = \frac{1}{\sqrt{\pi}}(\frac{2\mu}{r})^{\frac{1}{2}}\int_0^\infty((\frac{2\mu}{r})^{\frac{1}{\alpha}}\zeta t)^{-\frac{\alpha}{2}}\varphi((\frac{2\mu}{r})^{\frac{1}{\alpha}}\zeta t)H^{1,0}_{1,1}(\zeta)\frac{d\zeta}{\zeta}$$
$$= \frac{1}{2\sqrt{\pi}\sqrt{t^\alpha}}\int_0^\infty \frac{1}{\sqrt{\zeta^\alpha}}exp[-\mu^2 t^\alpha \zeta^\alpha - \frac{r^2}{4}t^{-\alpha}\zeta^{-\alpha}]H^{1,0}_{1,1}(\zeta)\frac{d\zeta}{\zeta}. \quad (24)$$

Letting $\sigma = \zeta^\alpha$ and using the properties (61) and (62) of the H-function, we obtain

$$G^{\alpha}_{\mu}(r,t)$$
$$= \frac{1}{2\alpha\sqrt{\pi}\sqrt{t^\alpha}}\int_0^\infty \sigma^{-\frac{3}{2}}exp[-\mu^2 t^\alpha\sigma - \frac{r^2}{4}t^{-\alpha}\sigma^{-1}]H^{1,0}_{1,1}\left(\sigma^{\frac{1}{\alpha}}\left|\begin{array}{c}(1,1)\\(1,\frac{1}{\alpha})\end{array}\right.\right)d\sigma$$
$$= \frac{1}{2\sqrt{\pi}\sqrt{t^\alpha}}\int_0^\infty \sigma^{-\frac{3}{2}}exp[-\mu^2 t^\alpha\sigma - \frac{r^2}{4}t^{-\alpha}\sigma^{-1}]H^{1,0}_{1,1}\left(\sigma\left|\begin{array}{c}(1,\alpha)\\(1,1)\end{array}\right.\right)d\sigma \quad (25)$$
$$= \frac{1}{2\sqrt{\pi}\sqrt{t^\alpha}}\int_0^\infty exp[-\mu^2 t^\alpha\sigma - \frac{r^2}{4}t^{-\alpha}\sigma^{-1}]H^{1,0}_{1,1}\left(\sigma\left|\begin{array}{c}(1-\frac{3\alpha}{2},\alpha)\\(-\frac{1}{2},1)\end{array}\right.\right)d\sigma.$$

Hence, the inverse Laplace transform of Eq. (11) leads to

$$u(\xi,t) = \int_0^\infty G^{\alpha}_{\mu}(|\xi-y|,t)u(y,0)dy \quad (26)$$

4. The complete solution of TFADE

In this section, we find the exact solution of the initial problem (1), (2) with $C_0(x) = C_0$. Using the Green function (25) and the initial condition (7), we have

$$u(\xi,t) = C_0\int_0^\infty G^{\alpha}_{\mu}(|\xi-y|,t)exp(-\mu y)dy$$
$$= \frac{C_0}{2\sqrt{\pi t^\alpha}}\int_0^\infty exp(-\mu y)dy$$
$$\times\int_0^\infty exp(-\mu^2 t^\alpha\sigma - \frac{1}{4}(\xi-y)^2 t^{-\alpha}\sigma^{-1})H^{1,0}_{1,1}\left(\sigma\left|\begin{array}{c}(1-\frac{3\alpha}{2},\alpha)\\(-\frac{1}{2},1)\end{array}\right.\right)d\sigma$$
$$= \frac{C_0}{2\sqrt{\pi t^\alpha}}exp(-\mu\xi)\int_0^\infty H^{1,0}_{1,1}\left(\sigma\left|\begin{array}{c}(1-\frac{3\alpha}{2},\alpha)\\(-\frac{1}{2},1)\end{array}\right.\right)d\sigma \quad (27)$$
$$\times\int_0^\infty exp\{-[\frac{\xi-y-2\mu t^\alpha\sigma}{2\sqrt{t^\alpha\sigma}}]^2\}dy.$$

Letting $\eta = \frac{2\mu t^\alpha \sigma - \xi + y}{2\sqrt{t^\alpha \sigma}}$, we get

$$u(\xi,t) = \frac{C_0}{2} exp(-\mu\xi) \int_0^\infty \sqrt{\sigma} H_{1,0}^{1,0}\left(\sigma \,\middle|\, \begin{matrix} (1-\frac{3\alpha}{2},\alpha) \\ (-\frac{1}{2},1) \end{matrix}\right) d\sigma \left[\frac{2}{\sqrt{\pi}} \int_{\eta_1}^\infty exp\{-\eta^2\} d\eta\right] \tag{28}$$

where $\eta_1 = \frac{2\mu t^\alpha \sigma - \xi}{2\sqrt{t^\alpha \sigma}}$.

Using the property (61) of the H-function, we obtain

$$u(\xi,t) = \frac{C_0}{2} exp(-\mu\xi) \int_0^\infty H_{1,1}^{1,0}\left(\sigma \,\middle|\, \begin{matrix} (1-\alpha,\alpha) \\ (0,1) \end{matrix}\right) [\mathrm{erfc}(\eta_1)] d\sigma \tag{29}$$

where the complementary error function, $\mathrm{erfc}(x)$, is defined as

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty exp(-x^2) dx. \tag{30}$$

Using Eq. (4), we obtain the complete solution for time fractional advection dispersion equation with $0 < \alpha \leq 1$:

$$C(x,t) = \frac{C_0}{2} \int_0^\infty H_{1,1}^{0,1}\left(\sigma \,\middle|\, \begin{matrix} (1-\alpha,\alpha) \\ (0,1) \end{matrix}\right) [\mathrm{erfc}(\eta_1)] d\sigma. \tag{31}$$

The function

$$H_{1,1}^{1,0}\left(z \,\middle|\, \begin{matrix} (1-\alpha,\alpha) \\ (0,1) \end{matrix}\right)$$

is a probability density. From Eqs (59), (60) and (56), we have

$$H_{1,1}^{1,0}\left(z \,\middle|\, \begin{matrix} (1-\alpha,\alpha) \\ (0,1) \end{matrix}\right) = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(1-\alpha-\alpha k)} \frac{z^k}{k!}, \qquad 0 < \alpha \leq 1. \tag{32}$$

5. Conclusions

The time fractional advection-dispersion equation is obtained from the classical advection-dispersion equation by replacing the first-order time derivative by a fractional derivative of order $\alpha (0 < \alpha \leq 1)$.Using variable transformation, the intermediate steps of Mellin and Laplace transforms, we derive the complete solution of this time fractional advection-dispersion equation. Its Green function includes a probability density function and a complementary error function.

Acknowledgements

This research has been supported by the ARC SPIRT grant C10024101 and the National Natural Science Foundation of China 10271098.

Appendix: Preliminaries

For the reader's convenience we present here certain ideas and the essential notions and notations concerning the Mellin and Laplace transforms, which are used in the paper.

Appendix A: The Mellin transform

The Mellin transform of a sufficiently well-behaved function φ on R^+ is defined as follows (Samko, Kilbas and Marichev, 1993):

$$\varphi^*(s) = M\{\varphi(t); s\} = \int_0^\infty t^{s-1}\varphi(t)dt \tag{33}$$

and its inverse is given by the formula

$$\varphi^*(t) = M^{-1}\{\varphi(t)^*(s); t\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \varphi^*(s)t^{-s}ds, \qquad \gamma = \mathrm{Re}(s). \tag{34}$$

The Mellin convolution relation is defined as

$$(h \circ \varphi)(t) = \int_0^\infty h(\frac{t}{z})\varphi(z)\frac{dz}{z}. \tag{35}$$

The Mellin convolution theorem with respect to (35) takes the form

$$(h \circ \varphi)^*(s) = h^*(s)\varphi^*(s). \tag{36}$$

Substituting the expression (36) instead of $\varphi^*(s)$ in (34) and taking (35) into account we obtain the Parseval relation

$$\int_0^\infty h(\frac{t}{z})\varphi(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} h^*(s)\varphi^*(s)t^{-s}ds. \tag{37}$$

If we denote by $\stackrel{M}{\longleftrightarrow}$ the correspondence between a function and its Mellin transform, then the following relations of the general types hold:

$$\varphi(at) \stackrel{M}{\longleftrightarrow} a^{-s}\varphi^*(s), \qquad a > 0, \tag{38}$$

$$t^a\varphi(t) \stackrel{M}{\longleftrightarrow} \varphi^*(s+a), \tag{39}$$

$$\varphi(t^m) \stackrel{M}{\longleftrightarrow} \frac{1}{|m|}\varphi^*(\frac{s}{m}), \qquad m \neq 0, \tag{40}$$

$$exp(-at^h - bt^{-h}) \stackrel{M}{\longleftrightarrow} \frac{2}{h}(\frac{b}{a})^{\frac{s}{2h}} K_{\frac{s}{h}}(2\sqrt{ab}),$$
$$\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, h > 0, \tag{41}$$

$$(t^2+b^2)^{-\frac{1}{2}\nu}K_\nu[a(t^2+b^2)^{\frac{1}{2}}] \stackrel{M}{\longleftrightarrow} a^{-\frac{s}{2}}2^{\frac{s}{2}-1}b^{\frac{s}{2}-\nu}\Gamma(\frac{s}{2})K_{\nu-\frac{s}{2}}(ab),$$

$$\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(s) > 0, \tag{42}$$

$$\int_0^\infty h(\frac{t}{z})\varphi(z)\frac{dz}{z} \overset{M}{\longleftrightarrow} h^*(s)\varphi^*(s), \tag{43}$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind (*Erde'lyi et al.*, 1954). The Mellin transform formulae of some functions and the properties of the Mellin transform can be found in *Erde'lyi et al.* (1954), Samko, Kilbas and Marichev (1993).

Appendix B: The Laplace transform

The Laplace transform of a function $\varphi(t), 0 < t < \infty$, is defined as follows (Samko, Kilbas and Marichev, 1993):

$$\widetilde{\varphi}(p) = L\{\varphi(t)\} = L\{\varphi(t); p\} = \int_0^\infty e^{-pt}\varphi(t)dt, \tag{44}$$

and its inverse is given by the formula

$$\varphi(t) = L^{-1}\{\widetilde{\varphi}(p); t\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\widetilde{\varphi}(p)dp, \quad \gamma = \mathrm{Re}(p) > p_0. \tag{45}$$

It follows that

$$L\{t^\nu; p\} = \frac{\Gamma(\nu+1)}{p^{\nu+1}}, \qquad \nu > -1. \tag{46}$$

One of the most useful properties of the Laplace transform is embodied in the convolution theorem (Churchill, 1944), which states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. Thus if $\widetilde{h}(s)$ and $\widetilde{\varphi}(s)$ are the Laplace transforms of $h(t)$ and $\varphi(t)$, respectively, then

$$L\{\int_0^t h(t-\tau)\varphi(\tau)d\tau\} = \widetilde{h}(p)\widetilde{\varphi}(p). \tag{47}$$

Now if φ is continuous, the fractional integral of order α of φ is

$$D^{-\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)d\tau, \qquad \alpha > 0 \tag{48}$$

which is a convolution integral. From (46), (47) and (48), we have

$$L\{D^{-\alpha}\varphi(t)\} = \frac{1}{\Gamma(\alpha)}L\{t^{\alpha-1}\}L\{\varphi(t)\} = p^{-\alpha}\widetilde{\varphi}(p), \qquad \alpha > 0. \tag{49}$$

The Mellin and Laplace transforms are related to each other by (Schneider and Wyss, 1989)

$$\varphi^*(s) = \frac{1}{\Gamma(1-s)}\int_0^\infty p^{-s}\widetilde{\varphi}(p)dp. \tag{50}$$

Appendix C: The H-function

An H-function is defined in terms of a Mellin-Bernes type integral as follows (Mathai and Saxena, 1978, Anh and Lenenko, 2001):

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_1,\alpha_1)\dots(a_p,\alpha_p) \\ (b_1,\beta_1)\dots(b_q,\beta_q) \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_L H^{*m,n}_{\ p,q}(s) z^{-s} ds, \quad (51)$$

where m, n, p and q are nonnegative integers such that $0 \leq n \leq p, 1 \leq m \leq q$ and empty products are interpreted as unity. The parameters $\alpha_1, ..., \alpha_p$ and $\beta_1, ..., \beta_q$ are positive real numbers, whereas $a_1, ..., a_p$ and $b_1, ..., b_q$ are complex numbers. The Fox or H-function is characterized by its Mellin transform:

$$H^{*m,n}_{\ p,q}(s) = H^{*m,n}_{\ p,q}\left[z \left| \begin{matrix} (a_j,\alpha_j) j = 1,\cdots,p \\ (b_i,\beta_i) i = 1,\cdots,q \end{matrix} \right.\right](s) = \frac{A(s)B(s)}{C(s)D(s)}, \quad (52)$$

where

$$A(s) = \prod_{j=1}^{m}\Gamma(b_j + \beta_j s), \quad B(s) = \prod_{j=1}^{m}\Gamma(1 - a_j - \alpha_j s),$$
$$C(s) = \prod_{j=m+1}^{q}\Gamma(1 - b_j - \beta_j s), \quad D(s) = \prod_{j=n+1}^{p}\Gamma(a_j + \alpha_j s). \quad (53)$$

In (51) $z^{-s} = exp\{-s\cdot\log|z| - i\arg(z)\}$ and $\arg(z)$ is not necessarily the principal value. The parameters are restricted by the condition $P(A) \cap P(B) = \emptyset$, where

$$P(A) = \{\text{poles of } \Gamma(1-a_i+\alpha_i s)\} = \{\frac{1 - a_i + k}{\alpha_i} \in C; i = 1,\cdots,n, k \in N_0\},$$

$$P(B) = \{\text{poles of } \Gamma(b_i + \beta_i s)\} = \{\frac{-b_i - k}{\beta_i} \in C; i = 1,\cdots,m, k \in N_0\},$$

$$N_0 = \{0, 1, \cdots\}.$$

The integral (51) converges if one of the following conditions holds (Hilfer, 2000; Anh and Lenenko, 2001):

$$L = L(c-i\infty, c+i\infty; P(A), P(B)), |argz| < \frac{w\pi}{2}, w > 0;$$

$$L = L(c-i\infty, c+i\infty; P(A), P(B)), |argz| < \frac{w\pi}{2}, w \geq 0, cR < -\mathrm{Re}(\gamma);$$

where

$$w = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=1}^{q}\beta_j,$$

$$R = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j,$$

$$\gamma = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2} + 1.$$

The H-functions are analytic for $z \neq 0$ and multivalued (single-valued on the Riemann surface of log$z$). The H-functions may be represented as (Hilfer, 2000; Anh and Lenenko, 2001):

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{c} (a_1,\alpha_1)\cdots(a_p,\alpha_p) \\ (b_1,\beta_1)\cdots(b_q,\beta_q) \end{array} \right.\right] = \sum_{i=1}^{m}\sum_{k=0}^{\infty} c_{ik}\frac{(-1)^k}{k!\beta_i} z^{\frac{b_i+k}{\beta_i}} \tag{54}$$

where

$$c_{ik} = \frac{\prod_{j=1,j\neq i}^{m} \Gamma(b_j - (b_i+k)\frac{\beta_j}{\beta_i}) \prod_{j=1}^{n} \Gamma(1-a_j+(b_i+k)\frac{\alpha_j}{\beta_i})}{\prod_{j=m+1}^{q} \Gamma(1-b_j+(b_i+k)\frac{\beta_j}{\beta_i}) \prod_{j=n+1}^{p} \Gamma(a_j-(b_i+k)\frac{\alpha_j}{\beta_i})} \tag{55}$$

whenever $R \geq 0$ and the poles in $P(A)$ are simple. Similarly,

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{c} (a_1,\alpha_1)...(a_p,\alpha_p) \\ (b_1,\beta_1)...(b_q,\beta_q) \end{array} \right.\right] = \sum_{i=1}^{m}\sum_{k=0}^{\infty} c_{ik}\frac{(-1)^k}{k!\alpha_i} z^{-\frac{1+a_i+k}{\alpha_i}}, \tag{56}$$

where

$$c_{ik} = \frac{\prod_{j=1}^{m} \Gamma(b_j + (1-a_i+k)\frac{\beta_j}{\alpha_i}) \prod_{j=1,j\neq i}^{n} \Gamma(1-a_j+(1-a_i+k)\frac{\alpha_j}{\alpha_i)})}{\prod_{j=m+1}^{q} \Gamma(1-b_j+(1-a_i+k)\frac{\beta_j}{\alpha_i}) \prod_{j=n+1}^{p} \Gamma(a_j+(1-a_i+k)\frac{\alpha_j}{\alpha_i})} \tag{57}$$

whenever $R \leq 0$ and the poles in $P(A)$ are simple.

In particular, if $R > 0$, we obtain from (54) that

$$\begin{aligned} H_{p,q}^{1,n}(z) &= H_{p,q}^{1,n}\left[z \left| \begin{array}{c} (a_1,\alpha_1)...(a_p,\alpha_p) \\ (b_1,\beta_1)...(b_q,\beta_q) \end{array} \right.\right] \\ &= \frac{1}{\beta_1}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B(s_k)}{C(s_k)D(s_k)} z^{-s_k}, \end{aligned} \tag{58}$$

where $s_k = -b_1 + k/\beta_1$. In the case $R < 0$, we obtain from (54) that

$$\begin{aligned} H_{p,q}^{m,1}(z) &= H_{p,q}^{m,1}\left[z \left| \begin{array}{c} (a_1,\alpha_1)...(a_p,\alpha_p) \\ (b_1,\beta_1)...(b_q,\beta_q) \end{array} \right.\right] \\ &= \frac{1}{\alpha_1}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{A(s_k)}{C(s_k)D(s_k)} z^{-s_k}, \end{aligned} \tag{59}$$

where $s_k = (k+1-\alpha_1)/\alpha_1$.

The following identities for the H-function are well-known:

$$z^k H_{p,q}^{m,n}\left[z\left|\begin{array}{l}(a_1,\alpha_1)...(a_p,\alpha_p)\\(b_1,\beta_1)...(b_q,\beta_q)\end{array}\right.\right]$$
$$= H_{p,q}^{m,n}\left[z\left|\begin{array}{l}(a_1+k\alpha_1,\alpha_1)...(a_p+k\alpha_p,\alpha_p)\\(b_1+k\beta_1,\beta_1)...(b_q+k\beta_q,\beta_q)\end{array}\right.\right], \tag{60}$$

$$H_{p,q}^{m,n}\left[z\left|\begin{array}{l}(a_1,\alpha_1)...(a_p,\alpha_p)\\(b_1,\beta_1)...(b_q,\beta_q)\end{array}\right.\right]$$
$$= kH_{p,q}^{m,n}\left[z^k\left|\begin{array}{l}(a_1,k\alpha_1)...(a_p,k\alpha_p)\\(b_1,k\beta_1)...(b_q,k\beta_q)\end{array}\right.\right]. \tag{61}$$

References

1. J.M. Angulo, M.D. Ruiz-Medina, V.V. Anh and W. Grecksch, *Fractional diffusion and fractional heat equation*, Adv. Appl. Prob. **32**, (2000), 1077-1099.
2. V.V. Anh and N.N. Leonenko, *Scaling laws for fractional diffusion-wave equations with singular data*, Statistics and Probability Letters, **48**, (2000), 239-252.
3. V.V. Anh and N.N. Leonenko, *Spectral analysis of fractional kinetic equations with random data*, J. Statist. Phys., **104**, (2001), 1349-1387.
4. M. Basu and D.P. Acharya, *On quadratic fractional generalized solid bi-criterion, he Korean J. Comp. Appl. Math.,* **2**, *2002, 131-144.*
5. *D.A. Benson., S. W. Wheatcraft and M. M. Meerschaert, Application of a fractional advection-despersion equation, Water Resour. Res.* **36(6)**, *(2000a), 1403-1412.*
6. *D.A. Benson, S. W. Wheatcraft and M. M. Meerschaert, The fractional-order governing equation of Levy motion, Water Resour. Res.,* **36(6)**, *(2000b), 1413-1423.*
7. *P. Biler, T. Funaki, W.A. Woyczynski, Fractal Burgers equation, I. Differential Equations,* **147**, *(1998), 1-38.*
8. *M. Caputo, The Green function of the diffusion of fluids in porous media with memory, Rend. Fis. Acc. Lincei (ser. 9),* **7**, *(1996), 243-250.*
9. *A.M.A. El-Sayed and M.A.E. Aly, Continuation theorem of fractionalorder evolutionary integral equations, the Korean J. Comp. Appl. Math.,* **2**, *2002, 525-534.*
10. *A. Erdelyi, Tables of Integral Transforms, Vol. I, McGraw-Hill, New York, 1954.*
11. *R. Giona and H.E. Roman, A theory of transport phenomena in disordered systems, Chem. Eng. J. bf 49, (1992), 1-10.*
12. *R. Gorenflo, Yu. Luchko and F. Mainardi, Analytical properties and applications of the Wright function, Fractional Calculus Appl. Anal.* **2**, *(1999), 383-414.*
13. *R. Gorenflo, Yu. Luchko and F. Mainardi, Wright function as scale-invariant solutions of the diffusion-wave equation , J. Comp. Appl. Math.* **118**, *(2000), 175-191.*
14. *R. Hilfer, Exact solutions for a class of fractal time random walks, Fractals,* **3**, *(1995), 211-216.*
15. *F. Liu, V. Anh and I. Turner, Numerical solution of the fractional-order advection-dispersion equation, Proceedings of the International Conference on Boundary and Interior Layers, Perth, Australia, (2002), 159-164.*

16. *F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, in : S. Rionero, T. Ruggeri (Eds.) , Waves and Stability in Continuous Media, World Scientific, Singapore, (1994), 246-251.*
17. *F. Mainardi, Fractional diffusive waves in viscoelastic solids in: J.L. Wagner and F.R. Norwood (Eds.), IUTAM Symposium - Nonlinear Waves in Solids, ASME/AMR, Fairfield NJ, (1995), 93-97.*
18. *F. Mainardi, Yu. Luchko and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, Fractional Calculus Appl. Anal.,* **4***, (2001).*
19. *K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley, New York, 1993.*
20. *K.B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, 1974.*
21. *I. Podlubny, Fractional Differential Equations, Academic Press, 1999.*
22. *A. Saichev and G. Zaslavsky, Fractional kinetic equations: solutions and applications, Chaos,* **7***, (1997), 753-764.*
23. *S.G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Newark, N J, 1993.*
24. *W.R. Schneider and W. Wyss, Fractional diffusion and wave equations, J. Math. Phys.* **30***, (1989), 134-144.*
25. *W. Wyss, The fractional diffusion equation, J. Math. Phys., 27, (1986), 2782-2785.*
26. *W. Wyss, The fractional Black-Scholes equation, Fractional Calculus Appl. Anal.* **3***, (2000), 51-61.*

Fawang Liu *received his MSc from Fuzhou University in 1982 and PhD from Trinity College, Dublin, in 1991. Since graduation, he has been working in computational and applied mathematics at Fuzhou University, Trinity College Dublin and University College Dublin, University of Queensland, Queensland University of Technology and Xiamen University. Now he is a Professor at Xiamen University. His research interest is numerical analysis and techniques for solving a wide variety of problems in applicable mathematics, including semiconductor device equations, microwave heating problems, gas-solid reactions, singular perturbation problem, saltwater intrusion into aquifer systems and fractional differential equations.*

(1) Department of Mathematics, Xiamen University, Xiamen 361005, China (2) School of Mathematical Sciences, Queensland University of Technology, Qld. 4001, Australia.
e-mail: fwliu@xmu.edu.cn

Vo Anh *received his PhD degree from the University of Tasmania, Australia, in 1978. He has been with Queensland University of Technology since 1984. His research interests include stochastic processes and random fields, fractional diffusion, environmental modelling financial modelling.*

School of Mathematical Sciences, Queensland University of Technology, Qld. 4001, Australia.
e-mail: v.anh@qut.edu.au

Ian Turner *is a senior lecturer at the School of Mathematical Science, QUT. He has extensive experience in the solution of systems of non-linear partial differential equations using the finite volume discretisation process and has written numerous journal publications in the field. He has been awarded outstanding paper awards from two international journals for his modelling work, with the most significantcontribution being the use of mathematical models for furthering the understanding of how microwaves interact with lossy materials during heating and drying processes. He also has considerable expertise in*

solving large sparse non-linear and linear systems via preconditioned Krylov based methods.

School of Mathematical Sciences, Queensland University of Technology, Qld. 4001, Australia.
e-mail: i.turner@qut.edu.au

Zhuang Pinghui *received his BSc and MSc from Fuzhou University in 1982 and 1988 respectively. Since graduation, he has been working in computational and applied mathematics at Xiamen University. Now he is an associate professor. His research interest is numerical analysis and techniques for solving singular perturbation problem and fractional differential equations, numerical simulation for saltwater intrusion into aquifer systems and computing.*

Department of Mathematics, Xiamen University, Xiamen 361005, China.
e-mail: zxy1104@xmu.edu.cn