ON SEMILINEAR STOCHASTIC FRACTIONAL DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

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Abstract. This paper introduces a class of semilinear stochastic fractional differential equations of Volterra type. The existence and uniqueness of their solutions is proved and some basic properties of the solutions are studied. A simulation scheme is proposed which converges uniformly in mean square for a special, but important, case.

1. Introduction

Fractional differential equations with noise input were studied in Gay and Heyde [14], Viano *et al.* [23], for example, in the Gaussian case. Okabe [20] and Inoue [15] considered a class of equations which contain the Stokes-Boussinesq-Langevin equation from hydrodynamics; some of these equations may be formulated in terms of fractional derivatives (Mainardi [18]). Anh *et al.* [2] proposed a class of fractional differential equations driven by Lévy noise and considered possible applications to finance and macroeconomics. Anh and McVinish [3] studied the sample path properties of this latter class of models and proposed a simulation scheme. Related to these models is the class of stochastic differential equations driven by fractional Brownian motion (Lin [17], Kleptsyna *et al.* [16]).

One motivating factor in the introduction of fractional derivatives in stochastic equations is to induce long memory into the dynamics of the resulting process. This is different from the approach in which long memory arises through the noise term as in stochastic differential equations driven by fractional Brownian motion with Hurst index $H > 1/2$. However, in all these models, the process is not assumed to display conditional heteroscedasticity, which is also an important property of observed data in many applications (Barndorff-Nielsen [5]).

In this paper we present a class of semilinear stochastic fractional differential equations obtained from the one-dimensional Itô stochastic differential equation by replacing the first-order time derivative by a linear fractional (in time) operator. Long memory is induced by the linear fractional operator under some condition (see Remark 1 below). On the other hand, conditional heteroscedasticity may be realised via the volatility factor of these equations. The existence and uniqueness of their solutions is proved. The main result of the paper is a representation theorem from which an approximation scheme is proposed to simulate the sample paths of the process for a special but important case. Uniform convergence (in probability) of this approximation scheme is proved using a combination of results from stochastic calculus and pathwise integration.

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2. Preliminaries

For a suitably regular function $f(t)$, its Riemann-Liouville fractional derivative is defined as

$$\mathcal{D}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{2.1}$$

$\alpha \in [n-1, n), n = 1, 2, \ldots$, and its Riemann-Liouville fractional integral is defined as

$$\mathcal{I}^{\alpha} f(t) = \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0. \tag{2.2}$$

(Samko *et al.* [22], Djrbashian [12], Podlubny [21]). We will widely use the notion of fractional Green function of a deterministic fractional differential equation of the form

$$\mathcal{L}y(t) = f(t), \tag{2.3}$$

where the linear differential operator $\mathcal{L}$ with constant coefficients is given by

$$\mathcal{L}y(t) = \left(A_n \mathcal{D}_t^{\beta_n} + \ldots + A_1 \mathcal{D}_t^{\beta_1} + A_0 \mathcal{D}_t^{\beta_0} \right) y(t), \tag{2.4}$$

$$\beta_n > \beta_{n-1} > \ldots > \beta_1 > \beta_0, \ n \geq 1. \tag{2.5}$$

As defined in Podlubny [21], p.150, the function $G(t-\tau)$ satisfying the following conditions:

a) $\mathcal{L}G(t-\tau) = 0$ for every $\tau \in (0, t)$;
b) $\lim_{\tau \to t-0} \mathcal{D}_t^{\beta_k - 1} G(t-\tau) = \delta_{k,n}$, $k = 0, 1, \ldots, n$, $\delta_{k,n}$ being the Kronecker delta;
c) $\lim_{\tau, t \to 0+, \tau < t} \mathcal{D}_t^{\beta_k} G(t-\tau) = 0$, $k = 0, 1, \ldots, n-1$

is called the Green function of equation (2.3). It was shown that this Green function is given by

$$\begin{aligned} G(t) &= \frac{1}{A_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0 + \ldots + k_{n-2} = m \\ k_0 \geq 0, \ldots, k_{n-2} \geq 0}} (m; k_0, \ldots, k_{n-2}) \\ &\quad \times \prod_{i=0}^{n-2} \left(\frac{A_i}{A_n} \right)^{k_i} t^{(\beta_n - \beta_{n-1})m + \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j - 1} \\ &\quad \times E^{(m)}_{\beta_n - \beta_{n-1}, \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j} \left(-\frac{A_{n-1}}{A_n} t^{\beta_n - \beta_{n-1}} \right), \end{aligned} \tag{2.6}$$

where $(m; k_0, \ldots, k_{n-2})$ denotes multinomial coefficients (Podlubny [21], p. 158) and $E_{\rho,\mu}(x)$ is the two-parameter Mittag-Leffler function (Djrbashian [12], pp. 1-6), which can be defined by the series expansion

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \ z \in \mathbb{C}, \ \rho > 0, \ \mu > 0. \tag{2.7}$$

Its k-th derivative $E^{(k)}_{\rho,\mu}(z)$ is obtained as

$$E^{(k)}_{\rho,\mu}(z) = \frac{d^k}{dz^k} E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(j+k)! z^j}{j! \Gamma(\rho j + \rho k + \mu)}. \tag{2.8}$$

The Green function solution of (2.3) is then given by

$$y(t) = \int_0^t G(t-s) f(s) \, ds \tag{2.9}$$

if the integral (2.9) exists. We should note that the Green function $G(t)$, $t \geq 0$, can also be given in terms of its Laplace transform

$$g(p) = \int_0^\infty e^{-pt} G(t) \, dt = \left(A_n p^{\beta_n} + \ldots + A_0 p^{\beta_0}\right)^{-1}, \quad p > 0. \tag{2.10}$$

Remark 1. *Anh* et al. [2] *consider $f(t)$ in (2.3) as Lévy noise, that is, the derivative in the distributional sense of Lévy motion $L(t)$. The integral then exists as an L_2- stochastic integral if $EL^2(1) < \infty$ and $\beta_n > 1/2$. If $\beta_0 < 1/2$ in addition to $\beta_n > 1/2$, we have then, as $t \to \infty$, that $X(t)$ converges to an asymptotic stationary solution*

$$X(t) = \int_{-\infty}^t G(t-s) \, dL(s) \tag{2.11}$$

with spectral density

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{2\pi} |g(i\omega)|^2 \\ &= \frac{\sigma^2}{2\pi} \frac{1}{\sum_{j=0}^n \sum_{k=0}^n A_j A_k |\omega|^{\beta_j + \beta_k} \cos\left(\pi\left(\beta_j - \beta_k\right)/2\right)}. \end{aligned} \tag{2.12}$$

It is clear that as $|\omega| \to 0$ the spectral density behaves as $O\left(|\omega|^{-2\beta_0}\right)$ and hence the solution possesses long-range dependence for $\beta_0 > 0$. As $|\omega| \to \infty$ the spectral density behaves as $O\left(|\omega|^{-2\beta_n}\right)$. The index β_0 in $O\left(|\omega|^{-2\beta_0}\right)$ is the Hurst index, while β_n in $O\left(|\omega|^{-2\beta_n}\right)$ is a fractal index, which indicates the degree of fractality of a path. In fact, the order $O\left(|\omega|^{-2\beta_n}\right)$ as $\omega \to \infty$ will specify the Hausdorff dimension of the path via an Abelian-Tauberian-type theorem (Bingham [6]*, Adler* [1]*, p. 204).*

Remark 2. *For $n = 1$ the fractional Green function is*

$$G(t) = A_1^{-1} t^{\beta_1 - 1} E_{\beta_1 - \beta_0, \beta_1}\left(\frac{-A_0}{A_1} t^{\beta_1 - \beta_0}\right) \mathbf{1}_{(0,\infty)}(t). \tag{2.13}$$

An integral representation of (2.13) can be determined from

$$E_{\rho,\mu}(-x^\rho) x^{\mu-1} = \frac{1}{\pi} \int_0^\infty \frac{\sin(\pi(\mu-\rho)) + \tau^\rho \sin(\pi\mu)}{1 + 2\cos(\pi\rho)\tau^\rho + \tau^{2\rho}} \tau^{\rho-\mu} e^{-x\tau} d\tau \tag{2.14}$$

for $\rho < 1$, $\mu \in (0, 1+\rho)$ and $x \in (0,\infty)$ (see Djrbashian [12]*). Hence, if $n = 1, \beta_1 \in [0,1]$ and $\beta_0 \in [0, \beta_1)$, the fractional Green function is completely monotonic. Anh and McVinish* [4] *extended this result to general n. They note that $G(t)$ is completely monotonic if and only if $\beta_n \leq 1$. If $n = 1$ and $(\beta_1, \beta_0) = (1, 0)$, then the inverse Laplace transform of $G(t)$ is $\delta(\lambda - A_0/A_1)/A_1$; otherwise it is given by*

$$\mu(d\lambda) = \frac{1}{\pi} \left[\frac{\sum_j A_j \lambda^{\beta_j} \sin\left(\pi\beta_j\right)}{\sum_k \sum_j A_k A_j \lambda^{\beta_k + \beta_j} \cos\left(\pi\left(\beta_k - \beta_j\right)\right)}\right] d\lambda. \tag{2.15}$$

A related property is hyperbolic complete monotonicity (Bondesson [7]*). A function* $f(\cdot)$ *is said to be hyperbolically completely monotone if the function* $f(uv)\, f(u/v)$ *is completely monotone in* $v+v^{-1}$ *for all* u*. From Theorem 4 of Anh and McVinish* [4] *it follows that the fractional Green function is hyperbolically completely monotone if* $\beta_n \leq 1/2$*. The completely monotone property of the fractional Green function will be used extensively throughout this paper. The hyperbolic completely monotone property will not be used further.*

3. Semilinear stochastic fractional differential equations

The class of processes we are concerned with is defined by the semilinear stochastic fractional differential equation

$$A_n\mathcal{D}^{\beta_n}X_t + \ldots + A_0\mathcal{D}^{\beta_0}X_t = b(t,X_t) + \sigma(t,X_t)\,\dot{W}_t, \tag{3.1}$$

where $\dot{W}_t$ is Gaussian white noise and b,σ satisfy the standard Lipschitz and linear growth conditions:

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K\,|x-y|\,, \tag{3.2}$$

$$|b(t,x)|^2 + |\sigma(t,x)|^2 \leq K^2\left(1+|x|^2\right). \tag{3.3}$$

We will also assume that there exist x_0 and y_0 such that $b(.,x_0)$ and $\sigma(.,y_0)$ belong to $\mathcal{L}_2$. The solution to (3.1) is interpreted as the solution to the integral equation

$$A_nX_t + A_{n-1}\mathcal{I}^{\beta_n-\beta_{n-1}}X_t + \ldots + A_0\mathcal{I}^{\beta_n-\beta_0}X_t = \mathcal{I}^{\beta_n}b(t,x_t) + \mathcal{I}^{\beta_n}\sigma(t,X_t)\,\dot{W}_t. \tag{3.4}$$

We follow Coutin and Decreusefond [10] and state that, by a solution to (3.4), we mean a real-valued, progressively measurable stochastic process $X = \{X_t, t\in I\}$ such that X belongs to $L^{1/H}(\Omega\times I, P\otimes dt)$, and for any t, X_t is almost surely a solution of (3.4). Here, if $\beta_n \geq 1$, then H is taken to be $1/2$; otherwise, if $\beta_n \in (1/2,1)$, then $H = 1/(\beta_n - 1/2)$. It is necessary to have $\beta_n > 1/2$ for the stochastic integral in (3.4) to exist.

The following theorem is a direct consequence of Coutin and Decreusefond [10], Decreusefond [11].

Theorem 1. *Under the above assumptions, there exists a unique continuous solution to the semilinear stochastic fractional differential equation (3.1) if* $\beta_n > 1/2$*.*

The rest of this paper focuses on the properties of the case $\beta_n = 1$. This is an important case since the solution will then have the semimartingale representation provided $\beta_{n-1} < 1/2$.

Theorem 2. *If* $\beta_n = 1, \beta_{n-1} < 1/2$*, then the solution to (3.1) has the semimartingale representation.*

Proof. Obviously, $\mathcal{I}^1\sigma(t,X_t)\,\dot{W}_t$ is a local martingale and $\mathcal{I}^1 b(t,X_t)$ is of bounded variation. We define a local Hölder exponent of X_t as

$$h_X(t) = \sup\left\{l : |X_t - X_s| \leq C\,|t-s|^l\right\} \tag{3.5}$$

for s sufficiently close to t. From Theorem 5.2 of Dudley and Norvaiša [13], any local martingale has finite $(2+\varepsilon)$-variation and hence from Lemma 4.3 of Dudley and Norvaiša

[13], the local Hölder exponent of a local martingale is greater than or equal to 1/2, almost everywhere. Lemma 1 of Anh and McVinish [3] then yields that, if $Y_t = \mathcal{I}^{\alpha} X_t$,

$$h_Y(t) \geq \min\left(h_X(t) + \alpha, 1 + \alpha\right). \tag{3.6}$$

Thus X_t has local Hölder exponents greater than or equal to 1/2, almost everywhere. If $\beta_{n-1} < 1/2$, then $\mathcal{I}^{1-\beta_i} X_t, i = 0, \ldots, n-1$, have local Hölder exponents greater than one, almost everywhere. It follows that $\mathcal{I}^{1-\beta_i} X_t, i = 0, \ldots, n-1$, is almost everywhere differentiable and hence of bounded variation. As a result, X_t has the semimartingale representation. ∎

4. Representation theorem

In this section we make use of the idea presented in Carmona and Coutin [8], Carmona *et al.* [9] in representing convolutions involving completely monotone functions as linear combinations of convolutions involving exponential functions. In the context of Gaussian processes with completely monotone autocorrelation functions, this amounts to representing the process as a linear combination of Ornstein-Uhlenbeck processes. This type of representation can then be used to derive a simulation scheme for the process. We first note that any solution of (3.4) must satisfy the integral equation

$$X_t = \int_0^t G(t-s)\, b(s, X_s)\, ds + \int_0^t G(t-s)\, \sigma(s, X_s)\, dW_s, \tag{4.1}$$

where $G(t)$ is the Green function of the linear fractional differential equation (3.1). The following proposition shows that we may replace the Green function by some suitable approximation (see the Appendix for the definition of the spaces $\mathcal{W}_p$ and other notations).

Proposition 1. *Define $\widetilde{X}_t$ as the solution to the integral equation*

$$\widetilde{X}_t = \int_0^t \tilde{G}(t-s)\, b\left(s, \tilde{X}_s\right) ds + \int_0^t \tilde{G}(t-s)\, \sigma\left(s, \tilde{X}_s\right) dW_s; \tag{4.2}$$

then $\tilde{X}_t \to X_t$ uniformly in mean square on $[0,T]$ if $\tilde{G} \to G$ in $\mathcal{W}_1$.

Proof. Define the martingales

$$M_t^{(1)} = \int_0^t \sigma(s, X_s)\, dW_s, \tag{4.3}$$

$$M_t^{(2)} = \int_0^t \left[\sigma(s, X_s) - \sigma\left(s, \tilde{X}_s\right)\right] dW_s. \tag{4.4}$$

From the triangle inequality, we get

$$\begin{aligned} \left|X_t - \tilde{X}_t\right| \leq\; & \left|\int_0^t \left[G(t-s) - \tilde{G}(t-s)\right] dM_s^{(1)}\right| + \left|\int_0^t \tilde{G}(t-s)\, dM_s^{(2)}\right| \\ & + \left|\int_0^t \left[\tilde{G}(t-s) - G(t-s)\right] b(s, X_s)\, ds\right| \\ & + \left|\int_0^t \tilde{G}(t-s)\left[b\left(s, \tilde{X}_s\right) - b(s, X_s)\right] ds\right|. \end{aligned} \tag{4.5}$$

We note as in Mikosch and Norvaiša [19] that, when they exist, the Itô stochastic integral and Riemann-Stieltjes integral take the same value. Hence, the Itô integrals in (4.5) may be interpreted as Riemann-Stieltjes integrals if the martingales have finite norm $\| \, . \, \|_{(\infty)}$. As this is true for martingales, we may apply the Love-Young inequality to get

$$\left| \int_0^t \left[G(t-s) - \tilde{G}(t-s) \right] dM_s^{(1)} \right| \leq \| G - \tilde{G} \|_{[1]} \sup_{0 \leq s \leq t} \left| M_s^{(1)} \right| \tag{4.6}$$

and

$$\left| \int_0^t \tilde{G}(t-s) \, dM_s^{(2)} \right| \leq \| \tilde{G} \|_{[1]} \sup_{0 \leq s \leq t} \left| M_s^{(2)} \right|. \tag{4.7}$$

In a similar fashion, we obtain for the Lebesgue integrals in (4.5)

$$\begin{aligned} \left| X_t - \tilde{X}_t \right| \quad \leq \quad & \| G - \tilde{G} \|_{[1]} \left(\sup_{0 \leq s \leq t} \left| M_s^{(1)} \right| + ct + c \int_0^t |X_u| \, du \right) + \| \tilde{G} \|_{[1]} \sup_{0 \leq s \leq t} \left| M_s^{(2)} \right| \\ & + \sup_{0 \leq s \leq t} \tilde{G}(s) \int_0^t \left| \tilde{X}_s - X_s \right| ds. \end{aligned} \tag{4.8}$$

From the Burkhölder inequality and the conditions on $\sigma(t, x)$,

$$E \left[\sup_{0 \leq s \leq t} \left| M_s^{(2)} \right|^2 \right] \leq \Lambda \int_0^t E \left| \sigma\left(s, \tilde{X}_s\right) - \sigma(s, X_s) \right|^2 ds \leq \Lambda c \int_0^t E \left| \tilde{X}_s - X_s \right|^2 ds \tag{4.9}$$

and

$$E \left[\sup_{0 \leq s \leq t} \left| M_s^{(1)} \right|^2 \right] \leq \Lambda \int_0^t E \left| \sigma(s, X_s) \right|^2 ds \leq \Lambda c \left(1 + \int_0^t E \left| X_s \right|^2 ds \right). \tag{4.10}$$

An application of Hölder's inequality then yields that

$$\begin{aligned} E \left(\sup_{0 \leq s \leq T} \left| \tilde{X}_s - X_s \right|^2 \right) \quad \leq \quad & c \left(\| G - \tilde{G} \|_{[1]}^2 \left[\int_0^T E \left| X_s \right|^2 ds + T \right] \right. \\ & \left. + \| \tilde{G} \|_{[1]}^2 \int_0^T E \left(\sup_{0 \leq u \leq T} \left| \tilde{X}_s - X_s \right|^2 \right) ds \right). \end{aligned} \tag{4.11}$$

Applying the Gronwall inequality to (4.11) we have

$$E \left(\sup_{0 \leq s \leq T} \left| \tilde{X}_s - X_s \right|^2 \right) \leq C \left(\| \tilde{G} \|_{[1]}, T \right) \| G - \tilde{G} \|_{[1]}^2 \tag{4.12}$$

and hence $\tilde{X}_t \to X_t$ uniformly in mean square. ∎

From Anh and McVinish [4], the fractional Green function $G(t)$ is completely monotonic and has an integral representation of the form (2.14). We follow Carmona *et al.* [9] and approximate the integral representation by a linear combination of exponentials:

$$\tilde{G}(t) = \sum e^{-\eta_i t} \mu \left\{ [\lambda_i, \lambda_{i+1}) \right\}, \quad \eta_i \in [\lambda_i, \lambda_{i+1}). \tag{4.13}$$

Carmona *et al.* [9] employ a quadrature rule and take

$$\eta_i = \frac{\int_{\lambda_i}^{\lambda_{i+1}} \lambda \mu(d\lambda)}{\int_{\lambda_i}^{\lambda_{i+1}} \mu(d\lambda)}; \tag{4.14}$$

however in our work we will simply take $\eta_i = \lambda_i$.

Proposition 2. *Let $\tilde{G}$ be defined by (4.13); then*

$$\| G - \tilde{G} \|_{[1]} \leq 2 \left(\int_{[r^{-M}, r^{N}]^{C}} \mu(d\lambda) + (r-1) G(0) \right). \tag{4.15}$$

Proof. Let $\hat{G}(t) = \int_{r^{-M}}^{r^{N}} e^{-\lambda t} \mu(d\lambda)$. The difference $\tilde{G}(t) - \hat{G}(t)$ is zero at $t = 0$ and monotonically increasing. The difference $\tilde{G} - \hat{G}$ can be bounded as

$$\left| \int_{r^{-M}}^{r^{N}} e^{-\lambda t} \mu(d\lambda) - \sum_{j=-M}^{N} e^{-r^{j} t} \mu\left\{ \left[r^{j}, r^{j+1} \right) \right\} \right| \tag{4.16}$$

$$\leq \sum_{j=-M}^{N} \int_{r^{j}}^{r^{j+1}} e^{-r^{j} t} \left| e^{-\left(\lambda - r^{j}\right) t} - 1 \right| \mu(d\lambda) \tag{4.17}$$

$$\leq (r-1) \sum_{j=-M}^{N} e^{-r^{j} t} r^{j} t \mu\left\{ \left[r^{j}, r^{j+1} \right) \right\} \tag{4.18}$$

$$\leq (r-1) \sum_{j=-M}^{N} \mu\left\{ \left[r^{j}, r^{j+1} \right) \right\} \leq (r-1) G(0). \tag{4.19}$$

The difference $G - \hat{G}$ is monotonically decreasing with maximum at $t = 0$ given by $\int_{[r^{-M}, r^{N}]^{C}} \mu(d\lambda)$. Applying the definition of the norm then completes the proof. ∎

This allows us to write the solution to (3.1) as the solution to the following coupled system of stochastic differential equations:

$$X_t = \sum_{\pi} Z_t(\lambda_i) \mu\{[\lambda_i, \lambda_{i+1})\} \tag{4.20}$$

$$\frac{dZ_t(\lambda_i)}{dt} = -\lambda_i Z_t(\lambda_i) + b(t, X_t) + \sigma(t, X_t) \frac{dW_t}{dt} \tag{4.21}$$

subject to the initial condition

$$Z_t(\lambda_i) = 0. \tag{4.22}$$

As an application of the representation theorem, we consider the Itô formula for continuous semimartingales. Let $f(t,x) : [0, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ be of the class $C^{1,2}$. Applying the Itô formula to $\widetilde{X}_t$ we have

$$\begin{aligned} f\left(t, \widetilde{X}_t\right) &= f\left(0, \widetilde{X}_0\right) + \int_0^t f_t\left(s, \widetilde{X}_s\right) ds \\ &\quad + \sum_i \mu\{[\lambda_i, \lambda_{i+1})\} \int_0^t f_x\left(s, \widetilde{X}_s\right) b\left(s, \widetilde{X}_s\right) ds \\ &\quad - \sum_i \mu\{[\lambda_i, \lambda_{i+1})\} \int_0^t f_x\left(s, \widetilde{X}_s\right) \lambda_i Z_s(\lambda_i) ds \\ &\quad + \sum_i \mu\{[\lambda_i, \lambda_{i+1})\} \int_0^t f_x\left(s, \widetilde{X}_s\right) \sigma\left(s, \widetilde{X}_s\right) dW_s \\ &\quad + \frac{1}{2} \sum_i \mu\{[\lambda_i, \lambda_{i+1})\} \sum_j \mu\{[\lambda_j, \lambda_{j+1})\} \int_0^t f_{xx}\left(s, \widetilde{X}_s\right) \sigma\left(s, \widetilde{X}_s\right) ds. \end{aligned} \tag{4.23}$$

Note that

$$Z_t(\lambda) = \int_0^t e^{-\lambda(t-s)} b\left(s, \widetilde{X}_s\right) ds + \int_0^t e^{-\lambda(t-s)} \sigma\left(s, \widetilde{X}_s\right) dW_s \tag{4.24}$$

and, if $G' \in L_2$, then

(4.25)

$$-\sum_i \mu\{[\lambda_i, \lambda_{i+1})\} \lambda_i Z_t(\lambda_i) \overset{m.s.}{\to} \int_0^t G'(t-s) b(s, X_s) ds + \int_0^t G'(t-s) \sigma(s, X_s) dW_s$$

as $\tilde{G} \to G$. The Lebesgue integrals converge almost surely from Propositions 1 and 2. The stochastic integral converges in mean square and hence we can choose a subsequence such that the stochastic integrals converge almost surely. Thus the left- and right-hand sides of the equation below are modifications of each other. As they are also continuous, the two processes are indistinguishable. The Itô formula for the case when $\beta_n = 1$ and $\beta_{n-1} < 1/2$ is then given by

$$\begin{aligned} f(t, X_t) = {} & f(0, X_0) + \int_0^t f_t(s, X_s) ds + G(0) \int_0^t f_x(s, X_s) b(s, X_s) ds \\ & + \int_0^t f_x(s, X_s) \int_0^s G'(s-u) b(u, X_u) du ds \\ & + \int_0^t f_x(s, X_s) \int_0^s G'(s-u) \sigma(u, X_u) dW_u ds \\ & + G(0) \int_0^t f_x(s, X_s) \sigma(s, X_s) dW_s \\ & + \frac{1}{2} G^2(0) \int_0^t f_{xx}(s, X_s) \sigma(s, X_s) ds. \end{aligned} \tag{4.26}$$

When $\beta_{n-1} \geq 1/2, X_t$ is not a semimartingale and the above Itô formula does not hold. In this case we need to apply a stochastic Fubini theorem to the stochastic integral in (4.23) and (4.1). From applying this stochastic Fubini theorem we obtain

$$\int_0^t f_x\left(s, \tilde{X}_s\right) \int_0^s \tilde{G}'(s-u) \sigma\left(u, \tilde{X}_u\right) dW_u ds$$

$$= \int_0^t \int_s^t \tilde{G}'(u-s) f_X\left(u, \tilde{X}_u\right) du \sigma\left(s, \tilde{X}_s\right) dW_s \tag{4.27}$$

Taking the limit and applying the same argument as in the semimartingale case we have the Itô formula

$$\begin{aligned} f(t,X_t) &= f(0,X_0)+\int_0^t f_t(s,X_s)\,ds+G(0)\int_0^t f_x(s,X_s)\,b(s,X_s)\,ds \\ &\quad+\int_0^t f_x(s,X_s)\int_0^s G'(s-u)\,b(u,X_u)\,duds \\ &\quad+\int_0^t \sigma(s,X_s)\int_s^t G'(u-s)\,f_x(u,X_u)\,dudW_s \\ &\quad+G(0)\int_0^t f_x(s,X_s)\,\sigma(s,X_s)\,dW_s \\ &\quad+\frac{1}{2}G^2(0)\int_0^t f_{xx}(s,X_s)\,\sigma(s,X_s)\,ds. \end{aligned} \tag{4.28}$$

We note that the two Itô formulae differ only in the terms (4.26) and (4.28) which are equivalent in the semimartingale case.

5. A simulation scheme

Our simulation scheme is based on the time discretisation of the system (4.20)-(4.21). The time discretisation of (4.20) - (4.21) is carried out in the following manner:

$$Z_0^{\Delta} = 0 \tag{5.1}$$

$$Z_{n\Delta}^{\Delta}(\lambda_i) = e^{-\lambda_i\Delta}Z_{(n-1)\Delta}^{\Delta}(\lambda_i)+\Delta b\left((n-1)\Delta, X_{(n-1)\Delta}^{\Delta}\right) +\sigma\left((n-1)\Delta, X_{(n-1)\Delta}^{\Delta}\right)\left[W_{n\Delta}-W_{(n-1)\Delta}\right] \tag{5.2}$$

$$X_{n\Delta}^{\Delta} = \sum_{\pi} Z_{n\Delta}^{\Delta}(\lambda_i)\,\mu\{[\lambda_i,\lambda_{i+1})\} \tag{5.3}$$

and Z_t^{Δ} for $t\in[n\Delta,(n+1)\Delta)$ is defined as $Z_{n\Delta}^{\Delta}$. Therefore, the approximation X_t^{Δ} is a piecewise constant approximation to X_t. Note that we do not use the standard Euler approximation in the simulations as we will require λ_i to take very large values for some i.

Theorem 3. *The approximation to X_t defined above converges uniformly in mean square provided*

$$\Delta^{1/2-\varepsilon}\int_K \lambda\mu(d\lambda)+(r-1)+\int_{K^C}\mu(d\lambda)\to 0 \tag{5.4}$$

for any $\varepsilon>0$, where μ is the inverse Laplace transform of the Green function, $K=\left(r^{-M},r^N\right)$.

Proof. For notational simplicity, we will assume that $b\equiv 0$ and $\sigma(t,x)=\sigma(x)$. The necessary change to the proof required to cover the case of large values is direct. For $t=n\Delta$ we may write (5.1)-(5.2) as the Riemann-Stieltjes integral

$$Z_{n\Delta}^{\Delta}=\int_0^{n\Delta}\sum_{m=1}^{n} e^{-\lambda\Delta(n-m)}\mathbf{1}_{\{s\in((m-1)\Delta,m\Delta]\}}\sigma\left(X_s^{\Delta}\right)dW_s. \tag{5.5}$$

The difference between $Z^{\Delta}_{n\Delta}$ and $Z_{n\Delta}$ is bounded as

$$\left|Z^{\Delta}_{n\Delta}-Z_{n\Delta}\right| \leq \left|\int_0^{n\Delta} f_{\Delta}(s)\left[\sigma(X_s)-\sigma\left(X^{\Delta}_s\right)\right]dW_s\right| \tag{5.6}$$

$$+\left|\int_0^{n\Delta} f_{\Delta}(s)-e^{-\lambda(n\Delta-s)}\sigma(X_s)\,dW_s\right|, \tag{5.7}$$

where

$$f_{\Delta}(s)=\sum_{m=1}^{n} e^{-\lambda\Delta(n-m)}\mathbf{1}_{\{s\in((m-1)\Delta,m\Delta]\}}. \tag{5.8}$$

Define the martingales

$$M_1(t) = \int_0^t \left[\sigma(X_s)-\sigma\left(X^{\Delta}_s\right)\right]dW_s, \tag{5.9}$$

$$M_2(t) = \int_0^t \sigma(X_s)\,dW_s. \tag{5.10}$$

Lemma 3 of Anh and McVinish [3] states that for any $q>1$

$$\| f_{\Delta}(s)-e^{-\lambda(n\Delta-s)} \|_{[q]}\leq C\left((\lambda\Delta)^{1-1/q}+\lambda\Delta\right). \tag{5.11}$$

From applying the Love-Young inequality,

$$\left|\int_0^{n\Delta} f_{\Delta}(s)-e^{-\lambda(n\Delta-s)}dM_2(s)\right|\leq C\| M_2(T)\|_{(2+\varepsilon)}\left(\lambda\Delta+(\lambda\Delta)^{1/2-\varepsilon}\right) \tag{5.12}$$

and from Theorem 5.2 of Dudley and Norvaiša [13], $\| M_2(T) \|_{(2+\varepsilon)}$ is finite, almost surely. The second stochastic integral may be bounded using the Love-Young inequality as in the proof of Proposition 1:

$$\left|\int_0^{n\Delta} f_{\Delta}(s)\,dM_1(s)\right|\leq \sup_{0\leq s\leq t}|M_1(s)|. \tag{5.13}$$

Hence,

$$\begin{aligned}\sup_{0\leq s\leq t}\left|X^{\Delta}_s-X_s\right| &\leq C\left(\sup_{0\leq s\leq t}|M_1(s)|+\| M_2(T)\|_{(2+\varepsilon)}\sum_i \mu\{[\lambda_i,\lambda_{i+1})\}\left(\lambda\Delta+(\lambda\Delta)^{1/2-\varepsilon}\right)\right)\\ &\leq C\left(\sup_{0\leq s\leq t}|M_1(s)|+\| M_2(T)\|_{(2+\varepsilon)}\Delta^{1/2-\varepsilon}\int_K \lambda\mu(d\lambda)\right).\end{aligned}$$

Using the conditional moment inequality

$$E(|X|\mid Y<z)\leq E|X|\Pr(Y<z)^{-1}, \tag{5.14}$$

we have for any $A>0$

$$\begin{aligned}&E\left(\sup_{0\leq s\leq t}\left|X^{\Delta}_s-X_s\right|^2 \mid\| M_2(T)\|_{(2+\varepsilon)}< A\right)\\ \leq\ & C\left\{\Pr\left(\| M_2(T)\|_{(2+\varepsilon)}< A\right)^{-1}E\left(\sup_{0\leq s\leq t}|M_1(s)|^2\right)+A^2\Delta^{1-\varepsilon}\left(\int_K\lambda\mu(d\lambda)\right)^2\right\}\\ \leq\ & C\left\{\Pr\left(\| M_2(T)\|_{(2+\varepsilon)}< A\right)^{-1}\int_0^t E\left(\sup_{0\leq s\leq t}\left|X^{\Delta}_s-X_s\right|^2\right)ds+A^2\Delta^{1-\varepsilon}\left(\int_K\lambda\mu(d\lambda)\right)^2\right\}.\end{aligned}$$

The function $E\left(\sup_{0\leq s\leq t}\left|X_s^{\Delta}-X_s\right|^2 \mid\| M_2(T)\|_{(2+\varepsilon)}< A\right)$ converges uniformly to $E\left(\sup_{0\leq s\leq t}\left|X_s^{\Delta}-X_s\right|^2\right)$ as $A\to\infty$. Then, by application of the Gronwall inequality,

$$E\left(\sup_{0\leq s\leq t}\left|X_s^{\Delta}-X_s\right|^2\right)=O\left(\delta_A+A^2\Delta^{1-\varepsilon}\int_K\lambda\mu(d\lambda)\right), \tag{5.15}$$

where $\delta_A\to 0$ as $A\to\infty$. As A is assumed to be large, but otherwise arbitrary, it follows that X_t^{Δ} converges uniformly in mean square to X_t provided (5.4) holds. ∎

6. Concluding remarks

In this paper we have introduced the class of semilinear stochastic fractional differential equations which can be obtained from the Itô stochastic differential equation by replacing the time derivative by a fractional (in time) linear operator. These equations possess a unique continuous solution. This paper has focused on the special case of $\beta_n=1$ which arises as the limit of a system of Itô stochastic differential equations. This representation theorem allows us to construct a change of variable formula even when the solution is not a semimartingale. It also allows us to construct an approximation scheme which converges uniformly in mean square.

It would be interesting to study the behaviour of the Fokker-Planck equation for the system of Itô stochastic differential equations (4.20) - (4.21) and to determine if there exists a limiting “fractional diffusion partial differential equation”. Such a result would be useful for application of these processes in financial modelling and, in particular, may allow us to construct a Black-Scholes-type formula.

7. Appendix

In this paper we have made considerable use of ideas from the theory of pathwise integration. In this appendix we will review the concepts required for the proof of the representation theorem and convergence of the approximation scheme. Firstly, we note that the integral

$$\int_a^b f(s)\,dM(s) \tag{7.1}$$

can be interpreted as a pathwise integral whenever f is of bounded $p-$variation and M is of bounded $q-$variation with $p^{-1}+q^{-1}>1$. When this holds, the integral exists
(i) in the Riemann-Stieltjes sense whenever the paths of f and M have no discontinuities at the same point;
(ii) in the Moore-Pollard-Stieltjes sense whenever the paths of f and M have no discontinuities at the same point and same side;
(iii) always in the sense defined by Young.
Let π be a point partition of the interval I on which a function f is defined. The p-variation $v_p(f)$ of f is defined as

$$v_p(f)=\sup_{\pi}\sum_{i=1}^{n}\left|f(t_{i+1})-f(t_i)\right|^p. \tag{7.2}$$

Define $\| f\|_{(p)}=v_p(f)^{1/p}$ and $\| f\|_{[p]}=\| f\|_{(p)}+\sup|f|$. The space of functions with finite $\|\,.\,\|_{[p]}$ is denoted by $\mathcal{W}_p$ and $\left(\mathcal{W}_p,\|\,.\,\|_{[p]}\right)$ defines a Banach space (see Theorem 4.2 of Dudley and Norvaiša [13]). Examples of stochastic processes with finite $p-$variation are martingales with $p > 2$ and Lévy motions without Brownian component for $q > p \geq 1$ such that

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge |x|^p)\, \nu\,(dx) < \infty \tag{7.3}$$

and ν is the Lévy measure.

A fundamental inequality in the theory of pathwise integration is the Love-Young inequality (see Dudley and Norvaiša [13], Proposition 4.26):

$$\left| \int_a^b f\,(s)\, dM\,(s) \right| \leq C_{p,q} \parallel f \parallel_{[p]} \parallel M \parallel_{(q)}, \tag{7.4}$$

where $C_{p,q} = \zeta\,(p^{-1} + q^{-1})$. The above results will also hold if $(p, q) = (1, \infty)$ or $(\infty, 1)$, in which case $C_{1,\infty} = C_{\infty,1} = 1$ (see Dudley and Norvaiša [13], Theorem 4.27).

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