Fractional-Order Regularization and Wavelet Approximation to the Inverse Estimation Problem for Random Fields

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The least-squares linear inverse estimation problem for random fields is studied in a fractional generalized framework. First, the second-order regularity properties of the random fields involved in this problem are analysed in terms of the fractional Sobolev norms. Second, the incorporation of prior information in the form of a fractional stochastic model, with covariance operator bicontinuous with respect to a certain fractional Sobolev norm, leads to a regularization of this problem. Third, a multiresolution approximation to the class of linear inverse problems considered is obtained from a wavelet-based orthogonal expansion of the input and output random models. The least-squares linear estimate of the input random field is then computed using these orthogonal wavelet decompositions. The results are applied to solving two important cases of linear inverse problems defined in terms of fractional integral operators.

Key Words: Fractional generalized random field; least-squares linear estimation; multiresolution analysis; regularization; stochastic inverse problem; wavelet

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Short Title:

Fractional-Order Regularization and Wavelets

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Abstract

1. INTRODUCTION

The problem of linear estimation of an input random field from the observation of a related output random field in a real system frequently arises in different fields of application such as medical imaging, geophysical prospecting, groundwater hydrology, physical chemistry, astronomy, etc. Formally, the real system is usually represented in terms of an operator that can be linear or non-linear. Different approaches have been introduced regarding the statistical solution to the inverse problem in infinite-dimensional spaces. Among others, the study of this problem in terms of Hilbert-space stochastic processes has been carried out, for example, in [15] and [24]; a method based on Hilbert-space-valued random variables is presented in [20]; the use of generalized random variables defined on distribution spaces is considered in [19]; see also the review paper on statistical inversion methods in [10]. A wavelet-based approximation to this problem is presented in [23], where a fractal-type prior model class (*multiscale* models) for the input is considered. In [3] a different discretization of a stochastic linear inverse problem is derived, in terms of wavelet-based orthogonal expansions of the random input and the related random output. The extension of this approach to the spatio-temporal case is given in [30].

In this paper, we introduce a fractional generalized approach to the least-squares linear inverse estimation problem for random fields. This allows us to define a wider class of solutions than the one associated with the usual L^2-topology of square-integrable functions. In particular, regularization of the problem is accomplished considering a prior model class $\mathcal{P}$ given by random fields with covariance operator bicontinuous with respect to a certain fractional Sobolev norm. This class includes generalized (improper) and ordinary random fields, respectively defined by stochastic fractional-order integro-differential and differential equations (see [29]). For example, fractional Brownian motion, and related random fields such as *fractional Riesz-Bessel motion*, introduced in [6] (see also [7] and [31]), are in class $\mathcal{P}$. These random fields are defined as solutions to fractional-order differential equations in terms of fractional powers of the negative Laplacian operator. The class of random fields with covariance operator defined by a positive rational function of a self-adjoint elliptic differential operator on $L^2(\mathbb{R}^d)$ with smooth coefficients (in the stationary case, random fields with positive fractional-order rational spectra) is also included in class $\mathcal{P}$ (see, for instance, [4] and [25]).

The study of the second-order regularity properties of the input and output random fields referred to the continuous scale of fractional Sobolev spaces allows the definition of the inverse of certain integral operators as a bounded operator. This is the case, for example, of strictly positive compact self-adjoint operators with respect to the L^2-topology, with decrease rate of the eigenvalues equivalent to the one presented by embeddings between fractional Sobolev spaces. A stable solution to the corresponding least-squares linear inverse estimation problem can then be derived. Here, we consider a class of integral operators for which the regularization method presented can be applied. Examples of integral operators in such a class can be found in [4] and [25]. Moreover, the wavelet-based orthogonal expansion of the input prior model as well as the transformed wavelet-based orthogonal expansion of the corresponding output provide a discretization of the problem. The elements of class $\mathcal{P}$ can be represented as a linear filter applied to white noise (see [4], [6], [29], and [31])). From the isometry between the space $L^2(\mathbb{R}^d)$ and the Hilbert space of random variables generated by any white-noise random field ε on $\mathbb{R}^d$, a multiresolution approximation to the space $L^2(\mathbb{R}^d)$ induces a 'random multiresolution-like approximation' to the Hilbert space of random variables generated by the input generalized random field. Truncation of these orthogonal expansions leads to a finite-dimensional formulation of the problem. The input and output random fields are thus represented in terms of an orthonormal vector of random coefficients and the corresponding vector of deterministic transformed wavelets coefficients. An approximation to the least-squares linear estimate of the input random field is computed by solving the finite-dimensional linear system of estimation equations defined in terms of the above deterministic coefficients. In the Gaussian case, under certain conditions, a reconstruction formula for the trajectories of the input random field is derived similar to the one provided by the wavelet-vaguelette decomposition introduced in [12] for the deterministic and ordinary cases.

In Section 2, we first refer to the ordinary formulation of the stochastic linear inverse problem, and introduce some fundamental results in relation to the fractional generalized framework, as well as a wavelet-based orthogonal expansion of fractional generalized random fields. We formulate in Section 3 the stochastic linear inverse problem in a fractional generalized framework, and derive a solution to this problem via its regularization based on the considered fractional generalized prior model class. Discretization of such a problem is derived in Section 4 in terms of the orthogonal decomposition of the input prior model and the corresponding output in terms of wavelets. Both representations lead to associated wavelet-based decompositions of the square-roots of the prior input covariance operator, the corresponding output, and their respective inverses. The least-squares linear estimate of the input random field is then calculated in terms of these wavelet decompositions. Truncation of the above series leads to a finite-dimensional approximation to the problem. In the case where the integral operator defining the problem does not have a continuous inverse with respect to any fractional Sobolev norm, the wavelet-based orthogonal expansion of the input prior model leads to a weak-sense approximation of the least-squares linear estimate. Finally, in Section 5, two important examples are described where the integral operator of the problem represents fractional integration. In Example 1, we assume that the input random field is of a fractal nature and, in particular, we consider fractional Brownian motion as a prior model to regularize the problem. Example 2 is formulated in terms of the Riesz kernel (the Abel transform is a particular case), which is homogeneous and preserves most of the scale-space local properties of wavelets (see [12]). In this case, the prior model is a fractional generalized random field defined by a fractional-order integral equation, in terms of the Bessel kernel.

2. PRELIMINARIES

In this section, we consider the basic formulation of the least-squares linear estimation method for ordinary stochastic linear inverse problems. To translate this problem into a fractional generalized framework, we introduce some fundamental results from [31] used below. Some preliminary results in relation to wavelet-based orthogonal expansions of fractional generalized random fields from [5], needed in Section 4, are also given.

Let $\{g(\mathbf{y}) : \mathbf{y} \in S\}$ and $\{f(\mathbf{z}) : \mathbf{z} \in S\}$ be two zero-mean second-order random fields defined on a domain $S \subseteq \mathbb{R}^d$ related by the following first-kind integral equation in the mean-square sense:

$$g(\mathbf{y}) = \int_S k(\mathbf{y}, \mathbf{z}) f(\mathbf{z}) d\mathbf{z} = K(f)(\mathbf{y}), \quad \mathbf{y} \in S_g \subseteq S, \tag{1}$$

where k denotes the kernel of the integral operator K transferring the information from f to g. By B_{gg}, B_{fg}, and B_{ff} we denote the covariance function of g, the cross-covariance function between f and g, and the covariance function of f, respectively. We assume that some prior information on B_{ff} is available. The least-squares linear estimation of f from the observation of g in a subset $S_g \subseteq S$ is summarized in the following paragraphs.

A linear estimate $\hat{f}_{\mathbf{z}}$ of $f(\mathbf{z})$, based on the information provided by the observation of g in a subset $S_g \subseteq S$, is defined by

$$\hat{f}_{\mathbf{z}} = \int_{S_g} l_{\mathbf{z}}(\mathbf{r}) g(\mathbf{r}) d\mathbf{r}, \tag{2}$$

where $l_{\mathbf{z}}(\cdot)$ is the weighting function summarizing the available information $\{g(\mathbf{z}) : \mathbf{z} \in S_g\}$ in the linear approximation of $f(\mathbf{z})$. From the Orthogonal Projection Theorem, the function $l_{\mathbf{z}}$ associated with the least-squares linear estimate of $f(\mathbf{z})$ satisfies the following equations:

$$B_{fg}(\mathbf{z},\mathbf{y}) = \int_{S_g} l_{\mathbf{z}}(\mathbf{r}) B_{gg}(\mathbf{r},\mathbf{y}) d\mathbf{r}, \qquad \mathbf{y} \in S_g. \tag{3}$$

Remark 2.1. In certain applications, there may exist some locations where the input random field is also observed (see, for example, [11], in the groundwater hydrology context). The available sample information then consists of the observations of the output g in a subset $S_g \subseteq S$, and of the input f in a subset $S_f \subseteq S$. A linear estimate $\tilde{f}_{\mathbf{z}}$ of $f(\mathbf{z})$ is defined in this case by

$$\tilde{f}_{\mathbf{z}} = \int_{S_g} l_{\mathbf{z}}^g(\mathbf{y}) g(\mathbf{y}) d\mathbf{y} + \int_{S_f} l_{\mathbf{z}}^f(\mathbf{y}) f(\mathbf{y}) d\mathbf{y},$$

where $l_{\mathbf{z}}^g(\cdot)$ and $l_{\mathbf{z}}^f(\cdot)$ now represent the weighting functions associated with the observation sets S_g and S_f, respectively, in the linear approximation of f at location $\mathbf{z}$ in $S \setminus S_f$. As before, in the case where $\tilde{f}_{\mathbf{z}}$ represents the least-squares linear estimate of $f(\mathbf{z})$, these functions satisfy the following equations:

$$\begin{aligned} B_{fg}(\mathbf{z},\mathbf{y}) &= \int_{S_g} l_{\mathbf{z}}^g(\mathbf{x}) B_{gg}(\mathbf{x},\mathbf{y}) d\mathbf{x} + \int_{S_f} l_{\mathbf{z}}^f(\mathbf{s}) B_{fg}(\mathbf{s},\mathbf{y}) d\mathbf{s}, \ \mathbf{y} \in S_g, \\ B_{ff}(\mathbf{z},\mathbf{v}) &= \int_{S_g} l_{\mathbf{z}}^g(\mathbf{x}) B_{gf}(\mathbf{x},\mathbf{v}) d\mathbf{x} + \int_{S_f} l_{\mathbf{z}}^f(\mathbf{s}) B_{ff}(\mathbf{s},\mathbf{v}) d\mathbf{s}, \ \mathbf{v} \in S_f. \quad (3) \end{aligned}$$

Usually, measurements are affected by additive noise. In this case, we assume that the observation model is given by

$$y(\mathbf{x}) = g(\mathbf{x}) + \nu(\mathbf{x}), \quad \mathbf{x} \in S_y \subseteq S, \tag{5}$$

where ν represents white noise with intensity σ, uncorrelated with the random input f. The least-squares linear estimate $\hat{f}_{\mathbf{z}}$ of $f(\mathbf{z})$, based on the information provided by the observations $\{y(\mathbf{x}) : \mathbf{x} \in S_y\}$, is then defined as

$$\hat{f}_{\mathbf{z}} = \int_{S_y} l_{\mathbf{z}}(\mathbf{r}) y(\mathbf{r}) d\mathbf{r}, \tag{6}$$

with function $l_{\mathbf{z}}(\cdot)$ satisfying the equations

$$B_{fy}(\mathbf{z},\mathbf{x}) = \int_{S_y} l_{\mathbf{z}}(\mathbf{r}) B_{yy}(\mathbf{r},\mathbf{x}) d\mathbf{r}, \quad \mathbf{x} \in S_y. \tag{7}$$

Here $B_{yy} = B_{gg} + B_{\nu\nu}$ represents the covariance function of the observation process y, and $B_{\nu\nu}$ the covariance function of the additive noise ν.

We now introduce some definitions and preliminary results from [31] on fractional generalized random fields.

Let $C_0^\infty(S)$ be the space of infinitely differentiable functions with compact support contained in S, with $S \subseteq \mathbb{R}^d$ being a bounded $C^\infty-$domain, that is, a bounded domain with C^∞ boundary. Let $H^\alpha(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$, be the fractional Sobolev space defined as the completion $\overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_\alpha}$ with respect to the associated norm

$$\|\phi\|_\alpha^2 = \int_{\mathbb{R}^d} (1 + |\boldsymbol{\lambda}|^2)^\alpha |\mathcal{F}(\phi)(\boldsymbol{\lambda})|^2 d\boldsymbol{\lambda}, \quad \phi \in H^\alpha(\mathbb{R}^d), \tag{8}$$

where $\mathcal{F}(\cdot)$ denotes the Fourier transform. In the following sections, we consider the fractional Sobolev spaces on a bounded $C^\infty-$domain S

$$\bar{H}^\alpha(S) = \left\{\phi \in H^\alpha\left(\mathbb{R}^d\right) : \mathrm{supp}\ \phi \subseteq \overline{S}\right\} = \overline{C_0^\infty(S)}^{\|\cdot\|_\alpha}.$$

The dual space of $\bar{H}^\alpha(S)$ is the space $H^{-\alpha}(S)$ of distributions defined on S that coincide with the restriction to S of a function of the space $H^{-\alpha}(\mathbb{R}^d)$. For $\alpha \in \mathbb{R}$, we denote by U_α and V_α the fractional dual Sobolev spaces $\bar{H}^\alpha(S)$ and $H^{-\alpha}(S)$, respectively.

The interpretation of the above-described least-squares linear inverse estimation problems in a fractional generalized framework is achieved via the second-order regularity properties of the input and output random fields referred to the continuous dual scales of fractional Sobolev spaces $\{U_\alpha,\ \alpha \in \mathbb{R}\}$ and $\{V_\alpha,\ \alpha \in \mathbb{R}\}$. Such properties are studied in terms of the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ of real-valued zero-mean random variables defined on the basic probability space $(\Omega, \mathcal{A}, P)$, with finite second-order moments, and with the inner product

$$< X, Y >_{\mathcal{L}^2(\Omega)} = E[XY], \quad X, Y \in \mathcal{L}^2(\Omega, \mathcal{A}, P). \tag{9}$$

The following concept of a fractional generalized random field, given in terms of the fractional Sobolev and the $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ norms, establishes a bridge between the topological properties induced by these two norms.

Definition 2.1. For $\alpha \in \mathbb{R}$, a random function $X_\alpha(\cdot)$ from U_α into $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ is said to be an $\alpha-$generalized random field ($\alpha-$GRF) if it is linear and continuous in the mean-square sense with respect to the $U_\alpha-$topology.

The minimum order α for which a generalized random function X can be defined as a continuous linear functional from U_α into $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ is referred to as the *minimum fractional singularity order* (respectively, $-\alpha$ defines the *maximum fractional regularity order*) of X. In the ordinary case, the singularity order is established in terms of the integrability in the mean-square sense of a random field in a fractional Sobolev space. Thus, if an ordinary random field $\mathcal{X}$ has minimum fractional singularity order $\alpha \in \mathbb{R}$, then $\bar{H}^{\alpha}(S)$ is the largest space where $\mathcal{X}$ is mean-square integrable or, equivalently, where its covariance operator $R_{\mathcal{X}}$ can be defined. That is,

$$(R_{\mathcal{X}}(h), h) = \int_S \int_S B_{\mathcal{X}\mathcal{X}}(\mathbf{z}, \mathbf{y}) h(\mathbf{y}) h(\mathbf{z}) d\mathbf{y} d\mathbf{z} < \infty, \quad \forall h \in \bar{H}^{\alpha}(S), \quad (10)$$

where $B_{\mathcal{X}\mathcal{X}}$ represents the covariance function of $\mathcal{X}$. Moreover, an ordinary random field has a non-positive singularity order, and the function h in Eq. (10) is a distribution. In this case, therefore, $B_{\mathcal{X}\mathcal{X}}$ belongs to the space of test functions where the distribution h is defined. In particular, for $-\alpha > d/2$, $B_{\mathcal{X}\mathcal{X}}$ is continuous and the random field $\mathcal{X}$ is continuous in the mean-square sense (see [33] on Embedding Theorems for fractional Sobolev spaces).

The minimum fractional singularity order of a fractional GRF X_{α} is determined by the regularity properties of the functions belonging to its reproducing kernel Hilbert space (RKHS) $\mathcal{H}(X_{\alpha})$, which is isometrically equivalent to the Hilbert subspace $H(X_{\alpha})$ of $\mathcal{L}^2(\Omega, \mathcal{A}, P)$. These spaces are defined as follows. The space $H(X_{\alpha})$ is defined as the closed span with respect to the mean-square norm of the subspace $\{X_{\alpha}(\varphi) : \varphi \in U_{\alpha}\}$ of $\mathcal{L}^2(\Omega, \mathcal{A}, P)$. The space $\mathcal{H}(X_{\alpha})$ is defined as the space of functions $u \in V_{\alpha}$ such that

$$u(\phi) = E[Y X_{\alpha}(\phi)], \quad \phi \in U_{\alpha}, \quad (11)$$

for a certain $Y \in H(X_{\alpha})$. The inner product in $\mathcal{H}(X_{\alpha})$ is defined as $< u, v >_{\mathcal{H}(X_{\alpha})} = E[YZ]$, where Y and Z are the elements of $H(X_{\alpha})$ defining u and v, respectively, as in Eq. (11). The spaces $H(X_{\alpha})$ and $\mathcal{H}(X_{\alpha})$ are therefore isometric under the relationship given by Eq. (11). The space $\mathcal{H}(X_{\alpha})$ also coincides with the closed span of $\{B_{X_{\alpha}X_{\alpha}}(\phi, \cdot) = E[X_{\alpha}(\phi) X_{\alpha}(\cdot)] : \phi \in U_{\alpha}\}$ with respect to the mean-square norm, where $B_{X_{\alpha}X_{\alpha}}(\cdot, \cdot)$ denotes the covariance function of X_{α}.

From the Kernel Theorem (see [16]), the covariance function $B_{X_{\alpha}X_{\alpha}}$ of the $\alpha-$GRF X_{α} can be represented as

$$B_{X_{\alpha}X_{\alpha}}(\varphi, \phi) = < (R_{X_{\alpha}} \varphi)^*, \phi >_{U_{\alpha}}, \quad \varphi, \phi \in U_{\alpha},$$

where $*$ stands for the duality between Hilbert spaces (the Riesz Representation Theorem), and $R_{X_{\alpha}}$ is the covariance operator of X_{α}, that is, a symmetric positive continuous linear operator from U_{α} into V_{α}. Hence, the

RKHS of X_α can be defined as

$$\mathcal{H}(X_\alpha) = \overline{\mathrm{sp}}^{\mathcal{L}^2(\Omega)} \left\{ R_{X_\alpha}(\phi) : \ \phi \in U_\alpha \right\}.$$

The regularization method we develop in the next section is based on the following concept of $\alpha-$duality introduced in [31]:

Definition 2.2. For $\alpha \in \mathbb{R}$, we say that an $\alpha-$GRF

$$\widetilde{X}_\alpha : V_\alpha \longrightarrow \mathcal{L}^2(\Omega, \mathcal{A}, P)$$

is the dual relative to U_α (or $\alpha-$dual) of the $\alpha-$GRF

$$X_\alpha : U_\alpha \longrightarrow \mathcal{L}^2(\Omega, \mathcal{A}, P)$$

if it satisfies:

(i) $H(X_\alpha) = H(\widetilde{X}_\alpha)$,
(ii) $< X_\alpha(\phi), \widetilde{X_\alpha}(v) >_{H(X_\alpha)} = < \phi, v^* >_{U_\alpha}$, for $\phi \in U_\alpha$, and $v \in V_\alpha$, with v^* being the dual element of v with respect to the $U_\alpha-$topology.

Note that the dual of $\widetilde{X}_\alpha$ relative to V_α is the $\alpha-$GRF X_α. In the above definition, $H(X_\alpha)$ and $H(\widetilde{X}_\alpha)$ represent, as before, the Hilbert spaces of random variables associated with X_α and $\widetilde{X}_\alpha$, respectively. We denote by $\mathcal{H}(\widetilde{X}_\alpha)$ the RKHS of the fractional GRF $\widetilde{X}_\alpha$ defined by

$$\mathcal{H}(\widetilde{X}_\alpha) = \overline{\mathrm{sp}}^{\mathcal{L}^2(\Omega)} \left\{ R_{\widetilde{X}_\alpha}(v)(\cdot) : \ v \in V_\alpha \right\}, \tag{12}$$

where $R_{\widetilde{X}_\alpha}$ represents the covariance operator of $\widetilde{X}_\alpha$. Under the existence of the $\alpha-$dual $\widetilde{X}_\alpha$, the covariance operator R_{X_α} of X_α is an isomorphism from U_α into V_α, and can be factorized as follows:

$$R_{X_\alpha} = \mathcal{T}\mathcal{T}', \tag{13}$$

where $\mathcal{T}$ is an isomorphism from $L^2(S)$ onto V_α, and $\mathcal{T}'$ represents the adjoint operator of $\mathcal{T}$ (see [31]). Conversely, X_α is the $\alpha-$dual of the random field $\widetilde{X}_\alpha$, and the covariance operator $R_{\widetilde{X}_\alpha}$ can also be factorized as

$$R_{\widetilde{X}_\alpha} = \tilde{\mathcal{T}}\tilde{\mathcal{T}}', \tag{14}$$

with ˜
T= $[\mathrm{T}']^{-1} = [\mathcal{T}^{-1}]'$. Thus, $R_{X_\alpha} = R_{\widetilde{X}_\alpha}^{-1}$.

Remark 2.2. Note that in the case where $R_{X_\alpha} : U_\alpha \longrightarrow V_\alpha$ is an isomorphism, its inverse operator defines the covariance operator of its dual fractional generalized random field. Definition 2.2. then provides a condition equivalent to the existence of the bounded inverse $R_{X_\alpha}^{-1} : V_\alpha \longrightarrow U_\alpha$ of R_{X_α}. In the case where the observation random field has duality order α, the least-squares linear estimation problem associated with f admits a solution in the space U_α as we show in Section 3.

Below, we finally describe the elements and results given in [5] for an orthogonal representation of an $\alpha-$GRF in terms of wavelets. Consider a multiresolution approximation of $L^2(S)$ in terms of a modified orthonormal basis of $L^2(\mathbb{R}^d)$ of wavelet functions with support contained in S (see, for example, [26]). Such an approximation leads to the orthogonal decomposition

$$L^2(S) = V_0 \bigoplus_{j\geq 0} W_j,$$

where V_0 represents the coarsest-scale space of interest generated by the orthonormal basis of scaling functions $\{\phi_{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\}$, and where W_j, $j \geq 0$, are closed subspaces of $L^2(S)$ respectively generated by the orthonormal wavelet bases $\{\psi_{j:\boldsymbol{\theta}} :\ \boldsymbol{\theta} \in \Lambda_j^S\}$, $j \geq 0$. Thus, a function f in $L^2(S)$ can be represented in terms of scaling coefficients with respect to the orthonormal basis $\{\phi_{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\}$, and wavelet coefficients (detail coefficients) with respect to the orthonormal bases $\{\psi_{j:\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Lambda_j^S\}$, $j \geq 0$, at different scales (see [22]). The following result provides a sufficient condition that ensures the convergence in the mean-square sense of a wavelet-based orthogonal expansion of a fractional GRF.

THEOREM 2.1. (see [5]) *Let X_α be an α-GRF satisfying the duality condition given in Definition 2.2. Then, X_α can be represented by the following orthogonal expansion in the mean-square sense:*

$$X_\alpha(\psi) = \sum_{\mathbf{k}\in\Gamma_0^S} X_\alpha\left(\varphi^{\mathbf{k}}\right)\varphi_{\mathbf{k}}(\psi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} X_\alpha\left(\gamma^{j,\boldsymbol{\theta}}\right)\gamma_{j,\boldsymbol{\theta}}(\psi),$$

for all $\psi \in U_\alpha$, where $\varphi_{\mathbf{k}} = \mathcal{T}(\phi_{\mathbf{k}})$, $\varphi^{\mathbf{k}} = \tilde{\mathcal{T}}(\phi_{\mathbf{k}})$, for all $\mathbf{k} \in \Gamma_0^S$, $\gamma_{j,\boldsymbol{\theta}} = \mathcal{T}(\psi_{j,\boldsymbol{\theta}})$, and $\gamma^{j,\boldsymbol{\theta}} = \tilde{\mathcal{T}}(\psi_{j,\boldsymbol{\theta}})$, for all $\boldsymbol{\theta} \in \Lambda_j^S$, $j \geq 0$. Here $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are defined as in (13) *and* (14)*, respectively.*

The random coefficients $\{X_\alpha(\varphi^{\mathbf{k}}) : \mathbf{k} \in \Gamma_0^S\} \cup \{X_\alpha(\gamma^{j,\boldsymbol{\theta}}) :\ \boldsymbol{\theta} \in \Lambda_j^S, j \geq 0\}$ are orthonormal in $H(X_\alpha)$.

A similar wavelet-based orthogonal expansion can be obtained for the $\alpha-$dual GRF $\widetilde{X}_\alpha$ of X_α.

COROLLARY 2.1. *Under the conditions of the above theorem, the following orthogonal expansion for $\widetilde{X}_\alpha$ is obtained:*

$$\widetilde{X}_\alpha(h) = \sum_{\mathbf{k}\in\Gamma_0^S} \widetilde{X}_\alpha(\varphi_{\mathbf{k}})\,\varphi^{\mathbf{k}}(h) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \widetilde{X}_\alpha\left(\gamma_{j,\boldsymbol{\theta}}\right)\gamma^{j,\boldsymbol{\theta}}(h),$$

for all $h \in [U_\alpha]^*$, where the functions $\left\{\varphi_{\mathbf{k}} : \mathbf{k}\in\Gamma_0^S\right\}$, $\left\{\varphi^{\mathbf{k}} : \mathbf{k}\in\Gamma_0^S\right\}$, $\left\{\gamma_{j,\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Lambda_j^S, j\geq 0\right\}$ and $\left\{\gamma^{j,\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Lambda_j^S, j\geq 0\right\}$ are defined as in Theorem 2.1, and the series converges in the mean-square sense. The random coefficients $\{\widetilde{X}_\alpha(\varphi_{\mathbf{k}}) : \mathbf{k}\in\Gamma_0^S\}\cup\{\widetilde{X}_\alpha\left(\gamma_{j,\boldsymbol{\theta}}\right) : \boldsymbol{\theta}\in\Lambda_j^S, j\geq 0\}$ are orthonormal in $H(\widetilde{X}_\alpha)$.

The systems $\{\varphi_{\mathbf{k}} : \mathbf{k}\in\Gamma_0^S\}\cup\{\gamma_{j,\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\}$ and $\{\varphi^{\mathbf{k}} : \mathbf{k}\in\Gamma_0^S\}\cup\{\gamma^{j,\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\}$ constitute dual Riesz bases. These bases provide scale-space local descriptions of the linear operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, respectively, which lead to the following series representations of the covariance operators R_{X_α} and $R_{\widetilde{X}_\alpha}$:

$$\begin{aligned}
R_{X_\alpha}(\psi)(\phi) &= \sum_{\mathbf{k}\in\Gamma_0^S}\varphi_{\mathbf{k}}(\psi)\,\varphi_{\mathbf{k}}(\phi) \\
&\quad +\sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S}\gamma_{j,\boldsymbol{\theta}}(\psi)\,\gamma_{j,\boldsymbol{\theta}}(\phi)\ ,\ \forall\psi,\phi\in U_\alpha, \\
R_{\widetilde{X}_\alpha}(h)(v) &= \sum_{\mathbf{k}\in\Gamma_0^S}\varphi^{\mathbf{k}}(h)\,\varphi^{\mathbf{k}}(v) \\
&\quad +\sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S}\gamma^{j,\boldsymbol{\theta}}(h)\,\gamma^{j,\boldsymbol{\theta}}(v)\ ,\ \forall h,v\in V_\alpha. \qquad (14)
\end{aligned}$$

Furthermore, the systems of orthonormal random coefficients in the above orthogonal expansions can also be interpreted as orthonormal bases of a 'random multiresolution-like approximation' of the spaces $H(X_\alpha)$ and $H(\widetilde{X}_\alpha)$, respectively. The dual Riesz bases $\left\{\varphi_{\mathbf{k}} :\ \mathbf{k}\in\Gamma_0^S\right\}\cup\left\{\gamma_{j,\boldsymbol{\theta}} :\ \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\right\}$ and $\left\{\varphi^{\mathbf{k}} :\ \mathbf{k}\in\Gamma_0^S\right\}\cup\left\{\gamma^{j,\boldsymbol{\theta}} :\ \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\right\}$ then correspond to the deterministic transformed wavelet coefficients of X_α and $\widetilde{X}_\alpha$, with respect to the orthonormal bases of random variables $\left\{X_\alpha(\varphi^{\mathbf{k}}) : \mathbf{k}\in\Gamma_0^S\right\}\cup\left\{X_\alpha(\gamma^{j,\boldsymbol{\theta}}) :\ \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\right\}$ and $\left\{\widetilde{X}_\alpha(\varphi_{\mathbf{k}}) :\ \mathbf{k}\in\Gamma_0^S\right\}\cup\left\{\widetilde{X}_\alpha(\gamma_{j,\boldsymbol{\theta}}) :\ \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\right\}$, respectively. Indeed, $\left\{\varphi_{\mathbf{k}} :\ \mathbf{k}\in\Gamma_0^S\right\}\cup\left\{\gamma_{j,\boldsymbol{\theta}} :\ \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\right\}$ and $\left\{\varphi^{\mathbf{k}} :\ \mathbf{k}\in\Gamma_0^S\right\}\cup\left\{\gamma^{j,\boldsymbol{\theta}} :\ \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\right\}$ constitute orthonormal bases of the RKHSs of X_α and $\widetilde{X}_\alpha$, respectively.

3. FRACTIONAL GENERALIZED LINEAR INVERSE ESTIMATION PROBLEM

The regularization method presented in this section is based on considering a fractional generalized framework to study the widest class of solutions associated with the least-squares linear inverse estimation problem described above. Each element in such a class, that is, a solution to a particular inverse problem under the general setup introduced in Section 2, is determined in the fractional Sobolev space that defines the domain of the covariance operator of the observation random field according to its minimum fractional singularity and duality orders (see Lemma 3.1). In addition, an important class $\mathcal{K}$ of integral operators for which this approach is applicable is provided in Theorem 3.1. The fractional generalized framework covers the ordinary and improper cases corresponding, respectively, to non-positive and positive minimum fractional singularity orders of the random fields involved in the integral equation (1).

Assume that the input and output random fields have minimum fractional singularity orders β and α, respectively, with $\beta \geq \alpha$ and $\alpha, \beta \in \mathbb{R}$. The system integral equation (1) is defined in a fractional generalized framework as

$$g_\alpha(\phi) = f_\beta(K'(\phi)), \quad \forall \phi \in U_\alpha = \bar{H}^\alpha(S), \tag{16}$$

where $K' : U_\alpha \longrightarrow U_\beta$, denotes the adjoint of the integral operator K defining such an equation. The fractional GRFs f_β and g_α respectively represent the random input and the corresponding random output. From Eq. (5), the fractional generalized observation model is given in the case of noisy data by

$$y_\alpha(\phi) = g_\alpha(\phi) + \nu_\alpha(\phi), \quad \forall \phi \in U_\alpha = \bar{H}^\alpha(S), \tag{17}$$

where g_α is defined as in Eq. (16). Here ν_α represents a fractional generalized white noise on the space U_α, that is, a fractional GRF defined on U_α with covariance function $B_{\nu_\alpha \nu_\alpha}$ given by

$$B_{\nu_\alpha \nu_\alpha}(\phi, \psi) = E\left[\nu_\alpha(\phi)\nu_\alpha(\psi)\right] = < \phi, \psi >_{U_\alpha}, \quad \forall \phi, \psi \in U_\alpha, \tag{18}$$

where $< \cdot, \cdot >_{U_\alpha}$ represents the inner product associated with the norm (8). The generalized observation models (16) and (17) are given in terms of the test functions of the space U_α with support contained in S. However, in the case where the available information is obtained from a subdomain S_g of S, the problem can be similarly handled considering the space of test functions $\bar{H}^\alpha(S_g)$.

Remark 3.1. Note that, from Definition 2.1, g_α and f_β are continuous in the mean-square sense, which implies that K' presents convenient regularity properties with respect to the involved fractional Sobolev norms.

Let $R_{f_\beta g_\alpha}$ be the covariance operator associated with the cross-covariance function $B_{f_\beta g_\alpha}$ between f_β and g_α. The adjoint $R'_{f_\beta g_\alpha}$ of $R_{f_\beta g_\alpha}$ is then defined by

$$< [R'_{f_\beta g_\alpha}(\phi)]^*, \varphi >_{U_\alpha} = < \phi, [R_{f_\beta g_\alpha}(\varphi)]^* >_{U_\beta} = B_{f_\beta g_\alpha}(\phi, \varphi),$$

for all $\phi \in U_\beta,\ \varphi \in U_\alpha$. Let R_{g_α} be the covariance operator associated with the covariance function $B_{g_\alpha g_\alpha}$ of g_α, defined by

$$< \varphi, [R_{g_\alpha}(\psi)]^* >_{U_\alpha} = < [R_{g_\alpha}(\varphi)]^*, \psi >_{U_\alpha} = B_{g_\alpha g_\alpha}(\varphi, \psi), \quad \forall \varphi, \psi \in U_\alpha.$$

For each $\phi \in U_\beta$, the least-squares linear estimate $\hat{f}_\beta(\phi)$ of $f_\beta(\phi)$, based on the information provided by $\{g_\alpha(\varphi) :\ \varphi \in U_\alpha\}$ (respectively, by $\{y_\alpha(\varphi) : \varphi \in U_\alpha\}$ in the case of noisy data), is defined in terms of the linear operator $\mathcal{L}' : U_\beta \longrightarrow U_\alpha$ as $\hat{f}_\beta(\phi) = g_\alpha(\mathcal{L}'(\phi))$ (respectively, $\hat{f}_\beta(\phi) = y_\alpha(\mathcal{L}'(\phi))$ minimizing the mean-square error

$$E\left[f_\beta(\phi) - \hat{f}_\beta(\phi)\right]^2.$$

From the Orthogonal Projection Theorem, in the case where the observation model is given by Eq. (16), $\mathcal{L}'$ satisfies the following linear system of estimation equations:

$$R'_{f_\beta g_\alpha}(\phi) \underset{V_\alpha}{=} R_{g_\alpha}(\mathcal{L}'\phi), \quad \forall \phi \in U_\beta, \tag{19}$$

that can also be written in terms of the corresponding adjoint operators as

$$R_{f_\beta g_\alpha}(\varphi) \underset{V_\beta}{=} \mathcal{L} R_{g_\alpha}(\varphi), \quad \forall \varphi \in U_\alpha. \tag{20}$$

In addition, $\mathcal{L}'$ is defined by

$$R'_{f_\beta y_\alpha}(\phi) \underset{V_\alpha}{=} R_{y_\alpha}(\mathcal{L}'\phi), \quad \forall \phi \in U_\beta, \tag{21}$$

or equivalently,

$$R_{f_\beta y_\alpha}(\varphi) \underset{V_\beta}{=} \mathcal{L} R_{y_\alpha}(\varphi), \quad \forall \varphi \in U_\alpha, \tag{22}$$

in the case of observation model given by Eq. (17). Here, as before, $R'_{f_\beta y_\alpha}$ represents the adjoint of the covariance operator $R_{f_\beta y_\alpha}$ associated with the cross-covariance function $B_{f_\beta y_\alpha}$ between f_β and y_α, and R_{y_α} represents the covariance operator associated with the covariance function $B_{y_\alpha y_\alpha}$ of y_α. In the ordinary case, the adjoint operator $\mathcal{L} : V_\alpha \longrightarrow V_\beta$ has kernel $l(\cdot, \cdot)$ given by Eq. (2) (respectively, by Eq. (7) in the case of noisy data).

The following lemma provides the definition of the solution to Eq. (19) (respectively, Eq. (21)), in terms of the covariance operator of the $\alpha-$dual of the information random field g_α (respectively, y_α).

LEMMA 3.1. *(i) Let g_α be the fractional generalized random output of Eq. (16). Assume that g_α has duality order α. Then, Eq. (19) has a unique stable solution in the RKHS of $\tilde{g}_\alpha$ defined by*

$$\mathcal{L}'(\phi) \underset{U_\alpha}{=} R_{\tilde{g}_\alpha} R'_{f_\beta g_\alpha}(\phi), \quad \forall \phi \in U_\beta, \tag{23}$$

where $R_{\tilde{g}_\alpha} : V_\alpha \longrightarrow U_\alpha$ denotes the covariance operator associated with the covariance function $B_{\tilde{g}_\alpha \tilde{g}_\alpha}$ of the $\alpha-$dual $\tilde{g}_\alpha$, and $R'_{f_\beta g_\alpha} : U_\beta \longrightarrow V_\alpha$ represents the adjoint of $R_{f_\beta g_\alpha} : U_\alpha \longrightarrow V_\beta$, associated with the cross-covariance function $B_{f_\beta g_\alpha}$ between f_β and g_α.

(ii) Similarly, in the case of noisy data, assume that y_α defined by Eq. (17) has duality order α. Then, the solution to Eq. (21) in the RKHS of $\tilde{y}_\alpha$ is defined by

$$\mathcal{L}'(\phi) \underset{U_\alpha}{=} R_{\tilde{y}_\alpha} R'_{f_\beta y_\alpha}(\phi), \quad \forall \phi \in U_\beta, \tag{24}$$

where $R_{\tilde{y}_\alpha}$ denotes the covariance operator of the $\alpha-$dual $\tilde{y}_\alpha$ of y_α, and $R'_{f_\beta y_\alpha}$ represents the adjoint of the covariance operator associated with the cross-covariance function $B_{f_\beta y_\alpha}$ between f_β and y_α. Such a solution is stable with respect to the considered fractional Sobolev geometries.

Proof. From Definition 2.2, the covariance operator of the $\alpha-$dual GRF $\tilde{g}_\alpha$ (respectively, $\tilde{y}_\alpha$) defines the continuous inverse of the covariance operator of the observation random field with respect to the fractional Sobolev norm $\|\cdot\|_\alpha$ on U_α (see [31]). Therefore, Eq. (23) (respectively, Eq. (24)) provides a stable solution to Eq. (19) (respectively, Eq. (21)) on U_β with respect to the fractional Sobolev norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$. From Eq. (12), such a solution lies in the RKHS of the $\alpha-$dual of the observation random field. ▌

From Lemma 3.1, the fractional duality order α determines the information provided by the observation of g_α useful for estimation of the input, by means of the test functions in the space U_α. In the ordinary case, the order α is non positive and, for each $\mathbf{z} \in S$, the weighting function $l(\mathbf{z}, \cdot)$ defining $\mathcal{L}$ is a distribution. That is, from fractional Sobolev Embedding Theorems (see [33]), in the case where $-\beta > d/2$, operator $\mathcal{L}$ defined by

$$\mathcal{L}(\psi) \underset{V_\beta}{=} R_{f_\beta g_\alpha} R_{\tilde{g}_\alpha}(\psi), \quad \forall \psi \in V_\alpha,$$

has a kernel l given by

$$\mathcal{L}(\psi)(\mathbf{z}) = l_{\mathbf{z}}(\psi) = \int_S l_{\mathbf{z}}(\mathbf{y})\psi(\mathbf{y})d\mathbf{y}, \forall\psi \in V_\alpha, \tag{25}$$

for each $\mathbf{z} \in S$. Eq. (25) provides the solution with minimum fractional singularity order to the corresponding ordinary least-squares linear inverse estimation problem.

We now consider the class $\mathcal{K}$ of integral operators K that commute with the Bessel potentials $\mathcal{I}_\gamma, \gamma > 0$, and satisfy that operator

$$K^{-1}\mathcal{I}_\gamma = \mathcal{I}_\gamma K^{-1}$$

is continuous with respect to the L^2-topology, associated with the space of square-integrable functions, for a certain positive $\gamma \in \mathbb{R}$. Here, we consider the weak-sense restriction of $\mathcal{I}_\gamma = (I - \Delta)^{-\gamma/2}$ to a bounded $C^\infty-$domain.

Remark 3.2. Since the Bessel potential is self-adjoint with respect to the inner product defined in the space of square-integrable functions $L^2(\mathbb{R}^d)$, each bounded function of such an operator commutes with it. However, the converse is false in general (see [9], pp. 145-148).

The class $\mathcal{K}$ includes, in particular, the subclass of integral operators defined as fractional-order positive rational functions of an elliptic and self-adjoint integer or fractional order differential operator commuting with the negative Laplacian operator on $L^2(\mathbb{R}^d)$, and with smooth coefficients. The direct estimation problem related with this class has been studied in [25], in the integer case, and in [4], in the fractional case. The subclass of fractional integral operators defined by Riesz kernels (see, for example, [12] and [32]) is also included in $\mathcal{K}$.

The following result provides a regularization method for stochastic linear inverse problems defined in terms of integral operators in $\mathcal{K}$.

THEOREM 3.1. *Assume that the integral operator K belongs to the class $\mathcal{K}$, and that the prior information can be represented in the form of a fractional GRF model in the class $\mathcal{P}$ (see Sections 1 and 2). Then, in the case where the output g in Eq.* (1) *has minimum fractional singularity order α, there exists a random input model $f_\beta \in \mathcal{P}$, with fractional duality order $\beta \geq \alpha$, which allows the inverse operator of K' to be defined as a continuous operator from U_β into U_α. Therefore, the inverse of the covariance operator of g_α in Eq.* (16) *(respectively, Eq.* (17)*) can be defined as a continuous operator from V_α into U_α. The solution to Eq.* (19) *can then be calculated as in Lemma 3.1. The solution to Eq.* (21) *is also given as in Lemma 3.1 in the case where*

$$\|R_{\nu_\alpha}\| < \|R_{\widetilde{g}_\alpha}\|^{-1}. \tag{26}$$

Proof. Let $\beta = \alpha+\gamma$, where $\alpha \in \mathbb{R}$ is the minimum fractional singularity order of the observation process, and $\gamma \in \mathbb{R}_+$ is the order defining K as an element of the class $\mathcal{K}$. In the case of data with no observation noise,

$$R_{g_\alpha} = KR_{f_\beta}K', \tag{27}$$

where R_{g_α} and R_{f_β} represent, respectively, the covariance operators of g_α and f_β. In the case of noisy data,

$$R_{y_\alpha} = KR_{f_\beta}K' + R_{\nu_\alpha}, \tag{28}$$

where R_{y_α} and R_{ν_α} represent, respectively, the covariance operators of y_α and ν_α. Since the minimum fractional duality order of the selected model f_β is β, the covariance operator R_{f_β} defines an isomorphism from U_β onto V_β, and then R_{f_β} can be factorized as in Eq. (13), that is,

$$R_{f_\beta} = \mathcal{T}_{f_\beta}\mathcal{T}'_{f_\beta}, \tag{29}$$

where $\mathcal{T}'_{f_\beta}$ is the adjoint of the isomorphism $\mathcal{T}_{f_\beta} : L^2(S) \longrightarrow V_\beta$. From Eq. (29) and the definition of the class $\mathcal{K}$, the operator $\mathcal{T}^{-1}_{f_\beta}K^{-1} : V_\alpha \longrightarrow L^2(S)$ is continuous:

$$\begin{aligned}
&\|\mathcal{T}^{-1}_{f_\beta}K^{-1}(\phi)\|_{L^2(S)} \le C_1\|K^{-1}(\phi)\|_{V_\beta} = C_1\|\mathcal{I}_\beta K^{-1}(\phi)\|_{L^2(S)}\\
&= C_1\|\mathcal{I}_{\alpha+\gamma}K^{-1}(\phi)\|_{L^2(S)} = C_1\|\mathcal{I}_\gamma K^{-1}\mathcal{I}_\alpha(\phi)\|_{L^2(S)}\\
&\le C_1C_2\|\mathcal{I}_\alpha(\phi)\|_{L^2(S)} = \|\phi\|_{V_\alpha}, \quad \phi \in V_\alpha,
\end{aligned}$$

where, as before, $\mathcal{I}_a = (I-\Delta)^{-a/2}$. Hence, its adjoint $(K^{-1})'(\mathcal{T}^{-1}_{f_\beta})'$ is continuous and, from Eqs. (27) and (29), $R^{-1}_{g_\alpha}$ is a continuous operator from V_α into U_α. Similarly, using Eq. (28), it can be proved that $R^{-1}_{y_\alpha}$ is continuous from the continuity of $R^{-1}_{g_\alpha}$ and condition (26) (see [18]). That is, the fractional GRFs g_α and y_α satisfy the duality condition, and Lemma 3.1 can be applied to obtain the solutions to Eqs. (19) and (21). ∎

As commented in the introduction, the considered prior model class $\mathcal{P}$ includes important cases of fractional stochastic models. In particular, the fractal-type model class considered, for example, in [23], constitutes an interesting example. The regularization properties of this class are very similar to those presented by the usual smoothness regularizers.

4. WAVELET-BASED ORTHOGONAL APPROXIMATION

The elements of class $\mathcal{P}$ admit a wavelet-based orthogonal representation as in Theorem 2.1. The orthogonal expansion of the input random field induces a transformed orthogonal expansion of the corresponding output. Both orthogonal expansions provide multiresolution approximations of the Hilbert spaces $H(f_\beta)$ and $H(g_\alpha)$ (in the sense pointed out in Sections 1 and 2). A discretization of the associated least-squares linear inverse estimation problem is also obtained in terms of such orthogonal expansions.

We first consider the case where the integral operator K is in the class $\mathcal{K}$, and study conditions under which the orthogonal expansion of the input random field in terms of wavelets induces an orthogonal expansion of the same type for the output random field. Under these conditions, in the Gaussian case and for $\alpha > d/2$, a reconstruction formula for the trajectories of the input random field can be derived using the orthogonal expansion of the output random field obtained in the following result.

LEMMA 4.1. *Assume the conditions of Theorem 3.1, and that operator K' is defined from U_α onto U_β. The fractional generalized input model f_β then admits the following orthogonal expansion:*

$$f_\beta(\phi) = \sum_{\mathbf{k}\in\Gamma_0^S} f_\beta\left(\varphi^{\mathbf{k}}\right)\varphi_{\mathbf{k}}(\phi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} f_\beta\left(\gamma^{j,\boldsymbol{\theta}}\right)\gamma_{j,\boldsymbol{\theta}}(\phi), \quad \forall\phi\in U_\beta, \tag{30}$$

where $\left\{\varphi_{\mathbf{k}} : \mathbf{k}\in\Gamma_0^S\right\} \cup \left\{\gamma_{j,\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Lambda_j^S,\, j\geq 0\right\}$ *and* $\left\{\varphi^{\mathbf{k}} : \mathbf{k}\in\Gamma_0^S\right\} \cup \left\{\gamma^{j,\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Lambda_j^S,\, j\geq 0\right\}$ *are dual Riesz bases respectively defined by*

$$\begin{aligned}\varphi_{\mathbf{k}} &= \mathcal{T}_{f_\beta}(\phi_{\mathbf{k}}), \qquad \varphi^{\mathbf{k}} = (\mathcal{T}_{f_\beta}^{-1})'(\phi_{\mathbf{k}}), \quad \mathbf{k}\in\Gamma_0^S, \text{ and}\\ \gamma_{j,\boldsymbol{\theta}} &= \mathcal{T}_{f_\beta}(\psi_{j,\boldsymbol{\theta}}), \qquad \gamma^{j,\boldsymbol{\theta}} = (\mathcal{T}_{f_\beta}^{-1})'(\psi_{j,\boldsymbol{\theta}}), \quad \boldsymbol{\theta}\in\Lambda_j^S,\, j\geq 0,\end{aligned}$$

with $R_{f_\beta} = \mathcal{T}_{f_\beta}\mathcal{T}_{f_\beta}'$. The corresponding fractional generalized output random field g_α can also be represented by

$$\begin{aligned} g_\alpha(\psi) &= \sum_{\mathbf{k}\in\Gamma_0^S} g_\alpha\left([K']^{-1}(\varphi^{\mathbf{k}})\right)[K(\varphi_{\mathbf{k}})](\psi)\\ &\quad + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} g_\alpha\left([K']^{-1}(\gamma^{j,\boldsymbol{\theta}})\right)[K(\gamma_{j,\boldsymbol{\theta}})](\psi) \qquad (30)\\ &= \sum_{\mathbf{k}\in\Gamma_0^S} f_\beta\left(\varphi^{\mathbf{k}}\right)\varphi_{\mathbf{k}}(K'\psi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} f_\beta\left(\gamma^{j,\boldsymbol{\theta}}\right)\gamma_{j,\boldsymbol{\theta}}(K'\psi), \qquad (31)\end{aligned}$$

for all $\psi\in U_\alpha$.

Proof. Under conditions of Theorem 3.1, since $K' : U_\alpha \longrightarrow U_\beta$ is assumed to be an onto mapping, the output g in Eq. (1) has minimum fractional singularity order α. Then, from Theorem 3.1, there exists a random input model $f_\beta \in \mathcal{P}$ with fractional duality order $\beta = \alpha + \gamma$, for which the fractional GRF model g_α associated with g has fractional duality order α. Therefore, from Theorem 2.1, f_β and g_α can be represented by the orthogonal expansions given by Eqs. (30) and (**??**), respectively. Finally, using the linear relationship between g_α and f_β given by Eq. (16), we obtain the transformed orthogonal expansion (32). ▮

Under the conditions of Lemma 4.1, the dual equation to Eq. (16) can be written as

$$\tilde{g}_\alpha(\psi) = \tilde{f}_\beta\left(K^{-1}(\psi)\right), \quad \forall \psi \in V_\alpha, \tag{33}$$

where $\tilde{f}_\beta$ and $\tilde{g}_\alpha$ respectively represent the $\beta-$dual and the $\alpha-$dual of the fractional GRFs f_β and g_α.

Remark 4.1. The orthogonal expansions (30) and (**??**) are given in terms of the same system of orthonormal random coefficients that provides a 'random multiresolution-like approximation' of the spaces $H(f_\beta)$ and $H(g_\alpha)$. Therefore, we can simultaneously generate the input and output random fields.

Remark 4.2. In the case of noisy data, under the conditions of Lemma 4.1, the observation process y_α has an $\alpha-$dual $\tilde{y}_\alpha$. From Theorem 2.1, a similar orthogonal expansion for y_α can then be derived.

Operator $\mathcal{L}'$ defining the least-squares linear estimate of the input random field can be represented in terms of the deterministic transformed wavelet coefficients appearing in the orthogonal expansions of the input random field and the $\alpha-$dual of the output random field, as we prove in the following result.

THEOREM 4.1. *Under the conditions of Lemma 4.1, the solution to Eq.* (19) *can be calculated from the following expansion:*

$$\begin{aligned}\mathcal{L}'(\phi)(\psi) &= \sum_{\mathbf{k}\in\Gamma_0^S} \varphi^{\mathbf{k}}(K^{-1}R'_{f_\beta g_\alpha}\phi)\varphi^{\mathbf{k}}(K^{-1}\psi) \\ &+ \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma^{j,\boldsymbol{\theta}}(K^{-1}R'_{f_\beta g_\alpha}\phi)\gamma^{j,\boldsymbol{\theta}}(K^{-1}\psi) \\ &= \sum_{\mathbf{k}\in\Gamma_0^S} \varphi_{\mathbf{k}}(\phi)\varphi^{\mathbf{k}}(K^{-1}\psi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma_{j,\boldsymbol{\theta}}(\phi)\gamma^{j,\boldsymbol{\theta}}(K^{-1}\psi), (33)\end{aligned}$$

for all $\phi \in U_\beta$ and $\psi \in V_\alpha$, where the systems of functions $\{\varphi_{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\} \cup \{\gamma_{j,\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Lambda_j^S,\, j \geq 0\}$ and $\{\varphi^{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\} \cup \{\gamma^{j,\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Lambda_j^S,\, j \geq 0\}$ are defined as in Lemma 4.1.

Proof. From the conditions assumed in Lemma 4.1, the $\beta-$dual $\tilde{f}_\beta$ of f_β, and the $\alpha-$dual $\tilde{g}_\alpha$ of g_α exist. Hence, from Eq. (15), the covariance operator of $\tilde{f}_\beta$ admits the following series representation:

$$R_{\tilde{f}_\beta}(\phi)(\psi) = \sum_{\mathbf{k}\in\Gamma_0^S} \varphi^{\mathbf{k}}(\phi)\varphi^{\mathbf{k}}(\psi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma^{j,\boldsymbol{\theta}}(\phi)\gamma^{j,\boldsymbol{\theta}}(\psi), \quad \forall \phi,\psi \in V_\beta. \tag{35}$$

In addition, from Eqs. (35) and (33), the covariance operator of the $\alpha-$dual $\tilde{g}_\alpha$ can then be represented as

$$\begin{aligned} R_{\tilde{g}_\alpha}(\phi)(\psi) &= \sum_{\mathbf{k}\in\Gamma_0^S} \varphi^{\mathbf{k}}\left(K^{-1}\phi\right)\varphi^{\mathbf{k}}\left(K^{-1}\psi\right) \\ &+ \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma^{j,\boldsymbol{\theta}}\left(K^{-1}\phi\right)\gamma^{j,\boldsymbol{\theta}}\left(K^{-1}\psi\right), \end{aligned} \tag{35}$$

for all $\phi,\psi\in V_\alpha$. From Lemma 3.1(i) and Eq. (36), the solution to Eq. (19) can be calculated as follows:

$$\begin{aligned} [\mathcal{L}'(\phi)](\psi) &= [R_{\tilde{g}_\alpha} R'_{f_\beta g_\alpha}(\phi)](\psi) \\ &= \sum_{\mathbf{k}\in\Gamma_0^S} \varphi^{\mathbf{k}}(K^{-1}R'_{f_\beta g_\alpha}\phi)\varphi^{\mathbf{k}}(K^{-1}\psi) \\ &+ \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma^{j,\boldsymbol{\theta}}(K^{-1}R'_{f_\beta g_\alpha}\phi)\gamma^{j,\boldsymbol{\theta}}(K^{-1}\psi), \end{aligned}$$

for all $\phi\in U_\beta$ and $\psi\in V_\alpha$. Now, for $\mathbf{k}\in\Gamma_0^S$ and $\boldsymbol{\theta}\in\Lambda_j^S$, $j\geq 0$,

$$\begin{aligned} \varphi_{\mathbf{k}}(\phi) &= \left(\mathcal{T}_{f_\beta}(\phi_{\mathbf{k}}),\phi\right) = \left(\phi_{\mathbf{k}},\mathcal{T}'_{f_\beta}(\phi)\right) = [\mathcal{T}'_{f_\beta}(\phi)](\phi_{\mathbf{k}}), \\ \varphi^{\mathbf{k}}(\psi) &= \left((\mathcal{T}_{f_\beta}^{-1})'(\phi_{\mathbf{k}}),\psi\right) = \left(\phi_{\mathbf{k}},\mathcal{T}_{f_\beta}^{-1}(\psi)\right) = [\mathcal{T}_{f_\beta}^{-1}(\psi)](\phi_{\mathbf{k}}). \end{aligned} \tag{36}$$

Similarly,

$$\begin{aligned} \gamma_{j,\boldsymbol{\theta}}(\phi) &= [\mathcal{T}'_{f_\beta}(\phi)](\psi_{j,\boldsymbol{\theta}}), \\ \gamma^{j,\boldsymbol{\theta}}(\psi) &= [\mathcal{T}_{f_\beta}^{-1}(\psi)](\psi_{j,\boldsymbol{\theta}}), \end{aligned} \tag{37}$$

for $\phi\in U_\beta$ and $\psi\in V_\beta$, where, as before, $\{\phi_{\mathbf{k}}:\mathbf{k}\in\Gamma_0^S\}\cup\{\psi_{j,\boldsymbol{\theta}}:\boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\}$ are the orthonormal scaling and wavelet bases of $L^2(S)$ previously considered. From Eqs. (37) and (38), the values $\{\varphi_{\mathbf{k}}(\phi):\mathbf{k}\in\Gamma_0^S\}\cup\{\gamma_{j,\boldsymbol{\theta}}(\phi):\boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0\}$ and $\{\varphi^{\mathbf{k}}(\psi):\mathbf{k}\in\Gamma_0^S\}\cup\{\gamma^{j,\boldsymbol{\theta}}(\psi):\boldsymbol{\theta}\in$

$\Lambda_j^S,\ j \geq 0\}$ represent, respectively, the coordinates of the functions $\mathcal{T}'_{f_\beta}(\phi)$ and $\mathcal{T}^{-1}_{f_\beta}(\psi)$ with respect to the othonormal basis $\{\phi_{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\} \cup \{\psi_{j,\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Lambda_j^S,\ j \geq 0\}$.

Furthermore,

$$
\begin{aligned}
\mathcal{R}_{g_\alpha} &= \mathcal{T}_{g_\alpha}\mathcal{T}'_{g_\alpha} = K\mathcal{T}_{f_\beta}\mathcal{T}'_{f_\beta}K', \\
R_{\tilde{g}_\alpha} &= (\mathcal{T}^{-1}_{g_\alpha})'\mathcal{T}^{-1}_{g_\alpha}, \quad \text{and} \\
R'_{f_\beta g_\alpha} &= \mathcal{T}_{g_\alpha}\mathcal{T}'_{f_\beta}.
\end{aligned} \tag{38}
$$

We finally obtain the last equality in Eq. (34), using Eqs. (37), (38) and (39), as follows:

$$
\begin{aligned}
\mathcal{L}'(\phi)(\psi) &= R_{\tilde{g}_\alpha}R'_{f_\beta g_\alpha}(\phi)(\psi) = \left((\mathcal{T}^{-1}_{g_\alpha})'\mathcal{T}^{-1}_{g_\alpha}\mathcal{T}_{g_\alpha}\mathcal{T}'_{\beta}(\phi), \psi\right) \\
&= \left(\mathcal{T}'_{f_\beta}(\phi), \mathcal{T}^{-1}_{g_\alpha}(\psi)\right) = \left(\mathcal{T}'_{f_\beta}(\phi), \mathcal{T}^{-1}_{f_\beta}(K^{-1}\psi)\right) \\
&= \sum_{\mathbf{k}\in\Gamma_0^S} \varphi_{\mathbf{k}}(\phi)\varphi^{\mathbf{k}}(K^{-1}\psi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma_{j,\boldsymbol{\theta}}(\phi)\gamma^{j,\boldsymbol{\theta}}(K^{-1}\psi).
\end{aligned}
$$

∎

From Remark 4.2, a series representation for the solution $\mathcal{L}'$ to Eq. (21) can be derived in a similar way to Theorem 4.1.

A finite-dimensional formulation of the least-squares linear inverse estimation problem considered here can be obtained by truncation, say at scale $j = M$, of the wavelet-based orthogonal expansions of the input and output random fields. In particular, the corresponding truncated series expansion to (34) provides a finite-dimensional approximation to the least-squares linear estimate of the input random field. Denoting by $N(0)$ the number of non-zero scaling coefficients, and by $Q(j)$ the number of non-zero wavelet coefficients at each scale j, the approximation of $\mathcal{L}'$ at scale $j = M$ is given by

$$
\begin{aligned}
\mathcal{L}'(\phi)(\psi) &= \sum_{r=1}^{N(0)} \varphi_{\mathbf{k}_r}(\phi)\varphi^{\mathbf{k}_r}(K^{-1}\psi) + \sum_{j=0}^{M-1}\sum_{u=1}^{Q(j)} \gamma_{j,\boldsymbol{\theta}_u}(\phi)\gamma^{j,\boldsymbol{\theta}_u}(K^{-1}\psi) \\
&= \left[\boldsymbol{\varphi}(\phi)^T, \boldsymbol{\gamma}(\phi)^T\right]\left[\tilde{\boldsymbol{\varphi}}(K^{-1}\psi)^T, \tilde{\boldsymbol{\gamma}}(K^{-1}\psi)^T\right]^T,
\end{aligned} \tag{39}
$$

for all $\phi \in U_\beta$ and $\psi \in V_\alpha$. Here, $[\cdot]^T$ means the transposition of $[\cdot]$, and

$$\begin{aligned}
\boldsymbol{\varphi}(\phi) &= \left[\varphi_{\mathbf{k}_1}(\phi), \cdots, \varphi_{\mathbf{k}_{N(0)}}(\phi)\right]^T, \\
\boldsymbol{\gamma}(\phi) &= \left[\gamma_{0,\boldsymbol{\theta}_1}(\phi), \cdots, \gamma_{0,\boldsymbol{\theta}_{Q(0)}}(\phi), \cdots, \gamma_{M-1,\boldsymbol{\theta}_1}(\phi), \cdots, \gamma_{M-1,\boldsymbol{\theta}_{Q(M-1)}}(\phi)\right]^T, \\
\tilde{\boldsymbol{\varphi}}(K^{-1}\psi) &= \left[\varphi^{\mathbf{k}_1}(K^{-1}\psi), \cdots, \varphi^{\mathbf{k}_{N(0)}}(K^{-1}\psi)\right]^T, \\
\tilde{\boldsymbol{\gamma}}(K^{-1}\psi) &= \left[\gamma^{0,\boldsymbol{\theta}_1}(K^{-1}\psi), \cdots, \gamma^{0,\boldsymbol{\theta}_{Q(0)}}(K^{-1}\psi), \cdots, \gamma^{M-1,\boldsymbol{\theta}_{Q(M-1)}}(K^{-1}\psi)\right]^T.
\end{aligned}$$

In the last result in this section, we refer to the case where the integral operator K is not necessarily in the class $\mathcal{K}$. The wavelet-based orthogonal expansion of the input random field model allows in this case the computation of a pointwise approximation on $K(V_\beta)$ to the functions in the image space of the operator $\mathcal{L}'$ defined by Eq. (19).

THEOREM 4.2. *Consider the fractional generalized observation model defined by Eq.* (16). *Assume that a model for the input random field is given by the fractional GRF f_β with fractional duality order β. Then, the image-space of the operator $\mathcal{L}'$ admits the following pointwise series approximation on $K(V_\beta)$:*

$$\mathcal{L}'(\phi)(K\psi) = \sum_{\mathbf{k}\in\Gamma_0^S} \varphi^{\mathbf{k}}(R_{f_\beta}\phi)\varphi^{\mathbf{k}}(\psi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma^{j,\boldsymbol{\theta}}(R_{f_\beta}\phi)\gamma^{j,\boldsymbol{\theta}}(\psi),$$

for all $\psi \in V_\beta$ and $\phi \in U_\beta$, with the system $\{\varphi^{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\} \cup \{\gamma^{j,\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Lambda_j^S, j \geq 0\}$ being defined as in Lemma 4.1.

Proof. From Eqs. (37) and (38), for $\psi \in V_\beta$, and $\phi \in U_\beta$, we obtain

$$\begin{aligned}
\mathcal{L}'(\phi)\,(K\psi) &= \left(\mathcal{T}_{f_\beta}^{-1}R_{f_\beta}(\phi), \mathcal{T}_{f_\beta}^{-1}(\psi)\right) \\
&= \sum_{\mathbf{k}\in\Gamma_0^S} \varphi^{\mathbf{k}}(R_{f_\beta}\phi)\varphi^{\mathbf{k}}(\psi) + \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} \gamma^{j,\boldsymbol{\theta}}(R_{f_\beta}\phi)\gamma^{j,\boldsymbol{\theta}}(\psi).
\end{aligned}$$

∎

5. APPLICATIONS

To illustrate the approach described in Sections 3 and 4, we study here two examples where the regularization method developed can be applied.

In both examples the integral operator K belongs to the class $\mathcal{K}$ and represents fractional integration. In Example 1, the input model is given by fractional Brownian motion. In Example 2, K is defined in terms of the Riesz kernel, and the generalized input model is defined in terms of a fractional integral equation involving a Bessel kernel. These two examples then cover important models in applications (see, for example, [12] and [23]).

5.1. Example 1

We first consider the following stochastic integral equation:

$$g = Kf, \tag{41}$$

with

$$\begin{aligned} K &= (I-\Delta)^{-(\nu+\rho)/2}, \\ f(\mathbf{z}) &= \left[(-\Delta)^{-\rho/2}\varepsilon\right](\mathbf{z}), \quad \mathbf{z} \in S \subseteq \mathbb{R}^d, \\ g(\mathbf{z}) &= \left[(I-\Delta)^{-\rho/2}X\right](\mathbf{z}), \quad \mathbf{z} \in S \subseteq \mathbb{R}^d, \end{aligned} \tag{41}$$

where $\rho, \nu \in \mathbb{R}_+$, S is a bounded $C^\infty-$domain, $(-\Delta)$ denotes the negative Laplacian operator on such a domain, ε is a Gaussian white noise on $L^2(S)$, and X is defined by the stochastic fractional-order differential equation

$$(I-\Delta)^{\nu/2}(-\Delta)^{\rho/2}X = \varepsilon. \tag{43}$$

The weak-sense solution to Eq. (43) has been studied in [6] as an important example of a fractional GRF, called *fractional Riesz-Bessel motion* (fRBm). For $\nu \geq 0$ and $0 < \rho < d$, the fRBm has fractional duality order $\nu + \rho$. In the case where $\nu + \rho > d/2$, the weak-sense solution to Eq. (43) on S becomes strong-sense.

Note that for different values of parameter ρ, the input random field model considered for f covers Brownian motion and fractional Brownian motion.

From Lemma 3.2 and Lemma 3.3 of [6], g has fractional duality order $\nu + 2\rho$. That is, g defines a fractional generalized random field $g_{-(\nu+2\rho)}$ on $U_{-(\nu+2\rho)} = \bar{H}^{-(\nu+2\rho)}(S)$ and its fractional generalized dual $\tilde{g}_{\nu+2\rho}$ exists on $V_{-(\nu+2\rho)} = H^{\nu+2\rho}(S)$. Therefore, its covariance operator $R_{g_{-(\nu+2\rho)}}$ defines an isomorphism from $\bar{H}^{-(\nu+2\rho)}(S)$ into $H^{\nu+2\rho}(S)$. In addition, from Theorem 9.5.10(a) of [14], it can be proved that $(-\Delta)^{-\rho/2}$, with $\rho \in (0, d)$, is bicontinuous as an operator from $\bar{H}^{-\rho}(S)$ into $L^2(S)$. The covariance operator R_f of f is then a bicontinuous operator from $\bar{H}^{-\rho}(S)$ into $H^\rho(S)$, and, from [17] (p. 164), R_f is an isomorphism from $\bar{H}^{-\rho}(S)$ into $H^\rho(S)$. The considered input model for f then defines a fractional GRF $f_{-\rho}$ on $U_{-\rho} = \bar{H}^{-\rho}(S)$ with fractional duality order ρ.

Since $K' = [(I-\Delta)^{-(\nu+\rho)/2}]' = \mathcal{I}'_{\nu+\rho}$, then K belongs to $\mathcal{K}$. Moreover, K' defines an isomorphism from $\bar{H}^{-(\nu+2\rho)}(S)$ onto $\bar{H}^{-\rho}(S)$, and the conditions of Lemma 4.1 are satisfied. Hence, from Theorem 4.1, operator $\mathcal{L}'$ defining the least-squares linear estimate of the input random field in Eqs. (41) and (42) admits a wavelet-based series expansion as in Eq. (34), with $\beta = -\rho$ and $\alpha = -(\nu+2\rho)$. For each $\phi \in \bar{H}^{-\rho}(S)$, the coefficients of $\mathcal{L}'(\phi)$ in such an expansion can be calculated as defined below.

Let $\{\phi_{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\} \cup \{\psi_{j,\boldsymbol{\theta}} : \ \boldsymbol{\theta} \in \Lambda_j^S, j \geq 0\}$ be an orthonormal scaling and wavelet basis of $L^2(S)$. For each $\mathbf{k} \in \Gamma_0^S$ and $\boldsymbol{\theta} \in \Lambda_j^S$, with $j \geq 0$,

$$\begin{aligned}
\varphi_{\mathbf{k}}(\phi) &= \int_{\mathbb{R}^d} \left(\frac{\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})}{|\boldsymbol{\lambda}|^\rho} \right) \mathcal{F}(\phi)(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \\
\varphi^{\mathbf{k}}(\phi^*) &= \int_{\mathbb{R}^d} \left(\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda}) |\boldsymbol{\lambda}|^\rho \right) \mathcal{F}(\phi^*)(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \\
\gamma_{j,\boldsymbol{\theta}}(\phi) &= \int_{\mathbb{R}^d} \left(\frac{\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})}{|\boldsymbol{\lambda}|^\rho} \right) \mathcal{F}(\phi)(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \\
\gamma^{j,\boldsymbol{\theta}}(\phi^*) &= \int_{\mathbb{R}^d} \left(\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda}) |\boldsymbol{\lambda}|^\rho \right) \mathcal{F}(\phi^*)(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \qquad (43)
\end{aligned}$$

for all $\phi \in \bar{H}^{-\rho}(S) = U_{-\rho}$, where $*$ stands for the duality between fractional Sobolev spaces with respect to $L^2(S)$, and $\mathcal{F}$ denotes the Fourier transform. Let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(S)$. For each $n \in \mathbb{N}$, we define

$$\begin{aligned}
\tilde{u}_n(\psi^*) &= \int_{\mathbb{R}^d} \frac{\mathcal{F}(u_n)(\boldsymbol{\lambda})\mathcal{F}(\psi^*)(\boldsymbol{\lambda})}{(1+|\boldsymbol{\lambda}|^2)^{(\nu+2\rho)/2}} d\boldsymbol{\lambda}, \\
\tilde{u}^n(\psi) &= \int_{\mathbb{R}^d} (1+|\boldsymbol{\lambda}|^2)^{(\nu+2\rho)/2} \mathcal{F}(u_n)(\boldsymbol{\lambda})\mathcal{F}(\psi)(\boldsymbol{\lambda}) d\boldsymbol{\lambda},
\end{aligned} \qquad (44)$$

for all $\psi \in V_{-(\nu+2\rho)} = H^{\nu+2\rho}(S)$. The systems $\{\tilde{u}_n : n \in \mathbb{N}\} \subseteq H^{\nu+2\rho}(S)$ and $\{\tilde{u}^n : n \in \mathbb{N}\} \subseteq \bar{H}^{-(\nu+2\rho)}(S)$ are dual Riesz bases with respect to $L^2(S)$. Eq. (34) is then calculated by using Parseval's identity as follows:

$$\begin{aligned}
\mathcal{L}'(\phi)(\tilde{u}_n) = \sum_{\mathbf{k}\in\Gamma_0^S} &\left[\int_{\mathbb{R}^d} \frac{\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})\mathcal{F}(\phi)(\boldsymbol{\lambda})}{|\boldsymbol{\lambda}|^\rho} d\boldsymbol{\lambda} \right] \left[\int_{\mathbb{R}^d} \frac{|\boldsymbol{\lambda}|^\rho \mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})\mathcal{F}(u_n)(\boldsymbol{\lambda})}{[1+|\boldsymbol{\lambda}|^2]^{\rho/2}} d\boldsymbol{\lambda} \right] \\
+ \sum_{j\geq 0} \sum_{\boldsymbol{\theta}\in\Lambda_j^S} &\left[\int_{\mathbb{R}^d} \frac{\mathcal{F}(\phi)(\boldsymbol{\lambda})\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})}{|\boldsymbol{\lambda}|^\rho} d\boldsymbol{\lambda} \right] \left[\int_{\mathbb{R}^d} \frac{|\boldsymbol{\lambda}|^\rho \mathcal{F}(u_n)(\boldsymbol{\lambda})\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})}{[1+|\boldsymbol{\lambda}|^2]^{\rho/2}} d\boldsymbol{\lambda} \right],
\end{aligned}$$

for $n \in \mathbb{N}$, and for each $\phi \in U_\rho = \bar{H}^{-\rho}(S)$.

From Lemma 4.1, we can generate the random variables $\{g_{-(\nu+2\rho)}(\mathcal{L}'\phi) : \phi \in U_{-\rho}\}$ (see Eq. (**??**)), obtaining the least-squares linear approximation of $\{f_{-\rho}(\phi) : \phi \in U_{-\rho}\}$. Then, for each $\phi \in U_{-\rho}$,

$$\begin{aligned}
\hat{f}_{-\rho}(\phi) &= g_{-(\nu+2\rho)}\left(\sum_{n\in\mathbb{N}} \mathcal{L}'(\phi)(\tilde{u}_n)\tilde{u}^n\right) \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} g^{\mathbf{k}}_{-(\nu+2\rho)}(\omega)\varphi_{\mathbf{k}}\left(\mathcal{I}'_{\nu+\rho}\left(\sum_{n\in\mathbb{N}} \mathcal{L}'(\phi)(\tilde{u}_n)\tilde{u}^n\right)\right) \\
&+ \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j} g^{j,\boldsymbol{\theta}}_{-(\nu+2\rho)}(\omega)\gamma_{j,\boldsymbol{\theta}}\left(\mathcal{I}'_{\nu+\rho}\left(\sum_{n\in\mathbb{N}} \mathcal{L}'(\phi)(\tilde{u}_n)\tilde{u}^n\right)\right), \quad (45)
\end{aligned}$$

where $\mathcal{I}_{\nu+\rho} = (I-\Delta)^{-(\nu+\rho)/2}$, and the random coefficients $\{g^{\mathbf{k}}_{-(\nu+2\rho)} : \mathbf{k} \in \Gamma_0^S\} \cup \{g^{j,\boldsymbol{\theta}}_{-(\nu+2\rho)} : \boldsymbol{\theta} \in \Lambda_j^S, j \geq 0\}$, which coincide with the random coefficients $\{f^{\mathbf{k}}_{-\rho} : \mathbf{k} \in \Gamma_0^S\} \cup \{f^{j,\boldsymbol{\theta}}_{-\rho} : \boldsymbol{\theta} \in \Lambda_j^S, j \geq 0\}$ of the wavelet-based orthogonal expansion of $f_{-\rho}$, are uncorrelated and all have the standard Gaussian distribution. The system $\{\varphi_{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\} \cup \{\gamma_{j,\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Lambda_j^S, j \geq 0\}$ is defined as in Eq. (44). Similarly, the input model $\{f_{-\rho}(\phi) : \phi \in U_\rho\}$ can be generated from Eq. (30), with $\beta = -\rho$ and $\{\varphi_{\mathbf{k}} : \mathbf{k} \in \Gamma_0^S\} \cup \{\gamma_{j,\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Lambda_j^S, j \geq 0\}$ defined as in Eq. (44). That is, for each $n \in \mathbb{N}$,

$$\begin{aligned}
f_{-\rho}(v^n) &= \sum_{\mathbf{k}\in\Gamma_0^S} f^{\mathbf{k}}_{-\rho}(\omega)\int_{\mathbb{R}^d} \frac{\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})\mathcal{F}(u_n)(\boldsymbol{\lambda})(1+|\boldsymbol{\lambda}|^2)^{\rho/2}}{|\boldsymbol{\lambda}|^\rho}d\boldsymbol{\lambda} \\
&+ \sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S} f^{j,\boldsymbol{\theta}}_{-\rho}(\omega)\int_{\mathbb{R}^d} \frac{\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})\mathcal{F}(u_n)(\boldsymbol{\lambda})(1+|\boldsymbol{\lambda}|^2)^{\rho/2}}{|\boldsymbol{\lambda}|^\rho}d\boldsymbol{\lambda},
\end{aligned}$$

where $\{v_n : n \in \mathbb{N}\} \subseteq H^\rho(S)$ and $\{v^n : n \in \mathbb{N}\} \subseteq \bar{H}^{-\rho}(S)$ are dual Riesz bases defined from the orthonormal basis $\{u_n : n \in \mathbb{N}\} \subseteq L^2(S)$ similarly to Eq. (45).

5.2. Example 2

We now consider the fractional generalized stochastic integral equation

$$g_\alpha(\phi) = f_\beta(K'(\phi)), \quad \forall \phi \in U_\alpha = \bar{H}^\alpha(S),$$

where $K = (-\Delta)^{-\rho/2}$ represents fractional integration. The model f_β considered here for the input is improper, and it is defined as the fractional generalized solution to the following fractional-order integral equation:

$$(I-\Delta)^{-\beta/2} f_\beta = \varepsilon,$$

where ε represents a Gaussian white noise. Then,

$$f_\beta(\phi) = (I-\Delta)^{\beta/2}\varepsilon(\phi), \quad \forall \phi \in U_\beta = \bar{H}^\beta(S),$$

with ε now interpreted as a generalized white noise on $L^2(S)$ (see Eq. (18)). The random field g_α is defined as the generalized solution to the fractional-order integro-differential equation

$$\left[(I-\Delta)^{-\beta/2}(-\Delta)^{\rho/2}\right] g_\alpha = \varepsilon,$$

where we denote $\alpha = \beta - \rho > 0$. Then,

$$g_\alpha(\phi) = \left[(-\Delta)^{-\rho/2}(I-\Delta)^{\beta/2}\right]\varepsilon(\phi), \quad \forall \phi \in U_\alpha = \bar{H}^\alpha(S).$$

The fractional singularity and duality orders of the model f_β are both equal to β. The fractional singularity and duality orders of g_α are both equal to α. As in the above example, a wavelet-based orthogonal expansion of f_β is calculated in terms of the following dual Riesz bases:

$$\begin{aligned}
\varphi_{\mathbf{k}}(\phi) &= \int_{\mathbb{R}^d} [1+|\boldsymbol{\lambda}|^2]^{\beta/2}\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})\mathcal{F}(\phi)(\boldsymbol{\lambda})d\boldsymbol{\lambda}, \quad \text{and} \\
\varphi^{\mathbf{k}}(\phi^*) &= \int_{\mathbb{R}^d} \left(\frac{\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})}{[1+|\boldsymbol{\lambda}|^2]^{\beta/2}}\right)\mathcal{F}(\phi^*)(\boldsymbol{\lambda})d\boldsymbol{\lambda}, \quad \mathbf{k}\in\Gamma_0^S; \\
\gamma_{j,\boldsymbol{\theta}}(\phi) &= \int_{\mathbb{R}^d} [1+|\boldsymbol{\lambda}|^2]^{\beta/2}\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})\mathcal{F}(\phi)(\boldsymbol{\lambda})d\boldsymbol{\lambda}, \quad \text{and} \\
\gamma^{j,\boldsymbol{\theta}}(\phi^*) &= \int_{\mathbb{R}^d} \left(\frac{\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})}{[1+|\boldsymbol{\lambda}|^2]^{\beta/2}}\right)\mathcal{F}(\phi^*)(\boldsymbol{\lambda})d\boldsymbol{\lambda}, \quad \boldsymbol{\theta}\in\Lambda_j^S,\ j\geq 0,
\end{aligned}$$

for all $\phi \in U_\beta$, where, as before, $\{\phi_{\mathbf{k}} : \mathbf{k}\in\Gamma_0^S\}\cup\{\psi_{j,\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Lambda_j^S, j\geq 0\}$ is an orthonormal scaling and wavelet basis. The series expansion of operator $\mathcal{L}'$ can then be computed as follows: For $\phi\in U_\beta$ and $n\in\mathbb{N}$,

$$\begin{aligned}
\mathcal{L}'(\phi)(\tilde{u}^n) &= \sum_{\mathbf{k}\in\Gamma_0^S}\left[\int_{\mathbb{R}^d}[1+|\boldsymbol{\lambda}|^2]^{\beta/2}\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})\mathcal{F}(\phi)(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right] \\
&\quad\times\left[\int_{\mathbb{R}^d}\frac{|\boldsymbol{\lambda}|^\rho[1+|\boldsymbol{\lambda}|^2]^{\alpha/2}\mathcal{F}(\phi_{\mathbf{k}})(\boldsymbol{\lambda})\mathcal{F}(u_n)(\boldsymbol{\lambda})}{[1+|\boldsymbol{\lambda}|^2]^{\beta/2}}d\boldsymbol{\lambda}\right] \\
&\quad+\sum_{j\geq 0}\sum_{\boldsymbol{\theta}\in\Lambda_j^S}\left[\int_{\mathbb{R}^d}[1+|\boldsymbol{\lambda}|^2]^{\beta/2}\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})\mathcal{F}(\phi)(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right] \\
&\quad\times\left[\int_{\mathbb{R}^d}\frac{|\boldsymbol{\lambda}|^\rho[1+|\boldsymbol{\lambda}|^2]^{\alpha/2}\mathcal{F}(\psi_{j,\boldsymbol{\theta}})(\boldsymbol{\lambda})\mathcal{F}(u_n)(\boldsymbol{\lambda})}{[1+|\boldsymbol{\lambda}|^2]^{\beta/2}}d\boldsymbol{\lambda}\right], \quad (46)
\end{aligned}$$

where $\{\tilde{u}_n : n \in \mathbb{N}\} \subseteq U_\alpha$ and $\{\tilde{u}^n : n \in \mathbb{N}\} \subseteq V_\alpha$ are dual Riesz bases with respect to $L^2(S)$, defined from an orthonormal basis $\{u_n : n \in \mathbb{N}\}$ of this space as in Eq. (45). From Eq. (47), the least-squares linear estimate $\hat{f}_\beta$ can be calculated similarly to Eq. (46).

6. CONCLUSION

The definition and study of deterministic or random functions with a very irregular local behaviour has traditionally been carried out by using the theory of distributions or generalized functions (see, for example, [16]). The usual concept of generalized random field on $C_0^\infty(S)$ is based on the theory of countably, nuclear, and rigged Hilbert spaces (see also [27], [28]). The concept of fractional GRF considered in this paper is based on the abstract definition of generalized random field on a Hilbert space (see, for example, [1][29]), and the theory of fractional Sobolev spaces. The fractional generalized approach allows to measure, in the continuous real scale, the degree of regularity or singularity in the mean-square sense of a random function. Furthermore, under the duality condition, the Sobolev geometry can be related to the geometry defined by the associated generalized covariance function via the RKHS. The duality condition is equivalent to defining the covariance operator as an isomorphism between appropriate fractional Sobolev spaces. The class of stochastic models satisfying such a condition includes important cases mentioned in Section 1. In this paper, we use such a class as the prior model class $\mathcal{P}$ to regularize stochastic linear inverse problems defined in terms of an element K of the class $\mathcal{K}$ (see Section 3). The fractional duality order of the generalized observation model determines the function space where the solution to the least-squares linear inverse estimation problem can be found. The computation of the least-squares linear estimate of the input random field is achieved, in this fractional generalized framework, using the wavelet-based series expansions derived in Section 4.

In practice, the application of the method can be summarized in the following steps:

Step 1. Measurement of the minimum fractional singularity order α of the observation process.

Step 2. Determination of the parameter γ associated with K in class $\mathcal{K}$.

Step 3. Selection of the fractional GRF model for the input random field from the class $\mathcal{P}$ according to Steps 1 and 2 to allow the inversion of K.

Step 4. Calculation of the wavelet-based series expansion of $\mathcal{L}'$ defining the least-squares linear estimate of the input.

Step 5. Approximation of the least-squares linear estimate of the input fractional GRF in terms of the truncated wavelet-based series expansion calculated in Step 4.

In the case where K is not in $\mathcal{K}$, Theorem 4.2 of Section 4 provides a weak-sense wavelet-based series approximation to $\mathcal{L}'$, provided that the prior information can be represented by an element of $\mathcal{P}$.

As the class $\mathcal{P}$ includes ordinary and generalized stochastic models, the presented approximation covers both cases, as we note in Sections 1 and 3. Moreover, this class contains models important in applications such as fractional Brownian motion, fractional Riesz-Bessel motion, fractional-order rational functions of self-adjoint elliptic differential operators of fractional order, etc. (see, for example, [4], [6], [7], [23], [25] and [35]). The class $\mathcal{K}$ also includes important fractional integral transforms, as mentioned in Section 3, such as the Riesz transform (in particular, the Abel transform) and the Bessel transform (see, for example, [12] and [32]).

Finally, it must be noted that in certain applications, for instance in groundwater hydrology, inverse problems usually involve compact operators, which do not have a continuous inverse with respect to the L^2-topology. However, a continuous inverse may exist with respect to a certain fractional Sobolev topology. That is the case of positive rational functions of elliptic and self-adjoint differential operators, and, in particular, of homogeneous and self-adjoint integral operators with a positive rational Fourier transform.

ACKNOWLEDGMENT

This work has been supported in part by projects PB96-1440 and BFM2000-1465 of the DGI, Spain, and by the Australian Research Council grant A89601825.

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