Robust Adaptive Control of Rigid Spacecraft Attitude Maneuvers

Aaron John Dando
Bachelor of Engineering (Aerospace Avionics)
Queensland University of Technology

Dissertation submitted to the Faculty of Built Environment and
Engineering at the Queensland University of Technology in partial
fulfilment of the requirements for the degree of
Doctor of Philosophy

School of Engineering Systems
Faculty of Built Environment and Engineering
Queensland University of Technology
July 2008

QUT

STATEMENT OF ORIGINAL AUTHORSHIP

"The work contained in this thesis has not been previously submitted to meet requirements for an award at this or any other higher education institution. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made."

Signed: _____

Date: _____

ABSTRACT

In this thesis novel feedback attitude control algorithms and attitude estimation algorithms are developed for a three-axis stabilised spacecraft attitude control system. The spacecraft models considered include a rigid-body spacecraft equipped with (i) external control torque devices, and (ii) a redundant reaction wheel configuration. The attitude sensor suite comprises a three-axis magnetometer and three-axis rate gyroscope assembly. The quaternion parameters (also called Euler symmetric parameters), which globally avoid singularities but are subject to a unity-norm constraint, are selected as the primary attitude coordinates. There are four novel contributions presented in this thesis.

The first novel contribution is the development of a robust control strategy for spacecraft attitude tracking maneuvers, in the presence of dynamic model uncertainty in the spacecraft inertia matrix, actuator magnitude constraints, bounded persistent external disturbances, and state estimation error. The novel component of this algorithm is the incorporation of state estimation error into the stability analysis. The proposed control law contains a parameter which is dynamically adjusted to ensure global asymptotic stability of the overall closed-loop system, in the presence of these specific system non-idealities. A stability proof is presented which is based on Lyapunov's direct method, in conjunction with Barbalat's lemma. The control design approach also ensures minimum angular path maneuvers, since the attitude quaternion parameters are not unique.

The second novel contribution is the development of a robust direct adaptive control strategy for spacecraft attitude tracking maneuvers, in the presence of dynamic model uncertainty in the spacecraft inertia matrix. The novel aspect of this algorithm is the incorporation of a composite parameter update strategy, which ensures global exponential convergence of the closed-loop system. A stability proof is presented which is based on Lyapunov's direct method, in conjunction with Barbalat's lemma. The exponential convergence results provided by this control strategy require persistently exciting reference

trajectory commands. The control design approach also ensures minimum angular path maneuvers.

The third novel contribution is the development of an optimal control strategy for spacecraft attitude maneuvers, based on a rigid body spacecraft model including a redundant reaction wheel assembly. The novel component of this strategy is the proposal of a performance index which represents the total electrical energy consumed by the reaction wheel over the maneuver interval. Pontraygin's minimum principle is applied to formulate the necessary conditions for optimality, in which the control torques are subject to time-varying magnitude constraints. The presence of singular sub-arcs in the state-space and their associated singular controls are investigated using Kelley's necessary condition. The two-point boundary-value problem (TPBVP) is formulated using Pontrayagin's minimum principle.

The fourth novel contribution is an attitude estimation algorithm which estimates the spacecraft attitude parameters and sensor bias parameters from three-axis magnetometer and three-axis rate gyroscope measurement data. The novel aspect of this algorithm is the assumption that the state filtering probability density function (PDF) is Gaussian distributed. This Gaussian PDF assumption is also applied to the magnetometer measurement model. Propagation of the filtering PDF between sensor measurements is performed using the Fokker-Planck equation, and Bayes theorem incorporates measurement update information. The use of direction cosine matrix elements as the attitude coordinates avoids any singularity issues associated with the measurement update and estimation error covariance representation.

PUBLICATIONS AND AWARDS

International Conference Papers

[1] Dando, A., and W. Enderle. (2005). The attitude control system concept for the Joint Australian Engineering Micro-Satellite (JAESat). Proceedings of the AIAA/USU Conference on Small Satellites, Utah State University, Logan, Utah, USA, August 8-11. Paper #SSC05-VII-6.

National Conference Papers

[2] A. Dando. (2005). JAESat micro-satellite attitude determination and control. Proceedings of the Australian Space Research Institute Conference, University of Queensland, Brisbane, Queensland, Australia, November 27.

[3] A. Dando. (2006). Attitude determination and control strategies for the JAESat micro-satellite. Proceedings of the Queensland University of Technology Built Environment and Engineering Design Theme Conference, QUT, Brisbane, Queensland, Australia, June 30.

Conference Paper Awards

[4] Best paper award in the Frank J. Redd Student Scholarship Competition, based on the work in [1].

[5] Best overall paper award in the Australian Space Research Institute Conference, based on the work in [2].

Journal Articles

[6] A. Dando. (2008). Nonlinear filtering for spacecraft attitude estimation based on the Fokker-Planck equation. In preparation.

[7] A. Dando. (2008). Direct adaptive control of spacecraft attitude maneuvers with exponential parameter convergence. In preparation.

[8] A. Dando. (2008). Minimum energy attitude maneuvers for spacecraft equipped with a redundant reaction wheel assembly. In preparation.

ACKNOWLEDGEMENTS

Table of Contents

List of Figures

List of Tables

Chapter 1   Introduction

This thesis considers the design of spacecraft attitude maneuvers and attitude estimation algorithms for general spacecraft mission applications. The specific dynamic model considered is a rigid-body spacecraft equipped with external control torque devices, and an attitude sensor suite. The latter consists of direction-type sensors (for example digital sun sensors, which provide a unit-vector measurement in the direction of the sun, or a magnetometer, which measures the earth's magnetic field) and a three-axis rate gyroscope assembly that measures the spacecraft angular rates. The main objective of this thesis is to investigate and propose novel robust adaptive control methods for rigid-body spacecraft attitude maneuvers. A secondary objective is to investigate and propose novel state estimation techniques for spacecraft attitude estimation.

This research is motivated by the future class of scientific spacecraft missions such as the NASA Beyond Einstein mission proposals[1] that will address specific questions regarding the structure and evolution of the universe. To meet stringent scientific objectives future spacecraft will require orders of magnitude improvements in guidance, navigation, and control (GNC) performance, which will in turn require radical advances in GNC spacecraft hardware and algorithms. The attitude control requirements will consist of high accuracy autonomous pointing and rapid slewing capabilities in the presence of environmental and systematic errors. The attitude estimation requirements will include the capability to provide high accuracy state estimates to the control algorithm so that rapid estimation filter convergence can occur even if there is poor filter initialisation. Additional attitude control system requirements may be introduced if there is a need to have coordinated control of multiple spacecraft in formation[2], which may well be necessary to satisfy the science objectives of missions such as LISA and Constellation X[1]. The formation GNC objectives can only be achieved when each spacecraft is carefully controlled to respond to the formation coordination commands. The control requirements are further complicated by the fact that each spacecraft will generally have a unique disturbance environment and actuator/sensor configuration.

1.1 Spacecraft Attitude Maneuvers

The general problem of large-angle three-axis spacecraft attitude maneuvers for rigid-body spacecraft has been the subject of considerable research[16,22,34,39]. The application of optimal control theory[8-12] in the design of feedback and open-loop controls for attitude maneuvers is extensively addressed in the literature[13-16]. The optimality criterion is dependant on the specific actuator configuration and spacecraft mission requirements, but useful cost functions include maneuver time, fuel consumption for gas jet thruster actuators, energy consumption for reaction wheel actuators, and more general indices such as spacecraft angular rate parameters. The optimal control may be computed using either open-loop or state feedback (closed-loop) methodologies. The closed-loop approach[13,14] requires the solution of the Hamilton-Jacobi-Bellman partial differential equation which suffers from the curse of dimensionality but offers the advantage of a robust closed-loop control framework. The open-loop approach[15,16] requires the solution of a nonlinear two-point boundary value (TPBV) problem[10] using iterative numerical methods[11]. In practice however open-loop maneuvers are sensitive to spacecraft dynamic model uncertainty and external disturbance torques, and are not suited to on-board real-time implementation. The design of state-feedback controls for performing spacecraft attitude maneuvers based on Lyapunov stability theory[17-20] has been extensively addressed in the literature[21-28]. These methods result in suboptimal designs that are compatible with real-time onboard implementation and ensure closed-loop stability of the overall system. More recently the interest has been in the application of robust adaptive control theory[35-38] to rigid spacecraft attitude maneuvers. Robustness in this context refers to the fact that the system is designed to perform in essentially the same manner despite the presence of system perturbations. These perturbations are due to dynamic model uncertainty, external bounded disturbance torques, sensor measurement error (state estimation error), and actuator failures. The effect of control torque magnitude and rate constraints must also be accounted for in the control law design. It is possible to design attitude maneuvers by combining open-loop reference trajectories using optimal control principles with state-feedback control laws using Lyapunov stability theory using a dynamic model inversion approach[39].

Parameterisation of the spacecraft attitude (orientation of the spacecraft body-fixed coordinate frame with respect to a suitable inertial frame) is a fundamental issue in the design of feedback control laws for spacecraft attitude maneuvers[3-5]. Shuster[6] presents a survey of various attitude parameterisations with their associated kinematic equations. The selection of a minimal attitude coordinate set such as the Euler angles leads to singularities in both the attitude parameters and kinematic equations at specific spacecraft orientations. This singular behaviour may be avoided by introducing an additional redundant parameter such as in the Euler symmetric parameters (also called the unit quaternion). Whilst several parameter options are available to the control engineer for designing attitude maneuvers the unit quaternion and the modified Rodrigues parameters (MRP) are the most commonly implemented attitude coordinates. Quaternions offer a globally non-singular description of spacecraft attitude which is free of geometric singularities; successive rotations are represented by a convenient quaternion multiplication rule, and they are well-suited to onboard real-time implementation since no trigonometric functions are present in the rotation matrix or kinematic equations. Since the MRP represent a minimal coordinate attitude description they exhibit singular behaviour as the spacecraft orientation approaches a complete rotation (± 360 degrees). Both the quaternion and the MRP parameterizations are based on the Euler axis/angle parameter set which follows directly from Euler's theorem on the rotational motion of rigid bodies[5]. As a result each coordinate set is non-unique which follows from the non-uniqueness property of the Euler axis/angle parameters. This limitation must be accounted for in the spacecraft attitude maneuver design to ensure that a minimum angular path maneuver is performed. This has been addressed in the literature for the MRP by switching between a standard parameter set and shadow set[7]. Maneuvers designed using sliding mode control[29-31] account for this non-uniqueness by introducing an additional term in the definition of the switching surfaces[32,34]. In general, however, minimum angle maneuver design has not been addressed explicitly in the literature for standard Lyapunov based attitude maneuver design. Wie and Barba[22] proposed an additional signum function term in the feedback control law design which results in a minimum angle maneuver but a suitable Lyapunov stability analysis was not performed.

An important practical design issue related to spacecraft control law design is control torque magnitude and rate saturation limits. Robinett et al.[24] developed a saturated function which rigorously enforces torque magnitude limits and was shown to be effective in numerical simulations. Although the controller transitions continuously at a touch point on the saturation boundary the major limitation of this work is the absence of a Lyapunov stability analysis showing convergence of the trajectory tracking errors to zero. In a related work Wie and Lu[25] considered the problem of attitude regulation maneuvers subject to rate-gyroscope and actuator constraints. A cascade-saturation controller was proposed which preserves the direction of the commanded torque. Akella et al.[26] considered the design of an angular rate independent control law for attitude tracking maneuvers utilising a passivity-based filter which rigorously enforces actuator magnitude and rate-saturation constraints. Implementation of the controller, however, requires knowledge of the spacecraft inertia parameters and the control gains are restricted by complex inequality constraints. References 27 and 28 consider the problem of torque magnitude constraints by introducing a control law structure such that the commanded control torque cannot exceed the limits regardless of the state trajectory.

Throughout the mission lifetime spacecraft operate in the presence of various disturbances including environmental torques (gravitational, aerodynamic, solar radiation, and magnetic) and non-environmental torques (for example mass expulsion torque). Cristi, et al.[40] considered the problem of disturbance rejection in a adaptive control framework by introducing an additional term in the commanded control torque. This term however is discontinuous across the sliding manifold and may cause control torque chattering which in turn may excite un-modelled high frequency attitude dynamics. Crassidis, et al.[34] proposed a saturation function to replace the function in Reference 40 to reject disturbance torques and to compensate for spacecraft inertia parameter uncertainty. This control law results in bounded trajectory tracking errors lying in a saturation boundary layer surrounding the sliding manifold. A σ-modification technique was applied to the model reference adaptive control (MRAC) of nonlinear systems by Akella and Junkins[41] which resulted in

bounded attitude trajectory tracking and parameter estimate errors. Schaub et al.[42] proposed a direct adaptive control framework for simultaneous disturbance estimation and rejection but the concept is limited to the case of constant magnitude disturbances. Boskovic et al.[43] considered an observer-based strategy for asymptotic rejection of state-dependant disturbances and different types of actuator failures. The main limitation of the proposed approach is that the disturbance estimates contain a sign function which may result in control chattering. In Reference 44 asymptotic tracking of a reference trajectory in the presence of disturbances is demonstrated without controller modification under the assumption of asymptotically decaying disturbances. More recently the literature has addressed techniques for asymptotic reference trajectory tracking whilst simultaneously rejecting bounded external disturbances. Boskovic, et al.[27] proposed a control law for robust attitude tracking maneuvers that takes into account control input saturation explicitly and achieves effective compensation of external disturbances with upper bounded magnitude and dynamic model uncertainty. A continuous approximate signum function is implemented as the controller with a time-varying control parameter which is adjusted dynamically to guarantee bounded closed-loop signals and asymptotic stability of the trajectory tracking errors. In a related work Wallsgrove and Akella[28] investigate a smooth analog of the variable structure design approach using continuous hyperbolic tangent functions which rigorously enforces control torque limits and achieves rejection of external disturbances. References 27 and 28 assume that the available control torque authority is sufficient to simultaneously track the reference trajectory motion and reject any external disturbance torques. The major drawback of implementing the dynamic control parameter approach is that the parameter may approach zero prior to convergence of the tracking errors therefore preventing an asymptotic stability result.

In the design of practical spacecraft attitude control systems the dynamic model is subject to uncertainty such as spacecraft inertia parameter uncertainty and external disturbances. The inertia uncertainty may be due to measurement error during the pre-launch testing phase, changes in the overall spacecraft system configuration (such as retrieval of a spacecraft by the space shuttle), or fuel

usage during the mission. Junkins et al.[39] and Cristi[40] et al. investigated a direct adaptive control framework for attitude maneuvers in the presence of inertia parameter uncertainty. The main limitation of this approach is that the algorithm does not ensure convergence of the adaptive inertia parameter estimates to their true values despite asymptotic converge of the trajectory tracking errors. In a related work Ahmed et al.[45] designed a periodic reference maneuver which allows identification of the spacecraft inertia parameters. Accurate estimation of the inertia parameters is possible using an indirect adaptive control scheme[46] which replaces the parameter update law with a real-time parameter estimation algorithm such as an extended Kalman filter. More recently the problem of spacecraft inertia uncertainty has been addressed in the literature by designing asymptotically stable feedback control laws which are independent of the inertia parameters[27,28].

An important practical issue in the design of spacecraft attitude maneuvers is the real-time estimation of the controller state variables based on sensor measurements. The problem of asymptotic tracking of spacecraft maneuvers in the presence of state estimation error has not been addressed in the literature and therefore remains an open problem. Singla et al.[47] developed a MRAC approach using a dead-zone technique to ensure boundedness of the model parameter estimates in the presence of sensor measurement error. More recently the interest has been in combined nonlinear observer and feedback control law design which ensures asymptotic stability of the closed-loop system under the assumption of perfect attitude sensor measurements (absence of measurement noise)[43,44,50-53]. Attitude maneuvers in the presence of constant bias on the rate gyroscope measurements were investigated by Thienel and Sanner[49] based on an early work by Boskovic et al.[48]. The observer is coupled with a nonlinear feedback control law based on the certainty equivalence principle and Lyapunov's direct method demonstrates asymptotic stability of the closed-loop system. In a related work Thienel and Sanner[50] proposed a nonlinear observer to estimate the angular rates of a non-cooperative target spacecraft during rendezvous-capture mission scenario. Boskovic et al.[43] Stepanyan and Hovakimyan[44] considered an observer-based strategy within a direct adaptive control framework. A fundamental limitation of nonlinear observer based

control law design is that the asymptotic stability properties of the trajectory tracking errors are compromised when the effect of realistic sensor models (sensor measurement noise resulting in non-zero bounded state estimation error) are considered.

To increase the autonomy of spacecraft mission operations an attitude control system needs to be capable of accommodating a diverse class of actuator failures without substantially affecting the performance of the overall system. Tandale and Valasek[52] investigated a fault tolerant MRAC strategy for the problem of actuator failures and control effector damage on redundantly actuated systems. The approach incorporates an actuator failure model in the controller so that different types of actuator failures can be identified as a change in the parameters of the failure model. Boskovic et al.[43] considered MRAC fault tolerant controller design while simultaneously rejecting state dependant disturbances using a variable structure observer.

The first novel contribution of this thesis is the development of feedback control strategy which provides global asymptotic trajectory tracking in the presence of sensor measurement error in addition to spacecraft inertia uncertainty, control torque magnitude limits, and external bounded disturbances. This controller may be considered an extension of the research by Wallsgrove and Akella[28] for rigid spacecraft attitude regulation maneuvers in which dynamic model uncertainty and torque magnitude limits are accounted for in the controller structure, and disturbance rejection of bounded persistent external disturbances is achieved using a dynamic control gain parameter. A stability analysis is performed using Lyapunov's direct method[17-20], in conjunction with Barbalat's lemma[19]. The novel control law proposed in this thesis considers the effect of additional state-dependent and upper bounded disturbance terms introduced into the Lyapunov stability analysis by state estimation error terms in the feedback control law and control gain parameter update law. A dynamic control gain parameter is implemented to ensure global asymptotic convergence of the trajectory tracking errors to zero and boundedness of the dynamic control parameter. Restrictions on the selection of the controller parameters are evaluated based on the expected reference trajectory motion, external

disturbance environment, and state estimation filter performance. Recommendations are also proposed regarding future research investigations.

The second novel contribution of this thesis is the development of an adaptive control strategy for spacecraft attitude tracking maneuvers. This controller may be considered an extension of the research by Cristi et al.[40] in which spacecraft inertia uncertainty is accounted for using a parameter update law. A bounded-gain-forgetting (BGF) composite parameter update strategy[19] is proposed, which ensures global exponential convergence of the trajectory tracking errors and parameter estimation errors to zero. This strategy utilises an auxiliary filter based on the exponential forgetting least-squares concept[19]. A stability analysis is performed using Lyapunov's direct method[17-20], in conjunction with Barbalat's lemma[19]. The exponential convergence results provided by this control strategy require persistently exciting[36,44,96] reference trajectory commands. Intelligent excitation[96] of the reference commands is also considered to ensure exponential smooth tracking of the reference trajectory.

The third novel contribution of this thesis is the development of an optimal control strategy for spacecraft attitude maneuvers. It is assumed that the rigid-body spacecraft is equipped with a redundant reaction wheel configuration. This optimal strategy may be considered an extension of the research by Skaar and Kraige[97,98] based on a minimum-power-squared performance index. In this thesis, a performance index is proposed which represents the total electrical energy consumed by the reaction wheels over the attitude maneuver interval. Pontryagin's minimum principle[9] is used to formulate the necessary conditions for optimality, in which the control torques are subject to time-varying magnitude constraints. Necessary conditions for the optimality of singular sub-arcs and associated singular controls are established using Kelley's necessary condition[8]. The two-point boundary value problem (TPBVP) is formulated using Pontryagin's minimum principle.

1.2 Spacecraft Attitude Estimation

The application of state estimation theory[53-56] in designing spacecraft attitude filters has been extensively studied[60,74,81,84,92]. Crassidis et al.[57] presents of current attitude estimation methods in the design of aerospace systems.

The most efficient attitude filters process direction-type noisy measurement vectors (for example digital sun sensor unit-vector measurement in the direction of the sun) using a single-point least-squares deterministic estimation method[58-64]. Attitude estimation based on the cost function for minimization proposed by Wahba in 1965 has become known as the Wahba's Problem[58]. Various solutions have been proposed to the Wahba problem in the literature[59] including the QUEST method[60], singular value decomposition method[61], and the Euler Q-method algorithm[62]. More recently Mortari[63] proposed an optimal linear attitude estimator based on the Gibbs vector attitude parameterisation which uses a different cost function to the Wahba function. Shuster[64] proved that the QUEST estimate of the spacecraft attitude is equivalent to a maximum likelihood approach for a specific measurement model. Although least-squares based filters are computationally fast and generally do require any information regarding sensor noise parameters or filter initialisation parameters, they are only capable of estimating the spacecraft attitude parameters and not the angular rate parameters. Furthermore state estimates are only available at discrete measurement points. More recently recursive techniques[65-67] have been developed which aim to incorporate QUEST-type measurement updates into a recursive filtering strategy to improve filter robustness and estimate additional parameters such as the spacecraft angular rates and sensor biases. Other techniques for estimating the spacecraft angular rates based on direction-type measurements without resorting to numerical differentiation of the attitude parameter estimates have also been proposed in the literature[68-72].

Sequential state estimation methods[53-56] are capable of full-state estimation (attitude and angular rate parameters) which for spacecraft applications are typically based on a combination of direction-type sensor measurements and rate gyroscope measurements. The gyroscope measurement may be replaced with a dynamic model for the spacecraft angular rates in certain applications.

The sequential estimation technique of Kalman filtering applied to spacecraft attitude estimation has been extensively studied in the literature[73-81] and offers improved estimation accuracy compared to deterministic least squares methods as well as imbedded covariance matrix information. The fact that all three parameter attitude representations are singular at certain orientations[6] has led to the implementation of the globally nonsingular quaternion representation in Kalman filtering applications. The quaternion parameters are subject to a unity norm constraint which must be enforced in the Kalman filter design. The multiplicative extended Kalman filter (EKF) parameterises the global quaternion with a four element global quaternion whist using a three element representation for the attitude estimation errors used in the filter update stage[74]. The reduced dimension error representation also avoids any issues associated with singularity of the estimation error covariance matrix. The major limitations of the EKF is the inherent assumption that the estimation state-space is linear and filtering probability density function (PDF), process noise, and sensor measurement noise are Gaussian distributed.

Various methods have been developed to overcome the inherent limitations of the extended Kalman filter[77,81,84]. Vathsal[77] proposed a nonlinear EKF based on star tracker and rate gyroscope information which includes second-order corrections to the state propagation and measurement residual equations. More recently the unscented EKF[81] was proposed for spacecraft attitude estimation which uses a set of carefully selected sample points to map the mean and covariance of the filtering PDF more accurately than the linearisation assumption of the standard EKF. Although the unscented EKF leads to faster convergence from poor initial filter conditions than the multiplicative EKF the zero and first order moments are not sufficient statistics to represent a general filtering PDF. A more general representation is possible using particle filtering methods[56,82-84] (also called sequential Monte Carlo methods) which refer to a set of algorithms implementing a recursive Bayesian model using simulation-based methods. Particle filtering implements a general numerical filtering PDF model represented by a discrete support structure and allows general non-Gaussian process noise and sensor measurement noise models. Oshman and Carmi[84] proposed a particle filter for attitude quaternion estimation which employs a

genetic algorithm to the estimate the gyroscope bias parameters to improve the algorithm efficiency. This filter was compared with various implementations of the EKF and demonstrated superior performance in terms of filter convergence time and robustness with respect to filter initial conditions. Limitations exists however with particle filtering methods which currently limit their applicability in practical systems[85] including approximations introduced to the optimal (minimum variance) filtering PDF support weights, the resampling procedure required to decrease particle degeneracy, and the large number of particles required to accurately represent the filtering PDF which requires large computational capabilities. Recently nonlinear observers have been applied to the spacecraft attitude estimation problem[48-51]. This approach is based on Lyapunov's direct method[17] and has the advantage of guaranteed convergence of the state estimation error to zero from any initial condition using ideal sensor measurement models.

The optimal solution of the nonlinear attitude filtering problem requires propagation of the state variable filtering PDF conditioned on the sensor measurement history[53]. It is well known that all practical filters are approximations to this ideal scenario. Finite dimensional filters have been proposed in the literature[53,86-91] which apply the Fokker-Planck equation (Kolmogorov's forward equation) to propagate a non-Gaussian PDF between measurements and incorporate measurement information using Bayes theorem. More recently Markley[92] developed an orthogonal filter which uses this approach based on earlier research[93-95]. The filtering PDF proposed by Markley consists of first-order terms in the attitude parameters, second-order terms and below in the bias parameters, and first-order correlation terms between the attitude and bias parameters.

The fourth novel contribution of this thesis is the development of an attitude estimation algorithm which provides real-time estimates of the spacecraft attitude parameters and sensor bias parameters. This algorithm may be considered a generalisation of the research by Markley[92] and Yau and Yau[94], and is based on an attitude sensor suite consisting of a three-axis magnetometer and a three-axis rate gyroscope assembly. The key contribution is the

implementation of a Gaussian filtering probability density function (PDF) as proposed in Reference 94 consisting of second-order terms and below in the attitude parameters and sensor bias parameters, and first-order correlation terms. A key objective will be to investigate whether the addition of the second-order attitude parameter terms improves the real-time state estimates based on the maximum aposteriori probability (MAP) principle. Nonlinear ordinary differential equations for the propagation of the filtering PDF parameters between measurements are developed using the Fokker-Planck equation, and the update of the filtering PDF parameters to incorporate the information in the magnetometer measurements is performed using Bayes theorem. Limitations of the attitude estimation algorithm are discussed and recommendations for future research are proposed.

1.3 References

[1] National Aeronautics and Space Administration, Beyond Einstein: From the big bang to black holes, NASA Structure and Evolution of the Universe Roadmap Report, January 2003.

[2] Wang, P.K.C., Hadaegh, F.Y., and K. Lau. (1999). Synchronized formation rotation and attitude control of multiple free-flying spacecraft. *The Journal of Guidance, Control, and Dynamics*, 22(1): 28-35.

[3] Wertz, J.R., ed. (1978). *Spacecraft attitude determination and control*. The Netherlands: D. Reidel Publishing Company.

[4] Hughes, P.C. (1986). *Spacecraft attitude dynamics*. Canada: John Wiley and Sons.

[5] Schaub, H., and J.L. Junkins. (2003). *Analytical mechanics of space systems*. USA: American Institute of Aeronautics and Astronautics.

[6] Shuster, M.D. (1993). A survey of attitude representations. *The Journal of the Astronautical Sciences*, 41(4): 439-517.

[7] Schaub, H., and J.L. Junkins. (1996). Stereographic orientation parameters for attitude dynamics: a generalization of the Rodrigues parameters. *The Journal of the Astronautical Sciences*, 44(1): 1-19.

[8] Bryson, A.E., and Y-C. Ho. (1969). *Applied optimal control: optimization, estimation, and control*. USA: Ginn and Company.

[9] Kirk, D.E. (1970). Optimal control theory: an introduction. USA: Prentice-Hall, Inc.

[10] Junkins, J.L., and J.D. Turner. (1986). *Optimal spacecraft rotational maneuvers*. The Netherlands: Elsevier Publishers.

[11] Stoer, J., and R. Bulirsch. (1980). Introduction to numerical analysis. USA: Springer-Verlag Inc.

[12] Gill, P.E., Murray, W., and M.H. Wright. (1981). *Practical optimization*. Great Britain: Academic Press, Inc.

[13] Carrington, C.K., and J.L. Junkins. (1986). Optimal nonlinear feedback control for spacecraft attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 9(1): 99-107.

[14] Vadali, S.R., and R. Sharma. (2006). Optimal finite-time feedback controllers for nonlinear systems with terminal constraints. *Journal of the Guidance, Control, and Dynamics*, 29(4): 921-928.

[15] Junkins, J.L., and J.D. Turner. (1980). Optimal continuous torque attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 3(3): 210-217.

[16] Redding, D.C., and N.J. Adams. (1987). Optimized rotation-axis attitude maneuver controller for the space shuttle orbiter. *The Journal of Guidance, Control, and Dynamics*, 10(1): 4-13.

[17] LaSalle, J., and S. Lefschetz. (1961). *Stability by Liapunov's direct method: with applications*. USA: Academic Press Inc.

[18] Vidyasagar, M. (1993). *Nonlinear systems analysis, 2nd Ed*. USA: Prentice-Hall, Inc.

[19] Slotine, J-J.E., and W. Li. (1991). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

[20] Skaar, S.B., and C.F. Ruoff, eds. (1994). *Teleoperation and robotics in space, Progress in Astronautics and Aeronautics Volume 161*. USA: American Institute of Aeronautics and Astronautics, Inc.

[21] Mortensen, R.E. (1968). A globally stable linear attitude regulator. *International Journal of Control*, 8(3): 297-302.

[22] Wie, B., and P.M. Barba. (1985). Quaternion feedback for spacecraft large-angle maneuvers. *The Journal of Guidance, Control, and Dynamics*, 8(3): 360-365.

[23] Tsiotras, P. (1996). Stabilization and optimality results for the attitude control problem. *The Journal of Guidance, Control, and Dynamics*, 19(4): 772-779.

[24] Robinett, R.D., et al. (1997). Lyapunov optimal saturated control for nonlinear systems. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1083-1088.

[25] Wie, B., and J. Lu. (1995). Feedback control logic for spacecraft eigenaxis rotations under slew rate and control constraints. *The Journal of Guidance, Control, and Dynamics*, 18(6): 1372-1379.

[26] Akella, M.R., Valdivia, A., and G.R. Kotamraju. (2005). Velocity-free attitude controllers subject to actuator magnitude and rate saturations. *The Journal of Guidance, Control, and Dynamics*, 28(4): 659-666.

[27] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2004). Robust tracking control design for spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 27(4): 627-633.

[28] Wallsgrove, R. J., and M.R. Akella. (2005). Globally stabilizing saturated attitude control in the presence of bounded unknown disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 957-963.

[29] Utkin, V.I. (1978). *Sliding modes and their application in variable structure systems*. Russia: MIR Publishers.

[30] Young, K-K., D., ed. (1993). *Variable structure control for robotics and aerospace applications*. The Netherlands: Elsevier Science Publishers B.V.

[31] Sabanovic, A., Fridman, L.M., and S. Spurgeon, eds. (2004). *Variable structure systems: from principles to implementation*. United Kingdom: The Institution of Electrical Engineers.

[32] Vadali, S.R. (1986). Variable-structure control of spacecraft large-angle maneuvers. *The Journal of Guidance, Control, and Dynamics*, 9(2): 235-239.

[33] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2001). Robust adaptive variable structure control of spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 24(1): 14-22.

[34] Crassidis, J.L., Vadali, S.R., and F.L. Markley (2000). Optimal variable-structure control tracking of spacecraft maneuvers. *Journal of Guidance, Control, and Dynamics*, 23(3): 564-566.

[35] Landau, Y.D. (1979). *Adaptive control: the model reference approach*. USA: Marcel Dekker.

[36] Narendra, K.S., and A.M. Annaswamy. (1989). *Stable adaptive systems*. USA: Prentice Hall, Inc.

[37] Kristic, M., Kanellakopoulos, I., and P.V. Kokotovic. (1995). *Nonlinear and adaptive control design*. USA: Wiley-Interscience.

[38] Iannou, P.A., and J. Sun. (1995). *Stable and robust adaptive control*. USA: Prentice-Hall, Inc.

[39] Junkins, J.L., Akella, M.R., and R.D. Robinett. (1997). Nonlinear adaptive control of spacecraft maneuvers. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1104-1110.

[40] Cristi, R., Burl, J., and N. Russo. (1994). Adaptive quaternion feedback regulation for eigenaxis rotations. *The Journal of Guidance, Control, and Dynamics*, 17(6): 1287-1291.

[41] Akella, M.R., and J.L. Junkins. (1998). Structured model reference adaptive control in the presence of bounded disturbances. *Proceedings of the AAS/AIAA Spaceflight Mechanics Meeting*, Monterey, California, USA, AAS Paper 98-121, 375-385.

[42] Schaub, H., Akella, M.R., and J.L. Junkins. (2001). Adaptive control of nonlinear attitude motions realizing linear closed loop dynamics. *The Journal of Guidance, Control, and Dynamics*, 24(1): 95-100.

[43] Boskovic, J.D., Bergstrom, S.E., and R.K. Mehra. (2005). Robust integrated flight control design under failures, damage, and state-dependent disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 902-917.

[44] Stepanyan, V., and N. Hovakimyan. (2007). Adaptive disturbance rejection controller for visual tracking of a maneuvering target. *The Journal of Guidance, Control, and Dynamics*, 30(4): 1090-1106.

[45] Ahmed, J. Coppola, V.T., and D.S. Bernstein. (1998). Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification. *The Journal of Guidance, Control, and Dynamics*, 21(5): 684-691.

[46] Paynter, S.J., and R.H. Bishop. (1997). Adaptive nonlinear attitude control and momentum management of spacecraft. *The Journal of Guidance, Control, and Dynamics*, 20(5): 1025-1032.

[47] Singla, P., Subbarao, K., and J.L. Junkins. (2006). Adaptive output feedback control for spacecraft rendezvous and docking under measurement uncertainty. *The Journal of Guidance, Control, and Dynamics*, 29(4): 892-902.

[48] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2000). A globally stable scheme for spacecraft control in the presence of sensor bias. *Proceedings of the IEEE Aerospace Conference*, Big Sky, Montana, USA, 505-511.

[49] Thienel, J.K., and R.M. Sanner. (2003). A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise. *IEEE Transactions on Automatic Control*, 48(11): 2011-2015.

[50] Thienel, J.K. and R.M. Sanner. (2007). Hubble space telescope angular velocity estimation during the robotic servicing mission. *The Journal of Guidance, Control, and Dynamics*, 30(1): 29-34.

[51] Seo, D., and M.R. Akella. (2007). Separation property of rigid-body attitude tracking control problem. The Journal of Guidance, Control, and Dynamics, 30(6): 1569-1576.

[52] Tandale, M.D., and J. Vasek. (2006). Fault-tolerant structured adaptive model inversion control. *The Journal of Guidance, Control, and Dynamics*, 29(3): 635-642.

[53] Jazwinski, A. (1970). *Stochastic processes and filtering theory*. USA, Academic Press, Inc.

[54] Gelb, A., ed. (1974). *Applied optimal estimation*. USA: MIT Press.

[55] Crassidis, J.L. and Junkins, J.L. (2003). *Optimal estimation of dynamic systems*. USA: CRC Press LLC.

[56] Ristic, B., Arulampalam, S., and N. Gordon. (2004). *Beyond the Kalman filter: particle filters for tracking applications*. USA: Artech House Publishers.

[57] Markley, F.L., Crassidis, J.L., and Y. Cheng. Survey of nonlinear attitude estimation methods. *The Journal of Guidance, Control, and Dynamics*, 30(1): 12-28.

[58] Wahba, G. (1965). A least squares estimate of spacecraft attitude. *SIAM Review*, 7(3): 409.

[59] Markley, F.L., and Mortari, D. (2000). Quaternion attitude estimation using vector observations. *The Journal of the Astronautical Sciences*, 48(2): 359-380.

[60] Shuster, M.D., and S.D. Oh. (1981). Three-axis attitude determination from vector observations. *The Journal of Guidance, Control, and Dynamics*, 4(1): 70-77.

[61] Markley, F.L. (1988). Attitude determination using vector observations and the singular value decomposition. *The Journal of the Astronautical Sciences*, 36(3): 245-258.

[62] Mortari, D. (1998). Euler-q algorithm for attitude determination from vector observations. *The Journal of Guidance, Control, and Dynamics*, 21(2): 328-334.

[63] Mortari, D. (2007). Optimal linear attitude estimator. *The Journal of Guidance, Control, and Dynamics*, 30(6): 1619-1627.

[64] Shuster, M.D. (1989). Maximum likelihood estimation of spacecraft attitude. *The Journal of the Astronautical Sciences*, 37(1): 79-88.

[65] Shuster, M.D. (1989). A simple Kalman filter and smoother for spacecraft attitude. *The Journal of the Astronautical Sciences*, 37(1): 89-106.

[66] Psiaki, M.L. (2000). Attitude-determination filtering via extended quaternion estimation. *The Journal of Guidance, Control, and Dynamics*, 23(2): 206-214.

[67] Choukroun, D., Bar-Itzhack, I.Y., and Y. Oshman. (2004). Optimal-REQUEST algorithm for attitude determination. *The Journal of Guidance, Control, and Dynamics*, 27(3): 418-425.

[68] Oshman, Y., and F.L. Markley. (1999). Sequential attitude and attitude-rate estimation using integrated-rate parameters. *The Journal of Guidance, Control, and Dynamics*, 22(3): 385-394.

[69] Harman, R.R., and I.Y. Bar-Itzhack. (1999). Pseudolinear and state-dependent Riccati equation filters for angular rate estimation. *The Journal of Guidance, Control, and Dynamics*, 22(5): 723-725.

[70] Bar-Itzhack. (2001). Classification of algorithms for angular velocity estimation. *The Journal of Guidance, Control, and Dynamics*, 24(3): 436-443.

[71] Azor, R., Bar-Itzhack, I.Y., Deutschmann, J.K., and R.R. Harman. (2001). Angular-rate estimation using delayed quaternion measurements. *The Journal of Guidance, Control, and Dynamics*, 24(2): 214-218.

[72] Rao, G.N., Alex, T.K., and M.S. Bhat. (2002). Incremental-angle and angular velocity estimation using a star sensor. *The Journal of Guidance, Control, and Dynamics*, 25(3): 433-441.

[73] Farrell, J.L. (1970). Attitude determination by Kalman filtering. *Automatica*, 6(5): 419-430.

[74] Lefferts, E.J., Markley, F.L., and M.D. Shuster. (1982). Kalman filtering for spacecraft attitude estimation. *The Journal of Guidance, Control, and Dynamics*, 5(5): 417-429.

[75] Bar-Itzhack, I.Y., and Y. Oshman. (1985). Attitude determination from vector observations: quaternion estimation. *IEEE Transactions on Aerospace and Electronic Systems*, 21(1): 128-136.

[76] Gai, E., Daly, K., Harrison, J., and L. Lemos. (1985). Star-sensor-based satellite attitude/attitude rate estimator. *The Journal of Guidance, Control, and Dynamics*, 8(5): 560-565.

[77] Vathsal, S. (1987). Spacecraft attitude determination using a second-order nonlinear filter. *The Journal of Guidance, Control, and Dynamics*, 10(6): 559-566.

[78] Fisher, H.L., Shuster, M.D., and T.E. Strikwerda, T.E. (1989). Attitude determination for the Star Tracker Mission. *Proceedings of the AAS/AIAA Astrodynamics Conference*, Vermont, USA. AAS Paper 89-365, 139-150.

20

[79] Shuster, M.D. (1990). Kalman filtering of spacecraft attitude and the QUEST model. *The Journal of the Astronautical Sciences*, 38(3): 377-393.

[80] Markley, F.L. (2003). Attitude error representations for Kalman filtering. *The Journal of Guidance, Control, and Dynamics*, 26(2): 311-317.

[81] Crassidis, J.L., and F.L. Markley. (2003). Unscented filtering for spacecraft attitude estimation. *The Journal of Guidance, Control, and Dynamics*, 26(4): 536-542.

[82] Cheng, Y., and J.L. Crassidis. (2004). Particle filtering for sequential spacecraft attitude estimation. Proceedings of the AIAA Guidance Navigation, and Control Conference, Rhode Island, USA, AIAA Paper 2004-5337.

[83] Rapoport, I., and Y. Oshman. (2005). Fault-tolerant particle filtering by using interacting multiple model-based Rao-Blackwellization. *The Journal of Guidance, Control, and Dynamics*, 28(6): 1171-1177.

[84] Oshman, Y., and A. Carmi. (2006). Attitude estimation from vector observations using genetic-algorithm-embedded quaternion particle filter. *The Journal of Guidance, Control, and Dynamics*, 29(4): 879-891.

[85] Daum, F., and J. Huang. (2004). Curse of dimensionality and particle filters. Proceedings of the IEEE Aerospace Conference, Big Sky, Montana, USA, 1979-1993.

[86] Kushner, H.J. (1967). Approximations to optimal nonlinear filters. *IEEE Transactions on Automatic Control*, 12(5): 546-556.

[87] Alspach, D.L., and H.W. Sorenson. (1972). Nonlinear Bayesian estimation using Gaussian sum approximations. *IEEE Transactions on Automatic Control*, 17(4): 439-448.

[88] Hempel, P.R. (1980). General expansion of the density for nonlinear filtering. *The Journal of Guidance, Control, and Dynamics*, 3(2): 166-171.

[89] Balakrishnan, S.N., and J.L. Speyer. (1989). Assumed density filter with application to homing missile guidance. *The Journal of Guidance, Control, and Dynamics*, 12(1): 4-12.

[90] Schmidt, G.C. (1993). Designing nonlinear filters based on Daum's theory. *The Journal of Guidance, Control, and Dynamics*, 16(2): 371-376.

[91] Beard, R., et al. (1999). Nonlinear projection filter based on Galerkin approximation. *The Journal of Guidance, Control, and Dynamics*, 22(2): 258-266.

[92] Markley, F.L. (2005). Attitude filtering on SO(3). *Proceedings of the AAS Malcolm D. Shuster Astronautics Symposium*, New York, USA AAS Paper 05-460, 149-174.

[93] Yau, S. S-T., Yau, S-T., and C. Yan. (2004). Linear filtering with nonlinear observations. *Proceedings of the IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, 2112-2117.

[94] Yau, S-T., and S. S-T. Yau. (2004). Nonlinear filtering and time varying Schrodinger equation I. *IEEE Transactions on Aerospace and Electronic Systems*, 40(1): 284-292.

[95] Yau, S. S-T., and L. Jia. (2005). Filtering problem with nonlinear observations and drift terms equal to gradient vector field plus affine vector field. *Proceedings of the IEEE Conference on Decision and Control*, Seville, Spain, 5865-5870.

[96] Cao, C., and N. Hovakimyan. (2005). Vision-based aerial tracking using intelligent excitation. *Proceedings of the American Control Conference*, Portland, Oregon, USA, 5091-5096.

[97] Skaar, S.B., and L.G. Kraige. (1982). Arbitrary large-angle satellite attitude maneuvers using an optimal reaction wheel power criterion. *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference*, San Diego, California, USA, August 1982.

[98] Skaar, S.B., and L.G. Kraige. (1982). Single-axis spacecraft attitude maneuvers using an optimal reaction wheel power criterion. *The Journal of Guidance, Control, and Dynamics*, 5(5): 543-544.

Chapter 2   Literature Review

2.1 Introduction

The two spacecraft mission requirements for an attitude control system (ACS) are (i) the ability to perform spacecraft attitude estimation, and (ii) the ability to perform spacecraft attitude maneuvers, based on a specific sensor/actuator configuration. The configuration detail is driven by the spacecraft mission and payload pointing accuracy requirements. In more complex spacecraft systems involving rigid and flexible elements additional optical pointing capabilities are required. It is envisioned that future spacecraft missions will require several orders of magnitude improvement in pointing and estimation capabilities, which are anticipated to come from revolutionary advances in algorithms and hardware design. Furthermore many future missions are anticipated to consist of multiple spacecraft flying in formation, requiring coordinated control techniques. For such systems the design of the attitude control system must be sufficiently general as each individual spacecraft in the formation may be subject to distinct environmental disturbance torques, sensor measurement noise, and hardware configurations. This literature review will provide a survey of current algorithms relating to rigid spacecraft attitude maneuvers and attitude estimation.

A key area of current interest in spacecraft attitude maneuver design is the incorporation of robust attitude controller characteristics such that highly accurate pointing and rapid maneuver capabilities are achieved in the presence of dynamic model uncertainty, control input saturation, bounded disturbance torques, sensor measurement noise, and control system failures. Key areas of current interest in spacecraft attitude estimation are improved robustness with respect to filter initialisation parameters, improved filter convergence rates, and generalisation of the process and sensor measurement noise models. In addition to these core issues, there is significant interest in (i) the selection of a suitable attitude parameter set to describe the spacecraft orientation, and (ii) the selection of the actuator type, which in turn impacts the spacecraft dynamic model.

The spacecraft attitude may be represented by several attitude parameterisations[A5-A8,A11,A13] including the direction cosine matrix, Euler angles, Euler axis/angle parameters, quaternions, Gibbs vector, and the modified Rodrigues parameters. The most common attitude parameter sets used in spacecraft attitude control system design are the quaternion (also called Euler symmetric parameters) and the modified Rodrigues parameters (MRP). Quaternions offer the advantage of (i) offering a globally non-singular description of spacecraft attitude which is free of geometric singularities, (ii) being able to represent successive rotations by a convenient quaternion multiplication rule, and (iii) being well-suited to onboard real-time implementation (since no trigonometric functions are present in the rotation matrix or kinematic equations). The quaternion parameterisation is, however, non-unique, a fact which follows from the non-uniqueness property of the Euler axis/angle parameter set. Consequently the quaternion $\mathbf{q}(t)$ and its negative counterpart $-\mathbf{q}(t)$ describe identical physical spacecraft orientations. This ambiguity must be accounted for in the spacecraft attitude maneuver design to ensure that a minimum angular path maneuver is realised. The modified Rodrigues parameters represent a minimal coordinate attitude description and accordingly exhibit singular behaviour as the spacecraft orientation approaches a complete rotation (±360 degrees). Singularities are avoided by switching between the standard MRP set $\sigma(t)$ corresponding to $\mathbf{q}(t)$ and a shadow set $\sigma_{\mathbf{s}}(t)$ corresponding to $-\mathbf{q}(t)$. This switching condition also ensures minimum angular path attitude maneuvers. Ultimately the selection of a suitable set of attitude parameters is left to the attitude control system designer.

The specific actuators selected for the attitude control system dictate the dynamic equations of motion[A1-A9,A11]. Control systems are either equipped with external control torque devices such as PWPM jet thrusters in which Euler's equations of motion apply, or internal torque devices (momentum exchange devices) such as reaction/momentum wheels and control moment gyroscopes in which a modified version of Euler's equations of motion apply. Both thrusters and control moment gyroscopes are capable of rapid attitude maneuvers, however the operation of the control moment gyros is complex and the thrusters

are not suitable for precision attitude stabilisation due to their discontinuous operation. Both reaction/momentum wheels and control moment gyros are capable of precision attitude control due to their smooth operating mode but require regular momentum dumping[A5].

Specification of the reference trajectory (desired spacecraft angular rates and attitude parameters) is important in designing spacecraft attitude maneuvers. The reference trajectory may be a general rotating reference motion (for example the WMAP mission all-sky scanning control mode[A15]), may be derived from real-time navigation information (for example earth-pointing spacecraft), or may be derived under optimal control considerations (for example time-optimal or fuel-optimal maneuvers). In general when the desired spacecraft motion has a fixed inertial orientation the attitude maneuver is called attitude regulation, while for tracking a rotating reference motion it is called attitude tracking. Control techniques may be applied to design a feedback control law such that the actual spacecraft attitude motion asymptotically tracks the reference trajectory. An alternate approach is dynamic model inversion[A14] which solves for a nominal open-loop control torque history corresponding to a specified reference trajectory, and a feedback control law is designed to determine perturbations to the open-loop torque commands to ensure closed-loop asymptotic stability.

2.2 Optimal Control Theory

2.2.1 Introduction

Bryson presents an historical overview of optimal control theory in the design of aerospace systems[B39]. Betts presents a survey of numerical methods for trajectory optimisation[B44] including dynamic programming, multiple shooting, and direct transcription methods. A number of textbooks on the general subject of optimal control are available[B1-B13]. The application of optimal control theory to spacecraft attitude maneuvers has been extensively studied over the last thirty years with the specific cases of fuel-optimal, time-optimal, and actuator-energy-optimal cases proving most useful in practical attitude control system design. It is important to note that no single performance index can encompass all mission

objectives. For example minimum-time and minimum-fuel requirements are competing performance measures. The spacecraft designer has the option to compute the optimal control torque history in either open-loop or closed-loop form. The former requires application of the necessary optimality conditions to formulate a two-point boundary-value problem (TPBVP) which is solved using a numerical optimization algorithm such as a multiple shooting algorithm[B9]. This approach however is not suitable for real-time onboard implementation and the pre-computed open-loop torque is generally sensitive to dynamic model errors and external disturbance torques. The result of the design process is a nonlinear open-loop state trajectory with a corresponding open-loop control torque history. Lyapunov stability techniques are typically used to develop a closed-loop feedback control law capable of asymptotically tracking the open-loop trajectory in the presence of spacecraft dynamic model uncertainty and external disturbance torques. The later closed-loop approach requires solution of the Hamilton-Jacobi-Bellman (HJB) partial differential equation which is generally difficult to solve but offers the advantage of state feedback control which is generally more robust to dynamic model errors and disturbances.

State-of-the art techniques for open-loop spacecraft trajectory are algorithms which apply the direct transcription method to decompose an original optimal control problem into a nonlinear programming (NLP) problem[B48]. Sequential quadratic programming (SQP) is used to solve the NLP problem to determine the optimal control history for a specific nonlinear performance index with nonlinear state and/or control variable constraints. This is achieved by using a sequence of quadratic programming (QP) sub-problems where the constraints of each QP sub-problem are local linearisations of the constraints in the original problem, and the objective function of the sub-problem is a local quadratic approximation to the Lagrangian function. Direct transcription has been applied to detumbling a spacecraft with three-axis external torques[B31], the design of optimal finite-thrust spacecraft transfer trajectories in an inverse-square gravitational field[B29], and the design of minimum-time, low-thrust interplanetary transfer trajectories[B38]. Kim and Mesbahi[B46] proposed a semi-definite programming algorithm for the rigid-body spacecraft reorientation problem which accounts for sun-avoidance constraints.

2.2.2 General Optimal Control

A compressive treatment of the open-loop optimal control problem for asymmetric rigid-body spacecraft rest-to-rest attitude maneuvers with continuous external control torques was given by Junkins and Turner[B15,B16]. The necessary conditions for optimality are generated using Pontrayagin's minimum principle[B1] based on a performance index involving the integral of a quadratic term in the control torques. Pontryagin's principle has the advantage that torque constraints are implicitly accounted for in the optimality analysis. A novel relaxation process is proposed which provides a more robust and efficient numerical solution to the two-point boundary value problem. Vadali and Junkins[B20,B22] studied optimal rotational maneuvers for a rigid spacecraft equipped with reaction wheels. Pontrayagin's principle is used to formulate the two-point boundary value problem for various performance indices involving the control torque and its higher-order derivatives leading to TPBVP which are solved using the method of particular solutions. The design of optimal feedback controls for spacecraft slew maneuvers was first investigated by Debs and Athans[B14] who presented an exact analytical solution of the HJB equation for the special case of quadratic performance index and linear controls. Dwyer and Sena[B19] solved the problem for rigid spacecraft with continuous external torques where the cost-to-go functional is expanded as a polynomial in the states, substituted into the HJB equation and solved recursively to generate the optimal control torque history. An alternate approach was provided by Carrington and Junkins[B21] in which the open-loop controls are expanded as a polynomial in the states and recursively solved for the polynomial coefficients. This approach uses Pontryagin's minimum principle to formulate the necessary conditions based on a quadratic performance index in the state and control variables. This idea was extended to the momentum transfer case of internal torques generated using reaction wheels in Reference B24. Optimal feedback control for rigid spacecraft equipped with reaction wheels was also investigated by Dwyer[B25] in which a linearising feedback transformation is applied to the state equations to produce an equivalent linear quadratic regulator (LQR) optimal control problem resulting in state feedback (closed-loop) controls. Tsiotras[B42] generalised this result using the passivity properties of the state equations to formulate closed-form partial solutions to the HJB equation for an axially symmetric rigid

spacecraft with two control torques. More recently Vadali and Sharma[B49] presented a dynamic programming approach based on a power series expansion of the performance index, for determining finite-time optimal feedback controllers for nonlinear systems with nonlinear terminal constraints.

2.2.3 Fuel Optimal Control

The problem of fuel-optimal attitude maneuvers was investigated for the Space Shuttle[B26] based on a linearised dynamic model for a three-axis reaction-control-jet actuator configuration. This Optimised Rotation-Axis (ORA) trajectory algorithm solves a two-point boundary value problem for fuel-optimal fixed-end-time maneuvers. A feedforward/feedback control structure is used to generate additional feedback jet commands to compensate for deviations from the nominal open-loop trajectory due to external disturbance torques. Seywald et al.[B35] extended this work by providing a complete analysis of the singular control cases for fuel-optimal control. Liu and Singh[B40] studied a weighted fuel/time performance index for optimal rest-to-rest maneuvers.

2.2.4 Time Optimal Control

A comprehensive survey of time-optimal attitude maneuvers over the period 1960-1990 for both rigid-body and flexible spacecraft is provided in Reference B36. Junkins, et al.[B17] developed a time-optimal attitude magnetic maneuver scheme for reorientation of the spin-axis for near-earth spin-stabilised spacecraft using Pontryagin's minimum principle. This algorithm was the first on-orbit implementation of an optimal control derived from Pontryagin's principle and was implemented on the NOVA family of navigation spacecraft. Direct application of the necessary conditions for optimality led to a twelfth-order nonlinear two-point boundary value problem which was reduced to fourth-order using a two-timescale approximation for real-time implementation. The optimal control is generated as a nonlinear bang-bang controller for the electromagnets. Another application of time-optimal control was investigated by Burdick et al.[B23] for pointing instruments at small celestial bodies such as comets and asteroids during high-velocity spacecraft encounters. The control scheme involves using a specified parabolic switching curve during high spacecraft

angular rates and a straight switching line when the attitude error estimates are within the controller deadband. The requirement is to select the parabolic switching line to minimise the transient response time resulting from dust impacts and other disturbances.

Li and Bainum[B28] studied minimum-time rest-to-rest maneuvers for an inertially symmetric rigid spacecraft with three-axis external control torques using an approximate continuation-type approach to determine optimal state and control trajectories. Bilimoria and Wie[B32] generalised this result to show that the eigenaxis (Euler rotation axis) rotation maneuver is not time-optimal in general and that the optimal solution is a bang-bang control torque structure with a significant nutational component. A multiple shooting algorithm is implemented to solve the TPBVP generating the optimal trajectories. A quasi-closed-form solution to the time-optimal rigid spacecraft reorientation problem was proposed by Byers and Vadali[B33] using a piecewise solution to the state transition matrix. This approach provides analytical solutions for the approximate switching functions of the open-loop control torques. Seywald and Kumar[B34] extends Reference B32 by investigating all possible singular optimal controls and higher-order necessary conditions for optimality of finite-order singular arcs are established using Goh's transformations of the associated accessory minimum problem. Shen and Tsiotras[B45] studied the minimum-time reorientation problem for a spacecraft equipped with two independent controls mounted perpendicular to the spacecraft symmetry axis. All possible structures including both singular and non-singular arcs are considered and an efficient method for numerically solving the optimal control problem based on a cascaded scheme using both direct and indirect methods is developed. A novel method of designing torque shaped reference maneuvers of the near-minimum-time (bang-bang) or near-minimum-fuel (bang-off-bang) type was proposed by Bell and Junkins[B37]. In this approach the instantaneous switching is replaced by controllably sharp spline switching to reduce excitation of flexible degrees of freedom. A dynamic inversion approach is used to develop a Lyapunov stable trajectory tracking law to provide robustness in the presence of model uncertainty and bounded disturbance torques. One major limitation of time-optimal control for spacecraft is that instantaneous switching in the bang-bang controls can excite flexible

body modes or unmodelled dynamics. As an alternative to the dynamic inversion approach Hurtado and Junkins[B43] proposed a novel performance index comprising a weighted combination of elapsed time and the first derivative of the control torques which results in smooth near-minimum-time optimal controls.

2.2.5 Energy Optimal Control

The problem of actuator-energy-optimal attitude maneuvers was studied by Skaar and Kraige[B18] who considered the single-axis slew maneuver of a rigid spacecraft equipped with a single reaction wheel. A performance index involving the integral of the square of the power consumed by the wheel motors was applied to ensure a unique control torque history.

2.2.6 Miscellaneous Optimal Control

Schaub et al.[B45] developed a universal attitude penalty function for which the spacecraft optimal control problem solutions are independent of attitude parameterisation and the function senses the shortest rotational path to the equilibrium state.

2.3   Lyapunov Stability Theory

2.3.1 Introduction

The application of Lyapunov's direct method[C1-C12] in designing rigid spacecraft attitude maneuvers has been extensively studied over the last thirty years. A suitable feedback control torque and positive definite Lyapunov function (similar to a performance index) must be selected by the spacecraft designer user such that the first-order time derivative of the Lyapunov function is negative definite. In the event that the Lyapunov derivative is negative semi-definite then LaSalle's invariance principle[C1,C12] or the more recent Mukherjee, R., and J.L. Junkins[C19] theorem may be applied to conclude global asymptotic stability with respect to the equilibrium state. Another powerful theorem which may be applied when the Lyapunov derivative is negative semi-definite is Barbalat's theorem[C4] which has been used extensively in adaptive control. In

contrast to open-loop optimal control theory the result of the design process is a nonlinear state feedback (closed-loop) control law capable of asymptotically tracking a general time-varying reference trajectory. Lyapunov based control laws are generally robust with respect to sources of model error such as dynamic model errors and bounded disturbances. In practice these sources comprise the ability of the spacecraft attitude motion to perfectly track the target reference motion. Prominent sources of bounded disturbances include environmental disturbance torques such as the gravity-gradient torque, miscellaneous external torques (for example from mass expulsion processes), and uncertain dynamic model terms. Another source of dynamic model error is sensor measurement error which is the result of attitude estimation error in the state variables of the feedback control law. The problem of torque saturation limits for simple magnitude and/or rate constraints imposes a physical constraint to the modelled system which must be explicitly accounted for in the Lyapunov analysis. The objective of state-of-the art analyses is to develop a Lyapunov stable control law which is sufficiently robust in the presence of model errors, bounded disturbances, sensor measurement error, and actuator failures.

Other control law design issues are apparent in practice, such as the requirement for apriori knowledge of certain model parameters and bounds on the reference motion, as well as any restricting assumptions used in the global asymptotic stability proof. A theoretical limitation of Lyapunov's direct method is that there is no clearly defined method available to formulate the Lyapunov function. For the application of spacecraft attitude maneuvers this is generally determined from kinetic and potential energy arguments. Oh et al.[C18] proposed a method of simplifying the control law design process by applying the work-energy rate principle from analytical dynamics to directly generate the Lyapunov function derivative. A novel application of Lyapunov stability theory was proposed by Tsiotras, et al.[C36] consisting of an integrated power/attitude control system for a spacecraft equipped with reaction/momentum wheels and thrusters. In this concept the momentum wheels are used as attitude control actuators providing reference trajectory tracking torques and as energy storage mechanisms which power to the spacecraft by accelerating/deaccelerating the wheels to either store or release kinetic energy. To demonstrate the maturity of the Lyapunov based

attitude control approach consider the Clementine lunar orbiter launched in 1994. The attitude control law for this approach was designed by Creamer et al.[C25] using extensions of the methodology presented in Reference C15. A novel aspect is the incorporation of an integral term in the perturbative feedback control law intended to null the reaction wheel motor friction-induced steady-state pointing errors. The reference attitude trajectory is a near-minimum-fuel profile producing bang-off-bang type reference control torques using the model inversion technique. Real-time attitude estimates of the state variables obtained from a combination of star camera and rate gyroscope sensor measurements are used directly in the perturbation feedback control law.

2.3.2 Standard Lyapunov Control

The first investigation into Lyapunov based attitude slew maneuver design for rigid spacecraft with continuous external torques was performed by Mortensen[C13] with the feedback control law linear in the spacecraft angular momenta and attitude quaternion error. The Lyapunov function consists of the spacecraft kinetic energy and a quadratic term in the quaternion error representing a pseudo potential energy term. Wie and Barba[C15] studied the same problem using a pulse-width pulse-frequency modulated (PWPM) reaction jet thruster actuator configuration, and introduced a signum function term in the feedback controls to ensure that the closest quaternion equilibrium point is selected during the attitude maneuver. Vadali and Junkins[C14] generalised the previous results for the case of attitude tracking maneuvers both with external control torques and reaction wheel actuators. Wen and Kreutz-Delgado[C17] investigated several globally asymptotically stable control laws with a common structure of proportional-derivative feedback plus some feedforward terms which are either model-independent, provide Coriolis torque compensation, or provide adaptive compensation. The problem of minimum rotational path maneuvers for the quaternion attitude parameterisation is addressed using a novel Lyapunov function candidate. The main limitation of this work is that the asymptotic stability proofs require restrictive assumptions in terms of the feedback control gains and apriori model information including the spacecraft inertia parameters and reference model trajectory. A dynamic inversion method

for designing Lyapunov stable control laws was studied by Junkins[C6] for the application of multi-body robotic space manipulators. This approach generates an open-loop reference control torque corresponding to a specific open-loop reference trajectory, and a perturbation feedback control torque is designed using Lyapunov stability theory to ensure closed-loop asymptotic stability in the ideal case of zero model errors. The design of feedback control laws based on the modified Rodrigues parameter kinematic representation was first investigated by Tsiotras[C21,C26] where a Lyapunov function with a kinetic energy term and natural logarithmic term in the attitude parameters leads to a feedback control law which is linear in the spacecraft angular rates and attitude parameters. A novel method of designing torque shaped reference maneuvers of the near-minimum-time (bang-bang) or near-minimum-fuel (bang-off-bang) type was proposed by Bell and Junkins[C22]. In this approach the instantaneous control switching is replaced by controllably sharp spline switching to reduce excitation of flexible degrees of freedom and unmodelled dynamics. A dynamic inversion approach is used to develop a reference trajectory tracking perturbative feedback control law to provide robustness in the presence of model errors and bounded disturbances. Schaub, et al.[C27] studied this problem for a landmark-tracking spacecraft equipped with reaction wheels. The concept of the switching between modified Rodrigues parameter sets is applied to ensure minimal rotational path maneuvers. A non-smooth feedback control law was developed by Tsiotras and Luo[C30] which significantly reduces the amount of control torque effort required to perform rest-to-rest attitude maneuvers for initial conditions near the equilibrium state. Hall, Tsiotras, and Shen[C31] also considered attitude slew maneuvers using the MRP coordinates but did not investigate parameterisation uniqueness. More recently, Wie et al.[C37] developed Lyapunov based control laws for agile rigid spacecraft equipped with control moment gyros capable of rapid retargeting and fast transient response.

2.3.3 Control Gain Selection

A time-consuming practical reality of Lyapunov control law design is finding suitable control gain matrices such that the system reorientation time satisfies spacecraft mission requirements. Wie, et al.[C16] showed that for exact spacecraft

inertia parameters a rest-to-rest slew maneuver about the Euler axis may be performed through suitable design of the feedback control gain matrices in the linear feedback controls. Robustness of the attitude slew maneuver with respect to spacecraft inertia parameter uncertainty is also investigated. Schaub et al.[C27] applied the root locus technique to design suitable feedback control gain parameters based on a linearised form of the MRP kinematic equations and principal-axis representation of the spacecraft inertia matrix.

2.3.4 Control Torque Limits

An important practical issue in the design of Lyapunov based feedback control laws is the presence of control torque saturation limits which has only been studied recently for spacecraft attitude maneuvers. Robinett et al.[C29] used Lyapunov's direct method to design stable saturated control laws for general nonlinear systems which transitions continuously across the saturation boundary. This method however does not ensure at least a negative semi-definite Lyapunov derivative during certain periods of torque saturation which may lead to system instability. Wie and Lu[C23] and Wie et al.[C37] developed a cascade saturation control law for rigid spacecraft slew maneuvers capable of handling actuator saturation, slew rate limits, and control bandwidth limits. Akella, et al.[C40] proposed a passivity-based feedback control law for attitude tracking of a rigid spacecraft subject to actuator magnitude and rate constraints. The stability proof is accomplished using a novel Lyapunov function candidate which includes hyperbolic trigonometric functions and a passive filter variable synthesised through a first-order stable differential equation. Inequality constraints involving apriori information such as upper bounds on the initial spacecraft angular rates and reference trajectory profiles govern the selection of the feedback control gains. State-of-the-art methods[C38,C42] approach the control torque saturation problem by carefully designing the structure of the feedback control law such that its magnitude cannot exceed the saturation limit.

2.3.5 Disturbance Rejection

Another important practical issue in Lyapunov control law design for spacecraft attitude maneuvers is the effect of bounded external disturbance torques and

dynamic model uncertainty. The model uncertainty terms may be considered an external disturbance in the Lyapunov analysis or may be addressed using an adaptive control approach. Solutions to this problem have only been proposed recently by Boskovic et al.[C38] and Wallsgrove and Akella[C42]. Both approaches implement a time-varying control gain parameter in the control law which is governed by a first-order nonlinear differential equation. The control parameter is accounted for in the Lyapunov stability analysis by adding a positive-definite term to the nominal Lyapunov function candidate. The only assumption in the analysis is that the available control torque authority is sufficient to simultaneously track the reference trajectory motion and reject any external disturbances.

2.3.6 Sensor Measurement Noise

The problem of attitude tracking maneuvers in the presence of sensor measurement noise remains an open problem. The robustness of Lyapunov based control laws with respect to sensor measurement noise have been demonstrated in References C38 and C40 based on unfiltered raw sensor measurements. At present, however, the effect of sensor measurement noise manifesting as state estimation error needs a rigorous Lyapunov stability analysis to show asymptotic stability.

2.3.7 Object Avoidance

In many space missions a basic mission requirement is the ability to perform frequent attitude maneuvers in order to retarget payload instrumentation. For missions with sensitive payloads such as cryogenically cooled infrared telescopes the maneuver must be achieved whilst simultaneously avoiding pointing the payload towards the sun or other infrared bright regions of the sky. McInnes[C20] proposed a novel Lyapunov function which includes a Gaussian artificial potential function to ensure bright object avoidance. Halbani[C34] developed a more complicated control algorithm capable of simultaneous bright object avoidance and maintenance of communication antenna contact with a ground station.

2.3.8 Angular Rate Independent Control

The problem of spacecraft attitude maneuver design without spacecraft angular rate feedback has emerged as an area of research within the past decade[C24,C32,C35,C39,C40]. This framework is motivated by practical considerations in which angular rate measurements may not be available due to cost limitations, implementations constraints, or rate gyroscope failures. All existing solutions are based on the passivity principle which results in the construction of a dynamic filter driven by the attitude parameters thus avoiding numerical differentiation of noise-corrupted attitude signals. More recently Subbarao and Akella[C39] proposed a novel velocity-free proportional-integral control law which is a generalisation of the existing passivity-based approaches.

2.3.9 Actuator Failures

To ensure increased autonomy of future spacecraft, their guidance, navigation and control systems must be capable of accommodating a large class of subsystem component failures and control effector damage without substantially affecting the performance and stability of the overall system. Boskovic et al.[C41] developed a decentralised failure detection, identification, and reconfiguration system to accurately detect different failures and appropriately reconfigure the control laws to maintain the closed-loop stability properties even in the presence of state-dependant disturbances. In a similar work Boskovic[C43] developed a theoretical framework for the retrofit reconfigurable control of nonlinear aerospace systems that compensates for control effector damage.

2.4 Sliding Mode Control Theory

Sliding mode control[D1-D5,D7,D9] (also called variable structure control) is an attractive alternative to Lyapunov-based spacecraft attitude maneuvers. Although the closed-loop stability analysis is performed identically for both types of control using Lyapunov's direct method, sliding mode control results in a reduced-order dynamic model and offers high levels of robustness in the presence of parameter uncertainty and dynamic model errors (most significantly characterised by bounded external disturbances). In the context of sliding mode control an additional term is generally added to the feedback control law to

counteract residual external torques arising from dynamic model uncertainty such as spacecraft inertia parameter uncertainty and bounded disturbances.

Excellent literature surveys of sliding mode control in industrial and aerospace applications are provided in References D5, D7, and D9. The design of spacecraft attitude maneuvers using sliding mode theory was first investigated by Vadali[D6] for a rigid-body spacecraft equipped with three-axis continuous external control torques. This work proposes a novel method for designing analytical sliding surfaces based on minimising a quadratic performance index in the spacecraft angular rates and attitude quaternion. An important feature of this sliding vector design process is the incorporation of an additional signum function term in the scalar component of the quaternion tracking error. This term accounts for the non-uniqueness of the quaternion attitude parameterisation thus allowing minimum path rotational maneuvers to be performed. Vadali developed a continuous feedback control law to force the system trajectory towards the sliding manifold and maintain ideal motion along the sliding manifold using the equivalent control method. Additional terms were added to the controls to account for the effect of dynamic model uncertainty and external disturbance torques. Dwyer and Sira-Ramirez[D8] motivated by Reference D6, proposed a general sliding-mode scheme based on nonlinear sliding manifolds and the Gibbs vector parameterisation of the spacecraft attitude. This design procedure, however, results in discontinuous controls which must be replaced with saturated controls to avoid chattering in the sliding dynamics at the expense of attitude tracking error. Chen and Lo[D10] presented an approach similar to Vadali[D6] and Dwyer and Sira-Ramirez[D8] based on the Gibbs vector attitude parameters. Although Vadali[D6] indirectly demonstrated via the kinematic equations that ideal motion on the sliding manifold is equivalent to zero quaternion and angular rate tracking error, Reference D10 proved this result using Lyapunov stability theory. An extension of this work is presented in Reference D11 where a modified version of standard sliding-mode control called smoothing model-reference sliding-mode control is proposed. This new scheme offers superior attitude tracking performance but the primary limitation of this work is that the minimum angular path maneuver issue has not been resolved due to the non-negative constraint on the scalar component of the

attitude quaternion corresponding to principal axis rotations up to 180 degrees. Crassidis and Markley[D12] were the first researchers to apply the modified Rodrigues parameters in the context of sliding-mode based attitude maneuvers; however their work did not consider the issue of minimum rotational path maneuvers.

Robinett and Parker[D13] applied sliding-mode control to attitude tracking maneuvers for a spacecraft equipped with reaction wheels and proposed a novel method of selecting the matrix coefficient of the discontinuous disturbance accommodation portion of the feedback control law in order to satisfy general performance requirements. Kim et al.[D14] combined standard sliding-mode control theory with disturbance accommodating control (signal synthesis adaptive control) for spacecraft attitude tracking maneuvers. The combined concept is more effective than traditional sliding-mode control since the steady-state pointing errors are reduced in the presence of external disturbance torques and the robustness of sliding mode is guaranteed in the range of actuator capability. In a similar work Kim and Crassidis[D15] conducted a comparative study of the disturbance accommodating sliding mode control[D14] and time-optimal control for a spacecraft equipped with PWPF modulated thrusters. More recently Crassidis et al.[D17] generalised the work of Vadali[D6] to design feedback controls which provide global asymptotic tracking of spacecraft maneuvers using either external control torques or reaction wheel internal torques. The novel aspects of this research work are that the multiplicative error quaternion definition is used for the reference trajectory tracking errors, and that the thickness of the sliding mode boundary layer in the presence of model errors and bounded external disturbances is analytically determined as a function of the dynamic model parameters, control gain parameters, and magnitude of the bounded disturbances.

An important practical aspect of sliding mode based spacecraft attitude maneuvers is accounting for the presence of control torque saturation. Boskovic, et al.[D16,D18] recently proposed an asymptotically stable control law for robust attitude stabilisation that takes into account control input saturation explicitly and achieves effective compensation of external disturbances and dynamic

model uncertainty. This work, however, only provides a Lyapunov stability analysis for spacecraft detumbling maneuvers (stabilisation of the spacecraft angular rates only) and the resulting feedback controls are discontinuous, which may lead to control chattering[D2]. These limitations were removed in a subsequent work by Boskovic[D19] where a continuous version of the sliding mode control design approach of Reference D18 is applied based on a full state Lyapunov analysis.

2.5 Adaptive Control Theory

2.5.1 Introduction

The application of adaptive control theory[E1-E11] in designing rigid spacecraft attitude maneuvers has been extensively studied over the last twenty years. This literature survey will concentrate on direct adaptive control in which the uncertain dynamic model parameters are updated using a parameter update law as opposed to indirect adaptive control in which the parameters are recursively estimated using a state estimator (for example a least-squares or Kalman filter algorithm). Robust adaptive control seeks to ensure system parameter stability, and even asymptotic stability, in the presence of torque saturation, bounded disturbances, sensor measurement error, and actuator failures.

2.5.2 Standard Adaptive Control

All spacecraft dynamic models are subject to uncertainty such as spacecraft inertia matrix uncertainty and environmental external disturbances. The inertia uncertainty may be due to measurement error during the pre-launch testing, changes in the overall spacecraft system configuration (such as retrieval of a spacecraft by the space shuttle), or fuel usage during the mission. For spacecraft equipped with thrusters it is assumed that the inertia parameters are constant throughout a spacecraft attitude maneuver despite the consumption of fuel. The purpose of adaptive control is to account for the dynamic model uncertainty appearing in the feedback control law by extending the effective state vector to include these uncertain parameters. The stability analysis is performed using Lyapunov's direct method such that direct adaptive control may be considered

an extension of standard Lyapunov control. Asymptotic convergence of the attitude tracking errors is possible despite the inertia parameter estimates not converging to their true values. Indirect adaptive control seeks not only to stabilise the system but to provide accurate parameter estimates as well. In addition to accounting for a single source of spacecraft dynamic model errors the design of robust adaptive controls which account for control torque saturation, bounded disturbances, and sensor measurement error have attracted considerable attention within the last ten years.

The first study of spacecraft attitude maneuvers based on direct adaptive control was performed by Slotine and Di Benedetto[E12]. This work considered a Lyapunov function consisting of positive terms in the sliding vector tracking error and the parameter estimation error. The stability analysis, however, did not account for uncertainty in the axial inertia parameters of the reaction wheels. A direct adaptive controller for the International Space Station (ISS) modelled as a collection of rigid-body segments was developed by Parlos and Sunkel[E13]. The dynamic model consists of a rigid-body spacecraft equipped with momentum exchange devices linearised about an equilibrium point of a gravity-gradient stabilised spacecraft. A full state feedback controller is developed incorporating gain-scheduled adaptation of the attitude gains to ensure acceptable attitude tracking performance in the presence of significant mass property variations. Boussalis et al.[E14] and Cristi et al.[E16,E18] also considered the problem of spacecraft inertia uncertainty using a direct adaptive control approach. An indirect adaptive control approach based on recursive least squares estimation was also considered and the necessary conditions for convergence of the parameter estimates were derived. Sheen and Bishop[E17] investigated two nonlinear adaptive control laws based on feedback linearisation for the attitude control and momentum management of the International Space Station equipped with control moment gyroscope actuators. The first parameter update law was driven by the tracking error for the first controller whilst information from both the tracking errors and estimation errors were used in the second controller. This work was extended in Reference E21 where an indirect adaptive nonlinear controller was developed. The control law is based on the theory of feedback linearization under a canonical state transformation of the original nonlinear

system. Accurate real-time spacecraft inertia parameter estimates are provided using an extended Kalman filter implementation with probing signals introduced to enhance observability. A novel method for nonlinear adaptive control of spacecraft near-minimum-time maneuvers of a rigid spacecraft equipped with reaction wheels was proposed by Junkins et al.[E22]. Open-loop control torques are developed which are cubic spline approximations of the near optimal minimum-time control torques in order to reduce the effects of flexible mode excitation. This approach allows the imposition of maximum saturation torque constraints. An inverse dynamics approach is used to solve for the open-loop maneuver control law and an adaptive feedback control law solves for the perturbations to the nominal open-loop torque commands that will ensure asymptotic tracking of the reference motion in the presence of spacecraft inertia parameter uncertainty (the uncertainty in the reaction wheel axial uncertainty is not considered). Ahmed et al.[E26] also studied rigid spacecraft attitude tracking maneuvers using external control torques and developed sufficient conditions to be satisfied by the reference attitude motion, such that the inertia parameter estimates convergence to their true values. Schaub et al.[E29,E32] developed a nonlinear adaptive control law for spacecraft attitude regulation which yields linear closed-loop dynamics in the presence of large inertia and external disturbance errors. Their research was motivated by the earlier work of Paielli and Bach[E15]. This approach allows classical linear control methodologies such as root-locus plots to be applied to the tracking error dynamics in accordance with design requirements such as control bandwidth and settling time. A limitation of this work is that the external disturbance torque is assumed constant. An integrated power/attitude control system based on indirect adaptive control was developed by Yoon and Tsiotras[E33] for spacecraft attitude tracking in the case of uncertain spin-axis direction of variable speed control moment gyroscopes based on an earlier work by Tsiotras et al.[E31]. A novel adaptive feedback controller only requiring spacecraft angular rate information was proposed by Miwa and Akella[E34].

2.5.3 Inertia-Independent Control Laws

Whilst traditional adaptive control seeks to directly account for dynamic model uncertainty using a parameter update law or real-time parameter estimation algorithm, more recent methods propose feedback controls which are independent of the spacecraft inertia parameters. Although these controllers cannot be strictly classified as adaptive controllers they do ensure that the spacecraft attitude maneuvers are robust to this form of dynamic model error as well as providing a simplified Lyapunov stability analysis environment. Inertia-independant globally asymptotically stable control laws have been proposed by Joshi et al.[E19], Boskovic et al.[E27,E30,E38], and Wallsgrove and Akella[E40].

2.5.4 Control Torque Limits

The problem of control torque saturation limits in the context of adaptive control has only been studied within the last ten years. Akella et al.[E24] considered a model reference adaptive control (MRAC) formulation with a boundary layer concept to account for actuator saturation. Tandale et al.[E36] developed a saturated control law based on a dynamic inversion approach. Lavretsky[E37,E43] proposed a μ-modification approach for stable adaptive control in the presence of control input constraints based on a linear dynamic model. More recently Boskovic et al.[E38], and Wallsgrove and Akella[E40] proposed a solution to the torque magnitude constraint problem by designing the feedback control law according to a bounded structure.

2.5.5 Disturbance Rejection

Traditional methods of direct adaptive control in the presence of bounded external disturbance torques include the dead-zone, parameter projection, σ-modification, switched σ-modification, e_1-modification techniques, or the concept of persistent excitation[E5,E10,E11]. The σ-modification technique was applied to the MRAC of nonlinear systems by Akella and Junkins[E25] which resulted in bounded attitude tracking error and parameter estimate errors. A problem associated with the σ-modification technique is a phenomenon known as bursting in which the tracking error may assume values higher than the order

of the disturbance for a short period of time. Subbarao and Junkins[E35] investigated a method of suppressing bursting behaviour using a dead-zone modification technique and applied the control algorithm to track agile aircraft maneuvers. Di Gennari[E20,E23] applied direct adaptive control to spacecraft attitude maneuvers including the effects of the gravity-gradient disturbance torque. Schaub et al.[E32] also proposed a direct adaptive control framework for disturbance rejection but this research is limited to constant magnitude disturbances. State-of-the-art techniques[E38,E40] are capable of asymptotic reference trajectory tracking whilst simultaneously rejecting bounded disturbance torques acting on the spacecraft. This is achieved by introducing a time-varying control law gain which effectively increases the dimension of the Lyapunov stability analysis.

2.5.6 Sensor Measurement Error

The problem of adaptive spacecraft attitude maneuvers in the presence of sensor measurement noise is an active area of research. Boskovic et al.[E28] considered attitude regulation maneuvers in the presence of a constant gyroscope bias on the angular rate measurements. The bias identification and state estimation processes was performed using a nonlinear adaptive observer. Singla et al.[E42] developed a MRAC approach to spacecraft rendezvous and docking maneuvers for a rigid spacecraft equipped with reaction wheels. A dead-zone technique is introduced to ensure boundedness of the model parameter estimates in the presence of sensor measurement error sources.

2.5.7 Miscellaneous

Other miscellaneous open problems in adaptive control of spacecraft maneuvers have been addressed in the literature. Tandale and Valasek[E41] derived a structured adaptive dynamic model inversion based control law for the problem of fault tolerance to actuator failures on redundantly actuated nonlinear model of an F-16 type aircraft. This approach incorporates an actuator failure model in the controller formulation so that an actuator failure can be identified as a change in the parameters of the failure model.

2.6 State Estimation Theory

The application of state estimation theory[F1-F5] in designing spacecraft attitude filters has been extensively studied over the last thirty years. Crassidis et al.[F6] presents an excellent survey of spacecraft attitude estimation methods in the design of aerospace systems. This literature review will survey only a small representative sample of the available literature on attitude filtering.

The most efficient attitude filters process direction-type measurement vectors (for example digital sun sensor or magnetometer) only using deterministic least-squares estimation methods[F7-F10]. Although these filters are computationally fast and do require any information regarding sensor noise parameters or filter initialisation parameters, they are only capable of estimating the spacecraft attitude parameters and not the angular rate parameters. In recent times various techniques have emerged to also estimate the spacecraft angular rates based on direction-type measurements[F11-F15] without resorting to a numerical differentiation of the attitude parameter estimates or sensor measurements. Sequential state estimation methods[F1-F5] are capable of full-state estimation (attitude and angular rate parameters) based on a combination of direction-type sensor measurements and rate gyroscope measurements. The sequential estimation technique of Kalman filtering has been extensively studied over the last thirty years and offers improved estimation accuracy compared to deterministic least squares methods as well as imbedded covariance matrix information. The basic form of the Kalman filter[F16] has been generalised to the extended Kalman filter (EKF) for nonlinear aerospace applications[F17-F23] to include the multiplicative EKF, the additive EKF, and more recently the unscented EKF which provides improved robustness for poor initial filter conditions. The main limitations of Kalman filtering methods are the assumption that the estimation state-space is linear and the attitude filtering condition probability density function (CPDF), process noise, and sensor measurement noise are all Gaussian distributed. Methods have been developed to improve the accuracy of the EKF by including high-order terms in the estimation state-space[F24].

Poor performance of Kalman filtering methods due to their inherent assumptions has led to development of nonlinear filters, most recently the unscented EKF and particle filtering methods, at the expense of increased computational burden. Markley[F27] developed an orthogonal filter based on earlier related research[F25,F26,F28] for a spacecraft equipped with a three-axis gyroscopes and three-axis magnetometer. This filter propagates a non-Gaussian filtering PDF using the Fokker-Planck equation (Kolmogorov's forward equation) and the state variable update stage is performed using Bayes theorem. The filtering PDF is a non-Gaussian function of the attitude matrix elements (redundant attitude parameters) and sensor bias parameters. The state estimates are extracted from the filtering PDF using a suitable numerical minimisation algorithm[F29,F30]. The results of the orthogonal filter were very similar to those generated using an EKF.

Particle filtering methods[F5,F31-F34] (also called sequential Monte Carlo methods) refer to a set of algorithms implementing a recursive Bayesian model using simulation-based methods. Cheng and Crassidis[F35] investigated particle filtering for spacecraft attitude estimation for an attitude sensor suite comprising a three-axis gyroscopes and star camera. The attitude filtering PDF is represented by a discrete support structure which continuously adapts in shape during recursive Bayesian estimation as opposed to a fixed functional form PDF whose parameters adapt. Particle filters allow flexible non-Gaussian process and sensor measurement noise distributions. Rapoport and Oshman[F36] studied particle filter methods and applied Rao-Blackwellization technique to reduce the size of the particle set for state estimation. Oshman and Carmi[F37] extended the results of Reference F35 to include an improved particle resampling filter stage and a comprehensive covariance analysis. A novel genetic algorithm is implemented to estimate the gyroscope bias parameters to improve the efficiency of the estimator. Simulation results of the particle filter show superior performance in terms of filter convergence time and robustness to filter initialisation compared to the Kalman filter variants. Although particle filters eliminate many of the restrictive assumptions associated with the Kalman filter, the issues of approximations to the optimal (minimum variance) filtering PDF support

weights and correct selection of a resampling procedure to decrease particle degeneracy are open problems.

2.7 General References

[A1] Meirovitch, L. (1970). *Methods of analytical dynamics*. USA: McGraw-Hill Inc.

[A2] Likins, P.W. (1973). *Elements of engineering mechanics*. USA: McGraw-Hill Inc.

[A3] Meriam, J., and L. Kraige. (1998). *Engineering Mechanics Volume 2: Dynamics*. 4th Ed. USA: John Wiley and Sons.

[A4] Greenwood, D. (2003). *Advanced mechanics*. United Kingdom: Cambridge University Press.

[A5] Wertz, J.R., ed. (1978). *Spacecraft attitude determination and control*. The Netherlands: D. Reidel Publishing Company.

[A6] Piscacane, V.L. and Moore, R.C.., ed. (1994). *Fundamentals of space systems*. USA: Oxford University Press.

[A7] Junkins, J.L., and J.D. Turner. (1986). *Optimal spacecraft rotational maneuvers*. The Netherlands: Elsevier Publishers.

[A8] Hughes, P.C. (1986). *Spacecraft attitude dynamics*. Canada: John Wiley and Sons.

[A9] Wiesel, W. (1989). *Spaceflight dynamics*. USA: McGraw-Hill Inc.

[A10] Wertz, J.R., and W.J. Larson, eds. (1999). *Space mission analysis and design, 3rd Ed*. USA: Microcosm Press.

[A11] Schaub, H., and J.L. Junkins. (2003). *Analytical mechanics of space systems*. USA: American Institute of Aeronautics and Astronautics.

[A12] Griffin, M., and J. French. (2004). *Space vehicle design, 2nd Ed*. USA: American Institute of Aeronautics and Astronautics.

[A13] Shuster, M.D. (1993). A survey of attitude representations. *The Journal of the Astronautical Sciences*, 41(4): 439-517.

[A14] Skaar, S.B., and C.F. Ruoff, eds. (1994). *Teleoperation and robotics in space, Progress in Astronautics and Aeronautics Volume 161*. USA: American Institute of Aeronautics and Astronautics, Inc.

[A15] Markley, F.L., et al. (2005). Attitude control system of the Wilkinson Microwave Anisotropy Probe. *The Journal of Guidance, Control, and Dynamics*, 28(3): 385-397.

2.8 Optimal Control References

[B1] Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., and E.F. Mishchenko. (1962). *The mathematical theory of optimal processes*. USA: John Wiley and Sons, Inc.

[B2] Leitmann, G., ed. (1962). *Optimization techniques: with applications to aerospace systems*. USA: Academic Press, Inc.

[B3] Lee, R.C.K. (1964). *Optimal estimation, identification, and control, Research Monograph No. 28*. USA: MIT Press.

[B4] Athans, M., and P.L. Falb. (1966). *Optimal control: an introduction to the theory and its applications*. USA: McGraw-Hill, Inc.

[B5] Bryson, A.E., and Y.-C. Ho. (1969). *Applied optimal control: optimization, estimation, and control*. USA: Ginn and Company.

[B6] Anderson, B.D.O., and J.B. Moore. (1971). *Linear optimal control*. USA: Prentice-Hall, Inc.

[B7] Kirk, D.E. (1970). Optimal control theory: an introduction. USA: Prentice-Hall, Inc.

[B8] Tabak, D., and B.C. Kuo. (1971). *Optimal control by mathematical programming*. USA: Prentice-Hall, Inc.

[B9] Stoer, J., and R. Bulirsch. (1980). Introduction to numerical analysis. USA: Springer-Verlag Inc.

[B10] Gill, P.E., Murray, W., and M.H. Wright. (1981). *Practical optimization*. Great Britain: Academic Press, Inc.

[B11] Bazaraa, M.S., Sherali, H.D., and C.M. Shetty. (1993). *Nonlinear programming: theory and algorithms, 2nd Ed*. Canada: John Wiley and Sons, Inc.

[B12] Naidu, D.S. (2003). *Optimal control systems*. USA: CRC Press LLC.

[B13] Junkins, J.L., and J.D. Turner. (1986). *Optimal spacecraft rotational maneuvers*. The Netherlands: Elsevier Publishers.

[B14] Debs, A.S., and M. Athans. (1969). On the optimal angular velocity control of asymmetric space vehicles. *IEEE Transactions on Automatic Control*, 14(1): 80-83.

[B15] Junkins, J.L., and J.D. Turner. (1978). Optimal continuous torque attitude maneuvers. *Proceedings of the AIAA/AAS Astrodynamics Conference*, Palo Alto, California, USA, AIAA Paper 78-1400, 1-21.

[B16] Junkins, J.L., and J.D. Turner. (1980). Optimal continuous torque attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 3(3): 210-217.

[B17] Junkins, J.L., Carrington, C.K., and C.E. Williams. (1981). Time-optimal magnetic attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 4(4): 363-368.

[B18] Skaar, S.B., and L.G. Kraige. (1982). Single-axis spacecraft attitude maneuvers using an optimal reaction wheel power criterion. *The Journal of Guidance, Control, and Dynamics*, 5(5): 543-544.

[B19] Dwyer, T.A.W. III, and R.P. Sena. (1982). Control of spacecraft slewing maneuvers. *Proceedings of the IEEE Conference on Decision and Control*, Orlando, Florida, USA, Dec 8-10, 1982.

[B20] Vadali, S.R., and J.L. Junkins. (1983). Spacecraft large angle rotational maneuvers with optimal momentum transfer. *The Journal of the Astronautical Sciences*, 31(2): 217-235.

[B21] Carrington, C.K., and J.L. Junkins. (1984). Nonlinear feedback control of spacecraft slew maneuvers. *The Journal of the Astronautical Sciences*, 32(1): 29-45.

[B22] Vadali, S.R., and J.L. Junkins. (1984). Optimal open-loop and stable feedback control of rigid spacecraft attitude maneuvers. *The Journal of the Astronautical Sciences*, 32(2): 105-122.

[B23] Burdick, G.M., Lin, H-S., and E.C. Wong. (1984). A scheme for target tracking and pointing during small celestial body encounters. *The Journal of Guidance, Control, and Dynamics*, 7(4): 450-457.

[B24] Carrington, C.K., and J.L. Junkins. (1986). Optimal nonlinear feedback control for spacecraft attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 9(1): 99-107.

[B25] Dwyer, T.A.W. III (1986). Exact nonlinear control of spacecraft slewing maneuvers with internal momentum transfer. *The Journal of Guidance, Control, and Dynamics*, 9(2): 240-246.

[B26] Redding, D.C., and N.J. Adams. (1987). Optimized rotation-axis attitude maneuver controller for the space shuttle orbiter. *The Journal of Guidance, Control, and Dynamics*, 10(1): 4-13.

[B27] Hargreaves, C.R., and S.W. Paris. (1987). Direct trajectory optimization using nonlinear programming and collocation. *The Journal of Guidance, Control, and Dynamics*, 10(4): 338-342.

[B28] Li, F., and P.M. Bainum. (1990). Numerical approach for solving rigid spacecraft minimum time attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 13(1): 38-45.

[B29] Enright, P.J., and B.A. Conway. (1991). Optimal finite-thrust spacecraft trajectories using collocation and nonlinear programming. *The Journal of Guidance, Control, and Dynamics*, 14(5): 981-985.

[B30] Enright, P.J., and B.A. Conway (1992). Discrete approximations to optimal trajectories using direct transcription and nonlinear programming. *The Journal of Guidance, Control, and Dynamics*, 15(4): 994-1002.

[B31] Herman, A.L., and B.A. Conway. (1992). Optimal spacecraft attitude control using collocation and nonlinear programming. *The Journal of Guidance, Control, and Dynamics*, 15(5): 1287-1289.

[B32] Bilimoria, K.D., and B. Wie. (1993). Time-optimal three-axis reorientation of a rigid spacecraft. *The Journal of Guidance, Control, and Dynamics*, 16(3): 446-452.

[B33] Byers, R.M., and S.R. Vadali. (1993). Quasi-closed-form solution to the time-optimal rigid spacecraft reorientation problem. *The Journal of Guidance, Control, and Dynamics*, 16(3): 453-461.

[B34] Seywald, H., and R.R. Kumar. (1993). Singular control in minimum time spacecraft reorientation. *The Journal of Guidance, Control, and Dynamics*, 16(4): 686-694.

[B35] Seywald, H., Kumar, R.R., Deshpande, S.S., and M.L. Heck. (1994). Minimum fuel spacecraft reorientation. *The Journal of Guidance, Control, and Dynamics*, 17(1): 21-29.

[B36] Scrivener, S.L., and R.C. Thompson. (1994). Survey of time-optimal attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 17(2): 225-233.

[B37] Bell, M.J., and J.L. Junkins. (1994). Near minimum-time three dimensional maneuvers of rigid and flexible spacecraft. *The Journal of the Astronautical Sciences*, 42(4): 421-438.

[B38] Tang, S., and B.A. Conway. (1995). Optimization of low-thrust interplanetary trajectories using collocation and nonlinear programming. *The Journal of Guidance, Control, and Dynamics*, 18(3): 599-604.

[B39] Bryson, A.E. (1996). Optimal control – 1950 to 1985. *IEEE Control Systems Magazine*, 16(3): 26-33.

[B40] Liu, S-W., and T. Singh. (1997). Fuel/time optimal control of spacecraft maneuvers. *Journal of the Guidance, Control, and Dynamics*, 20(2): 394-397.

[B41] Schaub, H., Junkins, J.L., and R.D. Robinett. (1997). New penalty functions and optimal control formulation for spacecraft attitude control problems. *Journal of the Guidance, Control, and Dynamics*, 20(3): 428-434.

[B42] Tsiotras, P. (1997). Optimal regulation and passivity results for axisymmetric rigid bodies using two controls. *The Journal of Guidance, Control, and Dynamics*, 20(3): 457-463.

[B43] Hurtado, J.E., and J.L. Junkins. (1998). Optimal near-minimum-time control. *The Journal of Guidance, Control, and Dynamics*, 21(1): 172-174.

[B44] Betts, J.T. (1998). Survey of numerical methods for trajectory optimization. *The Journal of Guidance, Control, and Dynamics*, 21(2): 193-207.

[B45] Shen, H., and P. Tsiotras. (1999). Time-optimal control of axisymmetric rigid spacecraft using two controls. *The Journal of Guidance, Control, and Dynamics*, 22(5): 682-694.

[B46] Kim, Y., and M. Mesbahi. (2004). Quadratically constrained attitude control via semidefinite programming. *IEEE Transactions on Automatic Control*, 49(5): 731-735.

[B47] Sharma, R., and A. Tewari. (2004). Optimal nonlinear tracking of spacecraft attitude maneuvers. *IEEE Transactions on Control Systems Technology*, 12(5): 677-682.

[B48] Gill, P.E., Murray, W., and M.A. Saunders. (2005). SNOPT: a SQP algorithm for large-scale constrained optimization. *SIAM Review*, 47(1): 99-131.

[B49] Vadali, S.R., and R. Sharma. (2006). Optimal finite-time feedback controllers for nonlinear systems with terminal constraints. *Journal of the Guidance, Control, and Dynamics*, 29(4): 921-928.

2.9 Lyapunov Control References

[C1] LaSalle, J., and S. Lefschetz. (1961). *Stability by Liapunov's direct method: with applications*. USA: Academic Press Inc.

[C2] Meirovitch, L. (1970). *Methods of analytical dynamics*. USA: McGraw-Hill, Inc.

[C3] Vidyasagar, M. (1993). *Nonlinear systems analysis, 2nd Ed*. USA: Prentice-Hall, Inc.

[C4] Slotine, J-J.E., and W. Li. (1991). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

[C5] Junkins, J.L., and Y. Kim. (1993). *Introduction to dynamics and control of flexible structures*. USA: American Institute of Aeronautics and Astronautics, Inc.

[C6] Skaar, S.B., and C.F. Ruoff, eds. (1994). *Teleoperation and robotics in space, Progress in Astronautics and Aeronautics Volume 161*. USA: American Institute of Aeronautics and Astronautics, Inc.

[C7] Sastry, S. (1999). *Nonlinear systems: analysis, stability, and control*. USA: Springer-Verlag, Inc.

[C8] Schaub, H., and J.L. Junkins. (2003). *Analytical mechanics of space systems*. USA: American Institute of Aeronautics and Astronautics, Inc.

[C9] Lyapunov, A.M. (1907). Problème général de la stabilité du movement. *Ann. Fac. Sci. Toulouse*, 9: 203-474.

[C10] Kalman, R.E., and J.E. Bertram. (1960). Control system analysis and design via the second method of Lyapunov, Part I: continuous-time systems. *Journal of Basic Engineering*, 82(2): 371-393.

[C11] Kalman, R.E., and J.E. Bertram. (1960). Control system analysis and design via the second method of Lyapunov, Part II: discrete-time systems. *Journal of Basic Engineering*, 82(2): 394-400.

[C12] LaSalle, J.P. (1960). Some extensions of Liapunov's second method. *IRE Transactions on Circuit Theory*, 7(4): 520-527.

[C13] Mortensen, R.E. (1968). A globally stable linear attitude regulator. *International Journal of Control*, 8(3): 297-302.

[C14] Vadali, S.R., and J.L. Junkins. (1984). Optimal open-loop and stable feedback control of rigid spacecraft attitude maneuvers. *The Journal of the Astronautical Sciences*, 32(2): 105-122.

[C15] Wie, B., and P.M. Barba. (1985). Quaternion feedback for spacecraft large-angle maneuvers. *The Journal of Guidance, Control, and Dynamics*, 8(3): 360-365.

[C16] Wie, B., Weiss, H., and A. Arapostathis. (1989). Quaternion feedback regulator for spacecraft eigenaxis rotations. *The Journal of Guidance, Control, and Dynamics*, 12(3): 375-380.

[C17] Wen, J.T-Y., and K. Kreutz-Delgado. (1991). The attitude control problem. *IEEE Transactions on Automatic Control*, 36(10): 1148-1162.

[C18] Oh, H.-S., Vadali, S.R., and J.L. Junkins. (1992). Use of the work-energy rate principle for designing feedback control laws. *The Journal of Guidance, Control, and Dynamics*, 15(1): 275-277.

[C19] Mukherjee, R., and J.L. Junkins. (1993). Invariant set analysis of the Hub-Appendage problem. *The Journal of Guidance, Control, and Dynamics*, 16(6): 1191-1193.

[C20] McInnes, C.R. (1994). Large angle slew maneuvers with autonomous sun vector avoidance. *The Journal of Guidance, Control, and Dynamics*, 17(4): 875-877.

[C21] Tsiotras, P. (1994). New control laws for the attitude stabilisation of rigid bodies. *Proceedings of the IFAC Symposium on Automatic Control in Aerospace*, 316-321.

[C22] Bell, M.J., and J.L. Junkins. (1994). Near minimum-time three dimensional maneuvers of rigid and flexible spacecraft. *The Journal of the Astronautical Sciences*, 42(4): 421-438.

[C23] Wie, B., and J. Lu. (1995). Feedback control logic for spacecraft eigenaxis rotations under slew rate and control constraints. *The Journal of Guidance, Control, and Dynamics*, 18(6): 1372-1379.

[C24] Lizarralde, F., and J.T. Wen. (1996). Attitude control without angular velocity measurement: a passivity approach. *IEEE Transactions on Automatic Control*, 41(3): 468-472.

[C25] Creamer, G., DeLaHunt, P., Gates, S., and M. Levenson. (1996). Attitude determination and control of Clementine during lunar mapping. *The Journal of Guidance, Control, and Dynamics*, 19(3): 505-511.

[C26] Tsiotras, P. (1996). Stabilization and optimality results for the attitude control problem. *The Journal of Guidance, Control, and Dynamics*, 19(4): 772-779.

[C27] Schaub, H., Robinett, R.D., and J.L. Junkins. (1996). Globally stable feedback laws for near-minimum-fuel and near minimum-time pointing maneuvers for a landmark-tracking spacecraft. *The Journal of the Astronautical Sciences*, 44(4): 443-466.

[C28] Junkins, J.L. (1997). Adventures on the interface of dynamics and control. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1058-1071.

[C29] Robinett, R.D., et al. (1997). Lyapunov optimal saturated control for nonlinear systems. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1083-1088.

[C30] Tsiotras, P., and J. Luo. (1997). Reduced effort control laws for underactuated rigid spacecraft. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1089-1095.

[C31] Hall, C.D., Tsiotras, P., and H. Shen. (1998). Tracking rigid body motion using thrusters and momentum wheels. *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference*, Boston, Massachusetts, USA, AIAA Paper 98-4471.

[C32] Tsiotras, P. (1998). Further passivity results for the attitude control problem. *IEEE Transactions on Automatic Control*, 43(11): 1597-1600.

[C33] Boskovic, J.D., Sai-Ming, Li, and R.K. Mehra. (1999). Globally stable adaptive tracking control design for spacecraft under input saturation. *Proceedings of the IEEE Conference on Decision and Control*, December 1999, 1952-1957.

[C34] Hablani, H.B. (1999). Attitude commands avoiding bright objects and maintaining communication with ground station. *The Journal of Guidance, Control, and Dynamics*, 22(6): 759-767.

[C35] Akella, M.R. (2001). Rigid body attitude tracking without angular velocity feedback. *Systems and Control Letters*, 42(4): 321-326.

[C36] Tsiotras, P., Shen, H., and C. Hall. (2001). Satellite attitude control and power tracking with energy/momentum wheels. *Journal of Guidance, Control, and Dynamics*, 24(1): 23-34.

[C37] Wie, B., Bailey, D., and C. Heiberg. (2002). Rapid multitarget acquisition and pointing control of agile spacecraft. *The Journal of Guidance, Control, and Dynamics*, 25(1): 96-103.

[C38] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2004). Robust tracking control design for spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 27(4): 627-633.

[C39] Subbarao, K., and M.R. Akella. (2004). Differentiator-free nonlinear proportional-integral controllers for rigid-body attitude stabilization. *The Journal of Guidance, Control, and Dynamics*, 27(6): 1092-1096.

[C40] Akella, M.R., Valdivia, A., and G.R. Kotamraju. (2005). Velocity-free attitude controllers subject to actuator magnitude and rate saturations. *The Journal of Guidance, Control, and Dynamics*, 28(4): 659-666.

[C41] Boskovic, J.D., Bergstrom, S.E., and R.K. Mehra. (2005). Robust integrated flight control design under failures, damage, and state-dependent disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 902-917.

[C42] Wallsgrove, R. J., and M.R. Akella. (2005). Globally stabilizing saturated attitude control in the presence of bounded unknown disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 957-963.

[C43] Boskovic, J.D., Prasanth, R., and R.K. Mehra. (2007). Retrofit fault-tolerant flight control design under control effector damage. *The Journal of Guidance, Control, and Dynamics*, 30(3): 703-712.

[C44] Wertz, J.R., ed. (1978). *Spacecraft attitude determination and control*. The Netherlands: D. Reidel Publishing Company.

[C45] Shuster, M.D. (1993). A survey of attitude representations. *The Journal of the Astronautical Sciences*, 41(4): 439-517.

[C46] Slotine, J.J.E., and W. Li. (1990). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

2.10 Sliding Mode Control References

[D1] Utkin, V.I. (1978). *Sliding modes and their application in variable structure systems*. Russia: MIR Publishers.

[D2] Slotine, J.J.E., and W. Li. (1990). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

[D3] Young, K-K., D., ed. (1993). *Variable structure control for robotics and aerospace applications*. The Netherlands: Elsevier Science Publishers B.V.

[D4] Sabanovic, A., Fridman, L.M., and S. Spurgeon, eds. (2004). *Variable structure systems: from principles to implementation*. United Kingdom: The Institution of Electrical Engineers.

[D5] Utkin, V.I. (1977). Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22(2): 212-222.

[D6] Vadali, S.R. (1986). Variable-structure control of spacecraft large-angle maneuvers. *The Journal of Guidance, Control, and Dynamics*, 9(2): 235-239.

[D7] DeCarlo, R.A., Zak, S.H., and G.P. Matthews. (1988). Variable structure control of nonlinear multivariable systems: a tutorial. *Proceedings of the IEEE*, 76(3): 212-232.

[D8] Dwyer, T.A.W., and H. Sira-Ramirez. (1988). Variable-structure control of spacecraft attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 11(3): 262-270.

[D9] Hung, J.Y., Gao, W., and J.C. Hung. (1993). Variable structure control: a survey. *IEEE Transactions on Industrial Electronics*, 40(1): 2-22.

[D10] Chen, Y-P., and S-C. Lo. (1993). Sliding-mode controller design for spacecraft attitude tracking maneuvers. *IEEE Transactions on Aerospace and Electronic Systems*, 29(4): 1328-1333.

[D11] Lo, S-C., and Y-P. Chen. (1995). Smooth sliding-mode control for spacecraft attitude tracking maneuvers. *The Journal of Guidance, Control, and Dynamics*, 18(6): 1345-1349.

[D12] Crassidis, J.L., and F.L. Markley. (1996). Sliding mode control using modified Rodrigues parameters. *The Journal of Guidance, Control, and Dynamics*, 19(6): 1381-1383.

[D13] Robinett, R.D., and G.G. Parker. (1997). Least squares sliding mode control tracking of spacecraft large angle maneuvers. *The Journal of the Astronautical Sciences*, 45(4): 433-450.

[D14] Kim, J., Kim, J., and J.L. Crassidis. (1998). Disturbance accommodating sliding mode controller for spacecraft attitude maneuvers. *Proceedings of the AAS/GSFC International Symposium on Space Flight Dynamics*, Greenbelt, Maryland, USA, AAS Paper 98-310, 141-153.

[D15] Kim, J., and J.L. Crassidis. (1998). A comparative study of sliding-mode control and time-optimal control. *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference*, Boston, Massachusetts, USA, AIAA Paper 98-4473, 316-323.

[D16] Boskovic, J.D., Sai-Ming, L., and R.K. Mehra. (1999). Globally stable adaptive tracking control design for spacecraft under input saturation. *Proceedings of the IEEE Conference on Decision and Control*, Phoenix, Arizona, USA, 1952-1957.

[D17] Crassidis, J.L., Vadali, S.R., and F.L. Markley (2000). Optimal variable-structure control tracking of spacecraft maneuvers. *Journal of Guidance, Control, and Dynamics*, 23(3): 564-566.

[D18] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2001). Robust adaptive variable structure control of spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 24(1): 14-22.

[D19] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2004). Robust tracking control design for spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 27(4): 627-633.

2.11 Adaptive Control References

[E1] Bellman, R. (1961). *Adaptive control processes: a guided tour*. USA: Princeton University Press.

[E2] Landau, Y.D. (1979). *Adaptive control: the model reference approach*. USA: Marcel Dekker.

[E3] Sastry, S., and M. Bodson. (1989). *Adaptive control: stability, convergence, and robustness*. USA: Prentice-Hall, Inc.

[E4] Astrom, K.J., and B. Wittenmark. (1989). *Adaptive control*. USA: Addison-Wesley Publishing.

[E5] Narendra, K.S., and A.M. Annaswamy. (1989). *Stable adaptive systems*. USA: Prentice-Hall, Inc.

[E6] Slotine, J.J.E., and W. Li. (1990). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

[E7] Krstic, M., Kanellakopoulos, I., and P.V. Kokotovic. (1995). *Nonlinear and adaptive control design*. USA: Wiley-Interscience.

[E8] Ioannou, P.A., and J. Sun. (1995). *Stable and robust adaptive control*. USA: Prentice-Hall, Inc.

[E9] Khalil, H.K. (2002). *Nonlinear systems*, 3rd ed. USA: Prentice-Hall, Inc.

[E10] Narendra, K.S., and A.M. Annaswamy. (1986). Robust adaptive control in the presence of bounded disturbances. *IEEE Transactions on Automatic Control*, 31(4): 306-315.

[E11] Narendra, K.S., and A.M. Annaswamy. (1987). A new adaptive law for robust adaptive control without persistent excitation. *IEEE Transactions on Automatic Control*, 32(2): 134-145.

[E12] Slotine, J-J.E., and M.D. Di Benedetto. (1990). Hamiltonian adaptive control of spacecraft. *IEEE Transactions on Automatic Control*, 35(7): 848-852.

[E13] Parlos, A.G., and J.W. Sunkel. (1992). Adaptive attitude control and momentum management for large-angle spacecraft maneuvers. *The Journal of Guidance, Control, and Dynamics*, 15(4): 1018-1028.

[E14] Boussalis, D., Bayard, D.S., and S.J. Wang. (1992). Adaptive spacecraft attitude control with application to space station. *Proceedings of the IEEE Conference on Control Applications*, Dayton, Ohio, USA, 440-447.

[E15] Paielli, R.A., and R.E. Bach. (1993). Attitude control with realization of linear error dynamics. *The Journal of Guidance, Control, and Dynamics*, 16(1): 182-189.

[E16] Cristi, R., and J. Burl. (1993). Adaptive eigenaxis rotations. *Proceedings of the European Control Conference*, The Netherlands, June 1993, 770-775.

[E17] Sheen, J-J., and R.H. Bishop. (1994). Adaptive nonlinear control of spacecraft. *Proceedings of the American Control Conference*, Baltimore, Maryland, USA, 2867-2871.

[E18] Cristi, R., Burl, J., and N. Russo. (1994). Adaptive quaternion feedback regulation for eigenaxis rotations. *The Journal of Guidance, Control, and Dynamics*, 17(6): 1287-1291.

[E19] Joshi, S.M., Kelkar, A.G., and J.T-Y. Wen. (1995). Robust attitude stabilization of spacecraft using nonlinear quaternion feedback. *IEEE Transactions on Automatic Control*, 40(10): 1800-1803.

[E20] Di Gennaro, S. (1995). Adaptive robust stabilization of rigid spacecraft in presence of disturbances. *Proceedings of the IEEE Conference on Decision and Control*, New Orleans, Louisiana, USA, 1147-1152.

[E21] Paynter, S.J., and R.H. Bishop. (1997). Adaptive nonlinear attitude control and momentum management of spacecraft. *The Journal of Guidance, Control, and Dynamics*, 20(5): 1025-1032.

[E22] Junkins, J.L., Akella, M.R., and R.D. Robinett. (1997). Nonlinear adaptive control of spacecraft maneuvers. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1104-1110.

[E23] Di Gennaro, S. (1997). Stabilization of rigid spacecraft with uncertainties and input saturation in a central gravitational field. *Proceedings of the IEEE Conference on Decision and Control*, San Diego, California, USA, 4204-4209.

[E24] Akella, M.R., Junkins, J.L., and R.D. Robinett. (1998). Structured model reference adaptive control with actuator saturation limits. *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference, Boston*, Massachusetts, USA, AIAA Paper 98-4472, 308-315.

[E25] Akella, M.R., and J.L. Junkins. (1998). Structured model reference adaptive control in the presence of bounded disturbances. *Proceedings of the AAS/AIAA Spaceflight Mechanics Meeting*, Monterey, California, USA, AAS Paper 98-121, 375-385.

[E26] Ahmed, J. Coppola, V.T., and D.S. Bernstein. (1998). Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification. *The Journal of Guidance, Control, and Dynamics*, 21(5): 684-691.

[E27] Boskovic, J.D., Sai-Ming, L., and R.K. Mehra. (1999). Globally stable adaptive tracking control design for spacecraft under input saturation. *Proceedings of the IEEE Conference on Decision and Control*, Phoenix, Arizona, USA, 1952-1957.

[E28] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2000). A globally stable scheme for spacecraft control in the presence of sensor bias. *Proceedings of the IEEE Aerospace Conference*, Big Sky, Montana, USA, 505-511.

[E29] Schaub, H., Akella, M.R., and J.L. Junkins. (2000). Adaptive realization of linear closed-loop tracking dynamics in the presence of large system model errors. *The Journal of the Astronautical Sciences*, 48(4): 537-551.

[E30] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2001). Robust adaptive variable structure control of spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 24(1): 14-22.

[E31] Tsiotras, P., Shen, H., and C. Hall. (2001). Satellite attitude control and power tracking with energy/momentum wheels. *Journal of Guidance, Control, and Dynamics*, 24(1): 23-34.

[E32] Schaub, H., Akella, M.R., and J.L. Junkins. (2001). Adaptive control of nonlinear attitude motions realizing linear closed loop dynamics. *The Journal of Guidance, Control, and Dynamics*, 24(1): 95-100.

[E33] Yoon, H., and P. Tsiotras. (2002). Spacecraft adaptive attitude and power tracking with variable speed control moment gyroscopes. *The Journal of Guidance, Control, and Dynamics*, 25(6): 1081-1090.

[E34] Miwa, H., and M.R. Akella. (2002). Global adaptive stabilization using output feedback for spacecraft attitude tracking. *Proceedings of the AAS/AIAA Spaceflight Mechanics Meeting*, San Antonio, Texas, USA, 345-357.

[E35] Subbarao, K., and J.. Junkins. (2003). Structured model reference adaptive control for a class of nonlinear systems. *The Journal of Guidance, Control, and Dynamics*, 26(4): 551-557.

[E36] Tandale, M.D., Subbarao, K., Valasek, J., and M.R. Akella. (2004). Structured adaptive model inversion control with actuator saturation constraints applied to tracking spacecraft maneuvers. *Proceedings of the American Control Conference*, Boston, Massachusetts, USA, 1031-1036.

[E37] Lavretsky, E., and N. Hovakimyan. (2004). Positive μ-modification for stable adaptation in the presence of input constraints. *Proceedings of the American Control Conference*, Boston, Massachusetts, USA, 2545-2550.

[E38] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2004). Robust tracking control design for spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 27(4): 627-633.

[E39] Yoon, H., and P. Tsiotras. (2005). Adaptive spacecraft attitude tracking control with actuator uncertainties. *Proceedings of the AIAA Guidance, Navigation, and Control Conference, San Francisco, California, USA, AIAA Paper 2005-6392*, 1031-1036.

[E40] Wallsgrove, R. J., and M.R. Akella. (2005). Globally stabilizing saturated attitude control in the presence of bounded unknown disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 957-963.

[E41] Tandale, M.D., and J. Vasek. (2006). Fault-tolerant structured adaptive model inversion control. *The Journal of Guidance, Control, and Dynamics*, 29(3): 635-642.

[E42] Singla, P., Subbarao, K., and J.L. Junkins. (2006). Adaptive output feedback control for spacecraft rendezvous and docking under measurement uncertainty. *The Journal of Guidance, Control, and Dynamics*, 29(4): 892-902.

[E43] Lavretsky, E., and N. Hovakimyan. (2007). Stable adaptation in the presence of actuator constraints with flight control applications. *The Journal of Guidance, Control, and Dynamics*, 30(2): 337-345.

2.12 State Estimation Theory References

[F1] Jazwinski, A. (1970). *Stochastic processes and filtering theory*. USA, Academic Press, Inc.

[F2] Gelb, A., ed. (1974). *Applied optimal estimation*. USA: MIT Press.

[F3] Sorenson, H.W. (1980). *Parameter estimation: principles and problems*. USA: Marcel Dekker, Inc.

[F4] Crassidis, J.L. and Junkins, J.L. (2003). *Optimal estimation of dynamic systems*. USA: CRC Press LLC.

[F5] Ristic, B., Arulampalam, S., and N. Gordon. (2004). *Beyond the Kalman filter: particle filters for tracking applications*. USA: Artech House Publishers.

[F6] Markley, F.L., Crassidis, J.L., and Y. Cheng. Survey of nonlinear attitude estimation methods. *The Journal of Guidance, Control, and Dynamics*, 30(1): 12-28.

[F7] Shuster, M.D., and S.D. Oh. (1981). Three-axis attitude determination from vector observations. *The Journal of Guidance, Control, and Dynamics*, 4(1): 70-77.

[F8] Markley, F.L. (1988). Attitude determination using vector observations and the singular value decomposition. *The Journal of the Astronautical Sciences*, 36(3): 245-258.

[F9] Shuster, M.D. (1989). Maximum likelihood estimation of spacecraft attitude. *The Journal of the Astronautical Sciences*, 37(1): 79-88.

[F10] Markley, F.L., and Mortari, D. (2000). Quaternion attitude estimation using vector observations. *The Journal of the Astronautical Sciences*, 48(2): 359-380.

[F11] Oshman, Y., and F.L. Markley. (1999). Sequential attitude and attitude-rate estimation using integrated-rate parameters. *The Journal of Guidance, Control, and Dynamics*, 22(3): 385-394.

[F12] Harman, R.R., and I.Y. Bar-Itzhack. (1999). Pseudolinear and state-dependent Riccati equation filters for angular rate estimation. *The Journal of Guidance, Control, and Dynamics*, 22(5): 723-725.

[F13] Bar-Itzhack. (2001). Classification of algorithms for angular velocity estimation. *The Journal of Guidance, Control, and Dynamics*, 24(3): 436-443.

[F14] Azor, R., Bar-Itzhack, I.Y., Deutschmann, J.K., and R.R. Harman. (2001). Angular-rate estimation using delayed quaternion measurements. *The Journal of Guidance, Control, and Dynamics*, 24(2): 214-218.

[F15] Rao, G.N., Alex, T.K., and M.S. Bhat. (2002). Incremental-angle and angular velocity estimation using a star sensor. *The Journal of Guidance, Control, and Dynamics*, 25(3): 433-441.

[F16] Kalman, R.E. (1960). A new approach to linear filtering and prediction problems. Journal of Basic Engineering, 82(1): 35-45.

68

[F17] Lefferts, E.J., Markley, F.L., and M.D. Shuster. (1982). Kalman filtering for spacecraft attitude estimation. *The Journal of Guidance, Control, and Dynamics*, 5(5): 417-429.

[F18] Bar-Itzhack, I.Y., and Y. Oshman. (1985). Attitude determination from vector observations: quaternion estimation. *IEEE Transactions on Aerospace and Electronic Systems*, 21(1): 128-136.

[F19] Gai, E., Daly, K., Harrison, J., and L. Lemos. (1985). Star-sensor-based satellite attitude/attitude rate estimator. *The Journal of Guidance, Control, and Dynamics*, 8(5): 560-565.

[F20] Vathsal, S. (1987). Spacecraft attitude determination using a second-order nonlinear filter. *The Journal of Guidance, Control, and Dynamics*, 10(6): 559-566.

[F21] Fisher, H.L., Shuster, M.D., and T.E. Strikwerda, T.E. (1989). Attitude determination for the Star Tracker Mission. *Proceedings of the AAS/AIAA Astrodynamics Conference*, Vermont, USA. AAS Paper 89-365, 139-150.

[F22] Shuster, M.D. (1990). Kalman filtering of spacecraft attitude and the QUEST model. *The Journal of the Astronautical Sciences*, 38(3): 377-393.

[F23] Markley, F.L. (2003). Attitude error representations for Kalman filtering. *The Journal of Guidance, Control, and Dynamics*, 26(2): 311-317.

[F24] Crassidis, J.L., and F.L. Markley. (2003). Unscented filtering for spacecraft attitude estimation. *The Journal of Guidance, Control, and Dynamics*, 26(4): 536-542.

[F25] Yau, S. S-T., Yau, S-T., and C. Yan. (2004). Linear filtering with nonlinear observations. *Proceedings of the IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, 2112-2117.

[F26] Yau, S-T., and S. S-T. Yau. (2004). Nonlinear filtering and time varying Schrodinger equation I. *IEEE Transactions on Aerospace and Electronic Systems*, 40(1): 284-292.

[F27] Markley, F.L. (2005). Attitude filtering on SO(3). *Proceedings of the AAS Malcolm D. Shuster Astronautics Symposium*, New York, USA AAS Paper 05-460, 149-174.

[F28] Yau, S. S-T., and L. Jia. (2005). Filtering problem with nonlinear observations and drift terms equal to gradient vector field plus affine vector field. *Proceedings of the IEEE Conference on Decision and Control*, Spain, 5865-5870.

[F29] Sorensen, D.C. (1982). Newton's method with a model trust region modification. *SIAM Journal of Numerical Analysis*, 19(2): 409-426.

[F30] Psiaki, M. L. (2000). Attitude-determination filtering via extended quaternion estimation. *The Journal of Guidance, Control, and Dynamics*, 23(2): 206-214.

[F31] Gordon, N.J., Salmond, D.J., and A.F. Smith. (1993). Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *Proceedings of the Institute of Electrical Engineers*, 140: 107-113.

[F32] Liu, J.S., and R. Cheng. (1998). Sequential Monte Carlo methods for dynamic systems. Journal of the American Statistical Association, 93(443): 1032-1044.

[F33] Doucet, A., Godsill, S., and C. Andrieu. (2000). On sequential Monte Carlo sampling methods for Bayesian filtering. *Journal of Statistics and Computing*, 10(3): 197-208.

[F34] Arulampalam, M.S., Maskell, S., Gordon, N., and T. Clapp. (2002). A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking. *IEEE Transactions on Signal Processing*, 50(2): 174-188.

[F35] Cheng, Y., and J.L. Crassidis. (2004). Particle filtering for sequential spacecraft attitude estimation. Proceedings of the AIAA Guidance Navigation, and Control Conference, Rhode Island, USA, AIAA Paper 2004-5337.

[F36] Rapoport, I., and Y. Oshman. (2005). Fault-tolerant particle filtering by using interacting multiple model-based Rao-Blackwellization. *The Journal of Guidance, Control, and Dynamics*, 28(6): 1171-1177.

[F37] Oshman, Y., and A. Carmi. (2006). Attitude estimation from vector observations using genetic-algorithm-embedded quaternion particle filter. *The Journal of Guidance, Control, and Dynamics*, 29(4): 879-891.

[F38] Markley, F.L., Cheng, Y., Crassidis, J.L., and Y. Oshman. Averaging quaternions. *The Journal of Guidance, Control, and Dynamics*, 30(4): 1193-1196.

[F39] Farrenkopf, R.L. (1978). Analytic steady-state accuracy solutions for two common spacecraft attitude estimators. *The Journal of Guidance, Control, and Dynamics*, 1(4): 282-284.

Chapter 3   Spacecraft Attitude Maneuvers Using Lyapunov Stability Theory

3.1 Introduction

This chapter will consider the design of feedback control laws based on Lyapunov's direct method[46] for rigid spacecraft attitude maneuvers which must track a general reference trajectory motion. In Section 3.2, the kinematic and dynamic equations of motion for attitude tracking maneuvers are presented. In Sections 3.3 & 3.4, a complete Lyapunov stability analysis is performed for the controller developed by Wie and associates[15,16] which ensures a minimum angular path to the reference trajectory. In Section 3.5, the effect of control torque limits in the stability analysis is considered according to Reference 29, while the effect of control torque and angular rate limits is addressed in Section 3.6 according to Reference 23. Section 3.7 addresses control torque limits and bounded disturbance torques according to Reference 38.

In Section 3.8 the research by Wallsgrove and Akella[42] is extended for attitude tracking maneuvers and to incorporate the effects of state estimation error in the stability analysis. This is one of the major novel contributions to existing literature provided in this thesis. The proposed control strategy provides global asymptotic trajectory tracking in the presence of sensor measurement error in addition to spacecraft inertia uncertainty, control torque magnitude limits, and external bounded disturbances. Dynamic model uncertainty and torque magnitude limits are accounted for in the controller structure and disturbance rejection of bounded persistent external disturbances is achieved using a dynamic control gain parameter. The novel control law proposed in this chapter considers the effect of additional state-dependent and upper bounded disturbance terms, introduced into the stability analysis by state estimation error terms in the feedback control law and control gain parameter update law. A dynamic control gain parameter is implemented to ensure asymptotic convergence of the trajectory tracking errors to zero and boundedness of the dynamic control parameter. Restrictions on the selection of the controller parameters are evaluated based on the expected reference trajectory motion, external disturbance environment, and state estimation filter performance. In

Section 3.9 MATLAB™ simulation results are presented to demonstrate the performance of the various control algorithms developed in Sections 3.3-3.8.

3.2 Spacecraft Dynamic and Kinematics

This section presents the kinematic and dynamic equations of motion necessary to model spacecraft attitude maneuvers. The spacecraft is modelled as a rigid body with actuators that provide three-axis external control torques. Consider an inertial-fixed coordinate frame $\{\mathbf{N}\} \equiv \{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3\}$ with an origin that is either fixed in inertial space or moving with a linear velocity relative to inertial space, and whose basis vectors are not rotating with respect to inertial space. Consider a spacecraft body-fixed coordinate frame $\{\mathbf{B}(t)\} \equiv \{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$ centered at a point P located in the rigid body and arbitrarily orientated with respect to the rigid body. The point P is not necessarily located at the center of mass of the spacecraft since this location may be unknown. Since $\{\mathbf{B}(t)\}$ is fixed with respect to the rigid body its angular velocity relative to inertial space will equal that of the rigid body spacecraft. Assume that an external body-fixed torque is produced about each spacecraft body-fixed axis using a cold-gas thruster system for example. To simplify the dynamic equations of motion center $\{\mathbf{B}(t)\}$ at the spacecraft mass center. A further simplification may be applied in which $\{\mathbf{B}(t)\}$ is aligned to realise a principal axis coordinate system but this assumption is not used in this work. Euler's equations of motion[44] for the rotational motion of a rigid spacecraft are given by

$$\frac{d}{dt}\boldsymbol{\omega}(t) = \mathbf{J}^{-1}\left[-\left[\boldsymbol{\omega}(t)\times\right]\mathbf{J}\boldsymbol{\omega}(t) + \mathbf{u}(t) + \mathbf{d}(t)\right] \tag{1}$$

where $\boldsymbol{\omega}(t) \in \mathbf{R}^{3\times1}$ contains the $\{\mathbf{B}(t)\}$ components of the angular velocity of the spacecraft (or equivalently spacecraft body-fixed frame) with respect to the inertial frame $\{\mathbf{N}\}$, $\mathbf{J} \in \mathbf{R}^{3\times3}$ is the inertia matrix of the spacecraft, and $\mathbf{u}(t) \in \mathbf{R}^{3\times1}$ are the $\{\mathbf{B}(t)\}$ components of the external control torque inputs, and $\mathbf{d}(t) \in \mathbf{R}^{3\times1}$ is an external torque term representing the effect of all un-modelled dynamics and external disturbances. The dominant source of external

disturbances is assumed to be the gravity-gradient torque $\mathbf{u}_{gg}(t)$ resulting from the non-uniform distribution of the earth's gravitational field over the rigid-body spacecraft. The $[\cdot \times]$ operator in Eq (1) is the vector cross-product operator defined by

$$[\boldsymbol{\omega}(t)\times] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad (2)$$

The spacecraft inertia matrix is a symmetric positive definite matrix which follows directly from the positive definite form of the rotational kinetic energy. In designing control laws based on the Lyapunov stability concepts the disturbance term in Eq (1) will be assumed to consist only of the gravity-gradient torque whose approximate functional form is specified in Reference 8. In addition, the spacecraft inertia parameters will be assumed to be known exactly.

For a spacecraft equipped with momentum exchange devices the spacecraft is not a rigid body and a modified version of Euler's equations must be applied. Consider an actuator configuration comprising n reaction wheels, where three wheels are aligned with $\{\mathbf{B}(t)\}$ and the fourth wheel is arbitrarily orientated. The dynamic equations of motion[44] are given by

$$\frac{d}{dt}\boldsymbol{\omega}(t) = -\mathbf{J}_s^{-1}\left\{[\boldsymbol{\omega}(t)\times]\mathbf{h}(t) + \widetilde{\mathbf{C}}^{\mathrm{T}}\mathbf{u}(t)\right\} \qquad (3)$$

$$\mathbf{h}(t) = \mathbf{J}^*\boldsymbol{\omega}(t) + \widetilde{\mathbf{C}}^{\mathrm{T}}\mathbf{J}_w\boldsymbol{\Omega}(t) \qquad (4)$$

$$\frac{d}{dt}\boldsymbol{\Omega}(t) = \mathbf{J}_w^{-1}\mathbf{u}(t) - \widetilde{\mathbf{C}}\frac{d}{dt}\boldsymbol{\omega}(t) \qquad (5)$$

where $\mathbf{h}(t) \in \mathbf{R}^{3\times 1}$ is the total angular momentum about the spacecraft mass center, $\mathbf{u}(t) \in \mathbf{R}^{3\times 1}$ is the internal control torque applied to the reaction wheels by their electric motors, $\boldsymbol{\Omega}(t) \in \mathbf{R}^{3\times 1}$ contains the reaction wheel angular

velocities, and $\widetilde{\mathbf{C}}(t) \in \mathbf{R}^{n \times 3}$ is a rotation matrix whose rows are the $\{\mathbf{B}(t)\}$ components of the n unit vectors along the reaction wheel spin axes. The remaining parameters include $\mathbf{J}^* \in \mathbf{R}^{3 \times 3}$ as the spacecraft inertia matrix relative to $\{\mathbf{B}(t)\}$ (with reaction wheels locked), $\mathbf{J_w} \in \mathbf{R}^{n \times n}$ as the reaction wheel axial inertia matrix defined by $\mathbf{J_w} = \mathrm{diag}\{J_{a,i}\}$ for $i = 1,...,n$, and $\mathbf{J_s} \in \mathbf{R}^{3 \times 3}$ as the effective system inertia matrix relative to $\{\mathbf{B}(t)\}$ defined by $\mathbf{J_s} = \left[\mathbf{J}^* - \widetilde{\mathbf{C}}^\mathrm{T} \mathbf{J_w} \widetilde{\mathbf{C}}\right]$. The objective is to develop a feedback control law, for either the system defined by Eq (1) or Eq (2), that is in the form $\mathbf{u}(t) \equiv \mathbf{u}(\mathbf{x}(t), t)$, where $\mathbf{x}(t)$ contains the state variables of the system. To solve for the n-dimensional reaction wheel control law $\mathbf{u}(t)$, based on the system of Eq (3), requires the solution of the equation $\mathbf{J_s}^{-1} \widetilde{\mathbf{C}}^\mathrm{T} \mathbf{u}(t) = \mathbf{f}(\mathbf{x}(t), t)$. This requires, however, the inversion of the 3 x n matrix $\mathbf{\Gamma} = \mathbf{J_s}^{-1} \widetilde{\mathbf{C}}^\mathrm{T}$, and it follows that the selection of the control law is underdetermined. Any suitable optimality criterion may used to select a particular control satisfying the equation, but a minimum torque criterion[6] which minimises the cost function $J = \mathbf{u}^\mathrm{T}(t)\mathbf{u}(t)$ results in the control law

$$\mathbf{u}(t) = \mathbf{\Gamma}^\mathrm{T} \left[\mathbf{\Gamma}\mathbf{\Gamma}^\mathrm{T}\right]^{-1} \mathbf{f}(\mathbf{x}(t), t) \tag{6}$$

In this thesis, however, the dynamic model defined by Eq (1) will be applied to spacecraft attitude maneuver design and not the dynamic model of Eqs (3)-(5).

The orientation of the spacecraft body-fixed coordinate frame $\{\mathbf{B}(t)\}$ with respect to an arbitrarily orientated inertial frame $\{\mathbf{N}\}$ at a particular instant in time t may be expressed using the orthonormal direction cosine matrix[44,45] $\mathbf{C}(t) \in \mathbf{R}^{3 \times 3}$ according to

$$\{\mathbf{B}(t)\} = \mathbf{C}(t)\{\mathbf{N}\} \tag{7}$$

where the direction cosine matrix is parameterised by the unit quaternion $\mathbf{q}(t) = [\mathbf{q}_{13}^\mathrm{T}(t) \quad q_4^\mathrm{T}(t)]^\mathrm{T} \in \mathbf{R}^{4 \times 1}$. The unit quaternion[44,45] is a non-minimal

representation of the spacecraft attitude which globally avoids singular orientations. Consequently the solution of the kinematic equations remains underdetermined such that the unit-norm constraint must be enforced at each instant in time. The components of the quaternion are directly obtained using Euler's theorem on the rotation of a rigid body based on a rotation $0 \leq \Phi(t) \leq 2\pi$ about the principal Euler rotation axis $\hat{\mathbf{e}}(t)$ according to

$$\mathbf{q}_{13}(t) = \hat{\mathbf{e}}(t)\sin(\Phi(t)/2) \tag{8}$$

$$q_4(t) = \cos(\Phi(t)/2) \tag{9}$$

The direction cosine matrix parameterised by the quaternion parameters $\mathbf{C}(t) \equiv \mathbf{C}(\mathbf{q}(t))$ is given by

$$\mathbf{C}(\mathbf{q}(t)) = \left(q_4^2(t) - \mathbf{q}_{13}^T(t)\mathbf{q}_{13}(t)\right)\mathbf{I}_{3\times3} + 2\mathbf{q}_{13}(t)\mathbf{q}_{13}^T(t) - 2q_4(t)\left[\mathbf{q}_{13}(t)\times\right] \tag{10}$$

where $\mathbf{I}_{3\times3} \in \mathbf{R}^{3\times3}$ denotes the 3 x 3 identity matrix, and the skew-symmetric matrix $\left[\mathbf{q}_{13}(t)\times\right]$ is defined with structure given by Eq (2). It follows from the definition of the quaternion that its components are not independent but satisfy the unit-norm constraint

$$\mathbf{q}^T(t)\mathbf{q}(t) = \mathbf{q}_{13}^T(t)\mathbf{q}_{13}(t) + q_4^2(t) = 1 \tag{11}$$

The quaternion parameters are not unique since $\mathbf{q}(t)$ (generated using the Euler axis/angle parameter sets $\{\hat{\mathbf{e}}(t), \Phi(t)\}$ and $\{-\hat{\mathbf{e}}(t), -\Phi(t)\}$), and $-\mathbf{q}(t)$ (generated using $\{\hat{\mathbf{e}}(t), \Phi(t) - 2\pi\}$ and $\{-\hat{\mathbf{e}}(t), 2\pi - \Phi(t)\}$) define the same physical orientation. This fact becomes important in the development of *minimum angular path* spacecraft rotational maneuvers based on the quaternion parameterisation. The kinematic equations of motion[44,45] for the quaternion parameters are given by

$$\frac{d}{dt}\mathbf{q}(t) = \frac{1}{2}\mathbf{\Omega}(\mathbf{\omega}(t))\mathbf{q}(t) = \frac{1}{2}\mathbf{\Xi}(\mathbf{q}(t))\mathbf{\omega}(t) \tag{12}$$

77

where the individual matrices $\boldsymbol{\Omega}(\boldsymbol{\omega}(t)) \in \mathbf{R}^{4\times4}$ and $\boldsymbol{\Xi}(\mathbf{q}(t)) \in \mathbf{R}^{4\times3}$ are defined as

$$\boldsymbol{\Omega}(\boldsymbol{\omega}(t)) = \begin{bmatrix} -[\boldsymbol{\omega}(t)\times] & \boldsymbol{\omega}(t) \\ -\boldsymbol{\omega}^{\mathrm{T}}(t) & 0 \end{bmatrix} \tag{13}$$

$$\boldsymbol{\Xi}(\mathbf{q}(t)) = \begin{bmatrix} [\mathbf{q}_{13}(t)\times] + q_4(t)\mathbf{I}_{3\times3} \\ -\mathbf{q}_{13}^{\mathrm{T}}(t) \end{bmatrix} \tag{14}$$

For spacecraft *attitude tracking maneuvers* the actual spacecraft motion must track a desired reference trajectory defined by a reference quaternion $\mathbf{q_d}(t) \in \mathbf{R}^{4\times1}$ and reference angular velocity $\boldsymbol{\omega_d}(t) \in \mathbf{R}^{3\times1}$. More explicitly the reference motion is defined by the orientation of a target coordinate frame $\{\mathbf{R}(t)\}$ with respect to an arbitrary inertial frame $\{\mathbf{N}\}$ denoted by $\mathbf{q_d}(t)$, and the $\{\mathbf{B}(t)\}$ components of the angular velocity of the reference trajectory coordinate frame $\{\mathbf{R}(t)\}$ with respect to the inertial frame $\{\mathbf{N}\}$ denoted by $\boldsymbol{\omega_d}(t)$. The tracking error is the difference between the actual spacecraft attitude motion and the reference trajectory defined by the error quaternion $\boldsymbol{\delta}\mathbf{q}(t) \in \mathbf{R}^{4\times1}$ denoting the orientation of $\{\mathbf{B}(t)\}$ with respect to $\{\mathbf{R}(t)\}$, and error angular velocity $\boldsymbol{\omega_e}(t) \in \mathbf{R}^{3\times1}$ denoting the $\{\mathbf{B}(t)\}$ components of the angular velocity of $\{\mathbf{B}(t)\}$ with respect to $\{\mathbf{R}(t)\}$. The error spacecraft attitude quaternion satisfies

$$\mathbf{C}(\boldsymbol{\delta}\mathbf{q}(t)) = \mathbf{C}(\mathbf{q}(t))\mathbf{C}^{\mathrm{T}}(\mathbf{q_d}(t)) \tag{15}$$

The error quaternion may be obtained using quaternion multiplication

$$\boldsymbol{\delta}\mathbf{q}(t) = \mathbf{q}(t) \otimes \mathbf{q_d}^{-1}(t) \tag{16}$$

where $\otimes$ is the quaternion multiplication operator[45] and the conjugate quaternion describing the orientation of $\{\mathbf{N}\}$ with respect to $\{\mathbf{R}(t)\}$ is defined by

$$\mathbf{q_d}^{-1}(t) = \begin{bmatrix} -\mathbf{q}_{\mathbf{d}13}^{\mathrm{T}}(t) & \mathbf{q}_{\mathbf{d}4}(t) \end{bmatrix}^{\mathrm{T}} \tag{17}$$

78

Performing the quaternion multiplication in Eq (16) gives

$$\delta\mathbf{q}(t) = \mathbf{\Lambda}(\mathbf{q_d}(t))\mathbf{q}(t) \tag{18}$$

where the $\mathbf{\Lambda}(\mathbf{q_d}(t)) \in \mathbf{R}^{4\times4}$ matrix is defined as

$$\mathbf{\Lambda}(\mathbf{q_d}(t)) = \begin{bmatrix} \mathbf{\Xi}^{\mathrm{T}}(\mathbf{q_d}(t)) \\ \mathbf{q_d^{\mathrm{T}}}(t) \end{bmatrix} \tag{19}$$

The kinematic equation for the error quaternion is obtained by using $\delta\mathbf{q}(t)$ and $\boldsymbol{\omega}_e(t)$ in Eq (12) which gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\delta\mathbf{q}(t) = \frac{1}{2}\mathbf{\Omega}(\boldsymbol{\omega}_e(t))\delta\mathbf{q}(t) = \frac{1}{2}\mathbf{\Xi}(\delta\mathbf{q}(t))\boldsymbol{\omega}_e(t) \tag{20}$$

The $\{\mathbf{B}(t)\}$ components of the error spacecraft angular rates are given by

$$\boldsymbol{\omega}_e(t) = \boldsymbol{\omega}(t) - \boldsymbol{\omega}_d(t) \tag{21}$$

where the reference trajectory angular velocity $\boldsymbol{\omega}_d(t)$ is expressed in the $\{\mathbf{B}(t)\}$ coordinate frame. In many applications the reference angular rate $\tilde{\boldsymbol{\omega}}_d(t)$ is available in $\{\mathbf{R}(t)\}$ such that a coordinate transformation is necessary given by $\boldsymbol{\omega}_d(t) = \mathbf{C}(\delta\mathbf{q}(t))\tilde{\boldsymbol{\omega}}_d(t)$. The reference angular rate motion $\boldsymbol{\omega}_d(t)$ in Eq (21) may be defined in terms of the reference attitude parameters

$$\boldsymbol{\omega}_d(t) = 2\left(q_{d4}(t)\dot{\mathbf{q}}_{d13}(t) - \dot{q}_{d4}(t)\mathbf{q}_{d13}(t) - 2[\mathbf{q}_{d13}(t)\times]\dot{\mathbf{q}}_{d13}(t)\right) \tag{22}$$

and differentiating Eq (22) gives the reference trajectory angular acceleration

$$\dot{\boldsymbol{\omega}}_d = 2\left(\mathbf{q}_{d4}\ddot{\mathbf{q}}_{d13} - \ddot{\mathbf{q}}_{d4}\mathbf{q}_{d13}\right) - 2[\mathbf{q}_{d13}\times]\ddot{\mathbf{q}}_{d13} \tag{23}$$

which may be related to the reference angular acceleration expressed in $\{\mathbf{N}\}$ using the Coriolis theorem. *Attitude regulation* may be considered a special case

of attitude tracking maneuvers in which the desired orientation is fixed with respect to the inertial coordinate frame and the desired spacecraft angular velocity is zero.

The dynamic equation of motion for the error spacecraft angular rates is obtained by substituting the first-order time derivative of Eq (21) into Eq (1) to obtain

$$\frac{d}{dt}\boldsymbol{\omega}_e(t) = \mathbf{J}^{-1}\left[-[\boldsymbol{\omega}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{J}\frac{d}{dt}\boldsymbol{\omega}_d(t) + \mathbf{u}(t) + \mathbf{d}(t)\right] \qquad (24)$$

Expanding the cross-coupling torque term in Eq (24) using Eq (21) gives

$$\frac{d}{dt}\boldsymbol{\omega}_e(t) = \mathbf{J}^{-1}\left[\mathbf{f}(\boldsymbol{\omega}_e(t),\boldsymbol{\omega}_d(t)) + \mathbf{g}(\boldsymbol{\omega}_d(t),\dot{\boldsymbol{\omega}}_d(t)) + \mathbf{u}(t) + \mathbf{d}(t)\right] \qquad (25)$$

where the auxiliary terms related to the reference trajectory motion are given by $\mathbf{f}(\boldsymbol{\omega}_e(t),\boldsymbol{\omega}_d(t)) \in \mathbf{R}^{3\times1} = -[\boldsymbol{\omega}_e(t)\times]\mathbf{J}\boldsymbol{\omega}_e(t) + \{[\mathbf{J}\boldsymbol{\omega}_d(t)\times] - [\boldsymbol{\omega}_d(t)\times]\mathbf{J}\}\boldsymbol{\omega}_e(t)$ and $\mathbf{g}(\boldsymbol{\omega}_d(t),\dot{\boldsymbol{\omega}}_d(t)) \in \mathbf{R}^{3\times1} = -[\boldsymbol{\omega}_d(t)\times]\mathbf{J}\boldsymbol{\omega}_d(t) - \mathbf{J}\dot{\boldsymbol{\omega}}_d(t)$. If the reference trajectory parameters $\widetilde{\boldsymbol{\omega}}_d(t)$ and $\dot{\widetilde{\boldsymbol{\omega}}}_d(t)$ are expressed in the reference trajectory coordinate frame $\{\mathbf{R}(t)\}$ then the $\mathbf{f}(\boldsymbol{\omega}_e(t),\boldsymbol{\omega}_d(t))$ term is replaced with

$\mathbf{f}(\boldsymbol{\omega}_e(t),\widetilde{\boldsymbol{\omega}}_d(t),\delta\mathbf{q}(t)) \in \mathbf{R}^{3\times1} = -[\boldsymbol{\omega}_e(t)\times]\mathbf{J}\boldsymbol{\omega}_e(t) + [\mathbf{J}\mathbf{C}(\delta\mathbf{q}(t))\widetilde{\boldsymbol{\omega}}_d(t)\times]\boldsymbol{\omega}_e(t)$

$-\{[\mathbf{C}(\delta\mathbf{q}(t))\widetilde{\boldsymbol{\omega}}_d(t)\times]\mathbf{J} + \mathbf{J}[\mathbf{C}(\delta\mathbf{q}(t))\widetilde{\boldsymbol{\omega}}_d(t)\times]\}$. The matrix term defined by $\mathbf{H}(\widetilde{\boldsymbol{\omega}}_d(t),\delta\mathbf{q}(t)) = [\mathbf{C}(\delta\mathbf{q}(t))\widetilde{\boldsymbol{\omega}}_d(t)\times]\mathbf{J} + \mathbf{J}[\mathbf{C}(\delta\mathbf{q}(t))\widetilde{\boldsymbol{\omega}}_d(t)\times]$ is a skew-symmetric matrix $\mathbf{H}(\widetilde{\boldsymbol{\omega}}_d(t),\delta\mathbf{q}(t)) = -\mathbf{H}^T(\widetilde{\boldsymbol{\omega}}_d(t),\delta\mathbf{q}(t))$ which satisfies the expression $\mathbf{x}^T(t)\mathbf{H}(\widetilde{\boldsymbol{\omega}}_d(t),\delta\mathbf{q}(t))\mathbf{x}(t) = 0$ for all $\mathbf{x}(t) \in \mathbf{R}^{3\times1}$. Similarly, based on the reference trajectory parameters, the $\mathbf{g}(\boldsymbol{\omega}_d(t),\dot{\boldsymbol{\omega}}_d(t))$ term is replaced with

$\mathbf{g}(\widetilde{\boldsymbol{\omega}}_d(t),\dot{\widetilde{\boldsymbol{\omega}}}_d(t),\delta\mathbf{q}(t)) \in \mathbf{R}^{3\times1} = -[\mathbf{C}(\delta\mathbf{q}(t))\widetilde{\boldsymbol{\omega}}_d(t)\times]\mathbf{J}\mathbf{C}(\delta\mathbf{q}(t))\widetilde{\boldsymbol{\omega}}_d(t) - \mathbf{J}\mathbf{C}(\delta\mathbf{q}(t))\dot{\widetilde{\boldsymbol{\omega}}}_d(t)$.

In Sections 3.5-3.9 spacecraft attitude maneuvers will be designed which account for control torque saturation and bounded disturbance torques acting on the system. The external control torque vector is bounded by

$\mathbf{u}(t) \in \mathbf{S_u} = \{\mathbf{u}(t) : |u_i(t)| \le u_{max} \ i = 1,2,3\}$. It is assumed that time-varying external disturbance torques of unknown functional form act on the spacecraft. The magnitude of these disturbances is upper bounded according to $\mathbf{d}(t) \in \mathbf{S_d} = \{\mathbf{d}(t) : \|\mathbf{d}(t)\|_1 = |d_1(t)| + |d_2(t)| + |d_3(t)| \le d_{max}\}$. It is assumed that internal torques and internal energy dissipation effects which decrease the system rotational kinetic energy are not present in the system. The upper bound on the norm of the spacecraft inertia matrix is given by $\|\mathbf{J}\|_2 \le \lambda_J$. The reference trajectory motion for the spacecraft angular rates represented by $\boldsymbol{\omega_d}(t)$ and $\boldsymbol{\dot{\omega}_d}(t)$ is upper bounded $\bar{v} = \sup_{t \ge 0}\{\|\boldsymbol{\omega_d}(t)\|_2^2 + \|\boldsymbol{\dot{\omega}_d}(t)\|_2\} = \bar{v}_1^2 + \bar{v}_2$. It follows that the reference trajectory terms in Eq (25) are bounded according to $\|\mathbf{g}(t)\|_2 \le \lambda_J \bar{v}$ and $\|\mathbf{H}(t)\boldsymbol{\omega_e}(t)\|_2 \le 2\lambda_J \bar{v}_1 \|\boldsymbol{\omega_e}(t)\|_2$.

The objective of the spacecraft attitude maneuver design process is to design an external control torque input $\mathbf{u}(t)$ for the nonlinear plant of Eqs (1) and (12) with reference trajectory tracking error given by Eqs (20) and (25), such that for all initial conditions, all $\mathbf{d}(t) \in \mathbf{S_d}$, and all $\mathbf{J}$ such that $\|\mathbf{J}\|_2 \le \lambda_J$, the tracking error asymptotically approaches zero. The control objective is therefore to ensure that $\lim_{t \to \infty} \boldsymbol{\omega_e}(t) = \mathbf{0}_{3 \times 1}$ and $\lim_{t \to \infty} \boldsymbol{\delta q}_{13}(t) = \mathbf{0}_{3 \times 1}$.

3.3 Control Law Design

This section considers the development of global asymptotically stable feedback control laws for large-angle three-axis spacecraft attitude maneuvers based on Lyapunov's direct method[46]. The novel aspect of this research is the development of a general feedback control law which drives the spacecraft attitude states to the desired equilibrium point in the shortest angular path. The introduction of additional terms in the control law to ensure minimum angular path maneuvers is necessary since the quaternion parameterization is based on the Euler axis/angle parameters which are not unique. Feedback controls for spacecraft equipped with external control torque actuators whose dynamics are governed by Eq (1) will be developed throughout this chapter. The reaction

wheel or control moment gyroscope case (internal control torques) can be developed using similar principles. A positive definite Lyapunov function and stabilizing feedback control law $\mathbf{u}(t) \equiv \mathbf{u}(\mathbf{x}(t), t)$ must be designed so that the resulting form of the Lyapunov function derivative is negative definite, implying globally asymptotic stability for the desired system equilibrium state $\mathbf{x}_{eq}(t)$.

For attitude tracking maneuvers the objective is to stabilize the spacecraft attitude motion relative to an arbitrarily rotating target coordinate frame. The desired equilibrium state for this time-varying target motion is specified as

$$\mathbf{x}_{eq} \equiv \mathbf{x}_d(t) = \begin{bmatrix} \mathbf{q}_d^T(t) & \boldsymbol{\omega}_d^T(t) \end{bmatrix}^T \tag{26}$$

based on the dynamic and kinematic equations of motion given by Eqs (1) and (12). An equivalent method is to transform the original state equations describing the actual spacecraft attitude motion into a set of state equations in terms of the tracking error attitude motion with state variables

$$\mathbf{x}_e \equiv \begin{bmatrix} \boldsymbol{\delta q}^T(t) & \boldsymbol{\omega}_e^T(t) \end{bmatrix}^T \tag{27}$$

governed by the tracking error dynamic and kinematic equations of Eq (20) and (24). It follows that reaching the target states defined by Eq (26) asymptotically is equivalent to reaching the origin of the tracking error state-space in Eq (27). By definition of the system equilibrium state, following feedback stabilization the spacecraft attitude motion will remain at the equilibrium state even in the absence of control torques provided that no further external or internal torques act on the spacecraft.

The traditional Lyapunov function candidate[13-15] for full-state spacecraft attitude maneuvers is given by

$$V(\mathbf{x}_e) \equiv V(\boldsymbol{\delta q}_{13}, \boldsymbol{\omega}_e) = \frac{1}{2}\boldsymbol{\omega}_e^T \mathbf{J}\boldsymbol{\omega}_e + \boldsymbol{\delta q}_{13}^T \mathbf{K} \boldsymbol{\delta q}_{13} \tag{28}$$

where $\mathbf{K}$ is a positive-definite diagonal weighting matrix, $\mathbf{J}$ is the spacecraft inertia matrix, $\boldsymbol{\omega}_e(t)$ is the tracking error angular rates, and $\boldsymbol{\delta q}_{13}(t)$ is the vector component of the tracking error quaternion. The Lyapunov function of Eq (28) represents the total energy (kinetic and potential) associated with the tracking error motion. For the case of attitude regulation the first term reduces to the kinetic energy of the spacecraft rotational motion. The Lyapunov function of Eq (28) satisfies the necessary conditions for global asymptotic stability since it is (i) positive definite over the entire state-space $\mathbf{V}(\mathbf{x}_e) > 0$ for $\mathbf{x}_e \neq \mathbf{0}_{6\times1}$ where $\mathbf{0}_{6\times1} \in \mathbf{R}^{6\times1}$ is a null column matrix, (ii) radially unbounded $\mathbf{V}(\mathbf{x}_e) \to \infty$ as $\|\mathbf{x}_e\|_2 \to \infty$, (iii) vanishes at the equilibrium state $\mathbf{V}(\mathbf{x}_e) = 0$ for $\mathbf{x}_e = \mathbf{0}_{6\times1}$, and (iv) has continuous partial derivatives with respect to all elements of $\mathbf{x}_e$. Note that although Eq (28) does not explicitly include the scalar component of the error quaternion it is implicitly included via the quaternion unity norm constraint defined in Eq (11). To develop a control law which provides the minimum angular path to the desired equilibrium state, first a control law is developed for the nominal equilibrium state, then for its negative counterpart, and a general control law capable of stabilisation with respect to either equilibrium point is synthesised. Hence in the Lyapunov stability analysis for the general control law the Lyapunov function must not explicitly contain positive definite terms in the scalar component of the error quaternion, unless this term vanishes to zero at either equilibrium state. Differentiation of Eq (28) with respect to time provides the expression

$$\dot{\mathbf{V}}(\mathbf{x}_e) = \boldsymbol{\omega}_e^T \mathbf{J}\dot{\boldsymbol{\omega}}_e + 2\boldsymbol{\delta q}_{13}^T \mathbf{K}\boldsymbol{\delta\dot{q}}_{13} \tag{29}$$

since the $\mathbf{K}$ weighting matrix is symmetric. Incorporating the tracking error dynamic equations of Eq (24) and kinematic equations of Eq (20) into Eq (29) gives

$$\dot{\mathbf{V}}(\mathbf{x}_e) = \boldsymbol{\omega}_e^T \left[-[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} + \mathbf{u}_{gg} + \mathbf{u} - \mathbf{J}\dot{\boldsymbol{\omega}}_d \right] + \delta q_4 \boldsymbol{\omega}_e^T \mathbf{K}\boldsymbol{\delta q}_{13} \tag{30}$$

Select a full-state feedback control law of the form

$$\mathbf{u}(t) = [\boldsymbol{\omega}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u}_{gg}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_d(t) - \mathbf{K}_1\boldsymbol{\omega}_e(t) - \delta q_4(t)\mathbf{K}_2\boldsymbol{\delta q}_{13}(t) \qquad (31)$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are positive definite diagonal control gain matrices. Substituting the control law of Eq (31) into Eq (30) results in the Lyapunov function derivative

$$\dot{V}(\mathbf{x}_e) = -\boldsymbol{\omega}_e^T\mathbf{K}_1\boldsymbol{\omega}_e - \delta q_4\boldsymbol{\omega}_e^T[\mathbf{K}_2 - \mathbf{K}]\boldsymbol{\delta q}_{13} \qquad (32)$$

If the control law error quaternion gain is selected as $\mathbf{K}_2 = \mathbf{K}$ the final form of the Lyapunov derivative is given by

$$\dot{V}(\mathbf{x}_e) = -\boldsymbol{\omega}_e^T\mathbf{K}_1\boldsymbol{\omega}_e \leq 0 \qquad (33)$$

which is a globally negative semi-definite function (semi-definite due to the absence of error quaternion terms) since the $\mathbf{K}_1$ is a positive definite matrix. Recall that Lyapunov's direct theorem requires that $\dot{V}(\mathbf{x}_e)$ is a globally negative definite function to prove global asymptotic stability of the equilibrium state.

Lasalle and Lefschetz's *maximum invariant subspace theorem* or *invariance principle* may be used to conclude global asymptotic stability when $V(\mathbf{x}_e)$ is positive definite and $\dot{V}(\mathbf{x}_e)$ is negative semi-definite each in a local bounded region $\mathbf{x}_e \in \Gamma$ containing the equilibrium point $\mathbf{x}_{eq}$. The set defined by $\mathbf{E} = \{\mathbf{x}_e \mid \mathbf{x}_e \in \Gamma, \dot{V}(\mathbf{x}_e) = 0\}$ for the Lyapunov derivative in Eq (33) is given by $\mathbf{E} = \{\mathbf{x}_e \mid \boldsymbol{\omega}_e = \mathbf{0}_{3\times1}\}$. The largest invariant set in $\mathbf{E}$ is the origin of the tracking error state-space $\mathbf{M} = \{\mathbf{x}_e \mid \mathbf{x}_e = \mathbf{0}_{6\times1}\}$ and all solutions of the tracking error equations of motion tend to the origin as $t \to \infty$. An alternative theory developed by Mukherjee and Junkins[19] may be used to conclude asymptotic stability of the equilibrium state. Let $\mathbf{Z}$ denote the set of points for which $\dot{V}(\mathbf{x}_e)$ vanishes. A sufficient condition for global asymptotic stability is that the first

(k-1) time derivatives of $V(\mathbf{x}_e)$ are zero on $\mathbf{Z}$ to some even order (k-1) and the kth nonzero Lyapunov function derivative evaluated on $\mathbf{Z}$ is of odd order and is negative definite (for all points on $\mathbf{Z}$). If the Lyapunov derivatives vanish on $\mathbf{Z}$ up to infinite order then the sufficient conditions for global asymptotic stability are that $\mathbf{V}(\mathbf{x}_e)$ is positive definite and $\mathbf{x}_{eq}$ is the only equilibrium point. Applying this theorem to the spacecraft attitude maneuver problem, the set $\mathbf{Z}$ of points for which the Lyapunov derivative vanishes is the set of all real values for the error quaternion $\boldsymbol{\delta q}_{13}(t)$ and zero values for the error angular rates $\boldsymbol{\omega}_e(t)$ defined by $\mathbf{Z} = \left\{ \mathbf{x}_e \mid \boldsymbol{\omega}_e = \mathbf{0}_{3\times1} \right\}$. The Lyapunov function derivatives up to third-order are given by Eq (33) and the following expressions

$$\ddot{V}(\mathbf{x}_e) = -2\boldsymbol{\omega}_e^\mathsf{T}\mathbf{K}_1\dot{\boldsymbol{\omega}}_e \tag{34}$$

$$\dddot{V}(\mathbf{x}_e) = -2\dot{\boldsymbol{\omega}}_e^\mathsf{T}\mathbf{K}_1\dot{\boldsymbol{\omega}}_e - 2\boldsymbol{\omega}_e^\mathsf{T}\mathbf{K}_1\ddot{\boldsymbol{\omega}}_e \tag{35}$$

Evaluating the Lyapunov derivatives for motion on $\mathbf{Z}$ gives zero for the first and second-order Lyapunov derivatives. Evaluating the third-order derivative on $\mathbf{Z}$ and substituting the tracking error dynamic equations given by $\dot{\boldsymbol{\omega}}_e = -\delta q_4 \mathbf{J}^{-1}\mathbf{K}_2\boldsymbol{\delta q}_{13}$ corresponding to closed-loop attitude motion on $\mathbf{Z}$ results in

$$\dddot{V}(\mathbf{x}) = -2\delta q_4^2 \boldsymbol{\delta q}_{13}^\mathsf{T}\left[\mathbf{K}_2^\mathsf{T}\mathbf{J}^{-1}\mathbf{K}_1\mathbf{J}^{-1}\mathbf{K}_2\right]\boldsymbol{\delta q}_{13} \quad \forall \mathbf{x} \in \mathbf{Z} \tag{36}$$

Since the second-order derivative vanishes on $\mathbf{Z}$ and the third-order derivative is negative definite on $\mathbf{Z}$ (and vanishes at the equilibrium point $\boldsymbol{\delta q}_{13}(t) = \mathbf{0}_{3\times1}$) provided that the matrix $\mathbf{B} = \mathbf{K}_2^\mathsf{T}\mathbf{J}^{-1}\mathbf{K}_1\mathbf{J}^{-1}\mathbf{K}_2$ is positive definite, then the sufficient conditions of the Mukherjee and Junkins theorem are satisfied. The control law of Eq (31) therefore provides global asymptotic stability of the desired equilibrium point. To show that $\mathbf{B}$ is positive definite first show that $\mathbf{B}_1 = \mathbf{J}^{-1}\mathbf{K}_1\mathbf{J}^{-1}$ is positive definite. Since $\mathbf{K}_1$ is a positive definite matrix by definition and the columns of $\mathbf{J}^{-1}$ are linearly independent then it follows that $\mathbf{B}_1$ is positive definite. A similar argument shows that $\mathbf{B}$ is positive definite.

It is important to note that since the error quaternion $\boldsymbol{\delta q}_{13}(t)$ is expressed as a quaternion of rotation then the Lyapunov function of Eq (28) will vanish when the spacecraft attitude coincides with the reference trajectory attitude $\mathbf{q_d}(t)$ or its negative counterpart $-\mathbf{q_d}(t)$ since both describe identical spacecraft attitudes. This highlights the fact that the attitude quaternion describing a specific physical orientation is not unique[45] since it is based on the Euler axis/angle parameters, and consequently either equilibrium point is valid in the Lyapunov stability analysis. The $\delta q_4(t)$ term in the control law of Eq (31) ensures the shortest angular path to the desired physical spacecraft attitude. Although this control law ensures a globally asymptotically stable system a limitation exists with respect to stabilisation (settling time) when the initial error quaternion component $\delta q_4(t_0)$ is close to zero at the commencement of the spacecraft attitude maneuver. It is therefore necessary to replace the $\delta q_4(t)$ term with either a $\text{sgn}[\delta q_4(t)]$ or $\text{sgn}[\delta q_4(t_0)]$ function due to the loss of control torque effectiveness when $\delta q_4(t_0)$ is close to zero. The signum function for an arbitrary state vector $\mathbf{x(t)} \in \mathbf{R}^{n \times 1}$ defined by

$$\text{sgn}[x_i(t)] = \begin{cases} +1 & x_i(t) > 0 \\ \text{undefined} & x_i(t) = 0 \\ -1 & x_i(t) < 0 \end{cases} \qquad i = 1,...,n \qquad (37)$$

is a singularity function since it has a discontinuous derivative evaluated at $x_i(t) = 0$. This discontinuity, however, is ignored by considering the derivative of $\text{sgn}[u_{us,i}(t)]$ to be zero for all $x_i(t)$ instead of zero for all $x_i(t) \neq 0$ and the unit impulse function at $x_i(t) = 0$. The modified version of the feedback control law of Eq (31) is given by

$$\mathbf{u}(t) = [\boldsymbol{\omega}(t) \times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u_{gg}}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_{\mathbf{d}}(t) - \mathbf{K}_1\boldsymbol{\omega}_{\mathbf{e}}(t) - \text{sgn}[\delta q_4(t)]\mathbf{K}_2\boldsymbol{\delta q}_{13}(t) \quad (38)$$

The control law of Eq (38) which includes a $\text{sgn}[\delta q_4(t)]$ term is almost identical to Eq (29) of Reference 16 which uses a $\text{sgn}[\delta q_4(t_0)]$ term instead.

MATLAB™ simulation results display near identical attitude motion for either control law. Furthermore the MATLAB™ simulations demonstrate that a positive initial condition for $\delta q_4(t_0)$ will result in the final condition given by $\lim_{t\to\infty}\delta q_4(t) = 1$, corresponding to the actual spacecraft attitude motion asymptotically tracking the reference trajectory $\mathbf{q_d}(t)$. A negative initial condition for $\delta q_4(t_0)$ will result in the attitude motion given by $\lim_{t\to\infty}\delta q_4(t) = -1$, corresponding to the $-\mathbf{q_d}(t)$ equilibrium point. Hence the actual equilibrium point to which the spacecraft attitude motion asymptotically converges is dictated by the polarity of the initial condition for $\delta q_4(t_0)$, or more generally by the polarity of $\delta q_4(t)$ for a critically damped response in the reference trajectory tracking error. The discontinuous nature of the control law of Eq (38) agrees with the results of Bhat and Bernstein[47] who demonstrate that rigid body spacecraft attitude motion cannot be globally asymptotically stabilised using a continuous control law.

To demonstrate Lyapunov stability of the closed-loop system based on the control law of Eq (38), the Lyapunov function candidate given by Eq (28) is still valid but the control law error quaternion gain in Eq (38) must be selected as $\mathbf{K}_2 = |\delta q_4(t)|\mathbf{K}$. If the $\mathbf{K}$ matrix is constant with respect to time then $\mathbf{K}_2(t)$ is a positive definite time-varying matrix whose elements approach the $\mathbf{K}$ matrix elements as the attitude motion converges to $\lim_{t\to\infty}\delta q_4(t) = \pm 1$. Alternatively if the $\mathbf{K}_2$ matrix is constant with respect to time it is necessary for $\mathbf{K}(t)$ to be a positive definite time-varying matrix which has the unique property that multiplying $\mathbf{K}(t)$ by $|\delta q_4(t)|$ results in a time-invariant matrix. The dependence of $\mathbf{K}(t)$ on time, however, needs to be accounted for when taking the derivative of the Lyapunov function given by Eq (28). When both the $\mathbf{K}$ weighting matrix in the Lyapunov function candidate and the $\mathbf{K}_2$ control gain matrix in Eq (38) are required to be constant with respect to time, then an alternative Lyapunov function candidate to Eq (28) is required to demonstrate asymptotic stability. This function is determined by considering separate Lyapunov stability analyses for each equilibrium point and then combining the results to produce a single

unified function. The Lyapunov function candidate for the $\mathbf{q_d}(t)$ equilibrium is given by

$$V(\mathbf{x_e}) \equiv V(\mathbf{q_e}, \boldsymbol{\omega_e}) = \frac{1}{2}\boldsymbol{\omega_e}^T \mathbf{J}\boldsymbol{\omega_e} + \left[\delta\mathbf{q}_{13}^T \mathbf{K}\delta\mathbf{q}_{13} + k(1 - \delta q_4)^2\right] \qquad (39)$$

where $\mathbf{K}$ is a positive definite weighting matrix given by $\mathbf{K} = k\mathbf{I}_{3\times3}$ and k is a positive scalar gain. It may be shown that the Lyapunov analysis based on Eq (39) results in the following feedback control law

$$\mathbf{u}(t) = \left[\boldsymbol{\omega}(t)\times\right]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u_{gg}}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_\mathbf{d}(t) - \mathbf{K}_1\boldsymbol{\omega_e}(t) - \mathbf{K}_2\delta\mathbf{q}_{13}(t) \qquad (40)$$

where $\mathbf{K}_1$ is a positive definite matrix and the error quaternion gain matrix is selected as $\mathbf{K}_2 = \mathbf{K} = k\mathbf{I}_{3\times3}$ to ensure asymptotic stability of the $\mathbf{q_d}(t)$ equilibrium point. The Lyapunov function candidate for the complementary equilibrium point $-\mathbf{q_d}(t)$ is formed by replacing the $(1 - \delta q_4)^2$ term in Eq (39) with $(1 + \delta q_4)^2$. The resulting Lyapunov analysis produces a control law given by

$$\mathbf{u}(t) = \left[\boldsymbol{\omega}(t)\times\right]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u_{gg}}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_\mathbf{d}(t) - \mathbf{K}_1\boldsymbol{\omega_e}(t) + \mathbf{K}_2\delta\mathbf{q}_{13}(t) \qquad (41)$$

Examining the common structural elements of Eqs (40) and (41) then the control law which ensures the minimum angular path to the equilibrium point is formed by replacing the positive/negative error quaternion feedback term with either $-\text{sgn}[\delta q_4(t_0)]k\delta\mathbf{q}_{13}(t)$ or $-\text{sgn}[\delta q_4(t)]k\delta\mathbf{q}_{13}(t)$ since $\delta q_4(t)$ will be either uniformly positive or uniformly negative for the entire duration of the spacecraft attitude maneuver depending on which equilibrium point provides the minimum angular path to the reference trajectory. This is the theoretical justification for the control law of Eq (38). For initial conditions on $\delta q_4(t_0)$ which are significantly larger than zero then $-\delta q_4(t_0)k\delta\mathbf{q}_{13}(t)$ or $-\delta q_4(t)k\delta\mathbf{q}_{13}(t)$ may be used instead as the error quaternion feedback term in the control law, leading to the result given by Eq (31). The candidate Lyapunov

function used to demonstrate asymptotic stability for the controller given by Eq (38) is formed by replacing the $(1 - \delta q_4)^2$ term in Eq (39) with a $(1 - |\delta q_4|)^2$ term. It is important to note that the presence of the $(1 - |\delta q_4|)^2$ term in the general minimum path Lyapunov function candidate implies that the function has a discontinuous derivative at $\delta q_4(t) = 0$ and therefore violates the necessary conditions required by Lyapunov's direct method. However, careful selection of the control gains in Eq (38) ensures that the state trajectory does not cross the $\delta q_4(t) = 0$ point at any time during the spacecraft maneuver and eventually converges to $\lim_{t \to \infty} \delta q_4(t) = \pm 1$, depending on the initial condition for $\delta q_4(t)$. Consequently, either the Lyapunov function candidate for the $\mathbf{q_d}(t)$ equilibrium or its complement (corresponding to the $-\mathbf{q_d}(t)$ equilibrium point) is valid throughout the attitude maneuver. Both of these Lyapunov functions have continuous partial derivatives with respect to the state variables.

The above Lyapunov stability result may be applied directly for the case of attitude slewing maneuvers (attitude regulation maneuvers) in which the desired spacecraft motion is inertial-pointing with zero angular rates (non-rotating target coordinate frame) such that the equilibrium state is time-invariant. The Lyapunov function candidate in this case is given by

$$V(\mathbf{x_e}) \equiv V(\boldsymbol{\delta q}_{13}, \boldsymbol{\omega}) = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega} + \boldsymbol{\delta q}_{13}^T \mathbf{K} \boldsymbol{\delta q}_{13} \tag{42}$$

and the control law of Eq (38) simplifies to

$$\mathbf{u}(t) = [\boldsymbol{\omega}(t) \times] \mathbf{J} \boldsymbol{\omega}(t) - \mathbf{u_{gg}}(t) - \mathbf{K}_1 \boldsymbol{\omega}(t) - \delta q_4(t) \mathbf{K}_2 \boldsymbol{\delta q}_{13}(t) \tag{43}$$

Selecting the control gain as $\mathbf{K}_2 = \mathbf{K}$ results in the Lyapunov function derivative

$$\dot{V}(\mathbf{x_e}) = -\boldsymbol{\omega}^T \mathbf{K}_1 \boldsymbol{\omega} \tag{44}$$

Note that identical results are obtained whether the nonlinear cross-coupling dynamic compensation term $[\omega(t)\times]\mathbf{J}\omega(t)$ in Eq (43) is retained or neglected, since this term produces a vector in a direction perpendicular to the spacecraft angular velocity vector. Global asymptotic stability of the equilibrium point can be shown using Eq (44) and the Mukherjee and Junkins theorem (or other suitable theorems). Modifications to the control law of Eq (43) are easily performed to circumvent the problem of severely degraded attitude maneuver settling time due to the error quaternion component of the control law not being effective until $\delta q_4(t)$ becomes sufficiently large.

The development of the nonlinear feedback control law of Eq (38) has been based on the assumption of zero dynamic model errors. Junkins and Kim[6] point out that guaranteeing stability in the presence of zero model errors is not a sufficient condition to guarantee stability of the actual plant having arbitrary model errors such as spacecraft inertia matrix uncertainty and external bounded disturbance torques. The procedure, however, assists in defining a region in gain space for which the best model of the nonlinear system is globally asymptotically stable. Determination of the actual controller gain values selected from this region in gain space is usually based on system requirements taking into account the disturbance torque environment, sensitivity of the system to model errors, desired system settling time parameters, control torque saturation, and sensor/actuator bandwidth limitations. State-of-the-art controllers designed using Lyapunov stability theory explicitly account for the effects of torque saturation, bounded external disturbance torques, sensor measurement error, and actuator failures. These issues will be discussed in further detail in Sections 3.7 & 3.8. In addition, if precise models of the external disturbance torque environment are available (including required spacecraft attitude parameter information) then the disturbance torques may be compensated directly in the feedback control law. The gravity-gradient torque compensation term in Eq (38) is based on this assumption. If this assumption is not valid then more sophisticated disturbance rejection techniques are required (see Section 3.7).

3.4 Determination of Feedback Control Law Gains

The specification of the control law proportional $\mathbf{K}_2$ and derivative $\mathbf{K}_1$ gain matrices under Lyapunov's direct method requires both matrices to be positive-definite. It is possible, however, to demonstrate that by selecting an alternate form of the Lyapunov function candidate this requirement can be relaxed. The alternate Lyapunov function is given by

$$V(\mathbf{x}_\mathbf{e}) = \frac{1}{2}\boldsymbol{\omega}_\mathbf{e}^\mathrm{T}\mathbf{K}^{-1}\mathbf{J}\boldsymbol{\omega}_\mathbf{e} + \boldsymbol{\delta q}_{13}^\mathrm{T}\boldsymbol{\delta q}_{13} \tag{45}$$

where the matrix product $\mathbf{K}^{-1}\mathbf{J}$ must be positive definite for $V(\mathbf{x}_\mathbf{e})$ to be a positive definite function. Proper selection of the controller gains will ensure that this term is positive definite. The derivative of Eq (45) incorporating the expression for the tracking error dynamics of Eq (24) is given by

$$\dot{V}(\mathbf{x}_\mathbf{e}) = \boldsymbol{\omega}_\mathbf{e}^\mathrm{T}\mathbf{K}^{-1}\left[-[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} + \mathbf{u}_\mathbf{gg} + \mathbf{u} - \mathbf{J}\dot{\boldsymbol{\omega}}_\mathbf{d}\right] + \delta q_4\boldsymbol{\omega}_\mathbf{e}^\mathrm{T}\boldsymbol{\delta q}_{13} \tag{46}$$

Select a full-state feedback control law of the form

$$\mathbf{u}(t) = [\boldsymbol{\omega}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u}_\mathbf{gg}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_\mathbf{d}(t) - \mathbf{K}_1\boldsymbol{\omega}_\mathbf{e}(t) - \delta q_4(t)\mathbf{K}_2\boldsymbol{\delta q}_{13}(t) \tag{47}$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are not necessarily positive definite weighting matrices whose characteristics are to be properly determined. The control law of Eq (47) results in the Lyapunov function derivative

$$\dot{V}(\mathbf{x}_\mathbf{e}) = -\boldsymbol{\omega}_\mathbf{e}^\mathrm{T}\mathbf{K}^{-1}\mathbf{K}_1\boldsymbol{\omega}_\mathbf{e} - \delta q_4\boldsymbol{\omega}_\mathbf{e}^\mathrm{T}\left[\mathbf{K}^{-1}\mathbf{K}_2 - \mathbf{I}_{3\times3}\right]\boldsymbol{\delta q}_{13} \tag{48}$$

Furthermore, if the control law error quaternion gain is selected as $\mathbf{K}_2 = \mathbf{K}$ the final form of the Lyapunov derivative is given by

$$\dot{V}(\mathbf{x}_\mathbf{e}) = -\boldsymbol{\omega}_\mathbf{e}^\mathrm{T}\mathbf{K}^{-1}\mathbf{K}_1\boldsymbol{\omega}_\mathbf{e} \tag{49}$$

which is a globally negative semi-definite function provided that $\mathbf{K}^{-1}\mathbf{K}_1$ is positive definite. Any suitable theorem may be applied to prove global asymptotic stability of the equilibrium point. The feedback control law of Eq (47) is identical to the originally proposed controller of Eq (38). A natural choice for the control gains to ensure that the matrix products $\mathbf{K}^{-1}\mathbf{J}$ and $\mathbf{K}^{-1}\mathbf{K}_1$ are positive definite is $\mathbf{K}_1 = d\mathbf{J}$ and $\mathbf{K}_2 = \mathbf{K} = k\mathbf{J}$ where d and k are positive scalar constants. This result is significant since the requirement for positive definite control matrices corresponding to the Lyapunov function of Eq (28) can be relaxed. The control law of Eq (47) should be replaced with Eq (38) to eliminate the settling time issue when the initial condition $\delta q_4(t_0)$ is close to zero. Modification of the Lyapunov function candidate given by Eq (45) in order to demonstrate asymptotic stability of the closed-loop system, based on the control law of Eq (38), follows in a manner identical to Section 3.3.

The scalar control gains d and k should be carefully selected to achieve the desired system closed-loop response for the spacecraft attitude maneuver. Wie et al.[16] have shown analytically that the choice of the control gains $\mathbf{K}_1 = d\mathbf{J}$ and $\mathbf{K}_2 = \mathbf{K} = k\mathbf{J}$ for the control law of Eq (47) results in a slew maneuver about the eigenaxis (Euler axis) for the case of rest-to-rest attitude maneuvers or if the initial eigenaxis is collinear with the spacecraft angular velocity vector. In the general case where initial rates are nonzero, and the spacecraft angular velocity and quaternion parameter vectors are not collinear then it may be demonstrated analytically that the Euler axis direction will vary until time t^* and the subsequent motion for $t \geq t^*$ will correspond to an eigenaxis rotation about $\mathbf{q}_{13}(t^*)$ which in general is different from $\mathbf{q}_{13}(t_0)$. A further advantage of selecting the feedback control law gains to be linear functions of spacecraft inertia matrix is that the damped response of the system is not affected when transferring the maneuver to a spacecraft with different inertia parameters. To show this, substitute the control law of Eq (38) into the tracking error dynamic equations of Eq (24) with the desired equilibrium point for a rest-to-rest attitude maneuver given by $\mathbf{x}_{eq} \equiv \mathbf{x}_d(t) = \begin{bmatrix} \pm \mathbf{q}_d^T(t) & \mathbf{0}_{3\times1}^T \end{bmatrix}^T$. This results in the closed-loop dynamic equations satisfying the linear form

$$\dot{\boldsymbol{\omega}}(t) = -d\boldsymbol{\omega}(t) - \text{sgn}[\delta q_4(t)]k\boldsymbol{\delta q}_{13}(t) \tag{50}$$

For rest-to-rest attitude maneuvers corresponding to pure eigenaxis rotations the quaternion parameters are defined as $\boldsymbol{\delta q}_{13}(t) = \sin(\Phi(t)/2)\hat{\mathbf{e}}$ and the spacecraft angular velocity is given by $\boldsymbol{\omega}(t) = \dot{\Phi}(t)\hat{\mathbf{e}}$ (where $\Phi(t)$ is the Euler angle and $\hat{\mathbf{e}}(t)$ is the Euler axis which remains constant during the attitude maneuver). Substituting these expressions into Eq (50) produces a second-order nonlinear ordinary differential equation in the Euler angle parameters

$$\ddot{\Phi} + d\dot{\Phi} + \text{sgn}[\delta q_4(t)]k\sin(\Phi/2) = 0 \tag{51}$$

Assume that the $\text{sgn}[\delta q_4(t)]$ term is unity throughout the attitude maneuver and for the purposes of selecting the control gains the $\sin(\Phi/2)$ term may be approximated by $(\Phi/2)$. This results in the second-order linear ODE given by

$$\ddot{\Phi} + d\dot{\Phi} + (k/2)\Phi = 0 \tag{52}$$

where the damping ratio ζ and the natural frequency ω_n satisfy $d = 2\zeta\omega_n$ and $k = 2\omega_n^2$. The selection of these two parameters is based on the desired system settling time and response characteristics. The settling time for the simplified closed-loop system described by Eq (52) is given by

$$t_s = 8/(\zeta\omega_n) \tag{53}$$

For the case of attitude tracking maneuvers the control law of Eq (38) results in closed-loop dynamic equations identical in form to Eq (50). It is difficult to analytically demonstrate that the attitude maneuver follows an eigenaxis rotation due to underlying assumptions established in Reference 16 which are generally not valid for attitude tracking maneuvers and the time-varying nature of the reference attitude trajectory. MATLAB[TM] simulations demonstrate, however, that the settling time expressions developed in this section may be used to calculate the feedback control law gains for the attitude tracking maneuver problem.

3.5 Modifications for Control Torque Limits

A general procedure for developing feedback control laws for attitude control systems subject to control torque saturation constraints is now considered. This requirement is motivated by the physical limitations of nonlinear mechanical systems. For example, momentum exchange devices such as reaction wheels and variable-speed control moment gyoscopes (VSCMG) operating at nominal speed have a maximum torque limit specified by their manufacturer. In this section only torque magnitude constraints are considered, and not torque rate magnitude constraints. The concepts presented in this section are not novel contributions to the literature, but were developed by Robinett et al.[29] and Wie and Lu[23].

Throughout this section it is assumed that the available external control torque for the ith spacecraft body-fixed axis is limited to $u_{min,i} \leq u_i(t) \leq u_{max,i}$. Robinett et al.[29] proposed the following feedback control law to account for torque saturation limits

$$u_i(t) = \begin{cases} u_{us,i}(t) & \left|u_{us,i}(t)\right| \leq u_{max,i} \\ u_{max,i}\, sgn\left[u_{us,i}(t)\right] & \left|u_{us,i}(t)\right| > u_{max,i} \end{cases} \qquad i = 1,2,3 \qquad (54)$$

where the unsaturated control law $\mathbf{u_{us}}(t)$ is given by Eq (38). The control law defined by Eq (54) ensures a continuous transition across the saturation boundary; it also prevents the possibility of control chattering due to the discontinuity at $u_{us,i}(t) = 0$ in the signum function that would result for example if a bang-bang control law of the form $u_i(t) = u_{max,i}\, sgn[u_{us,i}(t)]$ was implemented. An additional advantage of the controller given by Eq (54) is that it allows some elements of the control torque vector to become saturated whilst others are within the saturation limits. Junkins and Kim[6] point out that this feature differs from traditional gain scheduling and deadband methods, which typically reduce the feedback gains to keep all controls in the unsaturated range. Generalising the torque saturation function of Eq (54) to incorporate distinct upper and lower torque limits gives

$$
\text{sat}\big[u_{us,i}(t)\big] \equiv u_i(t) = \begin{cases} u_{max,i} & u_{us,i}(t) > u_{max,i} \\ u_{us,i}(t) & u_{min,i} \le u_{us,i}(t) \le u_{max,i} \\ u_{min,i} & u_{us,i}(t) < u_{min,i} \end{cases} \qquad i = 1,2,3 \qquad (55)
$$

Robinett et al.[29] demonstrated that if a bang-bang saturation control law similar to Eq (54) is used for spacecraft detumbling maneuvers then the Lyapunov function derivative is negative definite even during intervals of torque saturation. It will be shown, however, that this property does not hold for the general spacecraft attitude tracking problem. The Lyapunov stability results summarised in the Section 3.3 apply to regions of unsaturated control. For instances during the attitude maneuver in which at least one control is saturated the sufficient stability conditions may be derived using a procedure similar to Reference 29. The limitation is that for higher-dimensional systems this procedure often leads to overly conservative estimates of the stability region, or equivalently the region of control constraint violation will be typically larger than the region corresponding to positive $\dot{V}(\mathbf{x}_e)$.

The derivative of the Lyapunov function of Eq (28) with the $\mathbf{K}$ matrix replaced by $\mathbf{K}/|\delta q_4(t)|$ may be expressed as

$$
\dot{V}(\mathbf{x}_e) = \sum_{i=1}^{3} \omega_{e_i}\Big[\big(-[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} - \mathbf{J}\dot{\boldsymbol{\omega}}_d + \mathbf{u}_{gg}\big)_i + u_i\Big] + \text{sgn}[\delta q_4]\sum_{i=1}^{3} K_i \omega_{e_i}\delta q_{13_i} \qquad (56)
$$

Denoting the sets of saturated and unsaturated control indices by p and $r = 3 - p$ respectively, and substituting the control law expression of Eqs (38) and (54) into (56) leads to

$$
\dot{V}(\mathbf{x}_e) = \sum_{i=1}^{p} \omega_{e_i}\Big[\big(-[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} - \mathbf{J}\dot{\boldsymbol{\omega}}_d + \mathbf{u}_{gg}\big)_i + \text{sgn}[\delta q_4]K_i\delta q_{13_i} + u_{max_i}\,\text{sgn}(u_{us_i})\Big]
$$
$$
- \sum_{i=1}^{r} K_{1_i}\omega_{e_i}^2 \qquad (57)
$$

where K_{1_i} and $K_{2_i} = K_i$ for $i = 1,2,3$ are the diagonal elements of the positive definite matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}$ necessary to ensure global asymptotic stability of

the equilibrium state in the unconstrained control torque case. During periods of unsaturated control (all actuators are not saturated) the Lyapunov derivative is strictly negative implying that the system reference trajectory tracking error will decrease. For intervals in which at least one control torque component is saturated it is possible for the Lyapunov function derivative given by Eq (57) to be strictly positive, implying that the tracking errors will increase, or to be zero, corresponding to fixed trajectory tracking error energy. It is important to realise that during periods of control torque saturation the Lyapunov function derivative is not necessarily positive. In terms of overall global asymptotic stability results it is difficult to formulate an analytical expression encompassing the sufficient conditions to ensure a negative semi-definite or negative definite Lyapunov function derivative during periods of control saturation. This difficulty arises partly due to the complicated expression of Eqs (56) and (57). MATLAB[TM] simulations demonstrate, however, that control saturation occurs only for a fraction of the overall attitude maneuver time, such that the spacecraft rotational kinetic energy is eventually reduced, the controls subsequently operate in their linear range, and the system converges to the desired equilibrium state.

The above strategy for control saturation identifies which commanded control torque components are operating in the saturation range and limits the magnitude of the control torque. The controls operating in the linear range, however, are not modified such that overall direction of the commanded control torque is not preserved. An alternative strategy known as gain scheduling[23] identifies the maximum torque component and scales the torque magnitude, but retains the intended direction of the unsaturated controls. Based on this approach the control law is defined as

$$\mathbf{u}(t) = \text{sat}_\sigma \left[\mathbf{u}_{us}(t) \right] = \begin{cases} \mathbf{u}_{us}(t) & \sigma(\mathbf{u}_{us}(t)) \leq 1 \\ \mathbf{u}_{us}(t)/\sigma & \sigma(\mathbf{u}_{us}(t)) > 1 \end{cases} \tag{58}$$

where $\sigma(\mathbf{u}_{us}(t)) = \max\left(|u_{us,i}(t)| / u_{max\,i} \right)$ for $i = 1,...,m$ (where m is the number of external controls) is a positive scalar function of the unsaturated control given by Eq (38). Applying the control law of Eq (58) during periods of control

saturation in which at least control torque component is saturated $\sigma(\mathbf{u}_{us}(t)) > 1$ to the Lyapunov function candidate given by

$$V(\mathbf{x}_e) = \frac{1}{2}\sigma\boldsymbol{\omega}_e^T\mathbf{J}\boldsymbol{\omega}_e + \frac{1}{|\delta q_4|}\delta\mathbf{q}_{13}^T\mathbf{K}\delta\mathbf{q}_{13} \qquad (59)$$

and selecting the quaternion tracking error gain as $\mathbf{K}_2 = \mathbf{K}$ results in the Lyapunov function derivative

$$\dot{\mathbf{V}}(\mathbf{x}_e) = -\boldsymbol{\omega}_e^T\mathbf{K}_1\boldsymbol{\omega}_e - (1-\sigma)\boldsymbol{\omega}_e^T\left[-\left[\boldsymbol{\omega}\times\right]\mathbf{J}\boldsymbol{\omega} + \mathbf{u}_{gg} - \mathbf{J}\dot{\boldsymbol{\omega}}_d\right] \qquad (60)$$

The necessary conditions to ensure a negative definite Lyapunov function derivative during periods of control saturation may be established using Eq (60). A major practical limitation of the control law defined by Eq (58) is that it does not smoothly transition across the saturation boundary which introduces discontinuities into the control signal leading to abrupt changes in the spacecraft angular acceleration.

3.6 Modifications for Control Torque Limits and Angular Rate Limits

A general procedure for developing feedback control laws for attitude control systems subject to control torque limits and spacecraft angular rate constraints was developed by Wie and Lu[23]. This concept is useful for spacecraft equipped with low-rate saturation gyroscopes but will not be considered in this thesis.

3.7 Modifications for Control Torque Limits and Bounded Disturbance Torques

The objective of this section is to design feedback control laws for spacecraft attitude maneuvers subject to spacecraft inertia uncertainty, parameter control torque saturation, and bounded external disturbance torques. The issue of inertia uncertainty is avoided by designing the feedback control law independent of the inertia parameters. This avoids a direct adaptive control implementation of the control law which simplifies the Lyapunov stability analysis and avoids inertia

parameter update equations. The primary disadvantage of the control torque saturation strategy of Section 3.6 is that it is not possible to show that the Lyapunov function derivative is negative semi-definite during periods of torque saturation. An alternative approach to the problem of control torque limits is to design the structure of the feedback control laws such that the limits cannot be violated when the control law is evaluated over the entire state-space of the system or over all possible state trajectories. Thus the structure of the control law usually comprises a single function bounded by the torque limits, or a linear sum of terms which are collectively bounded by the torque limits. This approach, however, does not in general ensure that the unwanted terms resulting from the dynamic equations present in the Lyapunov function derivative are cancelled exactly. It is therefore necessary to implement a dynamic feedback control law in the sense that the control gains are not constant, but rather are driven by a nonlinear first-order ordinary differential equation. Since the control gain update law is a general nonlinear function of the state variables, the dimension of the system (and Lyapunov stability analysis) is effectively increased by one. The basic principle is similar to direct adaptive control where the spacecraft inertia parameter update law increases the state variable dimension. The cancellation terms are introduced into the Lyapunov function derivative by expanding the Lyapunov function candidate to include a positive definite term in the time-varying control gain. The flexibility of the dynamic control law methodology has a number of advantages in addition to accounting for torque limits. The first advantage is that a globally negative definite Lyapunov function derivative can be designed. The second advantage is this negative definite property can be ensured in the presence of practical system limitations such as bounded external disturbance torques, sensor measurement error, and actuator failures. This is achieved by upper bounding the terms in the Lyapunov function derivative inequality resulting from these effects and introducing an additional term in the dynamic control gain differential equation to cancel these unwanted terms. The concepts presented in this section are not novel contributions to the literature, but were developed by Boskovic et al.[38].

Currently two solutions have been proposed in the literature which address the problem of spacecraft attitude maneuvers using Lyapunov's direct method subject to external control torque saturation and bounded external disturbance

torques[38,42]. The solution of Boskovic et al.[38] considered the problem of spacecraft attitude tracking maneuvers. A full-state feedback control law was proposed by Boskovic, given by

$$u_i(t) = -u_{max} \frac{s_i(t)}{|s_i(t)| + k^2(t)\delta} \qquad i = 1,2,3 \tag{61}$$

where δ is a positive scalar constant, $k(t)$ is a time-varying control gain with properties to be defined, and u_{max} is the torque limit for all three control torque components such that $-u_{max} \leq u_i(t) \leq u_{max}$ for $i = 1,2,3$. Although state-dependant or time-varying torque limits may be implemented, either approach will complicate the Lyapunov stability analysis. Separate saturation limits $u_{max,i}$ for each control torque component or distinct lower/upper saturation limits $u_{min,i}$ and $u_{max,i}$ may also be implemented but this work assumes that a conservative torque limit $u_{max} = \min_i[u_{max,i}]$ is applied. The sliding vector $\mathbf{s}(t)$ (see Chapter 4.2) is defined by

$$\mathbf{s}(t) = \boldsymbol{\omega}_e(t) + k^2(t)\boldsymbol{\delta q}_{13}(t) \tag{62}$$

The following Lyapunov function candidate is applied

$$V(\mathbf{x}_e) = V(\boldsymbol{\omega}_e, \boldsymbol{\delta q}_{13}, k) = \frac{1}{2}\boldsymbol{\omega}_e^T \mathbf{J} \boldsymbol{\omega}_e + k^2 \left[\boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + (1 - \delta q_4)^2\right] + \frac{1}{2\gamma}k^2 \tag{63}$$

where γ is a positive scalar constant weighting factor. The following adjustment law for the time-varying control gain was proposed by Boskovic[38]

$$\dot{k}(t) = \frac{\gamma k(t)}{1 + 4\gamma[1 - \delta q_4(t)]}\left\{u_{max}\sum_{i=1}^{3}\left[\frac{\omega_{ei}(t)\delta q_i(t)}{|s_i(t)| + k^2(t)\delta} - \frac{|\omega_{ei}(t)|(1 + \delta)}{|\omega_{ei}(t)| + k^2(t)(1 + \delta)}\right]\right.$$
$$\left. - \boldsymbol{\omega}_e^T(t)\boldsymbol{\delta q}_{13}(t) - k^2\boldsymbol{\delta q}_{13}^T(t)\boldsymbol{\delta q}_{13}(t)\right\} \tag{64}$$

Using the dynamic equations of Eq (25) and the control law of Eq (61) it may be shown that the derivative of the Lyapunov function candidate of Eq (63) satisfies the inequality

$$\dot{V}(\boldsymbol{\omega}_e, \boldsymbol{\delta q}_{13}, \delta q_4, k) \le -\|\boldsymbol{\omega}_e\|_1 \left(u_{max} - \|\mathbf{g}\|_1 - \|\mathbf{d}\|_1 \right) - k^4 \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} \qquad (65)$$

where $\|\mathbf{g}\|_1$ is a term associated with the reference trajectory attitude motion and $\|\mathbf{d}\|_1$ is the point-wise L_1 norm of the external disturbance torques. The point-wise L_1 norm and L_2 norm (also called the Euclidean norm) of a general n-dimensional vector $\mathbf{x}(t)$ are defined as $\|\mathbf{x}(t)\|_1 = \sum\limits_{i=1}^{3} |x_i(t)|$ and $\|\mathbf{x}(t)\|_2 = \left[\sum\limits_{i=1}^{3} x_i^2(t) \right]^{1/2}$ respectively. It is assumed that $u_{max} \gg \|\mathbf{g}\|_1 + \|\mathbf{d}\|_1$, which essentially states that the available control torque authority is sufficient to simultaneously track the reference trajectory and reject any bounded disturbance torques. The limitation of this approach is that the rate of convergence of the state variables based on Eq (65) may only be increased by using a larger value of the control torque saturation limit u_{max}. In a strict sense the Lyapunov function derivative inequality of Eq (65) is a negative semi-definite function inequality due to the absence of $\delta q_4(t)$ terms. However since the components of the attitude quaternion must satisfy the unity-norm constraint of Eq (11) it may be argued that the asymptotic convergence of the attitude motion to $\lim_{t\to\infty} \boldsymbol{\delta q}_{13}(t) = 0$ implies that $\lim_{t\to\infty} \delta q_4(t) = 1$ based on the Lyapunov function of Eq (63). Barbalat's theorem is applied (see Reference 38) to show that the state trajectories converge to $\lim_{t\to\infty} \boldsymbol{\omega}_e(t) = 0$ and $\lim_{t\to\infty} k^2(t)\boldsymbol{\delta q}_{13}(t) = 0$. The fact that $\lim_{t\to\infty} k^2(t)\boldsymbol{\delta q}_{13}(t) = 0$ does not imply that $\lim_{t\to\infty} \boldsymbol{\delta q}_{13}(t) = 0$, for example if $\lim_{t\to\infty} k^2(t) = 0$. In addition, if $k(t)$ converges to zero before $\boldsymbol{\omega}_e(t)$, this may result in chattering in the control torque, since the control law of Eq (61) reduces to $u_i(t) = -u_{max} \text{sgn}[s_i(t)]$. However, the gain parameters in the control gain update equation of Eq (64) may be correctly adjusted to avoid the convergence of $k(t)$ to zero. Boskovic et

al.[38] provide an analysis to show that $k(t)$ remains above some positive constant for all time. A key feature of the control law and associated control gain update law given by Eqs (61) and (64) is that no spacecraft inertia parameter information is required. The bound on the reference trajectory term, however, requires at least a coarse knowledge of these parameters.

Similar to the methodology outlined in Section 3.3 it is necessary to modify the preceding results of this section to account for the non-uniqueness of the quaternion parameterisation of the spacecraft attitude. A feedback control law and associated control gain update law similar to Eqs (61) and (64) may be developed, which provide the minimum angular path to the desired equilibrium state. In Section 4.2, the sliding vector is specifically designed to provide a minimum angular path attitude maneuver by modifying its definition in Eq (62) according to

$$\mathbf{s}(t) = \boldsymbol{\omega}_e(t) + \mathrm{sgn}\big[\delta q_4(t)\big]k^2(t)\boldsymbol{\delta q}_{13}(t) \qquad (66)$$

The Lyapunov stability analysis provided by Eqs (61)-(65) may also be performed by defining a Lyapunov function candidate as the sum of a positive definite term in the sliding vector $\mathbf{s}(t)$ and a quadratic term in the dynamic control parameter $k(t)$.

3.8 Modifications for Control Torque Limits, Bounded Disturbance Torques, and Sensor Measurement Error

In Section 3.7 a nonlinear feedback control law and associated control gain update law were developed which ensures Lyapunov stability (with respect to a reference attitude trajectory) for spacecraft attitude maneuvers. This formulation accounts for the practical limitations of spacecraft inertia parameter uncertainty, control torque saturation, and bounded external torques acting on the spacecraft. This section aims to develop a control strategy to additionally account for the effects of sensor measurement error which manifests as state estimation error. The state estimates are provided using a suitable attitude filtering algorithm (for example a multiplicative extended Kalman filter). Despite attempts by

numerous researchers this problem has not been comprehensively solved in the literature. Modern spacecraft are typically equipped with an attitude sensor suite that provides observation unit-vector measurements in the direction of a reference object (for example digital sun sensors, magnetometers, and star sensors), and gyroscope measurements of the spacecraft angular rates. The measurements are processed by an attitude estimation filter to provide real-time estimates of the spacecraft attitude parameter and angular rates which are used as input to a state-feedback control law. Since the state estimation process is not perfect the state estimates will contain a certain level of estimation error which is dependent on the sensor noise levels. The state variable estimates $\hat{\mathbf{x}}(t)$ are given by

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) + \tilde{\mathbf{x}}(t) \tag{67}$$

where $\mathbf{x}(t)$ is the true value of the state vector and $\tilde{\mathbf{x}}(t)$ is the state estimation error which is upper bounded according to $\left\| \tilde{\mathbf{x}}(t) \right\|_2 \leq \left\| \tilde{\mathbf{x}}(t) \right\|_1 \leq \tilde{x}_{max}$ in a statistical sense. The general state vector in Eq (67) may be populated with a suitable attitude parameterisation (for example modified Rodrigues parameters, quaternion, or sliding vector) and the spacecraft angular rates. The results of Section 3.7 may be extended to account for sensor measurement error by replacing the true state variables with their estimates in the control law and control gain update law of Eqs (61) and (64) respectively. Preliminary investigations show, however, that the asymptotic stability result provided by Boskovic is difficult to demonstrate when state estimation error is present in the control design process.

This thesis will propose a novel algorithm to account for the effects of sensor measurement error based on previous research by Wallgrove and Akella[42]. This algorithm is one of the major novel contributions to the existing literature provided in this thesis. The first step in the methodology is to design an asymptotically stable feedback control law (and associated control gain update law) for spacecraft attitude maneuvers when bounded disturbances and sensor measurement noise are not present in the system. Both attitude regulation and

attitude tracking maneuvers will be considered. The structure of the proposed control law ensures that both spacecraft inertia parameter uncertainty and torque saturation will be taken into account. The next step is to account for the effect of bounded disturbances by modifying the control gain update law. The final step is to account for state estimation error by further modifying the control gain update law. In the proposed formulation the bounded disturbances and sensor measurement error effectively introduce upper bounded positive definite terms in the Lyapunov function candidate derivative, whose effect must be cancelled using the control gain update law.

3.8.1 Attitude Regulation Maneuvers

Consider the design of a feedback control law and control gain update equation for spacecraft attitude regulation maneuvers (time-invariant equilibrium state) based on the assumption that control torque limits are enforced but external disturbance torques and sensor measurement error are not present. Spacecraft inertia parameter uncertainty is accounted for by designing the feedback control law to be independent of the inertia parameters. Consider the Lyapunov function candidate

$$V(\mathbf{x_e}) \equiv V(\boldsymbol{\omega}, \boldsymbol{\delta q}_{13}, \delta q_4) = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{J}\boldsymbol{\omega} + \beta u_{max}\left[\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13} + \left(1 - \delta q_4\right)^2\right] \qquad (68)$$

which applies to the equilibrium point $\mathbf{q_d}(t)$ corresponding to the quaternion tracking error term $\delta q_4(t) = 1$. The parameters $\boldsymbol{\omega}(t)$ and $\boldsymbol{\delta q}(t)$ are the spacecraft angular rates and attitude quaternion associated with the reference trajectory tracking error defined in Section 3.2, β is a control gain parameter whose properties are to be defined, and u_{max} is the control torque upper limit such that $|u_i(t)| \leq u_{max}$ for $i = 1,2,3$. Differentiation of Eq (68) with respect to time and substitution of the state equations given by Eqs (1) and (20) yields

$$\dot{V}(\mathbf{x_e}) = \boldsymbol{\omega}^T\left[\beta u_{max}\boldsymbol{\delta q}_{13} + \mathbf{u}\right] \qquad (69)$$

Select a full-state feedback control law given by

$$\mathbf{u}(t) = -u_{max}\left[\beta\delta\mathbf{q}_{13}(t) + (1-\beta)\tanh\left(\frac{\omega(t)}{p^2}\right)\right] \tag{70}$$

where β is a control gain parameter with limits $0 < \beta < 1$ due to the control torque constraint, and p^2 is a control gain parameter (called the sharpness function) that is lower bounded above zero $0 < p_{min}^2 \le p^2 \in L_\infty$ and has a bounded derivative $\dot{p}^2 \in L_\infty$. The L_∞ norm of the general n-dimensional vector function $\mathbf{x}(t)$ is defined as $\|\mathbf{x}(t)\|_\infty = \max_{1 \le i \le n}\left\{\sup_{t \ge 0}|x_i(t)|\right\}$. The notation $\mathbf{x}(t) \in L_\infty$ means that the L_∞ norm for $\mathbf{x}(t)$ is bounded such that $\|\mathbf{x}(t)\|_\infty < \infty$. Observe the similarity in structure between the control law of Eq (70) and the attitude regulation maneuver control law of Eq (43) with the relevant terms eliminated. Substituting the control law of Eq (70) in Eq (69) gives

$$\dot{V}(\mathbf{x}_e) = -(1-\beta)u_{max}\omega^T\tanh\left(\frac{\omega}{p^2}\right) \tag{71}$$

which is a negative semi-definite function of the state variables. A disadvantage of this formulation is the lack of freedom in controlling the convergence of the state variables to the equilibrium point since the torque saturation limits are generally fixed by the spacecraft mission requirements. To conclude global asymptotic stability of the equilibrium point a suitable theorem (for example LaSalle's invariance principle) must be applied. In the absence of a suitable theorem the control gain parameter $p^2(t)$ may be allowed to vary with time according to the asymptotically stable first-order nonlinear ordinary differential equation

$$\frac{d}{dt}p(t) = -\gamma_p p(t)\delta\mathbf{q}_{13}^T(t)\delta\mathbf{q}_{13}(t) \tag{72}$$

where γ_p is a positive scalar constant. Since the update law of (72) is a function of the state variables the dimension of the state-space for the Lyapunov stability

analysis is effectively increased by one. As a result the following Lyapunov function candidate is proposed

$$V(\boldsymbol{\omega}, \boldsymbol{\delta q}_{13}, \delta q_4, p) = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{J}\boldsymbol{\omega} + \beta u_{max}\left[\boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + (1 - \delta q_4)^2\right] + \frac{1}{2\gamma_p}p^2 \quad (73)$$

Differentiation of Eq (73) and substituting into this expression the state equations given by Eqs (1) and (20), the feedback control law of Eq (70), and the control parameter update law of Eq (72) gives

$$\dot{V}(\mathbf{x}_e) = -(1 - \beta)u_{max}\boldsymbol{\omega}^T \tanh\left(\frac{\boldsymbol{\omega}}{p^2}\right) - p^2 \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} \quad (74)$$

which is a globally negative definite function of the state variables and the sufficient conditions for a globally asymptotically stable equilibrium point are satisfied.

This basic result demonstrates that a time-varying control parameter update law can introduce negative definite terms in the Lyapunov function derivative. This idea will become important when the effects of bounded external torques and sensor measurement error are taken into account in the Lyapunov stability analysis.

3.8.2 Attitude Tracking Maneuvers

Consider the extension of the above stability analysis for the case of spacecraft attitude tracking maneuvers. The tracking error state trajectory for the spacecraft angular rates is governed by Eq (25) and for the present analysis the external disturbance torque term is assumed to be zero. Consider the Lyapunov function candidate

$$V(\mathbf{x}_e) = \frac{1}{2}\boldsymbol{\omega}_e^T \mathbf{J}\boldsymbol{\omega}_e + \beta u_{max}\left[\boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + (1 - \delta q_4)^2\right] + \frac{1}{2\gamma_p}p^2 \quad (75)$$

where $\boldsymbol{\omega}_e(t)$ denotes the spacecraft angular rate trajectory tracking error. The derivative of Eq (75) using (i) the state equations of Eqs (20) and (25), (ii) the control law of Eq (70) with the spacecraft angular rate tracking error $\boldsymbol{\omega}_e(t)$ replacing the actual spacecraft angular rates $\boldsymbol{\omega}(t)$, (iii) and the control gain parameter update law of Eq (72), is given by

$$\dot{V}(\mathbf{x}_e) = -\boldsymbol{\omega}_e^T \mathbf{H} \boldsymbol{\omega}_e - \boldsymbol{\omega}_e^T \mathbf{g} - (1-\beta) u_{max} \boldsymbol{\omega}_e^T \tanh\left(\frac{\boldsymbol{\omega}_e}{p^2}\right) - p^2 \boldsymbol{\delta} \mathbf{q}_{13}^T \boldsymbol{\delta} \mathbf{q}_{13} \qquad (76)$$

which is an indefinite function due to the presence of the $\boldsymbol{\omega}_e^T \mathbf{g}$ term. This term may be converted to a positive definite form using the inequalities $\boldsymbol{\omega}_e^T \mathbf{g} \leq \|\boldsymbol{\omega}_e\|_2 \|\mathbf{g}\|_2$ and $\|\boldsymbol{\omega}_e\|_2 \|\mathbf{g}\|_2 \leq \|\boldsymbol{\omega}_e\|_1 \|\mathbf{g}\|_1 \leq \|\boldsymbol{\omega}_e\|_1 g_{max}$, where g_{max} is an upper bound on the $\mathbf{g}(t)$ term associated with the reference trajectory. The Lyapunov function derivative inequality that results from these inequality relationships is

$$\dot{V}(\mathbf{x}_e) \leq -\boldsymbol{\omega}_e^T \mathbf{H} \boldsymbol{\omega}_e + \|\boldsymbol{\omega}_e\|_1 g_{max} - (1-\beta) u_{max} \boldsymbol{\omega}_e^T \tanh\left(\frac{\boldsymbol{\omega}_e}{p^2}\right) - p^2 \boldsymbol{\delta} \mathbf{q}_{13}^T \boldsymbol{\delta} \mathbf{q}_{13} \qquad (77)$$

A negative definite inequality expression may be achieved by adding and subtracting the term $(1-\beta) u_{max} \|\boldsymbol{\omega}_e\|_1$ in Eq (77), which results in

$$\dot{V}(\mathbf{x}_e) \leq -\boldsymbol{\omega}_e^T \mathbf{H} \boldsymbol{\omega}_e + (1-\beta) u_{max} \sum_{i=1}^{3} |\omega_{ei}| \left[1 - \tanh\left(\frac{|\omega_{ei}|}{p^2}\right)\right] - p^2 \boldsymbol{\delta} \mathbf{q}_{13}^T \boldsymbol{\delta} \mathbf{q}_{13}$$
$$- \|\boldsymbol{\omega}_e\|_1 \left[(1-\beta) u_{max} - g_{max}\right] \qquad (78)$$

Wallsgrove and Akella[42] have shown that the hyperbolic tangent term in Eq (78) is upper bounded, independent of the spacecraft angular rate tracking error, according to

$$0 \leq (1-\beta) u_{max} \sum_{i=1}^{3} |\omega_{ei}| \left[1 - \tanh\left(\frac{|\omega_{ei}|}{p^2}\right)\right] \leq 3\alpha u_{max} (1-\beta) p^2 \qquad (79)$$

where α is a positive constant with properties outlined in Reference 42. Using the upper bound given by Eq (79) in Eq (78) results in the negative definite Lyapunov function derivative inequality

$$\dot{V}(\mathbf{x_e}) \leq -\boldsymbol{\omega}_e^T \mathbf{H}\boldsymbol{\omega}_e - \|\boldsymbol{\omega}_e\|_1 \left[(1-\beta)u_{max} - g_{max}\right] \\ - p^2 \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + 3\alpha(1-\beta)u_{max}p^2 \tag{80}$$

Defining $a^2 = (1-\beta)u_{max} - g_{max} > 0$ and $c^2 = 3\alpha(1-\beta)u_{max} > 0$, and eliminating the first quadratic term in the spacecraft angular rate tracking error using the inequality then Eq (80) is expressed as

$$\dot{V}(\mathbf{x_e}) \leq -a^2 \|\boldsymbol{\omega}_e\|_1 - p^2 \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + c^2 p^2 \tag{81}$$

The control law parameter β and the time-invariant control torque limit u_{max} must be selected to ensure that the a^2 and c^2 parameter inequalities are satisfied and the relevant state-dependent terms in Eq (81) are negative definite based on the reference attitude trajectory. In addition to the constraint $0 < \beta < 1$ due to the control torque limits and the $c^2 = 3\alpha(1-\beta)u_{max} > 0$ term, the constant control gain parameter must satisfy $\beta < 1 - (g_{max}/u_{max})$ to ensure that $a^2 = (1-\beta)u_{max} - g_{max} > 0$. This in turn ensures that angular rate tracking error term in Eq (81) is negative definite. Combining these two constraints results in an overall parameter constraint $0 < \beta < 1 - (g_{max}/u_{max}) \leq 1$, which is valid only if $g_{max} < u_{max}$, and reduces to $0 < \beta < 1$ in the absence of a time-varying reference trajectory (attitude regulation maneuvers). The state dependant positive definite term in Eq (78) has been transformed into a positive definite term which is independent of the state variables. This suggests the possibility of compensating for this term in the $p(t)$ control gain parameter update law. If this positive definite term is not compensated for in the Lyapunov stability analysis then the resulting state trajectory will not converge to the desired equilibrium state and a residual error set may be established to which the state variables $\boldsymbol{\delta q}(t)$ and $\boldsymbol{\omega}_e(t)$ and the sharpness parameter $p^2(t)$ converge. Following the approach of Wallsgrove and Akella[42] the upper bound on the Lyapunov function

107

derivative inequality specified by Eq (81) implies that $\dot{V}(\mathbf{x}_e) < 0$ for points along the state trajectory satisfying $\|\boldsymbol{\omega}_e\|_1 > a^{-2}c^2p^2$ and all points satisfying $\|\boldsymbol{\delta q}_{13}\|_2 > c$. Alternatively stated, the Lyapunov function derivative is negative outside a compact region surrounding the equilibrium point $\mathbf{q}_d(t)$ defined by the set $\mathbf{D} = \left\{ (\boldsymbol{\omega}_e, \boldsymbol{\delta q}_{13}) \mid \|\boldsymbol{\omega}_e\|_1 \le a^{-2}c^2p^2, \|\boldsymbol{\delta q}_{13}\| \le c \right\}$. When the spacecraft angular rates and/or attitude quaternion are sufficiently large in magnitude then the negative definite terms in Eq (81) dominate and the state trajectory tends towards the equilibrium point. When these variables become sufficiently small then the Lyapunov function derivative becomes positive definite and further progress towards the equilibrium point cannot be guaranteed. This concept defines the boundary of a residual set to which all trajectories eventually converge, implying that (i) the equilibrium point is globally stable in the sense of Lyapunov, and (ii) the state variables are bounded $\boldsymbol{\omega}_e(t)$, $\boldsymbol{\delta q}_{13}(t)$, $p\boldsymbol{\delta q}_{13}(t) \in L_\infty$. Furthermore, the boundedness of the tracking error quaternion $\boldsymbol{\delta q}_{13}(t)$ directly follows from the unity norm constraint of Eq (11), independent of the stability of the state trajectory. The boundedness of the sharpness parameter $p^2(t)$ can be shown using Eq (72) although in MATLAB$^{\text{TM}}$ simulations the update law parameters are selected such that $p^2(t)$ converges to a constant value before the state trajectory converges. Although the global boundedness of all state variables (including the sharpness function) are ensured based on Eqs (72) and (81), the desirable property of global asymptotic stability of the spacecraft angular rates and attitude quaternion must be demonstrated. Note that reducing the value of the sharpness function $p^2(t)$ reduces the size of the residual error set to which the state trajectory converges. This may lead, however, to increased sensitivity of the angular rate component of the nonlinear feedback control law in Eq (70). Asymptotic stability of the equilibrium point may be achieved by modifying the update law of Eq (72) to compensate for the positive definite term in Eq (81) according to

$$\frac{d}{dt}p(t) = -\gamma_p p(t) \left[\boldsymbol{\delta q}_{13}^{\text{T}}(t)\boldsymbol{\delta q}_{13}(t) + c^2 \right] \tag{82}$$

such that the Lyapunov function derivative inequality of Eq (81) reduces to

$$\dot{V}(\mathbf{x}_e) \leq -a^2 \left\| \boldsymbol{\omega}_e \right\|_1 - p^2 \boldsymbol{\delta} \mathbf{q}_{13}^T \boldsymbol{\delta} \mathbf{q}_{13} < 0 \tag{83}$$

Applying the fundamental principles of Lyapunov's direct method where $\dot{V}(\mathbf{x}_e) < 0$ shows that all state variables are upper bounded by the initial conditions of the state trajectory, such that $\boldsymbol{\omega}_e(t)$, $\boldsymbol{\delta} \mathbf{q}(t)$, $\delta q_4(t)$, $p(t) \in L_\infty$. Furthermore, the Lyapunov function is finite for all time $0 \leq V(\mathbf{x}_e(t_\infty)) \leq V(\mathbf{x}_e(t)) \leq V(\mathbf{x}_e(t_0)) < \infty$ and integrating Eq (83) for $t_0 \leq t \leq t_\infty$ yields

$$\infty > -\int_{t_0}^{t_\infty} \dot{V}(\mathbf{x}_e(\tau)) \, d\tau \geq a^2 \int_{t_0}^{t_\infty} \left\| \boldsymbol{\omega}_e(\tau) \right\|_1 d\tau + \int_{t_0}^{t_\infty} \left[p(\tau) \boldsymbol{\delta} \mathbf{q}_{13}(\tau) \right]^T \left[p(\tau) \boldsymbol{\delta} \mathbf{q}_{13}(\tau) \right] d\tau \tag{84}$$

which shows that $\boldsymbol{\omega}_e(t) \in L_1 \cap L_\infty$ and $p(t)\boldsymbol{\delta} \mathbf{q}_{13}(t) \in L_2 \cap L_\infty$. It follows directly from Eq (82) that the sharpness function update law derivative $\dot{p}(t) \in L_\infty$. Similarly $\dot{\boldsymbol{\omega}}_e(t) \in L_\infty$ since $\boldsymbol{\omega}_e(t)$, $\mathbf{u}(t)$, $\boldsymbol{\omega}_d(t)$, $\dot{\boldsymbol{\omega}}_d(t) \in L_\infty$, and $\frac{d}{dt}\left[p(t)\boldsymbol{\delta} \mathbf{q}_{13}(t) \right] \in L_\infty$ since $p(t)$, $\dot{p}(t)$, $\boldsymbol{\delta} \dot{\mathbf{q}}_{13}(t) \in L_\infty$. Applying Barbalat's lemma to this problem[46] results in

$$\lim_{t \to \infty} \boldsymbol{\omega}_e(t) = \mathbf{0}_{3 \times 1} \tag{85}$$
$$\lim_{t \to \infty} p(t)\boldsymbol{\delta} \mathbf{q}_{13}(t) = \mathbf{0}_{3 \times 1} \tag{86}$$

The conclusion that $\lim_{t \to \infty} p(t)\boldsymbol{\delta} \mathbf{q}_{13}(t) = \mathbf{0}_{3 \times 1}$ does not guarantee that the attitude maneuver objective $\lim_{t \to \infty} \boldsymbol{\delta} \mathbf{q}_{13}(t) = \mathbf{0}_{3 \times 1}$ will be achieved. The main issue is that the sharpness function $p^2(t)$ may converge to zero before the tracking error quaternion $\boldsymbol{\delta} \mathbf{q}_{13}(t)$ converges to zero. An analysis could be performed similar to Boskovic[38] to show that under certain conditions $p^2(t) \geq \bar{p}^2 > 0$ for $t \geq 0$ where $\bar{p}^2$ is a scalar constant providing a lower bound on the sharpness function. It may be difficult, however, to establish a set of conditions for which the lower

bound exists. The convergence rate of $p^2(t)$ can be adjusted using the parameters in Eq (82) such that it converges to a positive constant or converges to zero more slowly than $\delta\mathbf{q}_{13}(t)$ converges to zero.

3.8.3 Bounded Disturbance Torques

Consider the extension of the above stability analysis for the case of bounded external disturbance torques. In this case the functional form of the disturbance torque term in the dynamic equations of Eq (25) is not known but the upper bound given by $\|\mathbf{d}(t)\|_2 \leq \|\mathbf{d}(t)\|_1 \leq d_{max}$ on the magnitude of the disturbances is known. The Lyapunov stability analysis proceeds as in Section 3.8.2, based on the (i) Lyapunov function of Eq (75), (ii) the feedback control law of Eq (70), and (iii) the parameter update law of Eq (82). The indefinite term introduced by the bounded disturbance torque in the Lyapunov function derivative is converted to a strictly positive definite term according to the inequality $\boldsymbol{\omega}_e^T\mathbf{d} \leq \|\boldsymbol{\omega}_e\|_2\|\mathbf{d}\|_2 \leq \|\boldsymbol{\omega}_e\|_1\|\mathbf{d}\|_1 \leq \|\boldsymbol{\omega}_e\|_1 d_{max}$. The Lyapunov stability analysis is identical to that in Section 3.8.2 except that the constants in Eqs (82) and (83) are replaced with $a^2 = (1-\beta)u_{max} - g_{max} - d_{max} > 0$ and $c^2 = 3\alpha(1-\beta)u_{max}$. The assumption on the a^2 parameter loosely states that the control torque authority limited by $\|\mathbf{u}(t)\|_2 \leq u_{max}$ is sufficient to simultaneously track the desired motion $\mathbf{x_d}(t)$ and reject any disturbance torques with upper bound $\|\mathbf{d}(t)\|_2 \leq d_{max}$. Global asymptotic stability of the equilibrium state is demonstrated using Barbalat's lemma[42]. The control law parameter β and the control torque limit u_{max} must be selected to ensure that the a^2 and c^2 parameter inequalities are satisfied, and to ensure that Eq (83) is a negative definite inequality based on the expected disturbance torque environment and reference trajectory motion. The overall parameter constraint is given by $0 < \beta < 1 - [(g_{max} + d_{max})/u_{max}] \leq 1$, which is valid only if $g_{max} + d_{max} < u_{max}$, and reduces to $0 < \beta < 1$ in the absence of a time-varying reference trajectory and disturbance torques.

3.8.4 State Estimation Error

This remainder of this section is dedicated to extending the Lyapunov stability results of the previous sections to account for the effect of state estimation error. It is assumed that the true state variables are related to the state estimates according to Eq (67). In practice the measurement noise characteristics (in a statistical sense) of the attitude sensors are known prior to spacecraft launch. The state estimation error is assumed upper bounded according to $\left\|\widetilde{\mathbf{x}}(t)\right\|_2 \leq \left\|\widetilde{\mathbf{x}}(t)\right\|_1 \leq \widetilde{x}_{max}$ based on a suitable full-state estimator (for example a multiplicative extended Kalman filter). The estimation error upper bounds for the spacecraft attitude quaternion and angular rate trajectory tracking errors are given by $\left\|\boldsymbol{\delta}\widetilde{\mathbf{q}}_{13}(t)\right\|_1 \leq \widetilde{q}_{max}$ and $\left\|\widetilde{\boldsymbol{\omega}}_e(t)\right\|_1 \leq \widetilde{\omega}_{max}$ respectively based on Eq (18) and Eq (21). The methodology used to account for sensor measurement error is to use the Lyapunov stability results of the Sections 3.8.1-3.8.3 as a baseline design and investigate the effect of the additional terms introduced due to sensor measurement error. The additional terms are present since the true state variables appearing in the feedback control law of Eq (70) and the sharpness function update law of Eq (82) must be replaced by their corresponding state estimates. The feedback control law of Eq (70) with the state variables replaced by their estimates is given by

$$\mathbf{u}(t) = -u_{max}\left[\beta\boldsymbol{\delta}\hat{\mathbf{q}}_{13}(t) + (1-\beta)\tanh\left(\frac{\hat{\boldsymbol{\omega}}_e(t)}{p^2(t)}\right)\right] \qquad (87)$$

where $\boldsymbol{\delta}\hat{\mathbf{q}}_{13}(t)$ and $\hat{\boldsymbol{\omega}}_e(t)$ are the estimates of the spacecraft attitude quaternion and angular rates respectively, and β is a control parameter whose limits are to be defined. Expanding the control law of Eq (87) using Eq (67) gives

$$\mathbf{u} = -u_{max}\left[\beta(\boldsymbol{\delta}\mathbf{q}_{13} + \boldsymbol{\delta}\widetilde{\mathbf{q}}_{13}) + (1-\beta)\tanh\left(\frac{\boldsymbol{\omega}_e + \widetilde{\boldsymbol{\omega}}_e}{p^2}\right)\right] \qquad (88)$$

It is important to realise that the error in the quaternion appearing in Eq (88) is assumed to be additive. It is only valid to within first-order for small estimation

errors. The disadvantage with this approach is that an error quaternion may exist that violates the normality constraint of Eq (11). A multiplicative quaternion approach would alleviate this problem, but will result in a more complicated analysis. The hyperbolic tangent term in Eq (88) is simplified using the identity

$$\tanh(a + b) = \frac{1}{1 - \tanh(a)\tanh(b)}\left[\tanh(a) + \tanh(b)\right] \tag{89}$$

and the inequalities $\left\|\tanh\left(\boldsymbol{\omega}_e / p^2\right)\right\|_1 \leq 1$ and $\left\|\tanh\left(\widetilde{\boldsymbol{\omega}}_e / p^2\right)\right\|_1 < \widetilde{\chi}_{max} \leq 1$. Hence the denominator of the hyperbolic tangent identity is given by

$$\frac{-1}{1 - \tanh\left(\boldsymbol{\omega}_e / p^2\right)\tanh\left(\widetilde{\boldsymbol{\omega}}_e / p^2\right)} \leq \frac{-1}{1 + \left|\tanh\left(\boldsymbol{\omega}_e / p^2\right)\right|\left|\tanh\left(\widetilde{\boldsymbol{\omega}}_e / p^2\right)\right|} \leq -\frac{1}{2} \tag{90}$$

and the control law of Eq (88) reduces to

$$\mathbf{u} \leq -u_{max}\left[\beta\boldsymbol{\delta q}_{13} + \frac{1}{2}(1 - \beta)\tanh\left(\frac{\boldsymbol{\omega}_e}{p^2}\right)\right] - \frac{1}{2}(1 - \beta)u_{max}\tanh\left(\frac{\widetilde{\boldsymbol{\omega}}_e}{p^2}\right)$$
$$- \beta u_{max}\boldsymbol{\delta\widetilde{q}}_{13} \tag{91}$$

Similarly the update equation for the sharpness parameter given by Eq (72) is replaced with

$$\frac{d}{dt}p(t) = -\gamma_p p(t)\boldsymbol{\delta\hat{q}}_{13}^T(t)\boldsymbol{\delta\hat{q}}_{13}(t) \tag{92}$$

Additional state independent terms will be added to Eq (92) later in the stability analysis to compensate for the positive definite terms generated from the bounded disturbance torques and sensor measurement error. Using the estimation error expressions of Eq (67) in Eq (92) gives

$$\frac{d}{dt}p(t) = -\gamma_p p\left\{\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13} + 2\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta\widetilde{q}}_{13} + \boldsymbol{\delta\widetilde{q}}_{13}^T\boldsymbol{\delta\widetilde{q}}_{13}\right\} \tag{93}$$

It convenient to eliminate and modify terms in Eq (93) to simplify the stability analysis

$$\frac{d}{dt}p(t) \le -\gamma_p p\left\{\boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + 2\boldsymbol{\delta q}_{13}^T \boldsymbol{\delta \widetilde{q}}_{13}\right\} \le -\gamma_p p \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + 2\gamma_p p\left\|\boldsymbol{\delta \widetilde{q}}_{13}\right\|_1 \quad (94)$$

Hence, compared to the nominal design given by Eqs (75) and (76), there will be three additional terms appearing in the Lyapunov function derivative. Evaluating this derivative yields

$$\begin{aligned}
\dot{V}(\mathbf{x_e}) \le\ & -\boldsymbol{\omega}_e^T \mathbf{g} + \boldsymbol{\omega}_e^T \mathbf{d} - \frac{1}{2}(1-\beta)u_{max}\boldsymbol{\omega}_e^T \tanh\left(\frac{\boldsymbol{\omega}_e}{p^2}\right) - p^2 \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} \\
& -\frac{1}{2}(1-\beta)u_{max}\boldsymbol{\omega}_e^T \tanh\left(\frac{\boldsymbol{\widetilde{\omega}}_e}{p^2}\right) - \beta u_{max}\boldsymbol{\omega}_e^T \boldsymbol{\delta \widetilde{q}}_{13} + 2p^2\left\|\boldsymbol{\delta \widetilde{q}}_{13}\right\|_1 \quad (95)
\end{aligned}$$

To reduce Eq (95) further the indefinite terms are converted to positive definite, the term $(1-\beta)u_{max}\left\|\boldsymbol{\omega}_e\right\|_1$ is added and subtracted to Eq (95), and upper bounds on the reference trajectory, external disturbances and state estimation error are introduced. The hyperbolic tangent term in Eq (95) containing the spacecraft angular rate estimation error is upper bounded according to $\left\|\tanh\left(\boldsymbol{\widetilde{\omega}}_e / p^2\right)\right\|_1 < \widetilde{\chi}_{max} \le 1$ such that

$$\begin{aligned}
\dot{V}(\mathbf{x_e}) \le\ & \left\|\boldsymbol{\omega}_e\right\|_1 g_{max} + \left\|\boldsymbol{\omega}_e\right\|_1 d_{max} - \frac{1}{2}(1-\beta)u_{max}\sum_{i=1}^{3}\left|\omega_{e,i}\right|\tanh\left(\frac{\left|\omega_{e,i}\right|}{p^2}\right) - p^2 \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} \\
& + \frac{1}{2}(1-\beta)u_{max}\widetilde{\chi}_{max}\left\|\boldsymbol{\omega}_e\right\|_1 + \beta u_{max}\left\|\boldsymbol{\omega}_e\right\|_1 \widetilde{q}_{max} + 2p^2\widetilde{q}_{max} \\
& + \frac{1}{2}(1-\beta)u_{max}\left[\sum_{i=1}^{3}\left|\omega_{e,i}\right| - \left\|\boldsymbol{\omega}_e\right\|_1\right] \quad (96)
\end{aligned}$$

Combining the relevant terms in Eq (96) gives

$$\dot{V}(\mathbf{x_e}) \leq -\|\boldsymbol{\omega_e}\|_1 \left\{ \frac{1}{2}(1-\beta)u_{max}[1+2\tilde{q}_{max}-\tilde{\chi}_{max}]-g_{max}-d_{max}-u_{max}\tilde{q}_{max} \right\}$$
$$-p^2\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13}+\frac{1}{2}(1-\beta)u_{max}\sum_{i=1}^{3}|\omega_{e,i}|\left[1-\tanh\left(\frac{|\omega_{e,i}|}{p^2}\right)\right]+2p^2\tilde{q}_{max} \quad (97)$$

Using the upper bound on the hyperbolic tangent term given by Eq (79) to reduce Eq (97) further yields

$$\dot{V}(\mathbf{x_e}) \leq -\|\boldsymbol{\omega_e}\|_1 \left\{ \frac{1}{2}(1-\beta)u_{max}[1+2\tilde{q}_{max}-\tilde{\chi}_{max}]-g_{max}-d_{max}-u_{max}\tilde{q}_{max} \right\}$$
$$-p^2\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13}+p^2\left[\frac{3}{2}\alpha(1-\beta)u_{max}+2\tilde{q}_{max}\right] \quad (98)$$

Simplify the above expression by defining

$$a^2 = \left\{ \frac{1}{2}(1-\beta)u_{max}[1+2\tilde{q}_{max}-\tilde{\chi}_{max}]-g_{max}-d_{max}-u_{max}\tilde{q}_{max} \right\} > 0 \quad (99)$$

$$c^2 = \frac{3}{2}\alpha(1-\beta)u_{max}+2\tilde{q}_{max} > 0 \quad (100)$$

such that Eq (98) becomes

$$\dot{V}(\mathbf{x_e}) \leq -a^2\|\boldsymbol{\omega_e}\|_1 - p^2\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13} + c^2p^2 \quad (101)$$

The positive definite term in Eq (101) establishes a residual set to which the state trajectory will eventually converge. This term may be cancelled by modifying the sharpness parameter update equation of Eq (92)

$$\frac{d}{dt}p(t) = -\gamma_p p(t)\left[\boldsymbol{\delta\hat{q}}_{13}^T(t)\boldsymbol{\delta\hat{q}}_{13}(t)+c^2\right] \quad (102)$$

such that the Lyapunov function derivative inequality is given by

$$\dot{V}(\mathbf{x_e}) \leq -a^2\|\boldsymbol{\omega_e}\|_1 - p^2\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13} \quad (103)$$

Barbalat's lemma is applied to conclude global asymptotic stability of the equilibrium state[42]. The limits on the control law parameter β must be established to ensure that Eq (103) is a negative definite inequality based on the expected disturbance torque environment and state estimation error limits. Proceeding in the same manner as previous sections, the constraint $0 < \beta < 1$ is due to the control torque limit and the $\beta < 1 + \left[(4\tilde{q}_{max})/(3\alpha u_{max}) \right]$ constraint is due to the c^2 parameter inequality. However, since the parameters α, $\tilde{q}_{max}$, and u_{max} are strictly positive, the later constraint is not consistent with $0 < \beta < 1$ and can be ignored. Combining this result with the a^2 parameter inequality results in the constraint $0 < \beta < 1 - \lambda \le 1$ for the control gain parameter where $\lambda = (2/u_{max})(g_{max} + d_{max} + u_{max}\tilde{q}_{max})/(1 + 2\tilde{q}_{max} - \tilde{\chi}_{max})$. This constraint requires $(2/u_{max})(g_{max} + d_{max} + u_{max}\tilde{q}_{max}) < (1 + 2\tilde{q}_{max} - \tilde{\chi}_{max})$, and reduces to $0 < \beta < 1$ in the absence of a time-varying reference trajectory, disturbance torques, and state estimation error.

There are three parameters α, β, γ_p in addition to the update equation initial condition $p(t_0)$ that must be tuned to ensure suitable closed-loop response of the spacecraft attitude maneuver. In contrast the algorithm of Wallsgrove and Akella[42] requires tuning of five system parameters and two parameter gain update equations.

3.8.5 Minimum Angular Path Maneuvers

This section investigates the modification of the Lyapunov stability analysis presented in Sections 3.8.1-3.8.4 to account for the non-uniqueness property of the quaternion attitude coordinates. The objective is to develop a feedback control law similar to Eq (87) capable of automatically selecting the minimum angular path to the desired equilibrium point. In addition, a sharpness parameter update law similar to Eq (102) will need to be developed.

Consider the modification of the feedback control law given by Eq (70) and the sharpness parameter update law of Eq (82), in the absence of sensor

measurement error, to ensure minimum angular path maneuvers. The Lyapunov function candidate for the $\mathbf{q}_d(t)$ equilibrium point is given by Eq (75), and the control law of Eq (70) and update law of Eq (82) ensures an asymptotically stable closed-loop system as demonstrated in Sections 3.8.1 through 3.8.3. For the complementary equilibrium point $-\mathbf{q}_d(t)$ the Lyapunov function candidate is given by

$$V(\mathbf{x}_e) = \frac{1}{2}\boldsymbol{\omega}_e^T \mathbf{J}\boldsymbol{\omega}_e + \beta u_{max}\left[\boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + \left(1 + \delta q_4\right)^2\right] + \frac{1}{2\gamma_p}p^2 \qquad (104)$$

The feedback control law that ensures asymptotic stability is given by

$$\mathbf{u}(t) = -u_{max}\left[\left(1-\beta\right)\tanh\left(\frac{\boldsymbol{\omega}_e(t)}{p^2(t)}\right) - \beta\boldsymbol{\delta q}_{13}(t)\right] \qquad (105)$$

and the corresponding sharpness parameter update law is given by Eq (82). The Lyapunov function candidate corresponding to minimum angular path maneuvers (either equilibrium point $\pm\mathbf{q}_d(t)$) is given by

$$V(\mathbf{x}_e) = \frac{1}{2}\boldsymbol{\omega}_e^T \mathbf{J}\boldsymbol{\omega}_e + \beta u_{max}\left[\boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + \left(1 - |\delta q_4|\right)^2\right] + \frac{1}{2\gamma_p}p^2 \qquad (106)$$

See the discussion in Section 3.3 regarding the continuity of the partial derivatives of Eq (106) with respect to its arguments. To ensure asymptotic stability of the closed-loop system requires the sharpness parameter update law of Eq (82) and the general control law given by

$$\mathbf{u}(t) = -u_{max}\left[\left(1-\beta\right)\tanh\left(\frac{\boldsymbol{\omega}_e(t)}{p^2(t)}\right) + \beta\,\mathrm{sgn}\left[\delta q_4(t)\right]\boldsymbol{\delta q}_{13}(t)\right] \qquad (107)$$

In summary, to ensure minimum angular path attitude maneuvers (assuming zero state estimation error) (i) an additional $\mathrm{sgn}[\delta q_4(t)]$ term must be added to quaternion component of the control law given by Eq (70), and (ii) the sharpness parameter update law given by Eq (82) must not be modified. This

result is consistent with the developments in Section 3.3 where the control law of Eq (38) is a modification of Eq (40) to ensure minimum path maneuvers.

When the effects of sensor measurement error are taken into account both the feedback control law and the sharpness parameter update equation must be implemented with the true state variables replaced with their estimates. The Lyapunov stability analysis for the $\mathbf{q_d}(t)$ equilibrium point is provided in Section 3.8.4 and is summarised by Eqs (75), (87), (102), and (103). The Lyapunov function candidate for the $-\mathbf{q_d}(t)$ equilibrium point is given by Eq (104) and the feedback control law is a modified version of Eq (105) given by

$$\mathbf{u}(t) = -u_{max}\left[(1-\beta)\tanh\left(\frac{\hat{\boldsymbol{\omega}}_e(t)}{p^2(t)}\right) - \beta\boldsymbol{\delta}\hat{\mathbf{q}}_{13}(t)\right] \tag{108}$$

Following the procedure in Section 3.8.4 (summarised by Eqs (88) through (94)) leads to the Lyapunov function derivative

$$\dot{V}(\mathbf{x_e}) \leq -\boldsymbol{\omega}_e^T\mathbf{g} + \boldsymbol{\omega}_e^T\mathbf{d} - \frac{1}{2}(1-\beta)u_{max}\boldsymbol{\omega}_e^T\tanh\left(\frac{\boldsymbol{\omega}_e}{p^2}\right) - p^2\boldsymbol{\delta}\mathbf{q}_{13}^T\boldsymbol{\delta}\mathbf{q}_{13}$$
$$- \frac{1}{2}(1-\beta)u_{max}\boldsymbol{\omega}_e^T\tanh\left(\frac{\widetilde{\boldsymbol{\omega}}_e}{p^2}\right) + \beta u_{max}\boldsymbol{\omega}_e^T\boldsymbol{\delta}\widetilde{\mathbf{q}}_{13} + 2p^2\|\boldsymbol{\delta}\widetilde{\mathbf{q}}_{13}\|_1 \tag{109}$$

which is identical to the expression of Eq (95) except for the $\beta u_{max}\boldsymbol{\omega}_e^T\boldsymbol{\delta}\widetilde{\mathbf{q}}_{13}$ term. However, since this term is upper bounded in the Lyapunov stability analysis it follows that the remaining steps are identical to Section 3.8.4 (see Eqs (96) through (103)). The Lyapunov function candidate for the $\pm\mathbf{q_d}(t)$ equilibrium point is given by Eq (106) and the feedback control law is a modified version of Eq (107) given by

$$\mathbf{u}(t) = -u_{max}\left[(1-\beta)\tanh\left(\frac{\hat{\boldsymbol{\omega}}_e(t)}{p^2(t)}\right) + \beta\,\text{sgn}[\delta\hat{q}_4(t)]\boldsymbol{\delta}\hat{\mathbf{q}}_{13}(t)\right] \tag{110}$$

117

The Lyapunov stability analysis is performed using the procedure outlined in Section 3.8.4. The presence of the $\text{sgn}[\delta\hat{q}_4(t)]$ term in Eq (110) will produce additional terms in the Lyapunov function derivative compared to Eq (95). Expanding the control law given by Eq (110) using Eq (67) gives

$$\mathbf{u} = -u_{max}\left[(1-\beta)\tanh\left(\frac{\boldsymbol{\omega}_e + \widetilde{\boldsymbol{\omega}}_e}{p^2}\right) + \beta\,\text{sgn}[\delta\hat{q}_4](\boldsymbol{\delta q}_{13} + \boldsymbol{\delta\widetilde{q}}_{13})\right] \qquad (111)$$

Simplifying the hyperbolic tangent term in Eq (111) using the identity of Eq (89) and the inequality (90), and rearranging the quaternion term gives

$$\mathbf{u} \le -u_{max}\left[\frac{1}{2}(1-\beta)\tanh\left(\frac{\boldsymbol{\omega}_e}{p^2}\right) + \beta\,\text{sgn}[\delta q_4]\boldsymbol{\delta q}_{13}\right] - \frac{1}{2}(1-\beta)u_{max}\tanh\left(\frac{\widetilde{\boldsymbol{\omega}}_e}{p^2}\right)$$
$$- \beta u_{max}\,\text{sgn}[\delta\hat{q}_4]\boldsymbol{\delta\widetilde{q}}_{13} - \beta u_{max}\{\text{sgn}[\delta\hat{q}_4] - \text{sgn}[\delta q_4]\}\boldsymbol{\delta q}_{13} \qquad (112)$$

which the control law of Eq (107) with additional upper bounded terms due to the state estimation error. The reduction of the sharpness parameter update law due to the estimation error is identical to the procedure in Section 3.8.4 (see Eqs (92) through (94)). Using the Lyapunov candidate of Eq (106) and substituting all relevant expressions

$$\dot{V}(\mathbf{x}_e) \le -\boldsymbol{\omega}_e^T\mathbf{g} + \boldsymbol{\omega}_e^T\mathbf{d} - \frac{1}{2}(1-\beta)u_{max}\boldsymbol{\omega}_e^T\tanh\left(\frac{\boldsymbol{\omega}_e}{p^2}\right) - p^2\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13}$$
$$- \frac{1}{2}(1-\beta)u_{max}\boldsymbol{\omega}_e^T\tanh\left(\frac{\widetilde{\boldsymbol{\omega}}_e}{p^2}\right) - \beta u_{max}\,\text{sgn}[\delta\hat{q}_4]\boldsymbol{\omega}_e^T\boldsymbol{\delta\widetilde{q}}_{13}$$
$$- \beta u_{max}\{\text{sgn}[\delta\hat{q}_4] - \text{sgn}[\delta q_4]\}\boldsymbol{\omega}_e^T\boldsymbol{\delta q}_{13} + 2p^2\|\boldsymbol{\delta\widetilde{q}}_{13}\|_1 \qquad (113)$$

Compared to Eq (95) the Lyapunov function derivative inequality of Eq (113) has an additional term due to the generalisation of the control law given by Eq (110) for minimum angular path maneuvers. Furthermore, the $-\beta u_{max}\boldsymbol{\omega}_e^T\boldsymbol{\delta\widetilde{q}}_{13}$ term in Eq (95) has been replaced by a $-\beta u_{max}\,\text{sgn}[\delta\hat{q}_4]\boldsymbol{\omega}_e^T\boldsymbol{\delta\widetilde{q}}_{13}$ term, however both terms have identical upper bounds. The additional term in Eq (113) may be upper bounded according to

$$-\beta u_{max}\{sgn[\delta\hat{q}_4]-sgn[\delta q_4]\}\boldsymbol{\omega}_e^T\boldsymbol{\delta q}_{13} \le 2\beta u_{max}\|\boldsymbol{\omega}_e\|_1\|\boldsymbol{\delta q}_{13}\|_1 \le 2\beta u_{max}\|\boldsymbol{\omega}_e\|_1 \quad (114)$$

It is important to consider the special case in which the state estimator is accurately initialised and provides estimates such that $sgn[\delta\hat{q}_4]=sgn[\delta q_4]$ is valid throughout the entire attitude maneuver. In this case the additional term in Eq (113) is zero and the stability analysis is identical to Section 3.8.4. In the more general case the expression of Eq (113) is reduced using Eq (114) and the procedure defined by Eqs (96)-(103) in Section 3.8.4. Global asymptotic stability of the $\pm \mathbf{q_d}(t)$ equilibrium point is ensured provided the following terms are strictly positive

$$a^2 = \left\{\frac{1}{2}(1-\beta)u_{max}[1+2\tilde{q}_{max}-\tilde{\chi}_{max}]-g_{max}-d_{max}-u_{max}[\tilde{q}_{max}+2\beta]\right\} > 0 \quad (115)$$

$$c^2 = \frac{3}{2}\alpha(1-\beta)u_{max}+2\tilde{q}_{max} > 0 \quad (116)$$

The corresponding limits on the parameter β are given by $0 < \beta < \lambda \le 1$ where $\lambda = (2(g_{max}+d_{max})+u_{max}(\tilde{\chi}_{max}-1))/u_{max}(3+2\tilde{q}_{max}-\tilde{\chi}_{max})$. This constraint is valid provided that $(2(g_{max}+d_{max})+u_{max}(\tilde{\chi}_{max}-1)) < u_{max}(3+2\tilde{q}_{max}-\tilde{\chi}_{max})$, and reduces to $0 < \beta < 1$ in the absence of a time-varying reference trajectory, disturbance torques, and state estimation error.

3.9 Design Example

Several representative spacecraft attitude maneuver simulation results are presented in this section for both inertial pointing and earth-pointing scenarios to demonstrate the capabilities of the Lyapunov control law developed in Sections 3.2-3.8. The simulations were conducted using an attitude control system model developed in MATLAB™ Simulink®. The basic ACS model integrates Euler's equations of motion for the rigid-spacecraft using a fourth-order Runge-Kutta numerical integrator ODE4 in MATLAB™ Simulink® with a fixed step size of $\Delta t = 0.1 \sec$. An SGP4 numerical orbit propagator based on the two-line-elements (TLE) of the FEDSAT spacecraft provides a reference

800km sun-synchronous orbit. A key assumption is that for this particular orbit and the spacecraft configuration the gravity-gradient torque is the dominant source of environmental disturbance torques. The MATLAB™ line style designators for all simulation plots are 1-axis = solid line, 2-axis = dashed line, 3-axis = dash-dot line, and 4-axis = dotted line.

3.9.1 Attitude Regulation Maneuver

This example is a rest-to-rest attitude maneuver for an inertial-pointing spacecraft using the control law of Eq (38) with reference trajectory, initial conditions, and control gains defined in Table 3-1. The only disturbance torque considered is the gravity-gradient torque although the external control torques for this example are orders of magnitude larger than the gravity-gradient torque. Based on the analysis presented in Section 3.4, selecting the control gains only requires the initial condition for the tracking error quaternion. The initial Euler/Euler axis parameterisation given in Table 3-1 corresponds to a 90 deg rotation with respect to the minimum angle maneuver equilibrium point $-\mathbf{q_d}(t)$. For a settling time requirement of $t_s = 50$ sec and a critically damped response $\xi = 1$ the natural frequency is $\omega_n = 0.16$ rad/sec and the control gains are $d = 0.32$ and $k = 0.0512$.

Figure 3-1 through Figure 3-8 present the MATLAB™ simulation results of the rest-to-rest attitude maneuver. The error quaternion displays a critically damped response with the $\delta q_4(t)$ component settling to the desired minimum-angle maneuver equilibrium point, and the system settling time of 50 sec has been achieved to within the approximations of Section 3.4. The Euler axis (eigenaxis) unit-vector components corresponding to the error quaternion remain approximately constant through the attitude maneuver confirming Wie's theorem of an eigenaxis minimum-angle maneuver discussed in Section 3.4. It is interesting to observe in Figure 3-8 that the Lyapunov function derivative is always negative throughout the attitude rest-to-rest maneuver and eventually settles at the desired equilibrium point, despite the increase in magnitude of the tracking error rates early in the maneuver. Other control gain scenarios were

simulated by slightly modifying the control gains with respect to the critically damped maneuver and modifying the initial spacecraft angular rates to $\omega(t_0) = [0.2 \quad -0.2 \quad -0.2]^T$. The result was a marginally degraded settling time but the overall characteristics remained identical.

Parameters	Values
Spacecraft Inertia Matrix	$$\mathbf{J} = \begin{bmatrix} 600.28 & -3.57 & 0.17 \\ -3.57 & 611.44 & 0.25 \\ 0.17 & 0.25 & 507.90 \end{bmatrix}$$
Reference Trajectory	$$\mathbf{q_d} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$ $$\omega_\mathbf{d} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$
Initial Conditions	$$\mathbf{q}(t_0) = \begin{bmatrix} -\sqrt{1/6} & \sqrt{1/6} & -\sqrt{1/6} & -\sqrt{1/2} \end{bmatrix}^T$$ $$\hat{\mathbf{e}}(t_0) = \begin{bmatrix} -0.5774 & 0.5774 & -0.5774 \end{bmatrix}^T$$ $$\theta(t_0) = 270 \text{ deg}$$ $$\omega(t_0) = \begin{bmatrix} 0.0 & 0.0 & 0.0 \end{bmatrix}^T$$
Control Gains	$$\mathbf{K}_1 = 0.32\mathbf{J}$$ $$\mathbf{K}_2 = 0.0512\mathbf{J}$$

Table 3-1 Rest-to-Rest Attitude Maneuver
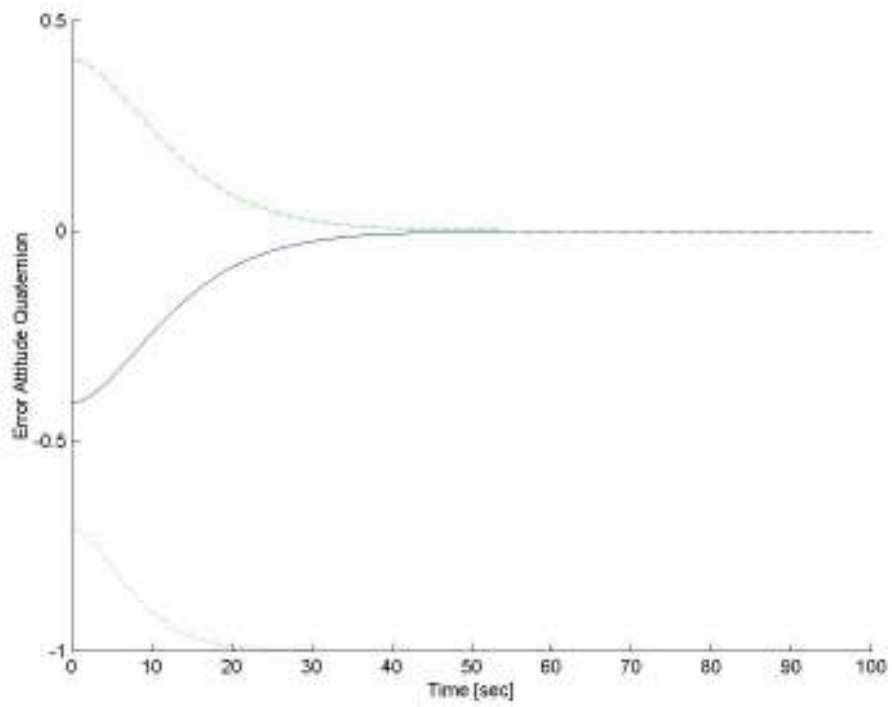
Figure 3-1 Spacecraft attitude quaternion tracking error
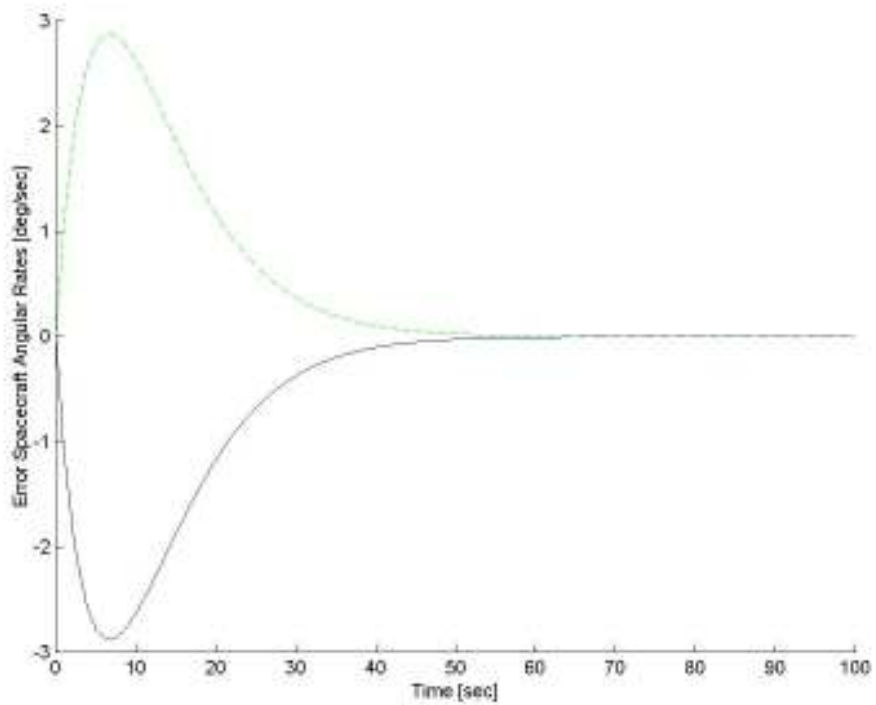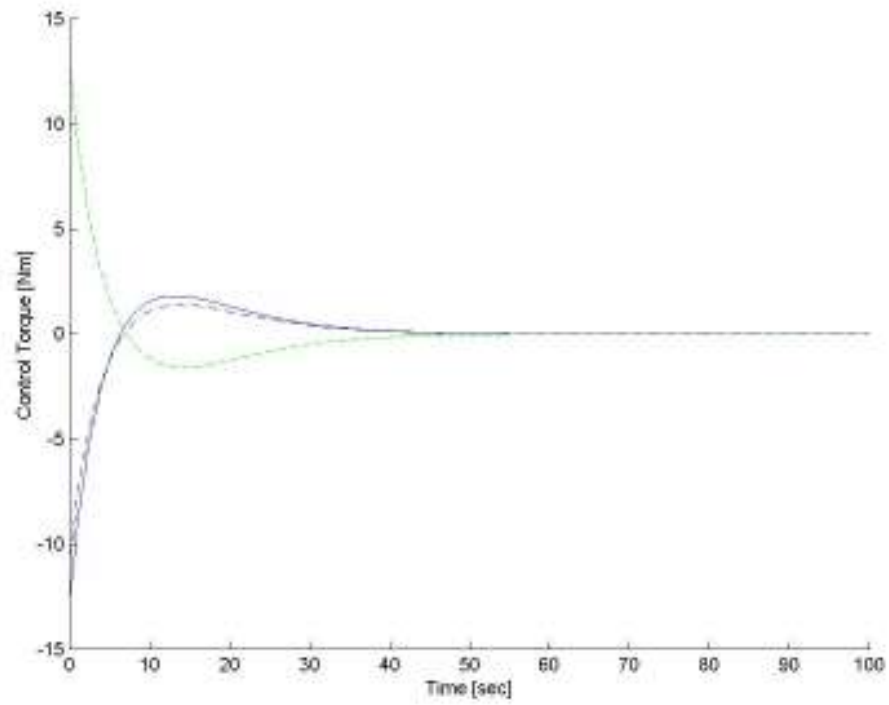

Figure 3-2 Spacecraft angular rates

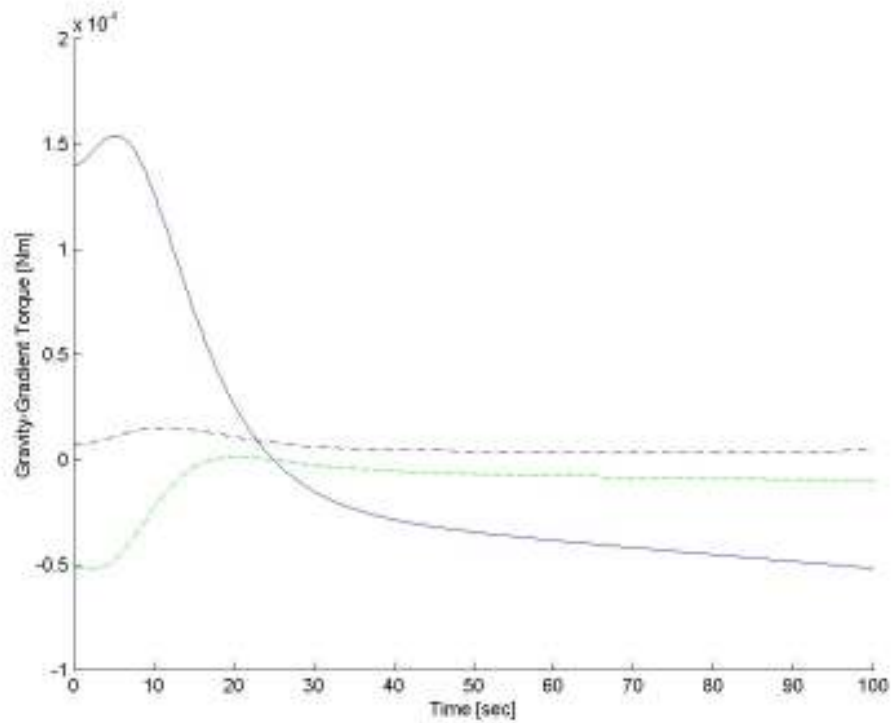**Figure 3-3 External control torques**

Figure 3-4 Gravity-gradient disturbance torques
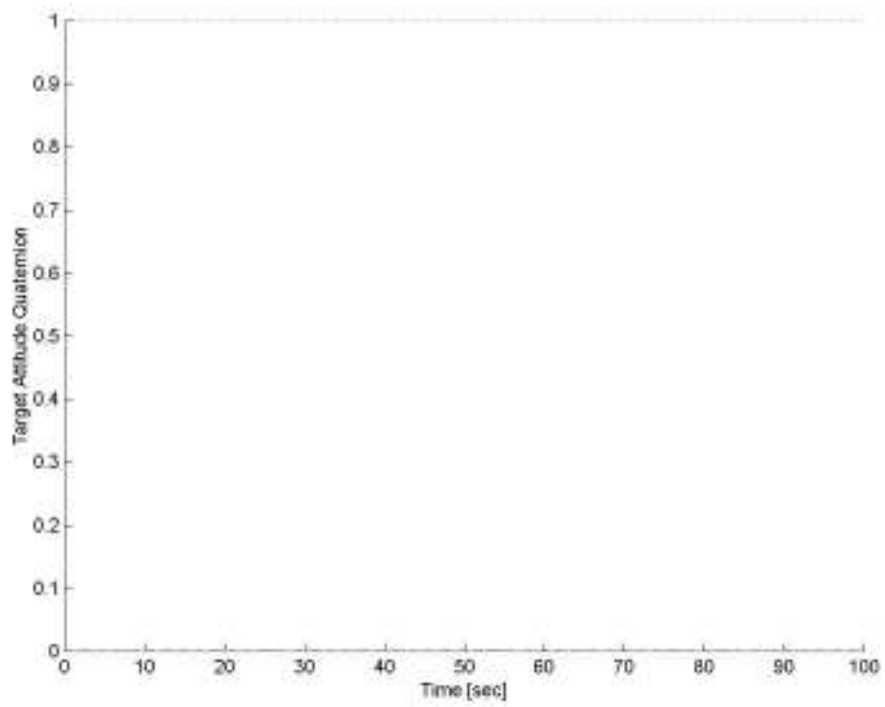
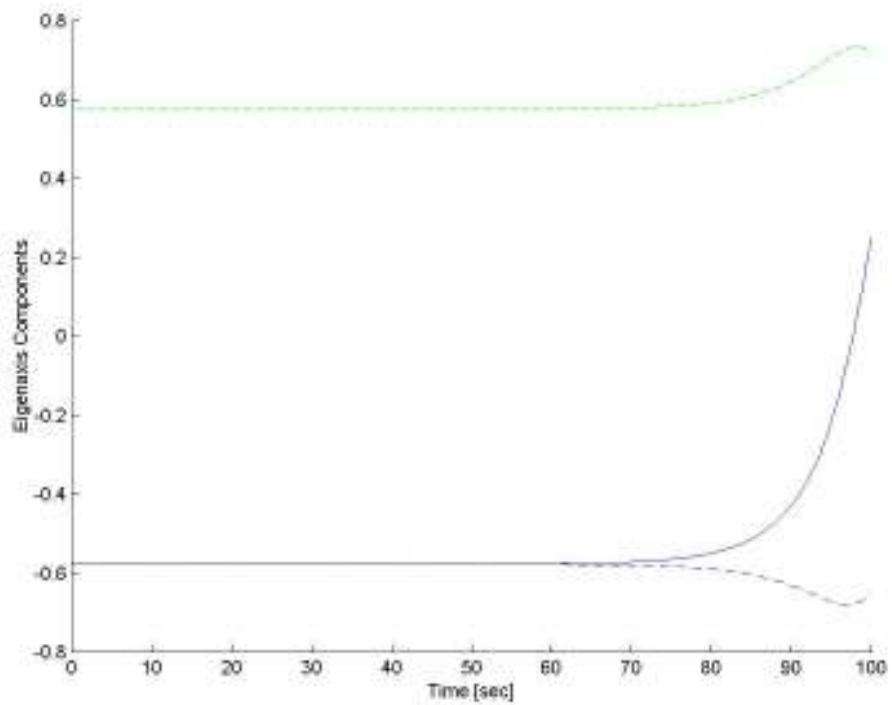123

Figure 3-5 Reference trajectory attitude quaternion

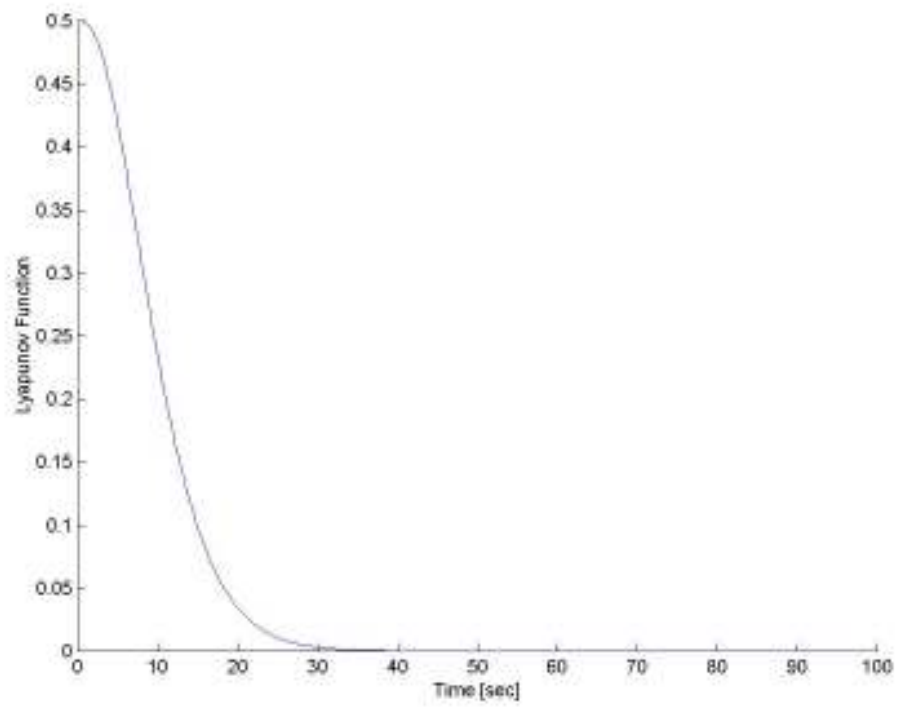**Figure 3-6 Euler axis components of tracking error quaternion**
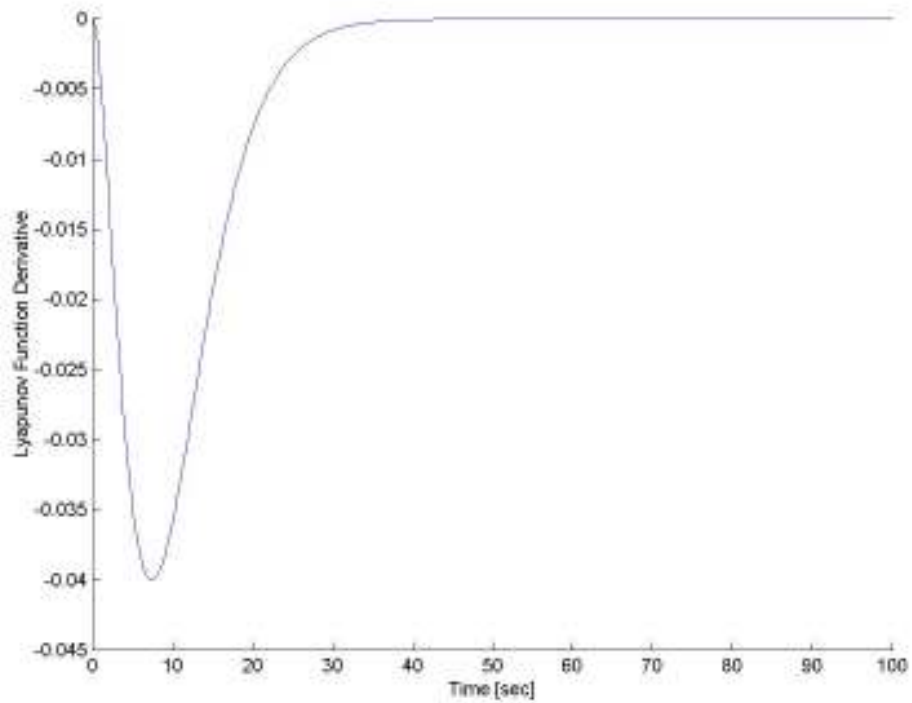
Figure 3-7 Lyapunov function

Figure 3-8 Lyapunov function derivative

125

3.9.2 Attitude Tracking Maneuver

This example is an attitude tracking maneuver for an earth-pointing spacecraft in which the desired trajectory is generated numerically based on the spacecraft navigation information supplied by an SGP4 orbit propagator. The initial Euler/Euler axis parameterisation given in Table 3-2 corresponds to a 179.9 deg rotation with respect to the minimum angle maneuver equilibrium point $\mathbf{q_d}(t)$.

Parameters	*Values*
Spacecraft Inertia Matrix	$\mathbf{J} = \begin{bmatrix} 600.28 & -3.57 & 0.17 \\ -3.57 & 611.44 & 0.25 \\ 0.17 & 0.25 & 507.90 \end{bmatrix}$
Reference Trajectory	Earth-pointing orientation based on SGP4 numerical orbit propagator
Initial Attitude	$\mathbf{q}(t_0) = \begin{bmatrix} 0.08592 & -0.6466 & 0.05430 & 0.7560 \end{bmatrix}^T$ $\mathbf{\hat{e}}(t_0) = \begin{bmatrix} 0.1313 & -0.9879 & -0.08297 \end{bmatrix}^T$ $\theta(t_0) = 81.8 \text{ deg}$ $\mathbf{q_d}(t_0) = \begin{bmatrix} 0.4188 & 0.5945 & -0.4747 & 0.4958 \end{bmatrix}^T$ $\mathbf{\hat{e}_d}(t_0) = \begin{bmatrix} 0.4823 & 0.6846 & -0.5466 \end{bmatrix}^T$ $\theta(t_0) = 120.6 \text{ deg}$ $\mathbf{\delta q}(t_0) = \begin{bmatrix} 0.0006165 & -0.7065 & 0.7077 & 0.0006176 \end{bmatrix}^T$ $\mathbf{\delta\hat{e}}(t_0) = \begin{bmatrix} 0.0006165 & -0.7065 & 0.7077 \end{bmatrix}^T$ $\delta\theta(t_0) = 179.9 \text{ deg}$ $(\alpha_0, \beta_0, \gamma_0) = \begin{bmatrix} 89.9 & 0.0 & 179.9 \end{bmatrix}^T \text{ deg}$
Initial Angular Rates	$\mathbf{\omega_d}(t_0) = \begin{bmatrix} 1.812 \times 10^{-7} & -1.038 \times 10^{-4} & -5.948 \times 10^{-2} \end{bmatrix}^T \text{ deg/sec}$ $\mathbf{\omega_e}(t_0) = \begin{bmatrix} 0.2 & -0.2 & -0.2 \end{bmatrix}^T \text{ deg/sec}$ $\mathbf{\omega}(t_0) = \begin{bmatrix} 0.2 & -0.2 & 0.2595 \end{bmatrix}^T \text{ deg/sec}$
Control Gains	$\mathbf{K}_1 = 0.32\mathbf{J}$ $\mathbf{K}_2 = 0.0512\mathbf{J}$

Table 3-2 Earth-Pointing Attitude Maneuver

Figure 3-9 through Figure 3-16 present the results of the earth-pointing attitude maneuver. The error quaternion displays a critically damped response with the $\delta q_4(t)$ component settling to the desired minimum-angle maneuver equilibrium point, and the system settling time of 50 sec has been achieved to within the approximations of Section 3.4. This example also demonstrates the effectiveness of replacing the control law of Eq (31) with Eq (38) for small values of $\delta q_4(t_0)$. From Figure 3-14, only the y- and z-axis eigenaxis unit-vector components remain constant. This demonstrates that Wie's theorem of an eigenaxis rotation for rest-to-rest maneuvers does not apply in general for attitude tracking maneuvers, although an identical control gain selection strategy may be applied. The gravity-gradient torque is constant in spacecraft body-fixed coordinates for an earth-pointing spacecraft which is observed in Figure 3-12. Since the desired equilibrium point corresponds to an earth-pointing configuration, the gravity-gradient torque provides a stabilising effect with respect to this equilibrium point. It is therefore not necessary to provide a term to counteract the gravity-gradient torque in the control law of Eq (38).
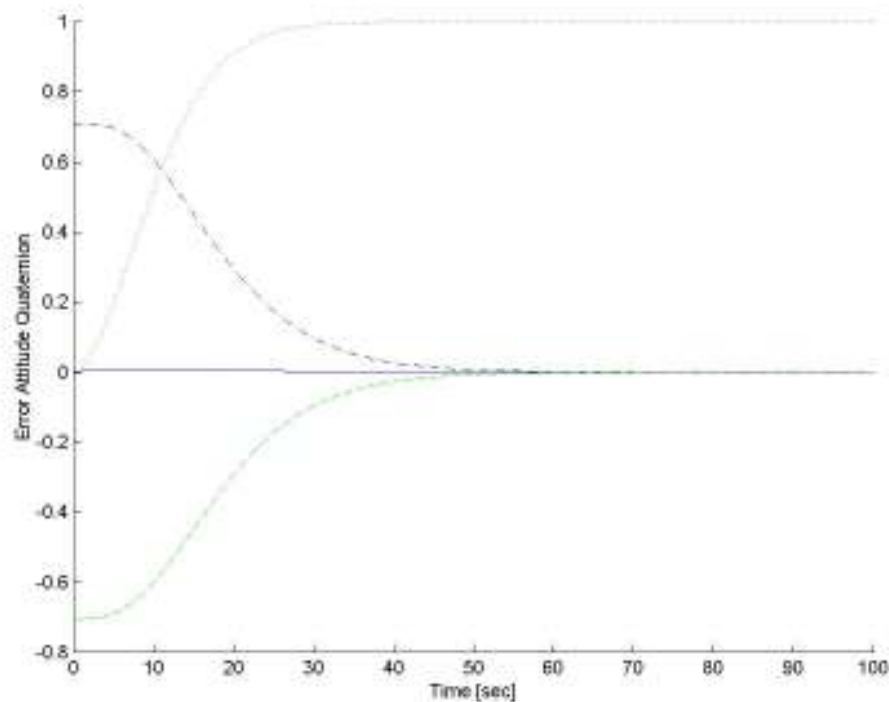

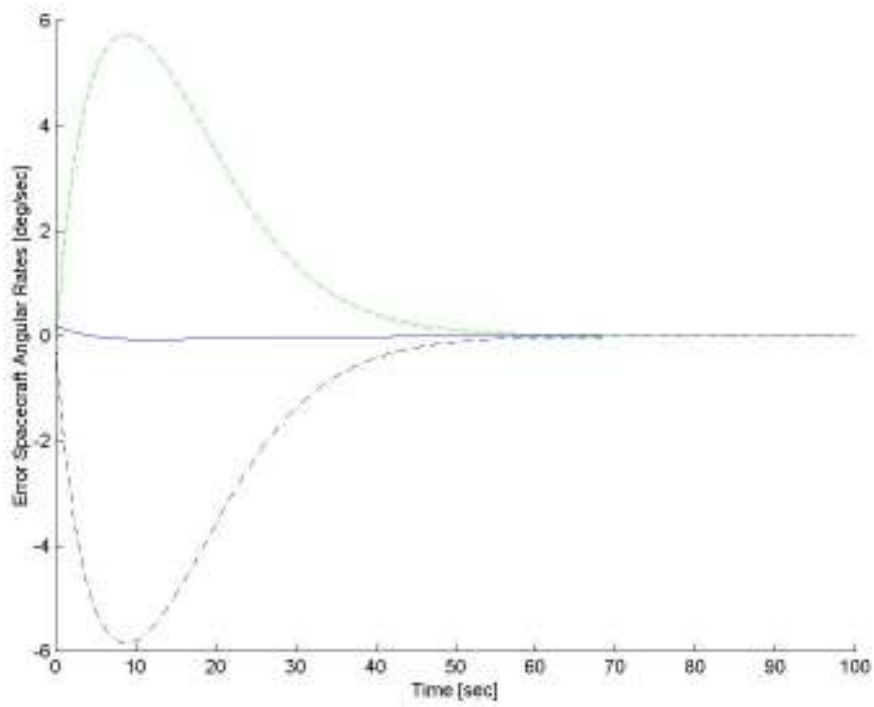**Figure 3-9 Spacecraft attitude quaternion tracking error**
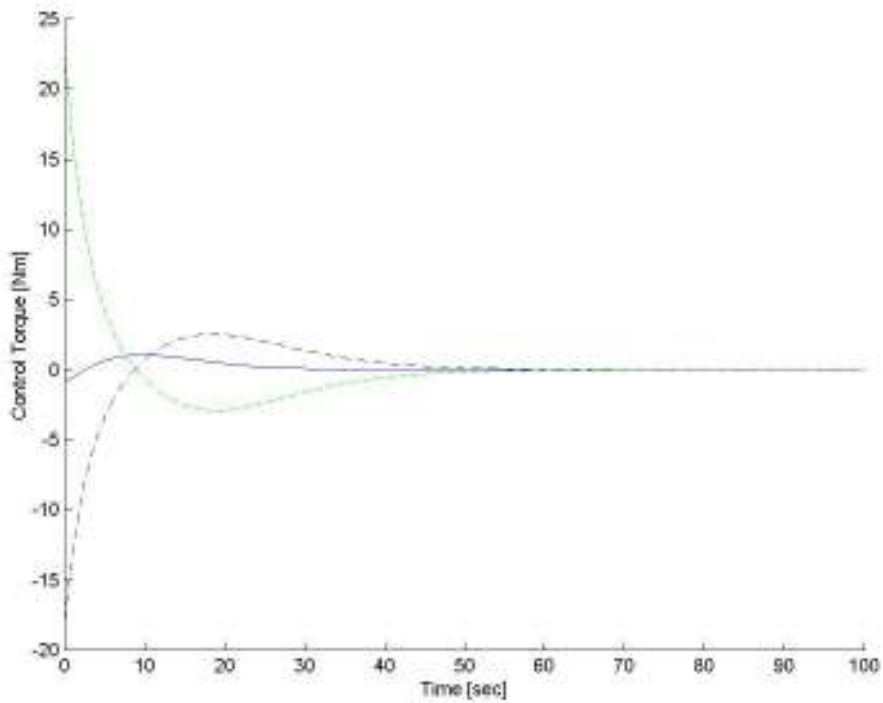
Figure 3-10 Spacecraft angular rate tracking error
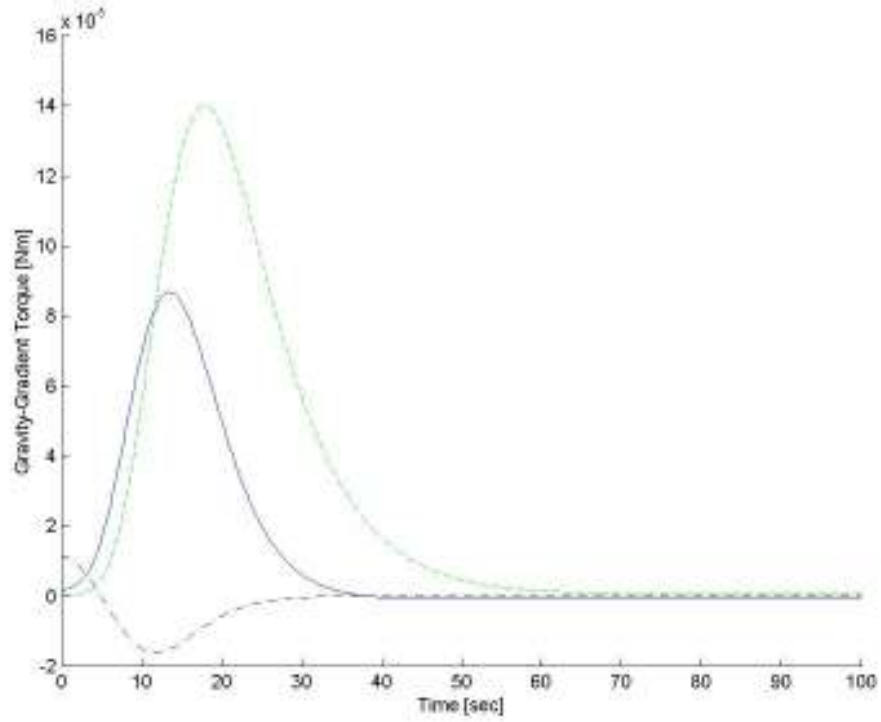

Figure 3-11 External control torques

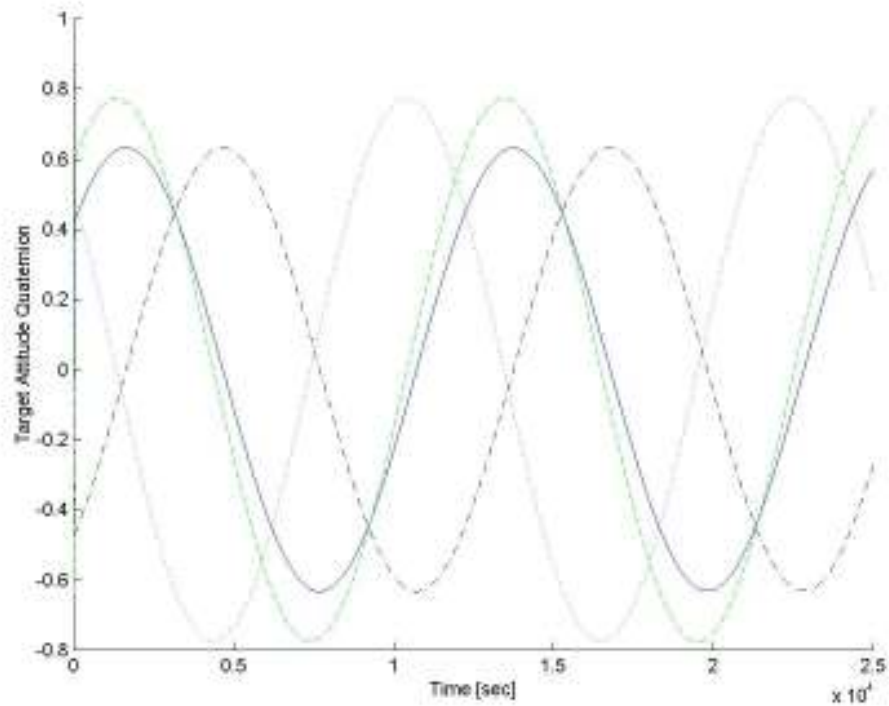**Figure 3-12 Gravity-gradient disturbance torques**
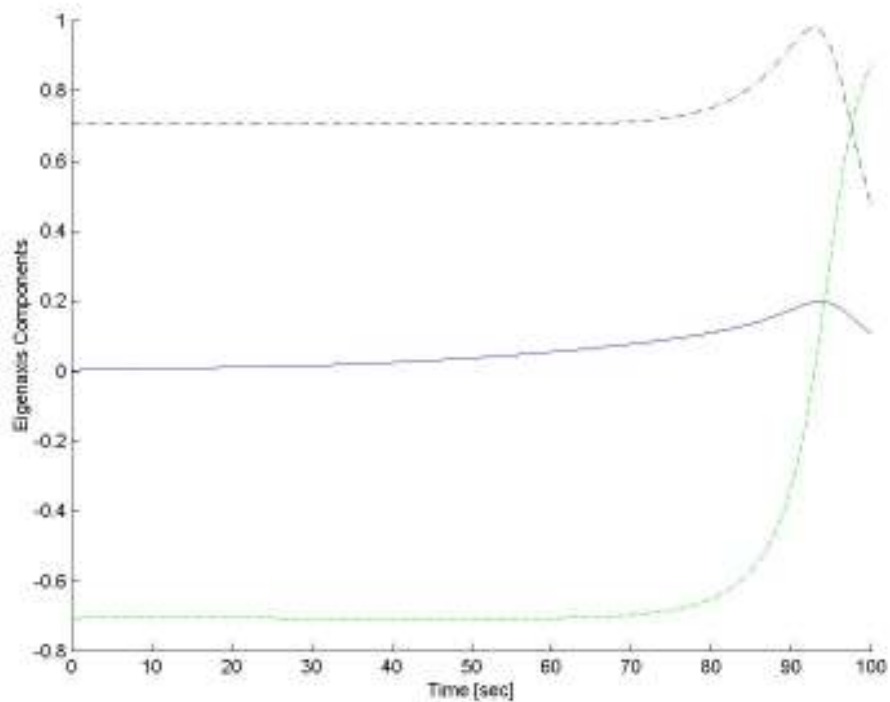

Figure 3-13 Reference trajectory attitude quaternion

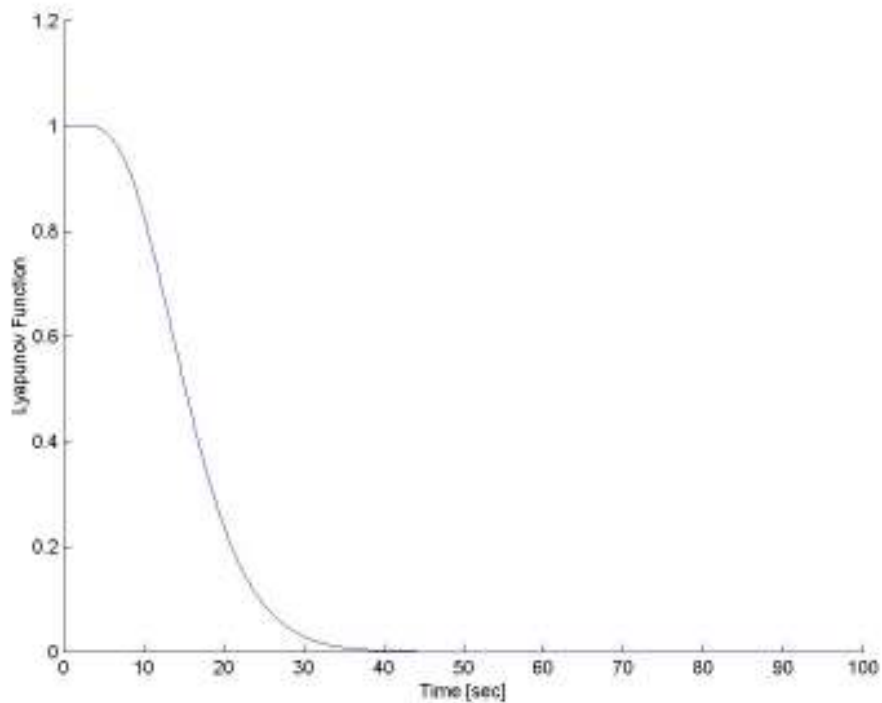**Figure 3-14 Euler axis components of tracking error quaternion**
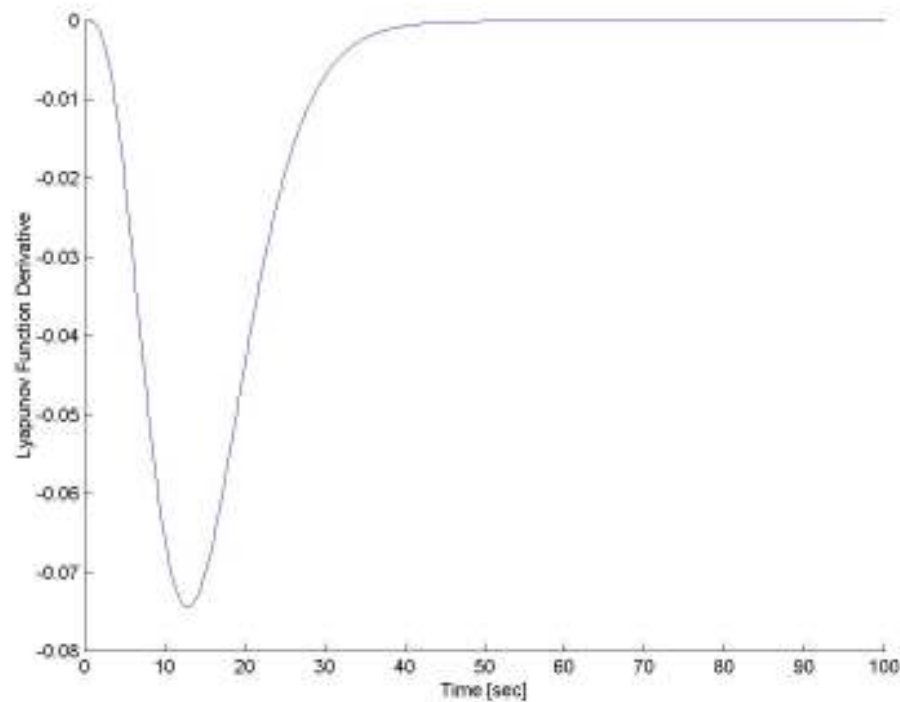

Figure 3-15 Lyapunov function

Figure 3-16 Lyapunov function derivative

3.9.3 Torque Magnitude Constraints

Example 3 repeats the rest-to-rest maneuver of Section 3.9.1 with the initial conditions defined in Table 3-1 but subject to external control torque saturation constraints. The individual upper and lower actuator torque limits are assumed equal and are given by $\mathbf{u}_{max} = \begin{bmatrix} 10 & 5 & 1 \end{bmatrix}^T$. In practice the control torque limits are specified by the actuator manufacturer. Figure 3-17 through Figure 3-24 clearly demonstrate that the system remains asymptotically stable and the settling time is only slightly degraded with respect to Example 1. Although the control torques in Figure 3-19 are saturated early in the attitude maneuver, the Lyapunov function derivative shown in Figure 3-24 remains negative during this period. It is expected that lowering the torque magnitude constraints may result in intervals during torque saturation in which the Lyapunov derivative is positive leading to a brief increase in trajectory tracking error. Figure 3-22 demonstrates that Wie's theorem of an eigenaxis rotation for rest-to-rest maneuvers does not apply when torque saturation limits are imposed.

Figure 3-17 Spacecraft attitude quaternion tracking error

Figure 3-18 Spacecraft angular rates
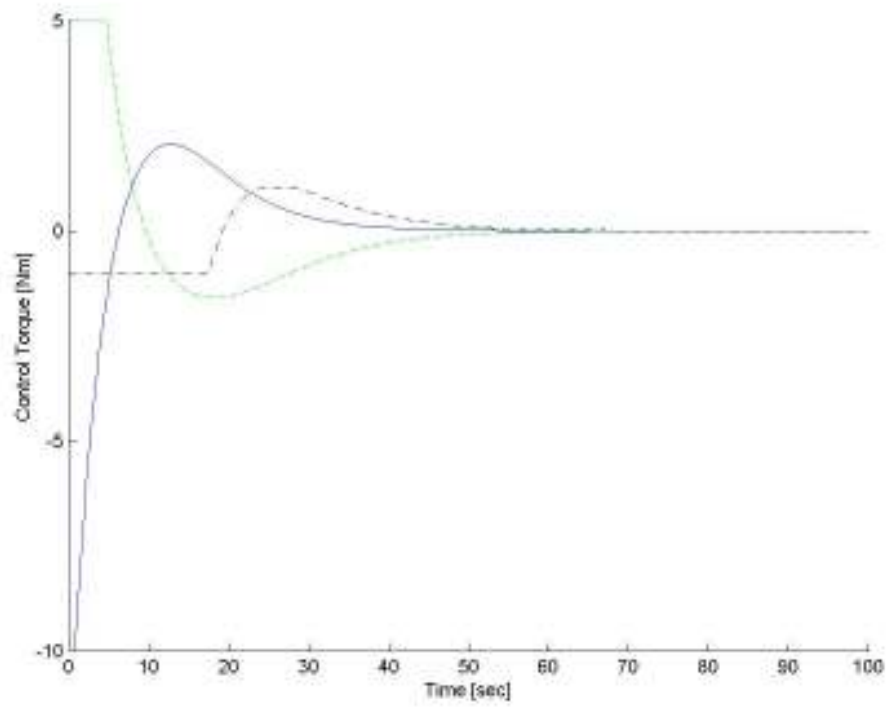
Figure 3-19 External control torques
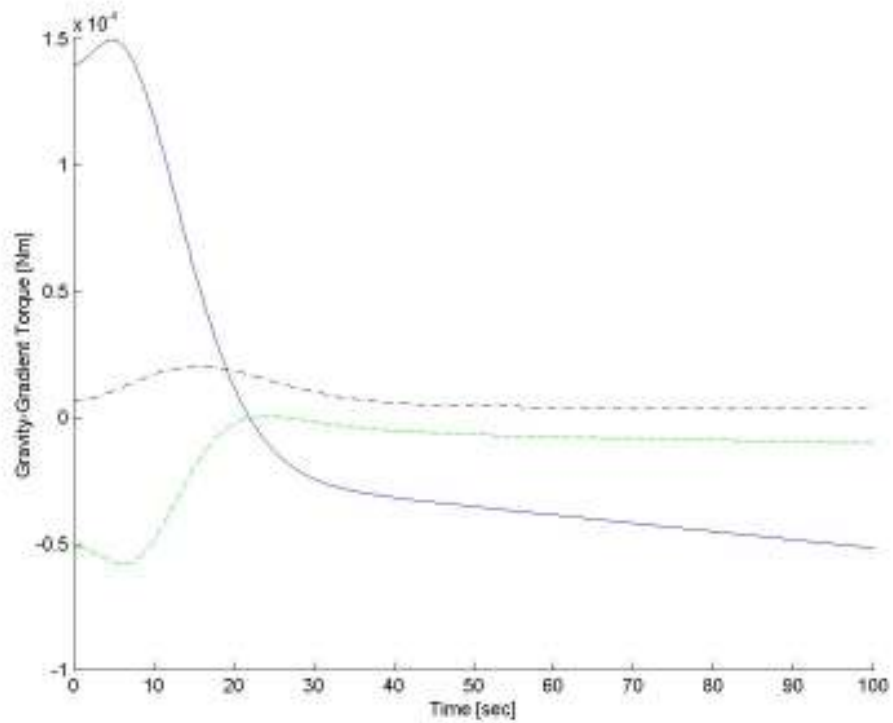
Figure 3-20 Gravity-gradient disturbance torques

133

Figure 3-21 Reference trajectory attitude quaternion

Figure 3-22 Euler axis components of tracking error quaternion
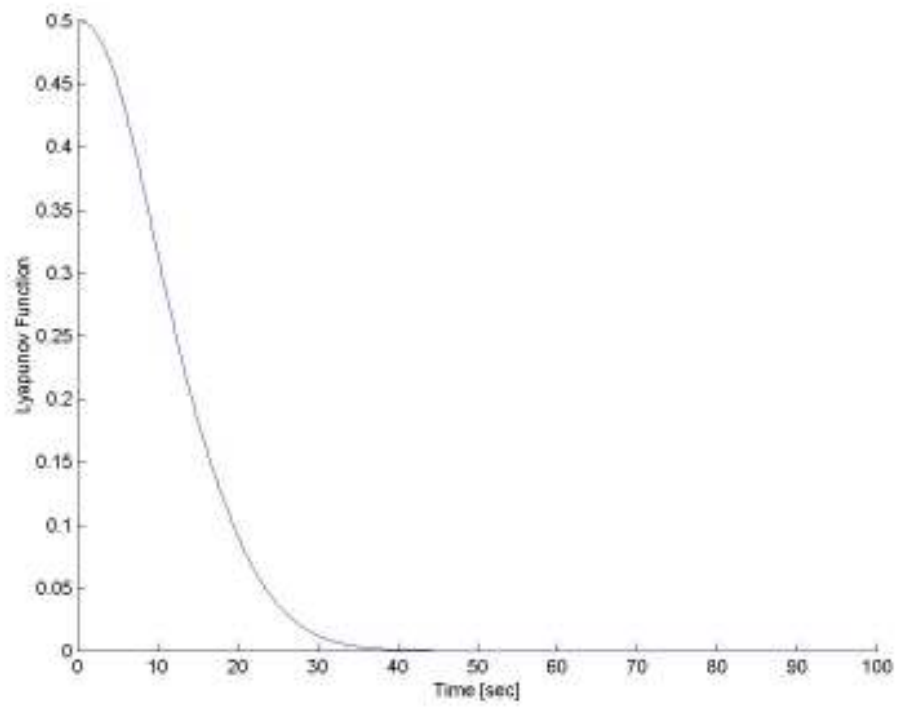
Figure 3-23 Lyapunov function
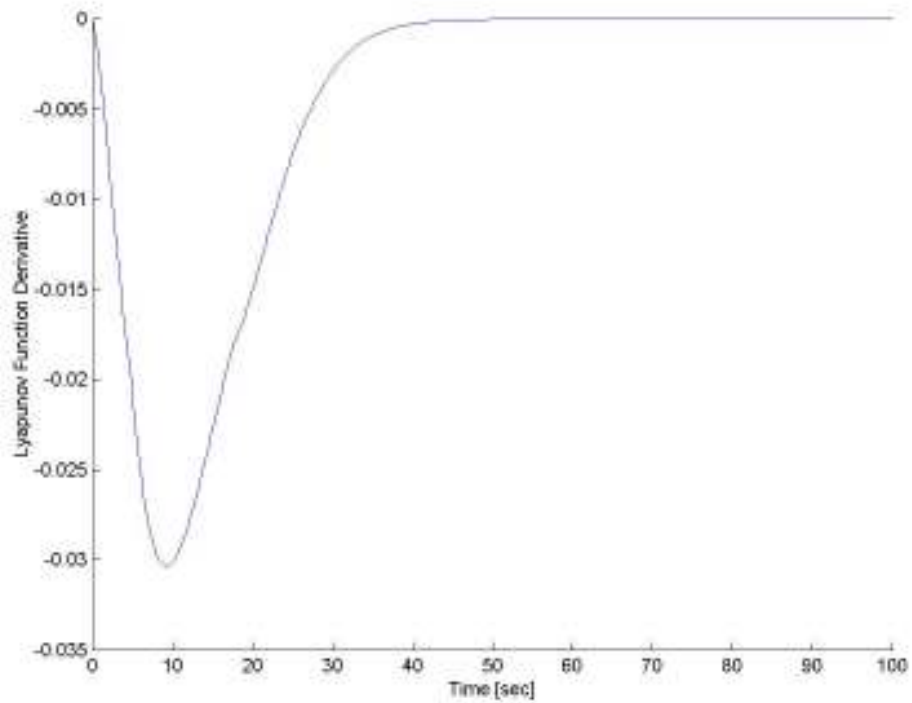
Figure 3-24 Lyapunov function derivative

135

3.9.4 Total Stability in the Presence of Disturbances

This example repeats the rest-to-rest maneuver of Section 3.9.1 with the initial conditions defined in Table 3-1, but subject to both a gravity-gradient disturbance torque and a sinusoidal miscellaneous external control torque given by

$$\mathbf{u_{misc}}(t) = \begin{bmatrix} \sin(0.5t) & 5\sin(0.1t) & -5\sin(t) \end{bmatrix}^T \tag{117}$$

It should be noted, however, that the miscellaneous torque is orders of magnitude larger than the gravity-gradient torque and will dominate the steady-state attitude trajectory tracking errors. Figure 3-25 and Figure 3-26 illustrate that the sinusoidal disturbance torque translates to steady-state pointing errors with the same sinusoidal frequency components as the general disturbance torque. Investigations were also performed using constant miscellaneous torque components which resulted in constant steady-state tracking errors and with square wave disturbance torques which resulted in a low frequency sinusoidal components. Figure 3-31 demonstrates that Wie's theorem of an eigenaxis rotation for rest-to-rest maneuvers is not valid when general disturbances are present in the system. The lower bounded Lyapunov function of Figure 3-32 and its derivative in Figure 3-33 clearly illustrate the concept of total stability[1] where a persistent disturbance torque acting on an otherwise ideal asymptotically stable system results in a residual set centered on the equilibrium point to which the state trajectory eventually converges. The magnitude of the residual set and proximity of the Lyapunov function to the state-space origin is proportional to the magnitude of the disturbance torque. Figure 3-33 illustrates that the Lyapunov derivative is strictly negative outside the residual set but is indefinite inside the set, such that the Lyapunov function may increase during specific intervals in the maneuver.
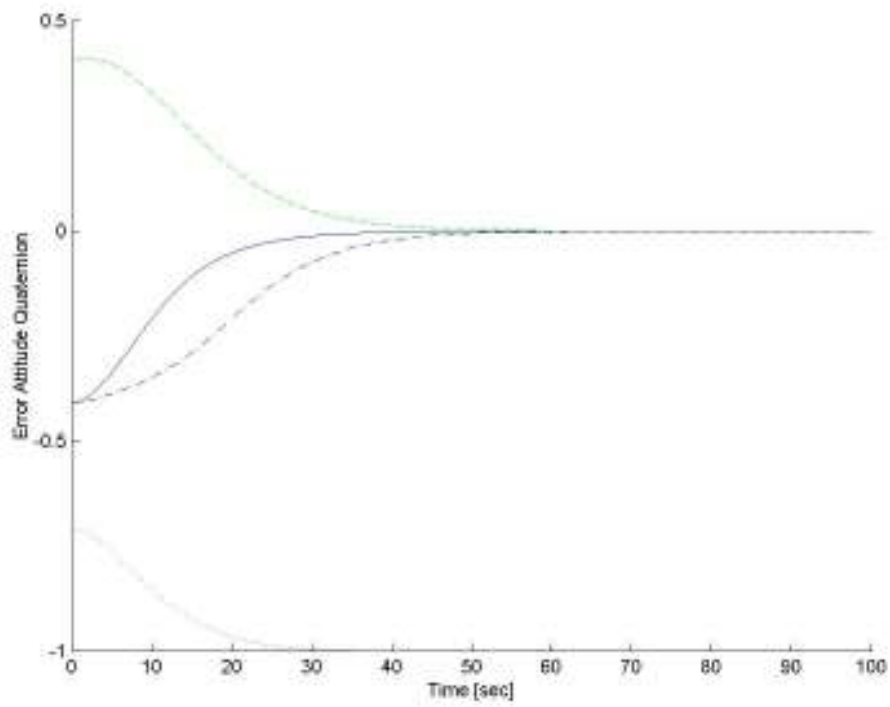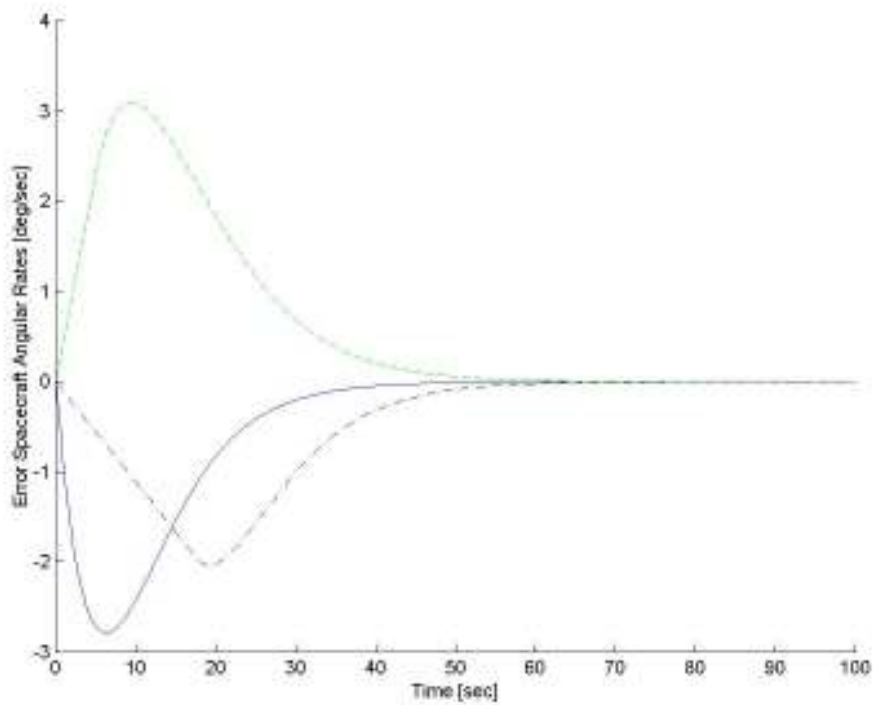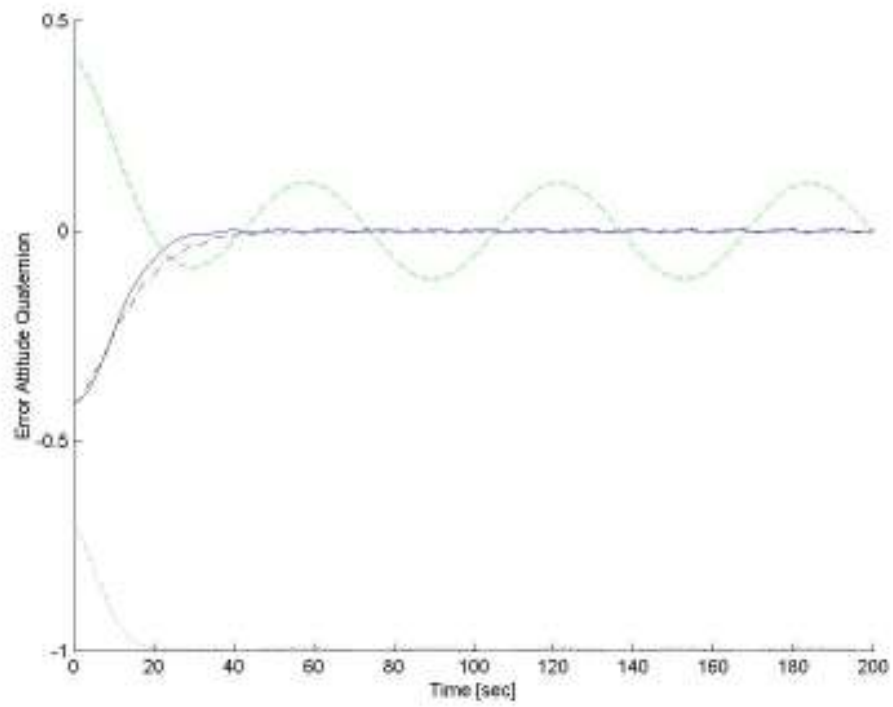
Figure 3-25 Spacecraft attitude quaternion tracking error
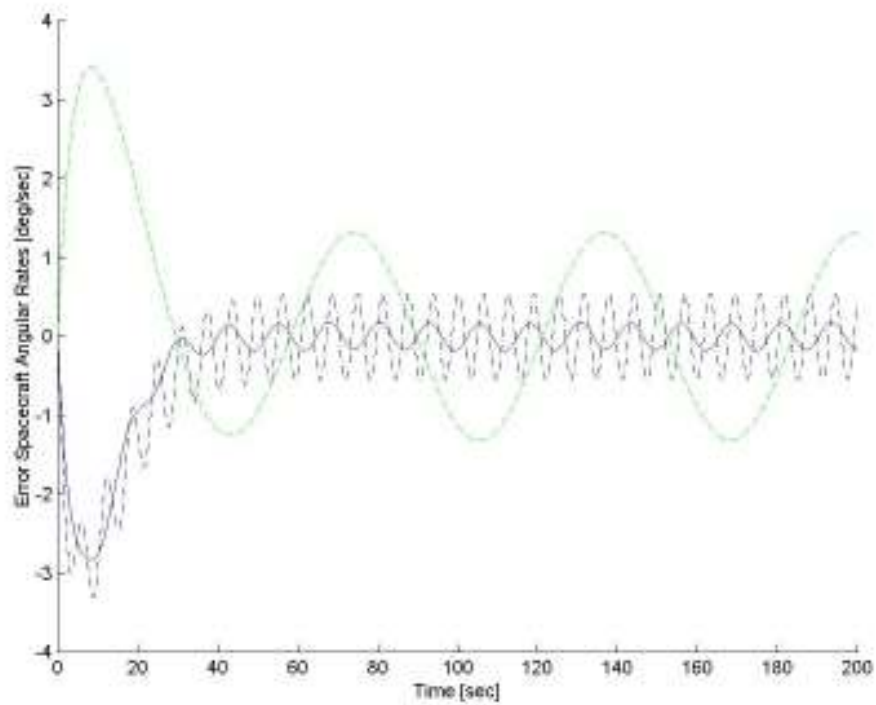
Figure 3-26 Spacecraft angular rates

137
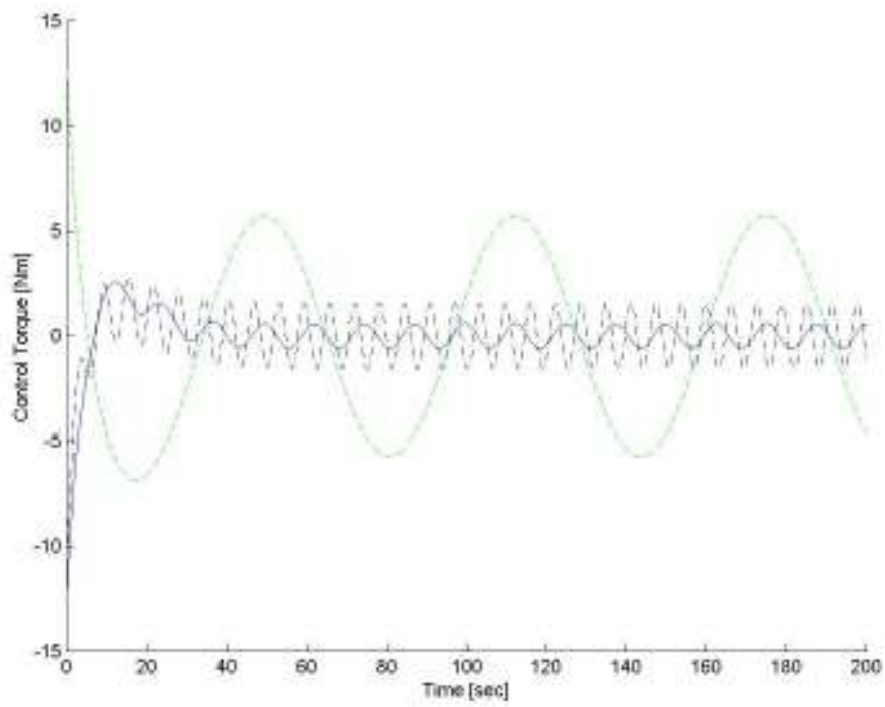
Figure 3-27 External control torques
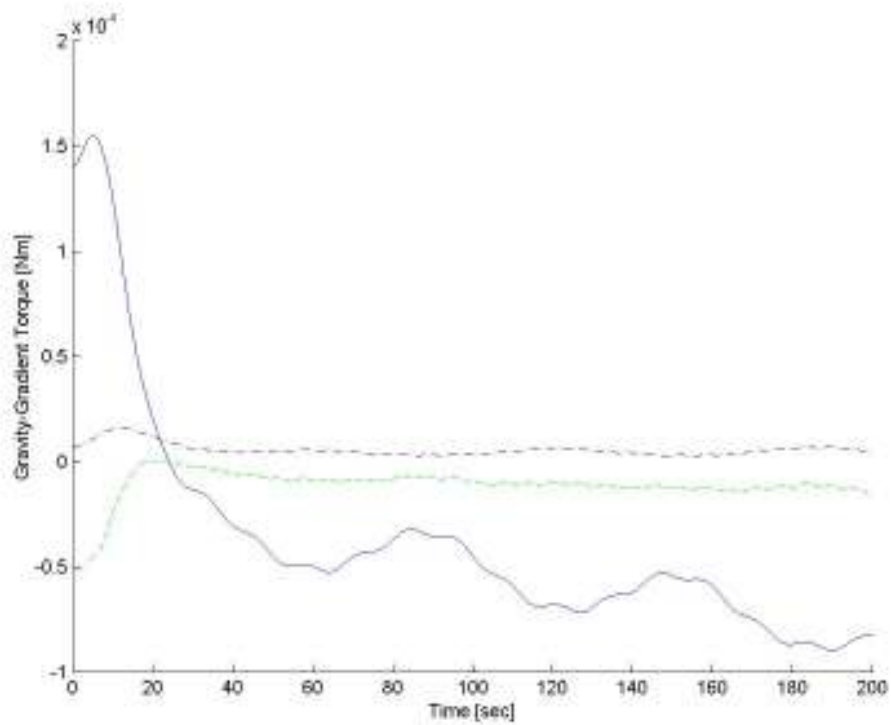
Figure 3-28 Gravity-gradient disturbance torques
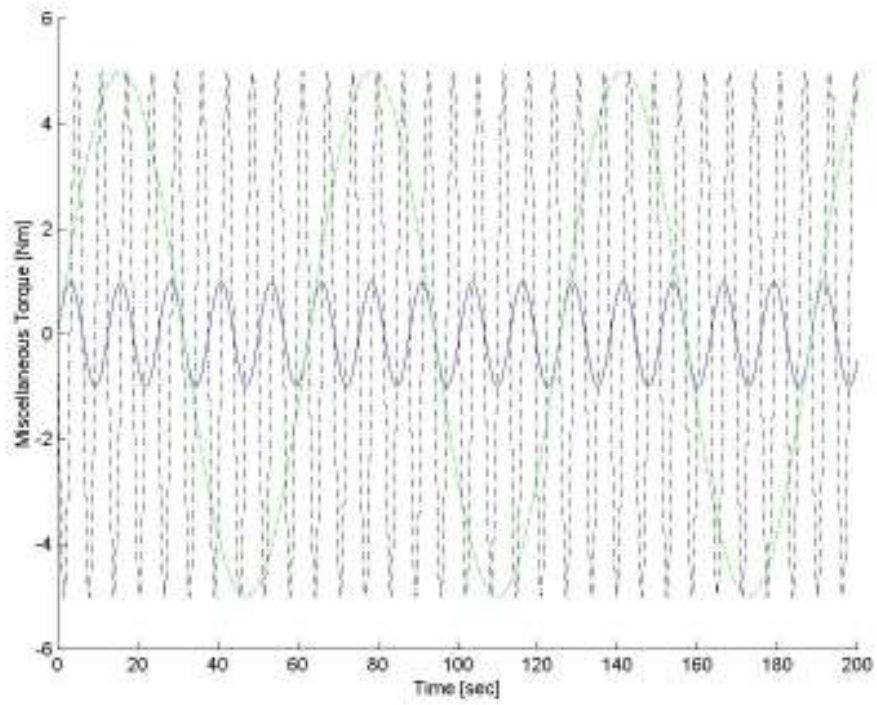
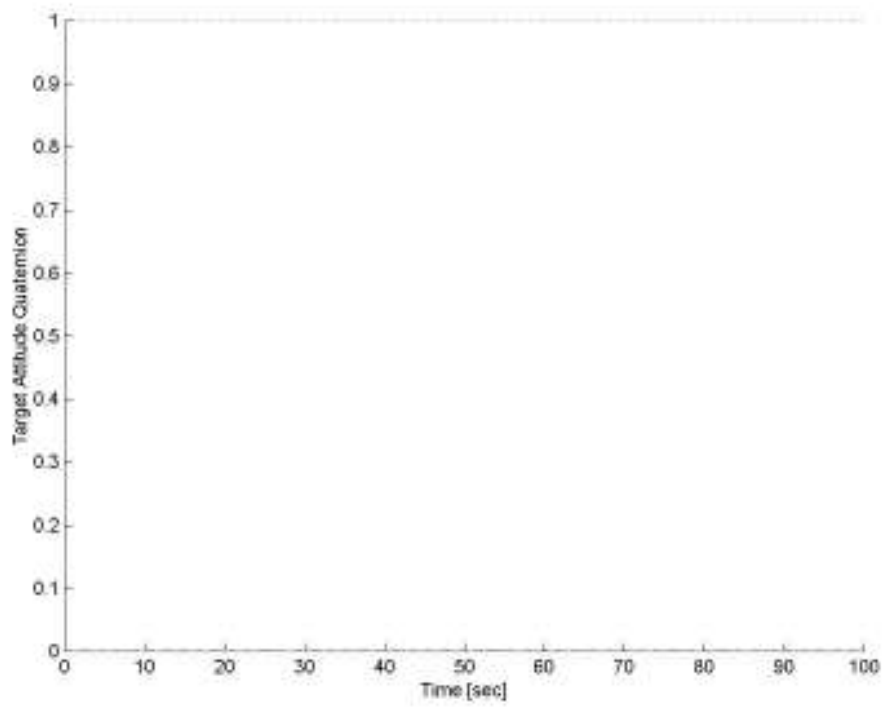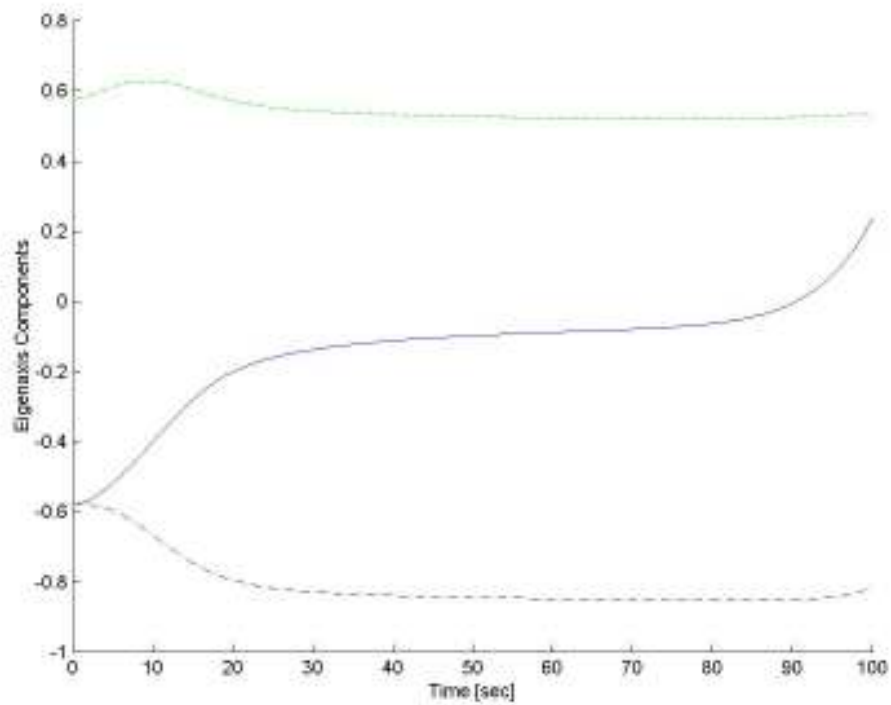**Figure 3-29 Miscellaneous disturbance torques**
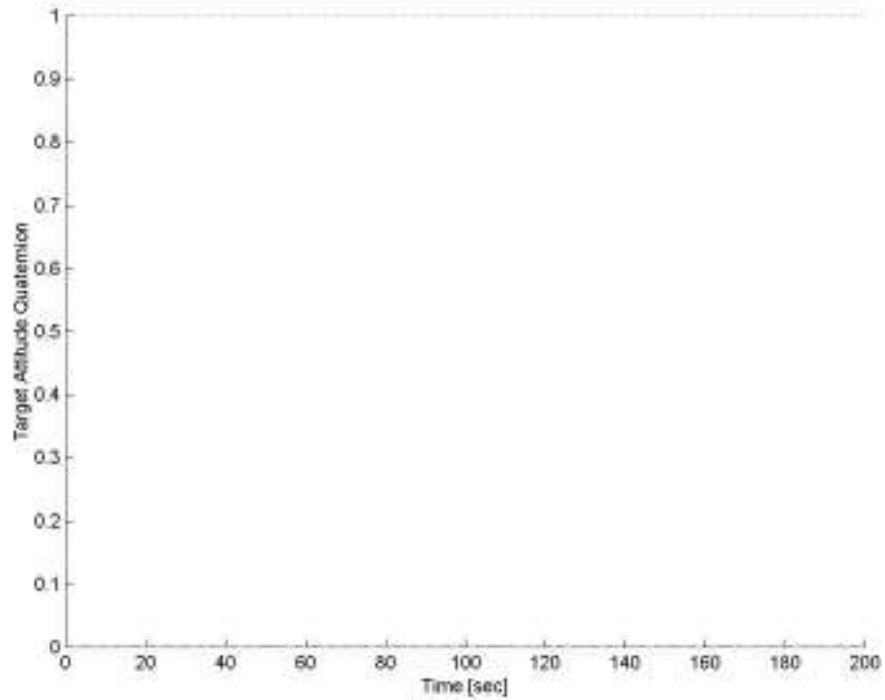
Figure 3-30 Reference trajectory attitude quaternion
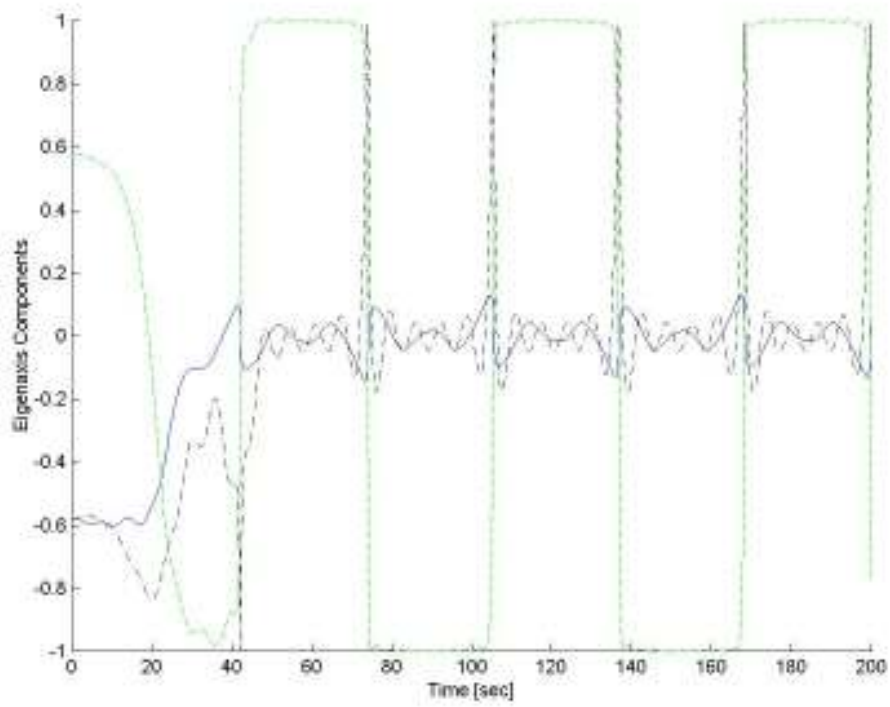
Figure 3-31 Euler axis components of tracking error quaternion
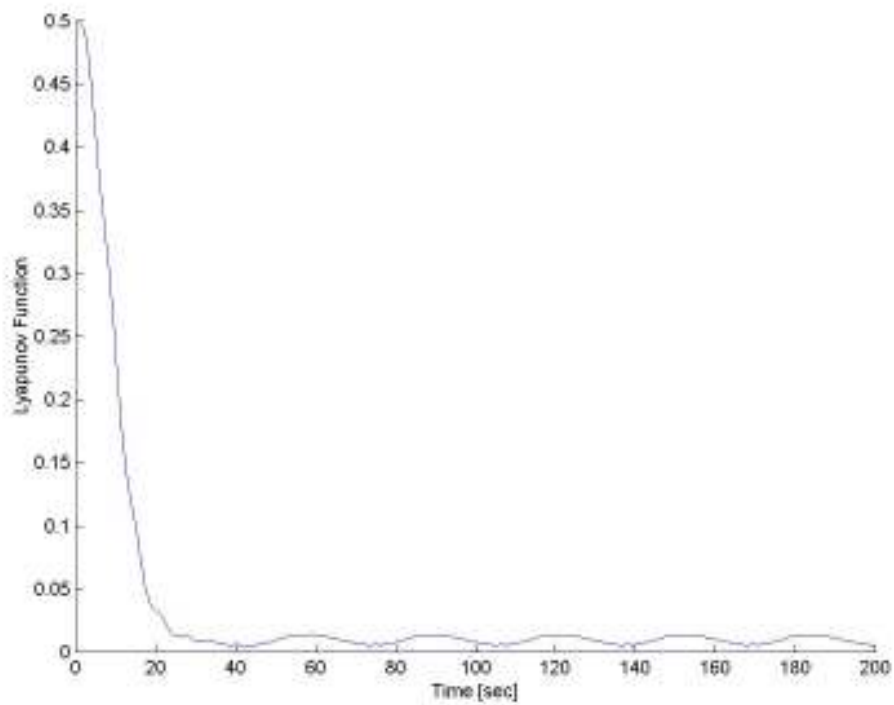
Figure 3-32 Lyapunov function
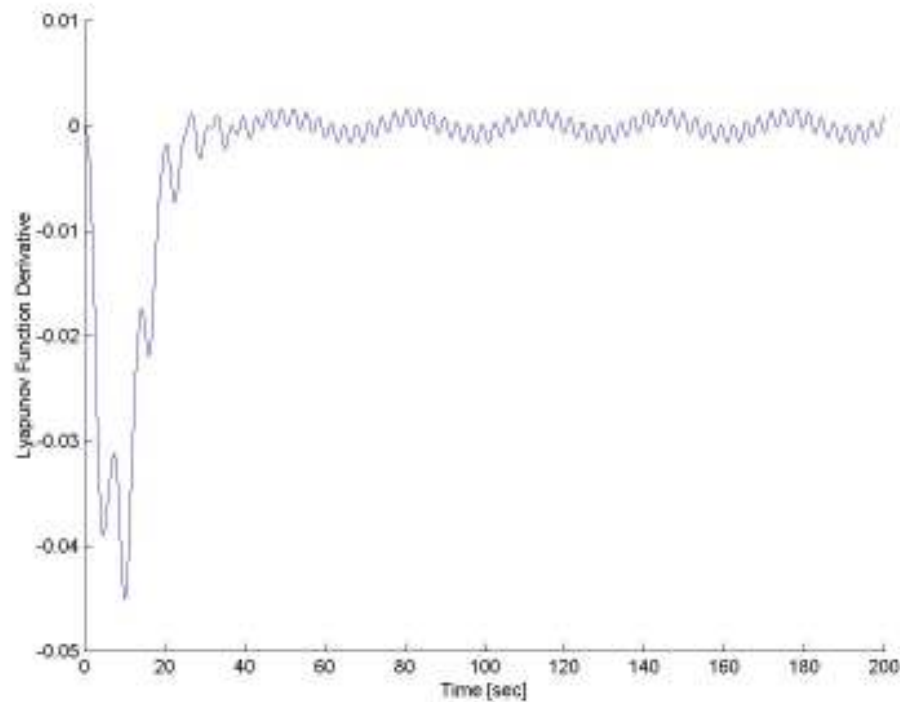
Figure 3-33 Lyapunov function derivative

3.9.5 Disturbance Torque Rejection

This example repeats the rest-to-rest maneuver of Section 3.9.1 with the different initial conditions shown in Table 3-3 and subject to the both a gravity-gradient torque and a miscellaneous torque given by Eq (117). The initial Euler axis/angle parameterisation corresponds to a 90 deg rotation with respect to the minimum angle maneuver equilibrium point $\mathbf{q_d}(t)$. The feedback control law is the disturbance rejection controller defined by Eq (61) with associated control gain update law given by Eq (64).

Figure 3-34 through Figure 3-43 present the results of the rest-to-rest attitude maneuver. The error quaternion settles to the correct minimum-angle maneuver equilibrium point with a settling time of 80 sec. The $\mathrm{sgn}[\delta q_4(t)]$ term in the sliding vector definition of Eq (11) in Chapter 4 has not been applied to this maneuver. Figure 3-40 demonstrates that Wie's theorem of an eigenaxis rotation for rest-to-rest maneuvers is not valid for the disturbance rejecting control

141

strategy of Eqs (61) and (64). Comparing the results of Section 3.9.4 with no disturbance rejection capabilities, the control strategy is effective in rejecting the presence of the sinusoidal disturbance torques. A key observation, however, is that the theoretical asymptotic stability result developed in Section 3.7 is not observed in the MATLAB™ simulation. Instead the feedback control law dilutes the effect of the miscellaneous disturbance torque on the steady-state trajectory tracking errors but does not completely reject the disturbance. Figure 3-34 and Figure 3-35 demonstrate that the bounded residual set is significantly smaller than for the uncompensated case in Section 3.9.4.

The time-varying control parameter $k(t)$ illustrated in Figure 3-43 converges to a finite positive steady-state value in approximately 80 sec which is reflected in the Lyapunov function of Figure 3-41. Furthermore this value is sufficiently large to ensure that control torque chattering does not occur. The effect of the disturbance torque in generating a residual set is not observed in Figure 3-41 due to the dominance of the control gain parameter $k(t)$ quadratic term in the Lyapunov function. The external control torques illustrated in Figure 3-36 are saturated during the initial 70 sec of the attitude maneuver and subsequently compensate exactly for the miscellaneous disturbance torques. It is interesting to observe that the adaptation of the control torques to the general disturbance environment occurs automatically. Although the disturbance rejection characteristics are excellent as illustrated in Figure 3-35 the spacecraft angular rate function displays sharp discontinuities. This characteristic may result in the excitation of unmodelled dynamics and flexible modes of the spacecraft which are undesirable effects. The discontinuity points in the angular rate function correspond to near-infinite derivatives which are clearly visible in the Lyapunov function derivative of Figure 3-42. One possible method of overcoming this discontinuous behaviour is to modify the control gain parameters to trade system convergence time and smoothness of the angular rate function. The main difficulty is to ensure that during the convergence of the control gain parameter $k(t)$ its value does not become small enough to create control torque chattering.

Parameters	Values
Spacecraft Inertia Matrix	$\mathbf{J} = \begin{bmatrix} 600.28 & -3.57 & 0.17 \\ -3.57 & 611.44 & 0.25 \\ 0.17 & 0.25 & 507.90 \end{bmatrix}$
Reference Trajectory	$\mathbf{q_d} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\boldsymbol{\omega_d} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
Initial Conditions	$\mathbf{q}(t_0) = \begin{bmatrix} \sqrt{1/6} & -\sqrt{1/6} & \sqrt{1/6} & \sqrt{1/2} \end{bmatrix}^T$ $\hat{\mathbf{e}}(t_0) = \begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \end{bmatrix}^T$ $\theta(t_0) = 90 \ \mathrm{deg}$ $\boldsymbol{\omega}(t_0) = [10.0 \quad -10.0 \quad 5.0]^T$
Control Gains	$u_{max} = 20 \ \mathrm{Nm}$ $\delta = 0.001$ $\gamma = 0.0001$ $k(t_0) = 5.0$

Table 3-3 Rest-to-Rest Attitude Maneuver (with disturbance rejection)
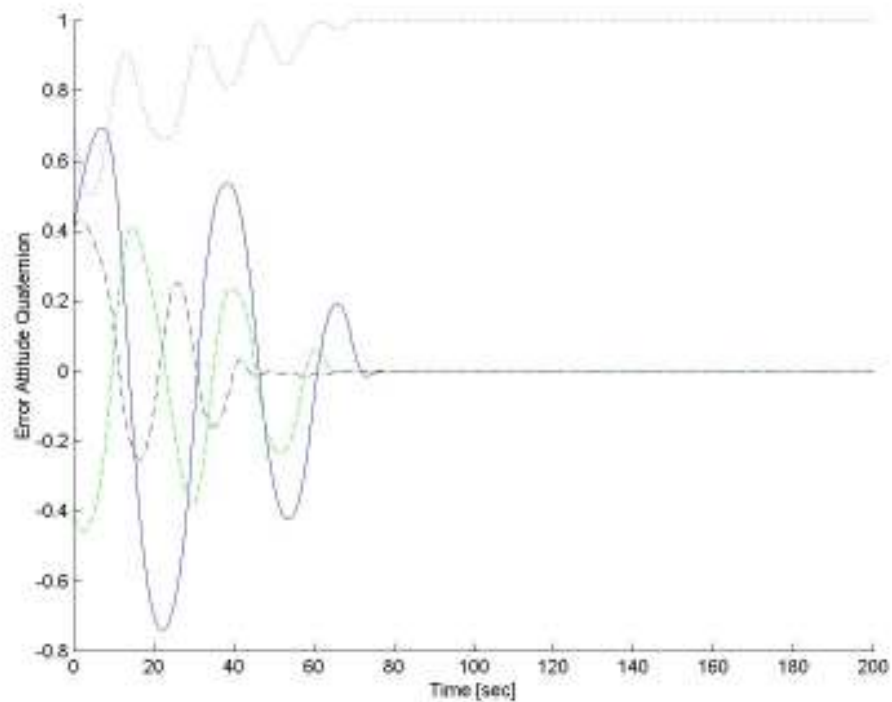
Figure 3-34 Spacecraft attitude quaternion tracking error
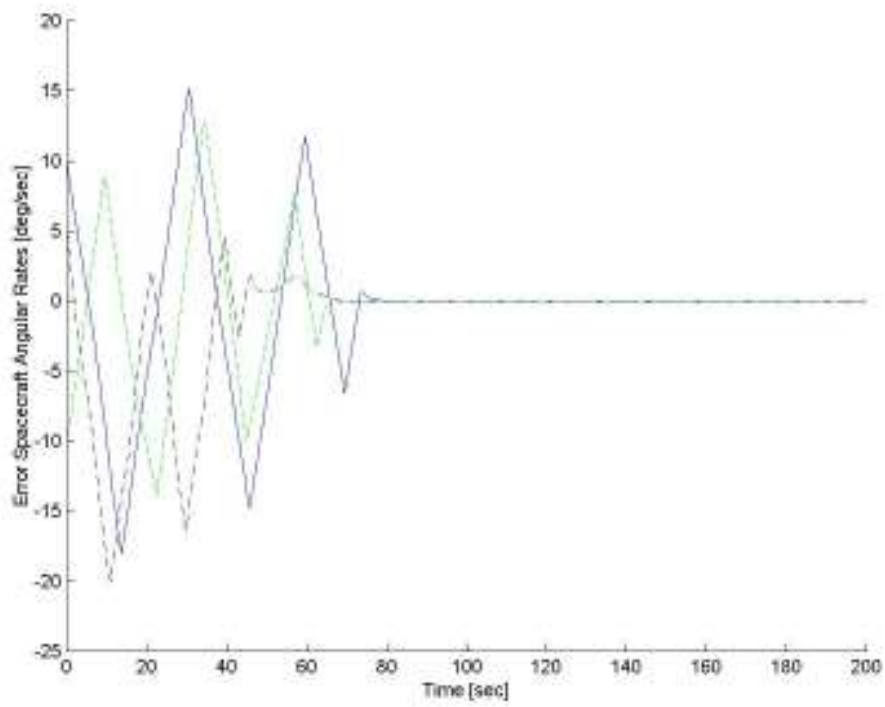
Figure 3-35 Spacecraft angular rates
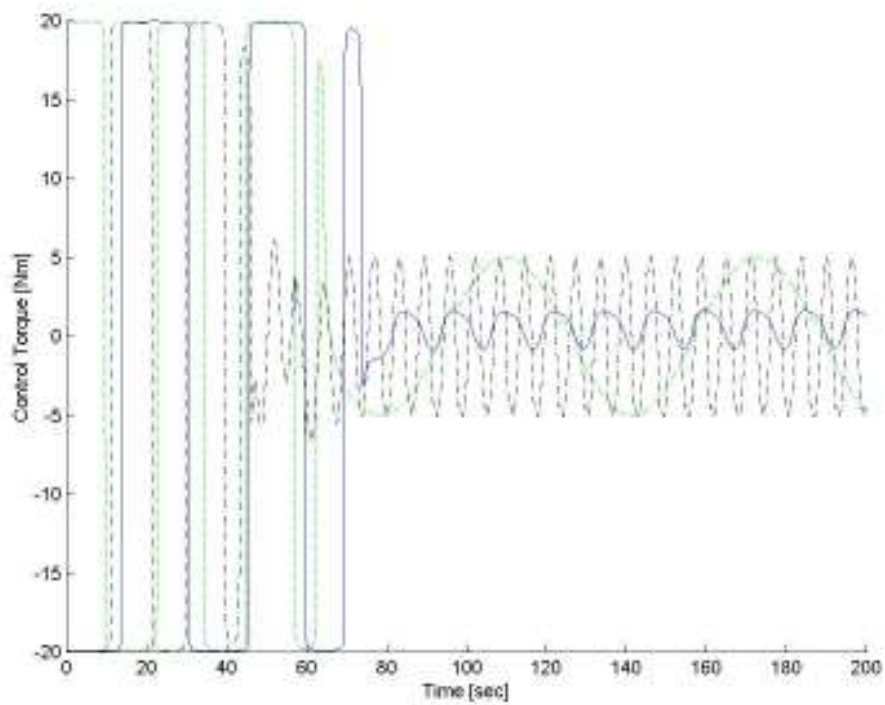
Figure 3-36 External control torques

Figure 3-37 Gravity-gradient disturbance torques

Figure 3-38 Miscellaneous disturbance torques

Figure 3-39 Reference trajectory attitude quaternion

Figure 3-40 Euler axis components of tracking error quaternion
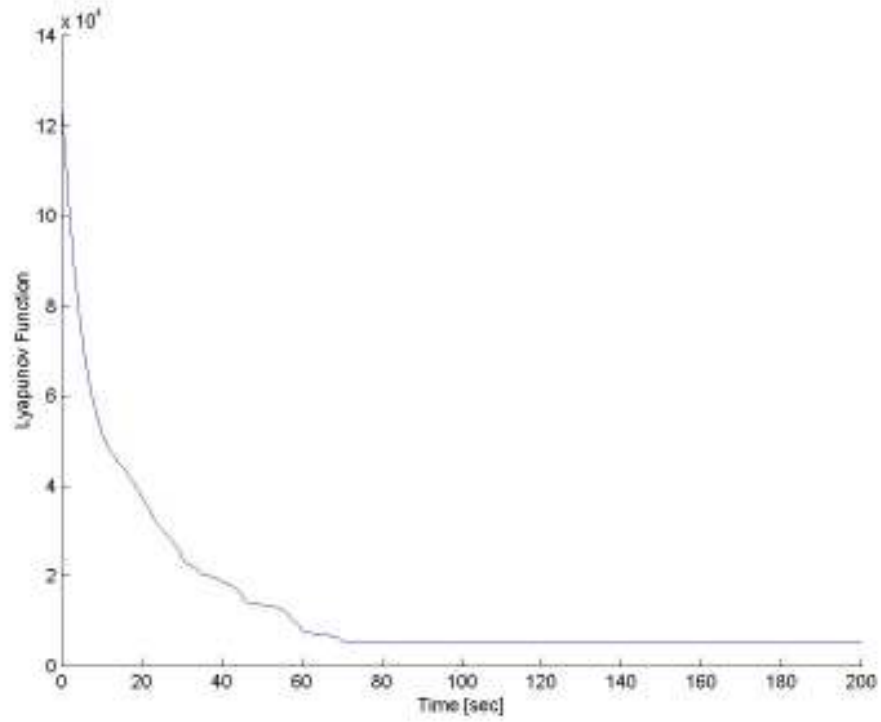
Figure 3-41 Lyapunov function

Figure 3-42 Lyapunov function derivative

147

Figure 3-43 Dynamic control parameter

3.9.6 Disturbance and State Estimation Error Rejection

This example repeats the rest-to-rest maneuver of Section 3.9.5 using the feedback control law and parameter update law defined by Eqs (87) and (102) respectively. The gravity-gradient disturbance torque and a miscellaneous torque given by Eq (117) are present in the simulation. The simulation initial conditions and control parameters are shown in Table 3-4.

The state estimation process is performed using a deterministic observer developed by Thienel and Sanner[48]. This design is based on exact measurements of the spacecraft attitude quaternion and gyroscope measurements produced using the gyroscope model in Reference 49. The gyroscope drift-rate ramp noise parameter is $\sigma_u = 1 \times 10^{-7}$ rad/sec$^{3/2}$, and the gyroscope measurement noise parameter is $\sigma_v = 1 \times 10^{-4}$ rad/sec$^{1/2}$. The estimation errors in the spacecraft attitude quaternion and angular rates are presented in Figure 3-54 and Figure 3-55 respectively. The presence of the gyroscope noise sources is evident in the

spacecraft angular rate estimates, and therefore enters the control torque through the angular rate tracking commands. Consequently, the trajectory tracking commands are superimposed with a stochastic process, which would lead to problems in practical applications. Furthermore, it is evident from Figure 3-46 that the gyroscope noise sources accumulate over time in the control torques. This scenario is unacceptable for practical applications and a recursive state estimator (for example an extended Kalman filter) should be implemented to provide smooth state estimates.

Parameters	*Values*
Spacecraft Inertia Matrix	$\mathbf{J} = \begin{bmatrix} 600.28 & -3.57 & 0.17 \\ -3.57 & 611.44 & 0.25 \\ 0.17 & 0.25 & 507.90 \end{bmatrix}$
Reference Trajectory	$\mathbf{q_d} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\mathbf{\omega_d} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
Initial Conditions	$\mathbf{q}(t_0) = \begin{bmatrix} \sqrt{1/6} & -\sqrt{1/6} & \sqrt{1/6} & \sqrt{1/2} \end{bmatrix}^T$ $\hat{\mathbf{e}}(t_0) = \begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \end{bmatrix}^T$ $\theta(t_0) = 90 \ \text{deg}$ $\mathbf{\omega}(t_0) = \begin{bmatrix} 10.0 & -10.0 & 5.0 \end{bmatrix}^T$ $\mathbf{b}(t_0) = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}^T \ \text{deg/sec}$ $\hat{\mathbf{b}}(t_0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \ \text{deg/sec}$ $\hat{\mathbf{q}}(t_0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$
Control Gains	$u_{max} = 20 \ \text{Nm}$ $\alpha = 0.3$ $\beta = 0.01$ $\gamma_p = 0.75$ $a^2 = 2.25$ $c^2 = 101$ $p(t_0) = 10.0$

Table 3-4 Rest-to-Rest Attitude Maneuver (with disturbance/state estimation rejection)

Figure 3-44 though Figure 3-53 present the results of the rest-to-rest attitude maneuver. The error quaternion settles to the desired equilibrium point with a

settling time of 200 sec. The $\text{sgn}[\delta\hat{q}_4(t)]$ term in the control law of Eq (110) has not been applied to this maneuver. Figure 3-50 demonstrates that Wie's theorem of an eigenaxis rotation is not valid for this control strategy.

Figure 3-45 clearly demonstrates the disturbance rejection capabilities of the control strategy defined by Eqs (87) and (102). Identical to the results in Section 3.9.5, the theoretical asymptotic stability result developed in Section 3.8 is not observed in the MATLAB™ simulation. However, the sharp discontinuities in the spacecraft angular rates of Figure 3-35 are not present in Figure 3-45. This is due to the selection of the system parameters which ensure smoother spacecraft angular rates, at the expense of increased system convergence time. However, since the maneuver time is larger, the dynamic control gain parameter $p(t)$ has a larger period of time to decrease. If its value becomes too small (prior to system convergence) then the spacecraft angular rate commands will be excessively amplified by the control law of Eq (87). Eventually, the hyperbolic tangent component of the control law will produce invalid values.
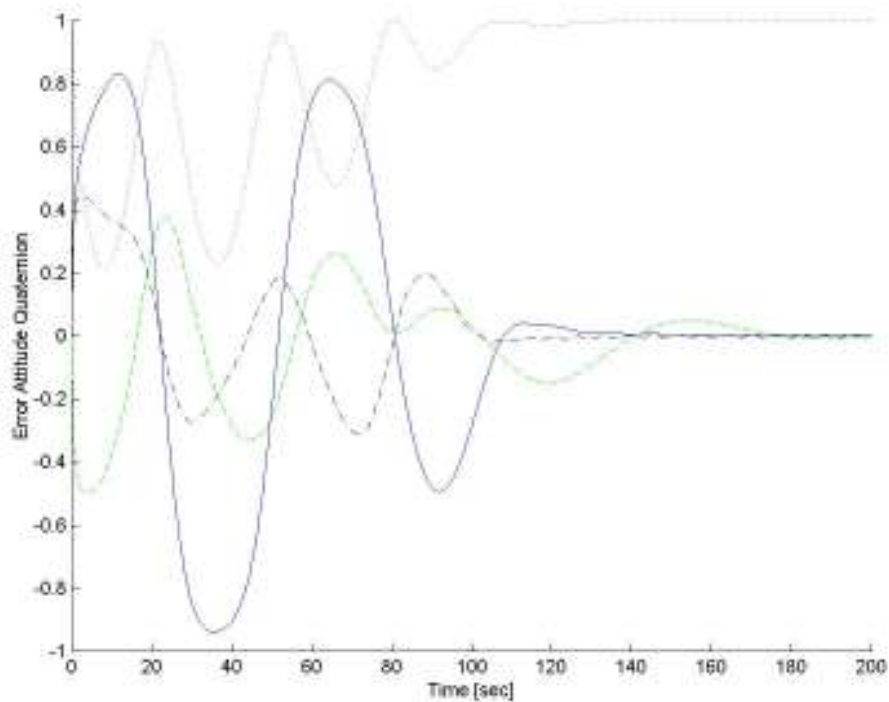
Figure 3-44 Spacecraft attitude quaternion tracking error
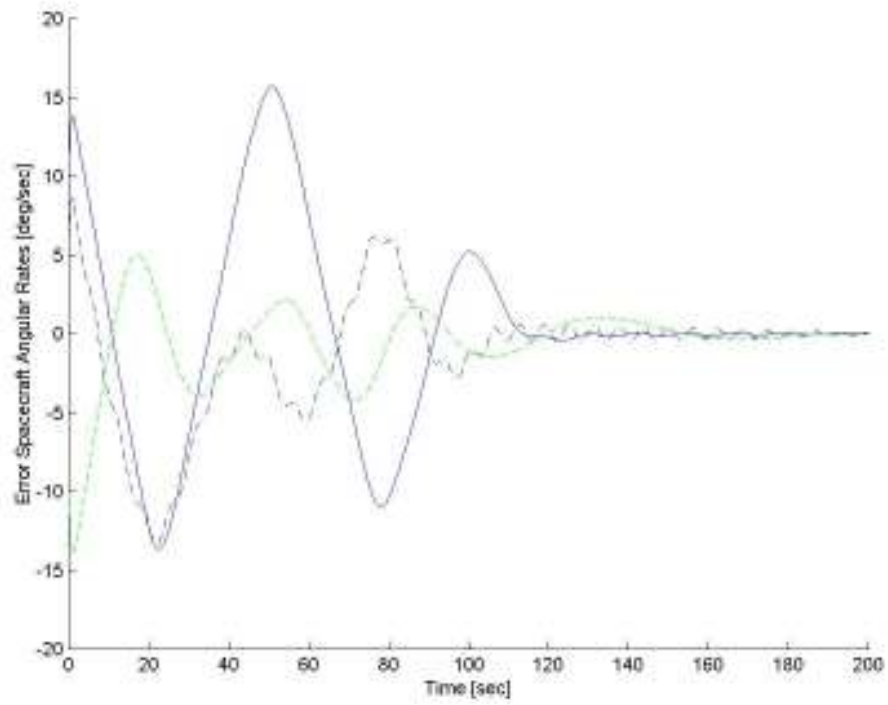
Figure 3-45 Spacecraft angular rates
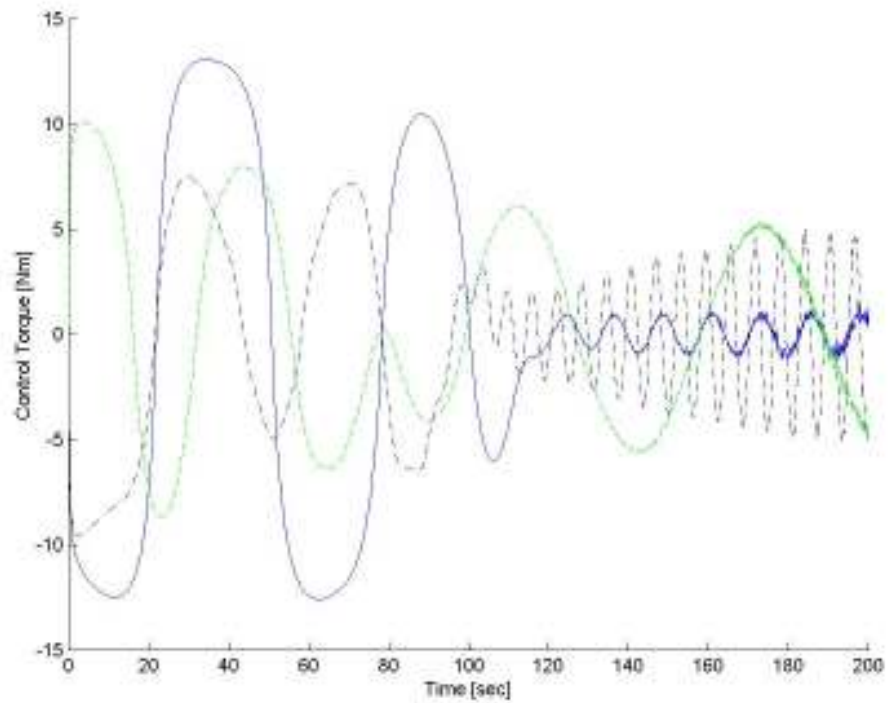
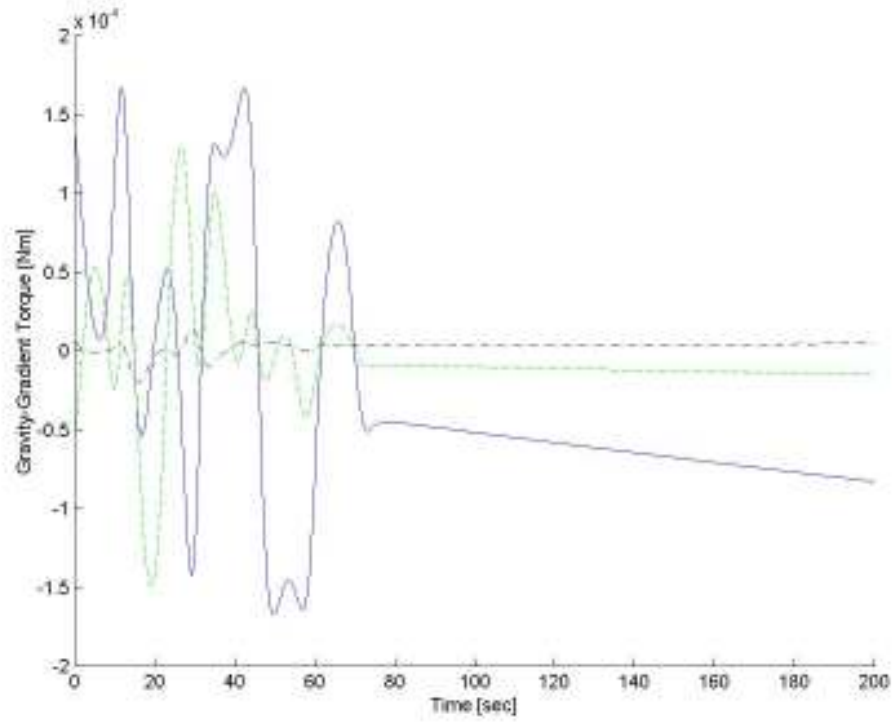**Figure 3-46 External control torques**
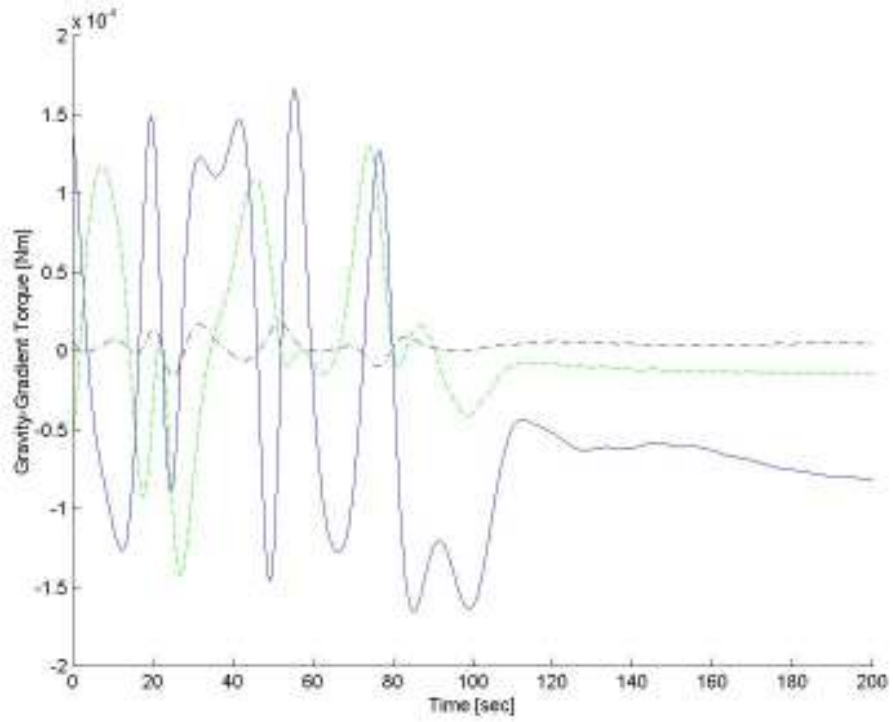
Figure 3-47 Gravity-gradient disturbance torques
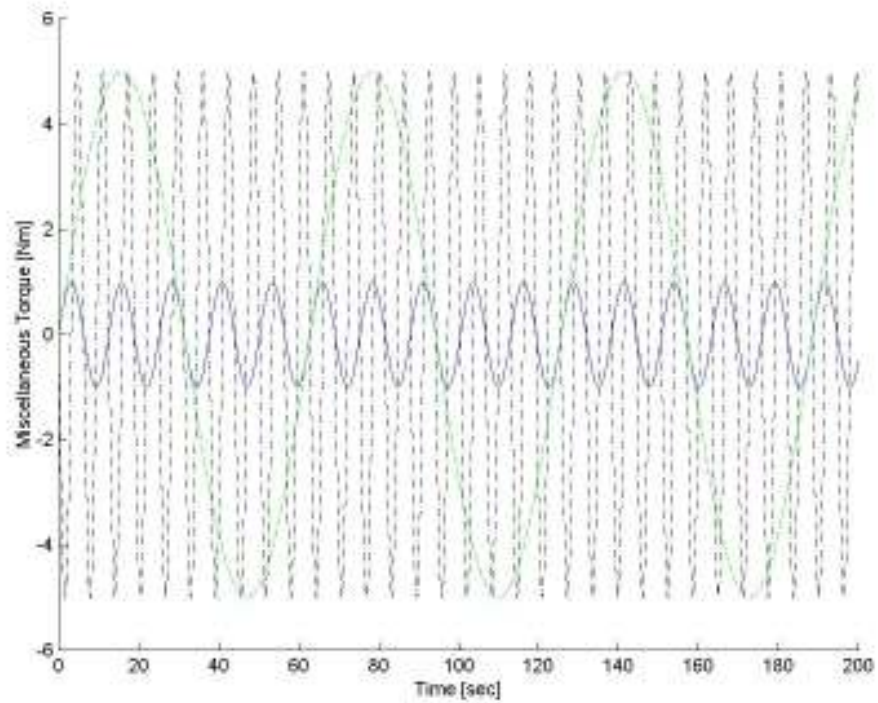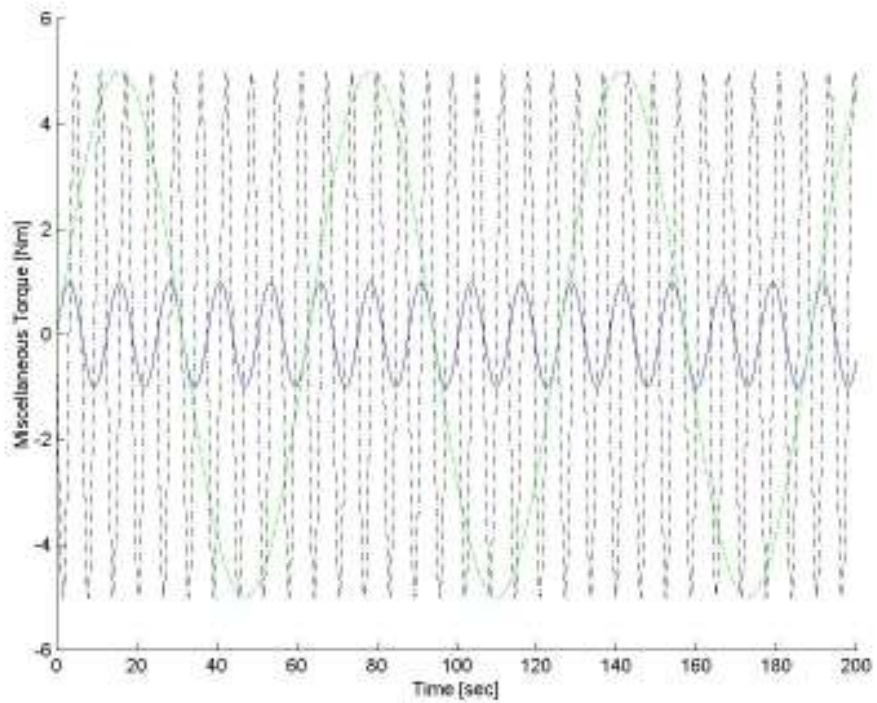

Figure 3-48 Miscellaneous disturbance torques
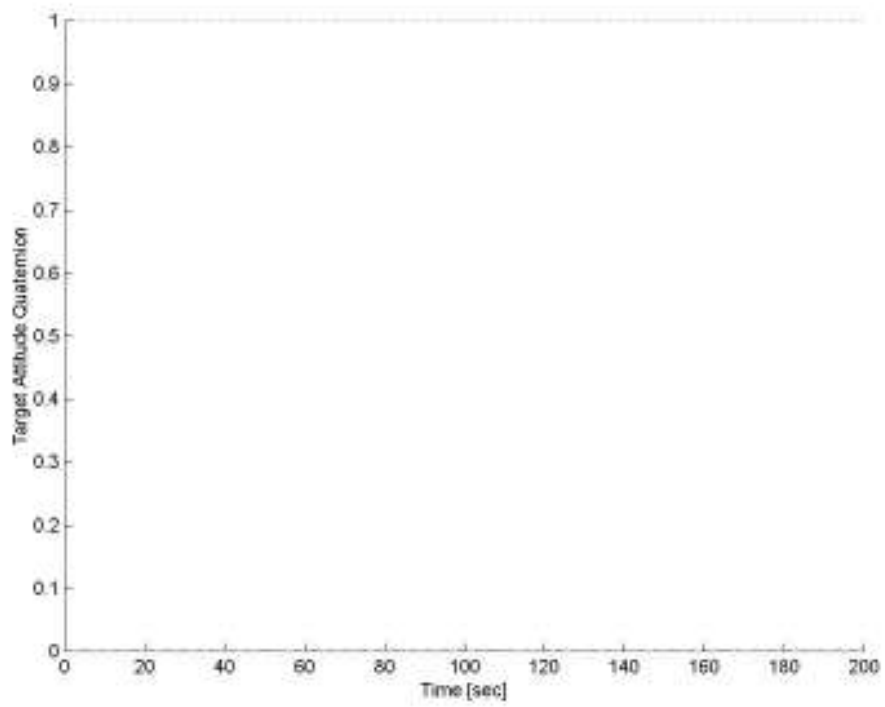
152
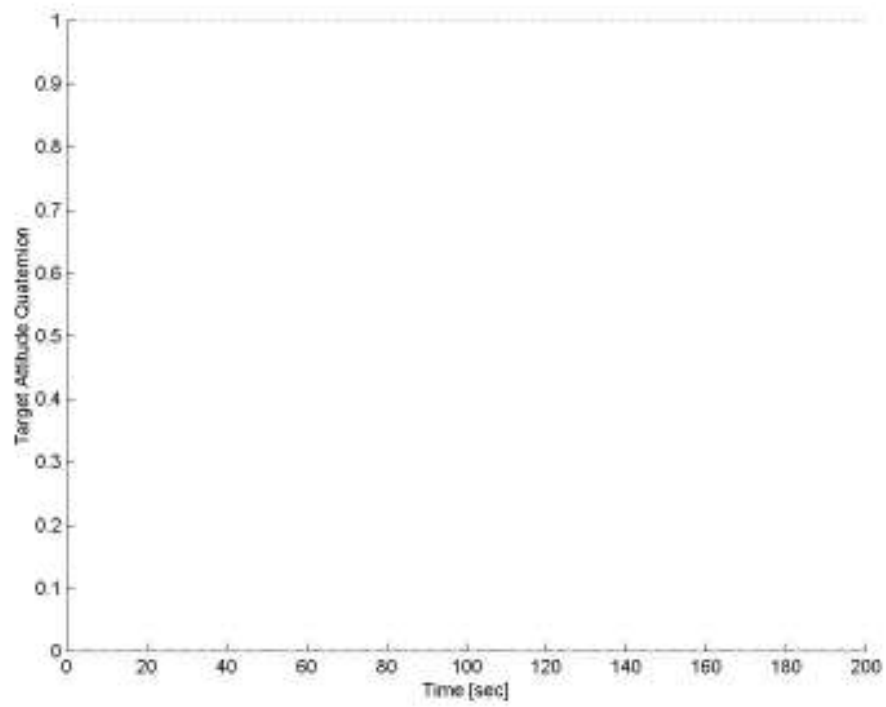
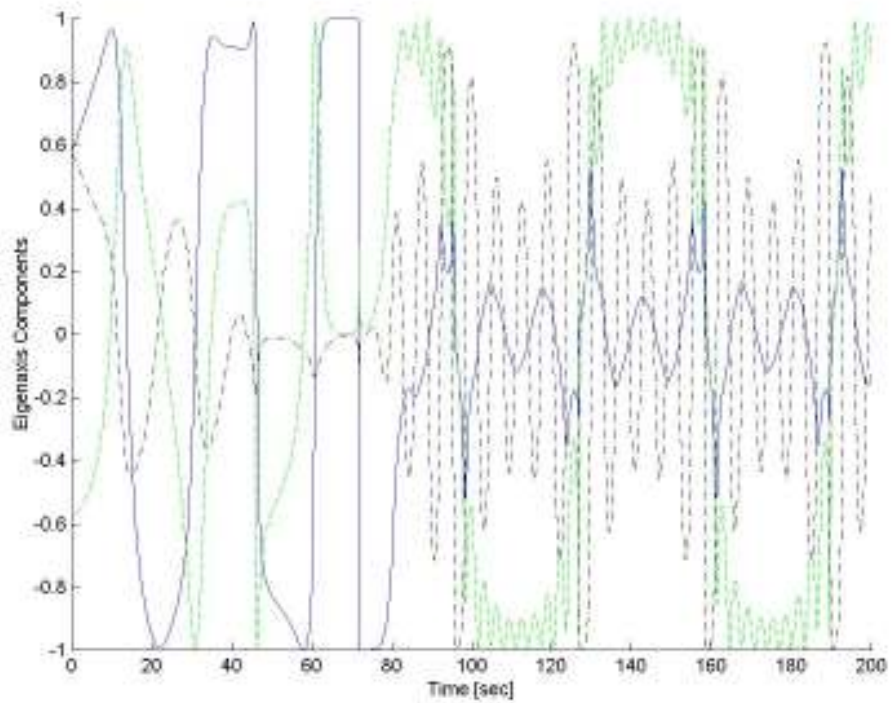**Figure 3-49 Reference trajectory attitude quaternion**
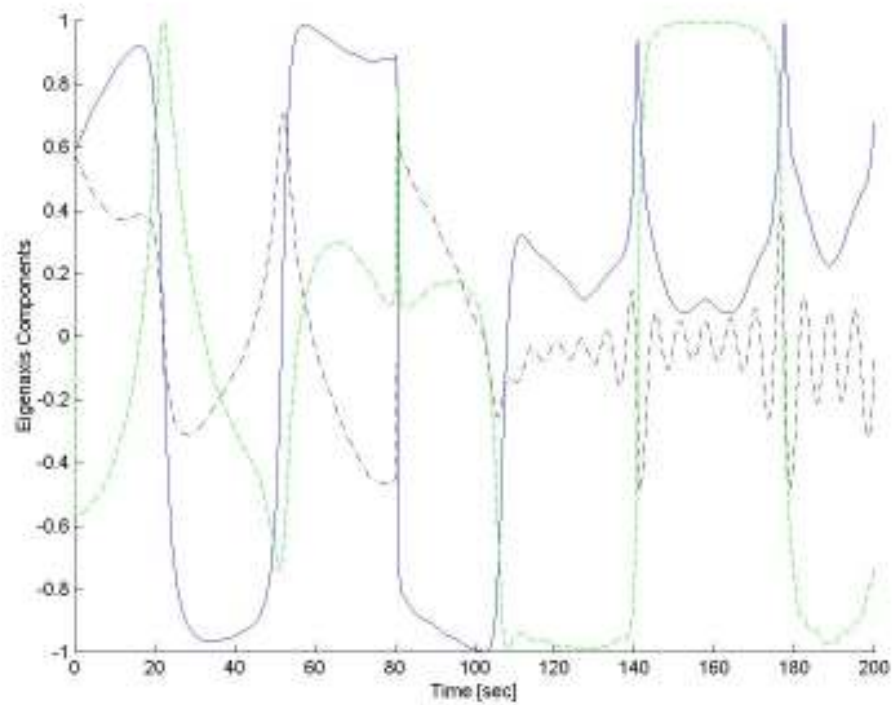
**Figure 3-50 Euler axis components of tracking error quaternion**
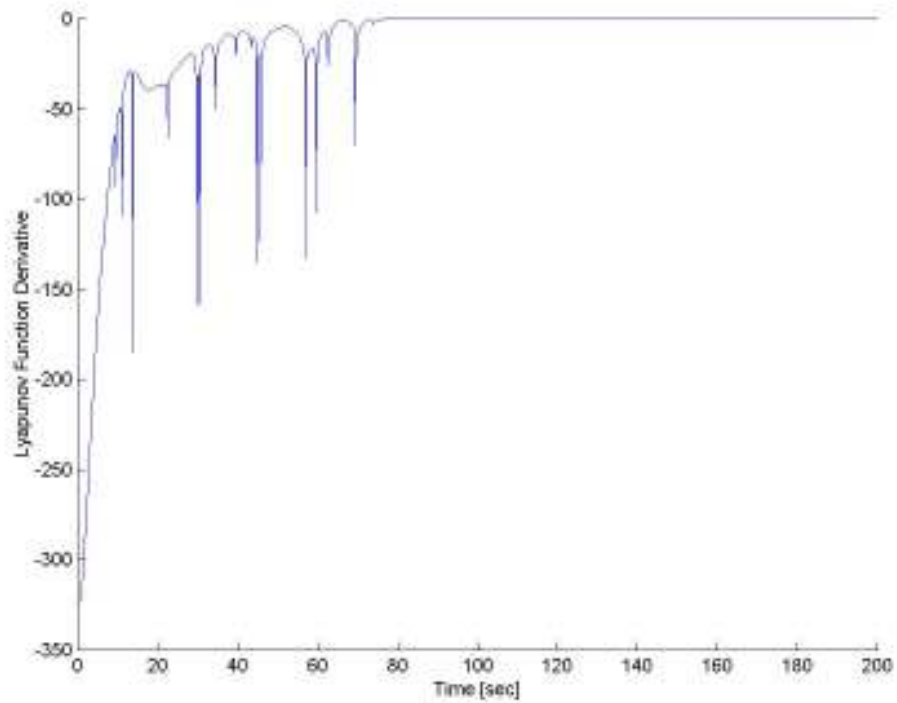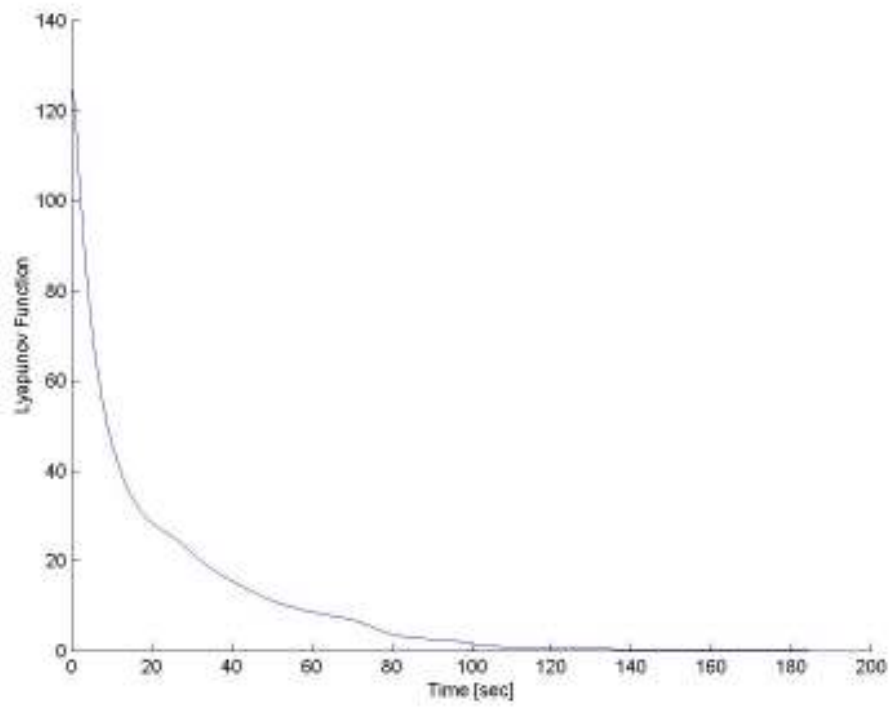
Figure 3-51 Lyapunov function
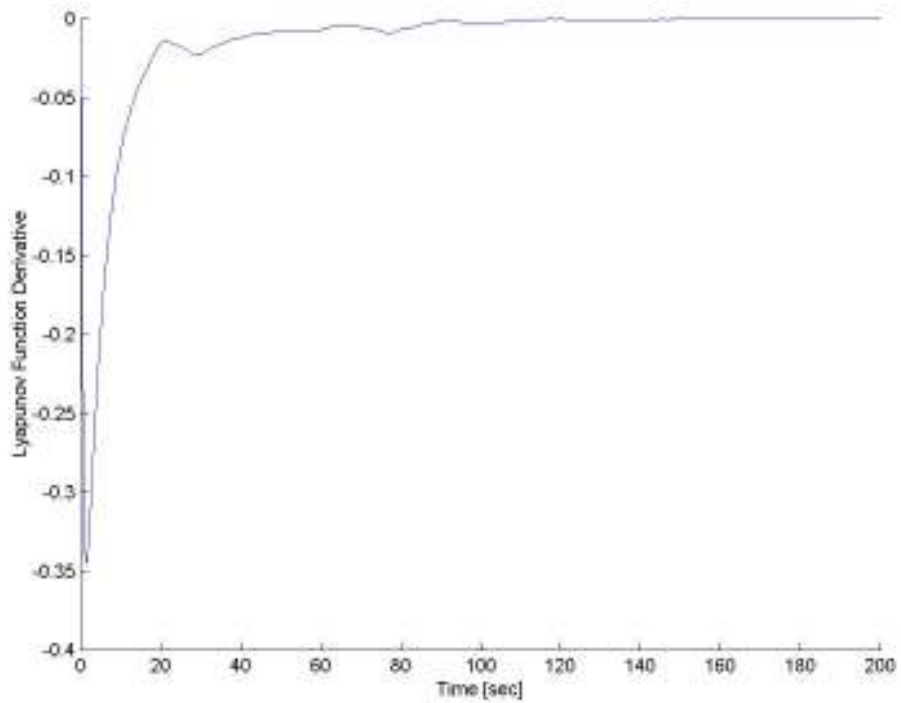
Figure 3-52 Lyapunov function derivative

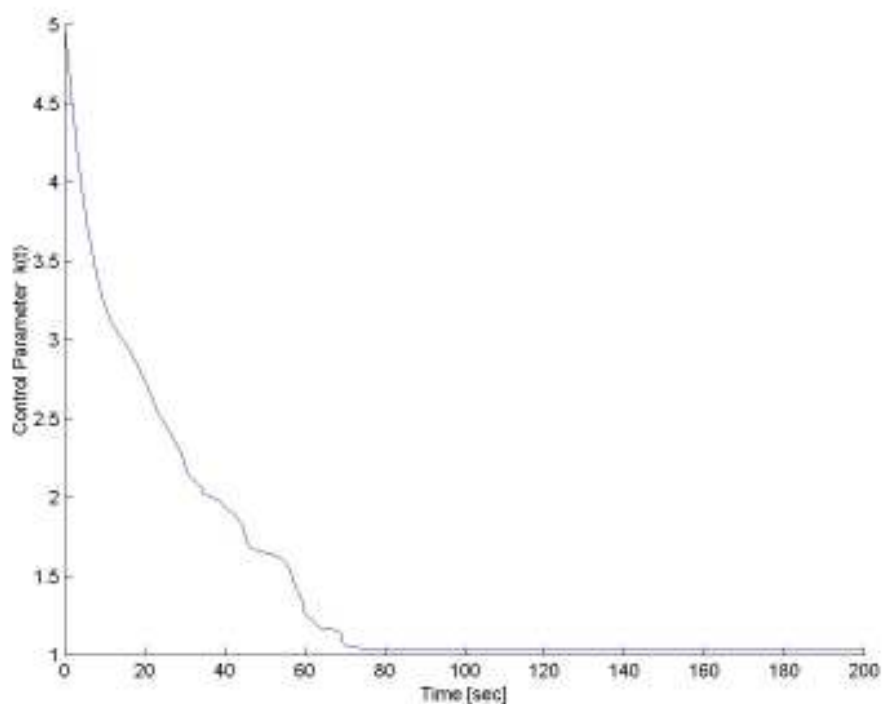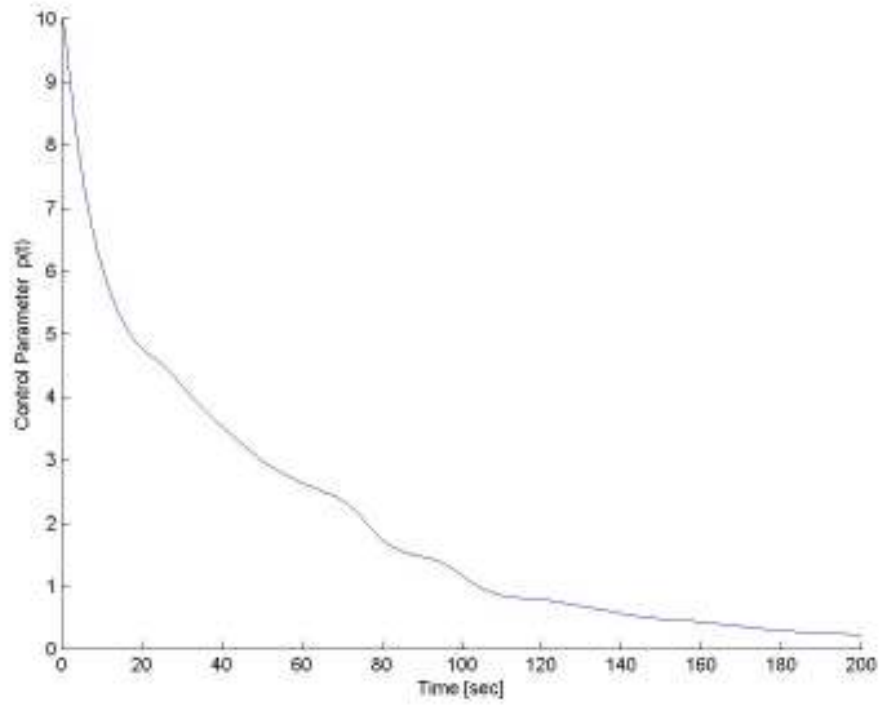**Figure 3-53 Dynamic control parameter**
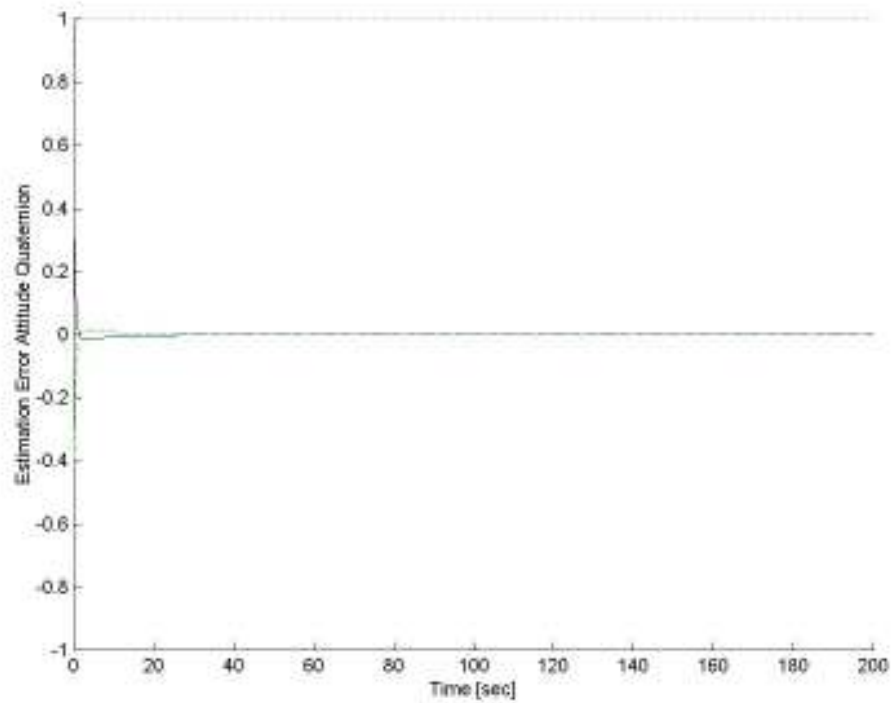
Figure 3-54 Spacecraft attitude quaternion observer estimation error

155
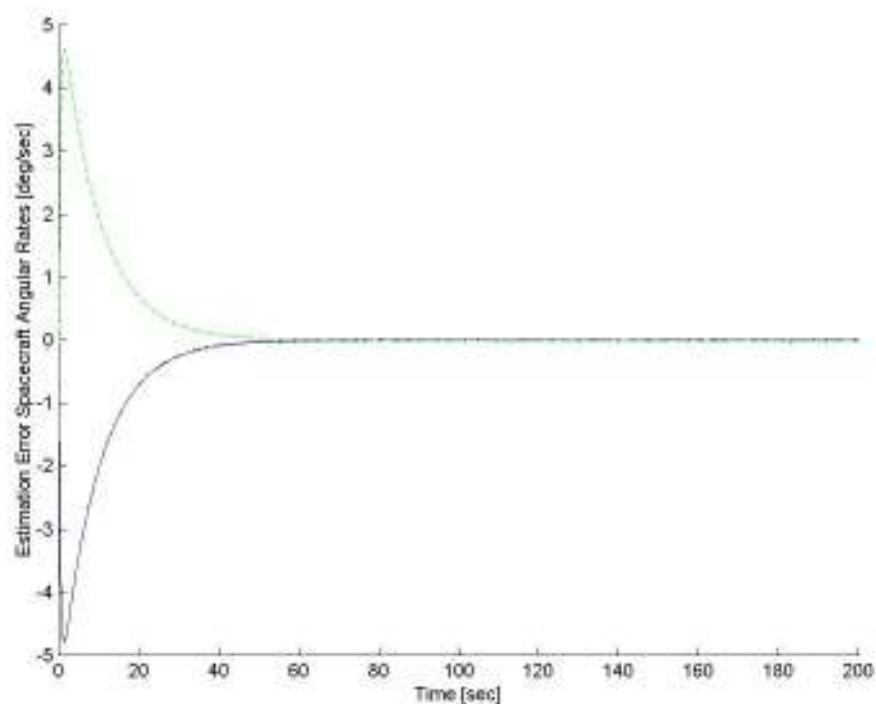
Figure 3-55 Spacecraft angular rates observer estimation error

3.10 Conclusion

In this chapter, various feedback control laws were developed for attitude tracking maneuvers based on Lyapunov's direct method. A novel control strategy was proposed which accounts for spacecraft inertia uncertainty, control torque magnitude saturation, bounded persistent external disturbances, and sensor measurement error. MATLAB™ simulation results were presented to demonstrate and validate the performance of the various control algorithms.

3.11 References

[1] LaSalle, J., and S. Lefschetz. (1961). *Stability by Liapunov's direct method: with applications*. USA: Academic Press Inc.

[2] Meirovitch, L. (1970). *Methods of analytical dynamics*. USA: McGraw-Hill.

[3] Vidyasagar, M. (1978). *Nonlinear systems analysis*. USA: Prentice-Hall, Inc.

[4] Slotine, J-J.E., and W. Li. (1991). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

[5] Junkins, J.L., and Y. Kim. (1993). *Introduction to dynamics and control of flexible structures*. USA: American Institute of Aeronautics and Astronautics, Inc.

[6] Skaar, S.B., and C.F. Ruoff, eds. (1994). *Teleoperation and robotics in space, Progress in Astronautics and Aeronautics Volume 161*. USA: American Institute of Aeronautics and Astronautics, Inc.

[7] Sastry, S. (1999). *Nonlinear systems: analysis, stability, and control*. USA: Springer-Verlag, Inc.

[8] Schaub, H., and J.L. Junkins. (2003). *Analytical mechanics of space systems*. USA: American Institute of Aeronautics and Astronautics, Inc.

[9] Lyapunov, A.M. (1907). Problème général de la stabilité du movement. *Ann. Fac. Sci. Toulouse*, 9: 203-474.

[10] Kalman, R.E., and J.E. Bertram. (1960). Control system analysis and design via the second method of Lyapunov, Part I: continuous-time systems. *Journal of Basic Engineering*, 82(2): 371-393.

[11] Kalman, R.E., and J.E. Bertram. (1960). Control system analysis and design via the second method of Lyapunov, Part II: discrete-time systems. *Journal of Basic Engineering*, 82(2): 394-400.

[12] LaSalle, J.P. (1960). Some extensions of Liapunov's second method. *IRE Transactions on Circuit Theory*, 7(4): 520-527.

[13] Mortensen, R.E. (1968). A globally stable linear attitude regulator. *International Journal of Control*, 8(3): 297-302.

[14] Vadali, S.R., and J.L. Junkins. (1984). Optimal open-loop and stable feedback control of rigid spacecraft attitude maneuvers. *The Journal of the Astronautical Sciences*, 32(2): 105-122.

[15] Wie, B., and P.M. Barba. (1985). Quaternion feedback for spacecraft large-angle maneuvers. *The Journal of Guidance, Control, and Dynamics*, 8(3): 360-365.

[16] Wie, B., Weiss, H., and A. Arapostathis. (1989). Quaternion feedback regulator for spacecraft eigenaxis rotations. *The Journal of Guidance, Control, and Dynamics*, 12(3): 375-380.

[17] Wen, J.T-Y., and K. Kreutz-Delgado. (1991). The attitude control problem. *IEEE Transactions on Automatic Control*, 36(10): 1148-1162.

[18] Oh, H.-S., Vadali, S.R., and J.L. Junkins. (1992). Use of the work-energy rate principle for designing feedback control laws. *The Journal of Guidance, Control, and Dynamics*, 15(1): 275-277.

[19] Mukherjee, R., and J.L. Junkins. (1993). Invariant set analysis of the Hub-Appendage problem. *The Journal of Guidance, Control, and Dynamics*, 16(6): 1191-1193.

[20] McInnes, C.R. (1994). Large angle slew maneuvers with autonomous sun vector avoidance. *The Journal of Guidance, Control, and Dynamics*, 17(4): 875-877.

[21] Tsiotras, P. (1994). New control laws for the attitude stabilisation of rigid bodies. *Proceedings of the IFAC Symposium on Automatic Control in Aerospace*, 316-321.

[22] Bell, M.J., and J.L. Junkins. (1994). Near minimum-time three dimensional maneuvers of rigid and flexible spacecraft. *The Journal of the Astronautical Sciences*, 42(4): 421-438.

[23] Wie, B., and J. Lu. (1995). Feedback control logic for spacecraft eigenaxis rotations under slew rate and control constraints. *The Journal of Guidance, Control, and Dynamics*, 18(6): 1372-1379.

[24] Lizarralde, F., and J.T. Wen. (1996). Attitude control without angular velocity measurement: a passivity approach. *IEEE Transactions on Automatic Control*, 41(3): 468-472.

[25] Creamer, G., DeLaHunt, P., Gates, S., and M. Levenson. (1996). Attitude determination and control of Clementine during lunar mapping. *The Journal of Guidance, Control, and Dynamics*, 19(3): 505-511.

[26] Tsiotras, P. (1996). Stabilization and optimality results for the attitude control problem. *The Journal of Guidance, Control, and Dynamics*, 19(4): 772-779.

[27] Schaub, H., Robinett, R.D., and J.L. Junkins. (1996). Globally stable feedback laws for near-minimum-fuel and near minimum-time pointing maneuvers for a landmark-tracking spacecraft. *The Journal of the Astronautical Sciences*, 44(4): 443-466.

[28] Junkins, J.L. (1997). Adventures on the interface of dynamics and control. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1058-1071.

[29] Robinett, R.D., et al. (1997). Lyapunov optimal saturated control for nonlinear systems. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1083-1088.

[30] Tsiotras, P., and J. Luo. (1997). Reduced effort control laws for underactuated rigid spacecraft. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1089-1095.

[31] Hall, C.D., Tsiotras, P., and H. Shen. (1998). Tracking rigid body motion using thrusters and momentum wheels. *Proceedings of the* AIAA/AAS Astrodynamics Specialist Conference, Boston, Massachusetts, USA, AIAA Paper 98-4471.

[32] Tsiotras, P. (1998). Further passivity results for the attitude control problem. *IEEE Transactions on Automatic Control*, 43(11): 1597-1600.

[33] Boskovic, J.D., Sai-Ming, Li, and R.K. Mehra. (1999). Globally stable adaptive tracking control design for spacecraft under input saturation. *Proceedings of the IEEE Conference on Decision and Control*, December 1999, 1952-1957.

[34] Hablani, H.B. (1999). Attitude commands avoiding bright objects and maintaining communication with ground station. *The Journal of Guidance, Control, and Dynamics*, 22(6): 759-767.

[35] Akella, M.R. (2001). Rigid body attitude tracking without angular velocity feedback. *Systems and Control Letters*, 42(4): 321-326.

[36] Tsiotras, P., Shen, H., and C. Hall. (2001). Satellite attitude control and power tracking with energy/momentum wheels. *Journal of Guidance, Control, and Dynamics*, 24(1): 23-34.

[37] Wie, B., Bailey, D., and C. Heiberg. (2002). Rapid multitarget acquisition and pointing control of agile spacecraft. *The Journal of Guidance, Control, and Dynamics*, 25(1): 96-103.

[38] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2004). Robust tracking control design for spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 27(4): 627-633.

[39] Subbarao, K., and M.R. Akella. (2004). Differentiator-free nonlinear proportional-integral controllers for rigid-body attitude stabilization. *The Journal of Guidance, Control, and Dynamics*, 27(6): 1092-1096.

[40] Akella, M.R., Valdivia, A., and G.R. Kotamraju. (2005). Velocity-free attitude controllers subject to actuator magnitude and rate saturations. *The Journal of Guidance, Control, and Dynamics*, 28(4): 659-666.

[41] Boskovic, J.D., Bergstrom, S.E., and R.K. Mehra. (2005). Robust integrated flight control design under failures, damage, and state-dependent disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 902-917.

[42] Wallsgrove, R. J., and M.R. Akella. (2005). Globally stabilizing saturated attitude control in the presence of bounded unknown disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 957-963.

[43] Boskovic, J.D., Prasanth, R., and R.K. Mehra. (2007). Retrofit fault-tolerant flight control design under control effector damage. *The Journal of Guidance, Control, and Dynamics*, 30(3): 703-712.

[44] Wertz, J.R., ed. (1978). *Spacecraft attitude determination and control*. The Netherlands: D. Reidel Publishing Company.

[45] Shuster, M.D. (1993). A survey of attitude representations. *The Journal of the Astronautical Sciences*, 41(4): 439-517.

[46] Slotine, J.J.E., and W. Li. (1990). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

[47] Bhat, S.P., and D.S. Bernstein. (1998). A topological obstruction to global asymptotic stabilization of rotational motion and the unwinding phenomenon. Proceedings of the American Control Conference, Philadelphia, Pennsylvania, USA, 2785-2789.

[48] Thienel, J.K., and R.M. Sanner. (2003). A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise. IEEE Transactions on Automatic Control, 48(11): 2011-2015.

[49] Lefferts, E.J., Markley, F.L., and M.D. Shuster. (1982). Kalman filtering for spacecraft attitude estimation. The Journal of Guidance, Control, and Dynamics, 5(5): 417-429.

Chapter 4   Spacecraft Attitude Maneuvers Using Sliding Mode Control Theory

4.1 Introduction

In Section 3.3 of Chapter 3, a Lyapunov stable feedback control law was designed for three-axis attitude maneuvering of rigid-body spacecraft. This control law was based on accurate knowledge of the spacecraft inertia matrix, state variable feedback information, and required the development of detailed functional models for all expected external disturbance torques, required for compensation in the feedback control law. In practice, the inertia parameters and state variables are not known exactly and the disturbance torque models may either not be available or available with relatively low accuracy. In addition, there will be dynamic model errors due to unanticipated disturbance torques and un-modelled dynamics such as actuator frictional effects. All sources of dynamic model uncertainty will contribute to the steady-state trajectory tracking error if not accounted for in the Lyapunov stability analysis. More recent methods (discussed in Sections 3.7 and 3.8 of Chapter 3) are generally capable of accounting for these effects through careful design of the feedback controller and associated control dynamic parameter update equation. In this chapter, the variable structure or sliding mode method will be applied to design Lyapunov stable control laws. These controllers provide robustness in the presence of spacecraft inertia uncertainty and bounded disturbance torques. The developments are not intended to supersede the results of Chapter 3, but rather provide an alternative approach for designing feedback control laws using the concept of a sliding vector. Chapter 5 will draw on the results of this chapter (along with several concepts in adaptive control theory) to develop Lyapunov stable controls which explicitly account for spacecraft inertia parameter uncertainty using an adaptive parameter update law.

The algorithms presented in Sections 4.2-4.4 are not novel contributions to the literature, but were developed in the early work by Vadali[6] and more recently by Crassidis, Vadali, and Markley[17]. Section 4.2 presents the design of a stable sliding manifold such that ideal spacecraft attitude motion on this manifold results in global asymptotic convergence of the trajectory tracking errors to

zero. Also in this section, a modification is made to the definition of the sliding manifold to ensure a minimum angular path maneuver to the sliding manifold. Section 4.3 considers a methodology for feedback control law design including a disturbance compensation term in the controller which mitigates the effects of spacecraft inertia matrix uncertainty and external disturbance torques. Section 4.4 provides an alternative design procedure based on the equivalent control principle[6,17]. In Section 4.5, MATLAB™ simulation results are presented to demonstrate the performance of the control algorithms in Section 4.3.

4.2 Sliding Manifold Design

To obtain the discontinuity surfaces a control law of the form $\boldsymbol{\omega}(t) = \boldsymbol{\omega}(\mathbf{q}_{13}(t))$ is sought which minimises the performance index

$$J(\boldsymbol{\omega}, \mathbf{q}_{13}) = \lim_{t \to \infty} \left[\frac{1}{2} \int_{t_s}^{t} \left\{ \rho \boldsymbol{\delta}\mathbf{q}_{13}^{T}(\tau) \boldsymbol{\delta}\mathbf{q}_{13}(\tau) + \boldsymbol{\omega}_{e}^{T}(\tau) \boldsymbol{\omega}_{e}(\tau) \right\} d\tau \right] \tag{1}$$

subject to the attitude kinematics of Eq (12) in Chapter 3, where $\boldsymbol{\delta}\mathbf{q}_{13}$ is the vector component of the tracking error quaternion defined by $\boldsymbol{\delta}\mathbf{q} = \mathbf{q} \otimes \mathbf{q}_{d}^{-1}$, $\boldsymbol{\omega}_{e}$ is the spacecraft angular rate tracking error defined by $\boldsymbol{\omega}_{e} = \boldsymbol{\omega} - \boldsymbol{\omega}_{d}$, ρ is a scalar weighting gain, and t_s is the time of arrival at the sliding manifold (sliding mode). Using the principles of optimal control theory the Hamiltonian associated with minimising Eq (1) subject to the kinematic equation constraints is defined as

$$H = \frac{1}{2} \rho \boldsymbol{\delta}\mathbf{q}_{13}^{T} \boldsymbol{\delta}\mathbf{q}_{13} + \frac{1}{2} \boldsymbol{\omega}_{e}^{T} \boldsymbol{\omega}_{e} + \boldsymbol{\lambda}^{T} \dot{\mathbf{q}} \tag{2}$$

where $\boldsymbol{\lambda}(t)$ is the co-state vector associated with the attitude kinematics. The necessary conditions for optimality are given by

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} \tag{3}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{q}} \tag{4}$$

$$\frac{\partial H}{\partial \boldsymbol{\omega}} = \mathbf{0}_{3\times1} \tag{5}$$

Using Eq (2) and Eqs (12), (18), (19), (21) in Chapter 3 the solution of Eqs (3)-(5) leads to the following two-point boundary value problem

$$\dot{\mathbf{q}} = \frac{1}{2}\boldsymbol{\Xi}(\mathbf{q})\boldsymbol{\omega} \tag{6}$$

$$\dot{\boldsymbol{\lambda}} = -\rho\boldsymbol{\Xi}(\mathbf{q_d})\boldsymbol{\Xi}^T(\mathbf{q_d})\mathbf{q} + \frac{1}{2}\boldsymbol{\Xi}(\boldsymbol{\lambda})\boldsymbol{\omega} \tag{7}$$

$$\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\Xi}^T(\mathbf{q})\boldsymbol{\lambda} + \boldsymbol{\omega_d} \tag{8}$$

It may be shown based on a suitable analytical analysis[6,17] that the following choice for the sliding manifold (intersection of the sliding surfaces) minimises the cost function of Eq (1)

$$\mathbf{s} = \boldsymbol{\omega_e} + k\boldsymbol{\delta q}_{13} = \mathbf{0}_{3\times1} \tag{9}$$

provided that $k = \pm\sqrt{\rho}$, where k is a scalar gain. The minimum value of the performance index of Eq (1) corresponding to this choice of sliding vector is given by[17]

$$J(\boldsymbol{\omega}^*, \mathbf{q}_{13}^*) = 2k[1 - \delta q_4(t_s)] \tag{10}$$

where k must be strictly positive in the evaluation of Eq (10). It is critical to note that for the quaternion parameterisation of the spacecraft attitude $\boldsymbol{\delta q}(t)$ and $-\boldsymbol{\delta q}(t)$ represent identical physical tracking error rotations, although the former gives the shortest angular distance to the sliding manifold whilst the later gives the longest distance. Hence more control energy may be required to maneuver the spacecraft to the reference trajectory based on the $-\boldsymbol{\delta q}(t)$ tracking error

commands. Vadali[6] and Crassidis et al.[17] proposed a modification to the sliding manifold defined by Eq (9) to ensure that the spacecraft follows the shortest possible path to the sliding manifold (and also to the reference trajectory)

$$\mathbf{s} = \boldsymbol{\omega}_e + k \operatorname{sgn}[\delta q_4(t_s)] \boldsymbol{\delta q}_{13} = \mathbf{0}_{3\times1} \tag{11}$$

where $\operatorname{sgn}[\cdot]$ is a signum function and it is assumed that $\delta q_4(t_s)$ is non-zero for a finite time. The performance index of Eq (1) evaluated for state trajectories satisfying Eq (11) is given by

$$J(\boldsymbol{\omega}^*, \mathbf{q}_{13}^*) = 2k\left[1 - |\delta q_4(t_s)|\right] \tag{12}$$

which provides a minimum value regardless of the quaternion tracking error representation. References 6 & 17 show that the $\operatorname{sgn}[\delta q_4(t_s)]$ term in Eq (11) may be replaced with $\operatorname{sgn}[\delta q_4(t)]$ (even before the sliding manifold is reached), which also produces a maneuver to the reference trajectory in the shortest distance. An alternative approach is to design the attitude maneuver using the modified Rodrigues parameter (MRP). This would result in a different functional form for the sliding manifold and the issue of minimum distance rotation maneuvers realised by switching between the normal MRP set and its shadow set. Substituting Eq (11) into the derivative of Eq (20) in Chapter 3 leads to the following kinematic equations for "ideal sliding motion" on the sliding manifold

$$\boldsymbol{\delta \dot{q}}_{13} = -\frac{1}{2}k|\delta q_4|\boldsymbol{\delta q}_{13} + \frac{1}{2}[\boldsymbol{\delta q}_{13} \times][2\boldsymbol{\omega}_d - k\operatorname{sgn}[\delta q_4(t_s)]\boldsymbol{\delta q}_{13}] \tag{13}$$

$$\delta \dot{q}_4 = \frac{1}{2}k\operatorname{sgn}[\delta q_4(t_s)](1 - \delta q_4^2) \tag{14}$$

The trajectory in state-space that slides on the sliding manifold (called the sliding mode) can be shown to be asymptotically stable using Lyapunov's direct method (see Chapter 3). The following candidate Lyapunov function is proposed

$$V(\boldsymbol{\delta q}_{13}) = \frac{1}{2}\boldsymbol{\delta q}_{13}^{\mathrm{T}}\boldsymbol{\delta q}_{13} \qquad (15)$$

Substituting Eq (13) into the derivative of Eq (15) leads to the following expression

$$\dot{V}(\boldsymbol{\delta q}_{13}) = -\frac{1}{2}k|\delta q_{4}|\boldsymbol{\delta q}_{13}^{\mathrm{T}}\boldsymbol{\delta q}_{13} \qquad (16)$$

which is clearly negative definite provided $k > 0$. This results shows that attitude tracking error $\boldsymbol{\delta q}_{13} \to \mathbf{0}_{3\times 1}$ as $t \to \infty$, and since the motion is on the sliding manifold defined by $\mathbf{s}(t) = \mathbf{0}_{3\times 1}$ it follows that $\boldsymbol{\omega}_{e}(t) \to \mathbf{0}_{3\times 1}$ as $t \to \infty$.

4.3 Control Law Design (Method 1)

This section considers the design of a state feedback control law for the entire system (including the dynamic equations) so that the closed-loop system can asymptotically track a reference trajectory defined by $\mathbf{x}_{\mathbf{d}}(t) = \begin{bmatrix}\mathbf{q}_{\mathbf{d}}^{\mathrm{T}}(t) & \boldsymbol{\omega}_{\mathbf{d}}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}$. Section 4.3 considers control law design based on Lyapunov's direct method in a manner similar to Chapter 3, whereas Section 4.4 will consider a more structured design approach based on the equivalent control method[5,6]. In many practical applications parameter uncertainty in the dynamic model or the presence of bounded external disturbances prevent perfect asymptotic tracking of the reference trajectory. The control law design process in this chapter will therefore explicitly account for these effects. This will be achieved by introducing an additional term in the control law which is specifically designed to cancel the effect of the external disturbances in the Lyapunov stability analysis.

4.3.1 Ideal Case

It was shown in Section 4.2 that the attitude kinematic motion on the sliding manifold defined by Eq (11) is asymptotically stable. To ensure that the spacecraft maneuvers to the reference trajectory in the shortest angular distance, a $\mathrm{sgn}[\delta q_{4}(t_{s})]$ function was introduced into the sliding vector definition of Eq

(11). A stability analysis was presented in Reference 17 which allows this $\text{sgn}[\delta q_4(t_s)]$ term to be replaced by $\text{sgn}[\delta q_4(t)]$ over the entire attitude maneuver. Taking the derivative of the sliding vector expression of Eq (11) augmented with the $\text{sgn}[\delta q_4(t)]$ term (and assuming this term is constant with respect to time) gives

$$\dot{\mathbf{s}} = \dot{\boldsymbol{\omega}}_e + k\,\text{sgn}[\delta q_4(t)]\boldsymbol{\delta\dot{q}}_{13} \qquad (17)$$

To prove that the spacecraft attitude motion convergences to the sliding manifold defined by $\mathbf{s}(t) = \mathbf{0}_{3\times 1}$ and remains on the manifold such that $\dot{\mathbf{s}}(t) = \mathbf{0}_{3\times 1}$ the following Lyapunov function candidate is proposed

$$V(\mathbf{s}) = \frac{1}{2}\mathbf{s}^T\mathbf{J}\mathbf{s} \qquad (18)$$

This function is a valid candidate since it vanishes at the equilibrium point $\mathbf{s}(t) = \mathbf{0}_{3\times 1}$ and is globally positive definite for $\mathbf{s}(t) \neq \mathbf{0}_{3\times 1}$ since the spacecraft inertia matrix $\mathbf{J}$ is positive definite. The Lyapunov derivative is given by

$$\dot{V}(\mathbf{s}) = \mathbf{s}^T\mathbf{J}\dot{\mathbf{s}} \qquad (19)$$

Substituting the expression for spacecraft rotational dynamics given by Eq (24) of Chapter 3 (where the gravity-gradient disturbance torque $\mathbf{u}_{gg}(t)$ and the general disturbance torque $\mathbf{d}(t)$ have been separated) into Eq (17) leads to

$$\mathbf{J}\dot{\mathbf{s}} = -[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} + \mathbf{u} + \mathbf{u}_{gg} + \mathbf{d} - \mathbf{J}\dot{\boldsymbol{\omega}}_d + k\,\text{sgn}[\delta q_4(t)]\mathbf{J}\boldsymbol{\delta\dot{q}}_{13} \qquad (20)$$

This section considers the case of zero external disturbances such that $\mathbf{u}_{gg}(t) = \mathbf{0}_{3\times 1}$ and $\mathbf{d}(t) = \mathbf{0}_{3\times 1}$. Substituting Eq (20) into the Lyapunov function derivative given by Eq (19) yields

$$\dot{V}(\mathbf{s}) = \mathbf{s}^T\left[-[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} + \mathbf{u} - \mathbf{J}\dot{\boldsymbol{\omega}}_d + k\,\text{sgn}[\delta q_4(t)]\mathbf{J}\boldsymbol{\delta\dot{q}}_{13}\right] \qquad (21)$$

168

Select the external control torque according to

$$\mathbf{u}(t) = -\mathbf{K}_1 \mathbf{s}(t) + [\boldsymbol{\omega}(t) \times] \mathbf{J} \boldsymbol{\omega}(t) - \mathbf{u}_{gg}(t) + \mathbf{J} \dot{\boldsymbol{\omega}}_d(t) - k \, \mathrm{sgn}[\delta q_4(t)] \mathbf{J} \delta \dot{\mathbf{q}}_{13}(t) \quad (22)$$

which ensures global asymptotic tracking of the reference trajectory due to the negative definite Lyapunov derivative given by

$$\dot{\mathbf{V}}(\mathbf{s}) = -\frac{1}{2} \mathbf{s}^{\mathrm{T}} \mathbf{K}_1 \mathbf{s} \quad (23)$$

such that $\mathbf{s}(t) \to \mathbf{0}_{3\times 1}$ as $t \to \infty$. This in turn implies that $\delta \mathbf{q}_{13}(t) \to \mathbf{0}_{3\times 1}$, $\delta q_4(t) \to \pm 1$ (depending on which equilibrium point is the minimum angular path), and $\boldsymbol{\omega}_e(t) \to \mathbf{0}_{3\times 1}$ as $t \to \infty$. The form of the control law given by Eq (22) is similar to the control law of Eq (38) in Chapter 3 developed using Lyapunov's direct method. Both controllers contain linear terms in $\delta \mathbf{q}_{13}(t)$ and $\boldsymbol{\omega}_e(t)$, as well as signum functions in the tracking error $\delta q_4(t)$ which ensure a minimum angle rotation to the equilibrium state. The key difference between the two control laws is that the sliding-mode control law of Eq (22) contains a $\delta \dot{\mathbf{q}}_{13}(t)$ term which introduces nonlinear terms in $\delta \mathbf{q}(t)$ and $\boldsymbol{\omega}_e(t)$. Note that the derivative term $\delta \dot{\mathbf{q}}_{13}(t)$ may be computed using the quaternion kinematic equations given by Eq (20) of Chapter 3, thus eliminating the need for numerical differentiation of the tracking error quaternion. An important advantage of the sliding-mode control design approach with respect to Lyapunov based methods is the effective reduction of the state-space dimension from $\mathbf{x}(t) \in \mathbf{R}^{7\times 1}$ to $\mathbf{x}(t) \in \mathbf{R}^{3\times 1}$.

4.3.2 Bounded Disturbance Torque Case

This section generalises the results of the previous section to consider the effect on the Lyapunov stability analysis of an external disturbance torque $\mathbf{d}(t)$ whose functional form is unknown, but whose upper limit is known. Throughout Chapter 3 and in Section 3.3.1 it was assumed that all parameters necessary to compensate for the gravity-gradient torque $\mathbf{u}_{gg}(t)$ were available. In this section

the gravity-gradient torque is included as part of the general disturbance torque term $\mathbf{d}(t)$. The control law given by Eq (22) is subsequently modified to mitigate the effect of all bounded disturbances in the Lyapunov stability analysis

$$\mathbf{u}(t) = -\mathbf{K}_1\mathbf{s}(t) + [\boldsymbol{\omega}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u}_{gg}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_d(t)$$
$$- k\,\mathrm{sgn}[\delta q_4(t)]\mathbf{J}\delta\dot{\mathbf{q}}_{13} + \boldsymbol{\tau}(t) \tag{24}$$

where $\boldsymbol{\tau}(t)$ is a disturbance rejection term specifically designed to counteract the effect of the all disturbance torques. To determine the required form of the disturbance rejection term consider the derivative of Eq (18) based on the modified control law of Eq (24) given by

$$\dot{\mathbf{V}}(\mathbf{s}) = -\frac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{K}_1\mathbf{s} + \frac{1}{2}\mathbf{s}^{\mathrm{T}}[\mathbf{d}+\boldsymbol{\tau}] \tag{25}$$

Assume that the disturbance torque is bounded according to $|d_i(t)| \le d_{max,i} < \Psi_{maxi}$ $i = 1,2,3$ for $t \ge 0$. This assumption requires that the total disturbance torque limits are available to the spacecraft attitude control system engineer prior to the spacecraft launch. In practice the worst-case-scenario magnitude of certain torques such as the gravity-gradient torque are available for a representative orbit and known spacecraft inertia matrix. However the effect of un-modelled disturbances or poorly modelled disturbances may lead to uncertainty in assessing the torque limits. It is therefore necessary to tune the gains governing the magnitude of the disturbance compensation term during spacecraft operations. A signum function whose argument is the sliding vector is proposed as the disturbance compensation term

$$\boldsymbol{\tau}(t) = -\boldsymbol{\Psi}_{max}\,\mathrm{sgn}[\mathbf{s}(t)] \tag{26}$$

where $\boldsymbol{\Psi}_{max} = \mathrm{diag}[\Psi_{maxi}]$ is a diagonal matrix of disturbance limits and the vector signum function $\mathrm{sgn}(\mathbf{s}) = [\mathrm{sgn}(s_1)\quad \mathrm{sgn}(s_2)\quad \mathrm{sgn}(s_3)]^{\mathrm{T}}$ has individual components defined by

$$\text{sgn}[s_i] = \begin{cases} +1 & s_i > 0 \\ 0 & s_i = 0 \\ -1 & s_i < 0 \end{cases} \qquad i = 1,2,3 \qquad (27)$$

To show that the disturbance rejection term of Eq (26) results in an asymptotically stable system it needs to be shown that the $\mathbf{s}^T(t)[\mathbf{d}(t) + \boldsymbol{\tau}(t)]$ term in Eq (25) is negative definite

$$\mathbf{s}^T[\mathbf{d} + \boldsymbol{\tau}] = \sum_{i=1}^{3} s_i \{d_i - \Psi_{\text{maxi}} \text{sgn}[s_i]\} = \sum_{i=1}^{3} |s_i|\{d_i \text{sgn}[s_i] - \Psi_{\text{maxi}}\} \qquad (28)$$

It follows from the disturbance torque limits that $\mathbf{s}^T(t)[\mathbf{d}(t) + \boldsymbol{\tau}(t)] < 0$ is a negative definite term and the Lyapunov derivative given by Eq (25) is therefore negative definite. Due to the disturbance compensation term of Eq (26) the presence of the $-\mathbf{K}_1\mathbf{s}(t)$ term in the control law of Eq (24) is not entirely necessary in proving the asymptotic stability result, it is does provide control over the rate of decrease of the Lyapunov function derivative through the $\mathbf{K}_1$ matrix.

Although the introduction of the disturbance compensation term $\boldsymbol{\tau}(t)$ allows us to conclude asymptotic stability, the primary disadvantage of this approach is that the signum function may create high-frequency chattering (high-frequency motion) in the control torques. This may in turn excite high-frequency dynamics neglected in the dynamic model (for example un-modelled structural modes, neglected time delays, etc.) and may eventually wear out mechanical hardware. This may be overcome by smoothing $\boldsymbol{\tau}(t)$ using a saturation function which offers a design trade-off between the steady-state trajectory tracking performance and dynamic model uncertainty, given the available control bandwidth. In this case, however, only Lyapunov stability (and not asymptotic stability) of the spacecraft attitude motion with respect to the equilibrium state can be guaranteed. Replacing the signum compensation term of Eq (26) with a saturation function

$$\boldsymbol{\tau}(t) = -\boldsymbol{\Psi}_{max} \text{sat}[\mathbf{s}(t)] \tag{29}$$

in which the definition of the saturation function is given by

$$\text{sat}[s_i] = \begin{cases} +1 & s_i > \varepsilon_i \\ \dfrac{s_i}{\varepsilon_i} & |s_i| \le \varepsilon_i \\ -1 & s_i < -\varepsilon_i \end{cases} \qquad i = 1,2,3 \tag{30}$$

where ε_i is a positive constant scalar called the saturation boundary thickness. The saturation function and its associated boundary layer is illustrated in Figure 4-1.

Based on the definition of the saturation function given by Eq (30) it is clear that the standard signum function is used when a sliding vector component lies outside the boundary layer defined by $|s_i| > \varepsilon_i$. It has been shown that the system is asymptotically stable under these conditions. When the spacecraft attitude motion is such that at least one sliding vector component lies within the boundary layer it is difficult to show that the Lyapunov function derivative of Eq (25) is negative definite since the term

$$\frac{1}{2}\mathbf{s}^T[\mathbf{d}+\boldsymbol{\tau}] = \frac{1}{2}\sum_{i \in p}^{3} s_i \left[d_i - \frac{\Psi_{max,i} s_i}{\varepsilon_i} \right] = -\frac{1}{2}\sum_{i=p}^{3} \frac{\Psi_{max,i}}{\varepsilon_i} s_i^2 + \frac{1}{2}\sum_{i=p}^{3} s_i d_i \tag{31}$$

for p control torques lying within the boundary layer is not negative definite in general. The thickness of the boundary layer may be appropriately designed by adjusting the ε_i parameter. The effect of using Eq (29) to compensate for the external disturbance torques is that state trajectories do not exactly move on the sliding manifold but are constrained to within the boundary layer defined by $|s_i| \le \varepsilon_i$ for $i = 1,2,3$.
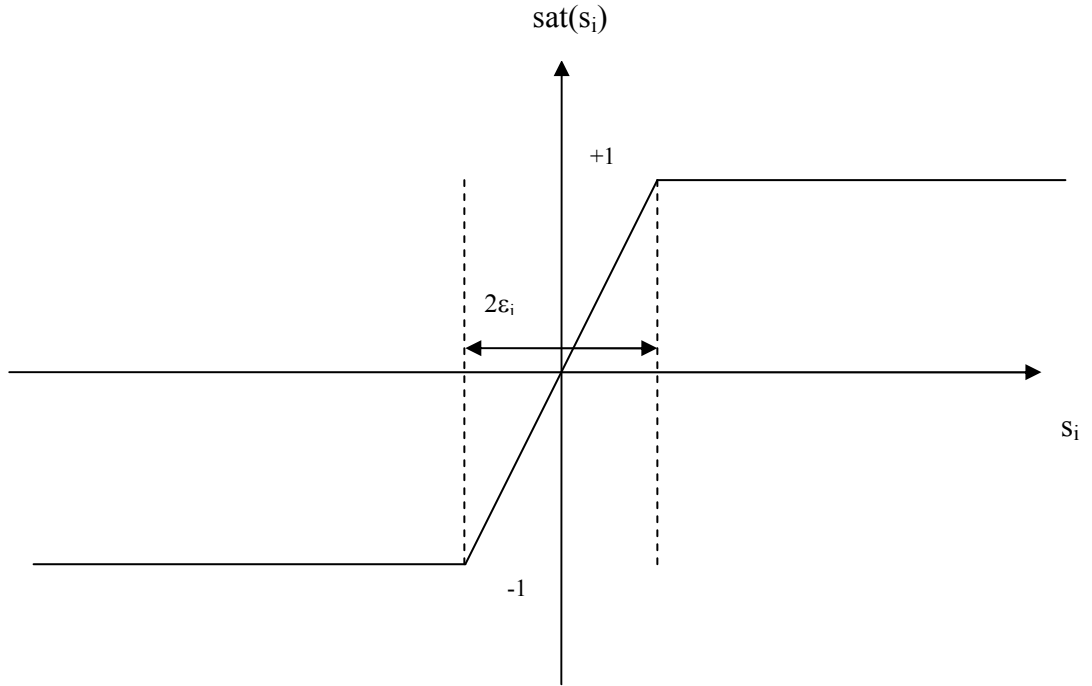
Figure 4-1 Saturation function with boundary layer $2\varepsilon_i$

4.3.3 Bounded Disturbance Torque and Modelling Uncertainty Case

In addition to the external disturbance torques appearing in the dynamic equations of motion the spacecraft is subject to model parameter uncertainty. This work will consider the case in which uncertainty present in the spacecraft inertia matrix is in the form

$$\mathbf{J} = \mathbf{J}^* + \Delta\mathbf{J} \tag{32}$$

where $\mathbf{J}$ is the true inertia matrix (constant over the spacecraft attitude maneuver), $\mathbf{J}^*$ is the best estimate of the inertia matrix, and $\Delta\mathbf{J}$ is the inertia matrix uncertainty. A modified version of the control law of Eq (24) must be applied under parameter uncertainty, in which the best estimate of the spacecraft inertia matrix is used

$$\begin{aligned}\mathbf{u}(t) = &-\mathbf{K}_1\mathbf{s}(t) + [\boldsymbol{\omega}(t)\times]\mathbf{J}^*\boldsymbol{\omega}(t) - \mathbf{u}_{gg}(t) + \mathbf{J}^*\dot{\boldsymbol{\omega}}_d(t) \\ &- k\,\mathrm{sgn}[\delta q_4(t)]\mathbf{J}^*\delta\dot{\mathbf{q}}_{13}(t) + \boldsymbol{\tau}(t)\end{aligned} \tag{33}$$

173

Substitute the parameter uncertainty expression of Eq (32) into Eq (33)

$$\mathbf{u} = -\mathbf{K}_1\mathbf{s} + [\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} - \mathbf{u}_{gg}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_d - k\,\mathrm{sgn}[\delta q_4(t)]\mathbf{J}\delta\dot{\mathbf{q}}_{13} + \boldsymbol{\tau} \qquad (34)$$
$$- [[\boldsymbol{\omega}\times]\Delta\mathbf{J}\boldsymbol{\omega} + \Delta\mathbf{J}\dot{\boldsymbol{\omega}}_d - k\,\mathrm{sgn}[\delta q_4(t)]\Delta\mathbf{J}\delta\dot{\mathbf{q}}_{13}]$$

Define an augmented torque term due to all external disturbance torques and spacecraft inertia uncertainty terms

$$\tilde{\mathbf{d}} = \mathbf{d} - [\boldsymbol{\omega}\times]\Delta\mathbf{J}\boldsymbol{\omega} - \Delta\mathbf{J}\dot{\boldsymbol{\omega}}_d + k\,\mathrm{sgn}[\delta q_4(t)]\Delta\mathbf{J}\delta\dot{\mathbf{q}}_{13} \qquad (35)$$

which has an upper limit given by $\left|\tilde{d}_i(t)\right| \le \tilde{d}_{max,i} < \tilde{\Psi}_{max,i}$. With the addition of spacecraft inertia uncertainty the Lyapunov function derivative given by Eq (25) is replaced with

$$\dot{V}(\mathbf{s}) = -\frac{1}{2}\mathbf{s}^T\mathbf{K}_1\mathbf{s} + \frac{1}{2}\mathbf{s}^T[\tilde{\mathbf{d}} + \boldsymbol{\tau}] \qquad (36)$$

A suitable saturation function may be designed to cancel the effect of the augmented disturbance torque term using a procedure similar to that described in Section 4.3.2.

4.3.4 Determination of Control Gains

An approach similar to that in Section 3.4 of Chapter 3 is applied to select the control gain matrix $\mathbf{K}_1$ and the sliding vector scalar gain k to achieve a desired closed-loop system response. In the disturbance free case the ordinary differential equation governing the evolution of the sliding vector is obtained by substituting the feedback control law of Eq (22) into Eq (20) which produces the stable first-order linear ODE

$$\dot{\mathbf{s}}(t) = -\mathbf{J}^{-1}\mathbf{K}_1\mathbf{s}(t) \qquad (37)$$

Any value of $\mathbf{K}_1$ may be used to ensure that $\mathbf{s}(t) \to \mathbf{0}_{3\times1}$ as $t \to \infty$ with larger values of the elements of $\mathbf{K}_1$ giving rise to faster convergence rates. The

Lyapunov stability analysis for the sliding manifold shows that once the spacecraft attitude motion is tracking the sliding manifold then the reference trajectory tracking errors eventually converge to the origin. Although an arbitrary choice of $\mathbf{K}_1$ achieves the desired tracking objectives it is desirable to design the system to achieve a desired closed-loop response. Consider the tracking error angular rate dynamics based on the control law of Eq (22) given by

$$\mathbf{J}\dot{\boldsymbol{\omega}}_e = -\mathbf{K}_1\boldsymbol{\omega}_e - k\mathbf{K}_1\operatorname{sgn}[\delta q_4(t)]\boldsymbol{\delta q}_{13} - k\operatorname{sgn}[\delta q_4(t)]\mathbf{J}\boldsymbol{\delta\dot{q}}_{13} \qquad (38)$$

By definition the vector component of the tracking error quaternion and angular rates are given by $\boldsymbol{\delta q}_{13}(t) = \sin(\theta(t)/2)\hat{\mathbf{e}}(t)$ and $\boldsymbol{\omega}_e(t) = \dot{\theta}(t)\hat{\mathbf{e}}(t)$ respectively, where $\theta(t)$ is the Euler angle of rotation and $\hat{\mathbf{e}}(t)$ is the Euler axis which is assumed constant in direction during the spacecraft attitude maneuver. Substituting these expressions into Eq (38) results in a second-order nonlinear ordinary differential equation in terms of the Euler angle

$$\ddot{\theta}\mathbf{J}\hat{\mathbf{e}} = -\dot{\theta}\mathbf{K}_1\hat{\mathbf{e}} - k\operatorname{sgn}[\delta q_4]\sin(\theta/2)\mathbf{K}_1\hat{\mathbf{e}} - k\operatorname{sgn}[\delta q_4](\dot{\theta}/2)\cos(\theta/2)\mathbf{J}\hat{\mathbf{e}} \qquad (39)$$

Select the control gain as $\mathbf{K}_1 = k_1\mathbf{J}$ and assume that the $\operatorname{sgn}[\delta q_4(t)]$ term is unity throughout the spacecraft attitude maneuver. Also assume that for the purposes of selecting the control gains the approximations $\sin(\theta/2) \approx \theta/2$ and $\cos(\theta/2) \approx 1$ are valid. The second-order linear ODE for the Euler angle given by Eq (39) reduces to

$$\ddot{\theta} + (k_1 + k/2)\dot{\theta} + (kk_1/2)\theta = 0 \qquad (40)$$

where the damping ratio ζ and the natural frequency ω_n satisfy $k_1 + k/2 = 2\zeta\omega_n$ and $kk_1 = 2\omega_n^2$. Solving these two simultaneous equations produces the differential equation $k_1^2 - 2\zeta\omega_n k_1 + \omega_n^2 = 0$ which has two solutions for k_1. Take the positive solution for k_1 since the Lyapunov function derivative must be negative definite and solve for the remaining control

parameter using $k = 4(\zeta\omega_n - k_1/2)$. The natural undamped frequency parameter ω_n is calculated using the settling time expression

$$t_s = 8/(\zeta\omega_n) \tag{41}$$

where the damping factor is $\varsigma = 1$ for a critically damped closed-loop response.

4.4 Control Law Design (Method 2)

In this section the equivalent control method[5-6] is used to develop a feedback control law that induces ideal sliding on the sliding manifold based on external control torque inputs in the absence of disturbances and dynamic model uncertainty. It will be shown using a Lyapunov stability analysis that the same control law can be used to force the state trajectory onto the sliding manifold[6]. A suitable disturbance compensation term will then be introduced into the control law which results in the state trajectories lying in a bounded region surrounding the sliding manifold. The control law development in this section is not intended to supersede the methodology introduced in Section 4.3 but rather provide an alternative approach for the controller design.

The kinematic and dynamic equations associated with the trajectory tracking error are given by Eq (20) and Eq (24) in Chapter 3. This may be expressed as a system of n equations linear in the m controls

$$\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t) \tag{42}$$

where $\mathbf{x}(t) \in \mathfrak{R}^{7\times1}$ is the state vector, $\mathbf{f}(\mathbf{x}(t)) \in \mathfrak{R}^{7\times1}$ is a nonlinear function of the state variables, $\mathbf{B} \in \mathfrak{R}^{7\times3}$ is the control coefficient matrix, and $\mathbf{u}(t) \in \mathfrak{R}^{3\times1}$ is the control torque vector. The specific structure for the spacecraft attitude maneuver problem are given by

$$\mathbf{x} = \begin{bmatrix} \delta\mathbf{q}^T & \mathbf{s}^T \end{bmatrix}^T \tag{43}$$

$$\mathbf{f(x)} = \begin{bmatrix} \dfrac{1}{2}\boldsymbol{\Xi}(\boldsymbol{\delta q})\boldsymbol{\omega}_e \\ -\mathbf{J}^{-1}[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} + \mathbf{J}^{-1}\mathbf{u}_{gg} - \dot{\boldsymbol{\omega}}_d + k\,\mathrm{sgn}[\delta q_4(t)]\boldsymbol{\delta\dot{q}}_{13} \end{bmatrix} \tag{44}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{3\times4} & \mathbf{J}^{-1} \end{bmatrix}^{\mathrm{T}} \tag{45}$$

based on the kinematic equations of Eqs (20) in Chapter 3 and the dynamic equations of Eq (20) without general external disturbance torques. During ideal sliding on the sliding manifold defined by $\mathbf{s}(t) = \mathbf{0}_{3\times1}$, the sliding vector derivative must also be zero. Applying the chain rule of differentiation to the sliding vector derivative

$$\dot{\mathbf{s}} = \mathbf{P}\dot{\mathbf{x}} = \mathbf{P}\mathbf{f(x)} + \mathbf{P}\mathbf{B}\mathbf{u}_{eq} = \mathbf{0}_{3\times1} \tag{46}$$

where $\mathbf{u}_{eq}(t)$ is the equivalent control torque and $\mathbf{P}$ is the sliding vector Jacobian matrix with respect to the states defined by

$$\mathbf{P} = \frac{\partial \mathbf{s}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial s_1}{\partial x_1} & \cdots & \dfrac{\partial s_1}{\partial x_7} \\ \vdots & & \vdots \\ \dfrac{\partial s_3}{\partial x_1} & \cdots & \dfrac{\partial s_1}{\partial x_7} \end{bmatrix} \tag{47}$$

Evaluating the Jacobian matrix based on the definition of the sliding vector in Eq (11) leads to

$$\mathbf{P} = \begin{bmatrix} k\,\mathrm{sgn}[\delta q_4(t)]\mathbf{I}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{I}_{3\times3} \end{bmatrix} \tag{48}$$

where the $\mathrm{sgn}[\delta q_4(t_s)]$ term in Eq (11) is replaced with $\mathrm{sgn}[\delta q_4(t)]$. The equivalent control torque can be obtained by rearranging Eq (45)

$$\mathbf{u}_{eq} = -[\mathbf{P}\mathbf{B}]^{-1}\mathbf{P}\mathbf{f(x)} \tag{49}$$

which has a unique solution provided that the matrix product $\mathbf{PB}$ is non-singular. Substituting Eqs (44), (45), and (48) into Eq (49) provides an expression for the equivalent control torque

$$\mathbf{u}_{eq}(t) = [\boldsymbol{\omega}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u}_{gg}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_{d}(t) - k\,\mathrm{sgn}[\delta q_4(t)]\mathbf{J}\delta\dot{\mathbf{q}}_{13}(t) \qquad (50)$$

where $\delta\dot{\mathbf{q}}_{13}(t)$ is given by Eq (20) of Chapter 3. The control law of Eq (50) is almost identical to the control law of Eq (22) except for the linear term in the sliding vector. Under the influence of additional external disturbance torques and dynamic model uncertainty the equivalent control torque given by Eq (50) will not be sufficient to exactly maintain the ideal sliding motion. Hence it is necessary to modify the equivalent control in order to account for these non-ideal effects such that the state trajectory will remain close to the sliding manifold. The modified control law is selected as

$$\mathbf{u}(t) = [\boldsymbol{\omega}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{u}_{gg}(t) + \mathbf{J}\dot{\boldsymbol{\omega}}_{d}(t) - k\,\mathrm{sgn}[\delta q_4(t)]\mathbf{J}\delta\dot{\mathbf{q}}_{13}(t) - \mathbf{JGv}(t) \qquad (51)$$

where $\mathbf{G} \in \Re^{3\times3}$ is a constant positive definite diagonal matrix and $\boldsymbol{v}(t)$ is a saturation function defined by

$$\upsilon_i(t) = \begin{cases} +1 & s_i > \varepsilon \\ \dfrac{s_i(t)}{\varepsilon} & |s_i| \le \varepsilon \\ -1 & s_i < -\varepsilon \end{cases} \qquad i = 1,2,3 \qquad (52)$$

where ε is a positive scalar constant which defines the thickness of the boundary layer. The saturation function of Eq (52) is used to cancel the effect of un-modelled dynamics and external disturbance torques whilst eliminating the possibility of chattering in the control torque. This will ensure that the spacecraft attitude motion remains within the boundary layer of the sliding manifold defined by $|s_i| \le \varepsilon$ for $i = 1,2,3$. The feedback control law of Eq (51) may be considered the best approximation of a discontinuous control law (due to the signum function) that would maintain $\mathbf{s}(t) = \mathbf{0}_{3\times1}$ in the presence of dynamic model errors and bounded disturbance torques. A global stability

analysis to show that the control law of Eq (51) forces the spacecraft attitude motion toward the sliding manifold is based on Lyapunov's direct method. Substituting the expression for spacecraft rotational dynamics given by Eq (24) of Chapter 3 and the feedback control law of Eq (51) into Eq (17) leads to

$$\dot{\mathbf{s}} = \mathbf{J}^{-1}\mathbf{d} - \mathbf{G}\upsilon \tag{53}$$

The following candidate Lyapunov function is proposed

$$V(\mathbf{s}) = \frac{1}{2}\mathbf{s}^{T}\mathbf{s} \tag{54}$$

which is similar to the Lyapunov function of Eq (18). Substituting Eq (53) into the derivative of Eq (54) produces the following expression

$$\dot{V}(\mathbf{s}) = -\mathbf{s}^{T}\left[-\mathbf{J}^{-1}\mathbf{d} + \mathbf{G}\upsilon\right] \tag{55}$$

which is negative definite function if the sliding vector is outside the boundary layer $|s_i| > \varepsilon$ for $i = 1,2,3$ and indefinite if $|s_i| \le \varepsilon$. The disturbance compensation term must be properly designed to specify the thickness of the boundary layer. This Lyapunov stability analysis shows that the control law of Eq (51) may be used to force the state trajectory towards the sliding manifold and track the sliding manifold indefinitely within a boundary layer defined by $|s_i| \le \varepsilon$ for $i = 1,2,3$.

In the special case of zero external disturbance torques the Lyapunov function derivative given by Eq (55) reduces to[17]

$$\dot{V}(\mathbf{s}) = -\mathbf{s}^{T}\mathbf{G}\upsilon \tag{56}$$

which is a negative definite function ensuring global asymptotic stability of the equilibrium point $\mathbf{s}(t) = \mathbf{0}_{3\times1}$.

4.5 Attitude Regulation Maneuver

Spacecraft attitude maneuver simulation results are presented for an inertial pointing spacecraft to demonstrate the capabilities of the sliding mode control law developed in Section 4.3. The simulations were conducted using an attitude control system model developed in MATLAB™ Simulink® with numerical integration performed using a fourth-order Runge-Kutta numerical integrator ODE4 with a fixed step size of $\Delta t = 0.1$ sec. The MATLAB™ line style designators for all simulation plots are 1-axis = solid line, 2-axis = dashed line, 3-axis = dash-dot line, and 4-axis = dotted line.

This design example is based on the control law of Eq (22) with simulation parameters defined in Table 4-1. The only environmental disturbance torque considered is the gravity-gradient torque which is compensated in Eq (22). The control gains are selected using the procedure outlined in Section 4.3.4. The initial Euler axis/angle parameterisation given in Table 4-1 corresponds to a 90 deg rotation with respect to the minimum angle maneuver equilibrium point $-\mathbf{q_d}(t)$. For a settling time requirement of $t_s = 50$ sec and a critically damped response $\xi = 1$ the natural frequency is $\omega_n = 0.16$ rad/sec and the control gains are $k_1 = 0.16$ and $k = 0.32$.

Figure 4-2 to Figure 4-9 present the results of the rest-to-rest attitude maneuver. The tracking error quaternion displays a critically damped response with the $\delta q_4(t)$ component settling to the desired equilibrium point, and the system settling time of 50 sec has been achieved to within the approximations introduced in Section 4.3.4. This demonstrates the $\text{sgn}[\delta q_4(t)]$ term present in the sliding vector defined by Eq (11) is correctly forcing the spacecraft attitude motion towards the correct minimum angular path equilibrium point. Figure 4-7 demonstrates that Wie's eigenaxis rotation theorem for rest-to-rest maneuvers discussed in Section 3.4 of Chapter 3 does not apply for sliding mode based attitude maneuvers. The Lyapunov function in Figure 4-8 and its derivative in Figure 4-9 are consistent with an asymptotically stable closed-loop system.

Parameters	*Values*
Spacecraft Inertia Matrix	$$\mathbf{J} = \begin{bmatrix} 600.28 & -3.57 & 0.17 \\ -3.57 & 611.44 & 0.25 \\ 0.17 & 0.25 & 507.90 \end{bmatrix}$$
Reference Trajectory	$$\mathbf{q_d} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$ $$\boldsymbol{\omega_d} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$
Initial Conditions	$$\mathbf{q}(t_0) = \begin{bmatrix} -\sqrt{1/6} & \sqrt{1/6} & -\sqrt{1/6} & -\sqrt{1/2} \end{bmatrix}^T$$ $$\hat{\mathbf{e}}(t_0) = \begin{bmatrix} -0.5774 & 0.5774 & -0.5774 \end{bmatrix}^T$$ $$\theta(t_0) = 270 \text{ deg}$$ $$\boldsymbol{\omega}(t_0) = \begin{bmatrix} 10.0 & -10.0 & 5.0 \end{bmatrix}^T$$
Control Gains	$$\mathbf{K_1} = 0.16\mathbf{J}$$ $$k = 0.32$$

Table 4-1 Rest-to-Rest Attitude Maneuvers
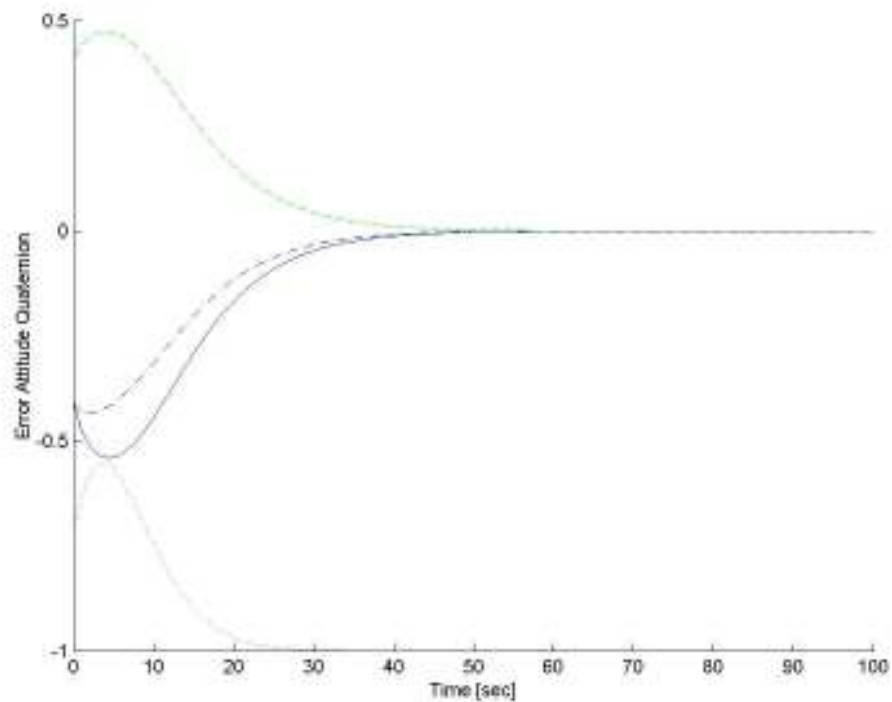
Figure 4-2 Spacecraft attitude quaternion tracking error
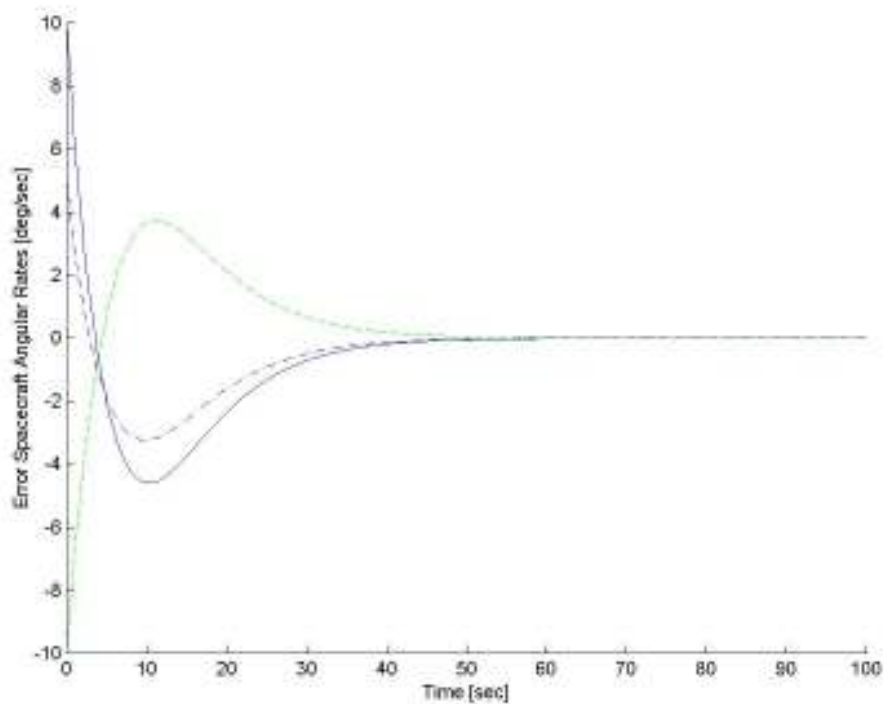
Figure 4-3 Spacecraft angular rates
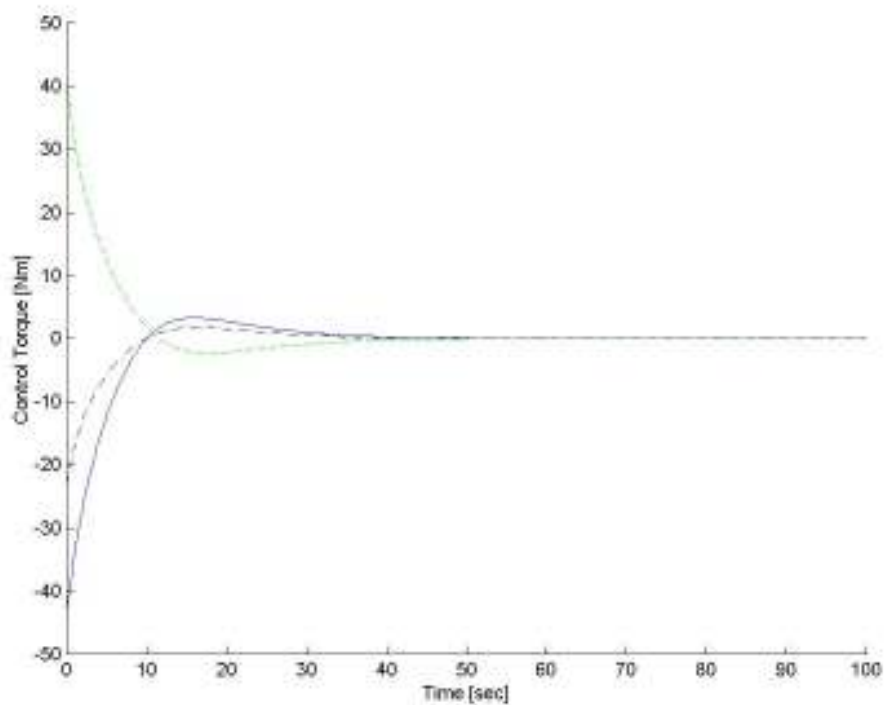
Figure 4-4 External control torques
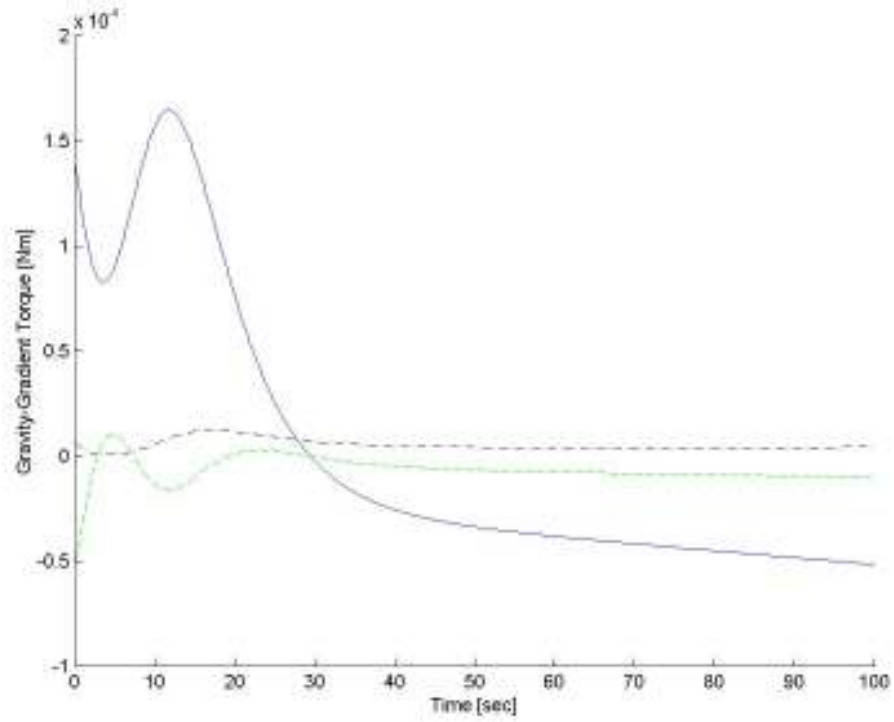
Figure 4-5 Gravity-gradient disturbance torques
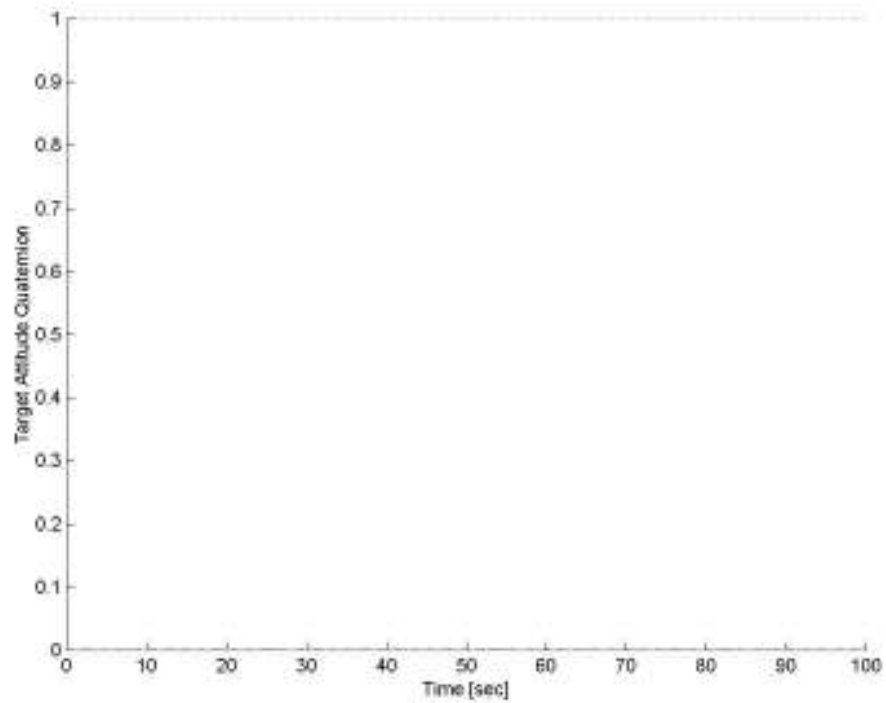
Figure 4-6 Reference trajectory attitude quaternion

183
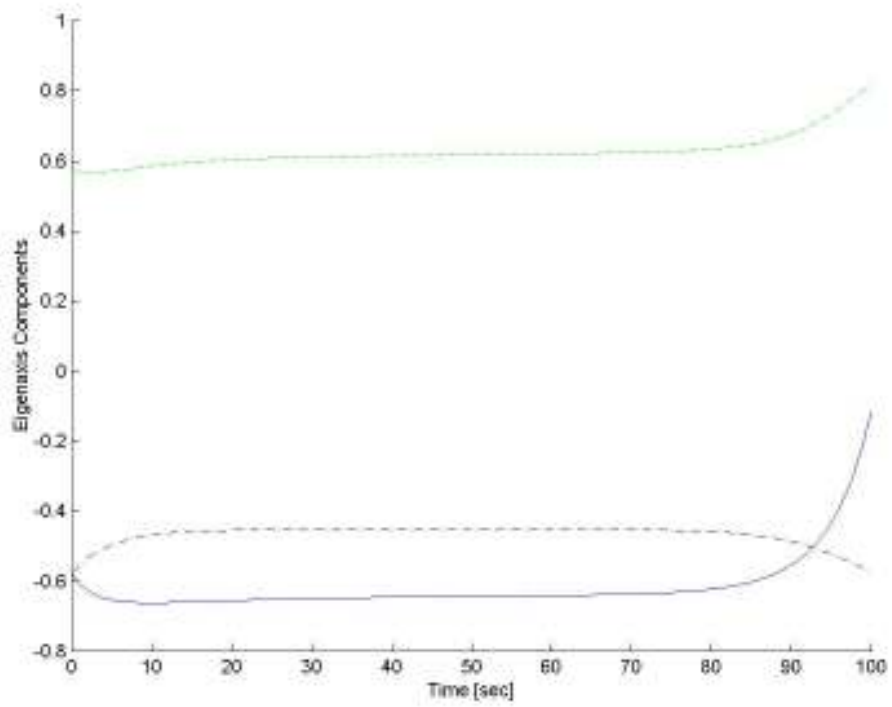
Figure 4-7 Euler axis components of tracking error quaternion
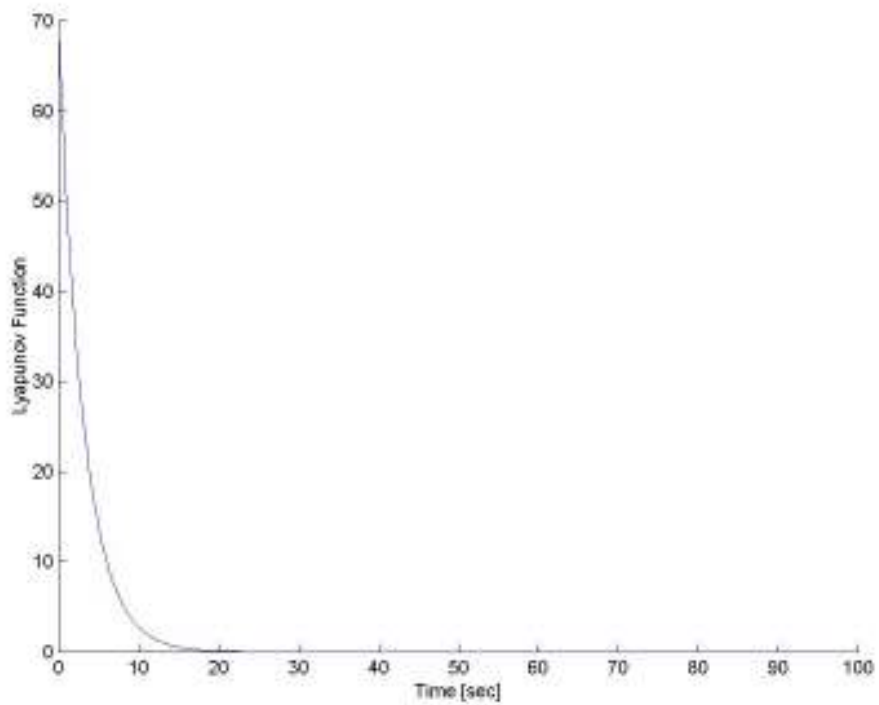
Figure 4-8 Lyapunov function
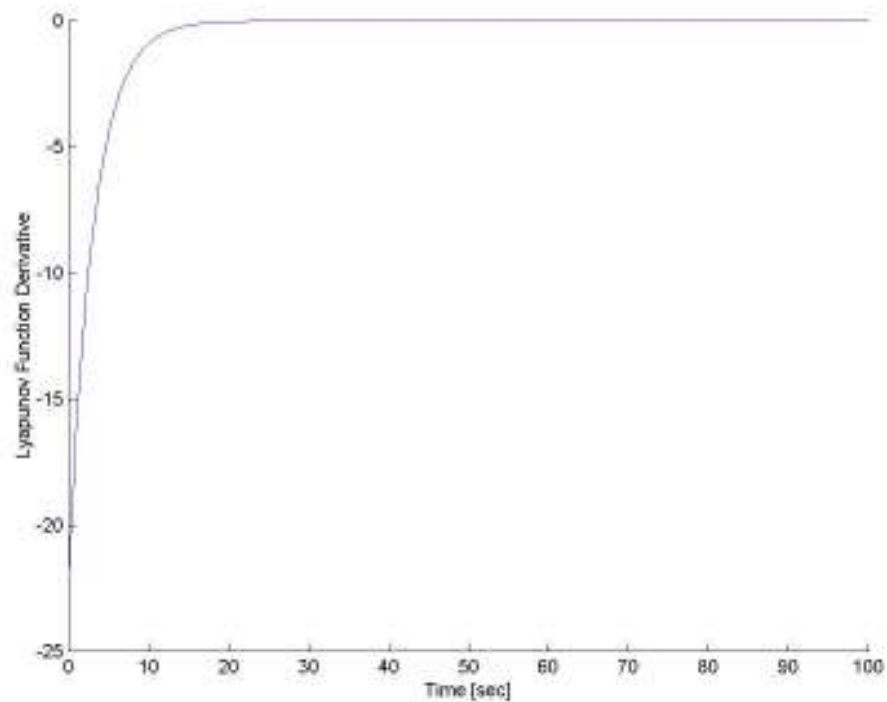
Figure 4-9 Lyapunov function derivative

4.6 Conclusion

In this chapter, two alternate sliding mode control laws were developed for attitude tracking maneuvers. A saturation function was introduced to compensate for the effects of spacecraft inertia uncertainty and bounded external disturbance torques. MATLAB™ simulation results were presented to demonstrate the performance of the fundamental control algorithm.

4.7 References

[1] Utkin, V.I. (1978). Sliding modes and their application in variable structure systems. Russia: MIR Publishers.

[2] Slotine, J.J.E., and W. Li. (1990). Applied nonlinear control. USA: Prentice-Hall, Inc.

[3] Young, K-K., D., ed. (1993). Variable structure control for robotics and aerospace applications. The Netherlands: Elsevier Science Publishers B.V.

185

[4] Sabanovic, A., Fridman, L.M., and S. Spurgeon, eds. (2004). Variable structure systems: from principles to implementation. United Kingdom: The Institution of Electrical Engineers.

[5] Utkin, V.I. (1977). Variable structure systems with sliding modes. IEEE Transactions on Automatic Control, 22(2): 212-222.

[6] Vadali, S.R. (1986). Variable-structure control of spacecraft large-angle maneuvers. The Journal of Guidance, Control, and Dynamics, 9(2): 235-239.

[7] DeCarlo, R.A., Zak, S.H., and G.P. Matthews. (1988). Variable structure control of nonlinear multivariable systems: a tutorial. Proceedings of the IEEE, 76(3): 212-232.

[8] Dwyer, T.A.W., and H. Sira-Ramirez. (1988). Variable-structure control of spacecraft attitude maneuvers. The Journal of Guidance, Control, and Dynamics, 11(3): 262-270.

[9] Hung, J.Y., Gao, W., and J.C. Hung. (1993). Variable structure control: a survey. IEEE Transactions on Industrial Electronics, 40(1): 2-22.

[10] Chen, Y-P., and S-C. Lo. (1993). Sliding-mode controller design for spacecraft attitude tracking maneuvers. IEEE Transactions on Aerospace and Electronic Systems, 29(4): 1328-1333.

[11] Lo, S-C., and Y-P. Chen. (1995). Smooth sliding-mode control for spacecraft attitude tracking maneuvers. The Journal of Guidance, Control, and Dynamics, 18(6): 1345-1349.

[12] Crassidis, J.L., and F.L. Markley. (1996). Sliding mode control using modified Rodrigues parameters. The Journal of Guidance, Control, and Dynamics, 19(6): 1381-1383.

[13] Robinett, R.D., and G.G. Parker. (1997). Least squares sliding mode control tracking of spacecraft large angle maneuvers. The Journal of the Astronautical Sciences, 45(4): 433-450.

[14] Kim, J., Kim, J., and J.L. Crassidis. (1998). Disturbance accommodating sliding mode controller for spacecraft attitude maneuvers. Proceedings of the AAS/GSFC International Symposium on Space Flight Dynamics, Greenbelt, Maryland, USA, AAS Paper 98-310, 141-153.

[15] Kim, J., and J.L. Crassidis. (1998). A comparative study of sliding-mode control and time-optimal control. Proceedings of the AIAA/AAS Astrodynamics Specialist Conference, Boston, Massachusetts, USA, AIAA Paper 98-4473, 316-323.

[16] Boskovic, J.D., Sai-Ming, L., and R.K. Mehra. (1999). Globally stable adaptive tracking control design for spacecraft under input saturation. Proceedings of the IEEE Conference on Decision and Control, Phoenix, Arizona, USA, 1952-1957.

[17] Crassidis, J.L., Vadali, S.R., and F.L. Markley (2000). Optimal variable-structure control tracking of spacecraft maneuvers. Journal of Guidance, Control, and Dynamics, 23(3): 564-566.

[18] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2001). Robust adaptive variable structure control of spacecraft under control input saturation. The Journal of Guidance, Control, and Dynamics, 24(1): 14-22.

[19] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2004). Robust tracking control design for spacecraft under control input saturation. The Journal of Guidance, Control, and Dynamics, 27(4): 627-633.

Chapter 5 Spacecraft Attitude Maneuvers Using Adaptive Control Theory

5.1 Introduction

This chapter considers the development of a direct adaptive control strategy[5,6] for spacecraft attitude tracking maneuvers. This strategy explicitly accounts for spacecraft inertia uncertainty, and may be considered an extension of the sliding mode control design in Chapter 4. The stability analysis will be based on Lyapunov's direct method, in conjunction with Barbalat's lemma[6]. The main outcomes of this chapter will be to develop an adaptive control strategy which provides exponentially convergent reference trajectory tracking and ensures exponential convergence of the inertia parameter estimates to their true values. The incorporation of sensor measurement error in the context of direct adaptive control will not be addressed in this thesis.

The inertia parameter convergence results in this chapter have not been addressed in the literature, and therefore represent one of the major novel contributions provided in this thesis. Section 5.2 presents the fundamental direct adaptive control strategy based on the research by Cristi et al.[18] for attitude regulation maneuvers and Ahmed et al.[26] for attitude tracking maneuvers. A feedback control law and associated parameter update law are proposed to ensure asymptotic trajectory tracking in the presence of parameter uncertainty. In Section 5.3, a constant gain parameter update law[6] is proposed which accounts for both trajectory tracking errors and output model prediction errors in the parameter update process. This approach requires direct noiseless measurements of the control torque inputs. Global asymptotic convergence of the tracking errors and parameter estimation errors is demonstrated via a Lyapunov stability analysis, provided that the reference trajectories are persistently exciting[5,6]. The design of a bounded-gain-forgetting (BGF) composite parameter update strategy is addressed in Section 5.4. This framework guarantees global exponential convergence of the trajectory tracking errors and parameter estimation errors, provided a persistency of excitation condition is satisfied. In Section 5.5, MATLAB™ simulation results are

presented to demonstrate the performance of the control algorithm in Section 5.2 and to investigate convergence of the inertia parameter adaptive estimates.

5.2 Adaptive Control Strategy

The closed-loop dynamic equations of motion are given by

$$\mathbf{J}\dot{\mathbf{s}}(t) = -[\boldsymbol{\omega}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) - \mathbf{J}\boldsymbol{\alpha}_r(t) + \mathbf{u}(t) \qquad (1)$$

where $\mathbf{J}$ is the inertia matrix, $\mathbf{s}(t) = \widetilde{\boldsymbol{\omega}}_c(t) + \lambda\,\mathrm{sgn}[\widetilde{\eta}_c(t)]\widetilde{\boldsymbol{\varepsilon}}_c(t) = \boldsymbol{\omega}(t) - \boldsymbol{\omega}_r(t)$ is the sliding vector, λ is a positive scalar constant, and the remaining parameters in Eq (1) are $\boldsymbol{\alpha}_r(t) = \mathbf{R}(\widetilde{\mathbf{q}}_c(t))\dot{\boldsymbol{\omega}}_{dR}(t) - [\widetilde{\boldsymbol{\omega}}_c(t)\times]\mathbf{R}(\widetilde{\mathbf{q}}_c(t))\boldsymbol{\omega}_{dR}(t) - \lambda\dot{\widetilde{\boldsymbol{\varepsilon}}}_c(t)$ and $\boldsymbol{\omega}_r(t) = \mathbf{R}(\widetilde{\mathbf{q}}_c(t))\boldsymbol{\omega}_{dR}(t) - \lambda\,\mathrm{sgn}[\widetilde{\eta}_c(t)]\widetilde{\boldsymbol{\varepsilon}}_c(t)$. The reference trajectory tracking errors are defined as $\widetilde{\boldsymbol{\omega}}_c(t) = \boldsymbol{\omega}(t) - \mathbf{R}(\widetilde{\mathbf{q}}_c(t))\boldsymbol{\omega}_{dR}(t)$ and $\widetilde{\mathbf{q}}_c(t) = \mathbf{q}(t)\otimes\mathbf{q}_d^{-1}(t)$, where $\otimes$ denotes the quaternion multiplication operator. The reference trajectories are defined by the commanded spacecraft angular rates $\boldsymbol{\omega}_{dR}(t)$ expressed in reference trajectory coordinates $\{\mathbf{R}(t)\}$, and the commanded spacecraft attitude quaternion $\mathbf{q}_d(t)$.

In Chapter 4, the control law of Eq (24) was selected to ensure asymptotic convergence of the trajectory tracking error $\mathbf{s}(t) \to \mathbf{0}_{3\times1}$ as $t \to \infty$. In the context of adaptive control it is assumed that the spacecraft inertia matrix $\mathbf{J}(t)$ is either completely unknown to the spacecraft designer, known to a certain level of certainty, or is time-varying with different load conditions. The true spacecraft inertia matrix is given by $\mathbf{J}(t) = \mathbf{J}^*(t) + \widetilde{\mathbf{J}}(t)$, where $\mathbf{J}^*(t)$ is the current best estimate of the inertia matrix and $\widetilde{\mathbf{J}}(t)$ is the inertia matrix uncertainty. The true inertia matrix may be constant or time-varying (due to fuel expenditure), but nin this research work it is assumed fixed throughout the duration of the spacecraft attitude maneuver. The spacecraft inertia parameter estimates are governed by a time-varying parameter update law to ensure that the control objectives are accomplished. It is therefore necessary from a practical viewpoint to replace the control law of Eq (24), without the gravity-

gradient disturbance torque $\mathbf{u}_{gg}(t)$ and the disturbance rejection term $\boldsymbol{\tau}(t)$, with the control law given by

$$\mathbf{u}(t) = -\mathbf{K}_{\mathbf{D}}\mathbf{s}(t) + \mathbf{J}^{*}\boldsymbol{\alpha}_{\mathbf{r}}(t) + [\boldsymbol{\omega}_{\mathbf{r}}(t)\times]\mathbf{J}^{*}\boldsymbol{\omega}(t) \tag{2}$$

The control law of Eq (2) may be expressed as

$$\mathbf{u}(t) = -\mathbf{K}_{\mathbf{D}}\mathbf{s}(t) + \mathbf{J}\boldsymbol{\alpha}_{\mathbf{r}}(t) + [\boldsymbol{\omega}_{\mathbf{r}}(t)\times]\mathbf{J}\boldsymbol{\omega}(t) + \tilde{\mathbf{u}}(t) \tag{3}$$

$$\tilde{\mathbf{u}}(t) = -\tilde{\mathbf{J}}\boldsymbol{\alpha}_{\mathbf{r}}(t) - [\boldsymbol{\omega}_{\mathbf{r}}(t)\times]\tilde{\mathbf{J}}\boldsymbol{\omega}(t) \tag{4}$$

where $\tilde{\mathbf{u}}(t)$ is an additional torque term related to the uncertainty in the spacecraft inertia matrix. For adaptive control applications, it is necessary to linearly parameterise $\tilde{\mathbf{u}}(t)$ in terms of the time-varying parameter error vector defined by

$$\tilde{\boldsymbol{\theta}}(t) = \begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{22} & \tilde{J}_{33} & \tilde{J}_{23} & \tilde{J}_{13} & \tilde{J}_{12} \end{bmatrix}^{\mathrm{T}} \tag{5}$$

Use the following identities to achieve the linear parameterisation based on an arbitrary vector $\mathbf{a}(t) \in \mathbf{R}^{3\times1}$

$$\tilde{\mathbf{J}}\mathbf{a} = \mathbf{L}(\mathbf{a})\tilde{\boldsymbol{\theta}} \tag{6}$$

$$\mathbf{L}(\mathbf{a}) = \begin{bmatrix} a_1 & 0 & 0 & 0 & a_3 & a_2 \\ 0 & a_2 & 0 & a_3 & 0 & a_1 \\ 0 & 0 & a_3 & a_2 & a_1 & 0 \end{bmatrix} \tag{7}$$

$$-[\mathbf{a}\times]\tilde{\mathbf{J}}\mathbf{a} = \mathbf{G}(\mathbf{a})\tilde{\boldsymbol{\theta}} \tag{8}$$

$$\mathbf{G}(\mathbf{a}) = -[\mathbf{a}\times]\mathbf{L}(\mathbf{a}) \tag{9}$$

It follows from definitions Eqs (6)-(9) that Eq (4) may be expressed in the elegant form

$$\widetilde{\mathbf{u}}(t) = \mathbf{\Phi}^{\mathrm{T}}(\mathbf{\omega}(t), \mathbf{\omega}_{\mathbf{r}}(t), \mathbf{\alpha}_{\mathbf{r}}(t))\widetilde{\mathbf{\theta}}(t) \tag{10}$$

$$\mathbf{\Phi}(\mathbf{\omega}(t), \mathbf{\omega}_{\mathbf{r}}(t), \mathbf{\alpha}_{\mathbf{r}}(t)) = -\left\{\mathbf{L}(\mathbf{\alpha}_{\mathbf{r}}(t)) + \left[\mathbf{\omega}_{\mathbf{r}}(t)\times\right]\mathbf{L}(\mathbf{\omega}(t))\right\}^{\mathrm{T}} \tag{11}$$

Using the control law of Eq (2) the closed-loop dynamic equations of motion are given by

$$\mathbf{J}\dot{\mathbf{s}}(t) = -\mathbf{K}_{\mathbf{D}}\mathbf{s}(t) + \left[\mathbf{J}\mathbf{\omega}(t)\times\right]\mathbf{s}(t) + \widetilde{\mathbf{u}}(t) \tag{12}$$

Select a Lyapunov candidate function given by

$$V(\mathbf{s}(t), \widetilde{\mathbf{\theta}}(t)) = \frac{1}{2}\mathbf{s}^{\mathrm{T}}(t)\mathbf{J}\mathbf{s}(t) + \frac{1}{2}\widetilde{\mathbf{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\widetilde{\mathbf{\theta}}(t) \tag{13}$$

which is an extension of Eq (18) in Chapter 4 to include a positive definite term in the inertia parameter error vector, where $\mathbf{\Gamma}^{-1}$ is a positive definite constant weighting matrix. The derivative of Eq (13) is given by

$$\dot{V}(\mathbf{s}(t), \widetilde{\mathbf{\theta}}(t)) = \mathbf{s}^{\mathrm{T}}(t)\mathbf{J}\dot{\mathbf{s}}(t) + \widetilde{\mathbf{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\dot{\widetilde{\mathbf{\theta}}}(t) \tag{14}$$

A parameter update law $\dot{\widetilde{\mathbf{\theta}}}(t)$ must be designed to update the estimates of the spacecraft inertia parameters $\mathbf{J}^{*}(t)$ such that the Lyapunov function derivative is negative semi-definite. Hence, the parameter update law only updates the inertia estimates to meet the control objectives. Consequently, convergence of the inertia parameter estimates to their true values is not guaranteed. To determine the specific structure of the update law substitute into Eq (14) the expression for the closed-loop sliding vector dynamics of Eq (12)

$$\dot{V}(\mathbf{s}(t), \widetilde{\mathbf{\theta}}(t)) = -\mathbf{s}^{\mathrm{T}}(t)\mathbf{K}_{\mathbf{D}}\mathbf{s}(t) + \mathbf{s}^{\mathrm{T}}(t)\widetilde{\mathbf{u}}(t) + \widetilde{\mathbf{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\dot{\widetilde{\mathbf{\theta}}}(t) \tag{15}$$

Select the parameter update law to cancel the $\mathbf{s}^{\mathrm{T}}(t)\widetilde{\mathbf{u}}(t)$ term in Eq (15) such that

$$\dot{\tilde{\boldsymbol{\theta}}}(t) = -\dot{\boldsymbol{\theta}}^*(t) = -\boldsymbol{\Gamma}\boldsymbol{\Phi}(\boldsymbol{\omega}(t), \boldsymbol{\omega}_{\mathbf{r}}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t))\mathbf{s}(t) \tag{16}$$

where $\dot{\boldsymbol{\theta}}^*(t)$ is the derivative of the inertia parameter estimate, and it is assumed that the true inertia parameters are time-invariant. Substituting Eq (16) into Eq (15) leads to

$$\dot{V}(\mathbf{s}(t), \tilde{\boldsymbol{\theta}}(t)) = -\mathbf{s}^{\mathrm{T}}(t)\mathbf{K}_{\mathbf{D}}\mathbf{s}(t) \tag{17}$$

which is identical to the Eq (25) of Chapter 4. The Lyapunov function derivative of Eq (17) may also be upper bounded according to

$$\dot{V}(\mathbf{s}(t), \tilde{\boldsymbol{\theta}}(t)) \le -k_{d}\mathbf{s}^{\mathrm{T}}(t)\mathbf{s}(t) \tag{18}$$

where $k_d = \lambda_{\min}(\mathbf{K}_{\mathbf{D}})$ is the minimum eigenvalue of the $\mathbf{K}_{\mathbf{D}}$ control matrix. The selection of control gains for a desired closed-loop system response also follows the procedure outlined in Section 4.3.4. The parameter update law of Eq (16) may be modified using the e_1-modification technique to compensate for the presence of external disturbance torques. Details of this technique may be found in Reference 25.

Applying the fundamental principles of Lyapunov's direct method, the negative semi-definite Lyapunov function derivative of Eq (17) guarantees that all state variables are upper bounded by the initial conditions of the state trajectory, such that $\mathbf{s}(t) \in L_{\infty}$ and $\tilde{\boldsymbol{\theta}}(t) \in L_{\infty}$. This in turn implies that $\tilde{\boldsymbol{\omega}}_{\mathbf{c}}(t) \in L_{\infty}$ and the components of the spacecraft attitude quaternion are bounded by definition such that $\tilde{\mathbf{q}}_{\mathbf{c}}(t) \in L_{\infty}$. The positive definiteness of the spacecraft inertia matrix $\mathbf{J}$ implies that $\mathbf{J} \in L_{\infty}$ and $\mathbf{J}^{-1} \in L_{\infty}$. The positive definiteness of the control matrix $\mathbf{K}_{\mathbf{D}}$ and the sliding vector gain λ imply that $\mathbf{K}_{\mathbf{D}} \in L_{\infty}$ and $\lambda \in L_{\infty}$. The reference trajectory parameters $\boldsymbol{\omega}_{\mathbf{dR}}(t) \in L_{\infty}$ and $\dot{\boldsymbol{\omega}}_{\mathbf{dR}}(t) \in L_{\infty}$ are bounded, and $\mathbf{q}_{\mathbf{d}}(t) \in L_{\infty}$ is also bounded by definition. The above results ensure that $\boldsymbol{\omega}(t) \in L_{\infty}$, $\boldsymbol{\alpha}_{\mathbf{r}}(t) \in L_{\infty}$, $\boldsymbol{\omega}_{\mathbf{r}}(t) \in L_{\infty}$, $\boldsymbol{\Phi}(\boldsymbol{\omega}(t), \boldsymbol{\omega}_{\mathbf{r}}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \in L_{\infty}$, and $\mathbf{q}(t) \in L_{\infty}$ is bounded by definition. Furthermore, the Lyapunov function is

finite $\forall t \geq t_0$ such that $0 \leq V(t_\infty) \leq V(t) \leq V(t_0) < \infty$ and integrating Eq (18) for $t \geq t_0$ yields

$$\infty \ > \ -\int_{t_0}^{t_\infty} \dot{V}(\tau) \ d\tau \ \geq \ k_d \int_{t_0}^{t_\infty} \mathbf{s}^T(\tau)\mathbf{s}(\tau) \ d\tau \qquad (19)$$

which shows that $\mathbf{s}(t) \in L_2 \cap L_\infty$. Since all terms in the closed-loop dynamics given by Eq (12) are bounded, this implies that $\dot{\mathbf{s}}(t) \in L_\infty$. Applying Barbalat's lemma[6] results in $\lim_{t \to \infty} \mathbf{s}(t) = \mathbf{0}_{3\times1}$, which implies that $\lim_{t \to \infty} \widetilde{\boldsymbol{\omega}}_c(t) = \mathbf{0}_{3\times1}$, $\lim_{t \to \infty} \widetilde{\boldsymbol{\varepsilon}}_c(t) = \mathbf{0}_{3\times1}$, and $\lim_{t \to \infty} \widetilde{\eta}_c(t) = \pm 1$ (depending on which equilibrium point is the minimum angular path).

5.3 Inertia Parameter Convergence (Method 1)

The objective of this section is to develop a composite adaptive update law for the inertia parameter estimates which guarantees asymptotic convergence of the inertia parameter estimates, in addition to asymptotic convergence of the trajectory tracking errors. Composite adaptation laws[6] will be developed, which are driven by both the trajectory tracking error $\mathbf{s}(t)$ and the output prediction error $\mathbf{e}(t)$. A constant gain composite adaptation law[6] is proposed

$$\dot{\widetilde{\boldsymbol{\theta}}}(t) = -\dot{\boldsymbol{\theta}}^*(t) = -\boldsymbol{\Gamma}\left[\boldsymbol{\Phi}(\boldsymbol{\omega}(t),\boldsymbol{\omega}_r(t),\boldsymbol{\alpha}_r(t))\mathbf{s}(t) + \mathbf{W}^T(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))\mathbf{e}(t)\right] \quad (20)$$

The control torque may be expressed as

$$\mathbf{u}(t) = \mathbf{Y}_1(\dot{\mathbf{s}}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))\boldsymbol{\theta}(t) \qquad (21)$$

which is considered the system output model obtained by linearly parameterising the left-hand side of the closed-loop dynamics given by Eq (1), with $\mathbf{Y}_1(\dot{\mathbf{s}}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t)) = \mathbf{L}(\dot{\mathbf{s}}(t)) + [\boldsymbol{\omega}(t)\times]\mathbf{L}(\boldsymbol{\omega}(t)) + \mathbf{L}(\boldsymbol{\alpha}_r(t))$. The output of Eq (21) cannot be utilised directly for inertia parameter estimation, due to the

presence of the unmeasurable sliding vector derivative $\dot{\mathbf{s}}(t)$. To avoid the dependence on this term, use a first-order filter defined by

$$\mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) = \frac{\alpha}{p + \alpha} \mathbf{Y}_1(\dot{\mathbf{s}}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \tag{22}$$

where p is the Laplace operator and α is a specified positive scalar constant. This leads to a new definition of the system output model given by

$$\mathbf{y}(t) = \mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \boldsymbol{\theta}(t) \tag{23}$$

The output prediction error is defined as the difference between the measured output $\mathbf{y}(t) = \mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \boldsymbol{\theta}(t)$ and the predicted output $\hat{\mathbf{y}}(t) = \mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \hat{\boldsymbol{\theta}}(t)$ such that

$$\mathbf{e}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t) = \mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \widetilde{\boldsymbol{\theta}}(t) \tag{24}$$

The derivative of Eq (13) using the control law of Eq (2) and the parameter update law of Eq (20) is derived as

$$\begin{aligned} \dot{V}(\mathbf{s}(t), \widetilde{\boldsymbol{\theta}}(t)) &= -\mathbf{s}^{\mathrm{T}}(t) \mathbf{K}_{\mathbf{D}} \mathbf{s}(t) \\ &\quad - \widetilde{\boldsymbol{\theta}}^{\mathrm{T}}(t) \mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \mathbf{W}^{\mathrm{T}}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \widetilde{\boldsymbol{\theta}}(t) \end{aligned} \tag{25}$$

Since the matrix product $\mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t)) \mathbf{W}^{\mathrm{T}}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_{\mathbf{r}}(t))$ is not positive definite in general, the Lyapunov function derivative of Eq (25) is negative semi-definite. Note that the introduction of the output error term in the parameter update law of Eq (20) results in an additional term in Eq (25) which is negative definite in the output prediction error. Intuitively, the form of Eq (25) indicates that the Lyapunov function $V(\mathbf{s}(t), \widetilde{\boldsymbol{\theta}}(t))$ will decrease when either the trajectory tracking error $\mathbf{s}(t)$ or the output prediction error $\mathbf{e}(t)$ is not zero. However, it is necessary to show asymptotic convergence of the trajectory tracking and the parameter estimation errors.

Applying the fundamental principles of Lyapunov's direct method, the negative semi-definite Lyapunov function derivative of Eq (25) guarantees that all state variables are upper bounded by the initial conditions of the state trajectory, such that $\mathbf{s}(t) \in L_\infty$ and $\widetilde{\boldsymbol{\theta}}(t) \in L_\infty$. This in turn implies that $\widetilde{\boldsymbol{\omega}}_{\mathbf{c}}(t) \in L_\infty$ and the components of the tracking error quaternion are bounded by definition such that $\widetilde{\mathbf{q}}_{\mathbf{c}}(t) \in L_\infty$. The positive definiteness of the spacecraft inertia matrix $\mathbf{J}$ implies that $\mathbf{J}^{-1} \in L_\infty$. The positive definiteness of the matrices $\mathbf{K_D}$ and $\boldsymbol{\Gamma}^{-1}$, and the sliding vector gain $\boldsymbol{\lambda}$ imply that $\mathbf{K_D} \in L_\infty$, $\boldsymbol{\Gamma} \in L_\infty$ and $\boldsymbol{\lambda} \in L_\infty$. The reference trajectory parameters $\boldsymbol{\omega_{dR}}(t) \in L_\infty$ and $\dot{\boldsymbol{\omega}}_{\mathbf{dR}}(t) \in L_\infty$ are bounded, and $\mathbf{q_d}(t) \in L_\infty$ is also bounded by definition. The results provided above ensure that $\boldsymbol{\omega}(t) \in L_\infty$, $\boldsymbol{\alpha_r}(t) \in L_\infty$, $\boldsymbol{\omega_r}(t) \in L_\infty$, $\boldsymbol{\Phi}(\boldsymbol{\omega}(t), \boldsymbol{\omega_r}(t), \boldsymbol{\alpha_r}(t)) \in L_\infty$, and $\mathbf{q}(t) \in L_\infty$ is bounded by definition. The output matrix is bounded $\mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha_r}(t)) \in L_\infty$ since $\mathbf{s}(t) \in L_\infty$, $\boldsymbol{\omega}(t) \in L_\infty$, $\boldsymbol{\alpha_r}(t) \in L_\infty$.

To apply Barblat's lemma first demonstrate the boundedness of $\ddot{V}(\mathbf{s}(t), \widetilde{\boldsymbol{\theta}}(t))$ since this will demonstrate that $\dot{V}(\mathbf{s}(t), \widetilde{\boldsymbol{\theta}}(t))$ is uniformly continuous. The derivative of Eq (25) is given by

$$\ddot{V}(\mathbf{s}(t), \widetilde{\boldsymbol{\theta}}(t)) = -2\mathbf{s}^\mathrm{T}(t)\mathbf{K_D}\dot{\mathbf{s}}(t) - 2\mathbf{e}^\mathrm{T}(t)\dot{\mathbf{e}}(t) \qquad (26)$$

Hence, to apply Barbalat's lemma it is necessary to show that $\dot{\mathbf{s}}(t)$ and $\dot{\mathbf{e}}(t)$ are bounded. The boundedness of all terms in the closed-loop dynamics of Eq (12) implies that $\dot{\mathbf{s}}(t) \in L_\infty$. It follows that $\dot{\widetilde{\boldsymbol{\omega}}}_{\mathbf{c}}(t) \in L_\infty$ and $\dot{\widetilde{\boldsymbol{\omega}}}_{\mathbf{c}}(t) \in L_\infty$ are bounded terms, and $\dot{\widetilde{\mathbf{q}}}_{\mathbf{c}}(t) \in L_\infty$ due to the boundedness of the $\widetilde{\boldsymbol{\omega}}_{\mathbf{c}}(t)$ and $\widetilde{\mathbf{q}}_{\mathbf{c}}(t)$ parameters. The output prediction error derivative is given by

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{W}}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha_r}(t))\widetilde{\boldsymbol{\theta}}(t) + \mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha_r}(t))\dot{\widetilde{\boldsymbol{\theta}}}(t) \qquad (27)$$

The output prediction error is bounded $\mathbf{e}(t) \in L_\infty$ since the terms $\mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha_r}(t))$ and $\widetilde{\boldsymbol{\theta}}(t)$ are bounded. The parameter update equation is

bounded $\tilde{\dot{\theta}}(t) \in L_\infty$ since $\mathbf{\Gamma}$, $\mathbf{\Phi}(\omega(t), \omega_r(t), \alpha_r(t))$, $\mathbf{W}(\mathbf{s}(t), \omega(t), \alpha_r(t))$, $\mathbf{s}(t)$, and $\mathbf{e}(t)$ are all bounded. It follows that the $\mathbf{W}(\mathbf{s}(t), \omega(t), \alpha_r(t)) \tilde{\dot{\theta}}(t)$ term in Eq (27) is bounded. The boundedness of $\dot{\mathbf{s}}(t)$, $\omega(t)$, and $\alpha_r(t)$ implies the boundedness of $\mathbf{Y}_l(\dot{\mathbf{s}}(t), \omega(t), \alpha_r(t))$ and that of the control torque given by Eq (21). Due to the fact that filter in Eq (22) is exponentially stable and strictly proper, and that $\mathbf{Y}_l(\dot{\mathbf{s}}(t), \omega(t), \alpha_r(t))$ is bounded, the $\dot{\mathbf{W}}(\mathbf{s}(t), \omega(t), \alpha_r(t)) \tilde{\dot{\theta}}(t)$ term is bounded. It follows that the $\dot{\mathbf{W}}(\mathbf{s}(t), \omega(t), \alpha_r(t)) \tilde{\theta}(t)$ term in Eq (27) is bounded. The above results guarantee the boundedness of the $\dot{\mathbf{e}}(t)$ parameter.

The boundedness of $\mathbf{K}_D$, $\mathbf{s}(t)$, $\mathbf{e}(t)$, $\dot{\mathbf{s}}(t)$, $\dot{\mathbf{e}}(t)$ implies the boundedness of $\ddot{V}(\mathbf{s}(t), \tilde{\theta}(t))$ and the uniform continuity of $\dot{V}(\mathbf{s}(t), \tilde{\theta}(t))$. Application of Barbalat's lemma[6] shows that $\dot{V}(\mathbf{s}(t), \tilde{\theta}(t)) \to 0$ as $t \to \infty$. Consequently, the trajectory tracking error $\mathbf{s}(t)$ and the output prediction error $\mathbf{e}(t)$ asymptotically convergence to zero as $t \to \infty$. The convergence of the tracking error $\lim_{t\to\infty} \mathbf{s}(t) = \mathbf{0}_{3\times1}$ guarantees that $\lim_{t\to\infty} \tilde{\omega}_c(t) = \mathbf{0}_{3\times1}$, $\lim_{t\to\infty} \tilde{\varepsilon}_c(t) = \mathbf{0}_{3\times1}$, and $\lim_{t\to\infty} \tilde{\eta}_c(t) = \pm 1$ (depending on which equilibrium point is the minimum angular path).

However, it is necessary to show that the parameter estimation error asymptotically converge to zero. This proof relies on the persistent excitation and uniform continuity of the reference trajectory parameters[45]. It well known the reference trajectory may be persistently excited by superimposing a sinusoidal signal on the reference trajectory, as discussed in Reference 45. The limitation, however, of this approach is that the spacecraft attitude maneuvers will track the perturbed trajectory. A possible method of overcoming this problem is using an intelligently exciting signal[40,46] which decays with the tracking and parameter estimation errors. Either approach however is valid to demonstrate asymptotic convergence of the inertia parameter estimates. The persistent excitation and uniform continuity of the reference trajectory parameters $\mathbf{q}_d(t)$ and $\omega_{dR}(t)$, implies the persistent excitation and uniform

continuity of the matrix $\mathbf{Y}_1(\dot{\mathbf{s}}_d(t), \boldsymbol{\omega}_{dR}(t), \boldsymbol{\alpha}_{rd}(t))$. This matrix is formed by replacing the true spacecraft attitude quaternion $\mathbf{q}(t)$ and angular rates $\boldsymbol{\omega}(t)$ with reference trajectory values in the original $\mathbf{Y}_1(\dot{\mathbf{s}}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_r(t))$ matrix. The persistent excitation and uniform continuity of $\mathbf{Y}_1(\dot{\mathbf{s}}_d(t), \boldsymbol{\omega}_{dR}(t), \boldsymbol{\alpha}_{rd}(t))$ in turn guarantees the persistent excitation of the matrix $\mathbf{W}(\mathbf{s}_d(t), \boldsymbol{\omega}_{dR}(t), \boldsymbol{\alpha}_{rd}(t))$, formed in a manner identical to the $\mathbf{Y}_1(\dot{\mathbf{s}}_d(t), \boldsymbol{\omega}_{dR}(t), \boldsymbol{\alpha}_{rd}(t))$ matrix. Provided that the tracking error $\mathbf{s}(t)$ components converges to zero, then the persistent excitation of $\mathbf{W}(\mathbf{s}_d(t), \boldsymbol{\omega}_{dR}(t), \boldsymbol{\alpha}_{rd}(t))$ implies the persistent excitation of the matrix $\mathbf{W}(\mathbf{s}(t), \boldsymbol{\omega}(t), \boldsymbol{\alpha}_r(t))$. The asymptotic convergence of the inertia parameter estimates to the true parameter values can be demonstrated by recognising that the parameter update law defined by Eq (20) represents an exponentially stable dynamic equation with convergent input $\boldsymbol{\Gamma}\boldsymbol{\Phi}(\boldsymbol{\omega}(t), \boldsymbol{\omega}_r(t), \boldsymbol{\alpha}_r(t))\mathbf{s}(t)$.

Therefore, the control law of Eq (2) and the composite and the constant gain composite adaptation law of Eq (20) ensure the asymptotic convergence to zero of both the trajectory tracking error and output prediction error. This is because composite adaptation uses information related to both these quantities. In contrast, the direct adaptive control strategy defined by Eq (2) and Eq (16) only guarantees asymptotic convergence to zero of the trajectory tracking errors. However, in addition to the more complex update law, the composite adaptive control strategy also requires the filtered torque of Eq (23). This torque is computable in practice because the control torque signals issued by the onboard computer are known.

5.4 Inertia Parameter Convergence (Method 2)

The objective of this section is to develop a composite adaptive update law for the inertia parameter estimates which guarantees exponential convergence of the inertia parameter estimates, in addition to exponential convergence of the trajectory tracking errors. However, before proceedings it is necessary to develop a theorem regarding exponential convergence of the trajectory tracking.

Consider the Lyapunov stability analysis given by Eqs (1) through (19), but with the choice of the control gain matrix $\mathbf{K_D} = k\mathbf{J}^*$. It may be shown that by modifying the matrix $\mathbf{\Phi}(\mathbf{\omega}(t), \mathbf{\omega_r}(t), \mathbf{\alpha_r}(t))$ defined by Eq (11) according to

$$\mathbf{\Phi}(\mathbf{s}(t), \mathbf{\omega}(t), \mathbf{\omega_r}(t), \mathbf{\alpha_r}(t)) = -\left\{ \mathbf{L}(\mathbf{\alpha_r}(t)) + [\mathbf{\omega_r}(t) \times] \mathbf{L}(\mathbf{\omega}(t)) - k\mathbf{L}(\mathbf{s}(t)) \right\}^T \quad (28)$$

then the Lyapunov function derivative becomes

$$\dot{V}(\mathbf{s}(t), \widetilde{\mathbf{\theta}}(t)) = -k\mathbf{s}^T(t)\mathbf{J}\mathbf{s}(t) \quad (29)$$

In the absence of spacecraft inertia uncertainty the choice of control gain matrix $\mathbf{K_D} = k\mathbf{J}^*$ results in an exponentially convergent system since $\dot{V}(\mathbf{s}(t)) = -2kV(\mathbf{s}(t))$ implies that $V(\mathbf{s}(t)) = V(\mathbf{s}(t_0))e^{-2kt}$. Furthermore, since $\mathbf{J}$ is a positive definite matrix then $(\alpha/2)\mathbf{s}^T(t)\mathbf{s}(t) \leq V(\mathbf{s}(t)) = V(\mathbf{s}(t_0))e^{-2kt}$, where $\alpha = \lambda_{min}(\mathbf{J})$ is the minimum eigenvalue of the spacecraft inertia matrix. This demonstrates the exponential convergence of $\mathbf{s}(t)$ to zero with convergence rate k. Furthermore, selecting $\lambda = k$ in the sliding vector definition, implies the exponential convergence of the tracking errors $\widetilde{\mathbf{\omega}}_c(t)$ and $\widetilde{\mathbf{\varepsilon}}_c(t)$ to zero, with convergence an identical convergence rate to $\mathbf{s}(t)$.

Consider the development of an adaptive control strategy which ensures exponentially convergence of the trajectory tracking errors and inertia parameter estimates. The stability analysis is similar to Section 5.3 except that a bounded-gain-forgetting (BFG) composite adaptation law[6] is proposed

$$\begin{aligned}
\dot{\widetilde{\mathbf{\theta}}}(t) = -\dot{\mathbf{\theta}}^*(t) &= -\mathbf{\Gamma}(t)\mathbf{\Phi}(\mathbf{\omega}(t), \mathbf{\omega_r}(t), \mathbf{\alpha_r}(t))\mathbf{s}(t) \\
&\quad - \mathbf{\Gamma}(t)\mathbf{W}^T(\mathbf{s}(t), \mathbf{\omega}(t), \mathbf{\alpha_r}(t))\mathbf{e}(t) \quad (30)
\end{aligned}$$

where $\mathbf{\Gamma}(t)$ is a uniformly positive definite weighting matrix, and the remaining other parameters are defined as in Section 5.3. This parameter estimation strategy uses an exponentially forgetting least-squares gain update given by

$$\frac{d}{dt}\mathbf{\Gamma}^{-1}(t) = -\beta(t)\mathbf{\Gamma}^{-1}(t) + \mathbf{W}^T(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))\mathbf{W}(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t)) \qquad (31)$$

where $\beta(t) = \beta_0\left[1 - \dfrac{\|\mathbf{\Gamma}\|_F}{k_0}\right]$ is a time-varying forgetting factor variable in which

β_0 and k_0 are positive scalar constants specifying the maximum forgetting rate

and the upper bound on the gain matrix $\mathbf{\Gamma}$ norm respectively, and $\|\mathbf{\Gamma}\|_F$ denotes

the Frobenius norm of the $\mathbf{\Gamma}$ matrix. This gain update strategy ensures that

$\beta(t) \geq 0$ and $\mathbf{\Gamma}(t) \leq k_0\mathbf{I}_{6\times 6}$ for any signal $\mathbf{W}(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))$, and that

$\beta(t) \geq \beta_1$, where β_1 is positive scalar constant, provided that

$\mathbf{W}(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))$ is a persistently exciting matrix.

The derivative of Eq (13) using the control law of Eq (2) (with gain matrix

$\mathbf{K}_D = k\mathbf{J}^*$) and the parameter update law of Eq (30) is derived as

$$\dot{V}(\mathbf{s}(t),\widetilde{\boldsymbol{\theta}}(t)) = -k\mathbf{s}^T(t)\mathbf{J}\mathbf{s}(t) - \frac{1}{2}\beta(t)\widetilde{\boldsymbol{\theta}}^T(t)\mathbf{\Gamma}^{-1}\widetilde{\boldsymbol{\theta}}(t)$$
$$- \widetilde{\boldsymbol{\theta}}^T(t)\mathbf{W}(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))\mathbf{W}^T(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))\widetilde{\boldsymbol{\theta}}(t) \qquad (32)$$

Since the matrix product $\mathbf{W}(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))\mathbf{W}^T(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_r(t))$ and the

constant $\beta(t)$ are not positive definite in general, the Lyapunov function

derivative of Eq (32) is negative semi-definite. Note that the parameter update

law of Eq (30) results in an additional term in Eq (32) in comparison with Eq

(25). Intuitively, the form of Eq (32) indicates that the Lyapunov function

$V(\mathbf{s}(t),\widetilde{\boldsymbol{\theta}}(t))$ will decrease when either the trajectory tracking error $\mathbf{s}(t)$ or the

output prediction error $\mathbf{e}(t)$ is not zero, or when the parameter estimation error

$\widetilde{\boldsymbol{\theta}}(t)$ is not zero provided that $\beta(t) > 0$. However, it is necessary to show

exponential convergence of the trajectory tracking errors and the parameter

estimation errors.

An analysis similar to that presented in Section 5.3 can be performed to show

that the parameters $\mathbf{s}(t)$, $\widetilde{\boldsymbol{\theta}}(t)$, $\mathbf{e}(t)$, $\dot{\mathbf{s}}(t)$, $\dot{\widetilde{\boldsymbol{\theta}}}(t)$, $\dot{\mathbf{e}}(t)$ are bounded. In addition,

the forgetting factor $\beta(t)$ and its derivative $\dot{\beta}(t)$ are bounded, since the weighting matrix $\mathbf{\Gamma}(t)$ and its derivative $\dot{\mathbf{\Gamma}}(t)$ are both bounded. The above results imply that $\ddot{V}(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t))$ is bounded since the derivative of Eq (32) is given by

$$\ddot{V}(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t)) = -2k\mathbf{s}^{\mathrm{T}}(t)\mathbf{J}\dot{\mathbf{s}}(t) - \frac{1}{2}\dot{\beta}(t)\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t)$$
$$- \beta(t)\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}}(t) - 2\mathbf{e}^{\mathrm{T}}(t)\dot{\mathbf{e}}(t) \tag{33}$$

The boundedness of $\ddot{V}(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t))$ implies that $\dot{V}(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t))$ is unirformly continuous. Application of Barbalat's lemma[6] shows that $\dot{V}(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t)) \to 0$ as $t \to \infty$. Consequently, the trajectory tracking error $\mathbf{s}(t)$, the output prediction error $\mathbf{e}(t)$, and the $\beta(t)\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t)$ term asymptotically convergence to zero as $t \to \infty$. The convergence of the tracking error $\lim_{t\to\infty}\mathbf{s}(t) = \mathbf{0}_{3\times1}$ guarantees that $\lim_{t\to\infty}\tilde{\boldsymbol{\omega}}_{\mathbf{c}}(t) = \mathbf{0}_{3\times1}$, $\lim_{t\to\infty}\tilde{\boldsymbol{\varepsilon}}_{\mathbf{c}}(t) = \mathbf{0}_{3\times1}$, and $\lim_{t\to\infty}\tilde{\eta}_{c}(t) = \pm1$ (depending on which equilibrium point is the minimum angular path).

However, it is necessary to show that the parameter estimation errors exponentially converge to zero. Asymptotic convergence may be demonstrated by recognising that the parameter update law of Eq (31) guarantees that $\mathbf{\Gamma}(t) \le k_0\mathbf{I}_{6\times6}$ for any signal $\mathbf{W}(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_{\mathbf{r}}(t))$, and that $\beta(t) \ge \beta_1$, where β_1 is positive scalar constant, provided that $\mathbf{W}(\mathbf{s}(t),\boldsymbol{\omega}(t),\boldsymbol{\alpha}_{\mathbf{r}}(t))$ is a persistently exciting matrix (see discussion on persistent excitation in Section 5.3). Consequently, the inequality

$$\beta(t)\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t) \ge \frac{\beta_1}{k_0}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t) \tag{34}$$

and the convergence of $\beta(t)\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\mathbf{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t)$ to zero imply that $\lim_{t\to\infty}\tilde{\boldsymbol{\theta}}(t) = \mathbf{0}_{6\times1}$. However, it is necessary to demonstrate the exponential convergence of both the

trajectory tracking and parameter estimation errors. The Lyapunov derivative of Eq (32) may be upper bounded according to

$$\dot{V}(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t)) \le -k\mathbf{s}^{T}(t)\mathbf{J}\mathbf{s}(t) - \frac{1}{2}\beta(t)\tilde{\boldsymbol{\theta}}^{T}(t)\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t) \qquad (35)$$

Defining a strictly positive constant $\gamma = \min\{2k,\beta_1\}$, then Eq (35) may be further reduced to

$$\dot{V}(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t)) \le -\gamma V(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t)) \qquad (36)$$

Therefore, the solution to the Lyapunov function of Eq (13) is upper bounded according to $V(\mathbf{s}(t),\tilde{\boldsymbol{\theta}}(t)) \le V(\mathbf{s}(t_0),\tilde{\boldsymbol{\theta}}(t_0))e^{-\gamma t}$. This in turn implies the exponential convergence of the trajectory tracking error $\mathbf{s}(t)$ and parameter estimation error $\tilde{\boldsymbol{\theta}}(t)$ to zero. The exponential convergence of the tracking errors $\tilde{\boldsymbol{\omega}}_c(t)$ and $\tilde{\boldsymbol{\varepsilon}}_c(t)$ to zero follows from the exponential convergence of $\mathbf{s}(t)$, provided $\lambda = k$ in the sliding vector definition. The introduction of a time-varying weighting parameter in the bounded-gain-forgetting composite adaptation law of Eq (30), allows a stronger result of exponential convergence to zero of the trajectory tracking errors and the parameter estimation errors.

5.5 Attitude Regulation Maneuver

Spacecraft attitude maneuver simulation results are presented for an inertial pointing spacecraft to demonstrate the capabilities of the direct adaptive control law developed in Section 5.2. The simulations were conducted using an attitude control system model developed in MATLAB™ Simulink® with numerical integration performed using a fourth-order Runge-Kutta numerical integrator ODE4 with a fixed step size of $\Delta t = 0.1 \ sec$. The MATLAB™ line style designators for all simulation plots are 1-axis = solid line, 2-axis = dashed line, 3-axis = dash-dot line, and 4-axis = dotted line.

Parameters	Values
Spacecraft Inertia Matrix	$\mathbf{J} = \begin{bmatrix} 600.28 & -3.57 & 0.17 \\ -3.57 & 611.44 & 0.25 \\ 0.17 & 0.25 & 507.90 \end{bmatrix}$ $\mathbf{J}^*(t_0) = \begin{bmatrix} 500 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 500 \end{bmatrix}$
Reference Trajectory	$\mathbf{q_d} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\boldsymbol{\omega_d} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
Initial Conditions	$\mathbf{q}(t_0) = \begin{bmatrix} -\sqrt{1/6} & \sqrt{1/6} & -\sqrt{1/6} & -\sqrt{1/2} \end{bmatrix}^T$ $\hat{\mathbf{e}}(t_0) = \begin{bmatrix} -0.5774 & 0.5774 & -0.5774 \end{bmatrix}^T$ $\theta(t_0) = 270 \text{ deg}$ $\boldsymbol{\omega}(t_0) = [10.0 \quad -10.0 \quad 5.0]^T$
Control Gains	$\mathbf{K}_1 = 0.16\mathbf{J}^*(t_0)$ $k = 0.32$ $\mathbf{K} = 0.5\mathbf{I}_{3\times3}$

Table 5-1 Adaptive control attitude maneuver

The control law of Eq (2) with the disturbance compensation term $\boldsymbol{\tau}(t) = \mathbf{0}_{3\times1}$ and inertia parameter update law of Eq (16) are implemented with reference trajectory, initial conditions, and control gains defined in Table 5-1. The only environmental disturbance torque considered is the gravity-gradient torque which is exactly compensated for in the control law design. The control gains are selected using the procedure outlined in Section 4.3.4 of Chapter 4 but the spacecraft inertia matrix present in the $\mathbf{K}_1$ control gain is replaced with the inertia matrix estimate at the initial simulation time.

Figure 5-1 through Figure 5-9 present the results of the rest-to-rest attitude maneuver which are similar to the Chapter 4 sliding mode design example. The error quaternion displays a critically damped response with the $\delta q_4(t)$ component settling to the desired minimum-angle maneuver equilibrium point,

and the system settling time of 50 sec has been achieved. This demonstrates the $\text{sgn}[\delta q_4(t)]$ term in the sliding vector defined by Eq (11) of Chapter 4 is correctly forcing the spacecraft attitude motion towards the correct minimum angle equilibrium point. Figure 5-6 demonstrates that Wie's theorem of an eigenaxis rotation for rest-to-rest maneuvers discussed in Section 3.4 of Chapter 3 does not apply for direct adaptive control based attitude maneuvers. Despite the spacecraft moment of inertia parameters in Figure 5-9 not converging to their correct values (which is reflected in the Lyapunov function of Figure 5-7) the trajectory tracking errors asymptotically converge to zero.
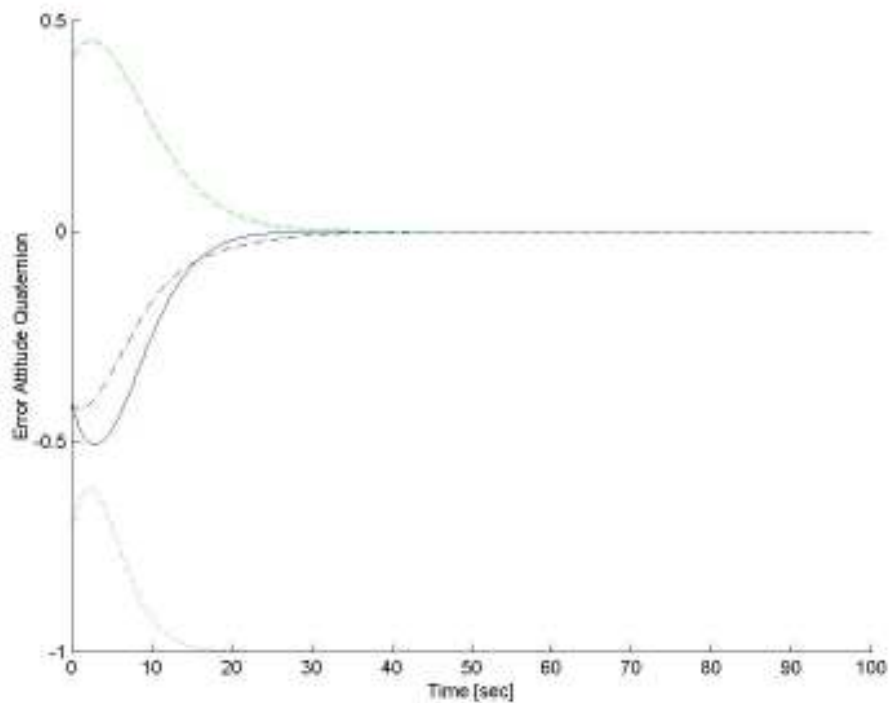
Figure 5-1 Spacecraft attitude quaternion tracking error
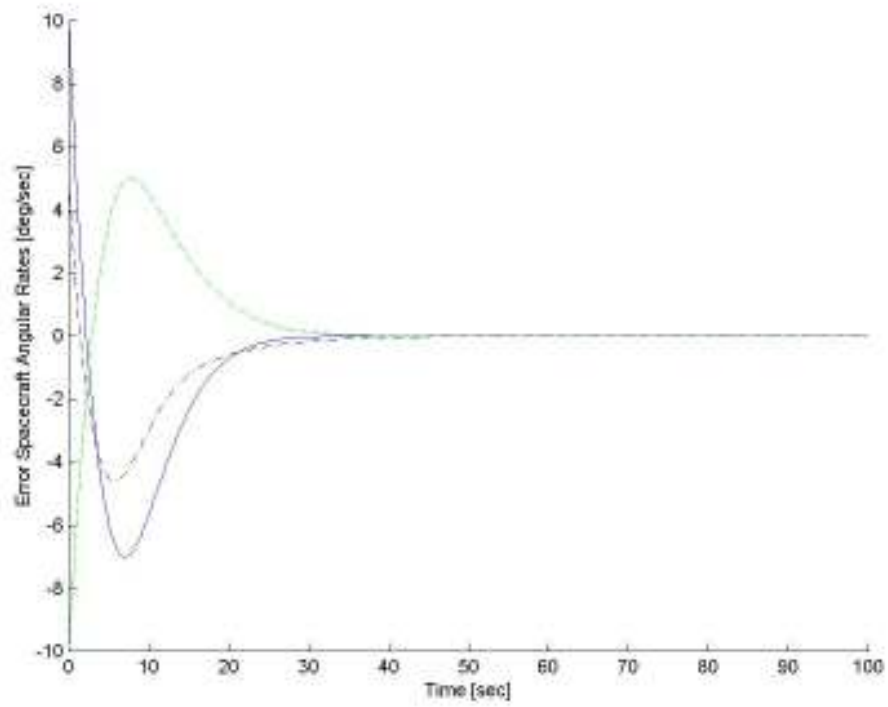
Figure 5-2 Spacecraft angular rates
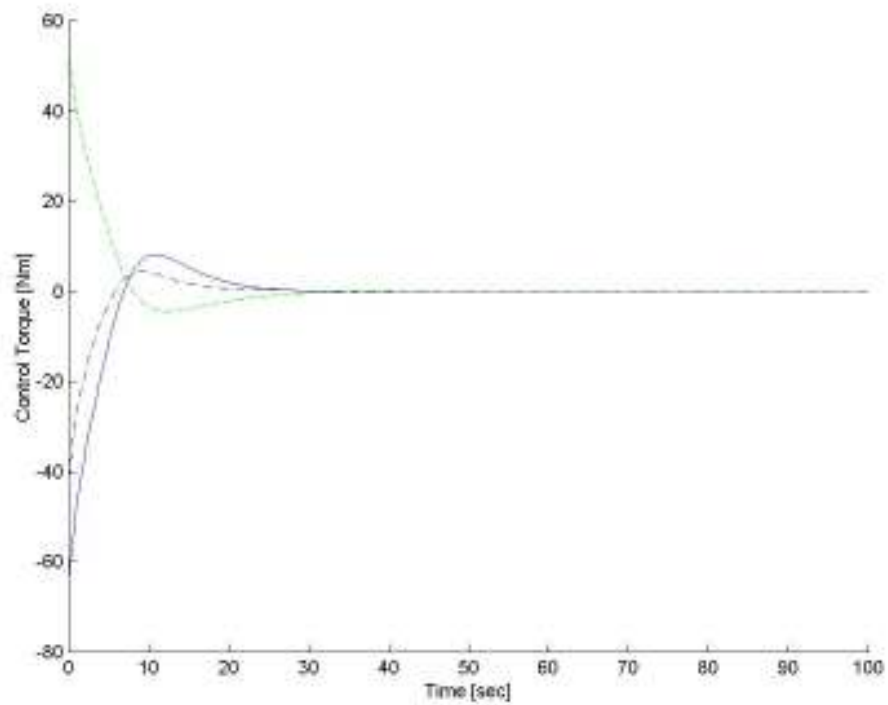
Figure 5-3 External control torques

205
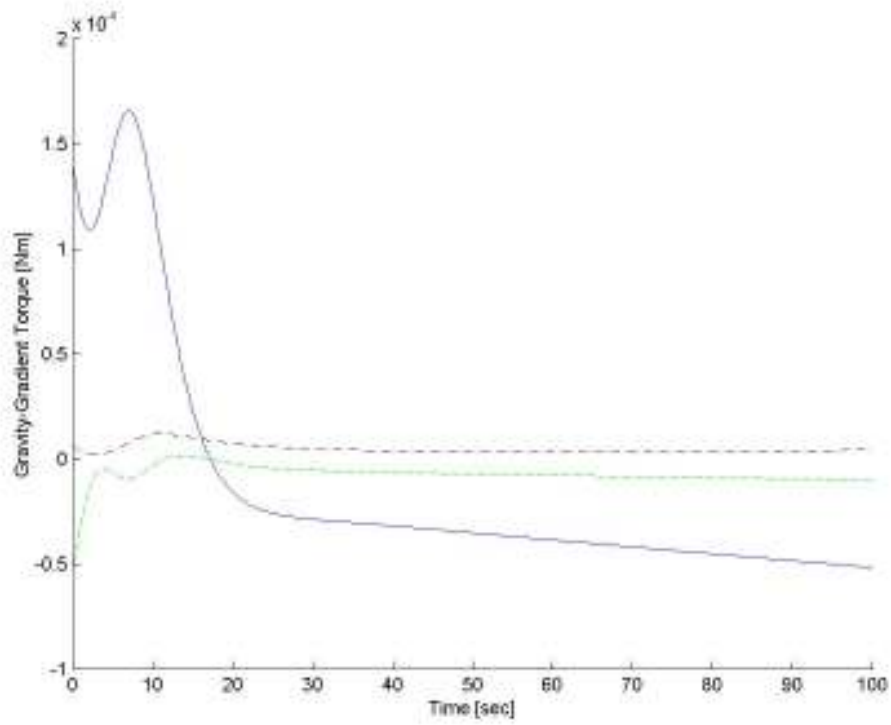
Figure 5-4 Gravity-gradient disturbance torques
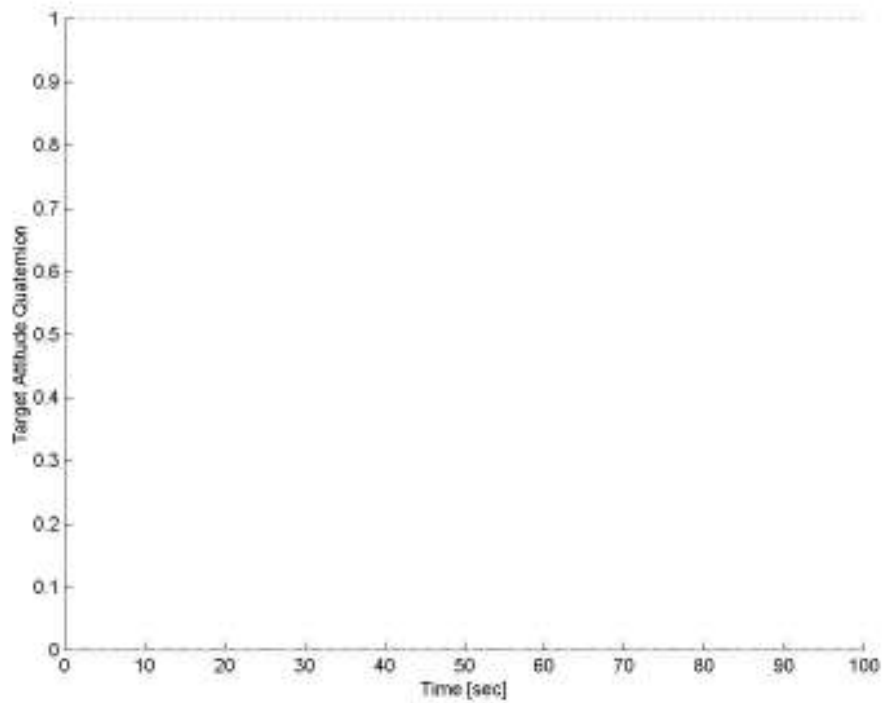
Figure 5-5 Reference trajectory attitude quaternion
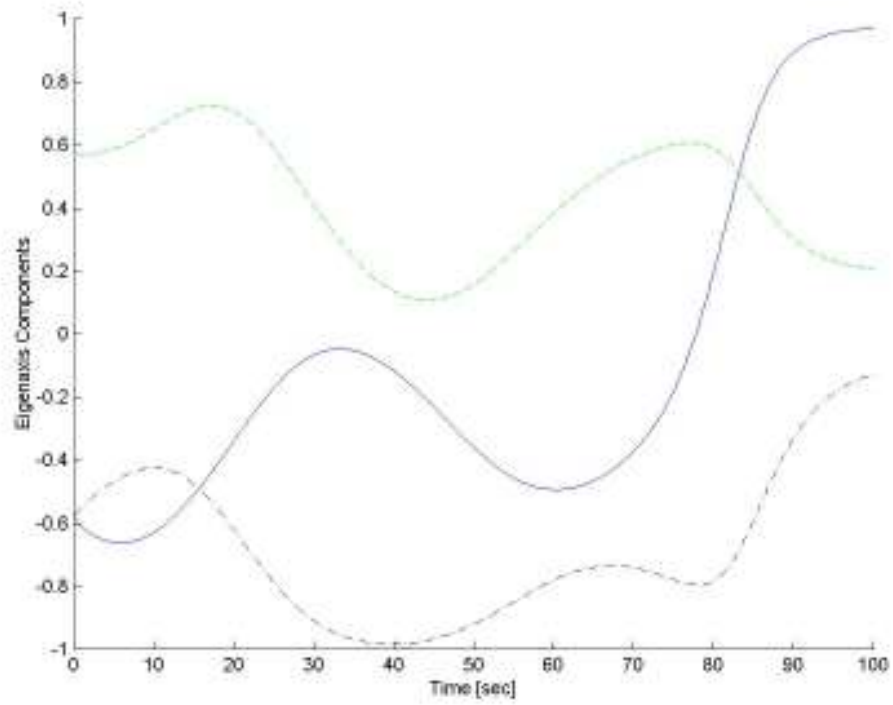
Figure 5-6 Euler axis components of tracking error quaternion
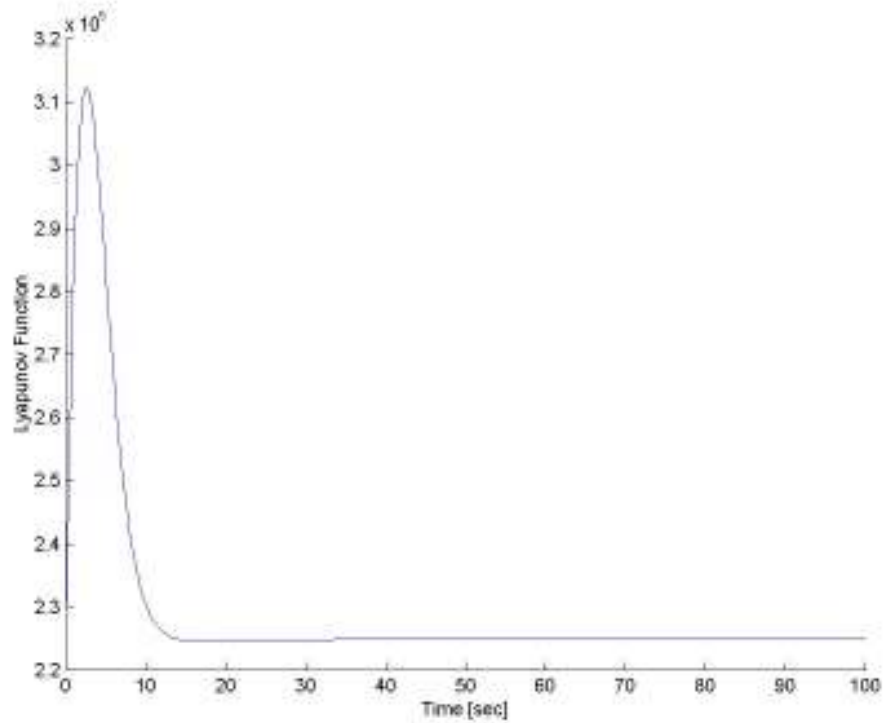
Figure 5-7 Lyapunov function
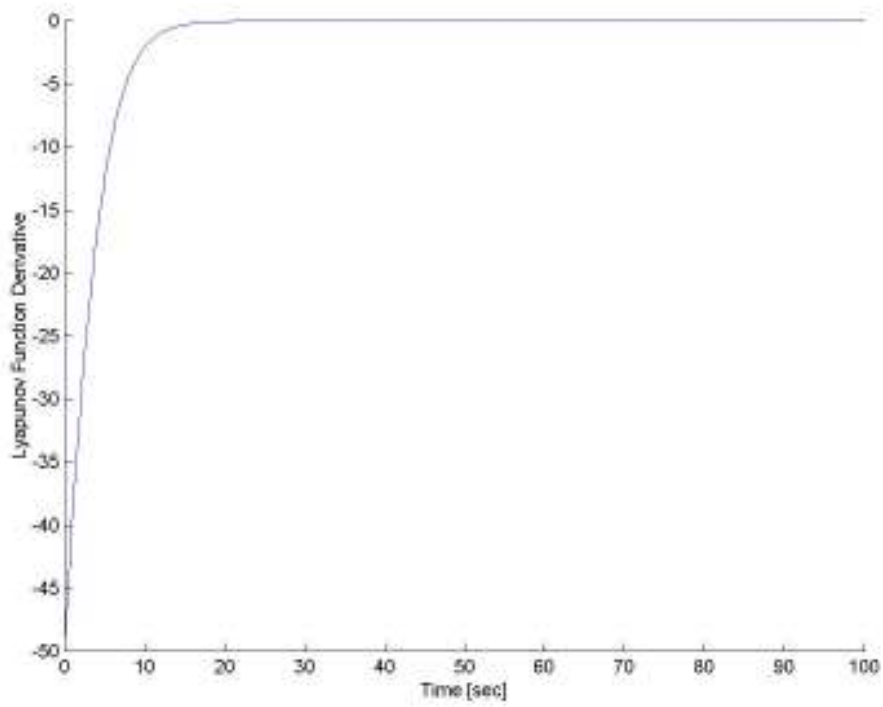
Figure 5-8 Lyapunov function derivative
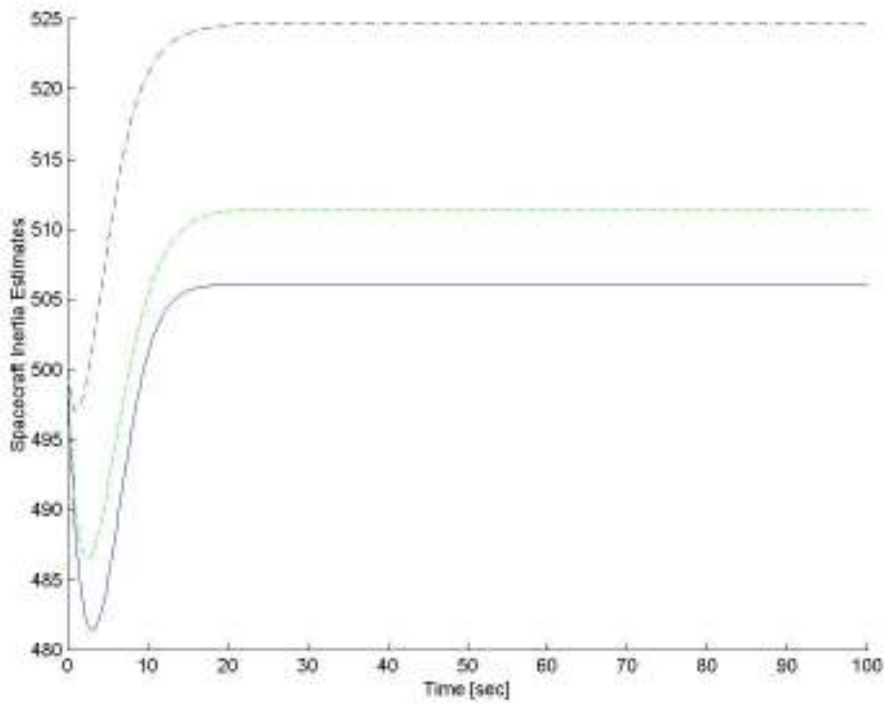
Figure 5-9 Spacecraft Moment of Inertia Estimates

5.6   Conclusion

In this chapter, a novel direct adaptive control strategy with composite parameter adaptation was developed for attitude tracking maneuvers. This strategy accounts for spacecraft inertia uncertainty, and guarantees exponential convergence to zero of the trajectory tracking errors and parameter estimation errors. MATLAB™ simulation results were presented to demonstrate the performance of the basic control strategy and investigate the convergence properties of the inertia parameter estimates.

5.7   References

[1] Bellman, R. (1961). *Adaptive control processes: a guided tour*. USA: Princeton University Press.

[2] Landau, Y.D. (1979). *Adaptive control: the model reference approach*. USA: Marcel Dekker.

[3] Sastry, S., and M. Bodson. (1989). *Adaptive control: stability, convergence, and robustness*. USA: Prentice-Hall, Inc.

[4] Astrom, K.J., and B. Wittenmark. (1989). *Adaptive control*. USA: Addison-Wesley Publishing.

[5] Narendra, K.S., and A.M. Annaswamy. (1989). *Stable adaptive systems*. USA: Prentice Hall, Inc..

[6] Slotine, J.J.E., and W. Li. (1990). *Applied nonlinear control*. USA: Prentice-Hall, Inc.

[7] Kristic, M., Kanellakopoulos, I., and P.V. Kokotovic. (1995). *Nonlinear and adaptive control design*. USA: Wiley-Interscience.

[8] Iannou, P.A., and J. Sun. (1995). *Stable and robust adaptive control*. USA: Prentice-Hall, Inc.

[9] Khalil, H.K. (2002). *Nonlinear systems*, 3rd ed. USA: Prentice-Hall, Inc.

[10] Narendra, K.S., and A.M. Annaswamy. (1986). Robust adaptive control in the presence of bounded disturbances. *IEEE Transactions on Automatic Control*, 31(4): 306-315.

[11] Narendra, K.S., and A.M. Annaswamy. (1987). A new adaptive law for robust adaptive control without persistent excitation. *IEEE Transactions on Automatic Control*, 32(2): 134-145.

[12] Slotine, J-J.E., and M.D. Di Benedetto. (1990). Hamiltonian adaptive control of spacecraft. *IEEE Transactions on Automatic Control*, 35(7): 848-852.

[13] Parlos, A.G., and J.W. Sunkel. (1992). Adaptive attitude control and momentum management for large-angle spacecraft maneuvers. *The Journal of Guidance, Control, and Dynamics*, 15(4): 1018-1028.

[14] Boussalis, D., Bayard, D.S., and S.J. Wang. (1992). Adaptive spacecraft attitude control with application to space station. *Proceedings of the IEEE Conference on Control Applications*, Dayton, Ohio, USA, 440-447.

[15] Paielli, R.A., and R.E. Bach. (1993). Attitude control with realization of linear error dynamics. *The Journal of Guidance, Control, and Dynamics*, 16(1): 182-189.

[16] Cristi, R., and J. Burl. (1993). Adaptive eigenaxis rotations. *Proceedings of the European Control Conference, The Netherlands, June 1993*, 770-775.

[17] Sheen, J-J., and R.H. Bishop. (1994). Adaptive nonlinear control of spacecraft. *Proceedings of the American Control Conference*, Baltimore, Maryland, USA, 2867-2871.

[18] Cristi, R., Burl, J., and N. Russo. (1994). Adaptive quaternion feedback regulation for eigenaxis rotations. *The Journal of Guidance, Control, and Dynamics*, 17(6): 1287-1291.

[19] Joshi, S.M., Kelkar, A.G., and J.T-Y. Wen. (1995). Robust attitude stabilization of spacecraft using nonlinear quaternion feedback. *IEEE Transactions on Automatic Control*, 40(10): 1800-1803.

[20] Di Gennaro, S. (1995). Adaptive robust stabilization of rigid spacecraft in presence of disturbances. *Proceedings of the IEEE Conference on Decision and Control*, New Orleans, Louisiana, USA, 1147-1152.

[21] Paynter, S.J., and R.H. Bishop. (1997). Adaptive nonlinear attitude control and momentum management of spacecraft. *The Journal of Guidance, Control, and Dynamics*, 20(5): 1025-1032.

[22] Junkins, J.L., Akella, M.R., and R.D. Robinett. (1997). Nonlinear adaptive control of spacecraft maneuvers. *The Journal of Guidance, Control, and Dynamics*, 20(6): 1104-1110.

[23] Di Gennaro, S. (1997). Stabilization of rigid spacecraft with uncertainties and input saturation in a central gravitational field. *Proceedings of the IEEE Conference on Decision and Control*, San Diego, California, USA, 4204-4209.

[24] Akella, M.R., Junkins, J.L., and R.D. Robinett. (1998). Structured model reference adaptive control with actuator saturation limits. *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference, Boston*, Massachusetts, USA, AIAA Paper 98-4472, 308-315.

[25] Akella, M.R., and J.L. Junkins. (1998). Structured model reference adaptive control in the presence of bounded disturbances. *Proceedings of the AAS/AIAA Spaceflight Mechanics Meeting*, Monterey, California, USA, AAS Paper 98-121, 375-385.

[26] Ahmed, J. Coppola, V.T., and D.S. Bernstein. (1998). Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification. *The Journal of Guidance, Control, and Dynamics*, 21(5): 684-691.

[27] Boskovic, J.D., Sai-Ming, L., and R.K. Mehra. (1999). Globally stable adaptive tracking control design for spacecraft under input saturation. *Proceedings of the IEEE Conference on Decision and Control*, Phoenix, Arizona, USA, 1952-1957.

[28] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2000). A globally stable scheme for spacecraft control in the presence of sensor bias. *Proceedings of the IEEE Aerospace Conference*, Big Sky, Montana, USA, 505-511.

[29] Schaub, H., Akella, M.R., and J.L. Junkins. (2000). Adaptive realization of linear closed-loop tracking dynamics in the presence of large system model errors. *The Journal of the Astronautical Sciences*, 48(4): 537-551.

[30] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2001). Robust adaptive variable structure control of spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 24(1): 14-22.

[31] Tsiotras, P., Shen, H., and C. Hall. (2001). Satellite attitude control and power tracking with energy/momentum wheels. *Journal of Guidance, Control, and Dynamics*, 24(1): 23-34.

[32] Schaub, H., Akella, M.R., and J.L. Junkins. (2001). Adaptive control of nonlinear attitude motions realizing linear closed loop dynamics. *The Journal of Guidance, Control, and Dynamics*, 24(1): 95-100.

[33] Yoon, H., and P. Tsiotras. (2002). Spacecraft adaptive attitude and power tracking with variable speed control moment gyroscopes. *The Journal of Guidance, Control, and Dynamics*, 25(6): 1081-1090.

[34] Miwa, H., and M.R. Akella. (2002). Global adaptive stabilization using output feedback for spacecraft attitude tracking. *Proceedings of the AAS/AIAA Spaceflight Mechanics Meeting*, San Antonio, Texas, USA, 345-357.

[35] Subbarao, K., and J.. Junkins. (2003). Structured model reference adaptive control for a class of nonlinear systems. *The Journal of Guidance, Control, and Dynamics*, 26(4): 551-557.

[36] Tandale, M.D., Subbarao, K., Valasek, J., and M.R. Akella. (2004). Structured adaptive model inversion control with actuator saturation constraints applied to tracking spacecraft maneuvers. *Proceedings of the American Control Conference*, Boston, Massachusetts, USA, 1031-1036.

[37] Lavretsky, E., and N. Hovakimyan. (2004). Positive µ-modification for stable adaptation in the presence of input constraints. *Proceedings of the American Control Conference*, Boston, Massachusetts, USA, 2545-2550.

[38] Boskovic, J.D., Li, S-M., and R.K. Mehra. (2004). Robust tracking control design for spacecraft under control input saturation. *The Journal of Guidance, Control, and Dynamics*, 27(4): 627-633.

[39] Yoon, H., and P. Tsiotras. (2005). Adaptive spacecraft attitude tracking control with actuator uncertainties. *Proceedings of the AIAA Guidance, Navigation, and Control Conference, San Francisco, California, USA, AIAA Paper 2005-6392*, 1031-1036.

[40] Cao, C., and N. Hovakimyan. (2005). Vision-based aerial tracking using intelligent excitation. *Proceedings of the American Control Conference, Portland, Oregon, USA, June 8-10,* 5091-5096.

[41] Wallsgrove, R. J., and M.R. Akella. (2005). Globally stabilizing saturated attitude control in the presence of bounded unknown disturbances. *The Journal of Guidance, Control, and Dynamics*, 28(5): 957-963.

[42] Tandale, M.D., and J. Vasek. (2006). Fault-tolerant structured adaptive model inversion control. *The Journal of Guidance, Control, and Dynamics*, 29(3): 635-642.

[43] Singla, P., Subbarao, K., and J.L. Junkins. (2006). Adaptive output feedback control for spacecraft rendezvous and docking under measurement uncertainty. *The Journal of Guidance, Control, and Dynamics*, 29(4): 892-902.

[44] Lavretsky, E., and N. Hovakimyan. (2007). Stable adaptation in the presence of actuator constraints with flight control applications. *The Journal of Guidance, Control, and Dynamics*, 30(2): 337-345.

[45] Stepanyan, V., and N. Hovakimyan. (2007). Adaptive disturbance rejection controller for visual tracking of a maneuvering target. *Journal of Guidance, Control, and Dynamics*, 30(4): 1090-1106.

[46] Cao, C., Hovakimyan, N., and J. Wang. (2007). Intelligent excitation for adaptive control with unknown parameters in reference input. *IEEE Transactions on Automatic Control*, 52(8): 1525-1532.

Chapter 6   Spacecraft Attitude Maneuvers Using Optimal Control Theory

6.1 Introduction

This chapter considers the development of an optimal control strategy[1-5] for three-axis spacecraft attitude maneuvers, subject to control torque constraints. The specific configuration to be considered is a rigid-body spacecraft equipped with a redundant reaction wheel assembly. For attitude maneuvers using reaction wheels the minimisation of the electrical energy consumed by the wheel motors is a useful optimal criterion. Vadali and Junkins[14-15] considered a class of optimal attitude maneuvers which minimise time integrals involving the reaction wheel motor torques and torque derivatives. Minimum motor torque maneuvers were investigated by Skaar and Kraige[12,13] using a performance index which is the integral of the reaction wheel motor power squared over the maneuver interval. Although the integral of power of the maneuver interval is a more suitable performance index, it represents the total mechanical work. The disadvantage is that the optimal criterion rewards wheel braking (or negative work) and penalises positive work. In this chapter, a performance index is proposed which represents the total electrical energy consumed by the reaction wheel motors over the attitude maneuver interval. A torque-constrained optimal spacecraft attitude maneuver based on this performance index has not been addressed in the literature, and therefore represents one of the major novel contributions provided in this thesis. Pontryagin's minimum principle[1,3,5] is used to formulate the necessary conditions for optimality, in which the control torques are subject to time-varying magnitude constraints. Necessary conditions for the optimality of finite-order singular arcs are established and the computation of corresponding singular controls is performed using Kelley's necessary condition[4]. The two-point boundary value problem (TPBVP) is formulated using Pontryagin's minimum principle.

Section 6.2 presents the spacecraft dynamic and kinematic models. A general formulation of the optimal control problem is outlined in Section 6.3. Sections 6.4 and 6.5 develop the necessary conditions for optimality of the minimum-torque and minimum-energy problems respectively, including the constrained

optimal control structure, singular controls, and the two-point boundary value problem.

6.2 Spacecraft Dynamic and Kinematic Model

This section specifies the dynamic and kinematic model for a rigid-body spacecraft equipped with m reaction wheels. A specific implementation for $m = 4$ has three wheels aligned with the spacecraft body-fixed coordinate frame, and a fourth wheel skewed at 45 degrees with respect to this frame.

6.2.1 Attitude Parameters

The spacecraft body-fixed coordinate frame $\{\mathbf{B}(t)\}$ is related to the inertial coordinate frame $\{\mathbf{N}\}$ via the direction cosine matrix[8-10] $\mathbf{C}(t) \in \mathbf{R}^{3 \times 3}$ (also called the spacecraft attitude matrix) according to

$$\{\mathbf{B}(t)\} = \mathbf{C}(t)\{\mathbf{N}\} \tag{1}$$

The unit quaternion $\mathbf{q}(t) = \begin{bmatrix} \mathbf{q}_{13}^T(t) & q_4(t) \end{bmatrix}^T \in \mathbf{R}^{4 \times 1}$ (also called the Euler symmetric parameters) is selected to parameterise the spacecraft attitude an any point in time t. The components of the unit quaternion are not independent but satisfy the unity-norm constraint

$$\sum_{i=1}^{4} q_i^2(t) = 1 \tag{2}$$

The quaternion parameters are related to the $\{\mathbf{B}(t)\}$ components of the spacecraft angular velocity $\boldsymbol{\omega}(t) \in \mathbf{R}^{3 \times 1}$ by the kinematic equations of motion[8-10]

$$\dot{\mathbf{q}}(t) = \frac{1}{2}\boldsymbol{\Xi}(\mathbf{q}(t))\boldsymbol{\omega}(t) = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}(t))\mathbf{q}(t) \tag{3}$$

The matrices $\boldsymbol{\Xi}(\mathbf{q}(t)) \in \mathbf{R}^{4 \times 3}$ and $\boldsymbol{\Omega}(\boldsymbol{\omega}(t)) \in \mathbf{R}^{4 \times 4}$ are defined by

$$\mathbf{\Xi}(\mathbf{q}(t)) = \begin{bmatrix} [\mathbf{q}_{13}(t)\times] + q_4(t)\mathbf{I}_{3\times3} \\ -\mathbf{q}_{13}^{T}(t) \end{bmatrix} \qquad (4)$$

$$\mathbf{\Omega}(\boldsymbol{\omega}(t)) = \begin{bmatrix} -[\boldsymbol{\omega}(t)\times] & \boldsymbol{\omega}(t) \\ -\boldsymbol{\omega}^{T}(t) & 0 \end{bmatrix} \qquad (5)$$

6.2.2 Spacecraft and Reaction Wheel Dynamics

The total system angular momentum $\mathbf{H}(t) \in \mathbf{R}^{3\times1}$ about point P (spacecraft mass center) is the sum of the spacecraft and reaction wheel momenta. Expressing the system angular momentum in $\{\mathbf{B}(t)\}$ gives

$$\mathbf{H}(t) = \mathbf{I}^{*}\boldsymbol{\omega}(t) + \widetilde{\mathbf{C}}^{T}\mathbf{J}\mathbf{\Omega}(t) \qquad (6)$$

where $\mathbf{I}^{*} \in \mathbf{R}^{3\times3}$ is the system inertia matrix relative to $\{\mathbf{B}(t)\}$ (with reaction wheels locked) , $\widetilde{\mathbf{C}} \in \mathbf{R}^{m\times3}$ is a coordinate transformation matrix whose rows are the three orthogonal $\{\mathbf{B}(t)\}$ components of the m unit vectors along the reaction wheel spin axes, $\mathbf{J} \in \mathbf{R}^{m\times m}$ is the reaction wheel axial inertia matrix defined by $\mathbf{J} = diag\{\mathbf{J}_{ai}\}$ for $i = 1,...,m$, and $\mathbf{\Omega}(t) \in \mathbf{R}^{m\times1}$ is a column matrix of the m reaction wheel angular velocities. The evolution of the system angular momentum is governed by Euler's equation of motion

$$\dot{\mathbf{H}}(t) + [\boldsymbol{\omega}(t)\times]\mathbf{H}(t) = \mathbf{N}(t) \qquad (7)$$

where $\mathbf{N}(t) \in \mathbf{R}^{3\times1}$ is the total external torque acting on the system about point P. The skew-symmetric matrix $[\boldsymbol{\omega}(t)\times] \in \mathbf{R}^{3\times3}$ is defined by

$$[\boldsymbol{\omega}(t)\times] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad (8)$$

217

Differentiating Eq (6) and substituting the result into Eq (7) gives the dynamic equations of motion for the overall system

$$\mathbf{I}^* \dot{\boldsymbol{\omega}}(t) + \widetilde{\mathbf{C}}^T \mathbf{J} \dot{\boldsymbol{\Omega}}(t) + [\boldsymbol{\omega}(t) \times] \mathbf{H}(t) = \mathbf{N}(t) \tag{9}$$

The equations of motion for the reaction wheels are given by

$$\mathbf{J} \dot{\boldsymbol{\Omega}}(t) + \mathbf{J} \widetilde{\mathbf{C}} \dot{\boldsymbol{\omega}}(t) = \mathbf{u}(t) \tag{10}$$

where the m components of the control torque vector $\mathbf{u}(t) \in \mathbf{R}^{m \times 1}$ are the axial torques applied to their respective wheels by the electric motors. The explicit dependence of Eq. (9) on the reaction wheel angular accelerations $\dot{\boldsymbol{\Omega}}(t)$ may be eliminated by multiplying Eq (10) by $\widetilde{\mathbf{C}}^T$ and substituting the result into Eq (9) to obtain[8,9]

$$\dot{\boldsymbol{\omega}}(t) = -\mathbf{J}_s^{-1} \left\{ [\boldsymbol{\omega}(t) \times] \mathbf{H}(t) + \widetilde{\mathbf{C}}^T \mathbf{u}(t) - \mathbf{N}(t) \right\} \tag{11}$$

where $\mathbf{J}_s = \left[\mathbf{I}^* - \widetilde{\mathbf{C}}^T \mathbf{J} \widetilde{\mathbf{C}} \right] \in \mathbf{R}^{3 \times 3}$ is an effective system inertia matrix relative to $\{\mathbf{B}(t)\}$. In general, optimal control theory allows the solution of the three orthogonal $\{\mathbf{B}(t)\}$ components of the control torque vector given by $\widetilde{\mathbf{u}}(t) = \mathbf{f}(\mathbf{x}(t), t)$ where $\mathbf{x}(t)$ contains the state variables of the system. To solve for the m-dimensional reaction wheel control law $\mathbf{u}(t)$, based on the system of Eq (10), requires the solution of the equation $\mathbf{J}_s^{-1} \widetilde{\mathbf{C}}^T \mathbf{u}(t) = \mathbf{f}(\mathbf{x}(t), t)$. This requires, however, the inversion of the 3 x m matrix $\boldsymbol{\Gamma} = \mathbf{J}_s^{-1} \widetilde{\mathbf{C}}^T$, and it follows that the selection of the control law is underdetermined. Any suitable optimality criterion may used to select a particular control satisfying the equation, but a minimum torque criterion[9] which minimises the cost function $J = \mathbf{u}^T(t)\mathbf{u}(t)$ results in the control law

$$\mathbf{u}(t) = \boldsymbol{\Gamma}^T \left[\boldsymbol{\Gamma} \boldsymbol{\Gamma}^T \right]^{-1} \mathbf{f}(\mathbf{x}(t), t) \tag{12}$$

6.2.3 State Transformation

In the absence of the external torques acting on the spacecraft, the system angular momentum remains fixed (magnitude and direction) in inertial space. A special inertial *momentum coordinate frame*[14] $\{\mathbf{G}\}$ is introduced in which the $\hat{\mathbf{g}}_2$ axis is aligned with the system angular momentum $\mathbf{H}(t)$. The remaining basis vectors $\hat{\mathbf{g}}_1$ and $\hat{\mathbf{g}}_3$ may be selected arbitrarily or defined by the direction of the $\hat{\mathbf{n}}_1$ inertial basis vector. Application of the momentum coordinate frame allows a reduction in the dimension of the system state variables. The two inertial coordinate frames are related by

$$\{\mathbf{N}\} = \mathbf{C}(\boldsymbol{\alpha})\{\mathbf{G}\} \tag{13}$$

where $\boldsymbol{\alpha}$ is a set of constant quaternion parameters. The orientation of the spacecraft body-fixed coordinate frame $\{\mathbf{B}(t)\}$ with respect to the momentum frame $\{\mathbf{G}\}$ is given by the projection

$$\{\mathbf{B}(t)\} = \mathbf{C}(\boldsymbol{\delta}(t))\{\mathbf{G}\} \tag{14}$$

where $\boldsymbol{\delta}(t)$ is a new set of attitude kinematic parameters replacing $\mathbf{q}(t)$. These two parameter sets are related according $\boldsymbol{\delta}(t) = \mathbf{q}(t) \otimes \boldsymbol{\alpha}$ which gives $\boldsymbol{\delta}(t) = [\boldsymbol{\Xi}(\boldsymbol{\alpha}) \quad \boldsymbol{\alpha}]\mathbf{q}(t)$. The new set of kinematic equations for $\boldsymbol{\delta}(t)$ are given by

$$\dot{\boldsymbol{\delta}}(t) = \frac{1}{2}\boldsymbol{\Xi}(\boldsymbol{\delta}(t))\boldsymbol{\omega}(t) = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}(t))\boldsymbol{\delta}(t) \tag{15}$$

The system angular momentum $\mathbf{H_G}$ expressed in the momentum frame $\{\mathbf{G}\}$ has a single component along the $\hat{\mathbf{g}}_2$ axis, equal to the magnitude of the system angular momentum H which remains constant in the absence of external torques. Using the inertial frame $\{\mathbf{N}\}$ components of system angular momentum $\mathbf{H_N} = [H_{n1} \quad H_{n2} \quad H_{n3}]^T$, the constant $\boldsymbol{\alpha}$ parameters are given by

$$\alpha_1 = -H_{n3} \left[\frac{H - H_{n2}}{2H(H_{n1}^2 + H_{n3}^2)} \right]^{\frac{1}{2}} \tag{16}$$

$$\alpha_2 = 0 \tag{17}$$

$$\alpha_3 = H_{n1} \left[\frac{H - H_{n2}}{2H(H_{n1}^2 + H_{n3}^2)} \right]^{\frac{1}{2}} \tag{18}$$

$$\alpha_4 = \left[\frac{H + H_{n2}}{2H} \right]^{\frac{1}{2}} \tag{19}$$

The system angular momentum $\mathbf{H}(t)$ is expressed in the spacecraft body-fixed frame $\{\mathbf{B}(t)\}$ as

$$\mathbf{H}(\boldsymbol{\delta}(t)) = \mathbf{C}(\boldsymbol{\delta}(t))\mathbf{H_G} = H \begin{bmatrix} 2(\delta_1\delta_2 + \delta_3\delta_4) \\ -\delta_1^2 + \delta_2^2 - \delta_3^2 + \delta_4^2 \\ 2(\delta_2\delta_3 - \delta_1\delta_4) \end{bmatrix} \tag{20}$$

The reaction wheel angular velocities are expressed as

$$\boldsymbol{\Omega}(t) = \mathbf{J}^{-1}\widetilde{\mathbf{C}}\left\{ \mathbf{H}(\boldsymbol{\delta}(t)) - \mathbf{I}^*\boldsymbol{\omega}(t) \right\} \tag{21}$$

The dependence of the system dynamic equations on the reaction wheel momenta is eliminated by substituting Eq (20) into Eq (11), for the case of zero external torques, to obtain

$$\dot{\boldsymbol{\omega}}(t) = -\mathbf{J}_s^{-1}\left\{ [\boldsymbol{\omega}(t)\times]\mathbf{H}(\boldsymbol{\delta}(t)) + \widetilde{\mathbf{C}}^{\mathrm{T}}\mathbf{u}(t) \right\} \tag{22}$$

The kinematic equations given by Eq (15) and the dynamic equations given by Eq (22) represent the state equations for the optimal control problem. The corresponding state variables are $\mathbf{x}(t) = \left[\boldsymbol{\delta}^{\mathrm{T}}(t) \quad \boldsymbol{\omega}^{\mathrm{T}}(t) \right]^{\mathrm{T}} \in \mathbf{R}^{7\times1}$.

6.3 General Optimal Control Formulation

This section considers the formulation of the necessary and sufficient conditions for the minimisation of a general cost function based on the state equations and boundary conditions specified in Section 6.2. The general problem[1-5] is to determine the optimal control torque $\mathbf{u}^*(t)$ which transfers the system described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{23}$$

from a set of initial conditions $\mathbf{x}(t_0)$ to a set of final conditions $\psi(\mathbf{x}(t_f))$ whilst minimising a cost function defined by

$$J(\mathbf{u}(t)) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \tag{24}$$

The optimal state trajectory corresponding to the optimal control history $\mathbf{u}^*(t)$ is denoted by $\mathbf{x}^*(t)$. For the spacecraft attitude maneuver problem, the final time t_f is fixed and all final conditions $\mathbf{x}(t_f)$ are fixed. There are no state variable constraints, but the control torque components are subject to simple magnitude constraints defined by

$$u_{min,i}(t) \le u_i(t) \le u_{max,i}(t) \qquad i = 1, ..., m \tag{25}$$

Pontryagin's minimum principle[1,3,5] is used to formulate the necessary conditions for $\mathbf{u}^*(t)$ to be a local extremal of the cost function defined by Eq (24). The Hamiltonian[5] is defined by

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)\dot{\mathbf{x}}(t) \tag{26}$$

where $\mathbf{p}(t)$ is a vector of co-states (also called Lagrange multipliers) associated with the differential equation constraints of Eq (23). The necessary conditions[5] are given by

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)}{\partial \mathbf{p}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad (27)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)}{\partial \mathbf{x}}$$

$$= -\frac{\partial g(\mathbf{x}^*(t), \mathbf{u}^*(t), t)}{\partial \mathbf{x}} - \left[\frac{\partial \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)}{\partial \mathbf{x}}\right]^T \mathbf{p}^*(t) \qquad (28)$$

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \le H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) \qquad (29)$$

The final necessary condition Eq (29) states that the optimal control $\mathbf{u}^*(t)$ must globally minimise the Hamiltonian at each point in time. In general, the optimal control $\mathbf{u}^*(t) = \mathbf{g}(\mathbf{x}^*(t), \mathbf{p}^*(t), t)$ is determined as a nonlinear function of the states and co-states (and in general time t) using Eq (29). This expression is substituted into the state equations defined by Eq (27) to form

$$\dot{\mathbf{x}}^*(t) = \mathbf{h}(\mathbf{x}^*(t), \mathbf{p}^*(t), t) \qquad (30)$$

Equations (30) and (28), in addition to the state variable boundary conditions, constitute a nonlinear two-point boundary-value problem (TPBVP) whose solution produces the optimal histories for $\mathbf{x}(t)$, $\mathbf{p}(t)$, and $\mathbf{u}(t)$.

6.4 Minimum Torque Maneuvers

This section considers minimum-torque[11,14,15] spacecraft attitude maneuvers which minimise the performance index

$$J(\mathbf{u}(t)) = \frac{1}{2} \int_{t_0}^{t_f} \mathbf{u}^T(\tau) \mathbf{W} \mathbf{u}(\tau) d\tau \qquad (31)$$

The control torque weighting matrix $\mathbf{W}$ is selected as either (i) $\mathbf{W} = \mathbf{I}_{m \times m}$ to emphasise each of the reaction wheel motor torques equally, or (ii) $\mathbf{W} = \widetilde{\mathbf{C}}\widetilde{\mathbf{C}}^T$ to emphasise the three spacecraft body-fixed components of the motor torques

equally (in this case $\mathbf{W}$ is not positive definite). The Hamiltonian for the cost function of Eq (31) is defined as

$$H = \frac{1}{2}\mathbf{u}^T(t)\mathbf{W}\mathbf{u}(t) + \boldsymbol{\gamma}^T(t)\dot{\boldsymbol{\delta}}(t) + \boldsymbol{\lambda}^T(t)\dot{\boldsymbol{\omega}}(t) \qquad (32)$$

where $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\lambda}(t)$ are the co-states associated with $\boldsymbol{\delta}(t)$ and $\boldsymbol{\omega}(t)$ respectively. The state equations $\dot{\boldsymbol{\delta}}(t)$ and $\dot{\boldsymbol{\omega}}(t)$ are given by Eqs (15) and (22). Application of Pontryagin's minimum principle requires that the co-states $\mathbf{p}(t) = \begin{bmatrix} \boldsymbol{\gamma}^T(t) & \boldsymbol{\lambda}^T(t) \end{bmatrix}^T \in \mathbf{R}^{7\times 1}$ satisfy

$$\dot{\boldsymbol{\gamma}}(t) = -\frac{\partial H(\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t),t)}{\partial \boldsymbol{\delta}}$$

$$= -\frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{\omega}(t))\boldsymbol{\gamma}(t) - \left[\frac{\partial \mathbf{H}(\boldsymbol{\delta}(t))}{\partial \boldsymbol{\delta}}\right]^T[\boldsymbol{\omega}(t)\times]\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t) \qquad (33)$$

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H(\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t),t)}{\partial \boldsymbol{\omega}} = -\frac{1}{2}\boldsymbol{\Xi}^T(\boldsymbol{\delta}(t))\boldsymbol{\gamma}(t) + [\mathbf{H}(\boldsymbol{\delta}(t)\times]\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t) \qquad (34)$$

where the system angular momentum $\mathbf{H}(\boldsymbol{\delta}(t)$ Jacobian matrix with respect to $\boldsymbol{\delta}(t)$ is defined as

$$\frac{\partial \mathbf{H}(\boldsymbol{\delta}(t))}{\partial \boldsymbol{\delta}} = 2\begin{bmatrix} \delta_2 & \delta_1 & \delta_4 & \delta_3 \\ -\delta_1 & \delta_2 & -\delta_3 & \delta_4 \\ -\frac{1}{2}\delta_4 & \frac{1}{2}\delta_3 & \frac{1}{2}\delta_2 & -\frac{1}{2}\delta_1 \end{bmatrix} \qquad (35)$$

The optimal control $\mathbf{u}^*(t)$ must minimise the Hamiltonian given by Eq (32) at each instant in time. If there are no control constraints the necessary condition to minimise H is $\partial H/\partial \mathbf{u} = \mathbf{0}_{m\times 1}$ which implies

$$\mathbf{u}_{us}(t) = 2(\mathbf{W} + \mathbf{W}^T)^{-1}\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t) \qquad (36)$$

where the weighting matrix $\mathbf{W}$ is generally not positive definite. Since the Hamiltonian for minimum-torque maneuvers given by Eq (32) is quadratic in the control variables the unbounded optimal control given by Eq (36) is a global minimum of the Hamiltonian. In addition the presence of the quadratic terms prevents singular controls and corresponding singular arcs in the state-space emerging during the spacecraft attitude maneuver. This is not the case for minimum-time or minimum-energy cost functions which have Hamiltonians that are linear in the controls as demonstrated in Section 6.5. Minimisation of the Hamiltonian subject to the time-dependent control constraints given by Eq (25) requires

$$u_i(t) = \begin{cases} u_{max,i}(t) & u_{us,i}(t) > u_{max,i}(t) \\ u_{us,i}(t) & u_{min,i}(t) < u_{us,i}(t) < u_{max,i}(t) \\ u_{min,i}(t) & u_{us,i}(t) < u_{min,i}(t) \end{cases} \qquad i = 1,...,m \qquad (37)$$

The control torques $\mathbf{u}(t)$ are eliminated from the system dynamic equations by substituting Eq (37) into Eq (22). The two-point boundary-value problem is defined by Eqs (15), (22), (33), and (34), and the boundary conditions $\mathbf{x}(t_0)$ and $\psi(\mathbf{x}(t_f))$. Numerical solution of the TPBVP generates the optimal profiles for $\boldsymbol{\delta}(t)$, $\boldsymbol{\omega}(t)$, $\boldsymbol{\gamma}(t)$, and $\boldsymbol{\lambda}(t)$. The optimal control torque profile $\mathbf{u}(t)$ is obtained directly from Eqs (36) and (37). The unsmooth nature of the optimal controls given by Eq (37) adds to the difficulty of finding a solution to the two-point boundary-value problem.

6.5 Minimum Energy Maneuvers

6.5.1 Necessary Conditions

This section considers minimum-energy[12,13] spacecraft attitude maneuvers which minimise the performance index

$$J(\mathbf{u}(t)) = \int_{t_0}^{t_f} \sum_{i=1}^{m} |u_i(\tau)| |\Omega_i(\tau)| \ d\tau \qquad (38)$$

representing a positive measure of the total electrical energy expenditure by the *m* reaction wheel configuration over the entire attitude maneuver interval. The Hamiltonian for the cost function of Eq (38) is defined as

$$H = \sum_{i=1}^{m} |u_i(t)| |\Omega_i(t)| + \boldsymbol{\gamma}^T(t)\dot{\boldsymbol{\delta}}(t) + \boldsymbol{\lambda}^T(t)\dot{\boldsymbol{\omega}}(t) \tag{39}$$

where $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\lambda}(t)$ are the co-states associated with $\boldsymbol{\delta}(t)$ and $\boldsymbol{\omega}(t)$ respectively. The state equations $\dot{\boldsymbol{\delta}}(t)$ and $\dot{\boldsymbol{\omega}}(t)$ are given by Eqs (15) and (22). Application of Pontryagin's minimum principle requires that the costates $\mathbf{p}(t)$ satisfy Eqs (33) and (34). The co-state equations for minimum-torque and minimum-energy maneuvers are identical since both cost functions are independent of the state variables $\boldsymbol{\delta}(t)$ and $\boldsymbol{\omega}(t)$. The reaction wheel angular velocities are effectively removed from the system dynamic equations through the introduction of an inertial momentum reference frame as described in Section 6.2.3. This is an important transformation since the Hamiltonian is not continuously differentiable with respect to the wheel angular velocities, leading to discontinuous co-state equations.

The optimal control $\mathbf{u}^*(t)$ must minimise the Hamiltonian given by Eq (39) at each instant in time. In contrast to minimum-torque maneuvers this Hamiltonian is linear in the control variables. This property implies that for unbounded controls the necessary condition $\partial H/\partial \mathbf{u} = \mathbf{0}_{m \times 1}$ gives no information regarding the optimal controls which minimise the Hamiltonian. For bounded controls subject to time-dependent control constraints of Eq (25) global minimisation of the Hamiltonian requires

$$u_i(t) = \begin{cases} u_{max}(t)\,\text{sgn}\left\{\left[\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right]_i\right\} & |\Omega_i(t)| < \left|\left[\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right]_i\right| \\ 0 & |\Omega_i(t)| > \left|\left[\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right]_i\right| \\ \text{singular} & |\Omega_i(t)| = \pm\left|\left[\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right]_i\right| \end{cases} \quad i = 1,...,m \tag{40}$$

where the torque constraint is given by $u_{max}(t) = u_{max,i}(t) = u_{min,i}(t)$ for $i = 1,...,m$. Furthermore, singular controls (and corresponding singular arcs in the state-space) may exist during the spacecraft attitude maneuver. Calculation of the singular controls is addressed in Section 6.5.2. The control torques $\mathbf{u}(t)$ are eliminated from the system dynamic equations by substituting Eq (40) into Eq (22). The two-point boundary-value problem is defined by Eqs (15), (22), (33), and (34), and the boundary conditions $\mathbf{x}(t_0)$ and $\psi(\mathbf{x}(t_f))$. Numerical solution of the TPBVP generates the optimal profiles for $\boldsymbol{\delta}(t)$, $\boldsymbol{\omega}(t)$, $\boldsymbol{\gamma}(t)$, and $\boldsymbol{\lambda}(t)$. The optimal control torque profile $\mathbf{u}(t)$ is obtained directly from Eq (40). The unsmooth nature of the optimal controls given by Eq (40) adds to the difficulty of finding a solution to the two-point boundary-value problem.

6.5.2 Singular Controls

This subsection investigates the singular controls[3-5] for the minimum-energy optimal control problem developed in Section 6.5.1. A control torque component $u_i(t)$ is called a singular control over a finite time interval $t \in [t_1, t_2]$ if the switching function $|\Omega_i(t)| = \pm\left|\left[\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right]_i\right|$ is satisfied on the interval. For time-optimal control problems[21-23] the switching function is independent of the control variables such that successive differentiation of the function with respect to time is required to obtain the specific functional form singular control. Furthermore, Seywald and Kumar[23] point out that not all possible combinations of singular and non-singular controls need to be considered. For example, in the case of one singular control, it may be assumed that $u_1(t)$ is singular and $u_2(t),...,u_m(t)$ are non-singular. For minimum-energy spacecraft attitude maneuvers, however, the switching function provides sufficient information to determine the singular controls. Consider the singular interval in state-space corresponding to

$$\Omega_i(t) + \left[\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right]_i = 0 \qquad (41)$$

The singular control is obtained substituting Eqs (10), (22), and (34) into the derivative of Eq (41), and rearranging to obtain

$$
\mathbf{u}_{singular}(t) = -\left[\mathbf{J}\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\widetilde{\mathbf{C}}^T + \mathbf{I}_{m \times m}\right]^{-1}\mathbf{J}\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\left\{\left[\boldsymbol{\omega}(t)\times\right]\mathbf{H}(\boldsymbol{\delta}(t))\right.
$$

$$
\left. -\frac{1}{2}\boldsymbol{\Xi}^T(\boldsymbol{\delta}(t))\boldsymbol{\gamma}(t) + \left[\mathbf{H}(\boldsymbol{\delta}(t))\times\right]\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right\} \qquad (42)
$$

Similarly, the singular controls corresponding to the singular interval $\Omega_i(t) - \left[\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right]_i = 0$ are given by

$$
\mathbf{u}_{singular}(t) = -\left[\mathbf{J}\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\widetilde{\mathbf{C}}^T + \mathbf{I}_{m \times m}\right]^{-1}\mathbf{J}\widetilde{\mathbf{C}}\mathbf{J}_s^{-1}\left\{\left[\boldsymbol{\omega}(t)\times\right]\mathbf{H}(\boldsymbol{\delta}(t))\right.
$$

$$
\left. +\frac{1}{2}\boldsymbol{\Xi}^T(\boldsymbol{\delta}(t))\boldsymbol{\gamma}(t) - \left[\mathbf{H}(\boldsymbol{\delta}(t))\times\right]\mathbf{J}_s^{-1}\boldsymbol{\lambda}(t)\right\} \qquad (43)
$$

6.6 Conclusion

In this chapter, a novel optimal control strategy was developed for minimum-energy spacecraft attitude maneuvers. The specific spacecraft configuration considered was a rigid-body spacecraft equipped with a redundant reaction wheel assembly. Bounded optimal controls were developed using Pontryagin's minimum principle, and an associated two-point boundary-problem was constructed. Expressions for singular controls corresponding to singular sub-arcs in the state-space were also developed.

6.7 References

[1] Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., and E.F. Mishchenko. (1962). *The mathematical theory of optimal processes*. USA: John Wiley and Sons, Inc.

[2] Leitmann, G., ed. (1962). *Optimization techniques: with applications to aerospace systems*. USA: Academic Press, Inc.

[3] Athans, M., and P.L. Falb. (1966). *Optimal control: an introduction to the theory and its applications*. USA: McGraw-Hill, Inc.

[4] Bryson, A.E., and Y.-C. Ho. (1969). *Applied optimal control: optimization, estimation, and control*. USA: Ginn and Company.

[5] Kirk, D.E. (1970). Optimal control theory: an introduction. USA: Prentice-Hall, Inc.

[6] Miele, A., and R.R. Iyer. (1970). General technique for solving nonlinear, two-point boundary-value problems via the method of particular solutions. *The Journal of Optimization Theory and Applications*, 5(5): 382-399.

[7] Lin, Y.Y., and L.G. Kraige. (1989). Enhanced techniques for solving the two-point boundary-value problem associated with the optimal attitude control of spacecraft. *The Journal of the Astronautical Sciences*, 37(1): 1-15.

[8] Wertz, J.R., ed. (1978). *Spacecraft attitude determination and control*. The Netherlands: D. Reidel Publishing Company.

[9] Junkins, J.L., and J.D. Turner. (1986). *Optimal spacecraft rotational maneuvers*. The Netherlands: Elsevier Publishers.

[10] Shuster, M.D. (1993). A survey of attitude representations. *The Journal of the Astronautical Sciences*, 41(4): 439-517.

[11] Junkins, J.L., and J.D. Turner. (1980). Optimal continuous torque attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 3(3): 210-217.

[12] Skaar, S.B., and L.G. Kraige. (1982). Arbitrary large-angle satellite attitude maneuvers using an optimal reaction wheel power criterion. *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference*, San Diego, California, USA, August 1982.

[13] Skaar, S.B., and L.G. Kraige. (1982). Single-axis spacecraft attitude maneuvers using an optimal reaction wheel power criterion. *The Journal of Guidance, Control, and Dynamics*, 5(5): 543-544.

[14] Vadali, S.R., and J.L. Junkins. (1983). Spacecraft large angle rotational maneuvers with optimal momentum transfer. *The Journal of the Astronautical Sciences*, 31(2): 217-235.

[15] Vadali, S.R., and J.L. Junkins. (1984). Optimal open-loop and stable feedback control of rigid spacecraft attitude maneuvers. *The Journal of the Astronautical Sciences*, 32(2): 105-122.

[16] Burdick, G.M., Lin, H-S., and E.C. Wong. (1984). A scheme for target tracking and pointing during small celestial body encounters. *The Journal of Guidance, Control, and Dynamics*, 7(4): 450-457.

[17] Carrington, C.K., and J.L. Junkins. (1986). Optimal nonlinear feedback control for spacecraft attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 9(1): 99-107.

[18] Dwyer, T.A.W. III (1986). Exact nonlinear control of spacecraft slewing maneuvers with internal momentum transfer. *The Journal of Guidance, Control, and Dynamics*, 9(2): 240-246.

[19] Redding, D.C., and N.J. Adams. (1987). Optimized rotation-axis attitude maneuver controller for the space shuttle orbiter. *The Journal of Guidance, Control, and Dynamics*, 10(1): 4-13.

[20] Li, F., and P.M. Bainum. (1990). Numerical approach for solving rigid spacecraft minimum time attitude maneuvers. *The Journal of Guidance, Control, and Dynamics*, 13(1): 38-45.

[21] Bilimoria, K.D., and B. Wie. (1993). Time-optimal three-axis reorientation of a rigid spacecraft. *The Journal of Guidance, Control, and Dynamics*, 16(3): 446-452.

[22] Byers, R.M., and S.R. Vadali. (1993). Quasi-closed-form solution to the time-optimal rigid spacecraft reorientation problem. *The Journal of Guidance, Control, and Dynamics*, 16(3): 453-461.

[23] Seywald, H., and R.R. Kumar. (1993). Singular control in minimum time spacecraft reorientation. *The Journal of Guidance, Control, and Dynamics*, 16(4): 686-694.

[24] Seywald, H., Kumar, R.R., Deshpande, S.S., and M.L. Heck. (1994). Minimum fuel spacecraft reorientation. *The Journal of Guidance, Control, and Dynamics*, 17(1): 21-29.

Chapter 7   Spacecraft Attitude Estimation

7.1 Introduction

The objective of spacecraft attitude determination is the real-time estimation of the spacecraft attitude parameters using on-board sensor measurements. Modern spacecraft missions generally utilise continuous-time gyroscope measurements of the spacecraft angular rates and discrete-time line-of-sight measurements using an attitude sensor suite. Typical line-of-sight sensors include star sensors, digital sun sensors, and magnetometers. The resulting stochastic differential equations (SDE) governing the evolution of the system state and discrete measurements are given by[1]

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), t)dt + \mathbf{G}(\mathbf{x}(t), t)d\boldsymbol{\beta}(t) \tag{1}$$

$$\mathbf{y}(t_k) = \mathbf{h}(\mathbf{x}(t_k), t_k) + \boldsymbol{\eta}(t_k) \tag{2}$$

where $\mathbf{x}(t)$ and $\boldsymbol{\beta}(t)$ are $\mathbf{R}^n$ random processes, and $\mathbf{y}(t_k)$ and $\boldsymbol{\eta}(t_k)$ are $\mathbf{R}^m$ random sequences. Furthermore $\boldsymbol{\beta}(t)$ has components which are independent Brownian motion processes, $\boldsymbol{\eta}(t_k)$ has components which are independent white Gaussian sequences, and $\boldsymbol{\beta}(t)$, $\boldsymbol{\eta}(t_k)$, and $\mathbf{x}(t_0)$ are assumed independent. The functions $\mathbf{f}(\mathbf{x}(t), t)$ and $\mathbf{h}(\mathbf{x}(t_k), t_k)$ are assumed to be $\mathbf{C}^\infty$ smooth.

The continuous-discrete attitude filtering problem consists of calculating an estimate of the state $\mathbf{x}(t)$ given a realisation of the sequence of observations $\mathbf{Y_k} = \{\mathbf{y}(t_0), ..., \mathbf{y}(t_k)\}$ up to and including time $t_k < t$ which is more concisely expressed as $\mathbf{Y_k} = \{\mathbf{y}(s) : t_0 \leq s \leq t\}$. To solve this problem it is necessary to (i) determine the evolution of the conditional probability density function (PDF) of the state $\mathbf{x}(t)$ conditioned on all prior measurements $\mathbf{Y_k}$ (called the filtering PDF), which is denoted by $\rho(\mathbf{x}(t), t | \mathbf{Y_k}) \equiv \rho(\mathbf{x}(t), t)$ and (ii) update this filtering PDF when discrete measurements become available. Between measurements

$t_{k-1} \leq t \leq t_k$ the filtering PDF $\rho(\mathbf{x}, t)$ satisfies the *Kolmogorov forward equation* or *Fokker-Planck equation*[1]

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\rho(\mathbf{x},t) f_i(\mathbf{x},t) \right]$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i x_j} \left[\rho(\mathbf{x},t) \left(\mathbf{G}(\mathbf{x},t) \mathbf{G}^T(\mathbf{x},t) \right)_{ij} \right] \qquad (3)$$

It may be shown that $\rho(\mathbf{x}, t)$ is constructed by normalising a conditional PDF $\sigma(\mathbf{x}, t)$ which satisfies the Duncan-Mortensen-Zakai equation[26]. This PDF is updated at time t_k to incorporate information in the measurement $\mathbf{y}(t_k)$ using Bayes theorem for probability densities

$$\sigma_{\mathbf{x}_k(+)|Y_k}(\mathbf{x},t) = \left[\frac{\int_{\Re^n} \sigma_{\mathbf{x}_k(+)|Y_k}(\mathbf{x},t) d^n \mathbf{x}}{\int_{\Re^n} \sigma_{\mathbf{y}_k|\mathbf{x}_k(-)}(\mathbf{x},t) \sigma_{\mathbf{x}_k(-)|Y_{k-1}}(\mathbf{x},t) d^n \mathbf{x}} \right] \sigma_{\mathbf{y}_k|\mathbf{x}_k(-)}(\mathbf{x},t) \sigma_{\mathbf{x}_k(-)|Y_{k-1}}(\mathbf{x},t) \qquad (4)$$

where $\sigma_{\mathbf{y}_k|\mathbf{x}_k(-)}(\mathbf{x},t)$ is the non-normalised PDF for the sensor measurements conditioned on the pre-measurement state. The $(-)$ and $(+)$ subscript symbols denote pre-measurement and post-measurement parameters respectively.

This chapter develops a novel attitude estimation algorithm that provides real-time estimates of the spacecraft attitude parameters and sensor bias parameters. The mission scenario is a spacecraft in low-earth orbit equipped with an attitude sensor suite consisting of a three-axis magnetometer and a three-axis rate gyroscope assembly. The objective is to solve the nonlinear attitude filtering problem described above through a real-time solution of the Fokker-Planck equation, based on an initial Gaussian filtering PDF, in terms of solutions of nonlinear ordinary differential equations (ODE). This algorithm may be considered an extension of the orthogonal attitude filter developed by Markley[27] and the nonlinear filtering algorithm developed by Yau and Yau[26]. Markley proposed a filtering PDF, parameterised by the attitude parameters and sensor bias parameters, consisting of first-order terms in the attitude parameters, first

and second-order terms in the bias parameters, and first-order correlations terms between the attitude and bias parameters. A major contribution of this thesis is the implementation of a Gaussian filtering PDF, as proposed in Reference 26, consisting of second-order terms (and below) in the attitude parameters and sensor bias parameters, and first-order correlation terms. A key objective will be to investigate whether the additional second-order attitude parameter terms better represents the filtering PDF, and improve the real-time state estimates.

In Section 7.2 a dynamic stochastic model is developed to describe the spacecraft attitude motion by augmenting the spacecraft kinematic equations of motion with Farrenkopf's gyroscope model[39]. Section 7.3 specifies the filtering PDF for the combined real-time estimation of the attitude parameters and sensor bias parameters, based on the maximum aposteriori probability (MAP) principle. The development of nonlinear ordinary differential equations for the propagation of the filtering PDF parameters between magnetometer measurements, using the Fokker-Planck equation, is considered in Section 7.4. The update of the filtering PDF parameters using Bayes theorem, to incorporate the information in the magnetometer measurements, is addressed in Section 7.5. In Section 7.6, the main limitations of the attitude estimation algorithm are discussed and recommendations for future research are proposed.

7.2 Spacecraft State Model

The orthogonal spacecraft attitude matrix $\mathbf{D}(t)$ satisfies the differential equation

$$\frac{d}{dt}\mathbf{D}(t) = -\big[\boldsymbol{\omega}(t)\times\big]\mathbf{D}(t) \tag{5}$$

where $\boldsymbol{\omega}(t)$ is the spacecraft angular velocity. The skew-symmetric matrix operator is defined by

$$\big[\boldsymbol{\omega}(t)\times\big] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{6}$$

233

To formulate a stochastic differential equation (SDE) for the attitude kinematic motion the true angular rate $\boldsymbol{\omega}(t)$ is augmented using Farrenkopf's gyroscope model[39]

$$\boldsymbol{\omega}_g(t) = \boldsymbol{\omega}(t) + \mathbf{x}_d(t) + \mathbf{G}_v \frac{d\mathbf{w}_v(t)}{dt} \tag{7}$$

where $\boldsymbol{\omega}_g(t)$ is the three-axis gyroscope measurement, $\mathbf{x}_d(t)$ is the gyroscope drift-rate bias, and $d\mathbf{w}_v(t)$ is a Wiener random process representing the gyroscope measurement noise with constant weighting matrix $\mathbf{G}_v$. Note that Farrenkopf's three-axis gyroscope model should be modified if more than three independent gyroscope measurements are available. Substituting into Eq (7) the expression for the infinitesimal angular rotation $d\boldsymbol{\omega}(t) = d\boldsymbol{\theta}(t)dt$ gives

$$d\boldsymbol{\theta}(t) = \left[\boldsymbol{\omega}_g(t) - \mathbf{x}_d(t)\right]dt - \mathbf{G}_v d\mathbf{w}_v(t) \tag{8}$$

Let $\mathbf{u}(t)$, $\mathbf{v}(t)$, and $\mathbf{w}(t)$ denote the columns of the attitude matrix

$$\mathbf{D}(t) = \begin{bmatrix} \mathbf{u}(t) & \mathbf{v}(t) & \mathbf{w}(t) \end{bmatrix} \tag{9}$$

Defining the attitude parameter states $\mathbf{x}_a(t) = \begin{bmatrix} \mathbf{u}^T(t) & \mathbf{v}^T(t) & \mathbf{w}^T(t) \end{bmatrix}^T$ and the matrix $\boldsymbol{\Gamma}_a(\mathbf{x}_a(t)) = \begin{bmatrix} [\mathbf{u}(t)\times] & [\mathbf{v}(t)\times] & [\mathbf{w}(t)\times] \end{bmatrix}$ provides the SDE for the attitude parameters

$$d\mathbf{x}_a(t) = -\boldsymbol{\Gamma}_a(\mathbf{x}_a(t))d\boldsymbol{\theta}(t) \tag{10}$$

Substituting into Eq (10) the expression for $d\boldsymbol{\theta}(t)$ from Eq (8) gives

$$d\mathbf{x}_a(t) = \boldsymbol{\Gamma}_a^T(\mathbf{x}_a(t))\left[\mathbf{x}_d(t) - \boldsymbol{\omega}_g(t)\right]dt + \boldsymbol{\Gamma}_a^T(\mathbf{x}_a(t))\mathbf{G}_v d\mathbf{w}_v(t) \tag{11}$$

The gyroscope drift-rate bias is assumed to obey the SDE

$$d\mathbf{x}_d(t) = \mathbf{G}_u d\mathbf{w}_u(t) \tag{12}$$

where $d\mathbf{w}_\mathbf{u}(t)$ is a Wiener random process representing the gyroscope drift-rate bias ramp noise. A similar SDE governs the magnetometer measurement bias parameters

$$d\mathbf{x}_\mathbf{m}(t) = \mathbf{G}_\mathbf{m} d\mathbf{w}_\mathbf{m}(t) \tag{13}$$

with the Weiner process $\mathbf{w}_\mathbf{m}(t)$ similarly defined. Defining the sensor bias parameter states $\mathbf{x}_\mathbf{b}(t) = \begin{bmatrix} \mathbf{x}_\mathbf{m}^T(t) & \mathbf{x}_\mathbf{d}^T(t) \end{bmatrix}^T$ and Wiener process $d\mathbf{w}_\mathbf{b}(t) = \begin{bmatrix} d\mathbf{w}_\mathbf{m}^T(t) & d\mathbf{w}_\mathbf{u}^T(t) \end{bmatrix}^T$ provides the SDE for the bias parameters

$$d\mathbf{x}_\mathbf{b}(t) = \begin{bmatrix} \mathbf{G}_\mathbf{m} & \mathbf{0}_{(N-3)\times 3} \\ \mathbf{0}_{3\times(N-3)} & \mathbf{G}_\mathbf{u} \end{bmatrix} d\mathbf{w}_\mathbf{b}(t) \tag{14}$$

The overall nonlinear filter state vector is defined as $\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_\mathbf{a}^T(t) & \mathbf{x}_\mathbf{b}^T(t) \end{bmatrix}^T$ which obeys the SDE

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t))dt + \mathbf{G}(\mathbf{x}(t))d\mathbf{w}(t) \tag{15}$$

where the relevant parameters are defined by

$$\mathbf{f}(\mathbf{x}(t)) = \begin{bmatrix} \mathbf{\Gamma}_\mathbf{a}^T(\mathbf{x}_\mathbf{a}(t))\begin{bmatrix} \mathbf{x}_\mathbf{d}(t) - \mathbf{\omega}_\mathbf{g}(t) \end{bmatrix} \\ \mathbf{0}_{(N-3)\times 1} \\ \mathbf{0}_{3\times 1} \end{bmatrix} \tag{16}$$

$$\mathbf{G}(\mathbf{x}(t)) = \begin{bmatrix} \mathbf{\Gamma}_\mathbf{a}^T(\mathbf{x}_\mathbf{a}(t))\mathbf{G}_\mathbf{v} & \mathbf{0}_{9\times(N-3)} & \mathbf{0}_{9\times 3} \\ \mathbf{0}_{(N-3)\times 3} & \mathbf{G}_\mathbf{m} & \mathbf{0}_{(N-3)\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times(N-3)} & \mathbf{G}_\mathbf{u} \end{bmatrix} \tag{17}$$

$$d\mathbf{w}(t) = \begin{bmatrix} d\mathbf{w}_\mathbf{v}^T(t) & d\mathbf{w}_\mathbf{m}^T(t) & d\mathbf{w}_\mathbf{u}^T(t) \end{bmatrix}^T \tag{18}$$

Markley points out that Eq (15) should be interpreted as a Stratonovich SDE rather than an Ito SDE. The main reason is that solutions of a Stratonovich SDE obey the usual laws of calculus including the product rule for differentiation,

whereas this is not true for an Ito SDE. This ensures that the orthogonality constraint on the spacecraft attitude matrix is preserved by a Stratonovich SDE. The dimensions for all vectors and matrices defined previously are as follows:

$\mathbf{x_a} \in \mathbf{R}^{9\times1}$, $\mathbf{x_a} \in \mathbf{R}^{N\times1}$, $\mathbf{\Gamma_a} \in \mathbf{R}^{3\times9}$, $\mathbf{x_d} \in \mathbf{R}^{3\times1}$, $\mathbf{\omega_g} \in \mathbf{R}^{3\times1}$, $\mathbf{G_v} \in \mathbf{R}^{3\times3}$, $\mathbf{w_v} \in \mathbf{R}^{3\times1}$,

$\mathbf{x_d} \in \mathbf{R}^{3\times1}$, $\mathbf{G_u} \in \mathbf{R}^{3\times3}$, $\mathbf{w_u} \in \mathbf{R}^{3\times1}$, $\mathbf{x_m} \in \mathbf{R}^{(N-3)\times1}$, $\mathbf{G_m} \in \mathbf{R}^{(N-3)\times(N-3)}$,

$\mathbf{w_m} \in \mathbf{R}^{(N-3)\times1}$, $\mathbf{x} \in \mathbf{R}^{(N+9)\times1}$, $\mathbf{f(x)} \in \mathbf{R}^{(N+9)\times1}$, $\mathbf{G(x)} \in \mathbf{R}^{(N+9)\times(N+3)}$, $\mathbf{w} \in \mathbf{R}^{(N+3)\times1}$.

7.3 Spacecraft State PDF

This subsection specifies the form of the joint conditional probability density function (CPDF) for the attitude parameters and sensor bias parameters. It also discusses one possible approach for simultaneously estimating these parameters. In Reference 27 Markley develops the orthogonal filter with individual conditional PDF's for the attitude parameters and bias parameters. The attitude parameter PDF consisted of linear (first-order) and lower terms in the attitude parameters, whilst the bias parameters were assumed to be Gaussian distributed. By definition the joint PDF is the multiplication of the individual attitude parameter and bias parameter PDF's in the case that all parameters are independent. This simple form, however, will not suffice if the parameters are correlated, and no theoretical methods exist to construct this correlated parameter PDF. Markley[27] points out that the term in the extended Kalman filter (EKF) for spacecraft attitude estimation[17] describing the correlation between the attitude parameters and bias parameters is linear in the components of the bias parameter vector and the attitude parameter vector. The orthogonal filter developed by Markley uses this approach to specify a joint PDF which includes a correlation term which is linear in the bias vector and the attitude matrix (attitude parameters). In this work the joint PDF for the attitude and bias parameters in the interval $t_k \leq t \leq t_{k+1}$ conditioned on all sensor measurements up to and including time t_k is assumed to be given by

$$\rho(\mathbf{x}(t), t | \mathbf{Y_k}) = \alpha \exp\left[\mathbf{x}^T(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}^T(t)\mathbf{x}(t) + c(t)\right] \qquad (19)$$

where α is a normalising constant to ensure unity probability over $\mathbf{R}^n$. The filtering PDF of Eq (19) includes linear correlation terms between attitude and bias parameters similar to Markley's joint PDF function[27]. It is assumed that the PDF parameters $\mathbf{A}(t)$, $\mathbf{b}(t)$, and $c(t)$ are the sufficient statistics necessary to represent the finite dimensional filtering PDF defined by Eq (19), such that no higher-order moments are required. In general the normalised PDF $\rho(\mathbf{x},t)$ may be constructed by normalising the PDF $\sigma(\mathbf{x},t)$ according to

$$\rho(\mathbf{x}(t),t|\mathbf{Y_k}) = \frac{\sigma(\mathbf{x_t},t|\mathbf{Y_k})}{\int\limits_{\mathbf{R}^n} \sigma(\mathbf{x_t},t|\mathbf{Y_k})d\mathbf{x_k}} \tag{20}$$

It can be shown that $\sigma(\mathbf{x}(t),t|\mathbf{Y_k})$ satisfies the Duncan-Mortensen-Zakai (DMZ) equation[26]. The log-likelihood function corresponding to $\sigma(\mathbf{x}(t),t|\mathbf{Y_k})$ is defined as

$$J(\mathbf{x}(t),t|\mathbf{Y_k}) = \ln[\sigma(\mathbf{x}(t),t|\mathbf{Y_k})] = \mathbf{x}^T(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}^T(t)\mathbf{x}(t) + c(t) \tag{21}$$

This function simplifies the measurement update process defined in Section 7.5 by decomposing Bayes update law into the linear superposition of the apriori filtering PDF and measurement model. A number of options exist for specifying the estimation criteria for the state estimate at any given point in time. The most common criteria include the conditional expectation (mean), maximum aposteriori probability (MAP) estimate, and the minimum mean-square error (MMSE) estimate. Markley[27] uses the singular value decomposition (SVD) to solve for the MAP estimate of a joint PDF defined in terms of the spacecraft attitude matrix and sensor bias parameters. This procedure, however, breaks down when the joint PDF is expanded to incorporate second-order terms in the attitude matrix elements as in Eq (19). This research uses the MAP estimates defined by

$$\hat{\mathbf{x}}_{t|k}^{\mathbf{MAP}} = \arg\max_{\mathbf{x}\in\mathbf{R}^n}[\rho(\mathbf{x}(t),t|\mathbf{Y_k})] = \arg\max_{\mathbf{x}\in\mathbf{R}^n}[J(\mathbf{x}(t),t|\mathbf{Y_k})] \tag{22}$$

subject to the nonlinear equality constraints $\mathbf{c}(\mathbf{x}) = \mathbf{0}_{3\times 1}$ which follow directly from the orthogonality property of the attitude matrix

$$\mathbf{c}(\mathbf{x}) = \begin{bmatrix} \mathbf{u}^T\mathbf{u} - 1 \\ \mathbf{v}^T\mathbf{v} - 1 \\ \mathbf{w}^T\mathbf{w} - 1 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 - 1 \\ x_4^2 + x_5^2 + x_6^2 - 1 \\ x_7^2 + x_8^2 + x_9^2 - 1 \end{bmatrix} \qquad (23)$$

Numerous techniques exist to solve the above nonlinear programming (NLP) problem specified by Eqs (22) and (23). The large dimension of the state vector, however, may prevent the use of a NLP based algorithm. In this case the problem may be solved by simultaneously solving the equations

$$\hat{\mathbf{x}}_{a,t|k}^{MAP} = \arg\max_{\mathbf{x}_a \in \mathbf{R}^n} \left[J(\mathbf{x}_a(t), \hat{\mathbf{x}}_b^{MAP}(t), t | \mathbf{Y}_k) \right] \qquad (24)$$

$$\hat{\mathbf{x}}_{b,t|k}^{MAP} = \arg\max_{\mathbf{x}_b \in \mathbf{R}^n} \left[J(\hat{\mathbf{x}}_a^{MAP}(t), \mathbf{x}_b(t), t | \mathbf{Y}_k) \right] \qquad (25)$$

subject to the nonlinear constraints

$$\mathbf{c}(\mathbf{x}) = \begin{bmatrix} x_{a,1}^2 + x_{a,2}^2 + x_{a,3}^2 - 1 \\ x_{a,4}^2 + x_{a,5}^2 + x_{a,6}^2 - 1 \\ x_{a,7}^2 + x_{a,8}^2 + x_{a,9}^2 - 1 \end{bmatrix} \qquad (26)$$

Since Eq (25) is the optimization of a state quadratic equation not subject to constraints on the state variables it may be solved analytically to give the necessary conditions

$$\hat{\mathbf{x}}_{b,t|k}^{MAP} = -\frac{1}{2}\mathbf{A}_4^{-1}\left\{ \left(\mathbf{A}_2 + \mathbf{A}_3^T\right)\hat{\mathbf{x}}_a^{MAP} + \mathbf{b}_2 \right\} \qquad (27)$$

The estimate in Eq (27) makes uses of the fact that $\mathbf{A}(t)$ is a symmetric matrix. The filtering PDF parameters are decomposed according to

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_3 \\ \mathbf{A}_2 & \mathbf{A}_4 \end{bmatrix} \qquad (28)$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_1^T & \mathbf{b}_2^T \end{bmatrix}^T \tag{29}$$

where the dimensions of the sub-matrices and sub-vectors are $\mathbf{A}_1 \in \mathbf{R}^{9 \times 9}$, $\mathbf{A}_2 \in \mathbf{R}^{N \times 9}$, $\mathbf{A}_3 \in \mathbf{R}^{9 \times N}$, $\mathbf{A}_4 \in \mathbf{R}^{N \times N}$, $\mathbf{b}_1 \in \mathbf{R}^{9 \times 1}$, and $\mathbf{b}_2 \in \mathbf{R}^{N \times 1}$. Due to the nonlinear constraints of Eq (26) an analytical solution for the MAP estimate given by Eq (24) is not available. A numerical algorithm such as that proposed by Psiaki[30] is required to evaluate the state estimate at any instant in time.

7.4 Propagation of State PDF

This section considers propagation of the filtering PDF $\rho(\mathbf{x}(t), t | \mathbf{Y_k})$ over the time interval $t_k \leq t \leq t_{k+1}$. Alternatively stated, this section considers propagating $\rho(\mathbf{x}(t_k), t_k | \mathbf{Y_k})$ to $\rho(\mathbf{x}(t_{k+1}), t_{k+1} | \mathbf{Y_k})$. This propagation stage in the attitude filter corresponds to the time interval between sensor measurements. The Fokker-Planck partial differential equation (PDE) which governs the evolution of the conditional PDF is used to derive equivalent analytical expressions in the form of nonlinear ordinary differential equations (ODE) for the parameters $\mathbf{A}(t)$, $\mathbf{b}(t)$, and $\mathbf{c}(t)$ which define the filtering PDF. Since the un-normalised version of the conditional PDF $\sigma(\mathbf{x}, t | \mathbf{Y_k}) \equiv \sigma(\mathbf{x}, t)$ defined in Section 7.3 is related to $\rho(\mathbf{x}, t | \mathbf{Y_k}) \equiv \rho(\mathbf{x}, t)$ by a constant term which is independent of the filter state, then $\sigma(\mathbf{x}, t)$ also satisfies the Fokker-Planck equation given by

$$
\begin{aligned}
\frac{\partial \sigma(\mathbf{x}, t)}{\partial t} &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left\{ \left[\mathbf{G}(\mathbf{x}) \mathbf{G}^T(\mathbf{x}) \right]_{ij} \sigma(\mathbf{x}, t) \right\} \\
&\quad - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ f_i(\mathbf{x}) \sigma(\mathbf{x}, t) \right\}
\end{aligned} \tag{30}
$$

Expanding the individual terms of the Fokker-Planck equation gives

$$
\frac{\partial \sigma(\mathbf{x}, t)}{\partial t} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\mathbf{G}(\mathbf{x}) \mathbf{G}^T(\mathbf{x}) \right]_{ij} \frac{\partial^2 \sigma(\mathbf{x}, t)}{\partial x_i \partial x_j}
$$

$$+ \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \frac{\partial \left[\mathbf{G(x)}\mathbf{G}^{\mathrm{T}}(\mathbf{x}) \right]_{ij}}{\partial x_{j}} - f_{i}(\mathbf{x}) \right\} \frac{\partial \sigma(\mathbf{x},t)}{\partial x_{i}}$$

$$+ \sum_{i=1}^{n} \left\{ \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^{2} \left[\mathbf{G(x)}\mathbf{G}^{\mathrm{T}}(\mathbf{x}) \right]_{ij}}{\partial x_{i} \partial x_{j}} - \frac{\partial f_{i}(\mathbf{x})}{\partial x_{i}} \right\} \sigma(\mathbf{x},t) \qquad (31)$$

The objective of the propagation stage is to use the Fokker-Planck equation to solve for $\sigma(\mathbf{x}(t),t|\mathbf{Y_{k}})$ over $t_{k} \leq t \leq t_{k+1}$ in the form specified in Section 7.3 and based on an initial conditional PDF at $t = t_{k}$ given by

$$\sigma_{0}(\mathbf{x}(t_{k}),t_{k}) = \exp \left[\mathbf{x}^{\mathrm{T}}(t_{k})\mathbf{A}(t_{k})\mathbf{x}(t_{k}) + \mathbf{b}^{\mathrm{T}}(t_{k})\mathbf{x}(t_{k}) + c(t_{k}) \right] \qquad (32)$$

Before evaluating the individual terms of the Fokker-Planck equation an expression for the symmetric matrix elements $\left[\mathbf{G(x)}\mathbf{G}^{\mathrm{T}}(\mathbf{x}) \right]_{ij}$ and a polynomial expression for $f_{i}(\mathbf{x})$, as well as their partial derivatives are required. Partial derivatives of the conditional PDF $\sigma(\mathbf{x},t)$ are also required. The analytical form of $\mathbf{G(x)}\mathbf{G}^{\mathrm{T}}(\mathbf{x})$ is given by

$$\begin{aligned} \mathbf{G(x)}\mathbf{G}^{\mathrm{T}}(\mathbf{x}) &= \begin{bmatrix} \mathbf{\Gamma}_{\mathbf{a}}^{\mathrm{T}}(\mathbf{x})\mathbf{Q_{v}}\mathbf{\Gamma}_{\mathbf{a}}(\mathbf{x}) & \mathbf{0}_{9\times(N-3)} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{(N-3)\times9} & \mathbf{Q_{m}} & \mathbf{0}_{(N-3)\times3} \\ \mathbf{0}_{3\times9} & \mathbf{0}_{3\times(N-3)} & \mathbf{Q_{u}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{\Gamma}_{\mathbf{a}}^{\mathrm{T}}(\mathbf{x})\mathbf{Q_{v}}\mathbf{\Gamma}_{\mathbf{a}}(\mathbf{x}) & \mathbf{0}_{9\times N} \\ \mathbf{0}_{N\times9} & \mathbf{Q_{b}} \end{bmatrix} \end{aligned} \qquad (33)$$

with the process noise power spectral density matrices defined by $\mathbf{Q_{v}} = \mathbf{G_{v}}\mathbf{G}_{\mathbf{v}}^{\mathrm{T}}$, $\mathbf{Q_{m}} = \mathbf{G_{m}}\mathbf{G}_{\mathbf{m}}^{\mathrm{T}}$, $\mathbf{Q_{u}} = \mathbf{G_{u}}\mathbf{G}_{\mathbf{u}}^{\mathrm{T}}$, and $\mathbf{Q_{b}}$ defined in Eq (33). The individual matrix dimensions are $\mathbf{G}\mathbf{G}^{\mathrm{T}} \in \mathbf{R}^{(N+9)\times(N+9)}$, $\mathbf{\Gamma}_{\mathbf{a}}^{\mathrm{T}}\mathbf{Q_{v}}\mathbf{\Gamma}_{\mathbf{a}} \in \mathbf{R}^{9\times9}$, $\mathbf{Q_{v}} \in \mathbf{R}^{3\times3}$, $\mathbf{Q_{m}} \in \mathbf{R}^{(N-3)\times(N-3)}$, $\mathbf{Q_{u}} \in \mathbf{R}^{3\times3}$, and $\mathbf{Q_{b}} \in \mathbf{R}^{N\times N}$. For the special case where $\mathbf{G_{v}} = \mathbf{I}_{3\times3}$, $\mathbf{G_{m}} = \mathbf{I}_{(N-3)\times(N-3)}$, and $\mathbf{G_{u}} = \mathbf{I}_{3\times3}$ the expression for $\mathbf{G(x)}\mathbf{G}^{\mathrm{T}}(\mathbf{x})$ reduces to

$$\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\Gamma}_{\mathbf{a}}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\Gamma}_{\mathbf{a}}(\mathbf{x}) & \mathbf{0}_{9\times N} \\ \mathbf{0}_{N\times 9} & \mathbf{I}_{N\times N} \end{bmatrix} \tag{34}$$

Simplifying the $\boldsymbol{\Gamma}_{\mathbf{a}}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\Gamma}_{\mathbf{a}}(\mathbf{x})$ matrix in Eq (34) using the orthogonality and unit-norm properties of the columns of the attitude matrix gives

$$\boldsymbol{\Gamma}_{\mathbf{a}}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\Gamma}_{\mathbf{a}}(\mathbf{x}) = -\begin{bmatrix} \mathbf{u}\mathbf{u}^{\mathrm{T}} - \mathbf{I}_{3\times 3} & \mathbf{v}\mathbf{u}^{\mathrm{T}} & \mathbf{w}\mathbf{u}^{\mathrm{T}} \\ \mathbf{u}\mathbf{v}^{\mathrm{T}} & \mathbf{v}\mathbf{v}^{\mathrm{T}} - \mathbf{I}_{3\times 3} & \mathbf{w}\mathbf{v}^{\mathrm{T}} \\ \mathbf{u}\mathbf{w}^{\mathrm{T}} & \mathbf{v}\mathbf{w}^{\mathrm{T}} & \mathbf{w}\mathbf{w}^{\mathrm{T}} - \mathbf{I}_{3\times 3} \end{bmatrix} \tag{35}$$

The expression for $\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij}$ is given by

$$\Theta_{ij}(\mathbf{x}) \equiv \left[\mathbf{G}(\mathbf{x})\mathbf{G}(\mathbf{x})\right]_{ij} = \left[\boldsymbol{\Gamma}_{\mathbf{a}}^{\mathrm{T}}(\mathbf{x})\mathbf{Q}_{\mathbf{v}}\boldsymbol{\Gamma}_{\mathbf{a}}(\mathbf{x})\right]_{ij} \qquad i,j = 1,...,9 \tag{36}$$

$$\left[\mathbf{G}(\mathbf{x})\mathbf{G}(\mathbf{x})\right]_{ij} = \left[\mathbf{Q}_{\mathbf{b}}\right]_{ij} \qquad i,j = 10,...,(N+9) \tag{37}$$

which may be expressed in the general form

$$\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n}\Gamma_{kl}^{ij}x_{k}x_{l} + \gamma^{ij} \qquad i,j = 1,...,(N+9) \tag{38}$$

The $\equiv$ operator in (36) refers to the definition of a new variable based on existing nomenclature. The coefficients γ^{ij} follow directly from the definition of $\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij}$

$$\gamma^{ij} = 0 \qquad i,j = 1,...,9 \tag{39}$$

$$\gamma^{ij} = \left[\mathbf{Q}_{\mathbf{b}}\right]_{ij} \qquad i,j = 10,...,(N+9) \tag{40}$$

The Γ_{kl}^{ij} coefficients are determined by expressing the $\Theta_{ij}(\mathbf{x})$ term in Eq (36) as a second-order polynomial in the states. By definition

$$\gamma_{ij}(\mathbf{x}) = \sum_{k=1}^{3}\left[\boldsymbol{\Gamma}_a^T(\mathbf{x})\mathbf{G}_v\right]_{ik}\left[\boldsymbol{\Gamma}_a^T(\mathbf{x})\mathbf{G}_v\right]_{jk} \qquad i,j=1,...,9 \qquad (41)$$

$$\left[\boldsymbol{\Gamma}_a^T(\mathbf{x})\mathbf{G}_v\right]_{ij} = \sum_{k=1}^{n}\Gamma_k^{ij}x_k \qquad i=1,...,9;\ j=1,...,3 \qquad (42)$$

The definition of the Γ_k^{ij} terms are listed in Table 7-1, with unlisted elements being equal to zero.

$\Gamma_2^{11}=-G_{v(3,1)}$	$\Gamma_3^{11}=G_{v(2,1)}$	$\Gamma_2^{12}=-G_{v(3,2)}$	$\Gamma_3^{12}=G_{v(2,2)}$	$\Gamma_2^{13}=-G_{v(3,3)}$	$\Gamma_3^{13}=G_{v(2,3)}$
$\Gamma_1^{21}=G_{v(3,1)}$	$\Gamma_3^{21}=-G_{v(1,1)}$	$\Gamma_1^{22}=G_{v(3,2)}$	$\Gamma_3^{22}=-G_{v(1,2)}$	$\Gamma_1^{23}=G_{v(3,3)}$	$\Gamma_3^{23}=-G_{v(1,3)}$
$\Gamma_1^{31}=-G_{v(2,1)}$	$\Gamma_2^{31}=G_{v(1,1)}$	$\Gamma_1^{32}=-G_{v(2,2)}$	$\Gamma_2^{32}=G_{v(1,2)}$	$\Gamma_1^{33}=-G_{v(2,3)}$	$\Gamma_2^{33}=G_{v(1,3)}$
$\Gamma_5^{41}=-G_{v(3,1)}$	$\Gamma_6^{41}=G_{v(2,1)}$	$\Gamma_5^{42}=-G_{v(3,2)}$	$\Gamma_6^{42}=G_{v(2,2)}$	$\Gamma_5^{43}=-G_{v(3,3)}$	$\Gamma_6^{43}=G_{v(2,3)}$
$\Gamma_4^{51}=G_{v(3,1)}$	$\Gamma_6^{51}=-G_{v(1,1)}$	$\Gamma_4^{52}=G_{v(3,2)}$	$\Gamma_6^{52}=-G_{v(1,2)}$	$\Gamma_4^{53}=G_{v(3,3)}$	$\Gamma_6^{53}=-G_{v(1,3)}$
$\Gamma_4^{61}=-G_{v(2,1)}$	$\Gamma_5^{61}=G_{v(1,1)}$	$\Gamma_4^{62}=-G_{v(2,2)}$	$\Gamma_5^{62}=G_{v(1,2)}$	$\Gamma_4^{63}=-G_{v(2,3)}$	$\Gamma_5^{63}=G_{v(1,3)}$
$\Gamma_8^{71}=-G_{v(3,1)}$	$\Gamma_9^{71}=G_{v(2,1)}$	$\Gamma_8^{72}=-G_{v(3,2)}$	$\Gamma_9^{72}=G_{v(2,2)}$	$\Gamma_8^{73}=-G_{v(3,3)}$	$\Gamma_9^{73}=G_{v(2,3)}$
$\Gamma_7^{81}=G_{v(3,1)}$	$\Gamma_9^{81}=-G_{v(1,1)}$	$\Gamma_7^{82}=G_{v(3,2)}$	$\Gamma_9^{82}=-G_{v(1,2)}$	$\Gamma_7^{83}=G_{v(3,3)}$	$\Gamma_9^{83}=-G_{v(1,3)}$
$\Gamma_7^{91}=-G_{v(2,1)}$	$\Gamma_3^{91}=G_{v(1,1)}$	$\Gamma_7^{92}=-G_{v(2,2)}$	$\Gamma_8^{92}=G_{v(1,2)}$	$\Gamma_7^{93}=-G_{v(2,3)}$	$\Gamma_8^{93}=G_{v(1,3)}$

Table 7-1 Γ_k^{ij} **Coefficients**

Based on the definition of Γ_k^{ij} in Eq (42) and Table 7-1, θ_{ij} may be expressed as a second-order polynomial in the state variables

$$\Theta_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n}\Gamma_{kl}^{ij}x_k x_l \qquad (43)$$

such that the coefficients Γ_{kl}^{ij} are given by

$$\Gamma_{kl}^{ij} = \sum_{m=1}^{3}\Gamma_k^{im}\Gamma_l^{jm} \qquad i,j=1,...,9 \qquad (44)$$

$$\Gamma_{kl}^{ij} = 0 \qquad i,j=10,...,(N+9) \qquad (45)$$

Expressions for the first and higher-order partial derivatives of $\left[\mathbf{G}(\mathbf{x})\mathbf{G}^T(\mathbf{x})\right]_{ij}$ are

$$\frac{\partial\left[\mathbf{G}(\mathbf{x})\mathbf{G}(\mathbf{x})\right]_{ij}}{\partial x_p} = \sum_{k=1}^{n}\left[\Gamma_{pk}^{ij}+\Gamma_{kp}^{ij}\right]x_k \qquad i,j=1,...,9;\ p=1,...,(N+9) \qquad (46)$$

$$\frac{\partial[\mathbf{G(x)G(x)}]_{ij}}{\partial x_p} = 0 \qquad\qquad i,j = 10,...,(N+9); \ p = 1,...,(N+9) \qquad (47)$$

$$\frac{\partial^2[\mathbf{G(x)G(x)}]_{ij}}{\partial x_p \partial x_q} = \Gamma^{ij}_{pq} + \Gamma^{ij}_{qp} \qquad\qquad i,j = 1,...,9; \ p,q = 1,...,(N+9) \qquad (48)$$

$$\frac{\partial^2[\mathbf{G(x)G(x)}]_{ij}}{\partial x_p \partial x_q} = 0 \qquad\qquad i,j = 10,...,(N+9); \ p,q = 1,...,(N+9) \qquad (49)$$

The expression for $\mathbf{f(x)}$ is given by

$$f_i(\mathbf{x}) = \mathbf{\Gamma_a^T(x_a)}[\mathbf{x_d} - \mathbf{\omega_g}] \qquad\qquad i = 1,...,9 \qquad (50)$$

$$f_i(\mathbf{x}) = 0 \qquad\qquad i = 10,...,(N+9) \qquad (51)$$

The nonzero component of $\mathbf{f(x)}$ given by Eq (50) may be expressed as a second-order polynomial

$$f_i(\mathbf{x}) = \sum_{j=1}^{n}\sum_{k=1}^{n} \Phi^i_{jk} x_j x_k + \sum_{j=1}^{n} \chi^i_j x_j \qquad\qquad i = 1,...,9 \qquad (52)$$

The definition of the Φ^i_{jk} and χ^i_j terms are listed in Table 7-2 and Table 7-3 respectively, with unlisted elements equal to zero. The matrix $\mathbf{\Lambda(\omega_g)} \in \mathbf{R}^{9\times9}$ constructed with elements $[\mathbf{\Lambda(\omega_g)}]_{ij} \equiv \chi^i_j$ is skew-symmetric. Expressions for the first and high-order partial derivatives of $f_i(\mathbf{x})$ are

$$\frac{\partial f_i(\mathbf{x})}{\partial x_p} = \sum_{j=1}^{n} \left[\Phi^i_{pj} + \Phi^i_{jp}\right]x_j + \chi^i_p \qquad\qquad i = 1,...,9; \ p = 1,...,(N+9) \qquad (53)$$

$$\frac{\partial f_i(\mathbf{x})}{\partial x_p} = 0 \qquad\qquad i = 10,...,(N+9); \ p = 1,...,(N+9) \qquad (54)$$

$$\frac{\partial^2 f_i(\mathbf{x})}{\partial x_p \partial x_q} = \Phi^i_{ab} + \Phi^i_{ba} \qquad\qquad i = 1,...,9; \ p,q = 1,...,(N+9) \qquad (55)$$

$$\frac{\partial^2 f_i(\mathbf{x})}{\partial x_a \partial x_b} = 0 \qquad\qquad i = 10,\ldots,(N+9); \;\; p,q = 1,\ldots,(N+9) \qquad (56)$$

i	(j,k)	
1	$\Phi^1_{2,15} = -1$	$\Phi^1_{3,14} = 1$
2	$\Phi^2_{1,15} = 1$	$\Phi^2_{3,13} = -1$
3	$\Phi^3_{1,14} = -1$	$\Phi^3_{2,13} = 1$
4	$\Phi^4_{5,15} = -1$	$\Phi^4_{6,14} = 1$
5	$\Phi^5_{4,15} = 1$	$\Phi^5_{6,13} = -1$
6	$\Phi^6_{4,14} = -1$	$\Phi^6_{5,13} = 1$
7	$\Phi^7_{8,15} = -1$	$\Phi^7_{9,14} = 1$
8	$\Phi^8_{7,15} = 1$	$\Phi^8_{9,13} = -1$
9	$\Phi^9_{7,14} = -1$	$\Phi^9_{8,13} = 1$

Table 7-2 Φ^i_{jk} **Coefficients**

i	j	
1	$\chi^1_2 = \omega_{g,3}$	$\chi^1_3 = -\omega_{g,2}$
2	$\chi^2_1 = -\omega_{g,3}$	$\chi^2_3 = \omega_{g,1}$
3	$\chi^3_1 = \omega_{g,2}$	$\chi^3_2 = -\omega_{g,1}$
4	$\chi^4_5 = \omega_{g,3}$	$\chi^4_6 = -\omega_{g,2}$
5	$\chi^5_4 = -\omega_{g,3}$	$\chi^5_6 = \omega_{g,1}$
6	$\chi^6_4 = \omega_{g,2}$	$\chi^6_5 = -\omega_{g,1}$
7	$\chi^7_8 = \omega_{g,3}$	$\chi^7_9 = -\omega_{g,2}$
8	$\chi^8_7 = -\omega_{g,3}$	$\chi^8_9 = \omega_{g,1}$
9	$\chi^9_7 = \omega_{g,2}$	$\chi^9_8 = -\omega_{g,1}$

Table 7-3 χ^i_j **Coefficients**

The expressions for the partial derivatives of $\sigma(\mathbf{x}, t)$ are given by

$$\frac{\partial \sigma(\mathbf{x}, t)}{\partial x_i} = \left\{ \sum_{j=1}^{n} \left(A_{ij} + A_{ji} \right) x_j + b_i \right\} \sigma(\mathbf{x}, t) \qquad i = 1,\ldots,(N+9) \qquad (57)$$

$$\frac{\partial^2 \sigma(\mathbf{x},t)}{\partial x_i \partial x_j} = \left\{ \left(A_{ij} + A_{ji} \right) + \left[\sum_{k=1}^{n} \left(A_{ik} + A_{ki} \right) x_k + b_i \right]^2 \right\} \sigma(\mathbf{x},t) \quad i,j = 1,\ldots,(N+9) \quad (58)$$

The remainder of this section will evaluate the individual terms of the Fokker-Planck equation given by Eq (31) using the expressions Eqs (33)-(58). The first term on the left-hand-side of Eq (31) is given by

$$\frac{\partial \sigma(\mathbf{x},t)}{\partial t} = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{dA_{ij}(t)}{dt} x_i x_j + \sum_{i=1}^{n} \frac{db_i(t)}{dt} x_i + \frac{dc(t)}{dt} \right\} \sigma(\mathbf{x},t) \quad (59)$$

The first term on the right-hand-side of Eq (31) is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\mathbf{G}(\mathbf{x}) \mathbf{G}^T(\mathbf{x}) \right]_{ij} \frac{\partial^2 \sigma(\mathbf{x},t)}{\partial x_i \partial x_j} =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n} \Gamma_{kl}^{ij} x_k x_l + \gamma^{ij} \right\} \left\{ \left(A_{ij} + A_{ji} \right) + \left[\sum_{p=1}^{n} \left(A_{ip} + A_{pi} \right) x_p + b_i \right]^2 \right\} \sigma(\mathbf{x},t) \quad (60)$$

Expanding the terms in Eq (60) gives

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\mathbf{G}(\mathbf{x}) \mathbf{G}^T(\mathbf{x}) \right]_{ij} \frac{\partial^2 \sigma(\mathbf{x},t)}{\partial x_i \partial x_j} =$$

$$\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(A_{ij} + A_{ji} \right) \Gamma_{kl}^{ij} x_k x_l + \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma^{ij} \left(A_{ij} + A_{ji} \right) \right\} \sigma(\mathbf{x},t)$$

$$+ \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \begin{bmatrix} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right) \Gamma_{kl}^{ij} x_k x_l x_p x_q \\ + \left(A_{ip} + A_{pi} \right) b_j \Gamma_{kl}^{ij} x_k x_l x_p \\ + \left(A_{jq} + A_{qj} \right) b_i \Gamma_{kl}^{ij} x_k x_l x_q + b_i b_j \Gamma_{kl}^{ij} x_k x_l \end{bmatrix} \right\} \sigma(\mathbf{x},t)$$

$$+ \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \begin{bmatrix} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right) \gamma^{ij} x_p x_q \\ + \left(A_{ip} + A_{pi} \right) b_j \gamma^{ij} x_p \\ + \left(A_{jq} + A_{qj} \right) b_i \gamma^{ij} x_q + b_i b_j \gamma^{ij} \end{bmatrix} \right\} \sigma(\mathbf{x},t) \quad (61)$$

Each term of Eq (61) is evaluated below. The auxiliary $\mathbf{E}$ matrix is defined as

$$E_{kl} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma_{kl}^{ij} \left(A_{ij} + A_{ji} \right) \tag{62}$$

Therefore,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \Gamma_{kl}^{ij} \left(A_{ij} + A_{ji} \right) x_k x_l = \sum_{i=1}^{n} \sum_{j=1}^{n} E_{ij} x_i x_j = \mathbf{x}^T \mathbf{E} \mathbf{x} \tag{63}$$

$$F \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma^{ij} \left(A_{ij} + A_{ji} \right) \tag{64}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n} \Gamma_{kl}^{ij} x_k x_l \right\} \left\{ \left[\sum_{k=1}^{n} \left(A_{ik} + A_{ki} \right) x_k + b_i \right]^2 \right\} =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \left\{ \begin{array}{l} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right) \Gamma_{kl}^{ij} x_k x_l x_p x_q \\ + \left(A_{ip} + A_{pi} \right) b_j \Gamma_{kl}^{ij} x_k x_l x_p \\ + \left(A_{jq} + A_{qj} \right) b_i \Gamma_{kl}^{ij} x_k x_l x_q \\ + b_i b_j \Gamma_{kl}^{ij} x_k x_l \end{array} \right\} \tag{65}$$

$$H_{klpq} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right) \Gamma_{kl}^{ij} \tag{66}$$

Thus,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right) \Gamma_{kl}^{ij} x_k x_l x_p x_q =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} H_{ijkl} x_i x_j x_k x_l \tag{67}$$

$$J_{klp} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \left(A_{ip} + A_{pi} \right) b_j \Gamma_{kl}^{ij} \tag{68}$$

Hence,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(A_{ip} + A_{pi} \right) b_j \Gamma_{kl}^{ij} x_k x_l x_p = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} J_{ijk} x_i x_j x_k \tag{69}$$

$$K_{klq} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \left(A_{jq} + A_{qj} \right) b_i \Gamma_{kl}^{ij} \tag{70}$$

Therefore,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(A_{jq} + A_{qj} \right) b_i \Gamma_{kl}^{ij} x_k x_l x_q = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} K_{ijk} x_i x_j x_k \tag{71}$$

$$M_{kl} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \Gamma_{kl}^{ij} \tag{72}$$

Thus,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} b_i b_j \Gamma_{kl}^{ij} x_k x_l = \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} x_i x_j \tag{73}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma^{ij} \left\{ \left[\sum_{k=1}^{n} \left(A_{ik} + A_{ki} \right) x_k + b_i \right]^2 \right\} =$$
$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \left\{ \begin{array}{l} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right)\gamma^{ij} x_p x_q \\ + \left(A_{ip} + A_{pi} \right) b_j \gamma^{ij} x_p \\ + \left(A_{jq} + A_{qj} \right) b_i \gamma^{ij} x_q + b_i b_j \gamma^{ij} \end{array} \right\} \sigma(\mathbf{x}, t) \tag{74}$$

$$N_{pq} \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right) \gamma^{ij} \tag{75}$$

Therefore,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(A_{ip} + A_{pi} \right)\left(A_{jq} + A_{qj} \right) \gamma^{ij} x_p x_q = \sum_{i=1}^{n} \sum_{j=1}^{n} N_{ij} x_i x_j \tag{76}$$

$$S_p \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \left(A_{ip} + A_{pi} \right) b_j \gamma^{ij} \tag{77}$$

247

Thus,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(A_{ip}+A_{pi}\right)b_{j}\gamma^{ij}x_{p}=\sum_{i=1}^{n}S_{i}x_{i} \tag{78}$$

$$T_{q}\equiv\sum_{i=1}^{n}\sum_{j=1}^{n}\left(A_{jq}+A_{qj}\right)b_{i}\gamma^{ij} \tag{79}$$

Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(A_{jq}+A_{qj}\right)b_{i}\gamma^{ij}x_{q}=\sum_{i=1}^{n}T_{i}x_{i} \tag{80}$$

$$Z\equiv\sum_{i=1}^{n}\sum_{j=1}^{n}b_{i}b_{j}\gamma^{ij} \tag{81}$$

In summary, the first term on the right-hand-side of the Fokker-Planck equation of Eq (31) is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\mathbf{G(x)G^{T}(x)}\right]_{ij}\frac{\partial^{2}\sigma(\mathbf{x},t)}{\partial x_{i}\partial x_{j}}=$$
$$\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}H_{ijkl}x_{i}x_{j}x_{k}x_{l}+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(J_{ijk}+K_{ijk}\right)x_{i}x_{j}x_{k}\\ +\sum_{i=1}^{n}\sum_{j=1}^{n}\left(E_{ij}+M_{ij}+N_{ij}\right)x_{i}x_{j}+\sum_{i=1}^{n}\left(S_{i}+T_{i}\right)x_{i}+\left(F+Z\right)\end{array}\right\}\sigma(\mathbf{x},t) \tag{82}$$

The second term on the right-hand-side of Eq (31) is given by

$$\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\frac{\partial\left[\mathbf{G(x)G^{T}(x)}\right]_{ij}}{\partial x_{j}}-f_{i}(\mathbf{x})\right\}\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}}=$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\left[\mathbf{G(x)G^{T}(x)}\right]_{ij}}{\partial x_{j}}\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}}-\sum_{i=1}^{n}f_{i}(\mathbf{x})\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}} \tag{83}$$

Each term of Eq (83) is evaluated as follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathsf{T}}(\mathbf{x})\right]_{ij}}{\partial x_{j}}\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}}=$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\sum_{k=1}^{n}\left[\Gamma_{jk}^{ij}+\Gamma_{kj}^{ij}\right]x_{k}\right\}\left\{\sum_{l=1}^{n}\left(A_{il}+A_{li}\right)x_{l}+b_{i}\right\}\sigma(\mathbf{x},t)=$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{\left(\Gamma_{jk}^{ij}+\Gamma_{kj}^{ij}\right)\left(A_{il}+A_{li}\right)x_{k}x_{l}+\left(\Gamma_{jk}^{ij}+\Gamma_{kj}^{ij}\right)b_{i}x_{k}\right\}\sigma(\mathbf{x},t) \tag{84}$$

$$\widetilde{E}_{kl}\equiv\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\Gamma_{jk}^{ij}+\Gamma_{kj}^{ij}\right)\left(A_{il}+A_{li}\right) \tag{85}$$

$$\widetilde{F}\equiv\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\Gamma_{jk}^{ij}+\Gamma_{kj}^{ij}\right)b_{i} \tag{86}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathsf{T}}(\mathbf{x})\right]_{ij}}{\partial x_{j}}\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}}=\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\widetilde{E}_{ij}x_{i}x_{j}+\sum_{i=1}^{n}\widetilde{F}_{i}x_{i}\right\}\sigma(\mathbf{x},t) \tag{87}$$

$$\sum_{i=1}^{n}f_{i}(\mathbf{x})\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}}=\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\Phi_{jk}^{i}x_{j}x_{k}+\sum_{j=1}^{n}\chi_{j}^{i}x_{j}\right\}\left\{\sum_{l=1}^{n}\left(A_{il}+A_{li}\right)x_{l}+b_{i}\right\}\sigma(\mathbf{x},t)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{\begin{array}{l}\Phi_{jk}^{i}\left(A_{il}+A_{li}\right)x_{j}x_{k}x_{l}+\Phi_{jk}^{i}b_{i}x_{j}x_{k}\\+\chi_{j}^{i}\left(A_{il}+A_{li}\right)x_{j}x_{l}+\chi_{j}^{i}b_{i}x_{j}\end{array}\right\}\sigma(\mathbf{x},t) \tag{88}$$

$$\widetilde{H}_{jkl}\equiv\sum_{i=1}^{n}\Phi_{jk}^{i}\left(A_{il}+A_{li}\right) \tag{89}$$

$$\widetilde{J}_{jk}\equiv\sum_{i=1}^{n}\Phi_{jk}^{i}b_{i} \tag{90}$$

$$\widetilde{K}_{jk}\equiv\sum_{i=1}^{n}\chi_{j}^{i}\left(A_{il}+A_{li}\right) \tag{91}$$

$$\widetilde{M}_{j}\equiv\sum_{i=1}^{n}\chi_{j}^{i}b_{i} \tag{92}$$

$$\sum_{i=1}^{n}f_{i}(\mathbf{x})\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}}=\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\widetilde{H}_{ijk}x_{i}x_{j}x_{k}+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\widetilde{J}_{ij}+\widetilde{K}_{ij}\right)x_{i}x_{j}\\+\sum_{i=1}^{n}\widetilde{M}_{i}x_{i}\end{array}\right\}\sigma(\mathbf{x},t) \tag{93}$$

In summary, the second term on the right-hand-side of the Fokker-Planck equation of Eq (31) is given by

$$\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\frac{\partial\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij}}{\partial x_{j}}-f_{i}(\mathbf{x})\right\}\frac{\partial\sigma(\mathbf{x},t)}{\partial x_{i}}=$$
$$\left\{-\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\widetilde{H}_{ijk}x_{i}x_{j}x_{k}+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\widetilde{E}_{ij}-\widetilde{J}_{ij}-\widetilde{K}_{ij}\right)x_{i}x_{j}+\sum_{i=1}^{n}\left(\widetilde{F}_{i}-\widetilde{M}_{i}\right)x_{i}\right\}\sigma(\mathbf{x},t) \quad (94)$$

The third term on the right-hand-side of Eq (31) is given by

$$\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\frac{\partial^{2}\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij}}{\partial x_{i}\partial x_{j}}-\frac{\partial f_{i}(\mathbf{x})}{\partial x_{i}}\right\}\sigma(\mathbf{x},t)=$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^{2}\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij}}{\partial x_{i}\partial x_{j}}\sigma(\mathbf{x},t)-\sum_{i=1}^{n}\frac{\partial f_{i}(\mathbf{x})}{\partial x_{i}}\sigma(\mathbf{x},t) \quad (95)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^{2}\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij}}{\partial x_{i}\partial x_{j}}\sigma(\mathbf{x},t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\Gamma_{ij}^{ij}+\Gamma_{ji}^{ij}\right\}\sigma(\mathbf{x},t) \quad (96)$$

$$\widetilde{N}\equiv\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\Gamma_{ij}^{ij}+\Gamma_{ji}^{ij}\right) \quad (97)$$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^{2}\left[\mathbf{G}(\mathbf{x})\mathbf{G}^{\mathrm{T}}(\mathbf{x})\right]_{ij}}{\partial x_{i}\partial x_{j}}\sigma(\mathbf{x},t)=\widetilde{N}\sigma(\mathbf{x},t) \quad (98)$$

$$\sum_{i=1}^{n}\frac{\partial f_{i}(\mathbf{x})}{\partial x_{i}}\sigma(\mathbf{x},t)=\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\left[\Phi_{ij}^{i}+\Phi_{ji}^{i}\right]x_{j}+\chi_{i}^{i}\right\}\sigma(\mathbf{x},t) \quad (99)$$

$$\widetilde{S}_{j}\equiv\sum_{i=1}^{n}\left(\Phi_{ij}^{i}+\Phi_{ji}^{i}\right) \quad (100)$$

$$\widetilde{T}\equiv\sum_{i=1}^{n}\chi_{i}^{i} \quad (101)$$

Hence,

$$\sum_{i=1}^{n} \frac{\partial f_i(\mathbf{x})}{\partial x_i} \sigma(\mathbf{x}, t) = \left\{ \sum_{i=1}^{n} \widetilde{S}_i x_i + \widetilde{T} \right\} \sigma(\mathbf{x}, t) \qquad (102)$$

In summary, the third term on the right-hand-side of the Fokker-Planck equation of Eq (31) is given by

$$\sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \frac{\partial^2 \left[\mathbf{G}(\mathbf{x}) \mathbf{G}^{T}(\mathbf{x}) \right]_{ij}}{\partial x_i \partial x_j} - \frac{\partial f_i(\mathbf{x})}{\partial x_i} \right\} \sigma(\mathbf{x}, t) = \left\{ \widetilde{N} - \sum_{i=1}^{n} \widetilde{S}_i x_i - \widetilde{T} \right\} \sigma(\mathbf{x}, t) \qquad (103)$$

Collecting all individual terms of the Fokker-Planck equation derived in Eqs (59)-(103) into a single equation yields

$$\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{dA_{ij}(t)}{dt} x_i x_j + \sum_{i=1}^{n} \frac{db_i(t)}{dt} x_i + \frac{dc(t)}{dt} \right\} \sigma(\mathbf{x}, t) =$$

$$\left[\begin{array}{c} \left\{ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} H_{ijkl} x_i x_j x_k x_l \right\} \\ + \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{1}{2} J_{ijk} + \frac{1}{2} K_{ijk} - \widetilde{H}_{ijk} \right) x_i x_j x_k \right\} \\ + \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{2} E + \frac{1}{2} M + \frac{1}{2} N + \widetilde{E} - \widetilde{J} - \widetilde{K} - \frac{1}{2} \widetilde{Z} \right) x_i x_j \right\} \end{array} \right] \sigma(\mathbf{x}, t)$$

$$+ \left\{ \left\{ \sum_{i=1}^{n} \left(\frac{1}{2} S + \frac{1}{2} T + \widetilde{F} - \widetilde{M} - \widetilde{S} \right) x_i \right\} + \left\{ \frac{1}{2} F + \frac{1}{2} Z + \frac{1}{2} \widetilde{N} - \widetilde{T} \right\} \right\} \sigma(\mathbf{x}, t) \qquad (104)$$

Since terms higher than second order in the state variables are not present on the left-hand-side of Eq (104) then the higher-order terms on the right-hand-side of Eq (104) must be neglected or approximated with a second-order Taylor series. In this work the higher order terms are neglected which will introduce error into the filtering PDF propagation process.

$$\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{dA_{ij}(t)}{dt} x_i x_j + \sum_{i=1}^{n} \frac{db_i(t)}{dt} x_i + \frac{dc(t)}{dt} \right\} \sigma(\mathbf{x}, t) =$$

$$\left\{\left\{\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{2}E+\frac{1}{2}M+\frac{1}{2}N+\widetilde{E}-\widetilde{J}-\widetilde{K}-\frac{1}{2}\widetilde{Z}\right)x_{i}x_{j}\right]\right\}\sigma(\mathbf{x},t)\right.$$

$$+\left\{\left[\sum_{i=1}^{n}\left(\frac{1}{2}S+\frac{1}{2}T+\widetilde{F}-\widetilde{M}-\widetilde{S}\right)x_{i}\right]+\left\{\frac{1}{2}F+\frac{1}{2}Z+\frac{1}{2}\widetilde{N}-\widetilde{T}\right\}\right\}\sigma(\mathbf{x},t) \quad (105)$$

Equating the filtering parameters in Eq (105) results in a system of nonlinear ordinary differential equations between sensor measurements $t_k \le t \le t_{k+1}$ given by

$$\frac{d\mathbf{A}(t)}{dt}=\frac{1}{2}\left\{\mathbf{E}+\mathbf{M}+\mathbf{N}+2\widetilde{\mathbf{E}}-2\widetilde{\mathbf{J}}-2\widetilde{\mathbf{K}}-\widetilde{\mathbf{Z}}\right\} \quad (106)$$

$$\frac{d\mathbf{b}(t)}{dt}=\frac{1}{2}\left\{\mathbf{S}+\mathbf{T}+2\widetilde{\mathbf{F}}-2\widetilde{\mathbf{M}}-2\widetilde{\mathbf{S}}\right\} \quad (107)$$

$$\frac{dc(t)}{dt}=\frac{1}{2}\left\{F+Z+\widetilde{N}-2\widetilde{T}\right\} \quad (108)$$

7.5 Sensor Measurement Update

This previous section considered propagation of the filtering PDF $\sigma(\mathbf{x},t)$ from $t_{k-1} \le t \le t_k$ to produce the un-normalised conditional PDF of the state variables at time t_k conditioned on all sensor measurement up to and including time t_{k-1}

$$\sigma(\mathbf{x}(t_k),t_k|\mathbf{Y_{k-1}})=\exp\left[\mathbf{x}^T\mathbf{A}(t_k)\mathbf{x}+\mathbf{b}^T(t_k)\mathbf{x}+c(t_k)\right] \quad (109)$$

$$J(\mathbf{x}(t_k),t_k|\mathbf{Y_{k-1}})=\mathbf{x}^T\mathbf{A}(t_k)\mathbf{x}+\mathbf{b}^T(t_k)\mathbf{x}+c(t_k) \quad (110)$$

This section considers improving the filtering PDF based on sensor measurements available at time t_k. The normalised and un-normalised versions of the sensor measurement PDFs conditioned on the pre-measurement state, denoted by $\rho(\mathbf{y_k}|\mathbf{x_k})$ and $\sigma(\mathbf{y_k}|\mathbf{x_k})$ respectively, are assumed to be Gaussian distributed

$$\rho(\mathbf{y_k}|\mathbf{x_k}) = \exp\left[-\frac{1}{2}[\mathbf{y_k} - \mathbf{D_k r_k} - \mathbf{x_m}]^T \mathbf{F_k^m}[\mathbf{y_k} - \mathbf{D_k r_k} - \mathbf{x_m}] + \alpha\right] \qquad (111)$$

$$\sigma(\mathbf{y_k}|\mathbf{x_k}) = \exp\left[-\frac{1}{2}[\mathbf{y_k} - \mathbf{D_k r_k} - \mathbf{x_m}]^T \mathbf{F_k^m}[\mathbf{y_k} - \mathbf{D_k r_k} - \mathbf{x_m}]\right] \qquad (112)$$

$$J(\mathbf{y_k}|\mathbf{x_k}) = -\frac{1}{2}[\mathbf{y_k} - \mathbf{D_k r_k} - \mathbf{x_m}]^T \mathbf{F_k^m}[\mathbf{y_k} - \mathbf{D_k r_k} - \mathbf{x_m}] \qquad (113)$$

where $\mathbf{y_k}$ is the tri-axial magnetometer measurement of the ambient geomagnetic field with constant bias $\mathbf{x_m}$ expressed in the spacecraft body-fixed coordinate frame and with Gaussian measurement errors specified by the measurement information matrix $\mathbf{F_k^m}$. The $\mathbf{D_k}$ term represents the true spacecraft attitude matrix, and α is a normalising constant to ensure unity probability over the state-space $\mathbf{R^n}$, i.e. $\int_{\mathbf{R^n}} \rho(\mathbf{y_k}|\mathbf{x_k})d\mathbf{x_k} = 1$. The normalised filtering PDF $\rho(\mathbf{x_k}|\mathbf{Y_k})$ is updated following a measurement $\mathbf{y_k}$ at time t_k using Bayes theorem

$$\rho(\mathbf{x_k}, t_k|\mathbf{Y_k}) = \frac{\rho(\mathbf{y_k}|\mathbf{x_k})\rho(\mathbf{x_k}, t_k|\mathbf{Y_{k-1}})}{\int_{\mathbf{R^n}} \rho(\mathbf{y_k}|\mathbf{x_k})\rho(\mathbf{x_k}, t_k|\mathbf{Y_{k-1}})d\mathbf{x_k}} \qquad (115)$$

An equivalent form of Bayes theorem using the un-normalised filtering and measurement PDFs is given by

$$\sigma(\mathbf{x_k}, t_k|\mathbf{Y_k}) = \left[\frac{\int_{\mathbf{R^n}} \sigma(\mathbf{x_k}, t_k|\mathbf{Y_k})d\mathbf{x_k}}{\int_{\mathbf{R^n}} \sigma(\mathbf{y_k}|\mathbf{x_k})\sigma(\mathbf{x_k}, t_k|\mathbf{Y_{k-1}})d\mathbf{x_k}}\right]\sigma(\mathbf{y_k}|\mathbf{x_k})\sigma(\mathbf{x_k}, t_k|\mathbf{Y_{k-1}}) \qquad (116)$$

Taking the natural logarithm of Eq (116) gives

$$J(\mathbf{x_k}, t_k|\mathbf{Y_k}) = J(\mathbf{x_k}, t_k|\mathbf{Y_{k-1}}) + J(\mathbf{y_k}|\mathbf{x_k}) + \ln\left[\frac{\int_{\mathbf{R^n}} \sigma(\mathbf{x_k}, t_k|\mathbf{Y_k})d\mathbf{x_k}}{\int_{\mathbf{R^n}} \sigma(\mathbf{y_k}|\mathbf{x_k})\sigma(\mathbf{x_k}, t_k|\mathbf{Y_{k-1}})d\mathbf{x_k}}\right] \qquad (117)$$

It is important to note that the final term on the RHS of Eq (117) is independent of the state variables and is therefore not required to compute the MAP state estimate discussed in Section 7.3. The Bayesian update can be performed providing that that each side of Eq (117) has the same functional dependence on the state variables, otherwise approximations will be required. The first step of the update stage is to express $J(\mathbf{y_k}|\mathbf{x_k})$ as a second-order polynomial in the state variables. Expanding Eq (113), noting that the measurement information matrix $\mathbf{F_k^m}$ is symmetric, gives

$$J(\mathbf{y_k}|\mathbf{x_k}) = -\frac{1}{2}\mathbf{y_k^T F_k^m y_k} - \frac{1}{2}\mathbf{x_m^T F_k^m x_m} + \mathbf{y_k^T F_k^m x_m}$$
$$+ [\mathbf{y_k} - \mathbf{x_m}]^T \mathbf{F_k^m D_k r_k} - \frac{1}{2}\mathbf{r_k^T D_k^T F_k^m D_k r_k} \qquad (118)$$

The remainder of this section evaluates each term in Eq (118) using the definition of the state vector $\mathbf{x} = \begin{bmatrix} \mathbf{u}^T & \mathbf{v}^T & \mathbf{w}^T & \mathbf{x_m}^T & \mathbf{x_d}^T \end{bmatrix}^T$ and the spacecraft attitude matrix $\mathbf{D} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$. The first term on the right-hand-side of Eq (118) is given by

$$\mathbf{y_k^T F_k^m y_k} = \sum_{i=1}^{n}\sum_{j=1}^{n} F_{k,ij}^m y_{k,i} y_{k,j} \qquad (119)$$

which is independent of the state variables. The second term on the right-hand-side of Eq (118) is given by

$$\mathbf{x_m^T F_k^m x_m} = \sum_{i=1}^{3}\sum_{j=1}^{3}\left[\mathbf{F_k^m}\right]_{ij} x_i x_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\widetilde{\mathbf{F}}_k^m\right]_{ij} x_i x_j \qquad (120)$$

$$\widetilde{\mathbf{F}}_k^m = \begin{bmatrix} \mathbf{0}_{9\times9} & \mathbf{0}_{9\times3} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{3\times9} & \mathbf{F_k^m} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times9} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \qquad (121)$$

The third term on the right-hand-side of Eq (118) is given by

$$\mathbf{y}_k^T \mathbf{F}_k^m \mathbf{x}_m = \sum_{i=1}^{3} \left[\mathbf{F}_k^m \mathbf{y}_k\right]_i x_{m,i} = \sum_{i=1}^{n} \left[\widetilde{\mathbf{F}}_k^m \widetilde{\mathbf{y}}_k\right]_i x_i \tag{122}$$

$$\widetilde{\mathbf{F}}_k^m \widetilde{\mathbf{y}}_k = \left[\mathbf{0}_{9\times1}^T \quad \left[\mathbf{F}_k^m \mathbf{y}_k\right]^T \quad \mathbf{0}_{3\times1}^T\right]^T \tag{123}$$

The fourth term on the right-hand-side of Eq (118) is given by

$$\left[\mathbf{y}_k^T - \mathbf{x}_m^T\right]\mathbf{F}_k^m \mathbf{A}_k \mathbf{r}_k = \mathbf{y}_k^T \mathbf{F}_k^m \mathbf{A}_k \mathbf{r}_k - \mathbf{x}_m^T \mathbf{F}_k^m \mathbf{A}_k \mathbf{r}_k \tag{124}$$

Each term of Eq (124) is evaluated below:

$$\mathbf{A}_k \mathbf{r}_k = \begin{bmatrix} r_{k,1}x_1 + r_{k,2}x_4 + r_{k,3}x_7 \\ r_{k,1}x_2 + r_{k,2}x_5 + r_{k,3}x_8 \\ r_{k,1}x_3 + r_{k,2}x_6 + r_{k,3}x_9 \end{bmatrix} \tag{125}$$

$$\mathbf{y}_k^T \mathbf{F}_k^m \mathbf{A}_k \mathbf{r}_k = \left[\mathbf{F}_k^m \mathbf{y}_k\right]^T \mathbf{A}_k \mathbf{r}_k =$$
$$\begin{Bmatrix} \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_1 r_{k,1}\right]x_1 + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_2 r_{k,1}\right]x_2 + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_3 r_{k,1}\right]x_3 \\ + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_1 r_{k,2}\right]x_4 + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_2 r_{k,2}\right]x_5 + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_3 r_{k,2}\right]x_6 \\ + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_1 r_{k,3}\right]x_7 + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_2 r_{k,3}\right]x_8 + \left[\left(\mathbf{F}_k^m \mathbf{y}_k\right)_3 r_{k,3}\right]x_9 \end{Bmatrix}$$
$$= \sum_{i=1}^{n} \alpha_j x_j \tag{126}$$

$$\alpha = \left[r_{k,1}\left[\mathbf{F}_k^m \mathbf{y}_k\right]^T \quad r_{k,2}\left[\mathbf{F}_k^m \mathbf{y}_k\right]^T \quad r_{k,3}\left[\mathbf{F}_k^m \mathbf{y}_k\right]^T \quad \mathbf{0}_{6\times1}^T\right]^T \tag{127}$$

$$\mathbf{F}_k^m \mathbf{x}_m = \begin{bmatrix} F_{k,11}^m x_{10} + F_{k,12}^m x_{11} + F_{k,13}^m x_{12} \\ F_{k,21}^m x_{10} + F_{k,22}^m x_{11} + F_{k,23}^m x_{12} \\ F_{k,31}^m x_{10} + F_{k,32}^m x_{11} + F_{k,33}^m x_{12} \end{bmatrix} \tag{128}$$

$$\mathbf{A}_k \mathbf{r}_k = \begin{bmatrix} r_{k,1}x_1 + r_{k,2}x_4 + r_{k,3}x_7 \\ r_{k,1}x_2 + r_{k,2}x_5 + r_{k,3}x_8 \\ r_{k,1}x_3 + r_{k,2}x_6 + r_{k,3}x_9 \end{bmatrix} \tag{129}$$

$$\mathbf{x_m^T F_k^m A_k r_k} = \left[\mathbf{F_k^m x_m}\right]^T \mathbf{A_k r_k} =$$

$$\left\{ \begin{array}{l} \left[r_{k,1} F_{k,11}^m x_1 x_{10} + r_{k,1} F_{k,12}^m x_1 x_{11} + r_{k,1} F_{k,13}^m x_1 x_{12} \right] + \\ \left[r_{k,1} F_{k,21}^m x_2 x_{10} + r_{k,1} F_{k,22}^m x_2 x_{11} + r_{k,1} F_{k,23}^m x_2 x_{12} \right] + \\ \left[r_{k,1} F_{k,31}^m x_3 x_{10} + r_{k,1} F_{k,32}^m x_3 x_{11} + r_{k,1} F_{k,33}^m x_3 x_{12} \right] + \\ \left[r_{k,2} F_{k,11}^m x_4 x_{10} + r_{k,2} F_{k,12}^m x_4 x_{11} + r_{k,2} F_{k,13}^m x_4 x_{12} \right] + \\ \left[r_{k,2} F_{k,21}^m x_5 x_{10} + r_{k,2} F_{k,22}^m x_5 x_{11} + r_{k,2} F_{k,23}^m x_5 x_{12} \right] + \end{array} \right\}$$

$$\left\{ \begin{array}{l} \left[r_{k,2} F_{k,31}^m x_6 x_{10} + r_{k,2} F_{k,32}^m x_6 x_{11} + r_{k,2} F_{k,33}^m x_6 x_{12} \right] + \\ \left[r_{k,3} F_{k,11}^m x_7 x_{10} + r_{k,3} F_{k,12}^m x_7 x_{11} + r_{k,3} F_{k,13}^m x_7 x_{12} \right] + \\ \left[r_{k,3} F_{k,21}^m x_8 x_{10} + r_{k,3} F_{k,22}^m x_8 x_{11} + r_{k,3} F_{k,23}^m x_8 x_{12} \right] + \\ \left[r_{k,3} F_{k,31}^m x_9 x_{10} + r_{k,3} F_{k,32}^m x_9 x_{11} + r_{k,3} F_{k,33}^m x_9 x_{12} \right] \end{array} \right\}$$

$$= \sum_{i=1}^n \sum_{j=1}^n \beta_{ij} x_i x_j \tag{130}$$

$$\boldsymbol{\beta} = \begin{bmatrix} & r_{k,1}\mathbf{F_k^m} & \\ \mathbf{0}_{9\times 9} & r_{k,2}\mathbf{F_k^m} & \mathbf{0}_{9\times 3} \\ & r_{k,3}\mathbf{F_k^m} & \\ \mathbf{0}_{3\times 9} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 9} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \tag{131}$$

In summary, the fourth term on the right-hand-side of the Fokker-Planck equation of Eq (118) is given by

$$\left[\mathbf{y_k^T} - \mathbf{x_m^T} \right] \mathbf{F_k^m A_k r_k} = -\sum_{i=1}^n \sum_{j=1}^n \beta_{ij} x_i x_j + \sum_{i=1}^n \alpha_j x_j \tag{132}$$

The individual components of the fifth term on the right-hand-side of Eq (118) are given by

$$\mathbf{F_k^m D_k r_k} = \begin{bmatrix} F_{k,11}^m & F_{k,12}^m & F_{k,13}^m \\ F_{k,21}^m & F_{k,22}^m & F_{k,23}^m \\ F_{k,31}^m & F_{k,32}^m & F_{k,33}^m \end{bmatrix} \begin{bmatrix} r_{k,1}x_1 + r_{k,2}x_4 + r_{k,3}x_7 \\ r_{k,1}x_2 + r_{k,2}x_5 + r_{k,3}x_8 \\ r_{k,1}x_3 + r_{k,2}x_6 + r_{k,3}x_9 \end{bmatrix} \tag{133}$$

$$\begin{aligned} \left[\mathbf{F_k^m D_k r_k} \right]_{1,1} &= F_{k,11}^m r_{k,1}x_1 + F_{k,12}^m r_{k,1}x_2 + F_{k,13}^m r_{k,1}x_3 + F_{k,11}^m r_{k,2}x_4 \\ &+ F_{k,12}^m r_{k,2}x_5 + F_{k,13}^m r_{k,2}x_6 + F_{k,11}^m r_{k,3}x_7 + F_{k,12}^m r_{k,3}x_8 + F_{k,13}^m r_{k,3}x_9 \end{aligned} \tag{134}$$

$$\left[\mathbf{F_k^m D_k r_k}\right]_{2,1} = F_{k,21}^m r_{k,1} x_1 + F_{k,22}^m r_{k,1} x_2 + F_{k,23}^m r_{k,1} x_3 + F_{k,21}^m r_{k,2} x_4$$
$$+ F_{k,22}^m r_{k,2} x_5 + F_{k,23}^m r_{k,2} x_6 + F_{k,21}^m r_{k,3} x_7 + F_{k,22}^m r_{k,3} x_8 + F_{k,23}^m r_{k,3} x_9 \tag{135}$$

$$\left[\mathbf{F_k^m D_k r_k}\right]_{3,1} = F_{k,31}^m r_{k,1} x_1 + F_{k,32}^m r_{k,1} x_2 + F_{k,33}^m r_{k,1} x_3 + F_{k,31}^m r_{k,2} x_4$$
$$+ F_{k,32}^m r_{k,2} x_5 + F_{k,33}^m r_{k,2} x_6 + F_{k,31}^m r_{k,3} x_7 + F_{k,32}^m r_{k,3} x_8 + F_{k,33}^m r_{k,3} x_9 \tag{136}$$

$$\mathbf{r_k^T D_k^T F_k^m D_k r_k} = \left[\mathbf{D_k r_k}\right]^T \mathbf{F_k^m D_k r_k} = \sum_{i=1}^{3} \left[\mathbf{D_k r_k}\right]_i \left[\mathbf{F_k^m D_k r_k}\right]_i$$

$$= \begin{bmatrix} F_{k,11}^m r_{k,1}^2 x_1^2 + F_{k,12}^m r_{k,1}^2 x_1 x_2 + F_{k,13}^m r_{k,1}^2 x_1 x_3 + F_{k,11}^m r_{k,1} r_{k,2} x_1 x_4 + F_{k,12}^m r_{k,1} r_{k,2} x_1 x_5 \\ + F_{k,13}^m r_{k,1} r_{k,2} x_1 x_6 + F_{k,11}^m r_{k,1} r_{k,3} x_1 x_7 + F_{k,12}^m r_{k,1} r_{k,3} x_1 x_8 + F_{k,13}^m r_{k,1} r_{k,3} x_1 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,21}^m r_{k,1}^2 x_2 x_1 + F_{k,22}^m r_{k,1}^2 x_2^2 + F_{k,23}^m r_{k,1}^2 x_2 x_3 + F_{k,21}^m r_{k,1} r_{k,2} x_2 x_4 + F_{k,22}^m r_{k,1} r_{k,2} x_2 x_5 \\ + F_{k,23}^m r_{k,1} r_{k,2} x_2 x_6 + F_{k,21}^m r_{k,1} r_{k,3} x_2 x_7 + F_{k,22}^m r_{k,1} r_{k,3} x_2 x_8 + F_{k,23}^m r_{k,1} r_{k,3} x_2 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,31}^m r_{k,1}^2 x_3 x_1 + F_{k,32}^m r_{k,1}^2 x_3 x_2 + F_{k,33}^m r_{k,1}^2 x_3^2 + F_{k,31}^m r_{k,1} r_{k,2} x_3 x_4 + F_{k,32}^m r_{k,1} r_{k,2} x_3 x_5 \\ + F_{k,33}^m r_{k,1} r_{k,2} x_3 x_6 + F_{k,31}^m r_{k,1} r_{k,3} x_3 x_7 + F_{k,32}^m r_{k,1} r_{k,3} x_3 x_8 + F_{k,33}^m r_{k,1} r_{k,3} x_3 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,11}^m r_{k,1} r_{k,2} x_4 x_1 + F_{k,12}^m r_{k,1} r_{k,2} x_4 x_2 + F_{k,13}^m r_{k,1} r_{k,2} x_4 x_3 + F_{k,11}^m r_{k,2}^2 x_4^2 + F_{k,12}^m r_{k,2}^2 x_4 x_5 \\ + F_{k,13}^m r_{k,2}^2 x_4 x_6 + F_{k,11}^m r_{k,2} r_{k,3} x_4 x_7 + F_{k,12}^m r_{k,2} r_{k,3} x_4 x_8 + F_{k,13}^m r_{k,2} r_{k,3} x_4 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,21}^m r_{k,1} r_{k,2} x_5 x_1 + F_{k,22}^m r_{k,1} r_{k,2} x_5 x_2 + F_{k,23}^m r_{k,1} r_{k,2} x_5 x_3 + F_{k,21}^m r_{k,2}^2 x_5 x_4 + F_{k,22}^m r_{k,2}^2 x_5^2 \\ + F_{k,23}^m r_{k,2}^2 x_5 x_6 + F_{k,21}^m r_{k,2} r_{k,3} x_5 x_7 + F_{k,22}^m r_{k,2} r_{k,3} x_5 x_8 + F_{k,23}^m r_{k,2} r_{k,3} x_5 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,31}^m r_{k,1} r_{k,2} x_6 x_1 + F_{k,32}^m r_{k,1} r_{k,2} x_6 x_2 + F_{k,33}^m r_{k,1} r_{k,2} x_6 x_3 + F_{k,31}^m r_{k,2}^2 x_6 x_4 + F_{k,32}^m r_{k,2}^2 x_6 x_5 \\ + F_{k,33}^m r_{k,2}^2 x_6^2 + F_{k,31}^m r_{k,2} r_{k,3} x_6 x_7 + F_{k,32}^m r_{k,2} r_{k,3} x_6 x_8 + F_{k,33}^m r_{k,2} r_{k,3} x_6 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,11}^m r_{k,1} r_{k,3} x_7 x_1 + F_{k,12}^m r_{k,1} r_{k,3} x_7 x_2 + F_{k,13}^m r_{k,1} r_{k,3} x_7 x_3 + F_{k,11}^m r_{k,2} r_{k,3} x_7 x_4 + F_{k,12}^m r_{k,2} r_{k,3} x_7 x_5 \\ + F_{k,13}^m r_{k,2} r_{k,3} x_7 x_6 + F_{k,11}^m r_{k,3}^2 x_7^2 + F_{k,12}^m r_{k,3}^2 x_7 x_8 + F_{k,13}^m r_{k,3}^2 x_7 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,21}^m r_{k,1} r_{k,3} x_8 x_1 + F_{k,22}^m r_{k,1} r_{k,3} x_8 x_2 + F_{k,23}^m r_{k,1} r_{k,3} x_8 x_3 + F_{k,21}^m r_{k,2} r_{k,3} x_8 x_4 + F_{k,22}^m r_{k,2} r_{k,3} x_8 x_5 \\ + F_{k,23}^m r_{k,2} r_{k,3} x_8 x_6 + F_{k,21}^m r_{k,3}^2 x_8 x_7 + F_{k,22}^m r_{k,3}^2 x_8^2 + F_{k,23}^m r_{k,3}^2 x_8 x_9 \end{bmatrix}$$

$$+ \begin{bmatrix} F_{k,31}^m r_{k,1} r_{k,3} x_9 x_1 + F_{k,32}^m r_{k,1} r_{k,3} x_9 x_2 + F_{k,33}^m r_{k,1} r_{k,3} x_9 x_3 + F_{k,31}^m r_{k,2} r_{k,3} x_9 x_4 + F_{k,32}^m r_{k,2} r_{k,3} x_9 x_5 \\ + F_{k,33}^m r_{k,2} r_{k,3} x_9 x_6 + F_{k,31}^m r_{k,3}^2 x_9 x_7 + F_{k,32}^m r_{k,3}^2 x_9 x_8 + F_{k,33}^m r_{k,3}^2 x_9^2 \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} x_i x_j \tag{137}$$

$$\boldsymbol{\kappa} = \begin{bmatrix} r_{k,1}^2 \mathbf{F_k^m} & r_{k,1} r_{k,2} \mathbf{F_k^m} & r_{k,1} r_{k,3} \mathbf{F_k^m} \\ r_{k,2} r_{k,1} \mathbf{F_k^m} & r_{k,2}^2 \mathbf{F_k^m} & r_{k,2} r_{k,3} \mathbf{F_k^m} \\ r_{k,3} r_{k,1} \mathbf{F_k^m} & r_{k,3} r_{k,2} \mathbf{F_k^m} & r_{k,3}^2 \mathbf{F_k^m} \end{bmatrix} \tag{138}$$

In summary, the log-likelihood function of Eq (118) is given by

$$
J(\mathbf{y_k}|\mathbf{x_k}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[-\frac{1}{2}\widetilde{F}_{k,ij}^{m} - \beta_{ij} - \frac{1}{2}\kappa_{ij}\right]x_i x_j + \sum_{i=1}^{n}\left[\left(\widetilde{F}_{k}^{m}\widetilde{y}_k\right)_i + \alpha_i\right]x_i
$$
$$
-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}F_{k,ij}^{m}y_{k,i}y_{k,j} \tag{139}
$$

which has an identical functional form to the state PDF of Eq (110). Substituting Eqs (110) and (138) into Eq (117) gives the Bayes measurement update equation

$$
\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(+)x_i x_j + \sum_{i=1}^{n}b_i(+)x_i + c(+) = \sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(-)x_i x_j + \sum_{i=1}^{n}b_i(-)x_i + c(-)
$$
$$
+\sum_{i=1}^{n}\sum_{j=1}^{n}\left[-\frac{1}{2}\widetilde{F}_{k,ij}^{m} - \beta_{ij} - \frac{1}{2}\kappa_{ij}\right]x_i x_j + \sum_{i=1}^{n}\left[\left(\widetilde{F}_{k}^{m}\widetilde{y}_k\right)_i + \alpha_i\right]x_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}F_{k,ij}^{m}y_{k,i}y_{k,j}
$$
$$
+ \ln\left[\frac{\displaystyle\int_{\mathbf{R^n}}\sigma(\mathbf{x_k},t_k|\mathbf{Y_k})d\mathbf{x_k}}{\displaystyle\int_{\mathbf{R^n}}\sigma(\mathbf{y_k}|\mathbf{x_k})\sigma(\mathbf{x_k},t_k|\mathbf{Y_{k-1}})d\mathbf{x_k}}\right] \tag{140}
$$

where the parameters $\mathbf{A}(+)$, $\mathbf{b}(+)$, and $c(+)$ correspond to the post-measurement filtering PDF $\sigma(\mathbf{x}(t_k),t_k|\mathbf{Y_k})$ and the parameters $\mathbf{A}(-)$, $\mathbf{b}(-)$, and $c(-)$ correspond to the pre-measurement filtering PDF $\sigma(\mathbf{x}(t_k),t_k|\mathbf{Y_{k-1}})$ at time t_k. The update measurement equation given by Eq (139) may be expressed generally as

$$
\mathbf{x}^T\mathbf{A}(+)\mathbf{x} + \mathbf{b}^T(+)\mathbf{x} + c(+) =
$$
$$
\mathbf{x}^T\mathbf{A}(-)\mathbf{x} + \mathbf{b}^T(-)\mathbf{x} + c(-) + \mathbf{x}^T\Delta\mathbf{A}\mathbf{x} + \Delta\mathbf{b}^T\mathbf{x} + \Delta\widetilde{c} + \lambda \tag{141}
$$

$$
\lambda = \ln\left[\frac{\displaystyle\int_{\mathbf{R^n}}\sigma(\mathbf{x_k},t_k|\mathbf{Y_k})d\mathbf{x_k}}{\displaystyle\int_{\mathbf{R^n}}\sigma(\mathbf{y_k}|\mathbf{x_k})\sigma(\mathbf{x_k},t_k|\mathbf{Y_{k-1}})d\mathbf{x_k}}\right] \tag{142}
$$

The equivalent update terms for the filtering PDF parameters are given by

$$\mathbf{A}(+) = \mathbf{A}(-) + \mathbf{\Delta A} \tag{143}$$

$$\mathbf{b}(+) = \mathbf{b}(-) + \mathbf{\Delta b} \tag{144}$$

$$\mathrm{c}(+) = \mathrm{c}(-) + \Delta \mathrm{c} \tag{145}$$

$$\mathbf{\Delta A} = -\frac{1}{2}\begin{bmatrix} r_{k,1}^2\mathbf{F_k^m} & r_{k,1}r_{k,2}\mathbf{F_k^m} & r_{k,1}r_{k,3}\mathbf{F_k^m} & 2r_{k,1}\mathbf{F_k^m} & \\ r_{k,2}r_{k,1}\mathbf{F_k^m} & r_{k,2}^2\mathbf{F_k^m} & r_{k,2}r_{k,3}\mathbf{F_k^m} & 2r_{k,2}\mathbf{F_k^m} & \mathbf{0}_{9\times 3} \\ r_{k,3}r_{k,1}\mathbf{F_k^m} & r_{k,3}r_{k,2}\mathbf{F_k^m} & r_{k,3}^2\mathbf{F_k^m} & 2r_{k,3}\mathbf{F_k^m} & \\ & \mathbf{0}_{3\times 9} & & \mathbf{F_k^m} & \mathbf{0}_{3\times 3} \\ & \mathbf{0}_{3\times 9} & & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \tag{146}$$

$$\mathbf{\Delta b} = \begin{bmatrix} r_{k,1}\left[\mathbf{F_k^m y_k}\right]^T & r_{k,2}\left[\mathbf{F_k^m y_k}\right]^T & r_{k,3}\left[\mathbf{F_k^m y_k}\right]^T & \left[\mathbf{F_k^m y_k}\right]^T & \mathbf{0}_{1\times 3} \end{bmatrix}^T \tag{147}$$

$$\Delta \mathrm{c} = -\frac{1}{2}\mathbf{y_k^T F_k^m y_k} + \lambda \tag{148}$$

Section 7.3 outlines a procedure for obtaining the MAP state estimates directly following a sensor measurement update.

7.6 Limitations of Filtering Algorithm

The major limitations of the Yau filtering algorithm exist in the propagation of the state PDF parameters and in the computation of the state estimate based on the MAP estimate. Incorporating additional second-order terms and linear correlation terms in the attitude parameters and gyroscope bias parameters can be achieved by assuming that the (N+3) independent PDF parameters are Gaussian distributed according to Eq (19). In fact the total number of free PDF parameters required is ½N(N + 3) + 3(N + 3) in comparison to the ½N(N + 3) + 9(N + 1) parameter PDF proposed by Markley[27]. The nonlinear ordinary differential equation governing the evolution of each parameter must be evaluated and propagated between successive sensor measurements. The large number of PDF parameters translates to a heavy computational burden which prohibits the real-time filtering of the sensor measurements. Another aspect of the algorithm which is computationally expensive is the state estimation stage. As discussed in Section 7.3 the MAP estimate of the filter states can be

equivalently posed as a constrained NLP problem. This requires using existing NLP software routines to numerically determine the global maximum of the filtering PDF subject to the three unity norm constraints on the columns of the spacecraft attitude matrix. Filtering simulations using the Optimization Toolbox in MATLAB™ show that this numerical optimisation process is either not possible with current computational capabilities as the process terminates at one hundred iterations without converging to a solution, or an alternative optimisation algorithm is required with greater computational efficiency. One possible method of improving the convergence rate is to simultaneously estimate the attitude parameters based on the most recent estimate of the bias parameters, and vice-versa. This procedure was used by Markley[27] for a filtering PDF containing a quadratic term in the bias parameters, a linear term in the attitude parameters, and linear correlation terms. Due to the specific structure of this PDF, the attitude parameters were estimated using the singular value decomposition (SVD) method, and the bias parameters were solved analytically. The more complex structure of the filtering PDF given by Eq (19) does not exhibit such elegance, and therefore it is unlikely that the convergence rate can be improved based on this philosophy.

A major limitation of the Yau filtering algorithm[26] is the higher-order terms generated by the Fokker-Planck equation, and the Gaussian approximation for the filtering PDF. The second-order partial derivative term in Eq (30) results in the introduction of higher-order state variable terms (than those contained in the PDF) into the propagation equations. As demonstrated in Section 7.4 these terms must be either neglected or approximated using a lower-order Taylor series approximation. Neglecting the terms may or may not be significant depending on the values of the state variables, whereas using a Taylor series approximation will further complicate the filtering algorithm by introducing additional terms into the propagation process. These approximation terms are based on the current state estimate, and will consequently be poor when knowledge of the true attitude is poor (for example during the initial filter convergence period). Even if it were possible to develop an equation similar to the Fokker-Planck equation which did not introduce higher order terms, the filter developed throughout this chapter is limited by the assumption that the

process noise and filtering PDF are Gaussian distributed. If this assumption is not suitable then alternative state estimation techniques such as particle filtering[5,37] may achieve faster convergence and possibly better steady-state estimation accuracy. Particle filtering does not assume a continuous filtering PDF with a specific functional form but rather approximates an arbitrary continuous distribution with a finite set of support weights. The primary advantage of this approach is the flexibility of the filtering PDF to adapt with time to varying structures which in general are not Gaussian distributions.

Further research is required to evaluate the Cramer-Rao lower bound[5,27,37] on the estimation error covariance from the Fisher information matrix, in order to asses the asymptotic statistical efficiency of the proposed attitude estimation algorithm.

7.7 Conclusion

This chapter presented a novel algorithm for spacecraft attitude estimation based on a sensor suite comprising a three-axis magnetometer and a three-axis rate gyroscope assembly. Computational limitations of the algorithm were discussed in detail and recommendations for future research were proposed.

7.8 References

[1] Jazwinski, A. (1970). *Stochastic processes and filtering theory*. USA, Academic Press, Inc.

[2] Gelb, A., ed. (1974). *Applied optimal estimation*. USA: MIT Press.

[3] Sorenson, H.W. (1980). *Parameter estimation: principles and problems*. USA: Marcel Dekker, Inc.

[4] Crassidis, J.L. and Junkins, J.L. (2003). *Optimal estimation of dynamic systems*. USA: CRC Press LLC.

[5] Ristic, B., Arulampalam, S., and N. Gordon. (2004). *Beyond the Kalman filter: particle filters for tracking applications*. USA: Artech House Publishers.

[6] Markley, F.L., Crassidis, J.L., and Y. Cheng. Survey of nonlinear attitude estimation methods. *The Journal of Guidance, Control, and Dynamics*, 30(1): 12-28.

[7] Shuster, M.D., and S.D. Oh. (1981). Three-axis attitude determination from vector observations. *The Journal of Guidance, Control, and Dynamics*, 4(1): 70-77.

[8] Markley, F.L. (1988). Attitude determination using vector observations and the singular value decomposition. *The Journal of the Astronautical Sciences*, 36(3): 245-258.

[9] Shuster, M.D. (1989). Maximum likelihood estimation of spacecraft attitude. *The Journal of the Astronautical Sciences*, 37(1): 79-88.

[10] Markley, F.L., and Mortari, D. (2000). Quaternion attitude estimation using vector observations. *The Journal of the Astronautical Sciences*, 48(2): 359-380.

[11] Oshman, Y., and F.L. Markley. (1999). Sequential attitude and attitude-rate estimation using integrated-rate parameters. *The Journal of Guidance, Control, and Dynamics*, 22(3): 385-394.

[12] Harman, R.R., and I.Y. Bar-Itzhack. (1999). Pseudolinear and state-dependent Riccati equation filters for angular rate estimation. *The Journal of Guidance, Control, and Dynamics*, 22(5): 723-725.

[13] Bar-Itzhack. (2001). Classification of algorithms for angular velocity estimation. *The Journal of Guidance, Control, and Dynamics*, 24(3): 436-443.

[14] Azor, R., Bar-Itzhack, I.Y., Deutschmann, J.K., and R.R. Harman. (2001). Angular-rate estimation using delayed quaternion measurements. *The Journal of Guidance, Control, and Dynamics*, 24(2): 214-218.

[15] Rao, G.N., Alex, T.K., and M.S. Bhat. (2002). Incremental-angle and angular velocity estimation using a star sensor. *The Journal of Guidance, Control, and Dynamics*, 25(3): 433-441.

[16] Kalman, R.E. (1960). A new approach to linear filtering and prediction problems. Journal of Basic Engineering, 82(1): 35-45.

[17] Lefferts, E.J., Markley, F.L., and M.D. Shuster. (1982). Kalman filtering for spacecraft attitude estimation. *The Journal of Guidance, Control, and Dynamics*, 5(5): 417-429.

[18] Bar-Itzhack, I.Y., and Y. Oshman. (1985). Attitude determination from vector observations: quaternion estimation. *IEEE Transactions on Aerospace and Electronic Systems*, 21(1): 128-136.

[19] Gai, E., Daly, K., Harrison, J., and L. Lemos. (1985). Star-sensor-based satellite attitude/attitude rate estimator. *The Journal of Guidance, Control, and Dynamics*, 8(5): 560-565.

[20] Vathsal, S. (1987). Spacecraft attitude determination using a second-order nonlinear filter. *The Journal of Guidance, Control, and Dynamics*, 10(6): 559-566.

[21] Fisher, H.L., Shuster, M.D., and T.E. Strikwerda, T.E. (1989). Attitude determination for the Star Tracker Mission. *Proceedings of the AAS/AIAA Astrodynamics Conference*, Vermont, USA. AAS Paper 89-365, 139-150.

[22] Shuster, M.D. (1990). Kalman filtering of spacecraft attitude and the QUEST model. *The Journal of the Astronautical Sciences*, 38(3): 377-393.

[23] Markley, F.L. (2003). Attitude error representations for Kalman filtering. *The Journal of Guidance, Control, and Dynamics*, 26(2): 311-317.

[24] Crassidis, J.L., and F.L. Markley. (2003). Unscented filtering for spacecraft attitude estimation. *The Journal of Guidance, Control, and Dynamics*, 26(4): 536-542.

[25] Yau, S. S-T., Yau, S-T., and C. Yan. (2004). Linear filtering with nonlinear observations. *Proceedings of the IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, 2112-2117.

[26] Yau, S-T., and S. S-T. Yau. (2004). Nonlinear filtering and time varying Schrodinger equation I. *IEEE Transactions on Aerospace and Electronic Systems*, 40(1): 284-292.

[27] Markley, F.L. (2005). Attitude filtering on SO(3). *Proceedings of the AAS Malcolm D. Shuster Astronautics Symposium*, New York, USA AAS Paper 05-460, 149-174.

[28] Yau, S. S-T., and L. Jia. (2005). Filtering problem with nonlinear observations and drift terms equal to gradient vector field plus affine vector field. *Proceedings of the IEEE Conference on Decision and Control*, Spain, 5865-5870.

[29] Sorensen, D.C. (1982). Newton's method with a model trust region modification. *SIAM Journal of Numerical Analysis*, 19(2): 409-426.

[30] Psiaki, M. L. (2000). Attitude-determination filtering via extended quaternion estimation. *The Journal of Guidance, Control, and Dynamics*, 23(2): 206-214.

[31] Gordon, N.J., Salmond, D.J., and A.F. Smith. (1993). Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *Proceedings of the Institute of Electrical Engineers*, 140: 107-113.

264

[32] Liu, J.S., and R. Cheng. (1998). Sequential Monte Carlo methods for dynamic systems. Journal of the American Statistical Association, 93(443): 1032-1044.

[33] Doucet, A., Godsill, S., and C. Andrieu. (2000). On sequential Monte Carlo sampling methods for Bayesian filtering. *Journal of Statistics and Computing*, 10(3): 197-208.

[34] Arulampalam, M.S., Maskell, S., Gordon, N., and T. Clapp. (2002). A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking. *IEEE Transactions on Signal Processing*, 50(2): 174-188.

[35] Cheng, Y., and J.L. Crassidis. (2004). Particle filtering for sequential spacecraft attitude estimation. Proceedings of the AIAA Guidance Navigation, and Control Conference, Rhode Island, USA, AIAA Paper 2004-5337.

[36] Rapoport, I., and Y. Oshman. (2005). Fault-tolerant particle filtering by using interacting multiple model-based Rao-Blackwellization. *The Journal of Guidance, Control, and Dynamics*, 28(6): 1171-1177.

[37] Oshman, Y., and A. Carmi. (2006). Attitude estimation from vector observations using genetic-algorithm-embedded quaternion particle filter. *The Journal of Guidance, Control, and Dynamics*, 29(4): 879-891.

[38] Markley, F.L., Cheng, Y., Crassidis, J.L., and Y. Oshman. Averaging quaternions. *The Journal of Guidance, Control, and Dynamics*, 30(4): 1193-1196.

[39] Farrenkopf, R.L. (1978). Analytic stead-state accuracy solutions for two common spacecraft attitude estimators. *The Journal of Guidance, Control, and Dynamics*, 1(4): 282-284.

Chapter 8   Conclusion

In this thesis novel feedback control strategies and attitude estimation algorithms have been developed for a three-axis stabilised spacecraft attitude control system.

The first novel contribution is the development of a robust feedback control strategy in Section 3.8 for tracking an arbitrary time-varying reference attitude trajectory, in the presence of dynamic model uncertainty in the spacecraft inertia matrix, actuator magnitude constraints, bounded persistent external disturbances, and state estimation error. The proposed control strategy contains a parameter which is dynamically adjusted to ensure global asymptotic stability of the overall closed-loop system. A stability proof was presented based on Lyapunov's direct method, in conjunction with Barbalat's lemma. Simulation results using MATLAB™ verify the performance of the proposed control algorithm. These results demonstrate nearly asymptotic rejection of the persistent disturbances and boundedness of the trajectory tracking errors, dependent upon the sensor noise levels. Further research includes assessing the performance of the control algorithm based on a recursive state estimation strategy.

The second novel contribution is the development of a direct adaptive control strategy in Chapter 5 for tracking an arbitrary time-varying reference attitude trajectory, in the presence of dynamic model uncertainty in the spacecraft inertia matrix. The proposed strategy incorporates a bounded-gain-forgetting (BGF) composite parameter update law with a dynamic weighting matrix, which guarantees global exponential stability of the overall closed-loop system. A stability proof was presented based on Lyapunov's direct method, in conjunction with Barbalat's lemma. Further research includes simulation studies to assess the capabilities and performance of the proposed adaptive control strategy.

The third novel contribution is the development of an optimal control strategy in Chapter 6 for spacecraft attitude maneuvers. The dynamic model consists of a

rigid-body spacecraft equipped with a redundant reaction wheel assembly. A performance index is proposed which represents the total electrical energy consumed by the reaction wheels over the attitude maneuver interval. Pontrayagin's minimum principle[9] is used to formulate the necessary conditions for optimality, in which the control torques are subject to time-varying magnitude constraints. Necessary conditions for the optimality of singular sub-arcs and associated singular controls are established using Kelley's necessary condition[8]. The two-point boundary value problem (TPBVP) is formulated using Pontryagin's minimum principle. Further research includes simulation studies to assess the capabilities and performance of the proposed optimal control strategy.

The fourth novel contribution is the development of an attitude estimation algorithm in Chapter 7 which estimates the spacecraft attitude parameters and sensor bias parameters from three-axis magnetometer and three-axis rate gyroscope measurement data. The algorithm assumes a Gaussian distributed filtering probability density function (PDF) and magnetometer measurement model. Ordinary differential equations were developed for propagation of the filtering PDF parameters between measurements based on the Fokker-Planck equation. Update equations were developed using Bayes theorem to incorporate information in the three-axis magnetometer measurements. Simulation results, however, were not obtained due to the heavy computational burden of the filtering and state estimation stages. Further research is required to evaluate the Cramer-Rao lower bound on the estimation error covariance, in order to assess the statistical efficiency of the proposed attitude estimation algorithm.