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Finite difference approximations for fractional Fokker-Planck equation ⋆

S. Chen [a,b], F. Liu [a,c,*], P. Zhuang [a] V. Anh [c]

[a] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, China*

[b] *Department of Mathematics, Quanzhou Normal University, Quanzhou 362000, China*

[c] *School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia*

Abstract

Fractional Fokker-Planck equation is used in many physical transport problems take place under the influence of an external force field. In this paper we examine some practical numerical methods to solve a class of initial-boundary value fractional Fokker-Planck equation on a finite domain. The solvability, stability, consistency, and convergence of these methods are discussed. The stability is proved by the energy method. Two numerical examples using these finite difference methods are also presented and compared with the exact analytical solution.

Key words: fractional Fokker-Planck equation, finite difference approximation, solvability, convergence, stability, the energy method.

1 Introduction

Brownian motion in the presence of an external force field $F(x) = -v'(x)$ is usually described in terms of the Fokker-Plank equation (FPE) (see [20])

$$\frac{\partial w}{\partial t} = \left[\frac{\partial}{\partial x}\frac{v'(x)}{m\eta_1} + K_1\frac{\partial^2}{\partial x^2}\right] w(x,t) \tag{1}$$

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* Corresponding author.
Email address: f.liu@qut.edu.au or fwliu@xmu.edu.cn (F. Liu).

which defines the probability density function (pdf) $w(x,t)$ to find the test particle at a certain position x at a given time t. In the (1), m denotes the mass of the particle, K_1 denotes the diffusion constant associated with the transport process, and the friction coefficient η_1 is a measure for the interaction of the particle with its environment. The basic properties of the FPE are the exponential decay of the modes, the Einstein relations which are intimately connected with the fluctuation-dissipation theorem and with linear response, and the Gaussian evolution in the force-free case (see [12], [13], and [14]). For example, in the force-free case, i.e., $v(x) = const$, the corresponding diffusion process is governed by Fick's second law, leading to the linear time dependence

$$< x^2(t) >= 2K_1 t \tag{2}$$

of the mean square displacement; this hallmark of Gaussian diffusion is a consequence of the central limit theorem. In a variety of systems one find that the (2) is violated. Instead, diffusion in such systems is characterized by the power-law time dependence

$$< x^2(t) >= \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha, \qquad \alpha \neq 1$$

of the mean square displacement (see [2], and [13]). This form is connected with broad, lévy-type transport statistics, ruled by the paramount generalized central limit theorem (see [2]). According to the value of the anomalous diffusion exponent α, one distinguishes subdiffusion ($0 < \alpha < 1$) and superdiffusion ($\alpha > 1$). In what follows, the first case is considered which is also referred to as dispersive transport (see [22]). Experimental evidence for such slow diffusion has been found for transport on percolation clusters (see [8]), a bead immersed in a polymeric network (see [1]), or for charge carrier transport in amorphous semiconductors (see [18], and [4]), just to mention a few (see [2], and [13]). Thereby the conventional FPE cannot describe the anomalous diffusion. As a model for subdiffusion in an external potential field $v(x)$, the fractional Fokker-Planck equation (FFPE)

$$\frac{\partial w}{\partial t} =_0 D_t^{1-\alpha} \left[\frac{\partial}{\partial x} \frac{v'(x)}{m\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right] w(x,t) \tag{3}$$

has recently been suggested (see [12], [24], [15], and [23]). Here, $K_\alpha > 0$ denotes the generalized diffusion coefficient of dimension $[K_\alpha] = cm^2 sec^{-\alpha}$, and η_α is the generalized friction coefficient with $[\eta_\alpha] = sec^{\alpha-2}$. The Eq. (3) uses a Riemann-Liouville fractional derivative of order $1-\alpha$, defined by

$${}_0D_t^{1-\alpha} w(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{w(x,s)}{(t-s)^{1-\alpha}} ds \tag{4}$$

where $0 \le \alpha < 1$ (see [17], [21], [16], and [6]), $\Gamma(x)$ is the Gamma function. It is worthwhile considering the example of a subdiffusive, harmonically bound particle, i.e. the subdiffusive motion in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$ which exerts a restoring force on the test particle. The FFPE can be rewritten as follows (see[14]):

$$\frac{\partial w}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}x + \frac{\partial^2}{\partial x^2}\right]w(x,t), \tag{5}$$

which is established according to the subdiffusive generalisation of the Ornstein-Uhlenbeck process. For convenience, let us only consider the case of K_α is constant. There have been some attempts on deriving numerical methods and analysis techniques for the anomalous subdiffusion equation. Yuste and Acedo [25] proposed an explicit finite difference method and a new Von Neumann-type stability analysis for the anomalous subdiffusion equation. Langlands and Henry [9] also investigated this problem and proposed an implicit numerical scheme (L1 approximation). However, effective numerical methods and supporting error analyses for the anomalous subdiffusion equation are still limited. Zhuang and Liu et al. [26] proposed a new implicit numerical method (INM) and two solution techniques for improving the order of convergence of the INM for the anomalous subdiffusion equation. Chen an Liu et al. [3] also proposed an implicit difference approximation (IDA) and a Fourier method for analyzing the stability and convergence of the IDA.

The FFPE have recently been treated by a number of authors. Liu et al. (see [10] and [11]) presented practical numerical methods to solve the space FFPE. Metzler et al. (see [12]) considered the FFPE(3) and the solution was obtained by using the separation of variables. However, the analytic solutions of most FFPE cannot be obtained explicitly, and published papers on the numerical solution of the FFPE(3) are sparse. This motivates us to consider effective numerical methods for the FFPE(3).

2 Some Implicit Approximations and Consistency

In this section we introduce some implicit approximations for solving the FFPE

$$\frac{\partial w}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{v'(x)}{m\eta_\alpha} + K_\alpha\frac{\partial^2}{\partial x^2}\right]w(x,t), \quad d \le x \le b, \quad 0 < t \le T, \tag{6}$$

subject to the initial condition

$$w(x,0) = \varphi(x), \quad d \le x \le b, \tag{7}$$

and the boundary conditions

$$w(d,t) = g_1(t), \quad w(b,t) = g_2(t), \quad 0 < t \leq T. \tag{8}$$

Let $t_n = n\tau$ denotes the integration time $t_n > 0$, $h = (b-d)/M$ is the grid size in space, where M is a positive integer, with $x_i = d + ih$ for $i = 0, \cdots, M$. Define w_i^n as the numerical approximation to $w(x_i, t_n)$. The initial condition is set by $w_i^0 = \varphi(x_i)$. The boundary conditions are set by $w_0^n = g_1(t_n)$ and $w_M^n = g_2(t_n)$.

For convenience, let $f(x) = \frac{v'(x)}{m\eta_\alpha}$. We assume that the problem (6)-(8) has a unique solution $w(x,t) \in C_{x,t}^{2,1}([d,b] \times [0,T])$, then can rewrite the (3) in the following equivalent form (see [13], [5], or Appendix)

$${}_0D_t^\alpha w(x,t) - \frac{w(x,0)t^{-\alpha}}{\Gamma(1-\alpha)} = \frac{\partial}{\partial x} f(x) w(x,t) + K_\alpha \frac{\partial^2}{\partial x^2} w(x,t). \tag{9}$$

To approximate the Eq. (9) we use the second order central difference scheme for the second order spatial derivative

$$\frac{\partial^2}{\partial x^2} w(x_i, t_n) \sim \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{h^2} + O(h^2).$$

In this section, we assume that the function $f(x)$ is not positive and decrease monotonously in the interval $[d,b]$. We introduce three kind implicit approximations.

(1) ***Grünwald−Letnikov*** **and the backward Euler implicit approximation**

We use the *Grünwald−Letnikov* definition (see [19]) to approximate the time fractional derivative:

$$\begin{aligned} &{}_0D_t^\alpha w(x_i, t_n) - \tfrac{w(x_i,0)t_n^{-\alpha}}{\Gamma(1-\alpha)} \\ &= {}_0D_t^\alpha (w(x,t) - w(x,0))|_{(x_i,t_n)} \\ &\sim \tau^{-\alpha} \sum_{k=0}^{n} u_k \left[w_i^{n-k} - w_i^0 \right] + O(\tau), \end{aligned}$$

where $u_k = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}$.

The first order spatial derivative is approximated using the backward difference scheme.

$$\frac{\partial}{\partial x}f(x_i)w(x_i,t_n) \sim \frac{f_i w_i^n - f_{i-1}w_{i-1}^n}{h} + O(h).$$

The first kind implicit approximation for the FFPE is determined by *Grünwald–Letnikov* and the backward difference approximation (GL-BDIA):

$$\begin{aligned} &\tau^{-\alpha}\left[w_i^n + \sum_{k=1}^{n-1} u_k w_i^{n-k} - \sum_{k=0}^{n-1} u_k w_i^0\right] \\ = &\frac{f_i w_i^n - f_{i-1}w_{i-1}^n}{h} + K_\alpha \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{h^2}, \\ &i = 1, 2, \cdots, M-1, \end{aligned} \tag{10}$$

$$w_i^0 = \varphi(x_i), \quad 1 \le i \le M, \tag{11}$$

$$w_0^n = g_1(t_n), \quad w_M^n = g_2(t_n), \quad n \ge 1. \tag{12}$$

The GL-BDIA defined by (10)-(12) is consistent with the FFPE, which gives a local truncation error of $O(\tau + h)$.

(2) L-1 approximation and the central difference implicit approximation

The time fractional derivative is approximated by the L-1 approximation (see [17]), which is valid for $0 \le \alpha < 1$. Explicitly, the L-1 approximation for the fractional derivative of order α with respect to time at $t = t_n$ is given by (see [17], [9])

$$\begin{aligned} &{}_0D_t^\alpha w(x_i,t_n) - \frac{w(x_i,0)t_n^{-\alpha}}{\Gamma(1-\alpha)} \\ \sim &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[w_i^n - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})w_i^k - a_{n-1}w_i^0\right] + O(\tau^{2-\alpha}) \end{aligned}$$

where $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$.
The first order spatial derivative is approximated using the central difference approximation.

$$\frac{\partial}{\partial x}f(x_i)w(x_i,t_n) \sim \frac{f_{i+1}w_{i+1}^n - f_{i-1}w_{i-1}^n}{2h} + O(h^2).$$

The second kind implicit approximation for the fractional Fokker-Planck equation is determined by L-1 approximation and the central difference approximation (L1-CDIA)

$$\begin{aligned}&\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[w_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})w_i^k-a_{n-1}w_i^0\right]\\&=\tfrac{f_{i+1}w_{i+1}^n-f_{i-1}w_{i-1}^n}{2h}+K_\alpha\tfrac{w_{i+1}^n-2w_i^n+w_{i-1}^n}{h^2},\\&\quad i=1,2,\cdots,M-1,\end{aligned}\tag{13}$$

$$w_i^0=\varphi(x_i),\quad 1\le i\le M,\tag{14}$$

$$w_0^n=g_1(t_n),\quad w_M^n=g_2(t_n),\quad n\ge 1.\tag{15}$$

The L1-CDIA defined by (13)-(15) is consistent with the FFPE, which gives a local truncation error of $O(\tau^{2-\alpha}+h^2)$.

(3) L-1 approximation and the backward difference implicit approximation

In the first kind implicit numerical method, we replace the *Grünwald − Letnikov* approximation with the L-1 approximation. Then we can establish the third implicit approximation (called L1-BDIA):

$$\begin{aligned}&\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[w_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})w_i^k-a_{n-1}w_i^0\right]\\&=\tfrac{f_iw_i^n-f_{i-1}w_{i-1}^n}{h}+K_\alpha\tfrac{w_{i+1}^n-2w_i^n+w_{i-1}^n}{h^2},\\&\quad i=1,2,\cdots,M-1,\end{aligned}\tag{16}$$

$$w_i^0=\varphi(x_i),\quad 1\le i\le M,\tag{17}$$

$$w_0^n=g_1(t_n),\quad w_M^n=g_2(t_n),\quad n\ge 1.\tag{18}$$

The L1-BDIA defined by (16)-(18) is consistent with the FFPE, which gives a local truncation error of $O(\tau^{2-\alpha}+h)$.

Remark 1: If the function $f(x)$ is not negative and decrease monotonously in the interval $[d,b]$, we replace the backward difference scheme with the forward difference scheme in the first kind implicit approximation. Then we can establish the *Grünwald − Letnikov* and the forward difference approximation (GL-FDIA):

$$\begin{aligned}&\tau^{-\alpha}\left[w_i^n+\sum_{k=1}^{n-1}u_kw_i^{n-k}-\sum_{k=0}^{n-1}u_kw_i^0\right]\\&=\tfrac{f_{i+1}w_{i+1}^n-f_iw_i^n}{h}+K_\alpha\tfrac{w_{i+1}^n-2w_i^n+w_{i-1}^n}{h^2},\\&\quad i=1,2,\cdots,M-1.\end{aligned}\tag{19}$$

$$w_i^0 = \varphi(x_i), \quad 1 \le i \le M, \tag{20}$$

$$w_0^n = g_1(t_n), \quad w_M^n = g_2(t_n), \quad n \ge 1. \tag{21}$$

The GL-FDIA defined by (19)-(21) is consistent with the FFPE, which gives a local truncation error of $O(\tau + h)$.

Remark 2: If the function $f(x)$ is not negative and decrease monotonously in the interval $[d, b]$, we replace the backward difference scheme with the forward difference scheme in the third kind implicit approximation. Then we can establish the L-1 approximation and the forward difference approximation (L1-FDIA):

$$\begin{aligned} &\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[w_i^n - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})w_i^k - a_{n-1}w_i^0\right] \\ &= \tfrac{f_{i+1}w_{i+1}^n - f_i w_i^n}{h} + K_\alpha \tfrac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{h^2}, \\ &\quad i = 1, 2, \cdots, M-1, \end{aligned} \tag{22}$$

$$w_i^0 = \varphi(x_i), \quad 1 \le i \le M, \tag{23}$$

$$w_0^n = g_1(t_n), \quad w_M^n = g_2(t_n), \quad n \ge 1. \tag{24}$$

The L1-FDIA defined by (22)-(24) is consistent with the FFPE, which gives a local truncation error of $O(\tau^{2-\alpha} + h)$.

3 Stability, Solvability and Convergence

Firstly, we introduce some relevant notations and properties (see [7]).
Suppose $u^n = \{u_i^n | 0 \le i \le M, n \ge 0\}$ and $v^n = \{v_i^n | 0 \le i \le M, n \ge 0\}$ are two grid functions. We introduce the following notations:

$$(u_i^n)_x = (u_{i+1}^n - u_i^n)/h, \quad (u_i^n)_{\bar{x}} = (u_i^n - u_{i-1}^n)/h,$$

$$(u^n, v^n) = \sum_{i=1}^{M-1} u_i^n v_i^n h, \qquad \|u^n\| = (u^n, u^n)^{1/2},$$

$$(u^n, v^n] = \sum_{i=1}^{M} u_i^n v_i^n h, \qquad \|u^n]| = (u^n, u^n]^{1/2}.$$

In addition, if $u_0^n = 0$ and $u_M^n = 0$, we have

$$(u^n, (v_{\bar{x}}^n)_x) = -(u_{\bar{x}}^n, v_{\bar{x}}^n], \tag{25}$$

$$\|u^n\|^2 \le \frac{l^2}{8}\|u^n_{\bar{x}}]|^2, \tag{26}$$

where $l = b - d$.

Secondly, we discuss the stability of these approximations.

Let us suppose that $\bar{w}_i^j$ $(i = 0, 1, 2, \cdots, M; j = 0, 1, 2, \cdots, n)$ is the approximate solution of corresponding difference scheme, $\varepsilon_i^j = \bar{w}_i^j - w_i^j$ $(i = 0, 1, 2, \cdots, M; j = 0, 1, 2, \cdots, n)$ denotes corresponding error.

Theorem 1 *(1)If the function $f(x)$ is not positive and decrease monotonously in the interval $[d, b]$, then GL-BDIA defined by (10)-(12) is unconditionally stable, i.e.,*

$$\|\varepsilon^n\|^2 \le \|\varepsilon^0\|^2.$$

(2)If the function $f(x)$ is constant in the interval $[d, b]$, then the L1-CDIA defined by (13)-(15) is unconditionally stable, i.e.,

$$\|\varepsilon^n\|^2 \le \|\varepsilon^0\|^2.$$

PROOF. (1) For the GL-BDIA defined by (10)-(12), its error satisfies

$$\begin{aligned} &\tau^{-\alpha}\left[\varepsilon_i^n + \sum_{k=1}^{n-1} u_k \varepsilon_i^{n-k} - \sum_{k=0}^{n-1} u_k \varepsilon_i^0\right] \\ &= \frac{f_i\varepsilon_i^n - f_{i-1}\varepsilon_{i-1}^n}{h} + K_\alpha \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{h^2}, \quad i = 1, 2, \cdots, M-1 \end{aligned} \tag{27}$$

and $\varepsilon_0^n = \varepsilon_M^n = 0$ $(\forall n \in \mathrm{N})$.

Multiplying (27) by $h\varepsilon_i^n$ and summing up for i from 1 to $M-1$, we obtain

$$\begin{aligned} &\tau^{-\alpha} h \sum_{i=1}^{M-1}\left[\varepsilon_i^n + \sum_{k=1}^{n-1} u_k \varepsilon_i^{n-k} - \sum_{k=0}^{n-1} u_k \varepsilon_i^0\right]\varepsilon_i^n \\ &= h\sum_{i=1}^{M-1} \frac{f_i\varepsilon_i^n - f_{i-1}\varepsilon_{i-1}^n}{h}\varepsilon_i^n + K_\alpha h \sum_{i=1}^{M-1} \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{h^2}\varepsilon_i^n. \end{aligned} \tag{28}$$

Eq. (28) can be rewritten as

$$\begin{aligned} &h\sum_{i=1}^{M-1}(\varepsilon_i^n)^2 = -h\sum_{i=1}^{M-1}\sum_{k=1}^{n-1} u_k \varepsilon_i^{n-k}\varepsilon_i^n + h\sum_{i=1}^{M-1}\sum_{k=0}^{n-1} u_k \varepsilon_i^0 \varepsilon_i^n \\ &+\tau^\alpha h\sum_{i=1}^{M-1} \frac{f_i\varepsilon_i^n - f_{i-1}\varepsilon_{i-1}^n}{h}\varepsilon_i^n + \tau^\alpha K_\alpha h\sum_{i=1}^{M-1} \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{h^2}\varepsilon_i^n. \end{aligned} \tag{29}$$

We have

$$h \sum_{i=1}^{M-1} (\varepsilon_i^n)^2 = \|\varepsilon^n\|^2. \tag{30}$$

Because $u_0 = 1, u_k = (1 - \frac{1+\alpha}{k})u_{k-1}$ and $\sum\limits_{k=0}^{\infty} u_k = 0$, we have $u_k < 0 \quad k = 1, 2, \cdots$, and $\sum\limits_{k=0}^{n-1} u_k > 0$.
From which it follows that

$$\begin{aligned} & -h \sum_{i=1}^{M-1} \sum_{k=1}^{n-1} u_k \varepsilon_i^{n-k} \varepsilon_i^n \\ & \leq -h \sum_{i=1}^{M-1} \sum_{k=1}^{n-1} u_k \tfrac{1}{2} \left[(\varepsilon_i^{n-k})^2 + (\varepsilon_i^n)^2 \right] \\ & = -\tfrac{1}{2} \sum_{k=1}^{n-1} u_k \|\varepsilon^{n-k}\|^2 - \tfrac{1}{2} \sum_{k=1}^{n-1} u_k \|\varepsilon^n\|^2 \end{aligned} \tag{31}$$

and

$$\begin{aligned} & h \sum_{i=1}^{M-1} \sum_{k=0}^{n-1} u_k \varepsilon_i^0 \varepsilon_i^n \\ & \leq h \sum_{i=1}^{M-1} \sum_{k=0}^{n-1} u_k \tfrac{1}{2} \left[(\varepsilon_i^0)^2 + (\varepsilon_i^n)^2 \right] \\ & = \tfrac{1}{2} \sum_{k=0}^{n-1} u_k \left[\|\varepsilon^0\|^2 + \|\varepsilon^n\|^2 \right]. \end{aligned} \tag{32}$$

Because $f(x) \leq 0$ and $f(x)$ decrease monotonously in $[d, b]$, we obtain

$$\begin{aligned} & \tau^\alpha h \sum_{i=1}^{M-1} \tfrac{f_i \varepsilon_i^n - f_{i-1} \varepsilon_{i-1}^n}{h} \varepsilon_i^n \\ & = \tau^\alpha \left[\sum_{i=1}^{M-1} f_i (\varepsilon_i^n)^2 - \sum_{i=1}^{M-1} f_{i-1} \varepsilon_i^n \varepsilon_{i-1}^n \right] \\ & \leq \tau^\alpha \left[\sum_{i=1}^{M-1} f_i (\varepsilon_i^n)^2 - \sum_{i=1}^{M-1} f_{i-1} \tfrac{(\varepsilon_i^n)^2 + (\varepsilon_{i-1}^n)^2}{2} \right] \\ & = \tau^\alpha \left[\sum_{i=1}^{M-1} f_i (\varepsilon_i^n)^2 - \tfrac{1}{2} \sum_{i=1}^{M-1} f_{i-1} (\varepsilon_i^n)^2 - \tfrac{1}{2} \sum_{i=1}^{M-1} f_{i-1} (\varepsilon_{i-1}^n)^2 \right] \\ & = \tau^\alpha \left[\sum_{i=1}^{M-1} f_i (\varepsilon_i^n)^2 - \tfrac{1}{2} \sum_{i=1}^{M-1} f_{i-1} (\varepsilon_i^n)^2 - \tfrac{1}{2} \sum_{i=1}^{M-2} f_i (\varepsilon_i^n)^2 \right] \\ & = \tau^\alpha \left[\tfrac{1}{2} \sum_{i=1}^{M-1} f_i (\varepsilon_i^n)^2 - \tfrac{1}{2} \sum_{i=1}^{M-1} f_{i-1} (\varepsilon_i^n)^2 + \tfrac{1}{2} f_{M-1} (\varepsilon_{M-1}^n)^2 \right] \\ & = \tau^\alpha \left[\tfrac{1}{2} \sum_{i=1}^{M-1} (f_i - f_{i-1}) (\varepsilon_i^n)^2 + \tfrac{1}{2} f_{M-1} (\varepsilon_{M-1}^n)^2 \right] \\ & \leq 0. \end{aligned} \tag{33}$$

Using (25) and (26), we obtain

$$\begin{aligned}
&\tau^\alpha K_\alpha h \sum_{i=1}^{M-1} \tfrac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{h^2} \varepsilon_i^n \\
&= \tau^\alpha K_\alpha h \sum_{i=1}^{M-1} \varepsilon_i^n ((\varepsilon_i^n)_{\bar{x}})_x \\
&= \tau^\alpha K_\alpha \left(\varepsilon^n, ((\varepsilon^n)_{\bar{x}})_x\right) \\
&= -\tau^\alpha K_\alpha \left((\varepsilon^n)_{\bar{x}}, (\varepsilon^n)_{\bar{x}}\right] \\
&= -\tau^\alpha K_\alpha \|(\varepsilon^n)_{\bar{x}}]|^2 \\
&\leq -\tfrac{8\tau^\alpha K_\alpha}{l^2} \|\varepsilon^n\|^2 \\
&\leq 0.
\end{aligned} \tag{34}$$

Taking into account (29)-(34) we conclude that

$$\|\varepsilon^n\|^2 \leq -\sum_{k=1}^{n-1} u_k \|\varepsilon^{n-k}\|^2 + \sum_{k=0}^{n-1} u_k \|\varepsilon^0\|^2. \tag{35}$$

It follows from (35) by induction that

$$\|\varepsilon^n\|^2 \leq \|\varepsilon^0\|^2, \qquad \forall n \in \mathrm{N} \tag{36}$$

In fact, for $n = 1$, (36) is fulfilled obviously. Suppose that

$$\|\varepsilon^k\|^2 \leq \|\varepsilon^0\|^2, \qquad k = 1, 2, \cdots, n-1,$$

then

$$\begin{aligned}
\|\varepsilon^n\|^2 &\leq -\sum_{k=1}^{n-1} u_k \|\varepsilon^{n-k}\|^2 + \sum_{k=0}^{n-1} u_k \|\varepsilon^0\|^2 \\
&\leq -\sum_{k=1}^{n-1} u_k \|\varepsilon^0\|^2 + \sum_{k=0}^{n-1} u_k \|\varepsilon^0\|^2 \\
&= \|\varepsilon^0\|^2.
\end{aligned}$$

According to (36), we obtain that the GL-BDIA defined by (10)-(12) is unconditionally stable

(2) For the the L1-CDIA defined by (13)-(15), its error satisfies

$$\begin{aligned}
&\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[\varepsilon_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \varepsilon_i^k - a_{n-1} \varepsilon_i^0\right] \\
&= \tfrac{f_{i+1}\varepsilon_{i+1}^n - f_{i-1}\varepsilon_{i-1}^n}{2h} + K_\alpha \tfrac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{h^2}, \\
&i = 1, 2, \cdots, M-1
\end{aligned} \tag{37}$$

and $\varepsilon_0^n = \varepsilon_M^n = 0$ $(\forall n \in \mathrm{N})$.

Multiplying (37) by $h\varepsilon_i^n$ and summing up for i from 1 to $M-1$, we obtain

$$\begin{aligned}&\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} h \sum_{i=1}^{M-1}\left[\varepsilon_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\varepsilon_i^k - a_{n-1}\varepsilon_i^0\right]\varepsilon_i^n \\ &= h\sum_{i=1}^{M-1}\tfrac{f_{i+1}\varepsilon_{i+1}^n - f_{i-1}\varepsilon_{i-1}^n}{2h}\varepsilon_i^n + K_\alpha h \sum_{i=1}^{M-1}\tfrac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{h^2}\varepsilon_i^n.\end{aligned} \tag{38}$$

Let $P = \tau^\alpha \Gamma(2-\alpha)$, we can rewrite (38) in the form

$$\begin{aligned}&h\sum_{i=1}^{M-1}(\varepsilon_i^n)^2 = h\sum_{i=1}^{M-1}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\varepsilon_i^k\varepsilon_i^n + a_{n-1}h\sum_{i=1}^{M-1}\varepsilon_i^0\varepsilon_i^n \\ &+Ph\sum_{i=1}^{M-1}\tfrac{f_{i+1}\varepsilon_{i+1}^n - f_{i-1}\varepsilon_{i-1}^n}{2h}\varepsilon_i^n + PK_\alpha h\sum_{i=1}^{M-1}\tfrac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{h^2}\varepsilon_i^n.\end{aligned} \tag{39}$$

Since $a_l > a_{l+1},\ l = 0, 1, \cdots$, we have

$$\begin{aligned}&h\sum_{i=1}^{M-1}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\varepsilon_i^k\varepsilon_i^n \\ &\le h\sum_{i=1}^{M-1}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\tfrac{1}{2}\left[(\varepsilon_i^k)^2 + (\varepsilon_i^n)^2\right] \\ &= \tfrac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|\varepsilon^k\|^2 + \tfrac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|\varepsilon^n\|^2 \\ &= \tfrac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|\varepsilon^k\|^2 + \tfrac{1}{2}(a_0 - a_{n-1})\|\varepsilon^n\|^2.\end{aligned} \tag{40}$$

We obtain

$$\begin{aligned}&a_{n-1}h\sum_{i=1}^{M-1}\varepsilon_i^0\varepsilon_i^n \\ &\le a_{n-1}h\sum_{i=1}^{M-1}\tfrac{1}{2}\left[(\varepsilon_i^0)^2 + (\varepsilon_i^n)^2\right] \\ &= \tfrac{1}{2}a_{n-1}\left[\|\varepsilon^0\|^2 + \|\varepsilon^n\|^2\right].\end{aligned} \tag{41}$$

Because $f(x) = C^*,\ x \in [d, b]$, where C^* is constant, we obtain

$$\begin{aligned}&Ph\sum_{i=1}^{M-1}\tfrac{f_{i+1}\varepsilon_{i+1}^n - f_{i-1}\varepsilon_{i-1}^n}{2h}\varepsilon_i^n \\ &= \tfrac{PC^*}{2}\sum_{i=1}^{M-1}(\varepsilon_i^n\varepsilon_{i+1}^n - \varepsilon_i^n\varepsilon_{i-1}^n) \\ &= \tfrac{PC^*}{2}(\varepsilon_{M-1}^n\varepsilon_M^n - \varepsilon_1^n\varepsilon_0^n) \\ &= 0.\end{aligned} \tag{42}$$

It follows from (34), we have

$$PK_\alpha h\sum_{i=1}^{M-1}\frac{\varepsilon_{i+1}^n-2\varepsilon_i^n+\varepsilon_{i-1}^n}{h^2}\varepsilon_i^n\le 0. \tag{43}$$

Taking into account (30) and (39)-(43) we conclude that

$$\|\varepsilon^n\|^2\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|\varepsilon^k\|^2+a_{n-1}\|\varepsilon^0\|^2. \tag{44}$$

It follows from (44) by induction that

$$\|\varepsilon^n\|^2\le\|\varepsilon^0\|^2,\qquad \forall n\in \mathrm{N}. \tag{45}$$

In fact, for $n=1$, (45) is fulfilled obviously. Suppose that

$$\|\varepsilon^k\|^2\le\|\varepsilon^0\|^2,\qquad k=1,2,\cdots,n-1,$$

then

$$\begin{aligned}&\|\varepsilon^n\|^2\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|\varepsilon^k\|^2+a_{n-1}\|\varepsilon^0\|^2\\&\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|\varepsilon^0\|^2+a_{n-1}\|\varepsilon^0\|^2\\&=(1-a_{n-1})\|\varepsilon^0\|^2+a_{n-1}\|\varepsilon^0\|^2\\&=\|\varepsilon^0\|^2.\end{aligned}$$

According to (45), we obtain that the L1-CDIA defined by (13)-(15) is unconditionally stable.

Using similar method above we have the following results:

Theorem 2 *(1)If the function $f(x)$ is not positive and decrease monotonously in the interval $[d,b]$, the L1-BDIA defined by (16)-(18) is unconditionally stable.*
(2)If the function $f(x)$ is not negative and decrease monotonously in the interval $[d,b]$, the GL-FDIA defined by (19)-(21) is unconditionally stable.
(3)If the function $f(x)$ is not negative and decrease monotonously in the interval $[d,b]$, the L1-FDIA defined by (22)-(24) is unconditionally stable.

Now we prove the solvability of these implicit difference approximations.

Theorem 3 *(1) If the function $f(x)$ is not positive and decrease monotonously in the interval $[d,b]$, then the GL-BDIA defined by (10)-(12) is uniquely solvable.*

(2) If the function $f(x)$ is constant in the interval $[d,b]$, then the L1-CDIA defined by (13)-(15) is uniquely solvable.
(3) If the function $f(x)$ is not positive and decrease monotonously in the interval $[d,b]$, the L1-BDIA defined by (16)-(18) is uniquely solvable.
(4) If the function $f(x)$ is not negative and decrease monotonously in the interval $[d,b]$, the GL-FDIA defined by (19)-(21) is uniquely solvable.
(5) If the function $f(x)$ is not negative and decrease monotonously in the interval $[d,b]$, the L1-FDIA defined by (22)-(24) is uniquely solvable.

PROOF. Since the proven method of the uniquely solvability of the above difference approximations is similar, so we only prove (1) the uniquely solvability of the GL-BDIA defined by (10)-(12).

Since the GL-BDIA is a system of linear algebraic equations at each time level. It suffices to show that the corresponding homogeneous equations:

$$\begin{aligned}&\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[w_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})w_i^k-a_{n-1}w_i^0\right]\\&=\tfrac{f_{i+1}w_{i+1}^n-f_{i-1}w_{i-1}^n}{2h}+K_\alpha\tfrac{w_{i+1}^n-2w_i^n+w_{i-1}^n}{h^2},\\&i=1,2,\cdots,M-1,\end{aligned}\tag{46}$$

$$w_i^0=0,\qquad 0\le i\le M,\tag{47}$$

$$w_0^n=0,\qquad w_M^n=0,\qquad n\ge 1\tag{48}$$

have only zero solution. From the proven procedure of Theorem 1, using (46) and (48), we have

$$\|w^n\|^2\le\|w^0\|^2,\qquad \forall n\in \mathrm{N}.\tag{49}$$

Using (47), we obtain

$$\|w^0\|^2=0.$$

According to (49), we have

$$\|w^n\|^2=0,\qquad n\ge 1.\tag{50}$$

It follows from (50) that

$$w_i^n=0,\qquad 1\le i\le M-1,\qquad n\ge 1.$$

Let $e_i^n=w(x_i,t_n)-w_i^n,\ e^n=\{e_i^n|\ 0\le i\le M,\ n\ge 0\ \}$ is a grid function. Now we give following theorem of convergence.

Theorem 4 *(1) If the function $f(x)$ is not positive and decrease monotonously in the interval $[d,b]$, then the GL-BDIA defined by (10)-(12) is convergence, and there exists a positive constant $C>0$, such that*

$$\|e^n\| \le C(\tau+h).$$

(2) If the function $f(x)$ is constant in the interval $[d,b]$, then the L1-CDIA defined by (13)-(15) is convergence, and there exists a positive constant $C>0$, such that

$$\|e^n\| \le C(\tau^{2-\alpha}+h^2).$$

PROOF. (1) According to the results from discussing consistency of the GL-BDIA defined by (10)-(12) in Section 3, we obtain the following error equation:

$$\begin{aligned}&\tau^{-\alpha}\left[e_i^n+\sum_{k=1}^{n-1}u_k e_i^{n-k}-\sum_{k=0}^{n-1}u_k e_i^0\right]\\&=\frac{f_i e_i^n-f_{i-1}e_{i-1}^n}{h}+K_\alpha\frac{e_{i+1}^n-2e_i^n+e_{i-1}^n}{h^2}+r_i^n,\quad i=1,2,\cdots,M-1,\end{aligned}\tag{51}$$

$$e_i^0=0,\qquad 1\le i\le M,\tag{52}$$

$$e_0^n=0,\qquad e_M^n=0,\qquad n\ge 0,\tag{53}$$

where $|r_i^n|\le C_1(\tau+h)$, C_1 is a positive constant.

Multiplying (51) by he_i^n and summing up for i from 1 to $M-1$, we obtain

$$\begin{aligned}&\tau^{-\alpha}h\sum_{i=1}^{M-1}\left[e_i^n+\sum_{k=1}^{n-1}u_k e_i^{n-k}-\sum_{k=0}^{n-1}u_k e_i^0\right]e_i^n\\&=h\sum_{i=1}^{M-1}\frac{f_i e_i^n-f_{i-1}e_{i-1}^n}{h}e_i^n+K_\alpha h\sum_{i=1}^{M-1}\frac{e_{i+1}^n-2e_i^n+e_{i-1}^n}{h^2}e_i^n+h\sum_{i=1}^{M-1}r_i^n e_i^n.\end{aligned}\tag{54}$$

Eq. (54) can be rewritten in the following form

$$\begin{aligned}&h\sum_{i=1}^{M-1}(e_i^n)^2=-h\sum_{i=1}^{M-1}\sum_{k=1}^{n-1}u_k e_i^{n-k}e_i^n+\tau^\alpha h\sum_{i=1}^{M-1}\frac{f_i e_i^n-f_{i-1}e_{i-1}^n}{h}e_i^n\\&+\tau^\alpha K_\alpha h\sum_{i=1}^{M-1}\frac{e_{i+1}^n-2e_i^n+e_{i-1}^n}{h^2}e_i^n+\tau^\alpha h\sum_{i=1}^{M-1}r_i^n e_i^n.\end{aligned}\tag{55}$$

It follows from the proven procedure of Theorem 1 that

$$h\sum_{i=1}^{M-1}(e_i^n)^2=\|e^n\|^2\tag{56}$$

and

$$-h\sum_{i=1}^{M-1}\sum_{k=1}^{n-1}u_k e_i^{n-k}e_i^n \leq -\tfrac{1}{2}\sum_{k=1}^{n-1}u_k\|e^{n-k}\|^2 - \tfrac{1}{2}\sum_{k=1}^{n-1}u_k\|e^n\|^2, \tag{57}$$

$$\tau^\alpha h\sum_{i=1}^{M-1}\frac{f_ie_i^n - f_{i-1}e_{i-1}^n}{h}e_i^n, \leq 0, \tag{58}$$

$$\tau^\alpha K_\alpha h\sum_{i=1}^{M-1}\frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{h^2}e_i^n \leq 0. \tag{59}$$

To discuss the right fourth term in (55), let us first discuss the coefficient u_k. For convenience, let us denote $v_n = -n^\alpha \sum\limits_{k=n}^{\infty} u_k$.

Because $\sum\limits_{k=0}^{\infty} u_k = 0$ and $\sum\limits_{k=0}^{n} u_k = \sum\limits_{k=0}^{n}\begin{pmatrix} k-\alpha-1 \\ k\end{pmatrix} = \begin{pmatrix} n-\alpha \\ n\end{pmatrix}$,

therefore $\quad -\sum\limits_{k=n}^{\infty} u_k = \sum\limits_{k=0}^{n-1} u_k = \begin{pmatrix} n-1-\alpha \\ n-1\end{pmatrix}$.

From which it follow that

$$\frac{-\sum\limits_{k=n}^{\infty} u_k}{-\sum\limits_{k=n+1}^{\infty} u_k} = \begin{pmatrix} n-1-\alpha \\ n-1\end{pmatrix} \div \begin{pmatrix} n-1-\alpha \\ n-1\end{pmatrix} = \frac{n}{n-\alpha} = 1 + \frac{\alpha}{n-\alpha} \tag{60}$$

For $x \in (-1,1)$, we have

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots.$$

Because $0 < \alpha < 1$, therefore we obtain

$$\begin{aligned} &\tfrac{(n+1)^\alpha}{n^\alpha} = (1+\tfrac{1}{n})^\alpha \\ &= 1 + \alpha\tfrac{1}{n} + \tfrac{\alpha(\alpha-1)}{2!}(\tfrac{1}{n})^2 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{3!}(\tfrac{1}{n})^3 + \cdots \\ &< 1 + \tfrac{\alpha}{n}. \end{aligned} \tag{61}$$

Using (60) and (61) we obtain

$$\frac{-\sum\limits_{k=n}^{\infty} u_k}{-\sum\limits_{k=n+1}^{\infty} u_k} > \frac{(n+1)^\alpha}{n^\alpha}.$$

From which it follows that

$$v_n > v_{n+1}. \tag{62}$$

Because

$$\frac{t^{-\alpha}}{\Gamma(1-\alpha)} =_0 D_t^{\alpha}(t-0)^0 = \lim_{\tau\to 0, n\tau=t} \tau^{-\alpha}\sum_{k=0}^{n} u_k,$$

for $t = 1$, we obtain

$$\lim_{n\to\infty} n^{\alpha}\sum_{k=0}^{n} u_k = \frac{1}{\Gamma(1-\alpha)}.$$

Thus we have

$$\begin{aligned}
&\lim_{n\to\infty} v_{n+1}\\
&= -\lim_{n\to\infty}(n+1)^{\alpha}\sum_{k=n+1}^{\infty} u_k\\
&= -\lim_{n\to\infty}(\tfrac{n+1}{n})^{\alpha} n^{\alpha}\sum_{k=n+1}^{\infty} u_k\\
&= -\lim_{n\to\infty} n^{\alpha}\sum_{k=n+1}^{\infty} u_k\\
&= \lim_{n\to\infty} n^{\alpha}\sum_{k=0}^{n} u_k\\
&= \tfrac{1}{\Gamma(1-\alpha)}.
\end{aligned} \tag{63}$$

Taking into account (63) and (63) we conclude that

$$v_n > \frac{1}{\Gamma(1-\alpha)}, \qquad n \in \mathrm{N}.$$

From which it follows that

$$-\sum_{k=n}^{\infty} u_k > \frac{1}{n^{\alpha}\Gamma(1-\alpha)}, \qquad n \in \mathrm{N}. \tag{64}$$

Using (64) we have

$$
\begin{aligned}
&\tau^\alpha h\sum_{i=1}^{M-1} r_i^n e_i^n\\
&\leq \tau^\alpha h\sum_{i=1}^{M-1}\left[\frac{\tau^\alpha}{-2\sum\limits_{k=n}^{\infty}u_k}(r_i^n)^2+\frac{-\sum\limits_{k=n}^{\infty}u_k}{2\tau^\alpha}(e_i^n)^2\right]\\
&=\frac{\tau^{2\alpha}h}{-2\sum\limits_{k=n}^{\infty}u_k}\sum_{i=1}^{M-1}(r_i^n)^2-\frac{1}{2}\sum_{k=n}^{\infty}u_k\|e^n\|^2\\
&\leq \frac{(\tau^{2\alpha}n^\alpha\Gamma(1-\alpha))}{2}h(M-1)C_1^2(\tau+h)^2-\frac{1}{2}\sum_{k=n}^{\infty}u_k\|e^n\|^2\\
&\leq \frac{lT^\alpha C_1^2\Gamma(1-\alpha)}{2}\tau^\alpha(\tau+h)^2-\frac{1}{2}\sum_{k=n}^{\infty}u_k\|e^n\|^2\\
&=\frac{C_2}{2}\tau^\alpha(\tau+h)^2-\frac{1}{2}\sum_{k=n}^{\infty}u_k\|e^n\|^2.
\end{aligned} \tag{65}
$$

Taking into account (65) and (55)-(59) we conclude that

$$\|e^n\|^2\leq -\sum_{k=1}^{n-1}u_k\|e^{n-k}\|^2+C_2\tau^\alpha(\tau+h)^2. \tag{66}$$

It follows from (66) by induction that

$$\|e^n\|^2\leq C_2(-\sum_{k=n}^{\infty}u_k)^{-1}\tau^\alpha(\tau+h)^2 \qquad \forall n\in \mathrm{N}. \tag{67}$$

In fact, for $n=1$, (66) is fulfilled obviously. Suppose that

$$\|e^k\|^2\leq C_2(-\sum_{s=k}^{\infty}u_s)^{-1}\tau^\alpha(\tau+h)^2, \qquad k=1,2,\cdots,n-1,$$

then

$$
\begin{aligned}
&\|e^n\|^2\leq -\sum_{k=1}^{n-1}u_k\|e^{n-k}\|^2+C_2\tau^\alpha(\tau+h)^2\\
&\leq\ -\sum_{k=1}^{n-1}u_kC_2(-\sum_{s=n-k}^{\infty}u_s)^{-1}\tau^\alpha(\tau+h)^2+C_2\tau^\alpha(\tau+h)^2\\
&\leq -\sum_{k=1}^{n-1}u_kC_2(-\sum_{s=n}^{\infty}u_s)^{-1}\tau^\alpha(\tau+h)^2+C_2\tau^\alpha(\tau+h)^2\\
&=(1+\sum_{k=n}^{\infty}u_k)C_2(-\sum_{s=n}^{\infty}u_s)^{-1}\tau^\alpha(\tau+h)^2+C_2\tau^\alpha(\tau+h)^2\\
&=C_2(-\sum_{s=n}^{\infty}u_s)^{-1}\tau^\alpha(\tau+h)^2.
\end{aligned}
$$

Using (64) and (67), we obtain

$$\begin{aligned}\|e^n\|^2 &\le C_2 n^\alpha \Gamma(1-\alpha)\tau^\alpha(\tau+h)^2\\ &\le C_2 T^\alpha \Gamma(1-\alpha)(\tau+h)^2\\ &= C^2(\tau+h)^2.\end{aligned}$$

Therefore, we obtain

$$\|e^n\| \le C(\tau+h).$$

(2) Now, we prove the convergence of the L1-CDIA defined by (13)-(15). According to the results from discussing consistency of the L1-CDIA in Section 3, we obtain the error equation.

$$\begin{aligned}&\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[e_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})e_i^k - a_{n-1}e_i^0\right]\\ &= \tfrac{f_{i+1}e_{i+1}^n - f_{i-1}e_{i-1}^n}{2h} + K_\alpha \tfrac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{h^2} + r_i^n,\\ &\quad i = 1, 2, \cdots, M-1,\end{aligned} \tag{68}$$

$$e_i^0 = 0, \qquad 1 \le i \le M, \tag{69}$$

$$e_0^n = 0, \qquad e_M^n = 0, \qquad n \ge 0 \tag{70}$$

where $|r_i^n| \le C_3(\tau^{2-\alpha}+h^2)$, C_3 is a positive constant.

Multiplying (68) by he_i^n and summing up for i from 1 to $M-1$, we obtain

$$\begin{aligned}&\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)} h \sum_{i=1}^{M-1}\left[e_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})e_i^k - a_{n-1}e_i^0\right]e_i^n\\ &= h\sum_{i=1}^{M-1}\tfrac{f_{i+1}e_{i+1}^n - f_{i-1}e_{i-1}^n}{2h}e_i^n + K_\alpha h \sum_{i=1}^{M-1}\tfrac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{h^2}e_i^n + h\sum_{i=1}^{M-1} r_i^n e_i^n.\end{aligned} \tag{71}$$

Eq. (71) can be rewritten in the following form

$$\begin{aligned}&h\sum_{i=1}^{M-1}(e_i^n)^2 = h\sum_{i=1}^{M-1}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})e_i^k e_i^n\\ &\quad + Ph\sum_{i=1}^{M-1}\tfrac{f_{i+1}e_{i+1}^n - f_{i-1}e_{i-1}^n}{2h}e_i^n\\ &\quad + PK_\alpha h\sum_{i=1}^{M-1}\tfrac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{h^2}e_i^n + Ph\sum_{i=1}^{M-1} r_i^n e_i^n.\end{aligned} \tag{72}$$

It follows from the proven procedure of Theorem 1 that

$$h\sum_{i=1}^{M-1}(e_i^n)^2 = \|e^n\|^2 \tag{73}$$

and

$$\begin{aligned} & h \sum_{i=1}^{M-1} \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) e_i^k e_i^n \\ & \leq \frac{1}{2} \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \|e^k\|^2 + \frac{1}{2}(a_0 - a_{n-1}) \|e^n\|^2, \end{aligned} \tag{74}$$

$$Ph \sum_{i=1}^{M-1} \frac{f_{i+1} e_{i+1}^n - f_{i-1} e_{i-1}^n}{2h} e_i^n = 0, \tag{75}$$

$$PK_\alpha h \sum_{i=1}^{M-1} \frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{h^2} e_i^n \leq 0. \tag{76}$$

To discuss the right fourth term in (72), let us first discuss the coefficient a_n. Since we have for $n \geq 2$

$$(1-\frac{1}{n})^{1-\alpha} = 1-\frac{1-\alpha}{n}+\frac{(1-\alpha)(-\alpha)}{2!}(-\frac{1}{n})^2+\frac{(1-\alpha)(-\alpha)(-\alpha-1)}{3!}(-\frac{1}{n})^3+\cdots$$

So we can obtain for $n \geq 2$

$$\begin{aligned} & a_{n-1} - \frac{1-\alpha}{n^\alpha} \\ & = n^{1-\alpha} \left[1 - \frac{1-\alpha}{n} - (1-\frac{1}{n})^{1-\alpha} \right] \\ & = n^{1-\alpha} \left[-\frac{(1-\alpha)(-\alpha)}{2!}(-\frac{1}{n})^2 - \frac{(1-\alpha)(-\alpha)(-\alpha-1)}{3!}(-\frac{1}{n})^3 - \cdots \right] \\ & > 0. \end{aligned}$$

For $n = 1$, we have $a_{n-1} - \frac{1-\alpha}{n^\alpha} = \alpha > 0$.
Therefore

$$a_{n-1} > \frac{1-\alpha}{n^\alpha}, \qquad n \in \mathrm{N}. \tag{77}$$

Using (77) we have

$$\begin{aligned}
&Ph\sum_{i=1}^{M-1} r_i^n e_i^n \\
&\le Ph\sum_{i=1}^{M-1}\left[\tfrac{P}{2a_{n-1}}(r_i^n)^2+\tfrac{a_{n-1}}{2P}(e_i^n)^2\right] \\
&= \tfrac{P^2h}{2a_{n-1}}\sum_{i=1}^{M-1}(r_i^n)^2+\tfrac{a_{n-1}h}{2}\sum_{i=1}^{M-1}(e_i^n)^2 \\
&\le \tfrac{(\Gamma(2-\alpha))^2\tau^{2\alpha}n^\alpha}{2(1-\alpha)}h(M-1)C_3^2(\tau^{2-\alpha}+h^2)^2+\tfrac{a_{n-1}}{2}\|e^n\|^2 \\
&\le \tfrac{(C_3\Gamma(2-\alpha))^2 lT^\alpha}{2(1-\alpha)}\tau^\alpha(\tau^{2-\alpha}+h^2)^2+\tfrac{a_{n-1}}{2}\|e^n\|^2 \\
&= \tfrac{C_4}{2}\tau^\alpha(\tau^{2-\alpha}+h^2)^2+\tfrac{a_{n-1}}{2}\|e^n\|^2.
\end{aligned} \tag{78}$$

According to (72)-(77) and (78), we get

$$\|e^n\|^2 \le \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|e^k\|^2+C_4\tau^\alpha(\tau^{2-\alpha}+h^2)^2. \tag{79}$$

It follows from (79) by induction that

$$\|e^n\|^2 \le C_4a_{n-1}^{-1}\tau^\alpha(\tau^{2-\alpha}+h^2)^2, \qquad \forall n\in \mathrm{N}. \tag{80}$$

In fact, for $n=1$, (80) is fulfilled obviously. Suppose that

$$\|e^k\|^2 \le C_4a_{k-1}^{-1}\tau^\alpha(\tau^{2-\alpha}+h^2)^2, \qquad k=1,2,\cdots,n-1,$$

then

$$\begin{aligned}
&\|e^n\|^2 \le \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|e^k\|^2+C_4\tau^\alpha(\tau^{2-\alpha}+h^2)^2 \\
&\le \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})C_4a_{k-1}^{-1}\tau^\alpha(\tau^{2-\alpha}+h^2)^2+C_4\tau^\alpha(\tau^{2-\alpha}+h^2)^2 \\
&\le \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})C_4a_{n-1}^{-1}\tau^\alpha(\tau^{2-\alpha}+h^2)^2+C_4\tau^\alpha(\tau^{2-\alpha}+h^2)^2 \\
&= (1-a_{n-1})C_4a_{n-1}^{-1}\tau^\alpha(\tau^{2-\alpha}+h^2)^2+C_4\tau^\alpha(\tau^{2-\alpha}+h^2)^2 \\
&= C_4a_{n-1}^{-1}\tau^\alpha(\tau^{2-\alpha}+h^2)^2.
\end{aligned}$$

According to (77) and (80), we can obtain

$$\begin{aligned}
&\|e^n\|^2 \le C_4\tfrac{n^\alpha}{1-\alpha}\tau^\alpha(\tau^{2-\alpha}+h^2)^2 \\
&\le \tfrac{C_4T^\alpha}{1-\alpha}(\tau^{2-\alpha}+h^2)^2 \\
&= C^2(\tau^{2-\alpha}+h^2)^2.
\end{aligned}$$

Therefore

$$\|e^n\| \le C(\tau^{2-\alpha} + h^2).$$

Similar to the proof of Theorem 4, we have the following results:

Theorem 5 ***(1)*** *If the function $f(x)$ is not positive and decreases monotonously in the interval $[d, b]$, the L1-BDIA defined by (16)-(18) is convergence, and there exists a positive constant $C > 0$, such that*

$$\|e^n\| \le C(\tau^{2-\alpha} + h).$$

(2) *If the function $f(x)$ is not negative and decreases monotonously in the interval $[d, b]$, the GL-FDIA defined by (19)-(21) is convergence, and there exists a positive constant $C > 0$, such that*

$$\|e^n\| \le C(\tau + h).$$

(3) *If the function $f(x)$ is not negative and decreases monotonously in the interval $[d, b]$, the L1-FDIA defined by (22)-(24) is convergence, and there exists a positive constant $C > 0$, such that*

$$\|e^n\| \le C(\tau^{2-\alpha} + h).$$

4 Numerical examples

To demonstrate the effectiveness of these methods for solving fractional Fokker-Planck equation, we consider the following two examples.

Example 1 Consider the following FFPE

$$\frac{\partial w(x,t)}{\partial t} ={}_0 D_t^{1-\alpha} \left[\frac{\partial}{\partial x}(-1) + \frac{\partial^2}{\partial x^2} \right] w(x,t)$$

$$0 \le x \le 1, \qquad t > 0,$$

subject to the initial condition

$$w(x,0) = x(1-x), \qquad 0 \le x \le 1,$$

and the boundary conditions

$$w(0,t) = -\frac{3t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \qquad t > 0.$$

$$w(1,t) = -\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \qquad t > 0,$$

where $f(x) = -1,\ x \in [0,1],\ K_{\alpha} = 1,\ l = 1.$
The exact solution of the above problem is

$$w(x,t) = x(1-x) + (2x-3)\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}$$

which may be verified by direct differentiation and substitution in the fractional differential equation, using the formula

$$_0D_t^{1-\alpha}[x^p] = \frac{\Gamma(p+1)}{\Gamma(p+\alpha)} x^{p+\alpha-1}.$$

Table 1
The maximum error $\|e^n\|$ for the GL-BDIA defined by (10)-(12) and the L1-CDIA defined by (13)-(15) and the effect of the grid size reduction at time $t = 100$ ($\alpha = 0.5$)

τ	h	The error $\|e^n\|$ for the the GL-BDIA (10)-(12)	The error $\|e^n\|$ for the L1-CDIA (13)-(15)
$\frac{1}{5}$	$\frac{1}{5}$	1.68e-002	2.69e-006
$\frac{1}{10}$	$\frac{1}{10}$	8.77e-003	1.02e-006
$\frac{1}{20}$	$\frac{1}{20}$	4.49e-003	3.98e-007
$\frac{1}{40}$	$\frac{1}{40}$	2.27e-003	1.59e-007

In Table 1, we using the GL-BDIA defined by (10)-(12) and the L1-CDIA defined by (13)-(15) in Example 1 with $\alpha = 0.5$ and $t = 100$ by setting $\tau = h = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, respectively. We compare the error $\|e^n\|$ of tthe GL-BDIA defined by (10)-(12) with the error $\|e^n\|$ of L1-CDIA defined by (13)-(15). It is observed that the numerical result coincides with the theoretical analysis.

Example 2 Consider the following FFPE

$$\frac{\partial w(x,t)}{\partial t} =_0 D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\left(\frac{1}{x+1}\right) + \frac{\partial^2}{\partial x^2}\right] w(x,t),$$

$$0 \le x \le 1, \qquad t > 0,$$

subject to the initial condition

$$w(x,0) = (x+1)^3, \qquad 0 \le x \le 1,$$

and the boundary conditions

$$w(0,t) = 1 + \frac{8t^\alpha}{\Gamma(1+\alpha)}, \qquad t > 0,$$
$$w(1,t) = 8 + \frac{16t^\alpha}{\Gamma(1+\alpha)}, \qquad t > 0,$$

where $f(x) = \frac{1}{x+1},\ x \in [0,1],\ K_\alpha = 1,\ l = 1.$
The exact solution of the above problem is

$$w(x,t) = (x+1)^3 + 8(x+1)\frac{t^\alpha}{\Gamma(1+\alpha)}.$$

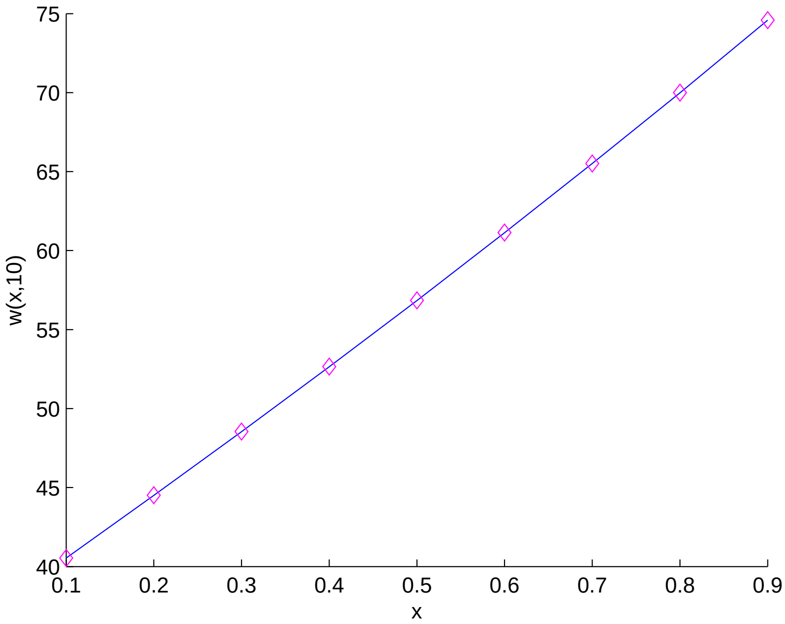

Fig. 1. A comparison of the numerical solution of the L1-FDIA defined by (22)-(24)) and exact solution for the Example 2.

In Figure 1 , we compute up to time level $t = 10$ by setting $\tau = h = 0.1$ and using L1-FDIA defined by (22)-(24)) for the Example 2 with $\alpha = 0.6$. The diamonds denote the results of the numerical tests and the solid lines correspond to the exact analytical solution. It is observed that the numerical solution (L1-FDIA) is in good agreement with exact solution.

5 Conclusion

In this paper several finite difference implicit approximations for the fractional Fokker-Planck equation in a bounded domain have been described and demonstrated. We prove that these finite difference implicit approximations are unconditionally stable and convergent. These techniques can be applied to solve fractional partial differential equations.

6 Appendix

To prove that the (3) is equivalent to the (9), let us first introduce some properties of the Riemann-Liouville fractional derivative (see[19]). For $0 < \alpha < 1$, the following conclusions hold

$$ {}_0D_t^{-\alpha}\left({}_0D_t^{\alpha}g(t)\right) = g(t) - \left[{}_0D_t^{\alpha-1}g(t)\right]_{t=0}\frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{81} $$

$$ {}_0D_t^{\alpha}\left({}_0D_t^{-\alpha}g(t)\right) = g(t) \tag{82} $$

and

$$ {}_0D_t^{\alpha-1}\left(\frac{dg}{dt}\right) = {}_0D_t^{\alpha}g(t) - \frac{g(0)t^{-\alpha}}{\Gamma(1-\alpha)} \tag{83} $$

Theorem 6 *If $w(x,t) \in C_{x,t}^{2,1}([d,b] \times [0,T]$, then can rewrite*

$$ \frac{\partial w}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}f(x) + K_\alpha\frac{\partial^2}{\partial x^2}\right]w(x,t) \tag{84} $$

in the following equivalent form

$$ {}_0D_t^{\alpha}w(x,t) - \frac{w(x,0)t^{-\alpha}}{\Gamma(1-\alpha)} = \left[\frac{\partial}{\partial x}f(x) + K_\alpha\frac{\partial^2}{\partial x^2}\right]w(x,t) \tag{85} $$

PROOF. (1) If (84) holds, then

$$ {}_0D_t^{\alpha-1}\left(\frac{\partial w}{\partial t}\right) = {}_0D_t^{\alpha-1}\left\{{}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}f(x) + K_\alpha\frac{\partial^2}{\partial x^2}\right]w(x,t)\right\} \tag{86} $$

Using (81) and (83), (86) becomes

$$
{}_0D_t^{\alpha} w(x,t) - \frac{w(x,0)t^{-\alpha}}{\Gamma(1-\alpha)} = \left[\frac{\partial}{\partial x} f(x) + K_\alpha \frac{\partial^2}{\partial x^2}\right] w(x,t)
- {}_0D_t^{-\alpha} \left[\frac{\partial}{\partial x} f(x) + K_\alpha \frac{\partial^2}{\partial x^2}\right] w(x,t)\Big|_{t=0} \frac{t^{-\alpha}}{\Gamma(1-\alpha)}
$$

If the function $g(t)$ is one times continuously differentiable in the closed interval $[0,t]$, then

$$
\begin{aligned}
{}_0D_t^{-\alpha} g(t)|_{t=0} &= \lim_{t\to 0+0} \tfrac{1}{\Gamma(\alpha)} \int_0^t \tfrac{g(s)}{(t-s)^{1-\alpha}} ds \\
&= \lim_{t\to 0+0} \left[\tfrac{t^\alpha}{\Gamma(1+\alpha)} g(0) + \tfrac{1}{\Gamma(1+\alpha)} \int_0^t g'(s)(t-s)^\alpha ds\right] \\
&= 0
\end{aligned}
$$

Since $w(x,t) \in C_{x,t}^{2,1}([d,b] \times [0,T]$, so we have

$$
-{}_0D_t^{-\alpha} \left[\frac{\partial}{\partial x} f(x) + K_\alpha \frac{\partial^2}{\partial x^2}\right] w(x,t)\Big|_{t=0} = 0
$$

Therefore (85) holds.
(2) If (85) holds, then

$$
{}_0D_t^{1-\alpha} \left[{}_0D_t^{\alpha} w(x,t) - \frac{w(x,0)t^{-\alpha}}{\Gamma(1-\alpha)}\right] ={}_0 D_t^{1-\alpha} \left[\frac{\partial}{\partial x} f(x) + K_\alpha \frac{\partial^2}{\partial x^2}\right] w(x,t)
$$

According to (82) and (83), therefore (84) holds.

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