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Generation of Bivariate Rayleigh and Nakagami-m Fading Envelopes

C. Tellambura, *Member, IEEE,* and A. D. S. Jayalath

Abstract—The letter presents an algorithm for generating bivariate Nakagami-m distributed fading envelopes with any desired power cross correlation. For this algorithm, the fading index m should be a positive integer ($m = 1$ for Rayleigh fading). Its applications include dual-branch selection combining diversity, dual-branch switch diversity systems, and wireless channel modeling.

Index Terms—Correlation, diversity, Nakagami-m fading, Rayleigh fading.

I. INTRODUCTION

DUAL DIVERSITY systems are perhaps the most widely used diversity techniques because of the diminishing returns yielded by higher order diversity systems. Rayleigh and Nakagami-m distributions are widely accepted for system performance studies. The effect of correlation on dual diversity systems has been analyzed by several authors, often deriving completely analytical solutions [1]. However, it is useful to simulate the performance of such systems and this requires the generation of bivariate Nakagami-m distributed random variables (RV's). This problem will be addressed in this letter. A recent letter [2] derived a procedure for the generation of two equal power Rayleigh-fading envelopes with any desired correlation-coefficient. The method of [2] was extended and improved in [3]. In this letter, we develop a more general procedure for the generation of two Rayleigh or Nakagami-m fading envelopes with any desired power cross-correlation. For this procedure, the fading index m can be any positive integer ($m = 1$ for Rayleigh fading) and the average powers need not necessarily be equal.

II. PROBLEM AND THE PROCEDURE

The pdf of a pair of correlated Nakagami-m distributed fading envelopes is given by [4], [5]

$$f(x, y) = \frac{4(xy)^m e^{-(\eta_2 x^2 + \eta_1 y^2)/\eta_1 \eta_2 (1-\rho)}}{\Gamma(m)\eta_1\eta_2(1-\rho)(\eta_1\eta_2\rho)^{(m-1)/2}}$$
$$\cdot I_{m-1}\left\{\frac{2\sqrt{\rho}\,xy}{\sqrt{\eta_1\eta_2}\,(1-\rho)}\right\} \quad (1)$$

where $x, y \geq 0$; $\eta_1 = \overline{x^2}/m$; $\rho = cov(x^2, y^2)/\sqrt{var(x^2)var(y^2)}$; and $m \geq 0.5$. Here ρ is the power correlation coefficient. Our problem is to generate (x, y) pairs distributed as above for integer m. For this, we modify the

well-known inverse transform method and develop a complete procedure for generating (x, y) pairs.

A. Inverse Transform Method

A general method for generating an RV x with continuous cdf $F(x)$ is the inverse transform method [6]:

1) Compute $\hat{x}$ the root of $F(x) = u$ where $u \sim U(0, 1)$

where $U(0, 1)$ is the uniform distribution from 0 to 1 and $a \sim b$ is read: "a is distributed as b." This method requires one to generate a uniform RV u in $(0, 1)$ and solve $F(x) = u$. It can easily be shown that the distribution of $\hat{x}$ is $F(x)$ [7]. The procedure can be repeated as many times as is necessary, but its usability depends on how easy it is to compute the root, $\hat{x}$. For instance, consider generating an exponential RV with $F(x) = 1 - e^{-x}$: from the above algorithm, if u is an uniform RV, then $\hat{x} = -\ln(1-u) = -\ln(u)$[1] is an exponential RV. By generating an uniform RV and using this relationship, an exponential RV can be generated. This method is particularly useful in this case since the above root can be obtained analytically. In general, $\hat{x}$ is computed using numerical techniques.

B. Modified Inverse Transform Method for a Pair of RV's

We now extend the above method for generating a pair of correlated RV's. Suppose the joint cdf $F(x, y)$, the marginal cdf $F_x(x)$ and the conditional cdf $F(y|x)$ are all known.

1) Generate $\hat{x} \sim F_x(x)$.
2) Compute $\hat{y}$ from $F(y|\hat{x}) = u$ where $u \sim U(0, 1)$.

It can readily be shown that the pair $(\hat{x}, \hat{y})$ is distributed as $F(x, y)$ (the heuristic explanation is that steps 1 and 2 generate two RV's distributed as $f(x)$ and $f(y|x)$ respectively; but $f(x, y) = f(x)f(y|x)$). Note that step 1 itself may use the inverse transform method. The cdf $F(y|x)$ is a function of x. The required steps are to generate $\hat{x} \sim F_x(x)$ and $u \sim U(0, 1)$ and to obtain $\hat{y}$ from both u and $\hat{x}$. So to apply the method to the problem at hand, we need to find the conditional cdf $F(y|x)$ and a method for inverting $F(y|x)$.

C. Complete Procedure

In (1), the marginal densities of x and y are given by the Nakagami distribution. For instance, the pdf of x is

$$f(x) = \frac{2}{\Gamma(m)\eta_1^m} x^{2m-1} e^{-x^2/\eta_1} \qquad x \geq 0 \qquad (2)$$

which is obtained by integrating (1) over y.

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[1] If $u \sim U(0, 1)$, then $(1 - u) \sim U(0, 1)$. Thus, $1 - u$ can be replaced by u.

From the definition [6, eqs. (7–14)], the conditional pdf is $f(y|x) = f(x, y)/f(x)$. Combining (1) and (2) thus yields

$$f(y|x) = \frac{2\eta_1^{m-1} y^m e^{-(\rho\eta_2 x^2 + \eta_1 y^2)/\eta_1\eta_2(1-\rho)}}{x^{m-1}\eta_2(1-\rho)(\eta_1\eta_2\rho)^{(m-1)/2}}$$

$$\cdot I_{m-1}\left\{\frac{2\sqrt{\rho}\,xy}{\sqrt{\eta_1\eta_2}\,(1-\rho)}\right\}. \tag{3}$$

Therefore, the conditional cdf is

$$F(y|x) = \int_0^y f(y|x)\,dy$$

$$= 1 - Q_m\left(x\sqrt{\frac{2\rho}{(1-\rho)\eta_1}}, \; y\sqrt{\frac{2}{(1-\rho)\eta_2}}\right) \tag{4}$$

where $Q_m(x, y)$ is the generalized Marcum Q) function and (4) follows from [8, eqs. (2-1-122)].

From step 2 in the modified inverse transform method, it follows that $\hat{y}$ is obtained by solving the equation

$$Q_m(a, t) = c \tag{5}$$

where $a = \hat{x}\sqrt{2\rho/[(1-\rho)\eta_1]}$; $t = y\sqrt{2/[(1-\rho)\eta_2]}$; and $0 \le c = 1 - u \le 1$. Since $Q_m(a, t)$ monotonically decreases from 1 to 0 as t varies from 0 to ∞, (5) has a unique solution. This root can be readily obtained by the use of the bisection method [9, pp. 353–354], which requires an initial interval that contains the root. If $Q_m(a, a) < c$, then the root is in $[0, a]$. If $Q_m(a, a) > c$, then we try the intervals $[a, 2a]$, $[2a, 4a]$ and so on, until the root is initially bracketed. Finally, the initial interval is halved repeatedly until the root is located with enough accuracy.

So the full procedure for generating the correlated Nakagami-m fading envelopes is as follows.

1) Generate

$$\hat{x} = \sqrt{-\eta_1 \sum_{j=1}^m \ln(1-u_j)} = \sqrt{-\eta_1 \ln\left(\prod_{j=1}^m u_j\right)}$$

where $u_j \sim U(0, 1)$ for $j = 1, \cdots, m$.
2) Compute $\hat{y}$ from $Q_m(a, t) = 1 - u$ where $u \sim U(0, 1)$, e.g., (5).

Step 1 follows from the fact that x^2 is a Gamma RV and, for integer m, is equal to the sum of m exponential RV's. The right-most formula saves the evaluation of the log function m times. Note that, as pointed out by an anonymous reviewer, Step 1 can also be replaced by the square root of the sum of $2m$ squared Gaussian RV's. Step 2 involves solving (5) for $\hat{y}$. The pair $(\hat{x}, \hat{y})$ is distributed according to the pdf in (1).

III. APPLICATIONS

A. Estimated Correlation Coefficient

To test the accuracy of the above algorithm, we generate $N = 10^5$ pairs of (x_j, y_j), $j = 1, \cdots, N$, using it for a given ρ. The correlation coefficient can be estimated as

$$\hat{\rho} = \sum_{j=1}^N (x_j^2 - \mu_1)(y_j^2 - \mu_2) \Big/$$

$$\sqrt{\sum_{j=1}^N (x_j^2 - \mu_1)^2 \sum_{j=1}^N (y_j^2 - \mu_2)^2} \tag{6}$$

TABLE I
ESTIMATED ρ

ρ	$m=1$		$m=2$		$m=3$	
	$\hat{\rho}$	%Error	$\hat{\rho}$	%Error	$\hat{\rho}$	%Error
0.1	0.103	3.0	0.1012	1.2	0.0969	-3.1
0.3	0.3036	1.2	0.2975	0.8	0.3064	2.1
0.5	0.4947	-1.0	0.5037	0.7	0.4956	-0.9
0.7	0.7004	0.06	0.7005	0.07	0.7008	0.11
0.9	0.9007	0.08	0.9002	0.02	0.9003	0.03

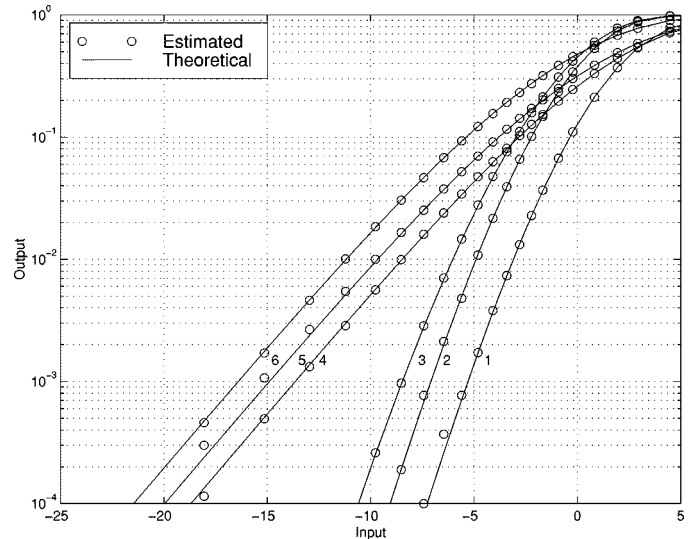

Fig. 1. Selection diversity combiner performance. For all curves, $\overline{y^2} = 1$. For curves 1–3, $m = 3$ and for curves 4–6 $m = 1$. Curve 1: $\overline{x^2} = 2$ and $\rho = 0.2$. Curve 2: $\overline{x^2} = 1$ and $\rho = 0.1$ Curve 3: $\overline{x^2} = 1$ and $\rho = 0.7$ Curve 4: $\overline{x^2} = 2$ and $\rho = 0.1$. Curve 5: $\overline{x^2} = 2$ and $\rho = 0.5$. Curve 6: $\overline{x^2} = 1$ and $\rho = 0.5$.

where μ_1 and μ_2 are the estimated means of x_j^2 and y_j^2 and the theoretical means $\overline{x^2}$ and $\overline{y^2}$ are normalized to unity. Table I shows the relationship between the actual correlation coefficient and the estimated one. Using the percentage relative error given by $100(\hat{\rho}-\rho)/\rho$, one can see that *extremely* accurate generation of correlated fading envelopes is possible—the relative error can be as low as 0.02%! Note that the relative error increases as ρ decreases but this trend can be overcome by increasing N.

B. Selection Diversity Combining

Selection diversity is the simplest and perhaps the most widely used diversity technique. It is, for example, a selection diversity technique that is used in the current digital cellular system IS-54. The output of a dual branch selection diversity combiner is given by $z = \max(x, y)$. The cdf of z can easily be obtained if the joint cdf of (x, y) is known: [6, eqs. (6-54)] $F(z) = F_{xy}(z, z)$. Tan and Beaulieu [5] derive an infinite series expression for the bivariate cdf $F_{xy}(x, y)$. Therefore, using [5, eq. (3)] the cdf of the output is given analytically by

$$F(z) = \frac{(1-\rho)^m}{\Gamma(m)} \sum_{k=0}^\infty \rho^k P\left(k+1, \frac{z^2}{\eta_1(1-\rho)}\right)$$

$$\cdot P\left(k+1, \frac{z^2}{\eta_2(1-\rho)}\right) \frac{\Gamma(m+k)}{k!} \tag{7}$$

where $P(\alpha, x)$ is defined in [5]. Fig. 1 shows the simulated system performance and that which was computed using (7). Again, an *almost exact* match occurs between the two. Note, however, that only 10^5 pairs have been generated. So the simulation points in the region $F(z) < 10^{-3}$ may not be estimated very accurately. Thus, the procedure derived in Section II-C can be used to study the performance of dual diversity systems.

IV. CONCLUSIONS

In this letter, we have developed a procedure for generating correlated Nakagami-m fading signal pairs. The fading severity index m can be any positive integer and the average signal powers need not necessarily be equal. This procedure will be useful for studying the effect of correlation on dual diversity systems such as selection combining, switch and stay, switch and examine, and so on. Several improvements to the procedure in Section II-C are possible. Its major shortcoming is the use of the bisection method in (5), which can be slow. First, (5) may be solved using a more efficient method such as the secant method. The Newton–Raphson method may also work since the derivative of $Q_m(a, t)$ is known. Second, the solving of (5) may be completely avoided. The rejection method [7, p. 565]

may be used to generate $F(y|x)$ instead. The improvements have not been pursued further because of the page limitation.

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